

PERIODIC HOMOGENIZATION OF NONSYMMETRIC LÉVY-TYPE PROCESSES

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In this paper we study homogenization problem for strong Markov processes on \mathbb{R}^d having infinitesimal generators

$$\begin{aligned} \mathcal{L}f(x) &= \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z| \leq 1\}}) k(x, z) \Pi(dz) \\ &\quad + \langle b(x), \nabla f(x) \rangle, \quad f \in C_b^2(\mathbb{R}^d) \end{aligned}$$

in periodic media, where Π is a nonnegative measure on \mathbb{R}^d that does not charge the origin 0, satisfies $\int_{\mathbb{R}^d} (1 \wedge |z|^2) \Pi(dz) < \infty$ and can be singular with respect to the Lebesgue measure on \mathbb{R}^d . Under a proper scaling we show the scaled processes converge weakly to Lévy processes on \mathbb{R}^d . The results are a counterpart of the celebrated work (*Asymptotic Analysis for Periodic Structures* (1978) North-Holland; *Ann. Probab.* **13** (1985) 385–396) in the jump-diffusion setting. In particular, we completely characterize the homogenized limiting processes when $b(x)$ is a bounded continuous multivariate 1-periodic \mathbb{R}^d -valued function, $k(x, z)$ is a nonnegative bounded continuous function that is multivariate 1-periodic in both x and z variables and, in spherical coordinate $z = (r, \theta) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}$,

$$\mathbb{1}_{\{|z| > 1\}} \Pi(dz) = \mathbb{1}_{\{r > 1\}} \varrho_0(d\theta) \frac{dr}{r^{1+\alpha}}$$

with $\alpha \in (0, \infty)$ and ϱ_0 being any finite measure on the unit sphere \mathbb{S}^{d-1} in \mathbb{R}^d . Different phenomena occur depending on the values of α ; there are five cases: $\alpha \in (0, 1)$, $\alpha = 1$, $\alpha \in (1, 2)$, $\alpha = 2$ and $\alpha \in (2, \infty)$.

1. Introduction. In the celebrated work [6, 7] the authors studied the periodic homogenization of a diffusion $X := (X_t)_{t \geq 0}$ on \mathbb{R}^d generated by the following second-order elliptic operator

$$\hat{\mathcal{L}}f(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial f(x)}{\partial x_i},$$

where the coefficients $(a_{ij}(x))_{1 \leq i, j \leq d}$ and $b(x) := (b_i(x))_{1 \leq i \leq d}$ are bounded and multivariate 1-periodic (i.e., they can be viewed as bounded functions defined on the d -dimensional torus $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$). Under the assumptions that each $a_{ij}(\cdot)$ has bounded second derivatives, each $b_i(\cdot)$ has bounded first derivatives, and $(a_{ij}(x))_{1 \leq i, j \leq d}$ is uniformly elliptic; they showed

$$\{X_t^\varepsilon - \varepsilon^{-1} \bar{b}t : t \geq 0\}$$

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converges weakly as $\varepsilon \rightarrow 0$ to a driftless Brownian motion with covariance matrix

$$\left(\int_{\mathbb{T}^d} \sum_{k,l=1}^d \left(\delta_{ki} - \frac{\partial \psi_i(x)}{\partial x_k} \right) a_{kl}(x) \left(\delta_{lj} - \frac{\partial \psi_j(x)}{\partial x_l} \right) \mu(dx) \right)_{1 \leq i, j \leq d}.$$

Here, $X_t^\varepsilon := \varepsilon X_{t/\varepsilon^2}$ for $t \geq 0$, $\mu(dx)$ is the unique invariant probability measure for the quotient process of X on \mathbb{T}^d , $\bar{b} = \int_{\mathbb{T}^d} b(x) \mu(dx)$, and $\psi \in C^2(\mathbb{R}^d)$ is the unique periodic solution to the equation

$$\hat{\mathcal{L}}\psi(x) = b(x) - \bar{b} \quad \text{on } \mathbb{T}^d$$

with $\mu(\psi) := \int_{\mathbb{R}^d} \psi(x) \mu(dx) = 0$. Here and below, we use $:=$ as a way of definition.

The goal of this paper is to study the periodic homogenization of jump diffusions whose infinitesimal generators are of the following form when acting on $C_b^2(\mathbb{R}^d)$:

$$(1.1) \quad \mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z| \leq 1\}}) k(x, z) \Pi(dz) + \langle b(x), \nabla f(x) \rangle.$$

Here, $\Pi(dz)$ is a nonnegative measure on \mathbb{R}^d that does not charge at the origin 0 and satisfies $\int_{\mathbb{R}^d} (1 \wedge |z|^2) \Pi(dz) < \infty$; $b(x)$ is a bounded continuous multivariate 1-periodic \mathbb{R}^d -valued function, and $k(x, z) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$ is a function that is bounded so that $x \mapsto k(x, z)$ is multivariate 1-periodic for each fixed $z \in \mathbb{R}^d$ and

$$(1.2) \quad \lim_{y \rightarrow x} \sup_{z \in \mathbb{R}^d} |k(y, z) - k(x, z)| = 0.$$

Since $k(\cdot, z)$ and $b(\cdot)$ are multivariate 1-periodic, it is easy to verify that $\mathcal{L}f$ is pointwisely well defined as a function on \mathbb{T}^d for every $f \in C^2(\mathbb{T}^d)$. By the maximum principle ($\mathcal{L}, C^2(\mathbb{T}^d)$) can be extended to a closed operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ on $C(\mathbb{T}^d)$. *Throughout the paper the following two assumptions are in force:*

(A1) *There exists a strong Markov process $X := ((X_t)_{t \geq 0}; (\mathbb{P}_x)_{x \in \mathbb{R}^d})$ associated with \mathcal{L} in the sense that, for every $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ with $\mathbb{R}_+ := [0, \infty)$,*

$$\left\{ f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial f}{\partial s}(s, X_s) + \mathcal{L}f(s, \cdot)(X_s) \right) ds, t \geq 0 \right\}$$

is a martingale under \mathbb{P}_x for all $x \in \mathbb{R}^d$ with respect to the natural filtration generated by X .

(A2) *Regarding X as an \mathbb{T}^d -valued process, the process X is exponentially ergodic in the sense that there exist a unique invariant probability measure $\mu(dx)$ and constants $\lambda_1, C_0 > 0$ so that*

$$(1.3) \quad \sup_{x \in \mathbb{T}^d} |\mathbb{E}_x f(X_t) - \mu(f)| \leq C_0 e^{-\lambda_1 t} \|f\|_\infty, \quad t > 0, f \in C_b(\mathbb{T}^d).$$

Assumption **(A1)** is satisfied when the martingale problem for the operator $(\mathcal{L}, C_b^2(\mathbb{R}^d))$ is well posed. The latter has been extensively investigated in the literature; see [11, 13–15, 27] and the references therein. See also [31], Theorem 3.1, for more recent study on the existence of a martingale solution associated with Lévy-type operators whose corresponding canonical process has the strong Markov property. Since \mathbb{T}^d is compact, Assumption **(A2)** is a direct consequence of the irreducibility (i.e., for any $t > 0, x \in \mathbb{R}^d$ and any nonempty open set $U \subset \mathbb{R}^d, \mathbb{P}_x(X_t \in U) > 0$) and the strong Feller property (i.e., for any $f \in B_b(\mathbb{R}^d)$ and $t > 0, x \mapsto \mathbb{E}_x f(X_t)$ is bounded and continuous) of the process X ; see [40], Theorem 1.1. Assumption **(A2)** also holds, if the process X admits a transition density function $p(t, x, y)$ with respect to the Lebesgue measure so that, for any $t > 0$, the function $(x, y) \mapsto p(t, x, y)$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d$ and that there is a nonempty open set $U \subset \mathbb{R}^d$ such that $p(t, x, y) >$

0 for all $t > 0$, $x \in \mathbb{R}^d$ and $y \in U$; see [6], page 365, Theorem 3.1, for a modification of Doebelin's celebrated result. The reader is referred to Section 7.2 for concrete examples on Assumptions **(A1)** and **(A2)**.

In order to deal with the scaling limit of X that requires recentering (which includes cases considered in Section 3.2, Section 4 and Section 5), we need one more assumption:

(A3) For every $f \in C(\mathbb{T}^d)$ with $\mu(f) = 0$, there exists a unique multivariate 1-periodic solution $\psi \in \mathcal{D}(\mathcal{L})$ to

$$(1.4) \quad \mathcal{L}\psi = f \quad \text{on } \mathbb{T}^d \text{ with } \mu(\psi) = 0$$

and

$$(1.5) \quad \|\psi\|_\infty + \|\nabla\psi\|_\infty \leq C_1 \|f\|_\infty,$$

where $C_1 > 0$ is independent of f .

There are a few literature on homogenization of nonlocal operators. We refer readers to [2, 3, 28, 34, 37] for the periodic homogenization results for stable-like operators or the operator with convolution-type kernels. The methods used in the these papers are analytic and called the corrector method. The probabilistic study of homogenization of periodic stable-like processes can be found in [20, 25, 36] via the characteristics of semimartingales, in [21, 23, 24] by SDE driven by Lévy processes or by Poisson random measures and in [22, 39] via the martingale problem method. A closely related topic is homogenization of nonlocal operators or jump diffusions in random media which typically requires a different approach than the periodic media case; see [35, 38], for example. Recently, we have studied homogenization of symmetric stable-like processes in stationary ergodic random media in [10].

The approach of this paper is different from all the papers mentioned above. We will use generator method combined with its connection to martingales. We summarize the novelties of our paper as follows:

(i) Our results reveal that the crucial ingredients for the homogenization of Lévy-type operators are the shape of large jumps for the jumping measures $\Pi(dz)$ and its limiting spherical measure on the unit sphere \mathbb{S}^{d-1} when expressed in spherical coordinates; see, for example, conditions (3.1), (4.1) and (5.1)–(5.3). Compared with the references mentioned above on the homogenization of nonlocal operators (which are mainly concerned on stable-like processes), our results work for more general jump processes with a large class of scaling factors; see the weighted function φ in Section 3, Example 7.2 and Remark 7.3. Moreover, the limiting processes of our homogenization results are also quite general, including *all* stable Lévy processes on \mathbb{R}^d that can be nonsymmetric, degenerate and have singular Lévy measures; see Example 1.1 below and examples in Section 7.1 for more details.

(ii) We will establish the periodic homogenization results for Lévy-type operators \mathcal{L} after suitable scalings that depend only on the tail of $\Pi(dz)$ which gives the rate of large jumps. Our results, in particular, cover the critical case; see Example 1.1(ii). To the best of our knowledge, this is the first time in literature that critical cases have been studied. Moreover, the Lévy measure Π can be singular with respect to the Lebesgue measure on \mathbb{R}^d , and its support can have zero Lebesgue measure.

(iii) Among all the results mentioned above, the process under investigation is either the unique strong solution of a stochastic differential equation or a Feller process on \mathbb{R}^d . In this paper the process is only assumed to solve the martingale problem of $(\mathcal{L}, C_b^2(\mathbb{R}^d))$ and have strong Markov property.

The main results of this paper are Theorems 3.2, 3.4, 4.1 and 5.1 as well as Theorems 6.5 and 6.7. We use the following example, a special case of these much more general results, for

illustration. We first introduce some notations which will be frequently used in the paper. By assumptions on k and Π (see the line immediately after (1.1)),

$$(1.6) \quad b_R(x) := \int_{\{1 < |z| \leq R\}} zk(x, z)\Pi(dz)$$

is well defined for every $R > 1$, and $b_R \in C_b(\mathbb{R}^d)$ is multivariate 1-periodic. Clearly, if $\int_{\{|z| > 1\}} |z|\Pi(dz) < \infty$, then

$$(1.7) \quad b_\infty(x) := \int_{\{|z| > 1\}} zk(x, z)\Pi(dz)$$

is also well defined and is the limit of $b_R(x)$ as $R \rightarrow \infty$. Let $\mathbb{R}_+ := [0, \infty)$ and \mathbb{S}^{d-1} be the unit sphere in \mathbb{R}^d . Denote by $\mathcal{D}([0, \infty); \mathbb{R}^d)$ the space of \mathbb{R}^d -valued right continuous functions having left limits on $[0, \infty)$ and equipped with the Skorohod topology.

EXAMPLE 1.1. Suppose that Assumptions (A1), (A2) and (A3) hold. Let $X := (X_t)_{t \geq 0}$ be the strong Markov process corresponding to the operator \mathcal{L} given by (1.1) with the jumping measure $\Pi(dz)$ such that

$$(1.8) \quad \mathbb{1}_{\{|z| > 1\}}\Pi(dz) = \mathbb{1}_{\{r > 1\}} \frac{1}{r^{1+\alpha}} dr \varrho_0(d\theta),$$

where $\alpha \in (0, \infty)$, $\varrho_0(d\theta)$ is a nonnegative finite measure on \mathbb{S}^{d-1} and (r, θ) denotes the spherical coordinates of $z \in \mathbb{R}^d$.

Denote by $\mu(dx)$ the stationary probability measure for the quotient process of X on \mathbb{T}^d . Define for any $R > 1$,

$$(1.9) \quad \bar{b} := \int_{\mathbb{T}^d} b(x)\mu(dx), \quad \bar{b}_R := \int_{\mathbb{T}^d} b_R(x)\mu(dx), \quad \bar{b}_\infty := \int_{\mathbb{T}^d} b_\infty(x)\mu(dx):$$

(i) Suppose that $k(x, z)$ is a bounded continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ so that $x \mapsto k(x, z)$ is multivariate 1-periodic for each fixed $z \in \mathbb{R}^d$ and $z \mapsto k(x, z)$ is multivariate 1-periodic for each fixed $x \in \mathbb{R}^d$. For any $\varepsilon \in (0, 1]$, define $(Y_t^\varepsilon)_{t \geq 0}$ by

$$Y_t^\varepsilon = \begin{cases} \varepsilon X_{t/\varepsilon^\alpha}, & 0 < \alpha < 1, \\ \varepsilon X_{t/\varepsilon^\alpha} - (\bar{b}_{1/\varepsilon} + \bar{b})t, & \alpha = 1, \\ \varepsilon X_{t/\varepsilon^\alpha} - \varepsilon^{1-\alpha}(\bar{b}_\infty + \bar{b})t, & 1 < \alpha < 2. \end{cases}$$

Then, the process $(Y_t^\varepsilon)_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$, as $\varepsilon \rightarrow 0$, to a (possibly nonsymmetric) α -stable Lévy process $(\bar{X}_t)_{t \geq 0}$ having jumping kernel $\frac{\bar{k}(\theta)}{r^{1+\alpha}} dr \varrho_0(d\theta)$, where $\bar{k} : \mathbb{S}^{d-1} \rightarrow \mathbb{R}_+$ is defined by

$$\bar{k}(\theta) := \int_{\mathbb{T}^d} \bar{k}(x, \theta)\mu(dx), \quad \theta \in \mathbb{S}^{d-1}$$

and $\bar{k} : \mathbb{R}^d \times \mathbb{S}^{d-1} \rightarrow \mathbb{R}_+$ satisfies that, for all $x \in \mathbb{R}^d$ and $\theta \in \mathbb{S}^{d-1}$,

$$(1.10) \quad \bar{k}(x, \theta) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T k(x, (r, \theta)) dr.$$

The infinitesimal generator of the Lévy process $(\bar{X}_t)_{t \geq 0}$ is given by

$$\bar{\mathcal{L}}f(x) = \begin{cases} \int_{\mathbb{R}^d} (f(x+z) - f(x))\bar{k}(z/|z|)\Pi_0(dz), & \alpha \in (0, 1), \\ \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z| \leq 1\}})\bar{k}(z/|z|)\Pi_0(dz), & \alpha = 1, \\ \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle)\bar{k}(z/|z|)\Pi_0(dz), & \alpha \in (1, 2), \end{cases}$$

where $\Pi_0(dz) := \mathbb{1}_{\{r > 0\}} \frac{1}{r^{1+\alpha}} dr \varrho_0(d\theta)$.

Furthermore, if the finite measure ϱ_0 on \mathbb{S}^{d-1} that does not charge on the set of rationally dependent $\theta \in \mathbb{S}^{d-1}$, then we can take $\bar{k}(\theta) \equiv \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} k(x, z) dz \mu(dx)$ for all $\theta \in \mathbb{S}^{d-1}$, which is a constant, in the statement above. Here, we call $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{S}^{d-1}$ is rationally dependent if there is some nonzero $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ so that $\langle m, \theta \rangle = \sum_{i=1}^d m_i \theta_i = 0$. Otherwise, we call θ rationally independent. When $d = 1$, $\mathbb{S}^0 = \{1, -1\}$ so every of its members is rationally independent. In particular, if ϱ_0 does not charge on singletons when $d = 2$ and does not charge on subsets of \mathbb{S}^{d-1} that are of Hausdorff dimension $d - 2$ when $d \geq 3$ (e.g., ϱ_0 is γ -dimensional Hausdorff measure with $\gamma \in (d - 2, d - 1]$), then ϱ_0 does not charge on the set of rationally dependent $\theta \in \mathbb{S}^{d-1}$ and so the result above holds with $\bar{k}(\theta) \equiv \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} k(x, z) dz \mu(dx)$. Moreover, if ϱ_0 is absolutely continuous with respect to the Lebesgue surface measure σ on \mathbb{S}^{d-1} with a bounded Radon–Nikodym derivative, then we can replace the joint continuity assumption on $k(x, z)$ by the continuity of the function $x \mapsto k(x, z)$ and condition (1.2).

(ii) When $\alpha = 2$, define

$$Y_t^\varepsilon := \varepsilon X_{\varepsilon^{-2}|\log \varepsilon|^{-1}t} - \varepsilon^{-1}|\log \varepsilon|^{-1}(\bar{b}_\infty + \bar{b})t, \quad t \geq 0.$$

Suppose that

$$(1.11) \quad k_0 := \lim_{|z| \rightarrow \infty} \int_{\mathbb{T}^d} k(x, z) \mu(dx) > 0.$$

Then, $(Y_t^\varepsilon)_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$, as $\varepsilon \rightarrow 0$, to Brownian motion $(B_t)_{t \geq 0}$ with the covariance matrix $A = \{a_{ij}\}_{1 \leq i, j \leq d}$ such that

$$(1.12) \quad a_{ij} := k_0 \int_{\mathbb{S}^{d-1}} \theta_i \theta_j \varrho_0(d\theta).$$

(iii) When $\alpha > 2$, define

$$Y_t^\varepsilon := \varepsilon X_{t/\varepsilon^2} - \varepsilon^{-1}(\bar{b}_\infty + \bar{b})t, \quad t \geq 0.$$

Then, $(Y_t^\varepsilon)_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$, as $\varepsilon \rightarrow 0$, to a d -dimensional Brownian motion $(B_t)_{t \geq 0}$ with the covariance matrix

$$(1.13) \quad A := \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (z + \psi(x + z) - \psi(x)) \otimes (z + \psi(x + z) - \psi(x)) k(x, z) \Pi(dz) \mu(dx).$$

Here, $\psi \in \mathcal{D}(\mathcal{L})$ is the unique periodic solution on \mathbb{R}^d to the following equation:

$$\mathcal{L}\psi(x) = -b_\infty(x) - b(x) + \bar{b}_\infty + \bar{b}, \quad x \in \mathbb{T}^d$$

with $\mu(\psi) = 0$.

One sufficient condition for **(A1)**, **(A2)** and **(A3)** to hold in this example is that $k(x, z)$ is bounded between two positive constants and that there is a constant $\beta \in (0, 1)$, so that $b(x) = (b_i(x))_{1 \leq i \leq d} \in C_b^\beta(\mathbb{R}^d)$,

$$\sup_{z \in \mathbb{R}^d} |k(x, z) - k(y, z)| \leq c_0 |x - y|^\beta, \quad x, y \in \mathbb{R}^d$$

for some $c_0 > 0$, and

$$\mathbb{1}_{\{|z| \leq 1\}} \Pi(dz) = \mathbb{1}_{\{|z| \leq 1\}} \frac{1}{|z|^{d+\alpha_0}} dz$$

for some $\alpha_0 \in (1, 2)$; see Propositions 7.5 and 7.6.

Motivated by the classical central limit theorem for stable laws (see, e.g., [17], page 161, Theorem 3.7.2; and page 164, Exercise 3.7.2), in order to study the limit behavior of the

scaled process $X^\varepsilon := (\varepsilon X_{t/\varepsilon^\alpha})_{t \geq 0}$, we do not need to recenter it when $\alpha \in (0, 1)$ but do need to recenter it when $\alpha \in [1, 2)$. Moreover, the normalizing factors of the centered terms are different in the critical case $\alpha = 1$ and in the case $\alpha \in (1, 2)$; see [26], Theorem 2.4, for related discussions on limit theorems for additive functions of a Markov chain to stable laws. A recent paper [25] studied the periodic homogenization for stable-like Feller processes under the centering condition on the drift term $b(x)$; see [25], Assumption (H4). We emphasize that such centering condition is commonly assumed in all the quoted papers above, except [21] which only considers symmetric α -stable Lévy noises with $\alpha \in (1, 2)$. (For the periodic homogenization for diffusion processes under the centering condition on the drift term $b(x)$, the reader is referred to [6].) In some sense, studying homogenization problem with general drift, as done in [7, 19] for diffusions, requires normalizing the center first which typically needs much more effort. Concerning the assertion (iii) in Example 1.1, the closely related works in [34, 36] deal with nonlocal operators with convolution-type kernels and Lévy-type operators without drift terms, respectively. The critical case corresponds to the assertion (ii). Note that in the critical case, in contrast to the diffusive case (iii), the scaling factor is $\varepsilon^{-2} |\log \varepsilon|^{-1}$ rather than the standard diffusive scaling ε^{-2} , and the corrector solution does not contribute to the diffusion coefficient of the limiting Brownian motion. Moreover, in this case, because of (1.12), the limiting Brownian motion may be degenerate even under the nondegeneracy assumption on $\mathbb{1}_{\{|z| \leq 1\}} \Pi(dz)$ which is different from the diffusive scaling case (iii) (see Remark 4.2 below). We should mention that all the limit processes have the scaling property; however, different from the cases (ii) and (iii), the limit processes considered in (i) is an α -stable Lévy process which can be nonsymmetric and singular, as the spherical measure $\varrho_0(d\theta)$ can be any finite measure on \mathbb{S}^{d-1} .

Finally, we emphasize that the results of our paper can be regarded as the counterpart in the jump-diffusion setting of the work by [7] which studied periodic homogenization for diffusion processes without assuming the zero averaging condition on the drift term (i.e., $\bar{b} = 0$) which was imposed in [6]. However, since we will treat general jump processes with a large class of scaling factors, there are essential differences which require new ideas. For example, it always takes the diffusion scaling in [7], while, as mentioned above, in the present paper the scaling and the limit process in the nondiffusive cases are determined by the asymptotic behavior of the jumping measure $\Pi(dz)$ at infinity. The α -stable scaling $\varepsilon^{-\alpha}$ with $\alpha \in (0, 2)$ in Example 1.1 is merely a special case (see Example 7.2). Furthermore, we do not assume the uniform ellipticity condition on the nonlocal operator \mathcal{L} of (1.1) in the sense that the support of the jumping measure $\mathbb{1}_{\{|z| > 1\}} \Pi(dz)$ can have zero Lebesgue measure and whose linear span can be a proper linear subspace of \mathbb{R}^d .

The remainder of this paper is organized as follows. In the next section we present an elementary lemma and some properties on the scaled processes under Assumptions (A1) and (A2). Sections 3 and 4 are devoted to the study of the limiting behaviors of the scaled processes under the jump scaling and the diffusive scaling, respectively. In Section 5 we consider homogenization in the critical cases. Roughly speaking, the limiting process is still Brownian motion, but the scaling factor is different from the standard diffusive scaling ε^{-2} . In Section 6 sufficient conditions are given for the key averaging assumption (3.6) of our main results to hold which are also of independent interest. With all the results above at hand, in Section 7.1 we give the proof of the assertions made in Example 1.1. Two more examples are given to illustrate the power of our main results. Sufficient conditions for Assumptions (A1)–(A3) to hold are presented in Section 7.2.

In this paper, for two positive functions f and g , $f \asymp g$ means that f/g is bounded between two positive constants, and $f \leq g$ means that f/g is bounded by a positive constant. We use $[a]$ to denote the largest integer not exceeding a . For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$. For any vector $x, y \in \mathbb{R}^d$, $x \otimes y$ denotes its tensor product which is equivalent to an $d \times d$ -matrix defined by $(x \otimes y)_{ij} = x_i y_j$ for $1 \leq i, j \leq d$.

2. Preliminaries.

2.1. *Elementary lemma.*

LEMMA 2.1. For $\psi \in C_b^1(\mathbb{R}^d; \mathbb{R}^d)$ and $\varepsilon > 0$, set

$$\Phi_\varepsilon(x) := x + \varepsilon\psi(x/\varepsilon), \quad \Theta_\varepsilon(x, z) := \psi(x/\varepsilon + z) - \psi(x/\varepsilon), \quad x \in \mathbb{R}^d.$$

Then, for any $f \in C_b^3(\mathbb{R}^d)$ and $M \in (0, \infty]$,

$$\begin{aligned} & f(\Phi_\varepsilon(x + \varepsilon z)) - f(\Phi_\varepsilon(x)) - \langle \nabla(f(\Phi_\varepsilon(\cdot)))(x), \varepsilon z \rangle \mathbb{1}_{\{|z| \leq M\}} \\ &= f(\Phi_\varepsilon(x) + \varepsilon z) - f(\Phi_\varepsilon(x)) - \langle \nabla f(\Phi_\varepsilon(x)), \varepsilon z \rangle \mathbb{1}_{\{|z| \leq M\}} \\ & \quad + \varepsilon \langle \nabla f(\Phi_\varepsilon(x)), \psi(x/\varepsilon + z) - \psi(x/\varepsilon) - \nabla \psi(x/\varepsilon) \cdot z \rangle \mathbb{1}_{\{|z| \leq M\}} \\ & \quad + \frac{\varepsilon^2}{2} \langle \nabla^2 f(\Phi_\varepsilon(x)), (2\Theta_\varepsilon(x, z) \otimes z) \mathbb{1}_{\{|z| \leq M\}} + \Theta_\varepsilon(x, z) \otimes \Theta_\varepsilon(x, z) \rangle + G_\varepsilon(x, z), \end{aligned}$$

where $G_\varepsilon(x, z)$ satisfies

$$\begin{aligned} |G_\varepsilon(x, z)| &\leq C_1 \varepsilon^3 \|\nabla^3 f\|_\infty (1 + \|\psi\|_\infty + \|\nabla \psi\|_\infty)^3 \\ & \quad \times (|z|^2 \mathbb{1}_{\{|z| \leq R\}} + |z| \mathbb{1}_{\{R < |z| \leq M\}} + \mathbb{1}_{\{|z| > M\}}) \\ & \quad + C_1 \varepsilon^2 \|\nabla^2 f\|_\infty \|\psi\|_\infty |z| \mathbb{1}_{\{R < |z| \leq M\}} + C_1 \|f\|_\infty \mathbb{1}_{\{|z| > M\}} \end{aligned}$$

for all $0 < R \leq M \leq \infty$ and some constant $C_1 > 0$ independent of $\varepsilon, M, R, x, z, f$ and ψ .

