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Recurrence of direct products of diffusion processes in random media having zero potentials

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Abstract

In this paper, we consider the recurrence of some multi-dimensional diffusion processes in random environments including zero potentials. Previous methods on diffusion processes in random environments are not applicable to the case of such environments. In main theorems, we obtain a sufficient condition to be recurrent for the product of a multi-dimensional diffusion process in semi-selfsimilar random environments and one-dimensional Brownian motion, and also more explicit sufficient conditions in the case of Gaussian random environments and random environments generated by Lévy processes. To prove them, we introduce an index which measures the strength of recurrence of symmetric Markov processes, and give some sufficient conditions for recurrence of direct products of symmetric diffusion processes. The index is given by the Dirichlet forms of the Markov processes.

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1 Introduction

Global properties of stochastic processes as well as related problems are important topics in both probability and potential theories. Among those, recurrence and transience of Markov processes have been studied by many authors under various probabilistic and analytic aspects in discrete and in continuous time. For instance, it is well-known that a d-dimensional Brownian motion consisting of d independent one-dimensional standard Brownian motions is recurrent if $d = 1, 2$, and transient otherwise. For more general diffusion processes, we have also many criteria for their recurrence and transience, but the criteria are not always so easy to be checked. In general, whether diffusion processes are recurrent or transient depends on their generators (see [5], [6], [7]). In this spirit, Ichihara [6] gave elegant criteria for the recurrence and transience of the

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diffusion process associated with a second order elliptic partial differential operator L on $\mathbb{R}^d$ defined by

$$L = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right), \tag{1.1}$$

where $a_{ij}(x)$ is a symmetric coefficient function such that the matrix $A(x) := (a_{ij}(x))_{1 \le i,j \le d}$ is strictly positive definite on $\mathbb{R}^d$.

Let $\mathcal{W}$ be the space of locally bounded Borel measurable functions on $\mathbb{R}^d$ vanishing at the origin and let $\mathcal{Q}$ be a probability measure on $\mathcal{W}$. In the present paper, an element of $\mathcal{W}$ is called an environment. Given an environment w, consider $\mathbf{Y}^w = (Y_w(t), P_x^w, x \in \mathbb{R}^d)$, the diffusion process with generator

$$\frac{1}{2}(\Delta - \nabla w \cdot \nabla) = \frac{1}{2} e^w \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left(e^{-w} \frac{\partial}{\partial x_k} \right).$$

It is well-known that $Y_w(t)$ can be constructed from the diffusion process $X_w(t)$ associated with (1.1) provided $a_{ij} = \frac{1}{2}\delta_{ij} e^{-w}$ through a random time change of $X_w(t)$. We call a stochastic process $\mathcal{Y}^w = (Y_w(t), \mathcal{Q} \otimes P_x^w, x \in \mathbb{R}^d)$ the diffusion process in a random environment. In the case where $d = 1$ and $(w, \mathcal{Q})$ is a Brownian environment, Brox [1] noticed that the process $\mathbf{Y}^w$ is a continuous version of Sinai's walk (see [16]) and showed that $Y_w(t)$ moves very slowly in some sense by the effect of the environment. Later, Brox's result was extended to a multi-dimensional diffusion process in a non-negative Lévy's Brownian environment (see [8], [12]).

Recurrence and transience of multi-dimensional diffusion processes in various random environments have been studied by many authors, in combining Ichihara's criteria with the ergodic aspects of measure preserving transformations on the random environments. The first result on this problem was obtained by Fukushima et al. in a one-dimensional Brownian environment (see [3]). Tanaka considered the diffusion process $\mathbf{Y}^w$ in a Lévy's Brownian environment and proved that it is to be recurrent for almost all environments in any dimension (see [19]), which made the effect of random environments on this problem quite transparent. After that, Tanaka's result was extended to a large class of multi-dimensional random environments (see [9], [11], [17], [18]). In [11], the authors considered multi-dimensional diffusion processes in multi-parameter random environments and studied their recurrence and transience. More precisely, the authors obtained some conditions for the dichotomy of recurrence and transience for d-dimensional diffusion process $Y_w(t) = (Y_w^1(t), Y_w^2(t), \cdots, Y_w^d(t))$ corresponding to the generator

$$\frac{1}{2} \sum_{k=1}^{d} e^{w(x_k)} \frac{\partial}{\partial x_k} \left(e^{-w(x_k)} \frac{\partial}{\partial x_k} \right), \tag{1.2}$$

where w is a one-dimensional (semi-)stable Lévy process whose values at different d points are regarded as constituting a multi-parameter environment. In their proof, the following property of the environments was crucial: for any $a_0 > 0$ and $\theta \ge 1$

$$\mathcal{Q} \left(\inf_{\sigma \in S^{d-1}} \sum_{j=1}^{d} w(\theta \sigma_j) > a_0 \right) > 0, \tag{1.3}$$

where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$. It turned out that the property (1.3) works well with Ichihara's test in studying the recurrence and transience of $Y_w(t)$. However, the property (1.3) does not hold if one component of w takes value identically zero. Indeed, for the two-dimensional direct product of diffusion process $(Y_w^1(t), B(t))$ given by the pair of the Brox's diffusion $Y_w^1(t)$ and a one-dimensional Brownian motion $B(t)$ independent of $Y_w^1(t)$, let $\widetilde{w}$ be the environment relative to $(Y_w^1(t), B(t))$. Then $\widetilde{w}(x_1, x_2) = w(x_1)$ and hence $\widetilde{w}(\sigma) = w(0) = 0$ for $\sigma := (0, 1) \in S^1$. In this sense, a $(d+1)$-dimensional diffusion process in d-parameter random environments

$$(Y_w(t), B(t)) := (Y_w^1(t), Y_w^2(t), \cdots, Y_w^d(t), B(t)) \tag{1.4}$$

with a one-dimensional Brownian motion $B(t)$ independent of $\{Y_w^j(t), j = 1, 2, \cdots, d\}$ is out of the framework of [11] (also of [18]).