PROOF. For any $f \in C_b^3(\mathbb{R}^d)$ and $M \in (0, \infty]$, we write

$$\begin{aligned} & f(\Phi_\varepsilon(x + \varepsilon z)) - f(\Phi_\varepsilon(x)) - \langle \nabla(f(\Phi_\varepsilon(\cdot)))(x), \varepsilon z \rangle \mathbb{1}_{\{|z| \leq M\}} \\ &= f(\Phi_\varepsilon(x) + \varepsilon z) - f(\Phi_\varepsilon(x)) - \langle \nabla f(\Phi_\varepsilon(x)), (\varepsilon z + \varepsilon \nabla \psi(x/\varepsilon) \cdot z) \rangle \mathbb{1}_{\{|z| \leq M\}} \\ &= [f(x + \varepsilon z + \varepsilon \psi(x/\varepsilon + z)) - f(x + \varepsilon \psi(x/\varepsilon + z))] \\ & \quad - \langle \nabla f(x + \varepsilon \psi(x/\varepsilon + z)), \varepsilon z \rangle \mathbb{1}_{\{|z| \leq M\}} \\ & \quad + [\langle \nabla f(x + \varepsilon \psi(x/\varepsilon + z)) - \nabla f(x + \varepsilon \psi(x/\varepsilon)), \varepsilon z \rangle \mathbb{1}_{\{|z| \leq M\}}] \\ & \quad + [f(x + \varepsilon \psi(x/\varepsilon + z)) - f(x + \varepsilon \psi(x/\varepsilon))] \\ & \quad - \langle \nabla f(x + \varepsilon \psi(x/\varepsilon)), \varepsilon \nabla \psi(x/\varepsilon) \cdot z \rangle \mathbb{1}_{\{|z| \leq M\}} \\ &=: \sum_{i=1}^3 I_i^\varepsilon. \end{aligned}$$

First, for fixed $x, z \in \mathbb{R}^d$ and $\varepsilon \in (0, 1)$, let

$$H_1(y) := f(x + y + \varepsilon z) - f(x + y) - \langle \nabla f(x + y), \varepsilon z \rangle \mathbb{1}_{\{|z| \leq M\}}.$$

Then,

$$\|H_1\|_\infty \leq \varepsilon^2 \|\nabla^2 f\|_\infty |z|^2 \mathbb{1}_{\{|z| \leq M\}} + \|f\|_\infty \mathbb{1}_{\{|z| > M\}},$$

and, for any $0 < R \leq M$,

$$\|\nabla H_1\|_\infty \leq \varepsilon^2 \|\nabla^3 f\|_\infty |z|^2 \mathbb{1}_{\{|z| \leq R\}} + \varepsilon \|\nabla^2 f\|_\infty |z| \mathbb{1}_{\{|z| > R\}}.$$

Thus,

$$\begin{aligned} I_1^\varepsilon &= H_1(\varepsilon \psi(x/\varepsilon + z)) = H_1(\varepsilon \psi(x/\varepsilon)) + [H_1(\varepsilon \psi(x/\varepsilon + z)) - H_1(\varepsilon \psi(x/\varepsilon))] \\ &=: f(\Phi_\varepsilon(x) + \varepsilon z) - f(\Phi_\varepsilon(x)) - \langle \nabla f(\Phi_\varepsilon(x)), \varepsilon z \rangle \mathbb{1}_{\{|z| \leq M\}} + G_{1,\varepsilon}(x, z). \end{aligned}$$

Furthermore, according to the mean value theorem, $G_{1,\varepsilon}(x, z)$ satisfies that

$$\begin{aligned} |G_{1,\varepsilon}(x, z)| &\leq \varepsilon \|\nabla H_1\|_\infty |\psi(x/\varepsilon + z) - \psi(x/\varepsilon)| \mathbb{1}_{\{|z| \leq M\}} + \|H_1\|_\infty \mathbb{1}_{\{|z| > M\}} \\ &\leq \varepsilon^3 \|\nabla^3 f\|_\infty \|\psi\|_\infty |z|^2 \mathbb{1}_{\{|z| \leq R\}} \\ &\quad + \varepsilon^2 \|\nabla^2 f\|_\infty \|\psi\|_\infty |z| \mathbb{1}_{\{R < |z| \leq M\}} + \|f\|_\infty \mathbb{1}_{\{|z| > M\}}. \end{aligned}$$

Second, by the Taylor expansion it holds that

$$\begin{aligned} I_2^\varepsilon &= \varepsilon^2 \langle \nabla^2 f(\Phi_\varepsilon(x)), \Theta_\varepsilon(x, z) \otimes z \rangle \mathbb{1}_{\{|z| \leq M\}} \\ &\quad + \frac{\varepsilon^3}{2} \langle \nabla^3 f(\Phi_\varepsilon(x) + \theta_0 \Theta_\varepsilon(x, z)), \Theta_\varepsilon(x, z) \otimes \Theta_\varepsilon(x, z) \otimes z \rangle \mathbb{1}_{\{|z| \leq M\}} \\ &=: \varepsilon^2 \langle \nabla^2 f(\Phi_\varepsilon(x)), \Theta_\varepsilon(x, z) \otimes z \rangle \mathbb{1}_{\{|z| \leq M\}} + G_{2,\varepsilon}(x, z), \end{aligned}$$

where $\theta_0 \in (0, 1)$ and

$$|G_{2,\varepsilon}(x, z)| \leq \varepsilon^3 \|\nabla^3 f\|_\infty (\|\psi\|_\infty \|\nabla \psi\|_\infty |z|^2 \mathbb{1}_{\{|z| \leq R\}} + \|\psi\|_\infty^2 |z| \mathbb{1}_{\{R < |z| \leq M\}})$$

for any $0 < R \leq M$.

Third, applying the Taylor expansion again and using the mean value theorem, we obtain

$$\begin{aligned} I_3^\varepsilon &= \varepsilon \langle \nabla f(\Phi_\varepsilon(x)), \psi(x/\varepsilon + z) - \psi(x/\varepsilon) - \nabla \psi(x/\varepsilon) \cdot z \rangle \mathbb{1}_{\{|z| \leq M\}} \\ &\quad + \frac{\varepsilon^2}{2} \langle \nabla^2 f(\Phi_\varepsilon(x) + \varepsilon \theta_1 \Theta_\varepsilon(x, z)), \Theta_\varepsilon(x, z) \otimes \Theta_\varepsilon(x, z) \rangle \\ &= \varepsilon \langle \nabla f(\Phi_\varepsilon(x)), \psi(x/\varepsilon + z) - \psi(x/\varepsilon) - \nabla \psi(x/\varepsilon) \cdot z \rangle \mathbb{1}_{\{|z| \leq M\}} \\ &\quad + \frac{\varepsilon^2}{2} \langle \nabla^2 f(\Phi_\varepsilon(x)), \Theta_\varepsilon(x, z) \otimes \Theta_\varepsilon(x, z) \rangle \\ &\quad + \frac{\varepsilon^3}{2} \theta_1 \langle \nabla^3 f(\Phi_\varepsilon(x) + \varepsilon \theta_1 \theta_2 \Theta_\varepsilon(x, z)), \Theta_\varepsilon(x, z) \otimes \Theta_\varepsilon(x, z) \otimes \Theta_\varepsilon(x, z) \rangle \\ &=: \varepsilon \langle \nabla f(\Phi_\varepsilon(x)), \psi(x/\varepsilon + z) - \psi(x/\varepsilon) - \nabla \psi(x/\varepsilon) \cdot z \rangle \mathbb{1}_{\{|z| \leq M\}} \\ &\quad + \frac{\varepsilon^2}{2} \langle \nabla^2 f(\Phi_\varepsilon(x)), \Theta_\varepsilon(x, z) \otimes \Theta_\varepsilon(x, z) \rangle + G_{3,\varepsilon}(x, z), \end{aligned}$$

where $\theta_1, \theta_2 \in (0, 1)$ and

$$\begin{aligned} |G_{3,\varepsilon}(x, z)| &\leq \varepsilon^3 \|\nabla^3 f\|_\infty (\|\psi\|_\infty \|\nabla \psi\|_\infty^2 |z|^2 \mathbb{1}_{\{|z| \leq R\}} + \|\psi\|_\infty^2 \|\nabla \psi\|_\infty |z| \mathbb{1}_{\{R < |z| \leq M\}} \\ &\quad + \|\psi\|_\infty^3 \mathbb{1}_{\{|z| > M\}}) \end{aligned}$$

for any $0 < R \leq M$.

Therefore, putting all the estimates above together, we prove the required assertion. \square

2.2. Consequences of (A1) and (A2). Let \mathcal{L} be the operator given by (1.1) with $\Pi(dz)$, $k(x, z)$ and $b(x)$ satisfying all the conditions below (1.1). We assume that Assumptions (A1) and (A2) in Section 1 also hold true. Denote by $X := (X_t)_{t \geq 0}$ the strong Markov process associated with the generator \mathcal{L} as in (A1).

Let $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a strictly increasing function such that $\lim_{r \rightarrow \infty} \rho(r) = \infty$. For $\varepsilon \in (0, 1)$, consider the scaled process

$$(2.1) \quad X^\varepsilon := \{\varepsilon X_{\rho(1/\varepsilon)t} : t \geq 0\}.$$

Clearly, X^ε is a strong Markov process on \mathbb{R}^d , and the associated generator is given by

$$(2.2) \quad \begin{aligned} \mathcal{L}^\varepsilon f(x) &= \rho(1/\varepsilon) \int_{\mathbb{R}^d} (f(x + \varepsilon z) - f(x) - \varepsilon \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z| \leq 1\}}) k(x/\varepsilon, z) \Pi(dz) \\ &\quad + \varepsilon \rho(1/\varepsilon) \langle b(x/\varepsilon), \nabla f(x) \rangle. \end{aligned}$$

See, for example, [10], Lemma 2.1. Since the coefficients of the generator \mathcal{L}^ε are multivariate ε -periodic, the process X^ε can be also viewed as an \mathbb{T}^d -valued process if $1/\varepsilon$ is an integer.

LEMMA 2.2. *Under Assumption (A2), we have the following two statements:*

(i) *For every $f \in C_b(\mathbb{T}^d)$ with $\mu(f) = 0$, any $0 < s < t$ and $x \in \mathbb{R}^d$,*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_x \left[\left| \int_s^t f(X_r^\varepsilon/\varepsilon) dr \right|^2 \right] = 0.$$

(ii) *Suppose that, for some $x \in \mathbb{R}^d$,*

$$(2.3) \quad \lim_{\varepsilon \rightarrow 0} \sup_{|s-t| \leq 1/\sqrt{\rho(1/\varepsilon)}} \mathbb{E}_x(|X_s^\varepsilon - X_t^\varepsilon| \wedge 1) = 0.$$

Then, for any bounded continuous function $F : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ and any $0 < s < t$,

$$(2.4) \quad \lim_{\varepsilon \rightarrow 0} \mathbb{E}_x \left[\left| \int_s^t F(X_r^\varepsilon/\varepsilon, X_r^\varepsilon) dr - \int_s^t \bar{F}(X_r^\varepsilon) dr \right|^2 \right] = 0,$$

where $\bar{F} : \mathbb{R}^d \rightarrow \mathbb{R}$ is defined by

$$\bar{F}(y) := \int_{\mathbb{T}^d} F(x, y) \mu(dx).$$

PROOF. (i) Let $f \in C_b(\mathbb{T}^d)$ be such that $\mu(f) = 0$. Then, for any $0 < s < t$ and $x \in \mathbb{R}^d$ and by the Markov property and (A2),

$$(2.5) \quad \begin{aligned} \mathbb{E}_x \left[\left| \int_s^t f(\varepsilon^{-1} X_r^\varepsilon) dr \right|^2 \right] &= 2 \mathbb{E}_x \left[\int_s^t \int_s^r f(X_{\rho(1/\varepsilon)r}) f(X_{\rho(1/\varepsilon)u}) du dr \right] \\ &= 2 \int_s^t \int_s^r \mathbb{E}_x [f(X_{\rho(1/\varepsilon)u}) \mathbb{E}_{X_{\rho(1/\varepsilon)u}} f(X_{\rho(1/\varepsilon)(r-u)})] du dr \\ &\leq \frac{\|f\|_\infty^2}{\rho(1/\varepsilon)} \int_s^t (1 - e^{-\lambda_1 \rho(1/\varepsilon)(r-s)}) dr \leq \frac{\|f\|_\infty^2 (t-s)}{\rho(1/\varepsilon)}. \end{aligned}$$

This along with the fact that $\lim_{\varepsilon \rightarrow 0} \rho(1/\varepsilon) = \infty$ yields the first desired assertion.

(ii) By the standard approximation, it suffices to prove (2.4) for $F(x, y) = f(x)g(y)$ with $f \in C_b(\mathbb{T}^d)$ and $g \in C_b^1(\mathbb{R}^d)$. Without loss of generality, we assume that $f \in C_b(\mathbb{T}^d)$ with $\mu(f) = 0$. For $\varepsilon > 0$, define $s_i = s + \frac{i(t-s)}{[\sqrt{\rho(1/\varepsilon)}]}$. Let

$$I_\varepsilon := \mathbb{E}_x \left[\left| \int_s^t f(X_r^\varepsilon/\varepsilon) g(X_r^\varepsilon) dr \right|^2 \right]$$

and

$$J_\varepsilon := \mathbb{E}_x \left[\left| \sum_{i=0}^{[\sqrt{\rho(1/\varepsilon)}]-1} \int_{s_i}^{s_{i+1}} f(X_r^\varepsilon/\varepsilon) dr \cdot g(X_{s_i}^\varepsilon) \right|^2 \right].$$

We can write

$$I_\varepsilon = \sum_{0 \leq i, j \leq [\sqrt{\rho(1/\varepsilon)}]-1} \int_{s_i}^{s_{i+1}} \int_{s_j}^{s_{j+1}} \mathbb{E}_x [f(X_r^\varepsilon/\varepsilon) f(X_u^\varepsilon/\varepsilon) g(X_r^\varepsilon) g(X_u^\varepsilon)] dr du$$

and

$$J_\varepsilon = \sum_{0 \leq i, j \leq [\sqrt{\rho(1/\varepsilon)}] - 1} \int_{s_i}^{s_{i+1}} \int_{s_j}^{s_{j+1}} \mathbb{E}_x [f(X_r^\varepsilon/\varepsilon) f(X_u^\varepsilon/\varepsilon) g(X_{s_i}^\varepsilon) g(X_{s_j}^\varepsilon)] dr du.$$

Note that, for every $0 \leq i, j \leq [\sqrt{\rho(1/\varepsilon)}] - 1$, $s_i \leq r \leq s_{i+1}$ and $s_j \leq u \leq s_{j+1}$,

$$\begin{aligned} & |\mathbb{E}_x [f(X_r^\varepsilon/\varepsilon) f(X_u^\varepsilon/\varepsilon) g(X_r^\varepsilon) g(X_u^\varepsilon)] - \mathbb{E}_x [f(X_r^\varepsilon/\varepsilon) f(X_u^\varepsilon/\varepsilon) g(X_{s_i}^\varepsilon) g(X_{s_j}^\varepsilon)]| \\ & \leq \|f\|_\infty^2 \|g\|_\infty (\mathbb{E}_x |g(X_r^\varepsilon) - g(X_{s_i}^\varepsilon)| + \mathbb{E}_x |g(X_u^\varepsilon) - g(X_{s_j}^\varepsilon)|) \\ & \leq 2\|f\|_\infty^2 \|g\|_\infty (\|g\|_\infty + \|\nabla g\|_\infty) (\mathbb{E}_x (|X_r^\varepsilon - X_{s_i}^\varepsilon| \wedge 1) + \mathbb{E}_x (|X_u^\varepsilon - X_{s_j}^\varepsilon| \wedge 1)) \\ & \leq c_1 \eta(\varepsilon), \end{aligned}$$

where $c_1 = 4(\|g\|_\infty + \|\nabla g\|_\infty) \|f\|_\infty^2 \|g\|_\infty$ and

$$\eta(\varepsilon) := \sup_{|r_1 - r_2| \leq (t-s)/[\sqrt{\rho(1/\varepsilon)}]} \mathbb{E}_x (|X_{r_1}^\varepsilon - X_{r_2}^\varepsilon| \wedge 1).$$

Thus,

$$|I_\varepsilon - J_\varepsilon| \leq c_1 \eta(\varepsilon) \sum_{0 \leq i, j \leq [\sqrt{\rho(1/\varepsilon)}] - 1} (s_{j+1} - s_j)(s_{i+1} - s_i) \leq c_1 \eta(\varepsilon) (t - s)^2.$$

Hence, $\lim_{\varepsilon \rightarrow 0} |I_\varepsilon - J_\varepsilon| = 0$. So it remains to show that $\lim_{\varepsilon \rightarrow 0} J_\varepsilon = 0$.

By the Cauchy-Schwarz inequality and (2.5),

$$\begin{aligned} J_\varepsilon & \leq [\sqrt{\rho(1/\varepsilon)}] \|g\|_\infty^2 \sum_{i=0}^{[\sqrt{\rho(1/\varepsilon)}] - 1} \mathbb{E}_x \left[\left| \int_{s_i}^{s_{i+1}} f(X_r^\varepsilon/\varepsilon) dr \right|^2 \right] \\ & \leq \frac{c_2 [\sqrt{\rho(1/\varepsilon)}]}{\rho(1/\varepsilon)} \|g\|_\infty^2 \|f\|_\infty^2 \sum_{i=0}^{[\sqrt{\rho(1/\varepsilon)}] - 1} (s_{i+1} - s_i) \leq c_2 \|f\|_\infty^2 \|g\|_\infty^2 \frac{(t - s)}{\sqrt{\rho(1/\varepsilon)}}. \end{aligned}$$

Since $\lim_{\varepsilon \rightarrow 0} \rho(1/\varepsilon) = \infty$, we get $\lim_{\varepsilon \rightarrow 0} J_\varepsilon = 0$. \square

3. Homogenization: Jump scalings. Let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ be the unit sphere on \mathbb{R}^d and $z := (r, \theta) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}$ be the spherical coordinate of $z \in \mathbb{R}^d \setminus \{0\}$. Throughout this section we assume that the jumping measure $\Pi(dz)$ in (1.1) has the following form on $\{|z| > 1\}$:

$$(3.1) \quad \mathbb{1}_{\{|z| > 1\}} \Pi(dz) := \mathbb{1}_{\{r > 1\}} \varrho(r, d\theta) dr = \mathbb{1}_{\{r > 1\}} \frac{\varrho_0(d\theta) + \kappa(r, d\theta)}{r\varphi(r)} dr,$$

where:

- (i) $\varrho_0(d\theta)$ is a e nonnegative finite measure on \mathbb{S}^{d-1} such that $\varrho_0(\mathbb{S}^{d-1}) > 0$;
- (ii) for every $r > 1$, $\kappa(r, d\theta)$ is a finite signed measure on \mathbb{S}^{d-1} so that, for any $r_0 > 1$,

$$(3.2) \quad \sup_{r \in [r_0, \infty)} |\kappa|(r, \mathbb{S}^{d-1}) < \infty, \quad \lim_{r \rightarrow \infty} |\kappa|(r, \mathbb{S}^{d-1}) = 0,$$

where, for each $r > 1$, $|\kappa|(r, d\theta)$ denotes the total variational measure of $\kappa(r, d\theta)$;

(iii) $\varphi : (1, \infty) \rightarrow \mathbb{R}_+$ is a strictly increasing function such that there are constants $\alpha \in (0, 2)$, $c_0 \in (0, 1]$ and $\eta_0 \in (0, \alpha \wedge |\alpha - 1| \wedge (2 - \alpha))$ when $\alpha \neq 1$ and $\eta_0 \in (0, 1/6)$ when $\alpha = 1$, so that, for any $1 < r \leq R$,

$$(3.3) \quad \lim_{\lambda \rightarrow \infty} \left| \frac{\varphi(\lambda r)}{\varphi(\lambda)} - r^\alpha \right| = 0, \quad c_0 (R/r)^{\alpha - \eta_0} \leq \frac{\varphi(R)}{\varphi(r)} \leq c_0^{-1} (R/r)^{\alpha + \eta_0}.$$

Define

$$(3.4) \quad \Pi_0(dz) := \mathbb{1}_{\{r>0\}} \frac{\varrho_0(d\theta) dr}{r^{1+\alpha}},$$

where $\alpha \in (0, 2)$ is the constant in (3.3). It is obvious that $\Pi_0(dz)$ satisfies the scaling property that $\Pi_0(sA) = s^{-\alpha} \Pi_0(A)$ for all $s > 0$ and $A \subset \mathbb{R}^d \setminus \{0\}$; however, since $\varrho_0(d\theta)$ may be nonsymmetric on \mathbb{S}^{d-1} , $\Pi_0(dz)$ can be nonsymmetric on \mathbb{R}^d .

We also suppose that for the function $k(x, z)$ in (1.1), there exists a bounded function $\bar{k} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ such that, for any $0 < r < R$ and $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying

$$(3.5) \quad \lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d, |z_1 - z_2| \leq \varepsilon} |f(x, z_1) - f(x, z_2)| = 0,$$

it holds that

$$(3.6) \quad \lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\{r \leq |z| \leq R\}} f(x, z) k(x/\varepsilon, z/\varepsilon) \Pi_0(dz) - \int_{\{r \leq |z| \leq R\}} f(x, z) \bar{k}(x/\varepsilon, z) \Pi_0(dz) \right| = 0.$$

REMARK 3.1. We make some comments on the assumptions above. In the following, φ is a function given in (3.3):

(1) Examples of functions satisfying (3.3) include $\varphi(r) = r^\alpha + r^\beta$ on $(1, \infty)$ for $0 < \beta \leq \alpha < 2$ and $\varphi(r) = r^\alpha \log(1 + r)$ on $(1, \infty)$ for $0 < \alpha < 2$. In fact, any strictly increasing function $\varphi(r)$ on $(1, \infty)$ of the form $\int_{\alpha_1}^{\alpha_2} r^\beta \nu(d\beta)$ satisfies condition (3.3), where $0 < \alpha_1 \leq \alpha_2 < 2$ and ν is a finite measure on $[\alpha_1, \alpha_2]$ so that α_2 is in the support of ν ; see Example 7.2 for the proof of this fact. Observe that the second condition in (3.3) is equivalent to that there is some $R_0 \geq 1$ so that, for all $R \geq r > R_0$,

$$c_0(R/r)^{\alpha-\eta_0} \leq \frac{\varphi(R)}{\varphi(r)} \leq c_0^{-1}(R/r)^{\alpha+\eta_0}.$$

Indeed, the statements in this section still hold if we replace $r > 1$ in (3.1) by $r > R_0$ for some $R_0 \geq 1$ and restrict φ , defined on (R_0, ∞) .

(2) Condition (3.2) means that the term $\kappa(r, \cdot)$ is a lower order perturbation as $r \rightarrow \infty$, and thus, by (3.1) the jumping measure $\Pi(dz)$ is comparable to $\frac{\varrho_0(d\theta) dr}{r\varphi(r)}$ for large $|z|$.

(3) Suppose that $\varphi_1(r)$ is a strictly increasing function on $(1, \infty)$ so that

$$(3.7) \quad \lim_{r \rightarrow \infty} \frac{\varphi_1(r)}{\varphi(r)} = 1.$$

Then, $\varphi_1(r)$ clearly satisfies (3.3), and we can rewrite $a(z)$ as

$$\mathbb{1}_{\{|z|>1\}} \Pi(dz) = \mathbb{1}_{\{r>1\}} \frac{\varrho_0(d\theta) + \tilde{\kappa}(r, d\theta)}{r\varphi_1(r)} dr$$

with

$$\tilde{\kappa}(r, d\theta) = \left(\frac{\varphi_1(r)}{\varphi(r)} - 1 \right) \varrho_0(d\theta) + \frac{\varphi_1(r)}{\varphi(r)} \kappa(r, d\theta).$$

Evidently, $\lim_{r \rightarrow \infty} |\tilde{\kappa}|(r, \mathbb{S}^{d-1}) = 0$ and $\sup_{r \in [r_0, \infty)} |\tilde{\kappa}|(r, \mathbb{S}^{d-1}) < \infty$ for all $r_0 > 1$. In other words, the representation of $\Pi(dz)$ in the form of (3.1) is invariant among the family of strictly increasing functions φ on $(1, \infty)$ that mutually satisfy the relation (3.7).

(4) Let $k : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ be the function satisfying the assumptions below (1.1). In view of (3.4), it is easy to see that, if for every $x \in \mathbb{R}^d$ and ϱ_0 -a.e. $\theta \in \mathbb{S}^{d-1}$, there is a constant $\bar{k}(x, \theta)$ so that

$$(3.8) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T k(x, (r, \theta)) \, dr = \bar{k}(x, \theta),$$

then (3.6) holds with $\bar{k}(x, z) := \bar{k}(x, z/|z|)$. Clearly, condition (3.8) holds, if for ϱ_0 -a.e. $\theta \in \mathbb{S}^{d-1}$, there exists a bounded measurable function $\bar{k}(\cdot, \theta) : \mathbb{R}^d \rightarrow \mathbb{R}_+$ such that

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} |k(x, (r/\varepsilon, \theta)) - \bar{k}(x, \theta)| = 0.$$

It is shown in Theorem 6.5 below that condition (3.8) holds for every $\theta \in \mathbb{S}^{d-1}$, and so condition (3.6) automatically holds for any finite measure ϱ_0 on \mathbb{S}^{d-1} , when $k(x, z)$ is bounded, continuous and multivariate 1-periodic in both x and z ; see Section 6 for more sufficient conditions for (3.6), including the information on $\bar{k}(x, \theta)$, when $z \mapsto k(x, z)$ is multivariate 1-periodic for any fixed $x \in \mathbb{R}^d$.

The purpose of this section is to consider the limiting behavior of the scaled process

$$(3.9) \quad X^\varepsilon = (X_t^\varepsilon)_{t \geq 0} := (\varepsilon X_{\varphi(1/\varepsilon)t})_{t \geq 0}.$$

Note that by (1.1) and (2.2), the generator of X^ε is given by

$$(3.10) \quad \begin{aligned} \mathcal{L}^\varepsilon f(x) &= \varphi(1/\varepsilon) \int_{\mathbb{R}^d} (f(x + \varepsilon z) - f(x) - \varepsilon \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z| \leq 1\}}) k(x/\varepsilon, z) \Pi(dz) \\ &\quad + \varepsilon \varphi(1/\varepsilon) \langle b(x/\varepsilon), \nabla f(x) \rangle. \end{aligned}$$

It turns out that the limiting behavior of X^ε as $\varepsilon \rightarrow 0$ depends on the value of α associated with the scaling function φ in (3.3). We will divide this section into two parts. One is to consider the invariance principle for X^ε that needs no recentering and the other that requires recentering. In some literature, invariance principle that requires recentering is called noncentral limit theorem; see, for instance, [21].

3.1. *Invariance principle without recentering:* $\alpha \in (0, 1)$. Recall that $\alpha \in (0, 2)$ is the constant in (3.3). In this subsection we will restrict ourselves to the case $\alpha \in (0, 1)$. Then, by (3.1) we have

$$(3.11) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon \varphi(\varepsilon^{-1}) &= 0, & \lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \left(\varepsilon \varphi(\varepsilon^{-1}) \int_1^{\delta/\varepsilon} \frac{1}{\varphi(r)} \, dr \right) &= 0, \\ \lim_{\delta \rightarrow 0} \sup_{\varepsilon \in (0,1)} \left(\delta \varepsilon \varphi(\varepsilon^{-1}) \int_1^{1/(\delta\varepsilon)} \frac{1}{\varphi(r)} \, dr \right) &= 0, \\ \lim_{\delta \rightarrow 0} \sup_{\varepsilon \in (0,1)} \left(\varphi(\varepsilon^{-1}) \int_{1/(\delta\varepsilon)}^\infty \frac{1}{r\varphi(r)} \, dr \right) &= 0, \\ \sup_{\varepsilon \in (0,1)} \left(\varepsilon \varphi(\varepsilon^{-1}) \int_1^{1/\varepsilon} \frac{1}{\varphi(r)} \, dr \right) &< \infty, & \sup_{\varepsilon \in (0,1)} \left(\varphi(1/\varepsilon) \int_{1/\varepsilon}^\infty \frac{1}{r\varphi(r)} \, dr \right) &< \infty. \end{aligned}$$

In particular,

$$\lim_{\delta \rightarrow 0} \sup_{\varepsilon \in (0,1)} \left(\delta^2 \varepsilon^2 \varphi(\varepsilon^{-1}) \int_1^{1/(\delta\varepsilon)} \frac{r}{\varphi(r)} \, dr \right) \leq \lim_{\delta \rightarrow 0} \sup_{\varepsilon \in (0,1)} \left(\delta \varepsilon \varphi(\varepsilon^{-1}) \int_1^{1/(\delta\varepsilon)} \frac{1}{\varphi(r)} \, dr \right) = 0.$$

In fact, for any $\varepsilon, \delta \in (0, 1)$,

$$\begin{aligned} \delta\varepsilon\varphi(\varepsilon^{-1}) \int_1^{1/(\delta\varepsilon)} \frac{1}{\varphi(r)} dr &= \delta \int_\varepsilon^{1/\delta} \frac{\varphi(1/\varepsilon)}{\varphi(r/\varepsilon)} dr \\ &\leq \delta \left(\int_\varepsilon^1 \frac{1}{r^{\alpha+\eta_0}} dr + \int_1^{1/\delta} \frac{1}{r^{\alpha-\eta_0}} dr \right) \leq \delta(1 + \delta^{-1-\eta_0+\alpha}), \end{aligned}$$

where we have used the second condition of (3.3) in the first inequality. So, by the fact $\eta_0 \in (0, \alpha \wedge (1 - \alpha))$, we obtain

$$\lim_{\delta \rightarrow 0} \sup_{\varepsilon \in (0, 1)} \left(\delta\varepsilon\varphi(\varepsilon^{-1}) \int_1^{1/(\delta\varepsilon)} \frac{1}{\varphi(r)} dr \right) = 0.$$

Other estimates in (3.11) can be proved similarly, and we omit the details.

The following is the main result of this section.

THEOREM 3.2. *Suppose that (3.1) and (3.6) hold. If (3.3) holds with $\alpha \in (0, 1)$, then the scaled process $(X_t^\varepsilon)_{t \geq 0}$ of (3.9) converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$, as $\varepsilon \rightarrow 0$, to an α -stable Lévy process $\bar{X} := (\bar{X}_t)_{t \geq 0}$ with Lévy measure $\bar{k}_0(z)\Pi_0(dz)$; that is, the generator of the α -stable Lévy process \bar{X} is given by*

$$\bar{\mathcal{L}}f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x))\bar{k}_0(z)\Pi_0(dz).$$

Here, $\Pi_0(dz)$ is the measure defined in (3.4) and $\bar{k}_0(z) := \int_{\mathbb{T}^d} \bar{k}(x, z)\mu(dx)$, where μ is the unique invariant probability measure of X on \mathbb{T}^d and $\bar{k}(x, z)$ is the function in (3.6).

To prove this theorem, we need the following property for the generator of the scaled process X^ε .

LEMMA 3.3. *Suppose that (3.1) and (3.6) hold and that $0 < \alpha < 1$. For every $f \in C_b^2(\mathbb{R}^d)$,*

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} |\mathcal{L}^\varepsilon f(x) - \bar{\mathcal{L}}^\varepsilon f(x)| = 0,$$

where

$$(3.12) \quad \bar{\mathcal{L}}^\varepsilon f(x) := \int_{\mathbb{R}^d} (f(x+z) - f(x))\bar{k}(x/\varepsilon, z)\Pi_0(dz)$$

with $\bar{k}(x, z)$ being the function in (3.6) and $\Pi_0(dz)$, defined by (3.4).

PROOF. By (3.10), for every $\varepsilon, \delta \in (0, 1)$ and $f \in C_b^2(\mathbb{R}^d)$,

$$\begin{aligned} \mathcal{L}^\varepsilon f(x) &= \varphi(1/\varepsilon) \int_{\{|z| \leq \delta/\varepsilon\}} (f(x+\varepsilon z) - f(x) - \langle \nabla f(x), \varepsilon z \rangle)k(x/\varepsilon, z)\Pi(dz) \\ &\quad + \varphi(1/\varepsilon) \int_{\{\delta/\varepsilon < |z| < 1/(\delta\varepsilon)\}} (f(x+\varepsilon z) - f(x))k(x/\varepsilon, z)\Pi(dz) \\ &\quad + \varphi(1/\varepsilon) \int_{\{|z| \geq 1/(\delta\varepsilon)\}} (f(x+\varepsilon z) - f(x))k(x/\varepsilon, z)\Pi(dz) \\ &\quad + \varepsilon\varphi(1/\varepsilon)\langle \nabla f(x), b_{\delta/\varepsilon}(x/\varepsilon) + b(x/\varepsilon) \rangle \\ &=: \sum_{i=1}^4 \mathcal{L}_i^{\varepsilon, \delta} f(x), \end{aligned}$$

where $b_{\delta/\varepsilon}(x)$ is defined by (1.6). We can write

$$\begin{aligned} \bar{\mathcal{L}}^\varepsilon f(x) &= \int_{\{|z|\leq\delta\}} (f(x+z) - f(x))\bar{k}(x/\varepsilon, z)\Pi_0(dz) \\ &\quad + \int_{\{\delta<|z|<1/\delta\}} (f(x+z) - f(x))\bar{k}(x/\varepsilon, z)\Pi_0(dz) \\ &\quad + \int_{\{|z|\geq 1/\delta\}} (f(x+z) - f(x))\bar{k}(x/\varepsilon, z)\Pi_0(dz) \\ &=: \sum_{i=1}^3 \bar{\mathcal{L}}_i^{\varepsilon, \delta} f(x). \end{aligned}$$

Since $\bar{k}(x, z)$ is bounded and $\alpha \in (0, 1)$, by (3.4) we have

$$|\bar{\mathcal{L}}_1^{\varepsilon, \delta} f(x)| \leq \|\nabla f\|_\infty \int_{\{|z|\leq\delta\}} |z|\Pi_0(dz) \leq \|\nabla f\|_\infty \int_0^\delta \frac{\varrho_0(\mathbb{S}^{d-1})}{r^\alpha} dr \leq \|\nabla f\|_\infty \delta^{1-\alpha}.$$

Applying the same argument to $|\bar{\mathcal{L}}_3^{\varepsilon, \delta} f(x)|$, we see that

$$\lim_{\delta \rightarrow 0} \sup_{\varepsilon \in (0, 1], x \in \mathbb{R}^d} (|\bar{\mathcal{L}}_1^{\varepsilon, \delta} f(x)| + |\bar{\mathcal{L}}_3^{\varepsilon, \delta} f(x)|) = 0.$$

On the other hand, according to (3.1),

$$(3.13) \quad \mathbb{1}_{\{|z|\geq 1\}}\Pi(dz) \leq \frac{\varrho_0(d\theta) + |\kappa|(r, d\theta)}{r\varphi(r)} \mathbb{1}_{\{r\geq 1\}} dr,$$

and so, by (3.2) we obtain that, for $0 < \varepsilon < \delta < 1$,

$$\begin{aligned} \sup_{x \in \mathbb{R}^d} |\mathcal{L}_1^{\varepsilon, \delta} f(x)| &\leq \|\nabla^2 f\|_\infty \varepsilon^2 \varphi(1/\varepsilon) \int_{\{|z|\leq\delta/\varepsilon\}} |z|^2 \Pi(dz) \\ &\leq \|\nabla^2 f\|_\infty \varepsilon^2 \varphi(1/\varepsilon) \left(\int_{\{|z|\leq 1\}} |z|^2 \Pi(dz) + \int_1^{\delta/\varepsilon} \frac{r}{\varphi(r)} dr \right) \\ &\leq \|\nabla^2 f\|_\infty \left(\varepsilon^2 \varphi(1/\varepsilon) \int_{\{|z|\leq 1\}} |z|^2 \Pi(dz) + \delta \varepsilon \varphi(1/\varepsilon) \int_1^{\delta/\varepsilon} \frac{1}{\varphi(r)} dr \right), \end{aligned}$$

$$\sup_{x \in \mathbb{R}^d} |\mathcal{L}_3^{\varepsilon, \delta} f(x)| \leq \|f\|_\infty \varphi(1/\varepsilon) \int_{\{|z|\geq 1/(\delta\varepsilon)\}} \Pi(dz) \leq \|f\|_\infty \varphi(1/\varepsilon) \int_{1/(\delta\varepsilon)}^\infty \frac{1}{r\varphi(r)} dr$$

and

$$\sup_{x \in \mathbb{R}^d} |b_{\delta/\varepsilon}(x/\varepsilon)| \leq \int_1^{\delta/\varepsilon} \frac{1}{\varphi(r)} dr.$$

Thus, by (3.11) and the fact that $b(x)$ is bounded, we have

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} (|\mathcal{L}_1^{\varepsilon, \delta} f(x)| + |\mathcal{L}_3^{\varepsilon, \delta} f(x)| + |\mathcal{L}_4^{\varepsilon, \delta} f(x)|) = 0.$$

Furthermore, due to (3.1),

$$\begin{aligned} \mathcal{L}_2^{\varepsilon, \delta} f(x) &= \varphi(1/\varepsilon) \int_{\{\delta<|z|<1/\delta\}} (f(x+z) - f(x))k(x/\varepsilon, z/\varepsilon)\Pi(d(z/\varepsilon)) \\ &= \varphi(1/\varepsilon) \int_\delta^{1/\delta} \int_{\mathbb{S}^{d-1}} (f(x+z) - f(x))k(x/\varepsilon, z/\varepsilon) \frac{\varrho_0(d\theta) + \kappa(r/\varepsilon, d\theta)}{r/\varepsilon \varphi(r/\varepsilon)} d(r/\varepsilon) \end{aligned}$$

$$\begin{aligned}
 &= \int_{\delta}^{1/\delta} \int_{\mathbb{S}^{d-1}} (f(x+z) - f(x))k(x/\varepsilon, z/\varepsilon) \frac{\varphi(1/\varepsilon)}{r\varphi(r/\varepsilon)} \varrho_0(d\theta) dr \\
 &\quad + \int_{\delta}^{1/\delta} \int_{\mathbb{S}^{d-1}} (f(x+z) - f(x))k(x/\varepsilon, z/\varepsilon) \frac{\varphi(1/\varepsilon)}{r\varphi(r/\varepsilon)} \kappa(r/\varepsilon, d\theta) dr \\
 &=: \mathcal{L}_{2,1}^{\varepsilon,\delta} f(x) + \mathcal{L}_{2,2}^{\varepsilon,\delta} f(x).
 \end{aligned}$$

By (3.2) and (3.3) as well as the dominated convergence theorem, we know that, for every fixed $\delta \in (0, 1)$,

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} |\mathcal{L}_{2,2}^{\varepsilon,\delta} f(x)| = 0.$$

Again, by (the first condition in) (3.3), (3.4) and (3.6) as well as the dominated convergence theorem, one can verify that, for every fixed $\delta \in (0, 1)$,

$$(3.14) \quad \lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} |\mathcal{L}_{2,1}^{\varepsilon,\delta} f(x) - \bar{\mathcal{L}}_2^{\varepsilon,\delta} f(x)| = 0.$$

Putting all the estimates together and letting $\varepsilon \rightarrow 0$ and then $\delta \rightarrow 0$, we get the desired assertion. \square

PROOF OF THEOREM 3.2. (1) Recall that \mathcal{L}^ε is the infinitesimal generator for the Markov process $X^\varepsilon := ((X_t^\varepsilon)_{t \geq 0}; (\mathbb{P}_x)_{x \in \mathbb{R}^d})$. For every $x \in \mathbb{R}^d$, $t > 0$, $f \in C_b^2(\mathbb{R}^d)$ and stopping time τ ,

$$(3.15) \quad \mathbb{E}_x f(X_{t \wedge \tau}^\varepsilon) = f(x) + \mathbb{E}_x \left[\int_0^{t \wedge \tau} \mathcal{L}^\varepsilon f(X_s^\varepsilon) ds \right].$$

For any $R > 1$, we write

$$\begin{aligned}
 \mathcal{L}^\varepsilon f(x) &= \varphi(1/\varepsilon) \int_{\{|z| \leq R/\varepsilon\}} (f(x + \varepsilon z) - f(x) - \langle \nabla f(x), \varepsilon z \rangle) k(x/\varepsilon, z) \Pi(dz) \\
 &\quad + \varphi(1/\varepsilon) \int_{\{|z| > R/\varepsilon\}} (f(x + \varepsilon z) - f(x)) k(x/\varepsilon, z) \Pi(dz) \\
 &\quad + \varepsilon \varphi(1/\varepsilon) \langle \nabla f(x), b_{R/\varepsilon}(x/\varepsilon) + b(x/\varepsilon) \rangle \\
 &=: \sum_{i=1}^3 I_i^{\varepsilon,R}(x),
 \end{aligned}$$

where $b_{R/\varepsilon}(x)$ is defined by (1.6). Using (3.13) and following the proof of Lemma 3.3, we have

$$\begin{aligned}
 \sup_{x \in \mathbb{R}^d} |I_1^{\varepsilon,R}(x)| &\leq \|\nabla^2 f\|_\infty \varepsilon^2 \varphi(1/\varepsilon) \left(\int_{\{|z| \leq 1\}} |z|^2 \Pi(dz) + \int_1^{R/\varepsilon} \frac{r}{\varphi(r)} dr \right), \\
 \sup_{x \in \mathbb{R}^d} |I_2^{\varepsilon,R}(x)| &\leq \|f\|_\infty \varphi(1/\varepsilon) \int_{R/\varepsilon}^\infty \frac{1}{r\varphi(r)} dr, \\
 \sup_{x \in \mathbb{R}^d} |I_3^{\varepsilon,R}(x)| &\leq \|\nabla f\|_\infty \varepsilon \varphi(1/\varepsilon) \left(\|b\|_\infty + \int_1^{R/\varepsilon} \frac{1}{\varphi(r)} dr \right).
 \end{aligned}$$

Hence, for every $R > 1$,

$$\begin{aligned}
 \sup_{x \in \mathbb{R}^d} |\mathcal{L}^\varepsilon f(x)| &\leq \|\nabla^2 f\|_\infty \varepsilon^2 \varphi(1/\varepsilon) \left(1 + \int_1^{R/\varepsilon} \frac{r}{\varphi(r)} dr\right) \\
 &+ \|f\|_\infty \varphi(1/\varepsilon) \int_{R/\varepsilon}^\infty \frac{1}{r\varphi(r)} dr \\
 &+ \|\nabla f\|_\infty \varepsilon \varphi(1/\varepsilon) \left(1 + \int_1^{R/\varepsilon} \frac{1}{\varphi(r)} dr\right).
 \end{aligned}
 \tag{3.16}$$

(2) In the following, for every $l > 0$, let $f_l \in C_b^3(\mathbb{R}^d)$ be such that

$$f_l(x) = \begin{cases} 0, & |x| \leq l/2, \\ 1, & |x| \geq l \end{cases}
 \tag{3.17}$$

and $\|\nabla^i f_l\|_\infty \leq l^{-i}$ for $0 \leq i \leq 3$. For any fixed $y \in \mathbb{R}^d$, we set $f_l^y(x) := f_l(x - y)$. Then, according to (3.16), we have

$$\begin{aligned}
 \sup_{x, y \in \mathbb{R}^d} |\mathcal{L}^\varepsilon f_l^y(x)| &\leq R^{-2} \varepsilon^2 \varphi(1/\varepsilon) \left(1 + \int_1^{R/\varepsilon} \frac{r}{\varphi(r)} dr\right) + \varphi(1/\varepsilon) \int_{R/\varepsilon}^\infty \frac{1}{r\varphi(r)} dr \\
 &+ R^{-1} \varepsilon \varphi(1/\varepsilon) \left(1 + \int_1^{R/\varepsilon} \frac{1}{\varphi(r)} dr\right).
 \end{aligned}$$

This along with (3.15) yields that, for any $T > 0$,

$$\begin{aligned}
 \mathbb{P}_0\left(\sup_{t \in [0, T]} |X_t^\varepsilon| > R\right) &\leq \mathbb{E}[f_R(X_{T \wedge \tau_R^\varepsilon})] = \mathbb{E}\left[\int_0^{T \wedge \tau_R^\varepsilon} \mathcal{L}^\varepsilon f_R(X_s^\varepsilon) ds\right] \\
 &\leq T \left[R^{-2} \varepsilon^2 \varphi(1/\varepsilon) \left(1 + \int_1^{R/\varepsilon} \frac{r}{\varphi(r)} dr\right) + \varphi(1/\varepsilon) \int_{R/\varepsilon}^\infty \frac{1}{r\varphi(r)} dr \right. \\
 &\quad \left. + R^{-1} \varepsilon \varphi(1/\varepsilon) \left(1 + \int_1^{R/\varepsilon} \frac{1}{\varphi(r)} dr\right) \right].
 \end{aligned}$$

Here and in what follows, $\tau_l^\varepsilon := \inf\{t > 0 : |X_t^\varepsilon - X_0^\varepsilon| > l\}$. Hence, according to (3.11), we have

$$\lim_{R \rightarrow \infty} \sup_{\varepsilon \in (0, 1)} \mathbb{P}_0\left(\sup_{t \in [0, T]} |X_t^\varepsilon| > R\right) = 0.
 \tag{3.18}$$

On the other hand, following the argument in (3.16), we can obtain that, for every $\theta \in (0, 1)$ and $y \in \mathbb{R}^d$,

$$\begin{aligned}
 \sup_{x \in \mathbb{R}^d} |\mathcal{L}^\varepsilon f_\theta^y(x)| &\leq \|\nabla^2 f_\theta^y\|_\infty \varepsilon^2 \varphi(1/\varepsilon) \left(1 + \int_1^{\theta/\varepsilon} \frac{r}{\varphi(r)} dr\right) + \|f_\theta^y\|_\infty \varphi(1/\varepsilon) \int_{\theta/\varepsilon}^\infty \frac{1}{r\varphi(r)} dr \\
 &+ \|\nabla f_\theta^y\|_\infty \varepsilon \varphi(1/\varepsilon) \left(1 + \int_1^{\theta/\varepsilon} \frac{1}{\varphi(r)} dr\right) \\
 &\leq \theta^{-2} \varepsilon \varphi(1/\varepsilon) + \varphi(1/\varepsilon) \int_{\theta/\varepsilon}^\infty \frac{1}{r\varphi(r)} dr + \theta^{-1} \varepsilon \varphi(1/\varepsilon) \int_1^{\theta/\varepsilon} \frac{1}{\varphi(r)} dr.
 \end{aligned}$$

It follows from (3.11) that, for every $\theta \in (0, 1)$,

$$\sup_{\varepsilon \in (0, 1)} \sup_{x, y \in \mathbb{R}^d} |\mathcal{L}^\varepsilon f_\theta^y(x)| \leq C(\theta) < \infty.$$

Therefore, for any stopping time τ with $\tau \leq T$ and any positive constant $\delta(\varepsilon)$ with $\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0$,

$$\begin{aligned} \mathbb{P}_0(|X_{\tau+\delta(\varepsilon)}^\varepsilon - X_\tau^\varepsilon| > \theta) &= \mathbb{E}_0[\mathbb{P}_{X_\tau^\varepsilon}(|X_{\delta(\varepsilon)}^\varepsilon - X_0^\varepsilon| > \theta)] \leq \mathbb{E}_0(\mathbb{E}_{X_\tau^\varepsilon} f_\theta(X_{\tau_\theta^\varepsilon \wedge \delta(\varepsilon)}^\varepsilon)) \\ &= \mathbb{E}_0\left[\mathbb{E}_{X_\tau^\varepsilon}\left(\int_0^{\tau_\theta^\varepsilon \wedge \delta(\varepsilon)} \mathcal{L}^\varepsilon f_\theta(X_s^\varepsilon) ds\right)\right] \leq C(\theta)\delta(\varepsilon) \end{aligned}$$

which implies

$$(3.19) \quad \lim_{\varepsilon \rightarrow 0} \mathbb{P}_0(|X_{\tau+\delta(\varepsilon)}^\varepsilon - X_\tau^\varepsilon| > \theta) = 0.$$

Due to (3.18) and (3.19) (see, e.g., [1], Theorem 1), we conclude that $\{X^\varepsilon\}_{\{\varepsilon \in (0,1]\}}$ is tight in $\mathcal{D}([0, \infty); \mathbb{R}^d)$.

(3) By (2) for any sequence $\{X^{\varepsilon_n}\}_{n \geq 1}$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, there is a subsequence $\{X^{\varepsilon_{n_k}}\}_{k \geq 1}$ (which we still denote by $\{X^{\varepsilon_n}\}_{n \geq 1}$ below for the notional simplicity) such that the distribution of $\{X^{\varepsilon_n}\}_{n \geq 1}$ on $\mathcal{D}([0, \infty); \mathbb{R}^d)$ equipped with the Skorohod topology converges weakly to a probability measure $\bar{\mathbb{P}}$ on $\mathcal{D}([0, \infty); \mathbb{R}^d)$. Let

$$(3.20) \quad \bar{\mathcal{L}}f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x))\bar{k}_0(z)\Pi_0(dz),$$

which is the infinitesimal generator of the Lévy process, as in the statement. In particular, the associated martingale problem for $(\bar{\mathcal{L}}, C_c^2(\mathbb{R}^d))$ is unique. Thus, it suffices to verify that, for any subsequence of $\{\varepsilon_n\}_{n \geq 1}$, the limit distribution $\bar{\mathbb{P}}$ is the same as that of the solution to the martingale problem for the operator $(\bar{\mathcal{L}}, C_c^2(\mathbb{R}^d))$.

Due to the fact that the distribution of $\{X^{\varepsilon_n}\}_{n \geq 1}$ converges weakly to $\bar{\mathbb{P}}$ in $\mathcal{D}([0, \infty); \mathbb{R}^d)$, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a series of stochastic processes $\{\tilde{X}^n\}_{n \geq 1}$ and \tilde{X} defined on it, such that the distribution of \tilde{X}^n under $\tilde{\mathbb{P}}$ is the same as that of X^{ε_n} under \mathbb{P}_0 for any $n \geq 1$, the distribution of \tilde{X} is the same as $\bar{\mathbb{P}}$ and \tilde{X}^n converges to \tilde{X} almost surely in $\mathcal{D}([0, \infty); \mathbb{R}^d)$.

Note again that $X^\varepsilon := ((X_t^\varepsilon)_{t \geq 0}; (\mathbb{P}^x)_{x \in \mathbb{R}^d})$ is a solution to the martingale problem for the operator $(\mathcal{L}^\varepsilon, C_c^2(\mathbb{R}^d))$. Then, for every $0 < s_1 < s_2, \dots < s_k < s \leq t$, $f \in C_c^2(\mathbb{R}^d)$ and $G \in C_b(\mathbb{R}^{dk})$,

$$\tilde{\mathbb{E}}\left[\left(f(\tilde{X}_t^n) - f(\tilde{X}_s^n) - \int_s^t \mathcal{L}^{\varepsilon_n} f(\tilde{X}_r^n) dr\right)G(\tilde{X}_{s_1}^n, \dots, \tilde{X}_{s_k}^n)\right] = 0.$$

According to (3.16), Lemma 3.3 and the dominated convergence theorem,

$$(3.21) \quad \lim_{n \rightarrow \infty} \tilde{\mathbb{E}}\left[\left(f(\tilde{X}_t^n) - f(\tilde{X}_s^n) - \int_s^t \tilde{\mathcal{L}}^{\varepsilon_n} f(\tilde{X}_r^n) dr\right)G(\tilde{X}_{s_1}^n, \dots, \tilde{X}_{s_k}^n)\right] = 0,$$

where $\tilde{\mathcal{L}}^\varepsilon f(x)$ is defined by (3.12). Set $F : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $\bar{F} : \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$F(x, y) := \int_{\mathbb{R}^d} (f(y+z) - f(y))\bar{k}(x, z)\Pi_0(dz), \quad \bar{F}(y) := \int_{\mathbb{T}^d} F(x, y)\mu(dx),$$

where $\bar{k}(x, z)$ is given by (3.6). Then, $\tilde{\mathcal{L}}^\varepsilon f(x) = F(x/\varepsilon, x)$ and $\bar{\mathcal{L}}f(x) = \bar{F}(x)$. Therefore,

$$\begin{aligned} &\tilde{\mathbb{E}}\left[\left|\int_s^t \tilde{\mathcal{L}}^{\varepsilon_n} f(\tilde{X}_r^n) dr - \int_s^t \bar{\mathcal{L}}f(\tilde{X}_r) dr\right|\right] \\ &\leq \tilde{\mathbb{E}}\left[\left|\int_s^t (F(\tilde{X}_r^n/\varepsilon_n, \tilde{X}_r^n) - \bar{F}(\tilde{X}_r^n)) dr\right|\right] + \tilde{\mathbb{E}}\left[\left|\int_s^t (\bar{F}(\tilde{X}_r^n) - \bar{F}(\tilde{X}_r)) dr\right|\right] \\ &=: I_1^n + I_2^n. \end{aligned}$$

Following the proof of (3.19), we have that, for any $\theta > 0$ and $\delta(\varepsilon) > 0$ with $\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}_0 \left(\sup_{0 \leq s \leq t \leq s + \delta(\varepsilon)} |X_t^\varepsilon - X_s^\varepsilon| > \theta \right) = 0.$$

Clearly,

$$\mathbb{E}_0 \left[\sup_{0 \leq s \leq t \leq s + \delta(\varepsilon)} |X_t^\varepsilon - X_s^\varepsilon| \wedge 1 \right] \leq \mathbb{P}_0 \left(\sup_{0 \leq s \leq t \leq s + \delta(\varepsilon)} |X_t^\varepsilon - X_s^\varepsilon| > \theta \right) + \theta.$$

By letting $\varepsilon \rightarrow 0$ first and then $\theta \rightarrow 0$ in the inequality above, we get that (2.3) holds as $\lim_{\varepsilon \rightarrow 0} \rho(1/\varepsilon) = \lim_{\varepsilon \rightarrow 0} \varphi(1/\varepsilon) = \infty$. Thus, by Lemma 2.2 and the fact that F is uniformly continuous,

$$(3.22) \quad \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left[\left| \int_s^t (F(\tilde{X}_r^n/\varepsilon_n, \tilde{X}_r^n) - \bar{F}(\tilde{X}_r^n)) dr \right|^2 \right] = 0,$$

and so $\lim_{n \rightarrow \infty} I_1^n = 0$. On the other hand, by the facts that \tilde{X}^n converges almost surely in $\mathcal{D}([0, \infty); \mathbb{R}^d)$ to \tilde{X} and $\bar{F} \in C_b(\mathbb{R}^d)$ as well as the dominated convergence theorem, it holds that $\lim_{n \rightarrow \infty} I_2^n = 0$. Thus, we obtain

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left[\left| \int_s^t \tilde{\mathcal{L}}^{\varepsilon_n} f(\tilde{X}_r^n) dr - \int_s^t \tilde{\mathcal{L}} f(\tilde{X}_r) dr \right| \right] = 0.$$

Putting the estimate above into (3.21) and letting $n \rightarrow \infty$, we get

$$\tilde{\mathbb{E}} \left[\left(f(\tilde{X}_t) - f(\tilde{X}_s) - \int_s^t \tilde{\mathcal{L}} f(\tilde{X}_r) dr \right) G(\tilde{X}_{s_1}, \dots, \tilde{X}_{s_k}) \right] = 0.$$

Thus, $(\tilde{X}, \tilde{\mathbb{P}})$ is a solution for the martingale problem $(\tilde{\mathcal{L}}, C_c^2(\mathbb{R}^d))$. This shows that \tilde{X} is a pure jump Lévy process with Lévy measure $\bar{k}_0(z)\Pi_0(dz)$. \square

3.2. *Invariance principle with recentering:* $\alpha \in [1, 2)$. In this subsection we are concerned with the case that $\alpha \in [1, 2)$ and will present scaling limit theorems that require recentering for X^ε of (3.9). Recall that $\Pi_0(dz)$ is defined by (3.4) and $\bar{k}_0(z) := \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \bar{k}(x, z)\mu(dx)$, where μ is the unique invariant probability measure of X on \mathbb{T}^d and $\bar{k}(x, z)$ is the function in (3.6). The following is the main result of this subsection.

THEOREM 3.4. *Suppose that (3.1) and (3.6) hold, and that Assumption (A3) is satisfied:*

(i) *Assume that (3.3) holds with $\alpha = 1$. Let*

$$Y_t^\varepsilon := X_t^\varepsilon - \varepsilon\varphi(1/\varepsilon)(\bar{b}_{1/\varepsilon} + \bar{b})t = \varepsilon(X_{\varphi(1/\varepsilon)t} - \varphi(1/\varepsilon)(\bar{b}_{1/\varepsilon} + \bar{b})t), \quad t \geq 0,$$

where $\bar{b}_{1/\varepsilon} := \int_{\mathbb{T}^d} b_{1/\varepsilon}(x)\mu(dx)$ and $\bar{b} := \int_{\mathbb{T}^d} b(x)\mu(dx)$. Then, as $\varepsilon \rightarrow 0$, $(Y_t^\varepsilon)_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$ to a Cauchy (i.e., 1-stable) Lévy process whose generator $\tilde{\mathcal{L}}$ is

$$(3.23) \quad \tilde{\mathcal{L}}f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z\mathbb{1}_{\{|z| \leq 1\}} \rangle) \bar{k}_0(z)\Pi_0(dz).$$

(ii) *Assume that (3.3) holds with $\alpha \in (1, 2)$. Let*

$$Y_t^\varepsilon := X_t^\varepsilon - \varepsilon\varphi(1/\varepsilon)(\bar{b}_\infty + \bar{b})t = \varepsilon(X_{\varphi(1/\varepsilon)t} - \varphi(1/\varepsilon)(\bar{b}_\infty + \bar{b})t), \quad t \geq 0,$$

where $\bar{b}_\infty := \int_{\mathbb{T}^d} b_\infty(x)\mu(dx)$. Then, as $\varepsilon \rightarrow 0$, $(Y_t^\varepsilon)_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$ to an α -stable Lévy process whose generator $\tilde{\mathcal{L}}$ is

$$\tilde{\mathcal{L}}f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle) \bar{k}_0(z)\Pi_0(dz).$$

Note that, when (3.3) holds with $\alpha \in (1, 2)$ (resp., $\alpha = 1$), $\lim_{\varepsilon \rightarrow 0} \varepsilon \varphi(1/\varepsilon) = \infty$ (resp., $\lim_{\varepsilon \rightarrow 0} \varepsilon \varphi(1/\varepsilon) > 0$). So in assumptions of Theorem 3.4, one really needs to recenter X^ε in order to have a limit.

To prove Theorem 3.4, we need two lemmas. The first one is analogous to Lemma 3.3. Recall that the infinitesimal generator \mathcal{L}^ε , given by (3.10), of the process X^ε can be written as

$$\mathcal{L}^\varepsilon f(x) = \mathcal{L}_0^\varepsilon f(x) + \varepsilon \varphi(1/\varepsilon) \langle \nabla f(x), b_{1/\varepsilon}(x/\varepsilon) + b(x/\varepsilon) \rangle$$

and

$$\mathcal{L}_0^\varepsilon f(x) := \varphi(1/\varepsilon) \int_{\mathbb{R}^d} (f(x + \varepsilon z) - f(x) - \langle \nabla f(x), \varepsilon z \rangle \mathbb{1}_{\{|z| \leq 1/\varepsilon\}}) k(x/\varepsilon, z) \Pi(dz).$$

Note that, according to (3.13),

$$\int_{\{|z| > 1\}} |z| k(x, z) \Pi(dz) \leq \int_1^\infty \frac{1}{\varphi(r)} dr,$$

and so we can define

$$(3.24) \quad b_\infty(x) = \int_{\{|z| > 1\}} z k(x, z) \Pi(dz),$$

provided

$$(3.25) \quad \int_1^\infty \frac{1}{\varphi(r)} dr < \infty.$$

In this case

$$\mathcal{L}^\varepsilon f(x) = \mathcal{L}_1^\varepsilon f(x) + \varepsilon \varphi(1/\varepsilon) \langle \nabla f(x), b_\infty(x/\varepsilon) + b(x/\varepsilon) \rangle,$$

where

$$(3.26) \quad \mathcal{L}_1^\varepsilon f(x) = \varphi(1/\varepsilon) \int_{\mathbb{R}^d} (f(x + \varepsilon z) - f(x) - \langle \nabla f(x), \varepsilon z \rangle) k(x/\varepsilon, z) \Pi(dz).$$

LEMMA 3.5. (i) For any $\varepsilon \in (0, 1)$ and $x \in \mathbb{R}^d$, define

$$(3.27) \quad \mathcal{L}_{0,x}^\varepsilon f(y) := \varphi(1/\varepsilon) \int_{\mathbb{R}^d} (f(y + \varepsilon z) - f(y) - \langle \nabla f(y), \varepsilon z \rangle \mathbb{1}_{\{|z| \leq 1/\varepsilon\}}) k(x/\varepsilon, z) \Pi(dz).$$

Suppose that (3.3) holds with $\alpha = 1$. Then, for every $f \in C_b^2(\mathbb{R}^d)$,

$$(3.28) \quad \lim_{\varepsilon \rightarrow 0} \sup_{x, y \in \mathbb{R}^d} |\mathcal{L}_{0,x}^\varepsilon f(y) - \tilde{\mathcal{L}}_{0,x}^\varepsilon f(y)| = 0,$$

where

$$\tilde{\mathcal{L}}_{0,x}^\varepsilon f(y) := \int_{\mathbb{R}^d} (f(y + z) - f(y) - \langle \nabla f(y), z \rangle \mathbb{1}_{\{|z| \leq 1\}}) \bar{k}(x/\varepsilon, z) \Pi_0(dz).$$

(ii) For any $\varepsilon \in (0, 1)$ and $x \in \mathbb{R}^d$, define

$$\mathcal{L}_{1,x}^\varepsilon f(y) = \varphi(1/\varepsilon) \int_{\mathbb{R}^d} (f(y + \varepsilon z) - f(y) - \langle \nabla f(y), \varepsilon z \rangle) k(x/\varepsilon, z) \Pi(dz).$$

Suppose that (3.3) holds with $1 < \alpha < 2$. Then, for every $f \in C_b^2(\mathbb{R}^d)$,

$$(3.29) \quad \lim_{\varepsilon \rightarrow 0} \sup_{x, y \in \mathbb{R}^d} |\mathcal{L}_{1,x}^\varepsilon f(y) - \tilde{\mathcal{L}}_{1,x}^\varepsilon f(y)| = 0,$$

where

$$\tilde{\mathcal{L}}_{1,x}^\varepsilon f(y) := \int_{\mathbb{R}^d} (f(y + z) - f(y) - \langle \nabla f(y), z \rangle) \bar{k}(x/\varepsilon, z) \Pi_0(dz).$$

PROOF. We only prove (ii), since the proof of (i) is similar and simpler.

Suppose that $1 < \alpha < 2$. Then, by (3.3) we have (3.25), and so $\mathcal{L}_{1,x}^\varepsilon f$ is well defined for any $\varepsilon \in (0, 1)$ and $x \in \mathbb{R}^d$. Moreover, according to (3.3) and $1 < \alpha < 2$,

$$(3.30) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \varphi(1/\varepsilon) &= 0, & \lim_{\delta \rightarrow 0} \left(\int_0^\delta \frac{r}{\varphi(r)} dr + \int_{1/\delta}^\infty \frac{1}{\varphi(r)} dr \right) &= 0, \\ \lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \left(\varepsilon^2 \varphi(1/\varepsilon) \int_1^{\delta/\varepsilon} \frac{r}{\varphi(r)} dr \right) &= 0, \\ \lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \left(\varepsilon \varphi(1/\varepsilon) \int_{1/(\delta\varepsilon)}^\infty \frac{1}{\varphi(r)} dr \right) &= 0. \end{aligned}$$

The proof for (3.30) is similar to that of (3.11), and we omit the details.

For every $\delta \in (0, 1)$ and $x \in \mathbb{R}^d$, we write

$$\begin{aligned} \mathcal{L}_{1,x}^\varepsilon f(y) &= \varphi(1/\varepsilon) \left(\int_{\{|z| \leq \delta/\varepsilon\}} + \int_{\{\delta/\varepsilon < |z| < 1/(\delta\varepsilon)\}} + \int_{\{|z| \geq 1/(\delta\varepsilon)\}} \right) \\ &\quad \times (f(y + \varepsilon z) - f(y) - \langle \nabla f(y), \varepsilon z \rangle) k(x/\varepsilon, z) \Pi(dz) \\ &=: \sum_{i=1}^3 \mathcal{L}_{1,x,i}^{\varepsilon,\delta} f(y) \end{aligned}$$

and

$$\begin{aligned} \bar{\mathcal{L}}_{1,x}^\varepsilon f(y) &= \left(\int_{\{|z| \leq \delta\}} + \int_{\{\delta < |z| < 1/\delta\}} + \int_{\{|z| \geq 1/\delta\}} \right) \\ &\quad \times (f(y + z) - f(y) - \langle \nabla f(y), z \rangle) \bar{k}(x/\varepsilon, z) \Pi_0(dz) \\ &=: \sum_{i=1}^3 \bar{\mathcal{L}}_{1,x,i}^{\varepsilon,\delta} f(y). \end{aligned}$$

By (3.4) and (3.30), it is obvious that

$$\begin{aligned} &\lim_{\delta \rightarrow 0} \sup_{\varepsilon \in (0,1)} \sup_{x,y \in \mathbb{R}^d} (|\bar{\mathcal{L}}_{1,x,1}^{\varepsilon,\delta} f(y)| + |\bar{\mathcal{L}}_{1,x,3}^{\varepsilon,\delta} f(y)|) \\ &\leq \lim_{\delta \rightarrow 0} \left(\|\nabla^2 f\|_\infty \int_0^\delta \frac{1}{r^{\alpha-1}} dr + \|f\|_\infty \int_{1/\delta}^\infty \frac{1}{r^{1+\alpha}} dr + \|\nabla f\|_\infty \int_{1/\delta}^\infty \frac{1}{r^\alpha} dr \right) = 0. \end{aligned}$$

On the other hand, according to (3.13), we have

$$\begin{aligned} \sup_{x,y \in \mathbb{R}^d} |\mathcal{L}_{1,x,1}^{\varepsilon,\delta} f(y)| &\leq \|\nabla^2 f\|_\infty \varepsilon^2 \varphi(1/\varepsilon) \int_{\{|z| \leq \delta/\varepsilon\}} |z|^2 \Pi(dz) \\ &\leq \|\nabla^2 f\|_\infty \varepsilon^2 \varphi(1/\varepsilon) \left(\int_{\{|z| \leq 1\}} |z|^2 \Pi(dz) + \int_1^{\delta/\varepsilon} \frac{r}{\varphi(r)} dr \right) \end{aligned}$$

and

$$\begin{aligned} \sup_{x,y \in \mathbb{R}^d} |\mathcal{L}_{1,x,3}^{\varepsilon,\delta} f(y)| &\leq \varphi(1/\varepsilon) \int_{\{|z| \geq 1/(\delta\varepsilon)\}} (\|f\|_\infty + \varepsilon \|\nabla f\|_\infty |z|) \Pi(dz) \\ &\leq (\|f\|_\infty + \|\nabla f\|_\infty) \\ &\quad \times \left(\varphi(1/\varepsilon) \int_{1/(\delta\varepsilon)}^\infty \frac{1}{r\varphi(r)} dr + \varphi(1/\varepsilon) \varepsilon \int_{1/(\delta\varepsilon)}^\infty \frac{1}{\varphi(r)} dr \right) \\ &\leq (\|f\|_\infty + \|\nabla f\|_\infty) \varphi(1/\varepsilon) \varepsilon \int_{1/(\delta\varepsilon)}^\infty \frac{1}{\varphi(r)} dr. \end{aligned}$$

These estimates along with (3.30) yield that

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \sup_{x, y \in \mathbb{R}^d} (|\mathcal{L}_{1,x,1}^{\varepsilon,\delta} f(y)| + |\mathcal{L}_{1,x,3}^{\varepsilon,\delta} f(y)|) = 0.$$

Following the argument for (3.14), we can also obtain that, for every fixed $\delta \in (0, 1)$,

$$\lim_{\varepsilon \rightarrow 0} \sup_{x, y \in \mathbb{R}^d} |\mathcal{L}_{1,x,2}^{\varepsilon,\delta} f(y) - \bar{\mathcal{L}}_{1,x,2}^{\varepsilon,\delta} f(y)| = 0.$$

Combining all the estimates above, by first letting $\varepsilon \rightarrow 0$ and then $\delta \rightarrow 0$, we get the assertion (3.29). \square

In the next lemma we use the convention $1/0 = \infty$.

LEMMA 3.6. *Suppose that Assumption (A3) holds. For any $\varepsilon \in [0, 1]$, let $\psi^\varepsilon \in \mathcal{D}(\mathcal{L})$ be the solution to*

$$(3.31) \quad \mathcal{L}\psi^\varepsilon(x) = -b_{1/\varepsilon}(x) - b(x) + \bar{b}_{1/\varepsilon} + \bar{b}, \quad x \in \mathbb{T}^d$$

with $\mu(\psi^\varepsilon) = 0$. Then,

$$(3.32) \quad \|\psi^\varepsilon\|_\infty + \|\nabla\psi^\varepsilon\|_\infty \leq 1 + \int_1^{1/\varepsilon} \frac{1}{\varphi(r)} dr.$$

PROOF. According to (3.13),

$$\sup_{x \in \mathbb{R}^d} |b_{1/\varepsilon}(x)| \leq \int_{\{1 < |z| \leq 1/\varepsilon\}} |z| \Pi(dz) \leq \int_1^{1/\varepsilon} \frac{1}{\varphi(r)} dr.$$

This along with the fact that $b(x) \in C_b(\mathbb{R}^d)$ and (A3) yields the desired assertion. \square

Now, we are in a position to present the following proof.

PROOF OF THEOREM 3.4. (1) Suppose that Assumption (A3) holds. We first assume that the solution ψ^ε of (3.31) satisfies that $\mu(\psi^\varepsilon) = 0$ and also $\psi^\varepsilon \in C^2(\mathbb{T}^d)$. Set $\Phi_\varepsilon(x) := x + \varepsilon\psi^\varepsilon(x/\varepsilon)$. Define

$$Z_t^\varepsilon := Y_t^\varepsilon + \varepsilon\psi^\varepsilon(X_t^\varepsilon/\varepsilon) = \Phi_\varepsilon(X_t^\varepsilon) - \varepsilon\varphi(1/\varepsilon)(\bar{b}_{1/\varepsilon} + \bar{b})t, \quad t \geq 0.$$

For $f \in C_b^2(\mathbb{R}^d)$, define

$$f_{\varepsilon,s}(x) := f(x - \varepsilon\varphi(1/\varepsilon)(\bar{b}_{1/\varepsilon} + \bar{b})s) \quad \text{and} \quad F_\varepsilon(s, x) = f_{\varepsilon,s}(\Phi_\varepsilon(x)).$$

Clearly, $f(Z_t^\varepsilon) = F_\varepsilon(t, X_t^\varepsilon)$. Since $X^\varepsilon := ((X_t^\varepsilon)_{t \geq 0}; (\mathbb{P}_x)_{x \in \mathbb{R}^d})$ is a solution to the martingale problem for the operator \mathcal{L}^ε , it holds that, for any $x \in \mathbb{R}^d$, $t > 0$, $f \in C_b^3(\mathbb{R}^d)$ and any stopping time τ ,

$$\mathbb{E}_x[f(Z_{t \wedge \tau}^\varepsilon)] = f(x + \varepsilon\psi^\varepsilon(\varepsilon^{-1}x)) + \mathbb{E}_x \left[\int_0^{t \wedge \tau} \left(\frac{\partial F_\varepsilon}{\partial s}(s, X_s^\varepsilon) + \mathcal{L}^\varepsilon F_\varepsilon(s, \cdot)(X_s^\varepsilon) \right) ds \right].$$

Note that

$$\frac{\partial F_\varepsilon}{\partial s}(s, x) = -\varepsilon\varphi(1/\varepsilon)\langle \nabla f_{\varepsilon,s}(\Phi_\varepsilon(x)), \bar{b}_{1/\varepsilon} + \bar{b} \rangle.$$

Applying Lemma 2.1 with $R = 1$ and $M = 1/\varepsilon$, we find that

$$\begin{aligned}
 & \mathcal{L}^\varepsilon F_\varepsilon(s, \cdot)(x) \\
 &= \varphi(1/\varepsilon) \int_{\mathbb{R}^d} (f_{\varepsilon,s}(\Phi_\varepsilon(x + \varepsilon z)) - f_{\varepsilon,s}(\Phi_\varepsilon(x)) - \langle \nabla(f_{\varepsilon,s}(\Phi_\varepsilon(\cdot)))(x), \varepsilon z \rangle \mathbb{1}_{\{|z| \leq 1\}}) \\
 & \quad \times k(x/\varepsilon, z) \Pi(dz) \\
 & \quad + \varepsilon \varphi(1/\varepsilon) \langle \nabla(f_{\varepsilon,s}(\Phi_\varepsilon(\cdot)))(x), b(x/\varepsilon) \rangle \\
 &= \varphi(1/\varepsilon) \int_{\mathbb{R}^d} (f_{\varepsilon,s}(\Phi_\varepsilon(x + \varepsilon z)) - f_{\varepsilon,s}(\Phi_\varepsilon(x)) - \langle \nabla(f_{\varepsilon,s}(\Phi_\varepsilon(\cdot)))(x), \varepsilon z \rangle \mathbb{1}_{\{|z| \leq 1/\varepsilon\}}) \\
 & \quad \times k(x/\varepsilon, z) \Pi(dz) \\
 & \quad + \varepsilon \varphi(1/\varepsilon) \langle \nabla(f_{\varepsilon,s}(\Phi_\varepsilon(\cdot)))(x), b_{1/\varepsilon}(x/\varepsilon) + b(x/\varepsilon) \rangle \\
 &= \varphi(1/\varepsilon) \int_{\mathbb{R}^d} (f_{\varepsilon,s}(\Phi_\varepsilon(x) + \varepsilon z) - f_{\varepsilon,s}(\Phi_\varepsilon(x)) - \langle \nabla f_{\varepsilon,s}(\Phi_\varepsilon(x)), \varepsilon z \rangle \mathbb{1}_{\{|z| \leq 1/\varepsilon\}}) \\
 & \quad \times k(x/\varepsilon, z) \Pi(dz) \\
 & \quad + \varepsilon \varphi(1/\varepsilon) \left\langle \nabla f_{\varepsilon,s}(\Phi_\varepsilon(x)), \int_{\mathbb{R}^d} (\psi^\varepsilon(x/\varepsilon + z) - \psi^\varepsilon(x/\varepsilon) \right. \\
 & \quad \left. - \langle \nabla \psi^\varepsilon(x/\varepsilon), z \rangle \mathbb{1}_{\{|z| \leq 1/\varepsilon\}}) k(x/\varepsilon, z) \Pi(dz) \right\rangle \\
 & \quad + \varepsilon \varphi(1/\varepsilon) \langle \nabla f_{\varepsilon,s}(\Phi_\varepsilon(x)), b_{1/\varepsilon}(x/\varepsilon) + b(x/\varepsilon) \rangle \\
 & \quad + \varepsilon \varphi(1/\varepsilon) \langle \nabla f_{\varepsilon,s}(\Phi_\varepsilon(x)), \nabla \psi^\varepsilon(x/\varepsilon) \cdot (b_{1/\varepsilon}(x/\varepsilon) + b(x/\varepsilon)) \rangle \\
 & \quad + \varphi(1/\varepsilon) \int_{\mathbb{R}^d} K_\varepsilon(x, z) k(x/\varepsilon, z) \Pi(dz) \\
 &= \mathcal{L}_{0,x}^\varepsilon f_{\varepsilon,s}(\Phi_\varepsilon(x)) + \varepsilon \varphi(1/\varepsilon) \langle \nabla f_{\varepsilon,s}(\Phi_\varepsilon(x)), (\mathcal{L}\psi^\varepsilon + b_{1/\varepsilon} + b)(x/\varepsilon) \rangle \\
 & \quad + \varphi(1/\varepsilon) \int_{\mathbb{R}^d} K_\varepsilon(x, z) k(x/\varepsilon, z) \Pi(dz),
 \end{aligned}$$

where \mathcal{L}^ε and $\mathcal{L}_{0,x}^\varepsilon$ are defined by (3.10) and (3.27), respectively, and $K_\varepsilon(x, z)$ satisfies that

$$\begin{aligned}
 & |K_\varepsilon(x, z)| \\
 & \leq (\varepsilon^3 \|\nabla^3 f\|_\infty + \varepsilon^2 \|\nabla^2 f\|_\infty) (1 + \|\psi^\varepsilon\|_\infty + \|\nabla \psi^\varepsilon\|_\infty)^3 \\
 & \quad \times (|z|^2 \mathbb{1}_{\{|z| \leq 1\}} + |z| \mathbb{1}_{\{1 < |z| \leq 1/\varepsilon\}} + \mathbb{1}_{\{|z| > 1/\varepsilon\}}) \\
 & \quad + \|f\|_\infty \mathbb{1}_{\{|z| > 1/\varepsilon\}}.
 \end{aligned}$$

According to all estimates above, (3.31), (3.13) and (3.32), we have that

$$\begin{aligned}
 & \frac{\partial F_\varepsilon}{\partial s}(s, x) + \mathcal{L}^\varepsilon F_\varepsilon(s, \cdot)(x) \\
 &= \mathcal{L}_{0,x}^\varepsilon f_{\varepsilon,s}(\Phi_\varepsilon(x)) \\
 (3.33) \quad & \quad + \varepsilon \varphi(1/\varepsilon) \langle \nabla f_{\varepsilon,s}(\Phi_\varepsilon(x)), (\mathcal{L}\psi^\varepsilon + b_{1/\varepsilon} + b - \bar{b}_{1/\varepsilon} - \bar{b})(x/\varepsilon) \rangle \\
 & \quad + \varphi(1/\varepsilon) \int_{\mathbb{R}^d} K_\varepsilon(x, z) k(x/\varepsilon, z) \Pi(dz) \\
 &= \mathcal{L}_{0,x}^\varepsilon f_{\varepsilon,s}(\Phi_\varepsilon(x)) + H_\varepsilon(x),
 \end{aligned}$$

where

$$\begin{aligned}
 \sup_{x \in \mathbb{R}^d} |H_\varepsilon(x)| &\leq \varepsilon^2 \varphi(1/\varepsilon) \left(\sum_{i=2}^3 \|\nabla^i f\|_\infty \right) \left(1 + \int_1^{1/\varepsilon} \frac{1}{\varphi(r)} dr \right)^3 \\
 (3.34) \quad &\times \left(1 + \int_1^{1/\varepsilon} \frac{1}{\varphi(r)} dr + \int_{1/\varepsilon}^\infty \frac{1}{r\varphi(r)} dr \right) \\
 &+ \varphi(1/\varepsilon) \|f\|_\infty \int_{1/\varepsilon}^\infty \frac{1}{r\varphi(r)} dr.
 \end{aligned}$$

(2) For any $l \geq 1$, let f_l be the function defined by (3.17). Then, for any $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
 |\mathcal{L}_{0,x}^\varepsilon f_l(y)| &\leq \varphi(1/\varepsilon) \left| \int_{\{|z| \leq 1/\varepsilon\}} (f_l(y + \varepsilon z) - f_l(y) - \langle \nabla f_l(y), \varepsilon z \rangle) k(x/\varepsilon, z) \Pi(dz) \right| \\
 &+ \varphi(1/\varepsilon) \left| \int_{\{1/\varepsilon < |z| \leq l/\varepsilon\}} (f_l(y + \varepsilon z) - f_l(y)) k(x/\varepsilon, z) \Pi(dz) \right| \\
 &+ \varphi(1/\varepsilon) \left| \int_{\{|z| > l/\varepsilon\}} (f_l(x + \varepsilon z) - f_l(x)) k(x/\varepsilon, z) \Pi(dz) \right| \\
 (3.35) \quad &\leq \|\nabla^2 f_l\|_\infty \varphi(1/\varepsilon) \varepsilon^2 \int_{\{|z| \leq 1/\varepsilon\}} |z|^2 \Pi(dz) \\
 &+ \|\nabla f_l\|_\infty \varepsilon \varphi(1/\varepsilon) \int_{\{1/\varepsilon < |z| \leq l/\varepsilon\}} |z| \Pi(dz) \\
 &+ \|f_l\|_\infty \varphi(1/\varepsilon) \int_{\{|z| > l/\varepsilon\}} \Pi(dz) \\
 &\leq l^{-2} \varepsilon^2 \varphi(1/\varepsilon) \left(1 + \int_1^{1/\varepsilon} \frac{r}{\varphi(r)} dr \right) + l^{-1} \varepsilon \varphi(1/\varepsilon) \int_{1/\varepsilon}^{l/\varepsilon} \frac{1}{\varphi(r)} dr \\
 &+ \varphi(1/\varepsilon) \int_{l/\varepsilon}^\infty \frac{1}{r\varphi(r)} dr.
 \end{aligned}$$

Let $F_{\varepsilon,l}(t, x) := f_l(\Phi_\varepsilon(x) - \varepsilon\varphi(1/\varepsilon)(\bar{b}_{1/\varepsilon} + \bar{b})t)$. Then, combining (3.33), (3.34) with (3.35), we find that

$$\begin{aligned}
 \sup_{x \in \mathbb{R}^d} \sup_{s > 0} &\left| \frac{\partial F_{\varepsilon,l}}{\partial s}(s, x) + \mathcal{L}^\varepsilon F_{\varepsilon,l}(s, \cdot)(x) \right| \\
 (3.36) \quad &\leq l^{-2} \varepsilon^2 \varphi(1/\varepsilon) \left(1 + \int_1^{1/\varepsilon} \frac{r}{\varphi(r)} dr + \left(\int_1^{1/\varepsilon} \frac{1}{\varphi(r)} dr \right)^4 + \left(\int_{1/\varepsilon}^\infty \frac{1}{r\varphi(r)} dr \right)^4 \right) \\
 &+ l^{-1} \varepsilon \varphi(1/\varepsilon) \left(\varepsilon + \int_{1/\varepsilon}^{l/\varepsilon} \frac{1}{\varphi(r)} dr + \varepsilon \left(\int_1^{1/\varepsilon} \frac{1}{\varphi(r)} dr \right)^4 + \varepsilon \left(\int_{1/\varepsilon}^\infty \frac{1}{r\varphi(r)} dr \right)^4 \right) \\
 &+ \varphi(1/\varepsilon) \int_{1/\varepsilon}^\infty \frac{1}{r\varphi(r)} dr.
 \end{aligned}$$

Note that in the arguments above for (3.36) we need that $\psi^\varepsilon \in C^2(\mathbb{T}^d)$. For general $\psi^\varepsilon \in \mathcal{D}(\mathcal{L})$ satisfying (3.31), there exists a sequence of function $\{\psi_k^\varepsilon\}_{k \geq 1} \subset C^2(\mathbb{T}^d)$ such that $\mu(\psi_k^\varepsilon) = 0$ for all $k \geq 1$ and

$$\lim_{k \rightarrow \infty} \sup_{x \in \mathbb{T}^d} (|\psi_k^\varepsilon(x) - \psi^\varepsilon(x)| + |\mathcal{L}\psi_k^\varepsilon(x) - \mathcal{L}\psi^\varepsilon(x)|) = 0.$$

This along with (1.5) yields that

$$\sup_{k \geq 1} (\|\psi_k^\varepsilon\|_\infty + \|\nabla \psi_k^\varepsilon\|_\infty) \leq 1 + \int_1^{1/\varepsilon} \frac{1}{\varphi(r)} dr.$$

For the arguments above we have used the facts that $\mathcal{L}\psi_k^\varepsilon \in C(\mathbb{T}^d)$ with $\mu(\mathcal{L}\psi_k^\varepsilon) = 0$ for all $k \geq 1$ and the solution to (1.4) with $f = \mathcal{L}\psi_k^\varepsilon$ is unique. By a standard approximation procedure, it is not difficult to verify that (3.36) still holds true for every $\psi^\varepsilon \in \mathcal{D}(\mathcal{L})$, satisfying (3.31).

We now assume that (3.3) holds with $\alpha = 1$. It follows from (3.3) that

$$\varphi(1/\varepsilon) \int_1^{1/\varepsilon} \frac{r}{\varphi(r)} dr \leq \int_1^{1/\varepsilon} r(\varepsilon r)^{-1-\eta_0} dr \leq \varepsilon^{-2}.$$

Estimating other terms in the right-hand side of (3.36) by the same way as above, we get

$$\lim_{R \rightarrow \infty} \sup_{\varepsilon \in (0,1)} \sup_{x \in \mathbb{R}^d} \sup_{s > 0} \left| \frac{\partial F_{\varepsilon,R}}{\partial s}(s, x) + \mathcal{L}^\varepsilon F_{\varepsilon,R}(s, \cdot)(x) \right| = 0.$$

Since for every $R > 1$ and $T > 0$,

$$\begin{aligned} \mathbb{P}_0 \left(\sup_{t \in [0, T]} |Z_t^\varepsilon| > R \right) &\leq \mathbb{E} f_R(Z_{T \wedge \tau_R^\varepsilon}) = \mathbb{E} \left[\int_0^{T \wedge \tau_R^\varepsilon} \left(\frac{\partial F_{\varepsilon,R}}{\partial s}(s, X_s^\varepsilon) + \mathcal{L}^\varepsilon F_{\varepsilon,R}(s, \cdot)(X_s^\varepsilon) \right) ds \right] \\ &\leq T \sup_{x \in \mathbb{R}^d} \sup_{s > 0} \left| \frac{\partial F_{\varepsilon,R}}{\partial s}(s, x) + \mathcal{L}^\varepsilon F_{\varepsilon,R}(s, \cdot)(x) \right|, \end{aligned}$$

we conclude that

$$(3.37) \quad \lim_{R \rightarrow \infty} \sup_{\varepsilon \in (0,1)} \mathbb{P}_0 \left(\sup_{t \in [0, T]} |Z_t^\varepsilon| > R \right) = 0.$$

According to the argument for (3.35), we can also obtain that, for every $\theta \in (0, 1)$,

$$\begin{aligned} &\sup_{x \in \mathbb{R}^d} \sup_{s > 0} \left| \frac{\partial F_{\varepsilon,\theta}}{\partial s}(s, x) + \mathcal{L}^\varepsilon F_{\varepsilon,\theta}(s, \cdot)(x) \right| \\ &\leq \theta^{-3} \varepsilon^2 \varphi(1/\varepsilon) \left(1 + \int_1^{1/\varepsilon} \frac{r}{\varphi(r)} dr + \left(\int_1^{1/\varepsilon} \frac{1}{\varphi(r)} dr \right)^4 \right) \\ &\quad + \varphi(1/\varepsilon) \int_{1/\varepsilon}^\infty \frac{1}{r\varphi(r)} dr. \end{aligned}$$

By this estimate and the fact that (3.3) holds with $\alpha = 1$, we have

$$\sup_{\varepsilon \in (0,1)} \sup_{x \in \mathbb{R}^d} \sup_{s > 0} \left| \frac{\partial F_{\varepsilon,\theta}}{\partial s}(s, x) + \mathcal{L}^\varepsilon F_{\varepsilon,\theta}(s, \cdot)(x) \right| \leq C(\theta)$$

for some constant $C(\theta) > 0$. This, together with the proof of (3.37), gives us that, for any increasing function $\delta(\varepsilon)$ with $\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0$ and stopping time τ with $\tau \leq T$,

$$(3.38) \quad \lim_{\varepsilon \rightarrow 0} \mathbb{P}_0(|Z_{\tau+\delta(\varepsilon)}^\varepsilon - Z_\tau^\varepsilon| > \theta) = 0.$$

Therefore, it follows from (3.37) and (3.38) as well as [1], Theorem 1, that the distribution of $\{Z^\varepsilon\}_{\varepsilon \in (0,1)}$ is tight in $\mathcal{D}([0, \infty); \mathbb{R}^d)$.

(3) Let $\{Z^{\varepsilon_n}\}_{n \geq 1}$ be a sequence of processes with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. There is a subsequence $\{Z^{\varepsilon_{n_k}}\}_{k \geq 1}$ (which will be still denoted by $\{Z^{\varepsilon_n}\}_{n \geq 1}$ below for simplicity) such that the distribution of Z^{ε_n} on $\mathcal{D}([0, \infty); \mathbb{R}^d)$ converges weakly under the Skorohod topology to a probability measure $\bar{\mathbb{P}}$ on $\mathcal{D}([0, \infty); \mathbb{R}^d)$. Note that

$$Y_t^\varepsilon = Z_t^\varepsilon + \varepsilon \psi^\varepsilon(X_t^\varepsilon/\varepsilon), \quad t \geq 0.$$

By (3.32), $\lim_{\varepsilon \rightarrow 0} \varepsilon \|\psi^\varepsilon\|_\infty = 0$. This implies that the distribution of $(Y_t^{\varepsilon_n})_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$ to $\bar{\mathbb{P}}$. Similar to the part (3) of the proof for Theorem 3.2, it suffices to verify that, for any subsequence $\{\varepsilon_n\}_{n \geq 1}$, the limit distribution $\bar{\mathbb{P}}$ is the same as that of the solution to the martingale problem for $\bar{\mathcal{L}}$ defined by (3.23).

For every $0 < s_1 < s_2, \dots < s_k < s \leq t$, $f \in C_b^2(\mathbb{R}^d)$ and $G \in C_b(\mathbb{R}^{dk})$, by (3.33),

$$\begin{aligned} & \mathbb{E} \left[\left(f(Z_t^\varepsilon) - f(Z_s^\varepsilon) - \int_s^t \left(\frac{\partial F_\varepsilon}{\partial r}(r, X_r^\varepsilon) + \mathcal{L}^\varepsilon F_\varepsilon(r, \cdot)(X_r^\varepsilon) \right) dr \right) G(Z_{s_1}^\varepsilon, \dots, Z_{s_k}^\varepsilon) \right] \\ &= \mathbb{E}_0 \left[\left(f(Z_t^\varepsilon) - f(Z_s^\varepsilon) - \int_s^t (\mathcal{L}_{0, X_r^\varepsilon}^\varepsilon f_{\varepsilon, r}(\Phi_\varepsilon(X_r^\varepsilon)) + H_\varepsilon(X_r^\varepsilon)) dr \right) G(Z_{s_1}^\varepsilon, \dots, Z_{s_k}^\varepsilon) \right] \\ &= 0. \end{aligned}$$

According to condition (3.3) with $\alpha = 1$ and (3.34), $\lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} |H_\varepsilon(x)| = 0$. Combining this with (3.28) further yields

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_0 \left[\left(f(Z_t^\varepsilon) - f(Z_s^\varepsilon) - \int_s^t \bar{\mathcal{L}}_{0, X_r^\varepsilon}^\varepsilon f(Z_r^\varepsilon) dr \right) G(Z_{s_1}^\varepsilon, \dots, Z_{s_k}^\varepsilon) \right] = 0.$$

Note that, by tracking the proof of Lemma 2.2, we can verify that if (2.3) holds with $(X_t^\varepsilon)_{t \geq 0}$ replaced by $(Z_t^\varepsilon)_{t \geq 0}$, then for every bounded continuous function $F : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $0 < s < t$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_x \left[\left| \int_s^t F(X_r^\varepsilon/\varepsilon, Z_r^\varepsilon) dr - \int_s^t \bar{F}(Z_r^\varepsilon) dr \right| \right] = 0,$$

where \bar{F} is the same function as in Lemma 2.2. Using this property and (3.38), we can follow the argument for (3.22) to obtain

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_0 \left[\left| \int_s^t \bar{\mathcal{L}}_{0, X_r^\varepsilon}^\varepsilon f(Z_r^\varepsilon) dr - \int_s^t \bar{\mathcal{L}} f(Z_r^\varepsilon) dr \right|^2 \right] = 0.$$

Hence,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_0 \left[\left(f(Z_t^\varepsilon) - f(Z_s^\varepsilon) - \int_s^t \bar{\mathcal{L}} f(Z_r^\varepsilon) dr \right) G(Z_{s_1}^\varepsilon, \dots, Z_{s_k}^\varepsilon) \right] = 0,$$

where $\bar{\mathcal{L}}$ is defined in (3.23). Notice further that the distribution of $\{Z^{\varepsilon_n}\}_{n \geq 1}$ converges weakly to $\bar{\mathbb{P}}$. According to the proof of Theorem 3.2 (in particular, by applying the Skorohod representation theorem), letting $\varepsilon = \varepsilon_n$ and taking $\varepsilon_n \rightarrow 0$ in the equation above give us

$$\bar{\mathbb{E}} \left[\left(f(Z_t) - f(Z_s) - \int_s^t \bar{\mathcal{L}} f(Z_r) dr \right) G(Z_{s_1}, \dots, Z_{s_k}) \right] = 0,$$

where $(Z_t)_{t \geq 0}$ denotes the coordinate process on $\mathcal{D}([0, \infty); \mathbb{R}^d)$ and $\bar{\mathbb{E}}$ denotes the expectation with respect to $\bar{\mathbb{P}}$. This implies that the distribution of $\bar{\mathbb{P}}$ is a solution to the martingale problem for the Lévy operator $\bar{\mathcal{L}}$. By now we have finished the proof for the assertion (i) of Theorem 3.4.

(4) Next, we assume that condition (3.3) holds with $\alpha \in (1, 2)$. In this case, $b_\infty(x)$ is well defined. According to Assumption (A3), let $\psi \in \mathcal{D}(\mathcal{L})$ be the unique solution to the following equation:

$$\mathcal{L}\psi(x) = -b_\infty(x) - b(x) + \bar{b}_\infty + \bar{b}, \quad x \in \mathbb{T}^d$$

with $\mu(\psi) = 0$. By the approximation argument as in Step (2), without loss of generality we can suppose that $\psi \in C^2(\mathbb{T}^d)$. For every $f \in C_b^3(\mathbb{R}^d)$, define

$$F_\varepsilon(s, x) := f(x + \varepsilon\psi(x/\varepsilon) - \varepsilon\varphi(1/\varepsilon)(\bar{b}_\infty + \bar{b})s).$$

Applying Lemma 2.1 with $R = 1$ and $M = \infty$ and following the same arguments for (3.33) and (3.36), we obtain

$$\frac{\partial F_\varepsilon}{\partial s}(s, x) + \mathcal{L}^\varepsilon F_\varepsilon(s, \cdot)(x) = \mathcal{L}_{1,x}^\varepsilon f_{\varepsilon,s}(\Phi_\varepsilon(x)) + H_\varepsilon(x),$$

where $\Phi_\varepsilon(x) = x + \varepsilon\psi(x/\varepsilon)$, $f_{\varepsilon,s}(x) = f(x - \varepsilon\varphi(1/\varepsilon)(\bar{b}_\infty + \bar{b})s)$, and H_ε satisfies that

$$\begin{aligned} \sup_{x \in \mathbb{R}^d} |H_\varepsilon(x)| &\leq \left(\sum_{i=2}^3 \|\nabla^i f\|_\infty \right) \varphi(1/\varepsilon) \varepsilon^2 \left(\int_{\{|z| \leq 1\}} |z|^2 \Pi(dz) + \int_1^\infty \frac{r}{\varphi(r)} dr \right) \\ &\leq \left(\sum_{i=2}^3 \|\nabla^i f\|_\infty \right) \varphi(1/\varepsilon) \varepsilon^2. \end{aligned}$$

In particular,

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} |H_\varepsilon(x)| = 0.$$

Using these estimates and repeating the proof for the assertion (i) (in particular, applying (3.29) instead of (3.28)), we obtain the assertion (ii) of Theorem 3.4. \square

4. Homogenization: Diffusive scaling. In this section we treat the case that the jumping measure for the nonlocal operator \mathcal{L} of (1.1) has a finite second moment, that is,

$$(4.1) \quad \int_{\mathbb{R}^d} |z|^2 \Pi(dz) < \infty.$$

Under this condition it is natural to conjecture that, after appropriate scaling, X would converge to Brownian motion. Thus, we will take the scaling function $\rho(r) = r^2$ in (2.1) and consider the limit of the scaled process $X^\varepsilon = (X_t^\varepsilon)_{t \geq 0} := (\varepsilon X_{t/\varepsilon^2})_{t \geq 0}$. Here is the main result of this section.

THEOREM 4.1. *Suppose that Assumption (A3) and (4.1) hold. Let*

$$Y_t^\varepsilon := X_t^\varepsilon - (\bar{b}_\infty + \bar{b})t/\varepsilon = \varepsilon(X_{t/\varepsilon^2} - (\bar{b}_\infty + \bar{b})t/\varepsilon^2), \quad t \geq 0.$$

Then, $(Y_t^\varepsilon)_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$, as $\varepsilon \rightarrow 0$, to Brownian motion with the covariance matrix A given by

$$A := \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (z + \psi(x+z) - \psi(x)) \otimes (z + \psi(x+z) - \psi(x)) k(x, z) \Pi(dz) \mu(dx),$$

where $\psi \in \mathcal{D}(\mathcal{L})$ is the unique solution to the following equation:

$$(4.2) \quad \mathcal{L}\psi(x) = -b_\infty(x) - b(x) + \bar{b}_\infty + \bar{b}, \quad x \in \mathbb{T}^d$$

such that $\mu(\psi) = 0$.

REMARK 4.2. Let $\{e_i : 1 \leq i \leq d\}$ be the standard orthonormal basis of \mathbb{R}^d . We claim that if the process X is irreducible and for each e_i , $1 \leq i \leq d$ there exists a sequence $\{z_k^i\}_{k \geq 1} \subset \text{supp}[\Pi]$ such that $z_k^i \neq 0$ for all $k \geq 1$ and

$$(4.3) \quad \lim_{k \rightarrow \infty} z_k^i = 0, \quad \lim_{k \rightarrow \infty} z_k^i / |z_k^i| = e_i, \quad 1 \leq i \leq d,$$

then the covariance matrix A in Theorem 4.1 above is nondegenerate. Indeed, for any $\xi \in \mathbb{R}^d$,

$$\langle A\xi, \xi \rangle = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \langle z + \psi(x+z) - \psi(x), \xi \rangle^2 \Pi(dz) \mu(dx).$$

Since the process X is irreducible, for any $t > 0$, $x \in \mathbb{R}^d$ and open set $U \subset \mathbb{R}^d$, $\mathbb{P}_x(X_t \in U) > 0$. Then, $\mu(U) = \int_{\mathbb{R}^d} \mathbb{P}_x(X_t \in U) \mu(dx) > 0$; that is, $\text{supp}[\mu] = \mathbb{T}^d$.

Now, assume that for some $0 \neq \xi \in \mathbb{R}^d$, $\langle A\xi, \xi \rangle = 0$. Note that under **(A3)**, by the dominated convergence theorem, $x \mapsto \int_{\mathbb{R}^d} \langle z + \psi(x+z) - \psi(x), \xi \rangle^2 \Pi(dz)$ is a continuous function. This along with the fact $\text{supp}[\mu] = \mathbb{T}^d$ yields that, for every $x \in \mathbb{T}^d$,

$$\int_{\mathbb{R}^d} \langle z + \psi(x+z) - \psi(x), \xi \rangle^2 \Pi(dz) = 0.$$

Without loss of generality, we assume that $\xi = (\xi_{(1)}, \dots, \xi_{(d)})$ with $\xi_{(1)} > 0$ (since $\xi \neq 0$). Let $\{z_k^1\}_{k \geq 1}$ be the sequence in the assumptions above. Then, we have

$$\langle \psi(x+z_k^1) - \psi(x), \xi \rangle = \langle -z_k^1, \xi \rangle, \quad k \geq 1, x \in \mathbb{T}^d.$$

By the mean value theorem and the fact that $\|\nabla \psi\|_\infty < \infty$ (see Lemma 3.6 with $\varepsilon = 0$), we have

$$\langle \nabla \psi(x), \xi \rangle, z_k^1/|z_k^1| + o(|z_k^1|/|z_k^1|) = \langle z_k^1/|z_k^1|, -\xi \rangle, \quad k \geq 1, x \in \mathbb{T}^d.$$

Letting $k \rightarrow \infty$ and using (4.3), we obtain

$$\partial_{x_1} \langle \psi(x), \xi \rangle = -\xi_{(1)} < 0, \quad x \in \mathbb{T}^d$$

which, obviously, contradicts with the fact that $x \mapsto \langle \psi(x), \xi \rangle$ is continuous and multivariate periodic. Therefore, we have $\langle A\xi, \xi \rangle > 0$ for every $\xi \neq 0$, and so A is nondegenerate.

We further note that the assumptions above, which guarantee that the covariance matrix A in Theorem 4.1 is nondegenerate, are weak in some sense. For example, let $d = 2$, $\Pi(dz) = \delta_{e_1}(dz)$, and $\xi = e_2$. Then, for any $x \in \mathbb{T}^d$, since ψ is multivariate periodic,

$$\int_{\mathbb{R}^2} \langle z + \psi(x+z) - \psi(x), \xi \rangle^2 \Pi(dz) = (\psi_{(2)}(x+e_1) - \psi_{(2)}(x))^2 = 0,$$

and so $\langle A\xi, \xi \rangle = 0$, where we write $\psi(x) = (\psi_{(1)}(x), \psi_{(2)}(x))$. Hence, the associated covariance matrix A in Theorem 4.1 is degenerate.

Note that the scaled process X^ε is a strong Markov process, whose generator is given by

$$\begin{aligned} \mathcal{L}^\varepsilon f(x) &= \varepsilon^{-2} \int_{\mathbb{R}^d} (f(x+\varepsilon z) - f(x) - \varepsilon \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z| \leq 1\}}) k(x/\varepsilon, z) \Pi(dz) \\ &\quad + \varepsilon^{-1} \langle \nabla f(x), b(x/\varepsilon) \rangle \\ &=: \mathcal{L}_1^\varepsilon f(x) + \varepsilon^{-1} \langle \nabla f(x), b_\infty(x/\varepsilon) + b(x/\varepsilon) \rangle. \end{aligned}$$

Here, $b_\infty(x)$ is defined by (3.24), and

$$(4.4) \quad \mathcal{L}_1^\varepsilon f(x) = \varepsilon^{-2} \int_{\mathbb{R}^d} (f(x+\varepsilon z) - f(x) - \langle \nabla f(x), \varepsilon z \rangle) k(x/\varepsilon, z) \Pi(dz).$$

LEMMA 4.3. For every $x \in \mathbb{R}^d$ and $f \in C_b^3(\mathbb{R}^d)$, define

$$(4.5) \quad \mathcal{L}_{1,x}^\varepsilon f(y) = \varepsilon^{-2} \int_{\mathbb{R}^d} (f(y+\varepsilon z) - f(y) - \langle \nabla f(y), \varepsilon z \rangle) k(x/\varepsilon, z) \Pi(dz).$$

Then,

$$(4.6) \quad \mathcal{L}_{1,x}^\varepsilon f(y) = \left\langle \nabla^2 f(y), \frac{1}{2} \int_{\mathbb{R}^d} (z \otimes z) k(x/\varepsilon, z) \Pi(dz) \right\rangle + G_{1,\varepsilon}(x, y),$$

where $G_{1,\varepsilon}(x, y)$ satisfies

$$(4.7) \quad \lim_{\varepsilon \rightarrow 0} \sup_{x, y \in \mathbb{R}^d} |G_{1,\varepsilon}(x, y)| = 0.$$

PROOF. According to the Taylor expansion, for any $f \in C_b^3(\mathbb{R}^d)$ and $R > 1$,

$$\begin{aligned} f(y + \varepsilon z) - f(y) - \langle \nabla f(y), \varepsilon z \rangle &= \begin{cases} \varepsilon^2 \langle \nabla^2 f(y), z \otimes z \rangle / 2 + \varepsilon^3 \langle \nabla^3 f(y + \theta_1 \varepsilon z), z \otimes z \otimes z \rangle / 6, & |z| \leq R, \\ \varepsilon^2 \langle \nabla^2 f(y + \theta_2 \varepsilon z), z \otimes z \rangle / 2, & |z| > R \end{cases} \\ &= \varepsilon^2 \langle \nabla^2 f(y), z \otimes z \rangle / 2 + H_\varepsilon(y, z), \end{aligned}$$

where $\theta_1, \theta_2 \in (0, 1)$ and

$$(4.8) \quad |H_\varepsilon(y, z)| \leq (\|\nabla^2 f\|_\infty + \|\nabla^3 f\|_\infty) (\varepsilon^3 |z|^3 \mathbb{1}_{\{|z| \leq R\}} + \varepsilon^2 |z|^2 \mathbb{1}_{\{|z| > R\}}).$$

In particular, (4.6) holds with

$$G_{1,\varepsilon}(x, y) = \varepsilon^{-2} \int_{\mathbb{R}^d} H_\varepsilon(y, z) k(x/\varepsilon, z) \Pi(dz).$$

By (4.8) it holds that

$$|G_{1,\varepsilon}(x, y)| \leq C_1 \left(\sum_{i=2}^3 \|\nabla^i f\|_\infty \right) \left(\varepsilon \int_{\{|z| \leq R\}} |z|^3 \Pi(dz) + \int_{\{|z| > R\}} |z|^2 \Pi(dz) \right)$$

with any $R > 0$ and some $C_1 > 0$ (which is independent of f, ε and R). Since $\int_{\mathbb{R}^d} |z|^2 \Pi(dz) < \infty$, first letting $\varepsilon \rightarrow 0$ and then $R \rightarrow \infty$ in the estimate above, we obtain (4.7). \square

PROOF OF THEOREM 4.1. We first assume that $\psi \in C^2(\mathbb{T}^d)$. Let

$$Z_t^\varepsilon = Y_t^\varepsilon + \varepsilon \psi(X_t^\varepsilon/\varepsilon) = X_t^\varepsilon + \varepsilon \psi(X_t^\varepsilon/\varepsilon) - \varepsilon^{-1}(\bar{b}_\infty + \bar{b})t, \quad t \geq 0.$$

Recall that the generator of the process X^ε is

$$\mathcal{L}^\varepsilon f(x) = \mathcal{L}_1^\varepsilon f(x) + \varepsilon^{-1} \langle \nabla f(x), b_\infty(x/\varepsilon) + b(x/\varepsilon) \rangle,$$

where $\mathcal{L}_1^\varepsilon f$ is defined by (4.4). Then, for every $f \in C_b^3(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t > 0$,

$$\mathbb{E}_x[f(Z_t^\varepsilon)] = \mathbb{E}_x[F_\varepsilon(t, X_t^\varepsilon)] = f(\Phi_\varepsilon(x)) + \mathbb{E}_x \left[\int_0^t \left(\frac{\partial F_\varepsilon}{\partial s}(s, X_s^\varepsilon) + \mathcal{L}^\varepsilon F_\varepsilon(s, \cdot)(X_s^\varepsilon) \right) ds \right],$$

where $F_\varepsilon(s, x) = f(\Phi_\varepsilon(x) - \varepsilon^{-1}(\bar{b}_\infty + \bar{b})s)$ and $\Phi_\varepsilon(x) = x + \varepsilon \psi(x/\varepsilon)$.

Let $f_{\varepsilon,s}(x) = f(x - \varepsilon^{-1}(\bar{b}_\infty + \bar{b})s)$. Then, $F_\varepsilon(s, x) = f_{\varepsilon,s}(\Phi_\varepsilon(x))$. Set $\Theta_\varepsilon(x, z) := \psi(x/\varepsilon + z) - \psi(x/\varepsilon)$ and

$$A(x) := \frac{1}{2} \int_{\mathbb{R}^d} (z + \psi(x+z) - \psi(x)) \otimes (z + \psi(x+z) - \psi(x)) k(x, z) \Pi(dz).$$

Applying Lemma 2.1 with $M = \infty$, $R \rightarrow 0$ and using (4.2) and Lemma 4.3, we can verify that

$$\begin{aligned} &\frac{\partial F_\varepsilon}{\partial s}(s, x) + \mathcal{L}^\varepsilon F_\varepsilon(s, \cdot)(x) \\ &= -\varepsilon^{-1} \langle \nabla f_{\varepsilon,s}(\Phi_\varepsilon(x)), \bar{b}_\infty + \bar{b} \rangle + \mathcal{L}_{1,x}^\varepsilon f_{\varepsilon,s}(\Phi_\varepsilon(x)) \\ &\quad + \varepsilon^{-1} \left\langle \nabla f_{\varepsilon,s}(\Phi_\varepsilon(x)), \int (\psi(x/\varepsilon + z) - \psi(x/\varepsilon) - \nabla \psi(x/\varepsilon) \cdot z) k(x/\varepsilon, z) \Pi(dz) \right\rangle \\ &\quad + \frac{1}{2} \left\langle \nabla^2 f_{\varepsilon,s}(\Phi_\varepsilon(x)), \int (2\Theta_\varepsilon(x, z) \otimes z + \Theta_\varepsilon(x, z) \otimes \Theta_\varepsilon(x, z)) k(x/\varepsilon, z) \Pi(dz) \right\rangle \end{aligned}$$

$$\begin{aligned}
 &+ H_1^\varepsilon(x) + \varepsilon^{-1} \langle \nabla f_{\varepsilon,s}(\Phi_\varepsilon(x)), b_\infty(x/\varepsilon) + b(x/\varepsilon) \rangle \\
 &+ \varepsilon^{-1} \langle \nabla f_{\varepsilon,s}(\Phi_\varepsilon(x)), \nabla \psi(x/\varepsilon) \cdot (b_\infty(x/\varepsilon) + b(x/\varepsilon)) \rangle \\
 = &\mathcal{L}_{1,x}^\varepsilon f_{\varepsilon,s}(\Phi_\varepsilon(x)) \\
 &+ \frac{1}{2} \left\langle \nabla^2 f_{\varepsilon,s}(\Phi_\varepsilon(x)), \int (2\Theta_\varepsilon(x, z) \otimes z + \Theta_\varepsilon(x, z) \otimes \Theta_\varepsilon(x, z)) k(x/\varepsilon, z) \Pi(dz) \right\rangle \\
 &+ H_1^\varepsilon(x) \\
 = &\frac{1}{2} \left\langle \nabla^2 f_{\varepsilon,s}(\Phi_\varepsilon(x)), \int (z + \Theta_\varepsilon(x, z)) \otimes (z + \Theta_\varepsilon(x, z)) k(x/\varepsilon, z) \Pi(dz) \right\rangle + H_2^\varepsilon(x) \\
 = &\langle \nabla^2 f_{\varepsilon,s}(\Phi_\varepsilon(x)), A(x/\varepsilon) \rangle + H_2^\varepsilon(x),
 \end{aligned}$$

where in the first and the second equalities $\mathcal{L}_{1,x}^\varepsilon$ is defined by (4.5). Here, H_1^ε satisfies

$$|H_1^\varepsilon(x)| \leq \varepsilon \left(\sum_{i=2}^3 \|\nabla^i f\|_\infty \right) \left(\int_{\{|z| \leq R\}} |z|^2 \Pi(dz) + \int_{\{|z| > R\}} |z| \Pi(dz) \right)$$

thanks to the fact that $\|\psi\|_\infty + \|\nabla \psi\|_\infty \leq 1$ under Assumption (A3) (see Lemma 3.6 with $\varepsilon = 0$) and $H_2^\varepsilon(x) = H_1^\varepsilon(x) + G_{1,\varepsilon}(x)$ with $G_{1,\varepsilon}$ as in Lemma 4.3. As explained in the proof of Theorem 3.4, the above estimate still holds true when $\psi \in \mathcal{D}(\mathcal{L})$.

Given these estimates, the rest of the proof is very similar to that of Theorem 3.4, so we omit it (see also the proof of Theorem 5.1 below). \square

5. Homogenization: Critical cases. Throughout this section, ϕ is a strictly positive and strictly decreasing function on \mathbb{R}_+ so that $\lim_{r \rightarrow 0} \phi(r) = \infty$, $\lim_{r \rightarrow 0} r^{-2} \phi(r)^{-1} = \infty$,

$$(5.1) \quad \lim_{\varepsilon \rightarrow 0} \left(\frac{\varepsilon \int_{\{|z| \leq 1/\varepsilon\}} |z|^3 \Pi(dz)}{\phi(\varepsilon)} + \frac{\int_{\{|z| > 1/\varepsilon\}} |z| \Pi(dz)}{\varepsilon \phi(\varepsilon)} \right) = 0,$$

and

$$(5.2) \quad \limsup_{\varepsilon \rightarrow 0} \frac{\int_{\{|z| \leq 1/\varepsilon\}} |z|^2 \Pi(dz)}{\phi(\varepsilon)} < \infty$$

with

$$(5.3) \quad A := \lim_{\varepsilon \rightarrow 0} \frac{\int_{\mathbb{T}^d} \int_{\{|z| \leq 1/\varepsilon\}} (z \otimes z) k(x, z) \Pi(dz) \mu(dx)}{\phi(\varepsilon)}$$

being a nonzero $d \times d$ -matrix.

We make four remarks on the assumptions above:

- (i) Since the matrix A is nonzero and $\lim_{\varepsilon \rightarrow 0} \phi(\varepsilon) = \infty$, $\int_{\mathbb{R}^d} |z|^2 \Pi(dz)$ has to be infinite.
- (ii) In (5.1), (5.2) and (5.3), the domain $\{|z| \leq 1/\varepsilon\}$ can be replaced by $\{r_0 \leq |z| \leq 1/\varepsilon\}$ for any fixed $r_0 \geq 1$.
- (iii) For $\mathbb{1}_{\{|z| \geq 1\}} \Pi(dz) = \mathbb{1}_{\{|z| \geq 1\}} |z|^{-(d+\alpha)} dz$ with $\alpha \in (0, 2)$, condition (5.2) holds with $\phi(\varepsilon) = \varepsilon^{\alpha-2}$ but condition (5.1) fails.
- (iv) For $\mathbb{1}_{\{|z| \geq 1\}} \Pi(dz) = \mathbb{1}_{\{|z| \geq 1\}} |z|^{-d-2} dz$, conditions (5.1) and (5.2) are satisfied with $\phi(\varepsilon) = \log(1 + 1/\varepsilon)$.

Under (5.1) and (5.2) we will take $\rho(\varepsilon) = \varepsilon^2/\phi(1/\varepsilon)$ in (2.1) which corresponds to the scaling function for critical cases in the setting of infinite second moments. The purpose of this section is to study the limit behavior of the scaled process $X^\varepsilon := (X_t^\varepsilon)_{t \geq 0}$, defined by $X_t^\varepsilon = \varepsilon X_{\varepsilon^{-2}\phi(\varepsilon)^{-1}t}$ for any $t > 0$.

THEOREM 5.1. *Suppose that Assumption (A3), (5.1), (5.2) and (5.3) hold. Let*

$$Y_t^\varepsilon := X_t^\varepsilon - \varepsilon^{-1} \phi(\varepsilon)^{-1} (\bar{b}_\infty + \bar{b})t = \varepsilon (X_{\varepsilon^{-2}\phi(\varepsilon)^{-1}t} - \varepsilon^{-2} \phi(\varepsilon)^{-1} (\bar{b}_\infty + \bar{b})t), \quad t \geq 0.$$

Then, $(Y_t^\varepsilon)_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$, as $\varepsilon \rightarrow 0$, to Brownian motion with the nonzero covariance matrix A , defined by (5.3).

For the scaled process $X^\varepsilon = (X_t^\varepsilon)_{t \geq 0} := (\varepsilon X_{\varepsilon^{-2}\phi(\varepsilon)^{-1}t})_{t \geq 0}$ as above, its infinitesimal generator is given by

$$\begin{aligned} \mathcal{L}^\varepsilon f(x) &= \frac{1}{\varepsilon^2 \phi(\varepsilon)} \int_{\mathbb{R}^d} (f(x + \varepsilon z) - f(x) - \langle \nabla f(x), \varepsilon z \rangle \mathbb{1}_{\{|z| \leq 1\}}) k(x/\varepsilon, z) \Pi(dz) \\ &\quad + \frac{1}{\varepsilon \phi(\varepsilon)} \langle b(x/\varepsilon), \nabla f(x) \rangle \\ &=: \mathcal{L}_1^\varepsilon f(x) + \frac{1}{\varepsilon \phi(\varepsilon)} \langle b_\infty(x/\varepsilon) + b(x/\varepsilon), \nabla f(x) \rangle, \end{aligned}$$

where

$$\mathcal{L}_1^\varepsilon f(x) = \frac{1}{\varepsilon^2 \phi(\varepsilon)} \int_{\mathbb{R}^d} (f(x + \varepsilon z) - f(x) - \langle \nabla f(x), \varepsilon z \rangle) k(x/\varepsilon, z) \Pi(dz).$$

Similar to Lemma 4.3, we have the following statement.

LEMMA 5.2. *For any $x, y \in \mathbb{R}^d$ and $f \in C_b^3(\mathbb{R}^d)$, let*

$$\mathcal{L}_{1,x}^\varepsilon f(y) = \varepsilon^{-2} \phi(\varepsilon)^{-1} \int_{\mathbb{R}^d} (f(y + \varepsilon z) - f(y) - \langle \nabla f(y), \varepsilon z \rangle) k(x/\varepsilon, z) \Pi(dz).$$

Suppose that (5.1) holds. Then, for every $x, y \in \mathbb{R}^d$ and $f \in C_b^3(\mathbb{R}^d)$,

$$\mathcal{L}_{1,x}^\varepsilon f(y) = \left\langle \nabla^2 f(y), \frac{1}{2\phi(\varepsilon)} \int_{\{|z| \leq 1/\varepsilon\}} (z \otimes z) k(x/\varepsilon, z) \Pi(dz) \right\rangle + G_{2,\varepsilon}(x, y),$$

where $G_{2,\varepsilon}(x, y)$ satisfies that

$$\lim_{\varepsilon \rightarrow 0} \sup_{x, y \in \mathbb{R}^d} |G_{2,\varepsilon}(x, y)| = 0.$$

PROOF. According to the Taylor expansion, for any $f \in C_b^3(\mathbb{R}^d)$ and $\varepsilon \in (0, 1)$,

$$\begin{aligned} &f(x + \varepsilon z) - f(x) - \langle \nabla f(x), \varepsilon z \rangle \\ &= \begin{cases} \frac{\varepsilon^2}{2} \langle \nabla^2 f(x), z \otimes z \rangle + \frac{\varepsilon^3}{6} \langle \nabla^3 f(x + \theta_1 \varepsilon z), z \otimes z \otimes z \rangle, & |z| \leq 1/\varepsilon, \\ \langle \nabla f(x + \theta_2 \varepsilon z), \varepsilon z \rangle - \langle \nabla f(x), \varepsilon z \rangle, & |z| > 1/\varepsilon \end{cases} \\ &= \frac{\varepsilon^2}{2} \langle \nabla^2 f(x), z \otimes z \rangle \mathbb{1}_{\{|z| \leq 1/\varepsilon\}} + H_\varepsilon(x, z), \end{aligned}$$

where $\theta_1, \theta_2 \in (0, 1)$ and

$$|H_\varepsilon(x, z)| \leq (\|\nabla f\|_\infty + \|\nabla^3 f\|_\infty) (\varepsilon^3 |z|^3 \mathbb{1}_{\{|z| \leq 1/\varepsilon\}} + \varepsilon |z| \mathbb{1}_{\{|z| > 1/\varepsilon\}}).$$

In particular, we have

$$\mathcal{L}_{1,x}^\varepsilon f(y) = \left\langle \nabla^2 f(y), \frac{1}{2\phi(\varepsilon)} \int_{\{|z| \leq 1/\varepsilon\}} (z \otimes z) k(x/\varepsilon, z) \Pi(dz) \right\rangle + G_{2,\varepsilon}(x, y),$$

where

$$G_{2,\varepsilon}(x, y) = \varepsilon^{-2} \phi(\varepsilon)^{-1} \int H_\varepsilon(y, z) k(x/\varepsilon, z) \Pi(dz).$$

Furthermore,

$$(5.4) \quad |G_{2,\varepsilon}(x, y)| \leq C_1 \left(\sum_{i=1}^3 \|\nabla^i f\|_\infty \right) \left(\frac{\varepsilon \int_{\{|z| \leq 1/\varepsilon\}} |z|^3 \Pi(dz)}{\phi(\varepsilon)} + \frac{\int_{\{|z| > 1/\varepsilon\}} |z| \Pi(dz)}{\varepsilon \phi(\varepsilon)} \right)$$

with some $C_1 > 0$ independent of f and ε . By (5.1) we can further obtain that

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} |G_{2,\varepsilon}(x)| \leq \lim_{\varepsilon \rightarrow 0} \left(\frac{\varepsilon \int_{\{|z| \leq 1/\varepsilon\}} |z|^3 \Pi(dz)}{\phi(\varepsilon)} + \frac{\int_{\{|z| > 1/\varepsilon\}} |z| \Pi(dz)}{\varepsilon \phi(\varepsilon)} \right) = 0.$$

The proof is finished. \square

PROOF OF THEOREM 5.1. (1) Let $\psi \in \mathcal{D}(\mathcal{L})$ be the unique solution to the following equation:

$$\mathcal{L}\psi(x) = -b_\infty(x) - b(x) + \bar{b}_\infty + \bar{b}, \quad x \in \mathbb{T}^d$$

with $\mu(\psi) = 0$. Let

$$Z_t^\varepsilon = Y_t^\varepsilon + \varepsilon \psi(X_t^\varepsilon/\varepsilon) = X_t^\varepsilon + \varepsilon \psi(X_t^\varepsilon/\varepsilon) - \varepsilon^{-1} \phi(\varepsilon)^{-1} (\bar{b}_\infty + \bar{b})t, \quad t \geq 0.$$

As explained in the proof of Theorem 3.4, without of loss generality we can assume that $\psi \in C^2(\mathbb{T}^d)$. Noticing that $((X_t^\varepsilon)_{t \geq 0}; (\mathbb{P}_x)_{x \in \mathbb{R}^d})$ is a solution to the martingale problem for the operator \mathcal{L}^ε , we obtain that, for every $f \in C_b^3(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t > 0$,

$$\mathbb{E}_x[f(Z_t^\varepsilon)] = f(\Phi_\varepsilon(x)) + \mathbb{E}_x \left[\int_0^t \left(\frac{\partial F_\varepsilon}{\partial s}(s, X_s^\varepsilon) + \mathcal{L}^\varepsilon F_\varepsilon(s, \cdot)(X_s^\varepsilon) \right) ds \right],$$

where $F_\varepsilon(s, x) = f(\Phi_\varepsilon(x) - \varepsilon^{-1} \phi(\varepsilon)^{-1} (\bar{b}_\infty + \bar{b})s)$ and $\Phi_\varepsilon(x) = x + \varepsilon \psi(x/\varepsilon)$.

Let $f_{\varepsilon,s}(x) = f(x - \varepsilon^{-1} \phi(\varepsilon)^{-1} (\bar{b}_\infty + \bar{b})s)$, $\Theta_\varepsilon(x, z) = \psi(x/\varepsilon + z) - \psi(x/\varepsilon)$ and

$$A_\varepsilon(x) = \frac{1}{2} \int_{\{|z| \leq 1/\varepsilon\}} (z \otimes z) k(x, z) \Pi(dz).$$

Applying Lemma 2.1 with $R = 1/\varepsilon$ and $M = \infty$ and using Lemma 5.2 and (5.4), we can verify that

$$\begin{aligned} & \frac{\partial F_\varepsilon}{\partial s}(s, x) + \mathcal{L}^\varepsilon F_\varepsilon(s, \cdot)(x) \\ &= \mathcal{L}_{1,x}^\varepsilon f_{\varepsilon,s}(\Phi_\varepsilon(x)) \\ &+ \frac{1}{2\phi(\varepsilon)} \left\langle \nabla^2 f_{\varepsilon,s}(\Phi_\varepsilon(x)), \int (2\Theta_\varepsilon(x, z) \otimes z + \Theta_\varepsilon(x, z) \otimes \Theta_\varepsilon(x, z)) k(x/\varepsilon, z) \Pi(dz) \right\rangle \\ &+ H_1^\varepsilon(x) \\ &= \frac{1}{2} \left\langle \nabla^2 f_{\varepsilon,s}(\Phi_\varepsilon(x)), \frac{1}{\phi(\varepsilon)} \int_{\{|z| \leq 1/\varepsilon\}} (z \otimes z) k(x/\varepsilon, z) \Pi(dz) \right\rangle + H_2^\varepsilon(x) \\ &= \left\langle \nabla^2 f_{\varepsilon,s}(\Phi_\varepsilon(x)), \frac{1}{\phi(\varepsilon)} A_\varepsilon(x/\varepsilon) \right\rangle + H_2^\varepsilon(x), \end{aligned}$$

where H_1^ε and H_2^ε satisfy

$$|H_i^\varepsilon(x)| \leq \left(\sum_{i=1}^3 \|\nabla^i f\|_\infty \right) \left[\frac{\varepsilon \left(\int_{\{0 < |z| \leq 1\}} |z|^2 \Pi(dz) + \int_{\{1 < |z| \leq 1/\varepsilon\}} |z|^3 \Pi(dz) \right)}{\phi(\varepsilon)} + \frac{\int_{\{|z| > 1/\varepsilon\}} |z| \Pi(dz)}{\varepsilon \phi(\varepsilon)} + \frac{1}{\phi(\varepsilon)} \left(\int_{\{|z| \leq 1\}} |z|^2 \Pi(dz) + \int_{\{|z| > 1\}} |z| \Pi(dz) \right) \right].$$

In particular, by (5.1), $\lim_{\varepsilon \rightarrow 0} \phi(\varepsilon) = \infty$ and $\int_{\{|z| > 1\}} |z| \Pi(dz) < \infty$, it holds that

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} |H_2^\varepsilon(x)| = 0.$$

(2) For any $l \geq 1$, let f_l be the function defined by (3.17) and $F_{\varepsilon,l}(s, x) = f_l(\Phi_\varepsilon(x) - \varepsilon^{-1} \phi(\varepsilon)^{-1}(\bar{b}_\infty + \bar{b})s)$. According to all the estimates above and $\sup_{\varepsilon \in (0,1)} \sup_{x \in \mathbb{R}^d} \frac{|A_\varepsilon(x)|}{\phi(\varepsilon)} < \infty$ (which is due to (5.2) and the boundedness of $k(x, z)$), we can get

$$\lim_{R \rightarrow \infty} \sup_{\varepsilon \in (0,1), x \in \mathbb{R}^d, s > 0} \left| \frac{\partial F_{\varepsilon,R}}{\partial s}(s, x) + \mathcal{L}^\varepsilon F_{\varepsilon,R}(s, \cdot)(x) \right| = 0$$

and

$$\sup_{\varepsilon \in (0,1), x \in \mathbb{R}^d, s > 0} \left| \frac{\partial F_{\varepsilon,\theta}}{\partial s}(s, x) + \mathcal{L}^\varepsilon F_{\varepsilon,\theta}(s, \cdot)(x) \right| \leq C(\theta), \quad \theta \in (0, 1).$$

Thus, following the proof of Theorem 3.4, we can obtain that $\{Z^\varepsilon\}_{\varepsilon \in (0,1)}$ is tight in $\mathcal{D}([0, \infty); \mathbb{R}^d)$.

(3) Recall that the generator of the process X^ε is \mathcal{L}^ε and, again, that

$$F_\varepsilon(s, x) = f(x + \varepsilon \psi(x/\varepsilon) - \varepsilon^{-1} \phi(\varepsilon)^{-1}(\bar{b}_\infty + \bar{b})s),$$

$$f_{\varepsilon,s}(x) = f(x - \varepsilon^{-1} \phi(\varepsilon)^{-1}(\bar{b}_\infty + \bar{b})s).$$

For every $0 < s_1 < s_2, \dots < s_k < s \leq t$, $f \in C_b^3(\mathbb{R}^d)$ and $G \in C_b(\mathbb{R}^{dk})$,

$$\begin{aligned} & \mathbb{E}_0 \left[\left(f(Z_t^\varepsilon) - f(Z_s^\varepsilon) - \int_s^t \left(\frac{\partial F_\varepsilon}{\partial r}(r, X_r^\varepsilon) + \mathcal{L}^\varepsilon F_\varepsilon(r, \cdot)(X_r^\varepsilon) \right) dr \right) G(Z_{s_1}^\varepsilon, \dots, Z_{s_k}^\varepsilon) \right] \\ &= \mathbb{E}_0 \left[\left(f(Z_t^\varepsilon) - f(Z_s^\varepsilon) - \int_s^t \left(\left\langle \nabla^2 f_{\varepsilon,r}(\Phi_\varepsilon(X_r^\varepsilon)), \frac{1}{\phi(\varepsilon)} A_\varepsilon(X_r^\varepsilon/\varepsilon) \right\rangle + H_2^\varepsilon(X_r^\varepsilon) \right) dr \right) \right. \\ & \quad \left. \times G(Z_{s_1}^\varepsilon, \dots, Z_{s_k}^\varepsilon) \right] \\ & \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Let

$$A_\varepsilon = \int_{\mathbb{T}^d} A_\varepsilon(x) \mu(dx) = \frac{1}{2} \int_{\mathbb{T}^d} \int_{\{|z| \leq \varepsilon^{-1}\}} (z \otimes z) k(x, z) \Pi(dz) \mu(dx).$$

Then, following the argument in (3.22), using Lemma 2.2 and the fact that

$$\sup_{\varepsilon \in (0,1)} \sup_{x \in \mathbb{R}^d} \frac{|A_\varepsilon(x)|}{\phi(\varepsilon)} < \infty,$$

we get that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_0 \left[\left| \int_s^t \left\langle \nabla^2 f_{\varepsilon,r}(\Phi_\varepsilon(X_r^\varepsilon)), \frac{1}{\phi(\varepsilon)} A_\varepsilon(X_r^\varepsilon/\varepsilon) \right\rangle dr - \int_s^t \left\langle \nabla^2 f_{\varepsilon,r}(\Phi_\varepsilon(X_r^\varepsilon)), \frac{A_\varepsilon}{\phi(\varepsilon)} \right\rangle dr \right|^2 \right] = 0.$$

Hence, putting all estimates together, we obtain

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_0 \left[\left(f(Z_t^\varepsilon) - f(Z_s^\varepsilon) - \int_s^t \left\langle \nabla^2 f(Z_r^\varepsilon), \frac{A_\varepsilon}{\phi(\varepsilon)} \right\rangle dr \right) G(Z_{s_1}^\varepsilon, \dots, Z_{s_k}^\varepsilon) \right] = 0.$$

Given this, the fact that $\lim_{\varepsilon \rightarrow 0} \frac{A_\varepsilon}{\phi(\varepsilon)} = \frac{A}{2}$ and the tightness of $\{Z^\varepsilon\}_{\varepsilon \in (0,1)}$ in $\mathcal{D}([0, \infty); \mathbb{R}^d)$, one can follow the proof of Theorem 3.4 to get the desired assertion. \square

6. Sufficient conditions for averaging assumption (3.6). In this section we present some sufficient conditions for the key averaging assumption (3.6) which is needed for the proof of the assertions in Example 1.1 and the two additional examples in Section 7.1. The main results of this section are Theorems 6.5 and 6.7.

Let $\Pi_0(dz)$ be defined by (3.4). Let $k(x, z)$ be a nonnegative bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ so that $x \mapsto k(x, z)$ is multivariate 1-periodic for each fixed $z \in \mathbb{R}^d$ and condition (1.2) holds. We will represent z in the spherical coordinate (r, θ) with $r = |z|$ and $\theta = z/|z|$ and will write $k(x, z)$ as $k(x, (r, \theta))$ as well.

PROPOSITION 6.1. *Suppose that, for every $x \in \mathbb{R}^d$ and Q_0 -a.e. $\theta \in \mathbb{S}^{d-1}$, there is a constant $\bar{k}(x, \theta)$ so that*

$$(6.1) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T k(x, (r, \theta)) dr = \bar{k}(x, \theta).$$

Then, for any bounded function $f : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying (3.5) and for $0 < r \leq R$,

$$(6.2) \quad \lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\{r \leq |z| \leq R\}} f(x, z) (k(x/\varepsilon, z/\varepsilon) - \bar{k}(x/\varepsilon, z/|z|)) \Pi_0(dz) \right| = 0.$$

PROOF. (1) Let Λ_0 denote the collection of all $\theta \in \mathbb{S}^{d-1}$ such that, for any $x \in \mathbb{R}^d$, there is a constant $\bar{k}(x, \theta)$ so that (6.1) holds. Now, we are going to show that the function $x \mapsto \bar{k}(x, \theta)$ is equicontinuous in x for all $\theta \in \Lambda_0$. Moreover, for every $\theta \in \Lambda_0$, the convergence in (6.1) is uniform in $x \in \mathbb{R}^d$.

For any $\varepsilon > 0$, by (1.2) there is a constant $\delta_0 > 0$ so that $|k(x, z) - k(y, z)| < \varepsilon$ for any $x, y, z \in \mathbb{R}^d$ with $|x - y| \leq \delta_0$. Thus, for $\theta \in \Lambda_0$,

$$|\bar{k}(x, \theta) - \bar{k}(y, \theta)| \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T |k(x, (r, \theta)) - k(y, (r, \theta))| dr \leq \varepsilon,$$

as long as $|x - y| \leq \delta_0$. In other words, $x \mapsto \bar{k}(x, \theta)$ is equicontinuous in x for all $\theta \in \Lambda_0$.

Let $\{y_k : 1 \leq k \leq N\} \subset \mathbb{T}^d$ be an δ_0 -net in \mathbb{T}^d meaning that, for every $x \in \mathbb{T}^d$, there is some $1 \leq j \leq N$ so that $|x - y_j| < \delta_0$. By (6.1), for each $\theta \in \Lambda_0$, there is some $T_0 \geq 1$ so that, for every $1 \leq k \leq N$,

$$\left| \frac{1}{T} \int_0^T k(y_k, (r, \theta)) dr - \bar{k}(y_k, \theta) \right| < \varepsilon \quad \text{for every } T \geq T_0.$$

For every $x \in \mathbb{T}^d$, there is some y_k so that $|x - y_k| < \delta_0$. Hence, for any $T \geq T_0$,

$$\begin{aligned} & \left| \frac{1}{T} \int_0^T k(x, (r, \theta)) dr - \bar{k}(x, \theta) \right| \\ & \leq \frac{1}{T} \int_0^T |k(x, (r, \theta)) - k(y_k, (r, \theta))| dr + \left| \frac{1}{T} \int_0^T k(y_k, (r, \theta)) dr - \bar{k}(y_k, \theta) \right| \\ & \quad + |\bar{k}(y_k, \theta) - \bar{k}(x, \theta)| \\ & \leq 3\varepsilon, \end{aligned}$$

where in the last inequality we used (1.2) again. This proves that, for each $\theta \in \Lambda_0$,

$$(6.3) \quad \lim_{T \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \left| \frac{1}{T} \int_0^T k(x, (r, \theta)) dr - \bar{k}(x, \theta) \right| = 0.$$

(2) Let f be a bounded function such that (3.5) is satisfied. For each $\varepsilon_0 > 0$, there is $\delta_1 \in (0, 1/4]$ so that $|f(x, z_1) - f(x, z_2)| < \varepsilon_0$, whenever $|z_1 - z_2| < \delta_1$. For $0 < r < R$, we divide $[r, R]$ into $N = 1 + [(R - r)/\delta_1]$ equal subintervals with partition points $\{t_0, t_1, \dots, t_N\}$ with $\Delta = t_k - t_{k-1} = (R - r)/N \in (0, \delta_1)$. By taking δ_1 smaller, if needed (which may depend on r, R and $\|f\|_\infty$), we can and do assume that

$$(6.4) \quad \sup_{x \in \mathbb{R}^d} \left| \frac{f(x, (s, \theta))}{s^{1+\alpha}} - \frac{f(x, (t_k, \theta))}{t_k^{1+\alpha}} \right| \leq \varepsilon_0 \quad \text{for } s \in [t_{k-1}, t_k], \theta \in \mathbb{S}^{d-1}.$$

Using spherical coordinates and Fatou’s lemma, we have, for any $0 < r \leq R$,

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\{r \leq |z| \leq R\}} f(x, z) (k(x/\varepsilon, z/\varepsilon) - \bar{k}(x/\varepsilon, z/|z|)) \Pi_0(dz) \right| \\ & \leq \limsup_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{S}^{d-1}} \int_r^R f(x, (s, \theta)) (k(x/\varepsilon, (s/\varepsilon, \theta)) - \bar{k}(x/\varepsilon, \theta)) \frac{1}{s^{1+\alpha}} ds \varrho_0(d\theta) \right| \\ & \leq \limsup_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \left| \sum_{k=1}^N \int_{t_{k-1}}^{t_k} f(x, (s, \theta)) (k(x/\varepsilon, (s/\varepsilon, \theta)) - \bar{k}(x/\varepsilon, \theta)) \frac{1}{s^{1+\alpha}} ds \right| \varrho_0(d\theta) \\ & \leq \limsup_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{S}^{d-1}} \sum_{k=1}^N \frac{f(x, (t_k, \theta))}{t_k^{1+\alpha}} \int_{t_{k-1}}^{t_k} (k(x/\varepsilon, (s/\varepsilon, \theta)) - \bar{k}(x/\varepsilon, \theta)) ds \varrho_0(d\theta) \right| \\ & \quad + 2\varepsilon_0 R \|k\|_\infty \varrho_0(\mathbb{S}^{d-1}) \\ & \leq \limsup_{\varepsilon \rightarrow 0} \sum_{k=1}^N \frac{\|f\|_\infty}{t_k^{1+\alpha}} \left(\int_{\mathbb{S}^{d-1}} t_{k-1} \sup_{x \in \mathbb{R}^d} \left| \frac{\varepsilon}{t_{k-1}} \int_0^{t_{k-1}/\varepsilon} k(x/\varepsilon, (s, \theta)) ds - \bar{k}(x/\varepsilon, \theta) \right| \varrho_0(d\theta) \right. \\ & \quad \left. + \int_{\mathbb{S}^{d-1}} t_k \sup_{x \in \mathbb{R}^d} \left| \frac{\varepsilon}{t_k} \int_0^{t_k/\varepsilon} k(x/\varepsilon, (s, \theta)) ds - \bar{k}(x/\varepsilon, \theta) \right| \varrho_0(d\theta) \right) \\ & \quad + 2\varepsilon_0 R \|k\|_\infty \varrho_0(\mathbb{S}^{d-1}) \\ & \leq \sum_{k=1}^N \frac{\|f\|_\infty}{t_k^{1+\alpha}} \left(\int_{\mathbb{S}^{d-1}} t_{k-1} \limsup_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \left| \frac{\varepsilon}{t_{k-1}} \int_0^{t_{k-1}/\varepsilon} k(x, (s, \theta)) ds - \bar{k}(x, \theta) \right| \varrho_0(d\theta) \right. \\ & \quad \left. + \int_{\mathbb{S}^{d-1}} t_k \limsup_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \left| \frac{\varepsilon}{t_k} \int_0^{t_k/\varepsilon} k(x, (s, \theta)) ds - \bar{k}(x, \theta) \right| \varrho_0(d\theta) \right) \end{aligned}$$

$$\begin{aligned}
 &+ 2\varepsilon_0 R \|k\|_\infty \varrho_0(\mathbb{S}^{d-1}) \\
 &= 2\varepsilon_0 R \|k\|_\infty \varrho_0(\mathbb{S}^{d-1}).
 \end{aligned}$$

Here in the third inequality, we have used (6.4) and the fact $\|\bar{k}\|_\infty \leq \|k\|_\infty$, while the last inequality follows from the fact that (6.3) holds for ϱ_0 -a.e. $\theta \in \mathbb{S}^{d-1}$. Since $\varepsilon_0 > 0$ is arbitrary, we get (6.2) immediately. \square

As in the proof of Proposition 6.1, in the following denote by Λ_0 the collection of all $\theta \in \mathbb{S}^{d-1}$ so that, for every, $x \in \mathbb{R}^d$, there is a constant $\bar{k}(x, \theta)$ such that (6.1) holds.

REMARK 6.2. Here are two simple cases so that $\theta \in \Lambda_0$:

(i) If $\theta \in \mathbb{S}^{d-1}$ has the property that $r \mapsto k(x, (r, \theta))$ is multivariate $T(x)$ -periodic for each $x \in \mathbb{R}^d$ (it can have different period for different $x \in \mathbb{R}^d$), then, clearly, (6.1) holds for any function $f(x, z)$ that satisfies (3.5) with $\bar{k}(x, \theta) = \frac{1}{T(x)} \int_0^{T(x)} k(x, (r, \theta)) dr$, and so $\theta \in \Lambda_0$.

(ii) A function $\varphi(r)$ on $[0, \infty)$ is said to be almost periodic if it is the uniform limit of some periodic functions (cf. [8], page 81). It follows from (i) above that if $\theta \in \mathbb{S}^{d-1}$ has the property that $r \mapsto k(x, (r, \theta))$ is almost periodic for each $x \in \mathbb{R}^d$, then (6.1) holds for any function $f(x, z)$ that satisfies (3.5), and so $\theta \in \Lambda_0$; see Lemma 6.4 and its proof below for more information.

Next, we will present some sufficient conditions for $\theta \in \Lambda_0$ under the periodicity of $z \mapsto k(x, z)$.

COROLLARY 6.3. Suppose, in addition, that, for each $x \in \mathbb{R}^d$, there is some $T := T(x) > 0$ so that $z \mapsto k(x, z)$ is multivariate T -periodic:

(i) $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{S}^{d-1}$ is in Λ_0 , if θ is pairwise rational in the sense that each θ_i/θ_j is a rational number whenever $\theta_j \neq 0$;

(ii) If ϱ_0 does not charge on the set of those $\theta \in \mathbb{S}^{d-1}$ that are not pairwise rational, then (6.2) holds for any function $f(x, z)$ that satisfies (3.5).

In particular, suppose that $\varrho_0(d\theta) = \delta_{\theta_0}(d\theta)$ for some rational point $\theta_0 = (m_1/n, \dots, m_d/n) \in \mathbb{S}^{d-1}$, where $n, m_1, \dots, m_d \in \mathbb{Z}$ and $\delta_{\theta_0}(d\theta)$ denotes the Dirac measure on \mathbb{S}^{d-1} . Then, (6.2) holds for any function $f(x, z)$ satisfying (3.5) with

$$\bar{k}(x, \theta) = \frac{1}{n} \int_0^n k(x, (r, \theta_0)) dr \quad \text{for all } \theta \in \mathbb{S}^{d-1}.$$

PROOF. (i) If the measure $\theta \in \mathbb{S}^{d-1}$ is pairwise rational, then there is some $r_0 > 0$ so that $r_0\theta$ has integer coordinates. For each $x \in \mathbb{R}^d$,

$$r \mapsto k(x, (r, \theta)) = k(x, (r\theta_1, \dots, r\theta_d))$$

is a bounded (r_0T) -periodic function on $[0, \infty)$, and so (6.1) holds with $\bar{k}(x, \theta) = \frac{1}{r_0T} \int_0^{r_0T} k(x, (r, \theta)) dr$ from Remark 6.2(i).

(ii) The assertion follows immediately from (i) and Proposition 6.1.

Having (i) and (ii) at hand, we can easily see the validity of the last assertion. \square

Recall that $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{S}^{d-1}$ is said to be rationally dependent if there is some nonzero $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ so that $\langle m, \theta \rangle = \sum_{i=1}^d m_i \theta_i = 0$. Otherwise, we call θ rationally independent. When $d = 1$, $\mathbb{S}^0 = \{1, -1\}$ so every its member is rationally independent.

LEMMA 6.4. *Suppose that $f(x) = f(x_1, \dots, x_d)$ is a continuous multivariate 1-periodic function on \mathbb{R}^d . Then, for each $\theta \in \mathbb{S}^{d-1}$, there is a constant $C(\theta)$ so that*

$$(6.5) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\theta t) dt = C(\theta).$$

Set

$$\Gamma_f = \left\{ \theta \in \mathbb{S}^{d-1} : C(\theta) = \int_{\mathbb{T}^d} f(x) dx \right\}.$$

Then, every rationally independent $\theta \in \mathbb{S}^{d-1}$ is in Γ_f . In particular, $\Gamma_f = \{1, -1\}$ if $d = 1$, $\mathbb{S}^1 \setminus \Gamma_f$ is countable if $d = 2$ and $\dim_H(\mathbb{S}^{d-1} \setminus \Gamma_f) \leq d - 2$ if $d \geq 3$. Here, \dim_H stands for the Hausdorff dimension.

PROOF. The result is trivial when $d = 1$. So we assume $d \geq 2$ in the rest of the proof. Let $\langle \cdot, \cdot \rangle$ denote the inner product in \mathbb{R}^d . Define $f(x) = \sum_{k \in \mathbb{Z}^d: |k| \leq N} c_k e^{i2\pi \langle k, x \rangle}$ with

$$c_k = \int_{\mathbb{T}^d} e^{-i2\pi \langle k, x \rangle} f(x) dx.$$

Then, for $\theta \in \mathbb{S}^{d-1}$,

$$f(\theta t) = \sum_{k \in \mathbb{Z}^d: |k| \leq N} c_k e^{i2\pi \langle k, \theta t \rangle}.$$

Clearly,

$$(6.6) \quad C(\theta) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\theta t) dt = \sum_{k \in \mathbb{Z}^d: |k| \leq N, \langle k, \theta \rangle = 0} c_k.$$

Note that for each nonzero $k \in \mathbb{Z}^d$ and $d \geq 2$, the set $\{\theta \in \mathbb{S}^{d-1} : \langle k, \theta \rangle = 0\}$ is a two-point set when $d = 2$ and has Hausdorff dimension $d - 2$ when $d \geq 3$. Noting also that

$$(6.7) \quad \mathbb{S}^{d-1} \setminus \Gamma_f \subset \{\theta \in \mathbb{S}^{d-1} : \langle k, \theta \rangle = 0 \text{ for some nonzero } k \in \mathbb{Z}^d\},$$

we have $\mathbb{S}^{d-1} \setminus \Gamma_f$ is a countable set when $d = 2$, and $\dim_H(\mathbb{S}^{d-1} \setminus \Gamma_f) \leq d - 2$ for $d \geq 3$.

Now, suppose that $f(x_1, \dots, x_d)$ is a continuous multivariate 1-period function on \mathbb{R}^d . It can then be viewed as a continuous function on \mathbb{T}^d . By the Stone–Weierstrass theorem it can be uniformly approximated by functions of the form $\sum_{k \in \mathbb{Z}^d: |k| \leq N} c_k e^{i2\pi \langle k, x \rangle}$ on \mathbb{R}^d ; see, for example, [8], page 26. Thus, for any $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{S}^{d-1}$,

$$f(\theta t) = f(\theta_1 t, \dots, \theta_d t)$$

can be approximated uniformly by the functions of the form $\sum_{k \in \mathbb{Z}^d: |k| \leq N} c_k e^{i2\pi \langle k, \theta t \rangle}$. It follows from (6.6) that, for any $\theta \in \mathbb{S}^{d-1}$, there is a constant $C(\theta)$ so that

$$(6.8) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\theta t) dt = C(\theta)$$

and (6.7) continues to hold for this f . In particular, $C(\theta) = \int_{\mathbb{T}^d} f(x) dx$ if $\theta \in \mathbb{S}^{d-1}$ is rationally independent. The assertion of the proposition now follows. \square

THEOREM 6.5. *Suppose that $k(x, z)$ is jointly continuous on $\mathbb{R}^d \times \mathbb{R}^d$ and $k(x, z)$ is multivariate 1-periodic both in x and in z . Then:*

- (i) $\Lambda_0 = \mathbb{S}^{d-1}$; that is, (6.1) holds for every $\theta \in \mathbb{S}^{d-1}$ and $x \in \mathbb{R}^d$ with some $\bar{k}(x, \theta)$.

(ii) *Let*

$$\bar{k}(x) = \int_{\mathbb{T}^d} k(x, z) dz, \quad x \in \mathbb{R}^d.$$

Then, for each $x \in \mathbb{R}^d$, $\bar{k}(x, \theta) = \bar{k}(x)$ for every rationally independent $\theta \in \mathbb{S}^{d-1}$. In particular, we have, for every $x \in \mathbb{R}^d$, $\bar{k}(x, 1) = \bar{k}(x, -1) = \bar{k}(x)$ when $d = 1$, $\{\theta \in \mathbb{S}^1 : \bar{k}(x, \theta) \neq \bar{k}(x)\}$ is countable when $d = 2$, and the Hausdorff dimension of $\{\theta \in \mathbb{S}^{d-1} : \bar{k}(x, \theta) \neq \bar{k}(x)\}$ is no larger than $d - 2$.

(iii) *Property (6.2) holds for any function $f(x, z)$ that satisfies (3.5).*

PROOF. This follows directly by applying Lemma 6.4 to function $z \mapsto k(x, z)$ and by Proposition 6.1. \square

REMARK 6.6. We present two explicit cases that Theorem 6.5 applies:

(i) Assume that $\Pi_0(dz)$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d , or, equivalently, ϱ_0 is absolutely continuous with respect to the Lebesgue surface measure σ on \mathbb{S}^{d-1} . Then, under the assumptions of Theorem 6.5, (6.2) holds with $\bar{k}(x, \theta) = \bar{k}(x) := \int_{\mathbb{T}^d} k(x, z) dz$ for all $x \in \mathbb{R}^d$ and $\theta \in \mathbb{S}^{d-1}$; that is,

$$(6.9) \quad \lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\{r \leq |z| \leq R\}} f(x, z)(k(x/\varepsilon, z/\varepsilon) - \bar{k}(x/\varepsilon))\Pi_0(dz) \right| = 0.$$

We emphasize that for this result we do not assume the boundedness of the Radon–Nikodym derivative $\frac{\varrho_0(d\theta)}{\sigma(d\theta)}$.

(ii) In fact, the conclusion (6.9) holds for any finite measure ϱ_0 on \mathbb{S}^{d-1} that does not charge on the set of rationally dependent $\theta \in \mathbb{S}^{d-1}$. In particular, if ϱ_0 does not charge on singletons when $d = 2$ and does not charge on subsets of \mathbb{S}^{d-1} that are of Hausdorff dimension $d - 2$ when $d \geq 3$ (e.g., ϱ_0 is γ -dimensional Hausdorff measure with $\gamma \in (d - 2, d - 1]$), then (6.9) holds for any function $f(x, z)$ that satisfies (3.5).

We can drop the continuous assumption on $z \mapsto k(x, z)$ in Theorem 6.5(iii) when the spherical measure ϱ_0 in Π_0 is absolutely continuous with respect to σ with bounded Radon–Nikodym derivative.

THEOREM 6.7. *Suppose that ϱ_0 is absolutely continuous with respect to the Lebesgue surface measure σ on \mathbb{S}^{d-1} with bounded Radon–Nikodym derivative. Let $k(x, z)$ be a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ such that $z \mapsto k(x, z)$ is 1-periodic for each fixed $x \in \mathbb{R}^d$ and (1.2) is true. Then, (6.2) holds for any function $f(x, z)$ satisfying (3.5) with $\bar{k}(x, \theta) = \bar{k}(x) := \int_{\mathbb{T}^d} k(x, z) dz$ for all $x \in \mathbb{R}^d$ and $\theta \in \mathbb{S}^{d-1}$.*

PROOF. Let $\varphi \geq 0$ is a smooth function with compact support in \mathbb{R}^d having $\int_{\mathbb{R}^d} \varphi(y) dy = 1$. For $\delta > 0$, let $\varphi_\delta(y) := \delta^{-d} \varphi(y/\delta)$. Define

$$k_\delta(x, z) = \int_{\mathbb{R}^d} k(x, z - y)\varphi_\delta(y) dy.$$

Clearly, $k_\delta(x, z)$ is a bounded, multivariate 1-periodic and continuous function on $\mathbb{R}^d \times \mathbb{R}^d$. Thus, by Remark 6.6,

$$(6.10) \quad \lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\{r \leq |z| \leq R\}} f(x, z)(k_\delta(x/\varepsilon, z/\varepsilon) - \bar{k}_\delta(x/\varepsilon))\Pi_0(dz) \right| = 0$$

for any function $f(x, z)$ that has property (3.5), where $\bar{k}_\delta(x) := \int_{[0,1]^d} k_\delta(x, z) dz$. Clearly,

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{T}^d} |k(x, z) - k_\delta(x, z)| dz = 0.$$

Condition (1.2) implies that the above convergence is uniform in $x \in \mathbb{R}^d$. Furthermore, $\bar{k}(x) := \int_{\mathbb{T}^d} k(x, z) dz$ is uniformly continuous in x , and $\bar{k}_\delta(x)$ converges to $\bar{k}(x)$ uniformly as $\delta \rightarrow 0$. Observe that by the fact that ϱ_0 is absolutely continuous with respect to σ on \mathbb{S}^{d-1} with bounded Radon–Nikodym derivative and the multivariate 1-periodicity of $(x, z) \mapsto k(x, z)$, it holds that, for any $0 < r < R$ and $\varepsilon \in (0, 1)$,

$$\begin{aligned} & \int_{\{r \leq |z| \leq R\}} |k(x/\varepsilon, z/\varepsilon) - k_\delta(x/\varepsilon, z/\varepsilon)| \Pi_0(dz) \\ & \leq c_1 r^{-(d+\alpha)} \int_{\{r \leq |z| \leq R\}} |k(x/\varepsilon, z/\varepsilon) - k_\delta(x/\varepsilon, z/\varepsilon)| dz \\ & \leq c_2 r^{-(d+\alpha)} |B(0, R) \setminus B(0, r)| \int_{\mathbb{T}^d} |k(x, z) - k_\delta(x, z)| dz, \end{aligned}$$

where c_1 and c_2 are two positive constants that are independent of $\varepsilon \in (0, 1)$, $\delta \in (0, 1)$ and $0 < r < R$. Thus, we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\{r \leq |z| \leq R\}} f(x, z) (k(x/\varepsilon, z/\varepsilon) - \bar{k}(x/\varepsilon)) \Pi_0(dz) \right| \\ & \leq \lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \|f\|_\infty \int_{\{r \leq |z| \leq R\}} |k(x/\varepsilon, z/\varepsilon) - k_\delta(x/\varepsilon, z/\varepsilon)| \Pi_0(dz) \\ & \quad + \lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\{r \leq |z| \leq R\}} f(x, z) (k_\delta(x/\varepsilon, z/\varepsilon) - \bar{k}_\delta(x/\varepsilon)) \Pi_0(dz) \right| \\ & \quad + \lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \|f\|_\infty |\bar{k}_\delta(x/\varepsilon) - \bar{k}(x/\varepsilon)| \Pi_0(r \leq |z| \leq R) \\ & \leq c_2 \|f\|_\infty r^{-(d+\alpha)} |B(0, R) \setminus B(0, r)| \sup_{x \in \mathbb{R}^d} \int_{\mathbb{T}^d} |k(x, z) - k_\delta(x, z)| dz \\ & \quad + \|f\|_\infty \Pi_0(r \leq |z| \leq R) \sup_{x \in \mathbb{R}^d} |\bar{k}_\delta(x) - \bar{k}(x)|. \end{aligned}$$

Letting $\delta \rightarrow 0$ in the right-hand side of the inequality above proves the result. \square

7. Examples and comments.

7.1. *Examples.* In this subsection we first give the proof of the assertions in Example 1.1 and then present two additional examples to further illustrate the applications of our main results. Example 1.1, together with two examples below, show that the periodic homogenization of jump processes is very different from that of diffusion processes. In the homogenization of jump processes, large jumps play a key role on the homogenized process. The scale function φ is determined by the tail of the jumping kernel.

PROOF OF EXAMPLE 1.1. (i) Suppose that $\alpha \in (0, 2)$ and that $k(x, z)$ is a bounded continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ so that $x \mapsto k(x, z)$ is multivariate 1-periodic for each fixed $z \in \mathbb{R}^d$, $z \mapsto k(x, z)$ is multivariate 1-periodic for each fixed $x \in \mathbb{R}^d$ and (1.2) is true. Clearly, $\varphi(r) := r^\alpha$ satisfies (3.3). Then, it is easy to see that $\Pi(dz)$, defined by (1.8), has the expression (3.1) with $\varrho_0(d\theta)$, $\varphi(r)$ given above and $\kappa(r, d\theta) \equiv 0$. Furthermore, we know by Theorem 6.5 that (3.6) holds with $\bar{k}(x, z) = \bar{k}(x, z/|z|)$, given by (1.10). Therefore, the claimed

assertions in this example follow readily from Theorem 3.2, Theorem 3.4, Remark 6.6(ii) and Theorem 6.7.

(ii) Suppose that $\alpha = 2$ and $\lim_{|z| \rightarrow \infty} \bar{k}(z) = k_0$, where $\bar{k}(z) = \int_{\mathbb{T}^d} k(x, z) \mu(dx)$. Then,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \left| \int_{\{|z| \leq 1/\varepsilon\}} (z \otimes z)(\bar{k}(z) - k_0) \Pi(dz) \right| = 0$$

and

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{\{1 \leq |z| \leq 1/\varepsilon\}} z_i z_j \Pi(dz) = \int_{\mathbb{S}^{d-1}} \theta_i \theta_j \varrho_0(d\theta).$$

Consequently,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{\{|z| \leq 1/\varepsilon\}} (z \otimes z) \bar{k}(z) \Pi(dz) = A,$$

where $A = \{a_{ij}\}_{1 \leq i, j \leq d}$ with

$$a_{ij} := k_0 \int_{\mathbb{S}^{d-1}} \theta_i \theta_j \varrho_0(d\theta).$$

Thus, (5.2) and (5.3) hold with $\phi(\varepsilon) = |\log \varepsilon|$. Furthermore, it is easy to see that (5.1) holds for $\phi(\varepsilon)$. Then, the assertion follows from Theorem 5.1.

(iii) If $\alpha > 2$, then $\int_{\mathbb{R}^d} |z|^2 \Pi(dz) < \infty$, so the desired assertion immediately follows from Theorem 4.1. \square

REMARK 7.1. We call a subset $\Gamma \subset \mathbb{R}^d$ an unbounded generalized cone, if $\lambda \Gamma \subset \Gamma$ for every $\lambda \geq 0$. Note that Γ can have several branches starting from the origin, and it can be nonsymmetric. Let $\sigma(d\theta)$ denote the Lebesgue surface measure on \mathbb{S}^{d-1} . If

$$\mathbb{1}_{\{|z| > 1\}} \Pi(dz) = \frac{1}{|z|^{d+\alpha}} \mathbb{1}_{\{|z| > 1: z \in \Gamma\}} dz = \frac{1}{r^{1+\alpha}} \mathbb{1}_{\{r > 1, \theta \in \Gamma\}} dr \sigma(d\theta)$$

for some generalized cone Γ with $\sigma(\Gamma \cap \mathbb{S}^{d-1}) > 0$ in Example 1.1(i), then the generator \mathcal{L}_0 of the limit process $(\bar{X}_t)_{t \geq 0}$ is given by

$$\bar{\mathcal{L}}_0 f(x) = \begin{cases} \int_{\mathbb{R}^d} (f(x+z) - f(x)) \frac{1}{|z|^{d+\alpha}} \mathbb{1}_\Gamma(z) dz, & \alpha \in (0, 1), \\ \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z| \leq 1\}}) \frac{1}{|z|^{d+1}} \mathbb{1}_\Gamma(z) dz, & \alpha = 1, \\ \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle) \frac{1}{|z|^{d+\alpha}} \mathbb{1}_\Gamma(z) dz, & \alpha \in (1, 2). \end{cases}$$

This gives us another concrete example that the jumping kernel limiting process $(\bar{X}_t)_{t \geq 0}$ can be degenerate.

In the following, we always suppose that Assumptions (A1), (A2) and (A3) hold and that $k(x, z)$ is a nonnegative bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ such that $x \mapsto k(x, z)$ is multivariate 1-periodic for each fixed $z \in \mathbb{R}^d$ and (1.2) is true. We refer the reader to Section 7.2 for conditions on small jumps of the jumping kernel such that all (A1), (A2) and (A3) are satisfied. Let $(X_t)_{t \geq 0}$ be the strong Markov process corresponding to the operator \mathcal{L} given by (1.1). Let $\mu(dx)$ be the stationary probability measure for the quotient process of X on \mathbb{T}^d . Let $b_R(x)$, $b_\infty(x)$, \bar{b}_R (with $R > 1$) and \bar{b}_∞ be defined by (1.6), (1.7) and (1.9), respectively. Let $\sigma(d\theta)$ denote the Lebesgue surface measure on \mathbb{S}^{d-1} .

EXAMPLE 7.2. Let $a_0(\theta)$ be a nonnegative bounded function defined on the unit sphere \mathbb{S}^{d-1} . Suppose that

$$\mathbb{1}_{\{|z|>1\}} \Pi(dz) = \frac{a_0(z/|z|)}{|z|^d \Phi(|z|)} \mathbb{1}_{\{|z|>1\}} dz,$$

where

$$(7.1) \quad \Phi(r) := \int_{\alpha_1}^{\alpha_2} r^\alpha \nu(d\alpha)$$

for constants $0 < \alpha_1 < \alpha_2 < 2$ and a nonnegative finite measure ν on $[\alpha_1, \alpha_2]$ such that $\alpha_2 \in \text{supp}[\nu]$ (i.e., $\nu((\alpha_2 - \varepsilon, \alpha_2]) > 0$ for any $\varepsilon > 0$). Suppose also that for every fixed $x \in \mathbb{R}^d$, $k(x, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is multivariate 1-periodic and satisfies (1.2).

For any $\varepsilon \in (0, 1]$, define $(X_t^\varepsilon)_{t \geq 0} = (\varepsilon X_{\Phi(1/\varepsilon)t})_{t \geq 0}$ and $(Y_t^\varepsilon)_{t \geq 0}$ by

$$Y_t^\varepsilon = \begin{cases} X_t^\varepsilon, & 0 < \alpha_2 < 1, \\ X_t^\varepsilon - \varepsilon \Phi(1/\varepsilon)(\bar{b}_{1/\varepsilon} + \bar{b})t, & \alpha_2 = 1, \\ X_t^\varepsilon - \varepsilon \Phi(1/\varepsilon)(\bar{b}_\infty + \bar{b})t, & 1 < \alpha_2 < 2. \end{cases}$$

Then, the process $(Y_t^\varepsilon)_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$, as $\varepsilon \rightarrow 0$, to a (possibly nonsymmetric) α_2 -stable process with jumping measure $\bar{k}_0 a_0(z/|z|) |z|^{-d-\alpha_2} dz$, where $\bar{k}_0 := \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} k(x, z) dz \mu(dx)$.

PROOF. Let $\varphi(r) = \Phi(r)$. Clearly, $\varphi(r)$ is a strictly increasing function on $(1, \infty)$. We claim that it satisfies condition (3.3) with $\alpha = \alpha_2$. For any $\eta \in (0, \alpha_2)$, since $\nu((\alpha_2 - \eta, \alpha_2]) > 0$,

$$\varphi(r) \asymp \int_{\alpha_2 - \eta}^{\alpha_2} r^\alpha \nu(d\alpha) \quad \text{for } r > 1.$$

Thus, for $r > 1$ and $\lambda \geq 1$,

$$\lambda^{\alpha_2 - \eta} \varphi(r) \leq \lambda^{\alpha_2 - \eta} \int_{\alpha_2 - \eta}^{\alpha_2} r^\alpha \nu(d\alpha) \leq \varphi(\lambda r) \leq \lambda^{\alpha_2} \int_{\alpha_2 - \eta}^{\alpha_2} r^\alpha \nu(d\alpha) \leq \lambda^{\alpha_2} \varphi(r).$$

Hence, we have shown that, for any $\eta \in (0, \alpha_2)$, there is a positive constant $c_0 = c_0(\eta) \leq 1$ so that

$$(7.2) \quad c_0 (R/r)^{\alpha_2 - \eta} \leq \frac{\varphi(R)}{\varphi(r)} \leq c_0^{-1} (R/r)^{\alpha_2} \quad \text{for any } R \geq r > 1.$$

Furthermore, for any $\eta > 0$ sufficiently small, clearly, we have, for every $r > 1$,

$$\limsup_{\lambda \rightarrow \infty} \frac{\varphi(\lambda r)}{\varphi(\lambda)} = \limsup_{\lambda \rightarrow \infty} \frac{\int_{\alpha_1}^{\alpha_2} \lambda^\alpha r^\alpha \nu(d\alpha)}{\int_{\alpha_1}^{\alpha_2} \lambda^\alpha \nu(d\alpha)} \leq r^{\alpha_2}.$$

On the other hand, since $\nu((\alpha_2 - \eta, \alpha_2]) > 0$,

$$\liminf_{\lambda \rightarrow \infty} \frac{\varphi(\lambda r)}{\varphi(\lambda)} \geq \liminf_{\lambda \rightarrow \infty} \frac{\int_{\alpha_2 - \eta}^{\alpha_2} \lambda^\alpha r^\alpha \nu(d\alpha)}{\int_{\alpha_2 - \eta}^{\alpha_2} \lambda^\alpha \nu(d\alpha)} \geq r^{\alpha_2 - \eta} \quad \text{for } r > 1.$$

Since the above holds for every sufficiently small $\eta > 0$, passing $\eta \rightarrow 0$ yields that $\liminf_{\lambda \rightarrow \infty} \varphi(\lambda r)/\varphi(\lambda) = r^{\alpha_2}$ for $r > 1$. Hence, we get

$$\lim_{\lambda \rightarrow \infty} \frac{\varphi(\lambda r)}{\varphi(\lambda)} = r^{\alpha_2} \quad \text{for } r \geq 1.$$

This, together with (7.2), proves the claim that (3.3) holds with α_2 in place of α there. On the other hand, it follows from Theorem 6.7 that (3.6) holds with $\bar{k}(x, z) = \bar{k}(x) := \int_{\mathbb{T}^d} k(x, u) du$

for all $x, z \in \mathbb{R}^d$. The desired assertions now follows from Theorems 3.2 and 3.4, after noticing that $\Pi(dz)$ has the representation (3.1) with $\varrho_0(d\theta) = a_0(\theta)\sigma(d\theta)$ and $\kappa(r, d\theta) \equiv 0$, where $\sigma(d\theta)$ denotes the Lebesgue surface measure on \mathbb{S}^{d-1} . \square

REMARK 7.3. (1) If $\nu(d\eta) = \delta_{\{\alpha\}}(d\eta) + \delta_{\{\beta\}}(d\eta)$ with $0 < \beta < \alpha < 2$ in (7.1), then $\mathbb{1}_{\{|z|>1\}}\Pi(dz)$ in Example 7.2 is reduced to

$$\frac{a_0(z/|z|)}{|z|^{d+\alpha} + |z|^{d+\beta}} \mathbb{1}_{\{|z|>1\}} dz.$$

In this case, $\mathbb{1}_{\{|z|>1\}}\Pi(dz)$ admits the expression (3.1) with $\varphi(r) = r^\alpha + r^\beta$, $\varrho_0(d\theta) = a_0(\theta)\sigma(d\theta)$ and $\kappa(r, d\theta) \equiv 0$. If we take $\varphi_1(r) = r^\alpha$, $\varrho_0(d\theta) = a_0(\theta)\sigma(d\theta)$ and $\kappa_1(r, d\theta) := \frac{r^\beta}{r^\alpha+r^\beta}a_0(\theta)\sigma(d\theta)$, then $\mathbb{1}_{\{|z|>1\}}\Pi(dz)$ can be also represented by (3.1) with φ_1 and $\kappa_1(r, d\theta)$ in place of φ and $\kappa(r, d\theta)$. Thus, the homogenization result for X holds with both φ and φ_1 as its time scaling function.

(2) If $\nu(d\eta)$ is the Lebesgue measure on $[\alpha/2, \alpha]$ for some $\alpha \in (0, 2)$, then

$$\mathbb{1}_{\{|z|>1\}}\Pi(dz) = \frac{a_0(z/|z|)}{|z|^d(|z|^\alpha - |z|^{\alpha/2}) \log |z|} \mathbb{1}_{\{|z|>1\}} dz.$$

In this case, $\mathbb{1}_{\{|z|>1\}}\Pi(dz)$ admits the expression (3.1) with $\varphi(r) = \Phi(r) = (r^\alpha - r^{\alpha/2}) \log r$, $\varrho_0(d\theta) = a_0(\theta)\sigma(d\theta)$ on \mathbb{S}^{d-1} and $\kappa(r, d\theta) \equiv 0$. If we take

$$\varphi_1(r) = r^\alpha \log r \mathbb{1}_{\{r>1\}},$$

then, clearly, $\lim_{r \rightarrow \infty} \varphi_1(r)/\varphi(r) = 1$. Thus, by Remark 3.1, $\mathbb{1}_{\{|z|>1\}}\Pi(dz)$ can also be represented by (3.1) with φ_1 in place of φ (but with different $\kappa(r, d\theta)$); that is, we can write

$$\mathbb{1}_{\{|z|>1\}}\Pi(dz) = \mathbb{1}_{\{|z|>1\}} \frac{a_0(\theta) + \kappa_1(r, d\theta)}{r^{1+\alpha} \log r} \sigma(d\theta) dr,$$

where $\kappa_1(r, \theta)$ satisfies (3.2). In particular, the homogenization result for X holds with both φ and φ_1 as its time scaling function.

(3) Similar to these of Example 1.1, we can get the assertion when the jumping measure $\Pi(dz)$ enjoys the form

$$\mathbb{1}_{\{|z|>1\}}\Pi(dz) = \mathbb{1}_{\{|z|>1\}} \frac{a_0(z/|z|)}{|z|^{d+\alpha} \log |z|} dz$$

with $\alpha \geq 2$. In details, when $\alpha = 2$, define

$$Y_t^\varepsilon := \varepsilon X_{\varepsilon^{-2}(\log |\log \varepsilon|)^{-1}t} - \varepsilon^{-1}(\log |\log \varepsilon|)^{-1}(\bar{b}_\infty + \bar{b})t, \quad t \geq 0.$$

Suppose that (1.11) holds for some $k_0 > 0$. Then, as $\varepsilon \rightarrow 0$, $(Y_t^\varepsilon)_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$ to Brownian motion $(B_t)_{t \geq 0}$ with the covariance matrix $A = \{a_{ij}\}_{1 \leq i, j \leq d}$, where

$$a_{ij} = k_0 \int_{\mathbb{S}^{d-1}} \theta_i \theta_j a_0(\theta) \sigma(d\theta).$$

When $\alpha > 2$, we define

$$Y_t^\varepsilon := \varepsilon X_{t/\varepsilon^2} - \varepsilon^{-1}(\bar{b}_\infty + \bar{b})t, \quad t \geq 0.$$

Then, as $\varepsilon \rightarrow 0$, $(Y_t^\varepsilon)_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$ to Brownian motion $(B_t)_{t \geq 0}$ with the covariance matrix A , defined by (1.13).

The following example is concerned with the homogenization for jump process with a singular jumping kernel.

EXAMPLE 7.4. Suppose that

$$(7.3) \quad \mathbb{1}_{\{|z|>1\}} \Pi(dz) = \sum_{i=1}^d \frac{1}{r^{1+\alpha}} \delta_{e_i}(d\theta) \mathbb{1}_{\{|r|>1\}} dr,$$

where $\{e_i\}_{i=1}^d$ is the standard orthonormal basis of \mathbb{R}^d and $\delta_{\theta_0}(d\theta)$ denotes the Dirac measure on \mathbb{S}^{d-1} concentrated at $\theta_0 \in \mathbb{S}^{d-1}$:

(i) Suppose that $z \mapsto k(x, z)$ is multivariate 1-periodic for each fixed $x \in \mathbb{R}^d$. For any $\varepsilon \in (0, 1)$, define $(Y_t^\varepsilon)_{t \geq 0}$ by

$$Y_t^\varepsilon = \begin{cases} \varepsilon X_{t/\varepsilon^\alpha}, & 0 < \alpha < 1, \\ \varepsilon X_{t/\varepsilon^\alpha} - (\bar{b}_{1/\varepsilon} + \bar{b})t, & \alpha = 1, \\ \varepsilon X_{t/\varepsilon^\alpha} - \varepsilon^{1-\alpha}(\bar{b}_\infty + \bar{b})t, & 1 < \alpha < 2. \end{cases}$$

Then, the process $(Y_t^\varepsilon)_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$, as $\varepsilon \rightarrow 0$, to a nonsymmetric α -stable process $(\bar{X}_t)_{t \geq 0}$ with infinitesimal generator \mathcal{L}_0 as follows:

$$\mathcal{L}_0 f(x) = \begin{cases} \sum_{i=1}^d \int_0^\infty (f(x + z_i e_i) - f(x)) \frac{\bar{k}_i}{z_i^{1+\alpha}} dz_i, & \alpha \in (0, 1), \\ \sum_{i=1}^d \int_0^\infty \left(f(x + z_i e_i) - f(x) - \frac{\partial f(x)}{\partial x_i} \cdot z_i \mathbb{1}_{\{0 < z_i \leq 1\}} \right) \frac{\bar{k}_i}{z_i^{1+\alpha}} dz_i & \alpha = 1, \\ \sum_{i=1}^d \int_0^\infty \left(f(x + z_i e_i) - f(x) - \frac{\partial f(x)}{\partial x_i} \cdot z_i \right) \frac{\bar{k}_i}{z_i^{1+\alpha}} dz_i & \alpha \in (1, 2), \end{cases}$$

where

$$\bar{k}_i := \int_{\mathbb{T}^d} \int_0^1 k(x, (0, \dots, z_i, \dots, 0)) dz_i \mu(dx).$$

(ii) When $\alpha = 2$, we define

$$Y_t^\varepsilon := \varepsilon X_{\varepsilon^{-2}|\log \varepsilon|^{-1}t} - \varepsilon^{-1}|\log \varepsilon|^{-1}(\bar{b}_\infty + \bar{b})t, \quad t \geq 0.$$

Suppose that (1.11) holds for some $k_0 > 0$. Then, as $\varepsilon \rightarrow 0$, $(Y_t^\varepsilon)_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$ to Brownian motion $(B_t)_{t \geq 0}$ with the covariance matrix $A := k_0 I_{d \times d}$, where $I_{d \times d}$ denotes the $d \times d$ identity matrix.

(iii) When $\alpha > 2$, we define

$$Y_t^\varepsilon := \varepsilon X_{t/\varepsilon^2} - \varepsilon^{-1}(\bar{b}_\infty + \bar{b})t, \quad t \geq 0.$$

Then, as $\varepsilon \rightarrow 0$, $(Y_t^\varepsilon)_{t \geq 0}$ converges weakly in $\mathcal{D}([0, \infty); \mathbb{R}^d)$ to Brownian motion $(B_t)_{t \geq 0}$ with the covariance matrix A , defined by (1.13).

PROOF. By (7.3) we know that (3.1) holds with $\varphi(r) = r^\alpha$, $\varrho_0(d\theta) = \sigma_0(d\theta)$ and $\kappa(r, d\theta) \equiv 0$. According to Corollary 6.3 below we know that (3.6) holds with $\bar{k}(x, e_i) = \int_0^1 k(x, (0, \dots, z_i, \dots, 0)) dz_i$ for $1 \leq i \leq d$, and $\bar{k}(x, \theta) = 0$ for any $\theta \in \mathbb{S}^{d-1} \setminus \{e_i\}_{1 \leq i \leq d}$. Hence the desired assertion in (i) follows from Theorems 3.2 and 3.4. The proofs of (ii) and (iii) are similar to these of Example 1.1. \square

7.2. *Comments on assumptions (A1), (A2) and (A3).* Assumptions (A1), (A2) and (A3) are closely related with recent developments on the fundamental solution of the Lévy-type operators. For example, in [14] the authors considered the following Lévy-type operator on \mathbb{R}^d :

$$\mathcal{L}f(x) = \lim_{\delta \rightarrow 0} \int_{\{|z| > \delta\}} (f(x+z) - f(x)) \frac{k(x,z)}{|z|^{d+\alpha}} dz,$$

where $0 < k_1 \leq k(x, z) \leq k_2$, $k(x, z) = k(x, -z)$ and $|k(x, z) - k(y, z)| \leq k_3|x - y|^\beta$ for some constants $k_i > 0$ ($i = 1, 2, 3$) and $\beta \in (0, 1)$. Later, the results of [14] are extended to time-dependent cases in [15] such that the symmetric assumption in z for the function $k(x, z)$ are not required; moreover, the corresponding results for the perturbation by a drift term $b(x)$ belonging to some Kato’s class when $\alpha \in (1, 2)$ are also considered there; see [15], Theorem 1.5. For the critical case (i.e., $\alpha = 1$), one can refer to [41]; see [11, 27, 29] and the references therein for more recent progress on this topic, including the case that a large class of symmetric Lévy processes are considered instead of rotationally symmetric α -stable processes, and the case that the index function $\alpha(x)$ depends on x .

PROPOSITION 7.5. *Let \mathcal{L} be the operator given by (1.1) such that the coefficients satisfy all the assumptions below (1.1), $k(x, z)$ is bounded between two positive constants and that there is a constant $\beta \in (0, 1)$ so that $b(x) \in C_b^\beta(\mathbb{R}^d)$ and*

$$\sup_{z \in \mathbb{R}^d} |k(x, z) - k(y, z)| \leq c_0|x - y|^\beta, \quad x, y \in \mathbb{R}^d$$

for some $c_0 > 0$. Assume that

$$(7.4) \quad \mathbb{1}_{\{|z| \leq 1\}} \Pi(dz) = \mathbb{1}_{\{|z| \leq 1\}} \frac{1}{|z|^{d+\alpha_0}} dz$$

for some $\alpha_0 \in (0, 2)$. For $\alpha_0 \in (0, 1)$, we assume, in addition, that $b(x) = \int_{\{|z| \leq 1\}} z \frac{k(x,z)}{|z|^{d+\alpha}} dz$; for $\alpha_0 = 1$, we assume, in addition, that $k(x, z) = k(x, -z)$ for all $x, z \in \mathbb{R}^d$. Then, assumptions (A1) and (A2) are satisfied.

PROOF. For simplicity, we only prove the case that $\alpha_0 \in (1, 2)$, since the proofs of the cases $\alpha_0 \in (0, 1)$ and $\alpha_0 = 1$ are similar:

(1) We first assume that $\Pi(dz) = |z|^{-d-\alpha_0} dz$. According [15], Theorem 1.5 (see also [27], Theorem 1.4), there is a unique fundamental solution $p(t, x, y) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ of the operator \mathcal{L} . Then, the existence and the uniqueness of the Feller process $X := ((X_t)_{t \geq 0}; (\mathbb{P}^x)_{x \in \mathbb{R}^d})$ associated with the operator \mathcal{L} was mentioned in [15], Remark 1.6. In particular, $p(t, x, y)$ is the transition density function of the process X with respect to the Lebesgue measure. By two-sided estimates and gradient estimates of $p(t, x, y)$ stated in [15], Theorem 1.5, (i) and (vi), we can easily see that the process X is irreducible and enjoys the strong Feller property; that is, the associated semigroup $(P_t)_{t \geq 0}$ maps measurable bounded functions into continuous bounded functions.

Concerning assumption (A1), it is clear that the probability law of X solves the martingale problem for $(\mathcal{L}, C_c^\infty(\mathbb{R}^d))$ in the sense that, for every $f \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$f(X_t) - f(x) - \int_0^t \mathcal{L}f(X_s) ds, \quad t \geq 0$$

is a \mathbb{P}^x -martingale; see [15], Remark 1.2(iv). By our assumptions on $k(x, z)$ and $b(x)$ again and the process X being conservative (see [15], Theorem 1.1, (iv)), X solves the martingale problem for $(\mathcal{L}, C_b^2(\mathbb{R}^d))$ as well. If we regard X as a \mathbb{T}^d -valued process, then the associated

semigroup is still irreducible and has the strong Feller property by the statements above; see, for example, the proof of [21], Proposition 1, or the argument of [36], Section 4. This along with [40], Theorem 1.1, gives us Assumption **(A2)**.

(2) Let \mathcal{L} be the operator given in Proposition 7.5. Set $\hat{\Pi}(dz) := |z|^{-d-\alpha_0} dz$ and

$$(7.5) \quad \mathcal{L}_0 f(x) := \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z| \leq 1\}}) k(x, z) \hat{\Pi}(dz) + \langle b(x), \nabla f(x) \rangle.$$

Then,

$$\mathcal{L} f(x) = \mathcal{L}_0 f(x) + \mathcal{A} f(x),$$

where

$$(7.6) \quad \mathcal{A} f(x) = \int_{\{|z| > 1\}} (f(x+z) - f(x)) k(x, z) (\Pi(dz) - \hat{\Pi}(dz)).$$

It is clear that, under assumptions on $k(x, z)$ and $\Pi(dz)$, there is a constant $c_1 > 0$ such that $\|\mathcal{A} f\|_\infty \leq c_1 \|f\|_\infty$ for all $f \in B_b(\mathbb{R}^d)$. By bounded perturbation results for martingale problems, one can deduce that assumption **(A1)** holds for the operator \mathcal{L} ; see, for example, [18], Chapter 4, Section 10, page 253.

By the proof of Proposition 6 and Remark 8 in [12], we know that the process $(X_t)_{t \geq 0}$ is irreducible. On the other hand, let $(P_t)_{t \geq 0}$ and $(P_t^0)_{t \geq 0}$ be the semigroups associated with the operator $(\mathcal{L}, C_b^2(\mathbb{R}^d))$ and $(\mathcal{L}_0, C_b^2(\mathbb{R}^d))$, respectively. It holds that

$$P_t f = P_t^0 f + \int_0^t P_s^0 \mathcal{A} P_{t-s} f ds, \quad t > 0, f \in B_b(\mathbb{R}^d).$$

This, along with the fact that $(P_t^0)_{t \geq 0}$ has the strong Feller property, yields that $(P_t)_{t \geq 0}$ also enjoys the strong Feller property. Thus, Assumption **(A2)** holds, thanks to [40], Theorem 1.1, again. \square

PROPOSITION 7.6. *Let \mathcal{L} be the operator given in Proposition 7.5. If $\alpha_0 \in (1, 2)$, then assumption **(A3)** is also satisfied.*

PROOF. Similar to the proof of Proposition 7.5, let \mathcal{L}_0 be defined by (7.5). We write

$$\mathcal{L}_0 f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z \rangle) k(x, z) \hat{\Pi}(dz) + \langle \hat{b}_\infty(x), \nabla f(x) \rangle,$$

where

$$\hat{\Pi}(dz) := \frac{1}{|z|^{d+\alpha_0}} dz, \quad \hat{b}_\infty(x) := b(x) + \int_{\{|z| > 1\}} z \frac{k(x, z)}{|z|^{d+\alpha_0}} dz.$$

Let $p_0(t, x, y)$ and P_t^0 be the fundamental solution and Markov semigroup associated with \mathcal{L}_0 , respectively. Note that $\hat{b}_\infty \in C_b^\beta(\mathbb{R}^d; \mathbb{R}^d)$. By [15], Theorem 1.5, for any $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$,

$$(7.7) \quad \begin{aligned} p_0(t, x, y) &\leq \frac{c_0 t}{(t^{1/\alpha_0} + |x - y|)^{d+\alpha_0}}, \\ |\nabla_x p_0(t, \cdot, y)(x)| &\leq \frac{c_0 t^{1-1/\alpha_0}}{(t^{1/\alpha_0} + |x - y|)^{d+\alpha_0}}. \end{aligned}$$

(Note that in our setting we can take $\eta = 0$ in [14], Theorem 1.5; see also [27], Theorem 1.4.) For any $\lambda > 0$, let R_λ^0 be the λ -resolvent of the semigroup $(P_t^0)_{t \geq 0}$, that is,

$$R_\lambda^0 f(x) := \int_0^\infty e^{-\lambda t} P_t^0 f(x) dt, \quad f \in C(\mathbb{T}^d), x \in \mathbb{R}^d.$$

According to (7.7), we can see that R_λ^0 is an operator such that $R_\lambda^0 : C(\mathbb{T}^d) \rightarrow C^1(\mathbb{T}^d)$ so that

$$\|R_\lambda^0 f\|_\infty + \|\nabla R_\lambda^0 f\|_\infty \leq \frac{c_1}{\lambda} \|f\|_\infty, \quad f \in C(\mathbb{T}^d),$$

where c_1 is a positive constant independent of λ and f . Here, we used the fact that $\alpha_0 \in (1, 2)$. Furthermore, It is well known that $R_\lambda^0 = (\lambda - \mathcal{L}_0)^{-1}$. Thus, $(\lambda - \mathcal{L}_0)^{-1} : C(\mathbb{T}^d) \rightarrow C^1(\mathbb{T}^d)$ and

$$(7.8) \quad \|(\lambda - \mathcal{L}_0)^{-1} f\|_\infty + \|\nabla(\lambda - \mathcal{L}_0)^{-1} f\|_\infty \leq \frac{c_1}{\lambda} \|f\|_\infty, \quad f \in C(\mathbb{T}^d).$$

Let $\mathcal{A}f$ be defined by (7.6). By the assumption on $k(x, z)$, $\mathcal{A} : C(\mathbb{T}^d) \rightarrow C(\mathbb{T}^d)$ satisfies that

$$(7.9) \quad \|\mathcal{A}f\|_\infty \leq c_2 \|f\|_\infty, \quad f \in C(\mathbb{T}^d).$$

Note that $\mathcal{L} = \mathcal{L}_0 + \mathcal{A}$. Then, for each $\lambda > 0$,

$$(\lambda - \mathcal{L})^{-1} = (\lambda - \mathcal{L}_0)^{-1} (1 - \mathcal{A}(\lambda - \mathcal{L}_0)^{-1})^{-1}.$$

Therefore, combining (7.8) with (7.9), we find that, for every $\lambda > \lambda_0 := c_1 c_2 > 0$, $(\lambda - \mathcal{L})^{-1} : C(\mathbb{T}^d) \rightarrow C^1(\mathbb{T}^d)$ is well defined such that

$$(7.10) \quad \|(\lambda - \mathcal{L})^{-1} f\|_\infty + \|\nabla(\lambda - \mathcal{L})^{-1} f\|_\infty \leq c_3(\lambda) \|f\|_\infty, \quad \lambda > \lambda_0, f \in C(\mathbb{T}^d).$$

For every $f \in C(\mathbb{T}^d)$ with $\mu(f) = 0$, let $\psi_f := -\int_0^\infty P_t f dt$ which is well defined by (1.3). Moreover, $\psi_f \in \mathcal{D}(\mathcal{L})$, $\mathcal{L}\psi_f = f$, $\mu(\psi_f) = 0$ and $\|\psi_f\|_\infty \leq c_4 \|f\|_\infty$. In particular, for every $\lambda > \lambda_0$, it holds that

$$\psi_f = (\lambda - \mathcal{L})^{-1} (\lambda \psi_f - f).$$

Hence, by (7.10), for any $\lambda > \lambda_0$, we obtain

$$\|\psi_f\|_\infty + \|\nabla \psi_f\|_\infty \leq c_3(\lambda) \|\lambda \psi_f - f\|_\infty \leq c_5(\lambda) \|f\|_\infty.$$

Let $(X_t^x)_{t \geq 0}$ be the process associated with the martingale problem for $(\mathcal{L}, C_b^2(\mathbb{R}^d))$ with initial value x . Let $f \in C(\mathbb{T}^d)$ such that $\mu(f) = 0$. Then, for every $\psi \in \mathcal{D}(\mathcal{L})$ satisfying $\mathcal{L}\psi = f$ and $\mu(\psi) = 0$, we have $\mathbb{E}[\psi(X_t^x)] = \psi(x) + \int_0^t \mathbb{E}[f(X_s^x)] ds$. Letting $t \rightarrow \infty$ and applying (1.3), we get $\psi(x) = -\int_0^\infty P_s f(x) ds$. This means there exists a unique $\psi \in \mathcal{D}(\mathcal{L})$ satisfying $\mathcal{L}\psi = f$. Therefore, according to all the conclusions above, we prove that Assumption **(A3)** holds. \square

REMARK 7.7. For simplicity, in Proposition 7.5 we require $\Pi(dz)$ and $b(x)$ to have special forms; for instance, $\Pi(dz)$ satisfies (7.4) and $b(x) = \int_{\mathbb{R}^d} z \frac{k(x,z)}{|z|^{d+\alpha_0}} dz$ when $\alpha_0 \in (0, 1)$. These conditions are used to verify Assumption **(A3)** under minimal regularity requirements on $k(x, z)$ and $b(x)$. Indeed, under more general assumptions on $k(x, z)$ and $b(x)$ (i.e., it is not required that $b(x) = \int_{\mathbb{R}^d} z \frac{k(x,z)}{|z|^{d+\alpha_0}} dz$ when $\alpha_0 \in (0, 1)$), we can still verify **(A1)**, **(A2)** (see [31] for details) and weaken (1.5) into

$$(7.11) \quad \|\psi\|_\infty + \|\nabla \psi\|_\infty \leq C_1 \|f\|_{C^\beta}.$$

Then, under the conditions above, Theorems 3.4, 4.1 and 5.1 still hold true with some small modifications in their proofs. We note that (7.11) is closely related to the Schauder estimates for Lévy-type operators; see [4, 5, 16, 30, 32] and references therein for more details.

Moreover, suppose that $k \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $b \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^d)$. Then, by using the theory of pseudo-differential operators, we can prove the existence of the Feller process $X := ((X_t)_{t \geq 0}; (\mathbb{P}_x)_{x \in \mathbb{R}^d})$ associated with $(\mathcal{L}, C_c^\infty(\mathbb{R}^d))$, and, moreover, the process X can

be written explicitly via a solution of a stochastic differential equation with jumps; see [9], Chapter 3, for more details. Hence, we may obtain the following estimates for the associated semigroup $(P_t)_{t \geq 0}$ through the Bismut-type formula (see [33] and references therein):

$$\|\nabla P_t f\|_\infty \leq c_1 \|\nabla f\|_\infty, \quad 0 < t \leq 1,$$

$$\|\nabla P_t f\|_\infty \leq c_2 \|f\|_\infty, \quad 1/2 < t < 1.$$

According to these estimates and (1.3), we can find that, for every $f \in C^1(\mathbb{T}^d)$ with $\mu(f) = 0$, there exists a unique $\psi \in \mathcal{D}(\mathcal{L})$ such that $\mathcal{L}\psi = f$, $\mu(\psi) = 0$ and

$$\|\psi\|_\infty + \|\nabla\psi\|_\infty \leq c_3 (\|f\|_\infty + \|\nabla f\|_\infty).$$

This also suffices to prove Theorems 3.4, 4.1 and 5.1 with some modifications in the proofs.

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