The purpose of this paper is to study the recurrence of some multi-dimensional diffusion processes in random environments including zero potentials. For this, we introduce a criterion for the recurrence of direct products of symmetric Markov processes motivated by Okura [13]. In the criterion, the index induced by the Dirichlet forms plays an important role as representing the strength of recurrence of the associated Markov processes. The criterion works well in the case of diffusion processes in random environments, and we are able to show that the diffusion processes in semi-selfsimilar random environments have very strong recurrence in sense of the index. As a result, we can show the recurrence of direct products of Markov processes given by the pair of a d-dimensional diffusion process in almost all environments having usual randomness, and a one-dimensional Brownian motion (see Theorems 2.1, 2.2 and 2.4).

2 Main results

Now, we state our framework and the main results of the present paper.

Let $\mathcal{W}$ be the space of locally bounded and Borel measurable functions on $\mathbb{R}^d$ with the topology generated by the uniform convergence on compact sets. Let $\mathscr{B}(\mathcal{W})$ be the Borel σ-field of $\mathcal{W}$ and $\mathcal{Q}$ be a probability measure on $(\mathcal{W}, \mathscr{B}(\mathcal{W}))$. We call an element $w \in \mathcal{W}$ an environment and assume that $\mathcal{Q}(w(0) = 0) = 1$. For given $w \in \mathcal{W}$, let $\mathbf{X}^w = (X_w(t), P_x^w, x \in \mathbb{R}^d)$ be the diffusion process associated with the generator (1.1) provided $a_{ij} = \frac{1}{2}\delta_{ij}e^{-w}$, equivalently, associated with the strongly local Dirichlet form $(\mathcal{E}^w, \mathcal{F}^w)$ defined by

$$\begin{cases} \mathcal{F}^w := \left\{ f \in L^2(\mathbb{R}^d; e^{-w}\mathrm{d}x) : \dfrac{\partial f}{\partial x^i} \in L^2(\mathbb{R}^d; e^{-w}\mathrm{d}x), i = 1, 2, \ldots, d \right\}, \\ \mathcal{E}^w(f, g) := \dfrac{1}{2} \displaystyle\int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla g(x)\, e^{-w(x)}\mathrm{d}x, \quad f, g \in \mathcal{F}^w, \end{cases} \tag{2.1}$$

where the derivatives $\partial f / \partial x_i$ are taken in the sense of Schwartz distributions.

For $r > 1$ and $\alpha > 0$, let T be a mapping from $\mathcal{W}$ to $\mathcal{W}$ defined by $Tw(x) = r^{-\alpha}w(rx)$ for $x \in \mathbb{R}^d$. We assume that

$$\mathcal{Q}(A) = \mathcal{Q}(TA), \quad A \in \mathscr{B}(\mathcal{W}), \tag{2.2}$$

which implies that T is a measure preserving transformation of $\mathcal{Q}$. We call a space $(\mathcal{W}, \mathscr{B}(\mathcal{W}), \mathcal{Q})$ satisfying the condition (2.2) an α-semi-selfsimilar random environment.

Our first result concerns a sufficient condition to be recurrent for the direct product of a d-dimensional diffusion process in semi-selfsimilar random environments and a one-dimensional Brownian motion.

Theorem 2.1. *Assume that T is weakly mixing and*

$$\mathcal{Q}\left(\overline{w}(1, r^2) - \underline{w}(0, 1) < 2\underline{w}(1, r^2)\right) > 0, \tag{2.3}$$

where $\overline{w}$ and $\underline{w}$ are given by

$$\overline{w}(a, b) := \sup_{a \le |x| \le b} w(x), \quad \underline{w}(a, b) := \inf_{a \le |x| \le b} w(x)$$

for $a, b \geq 0$. Then, the $(d+1)$-dimensional direct product of diffusion process $(X_w(t), B(t))$ given by the pair of the d-dimensional diffusion process $X_w(t)$ and a one-dimensional Brownian motion $B(t)$ independent of $X_w(t)$ is recurrent for almost all environments.

It is known that random environments make a strong effect for the recurrence of the diffusion process in random environments. Indeed, diffusion processes in various random environments are recurrent in any dimension under some natural conditions (cf. [9], [11], [17], [18]). However, we cannot apply the previous methods to prove Theorem 2.1, because the component $B(t)$ of the product process $(X_w(t), B(t))$ has no effect of environments. As an alternative to the previous method, we employ a new criterion for the recurrence of direct products of symmetric Markov processes based on the theory of Dirichlet forms (Proposition 3.1 below). The criterion will be applied to diffusion processes in random environments together with Proposition 4.2, which plays a key role for the proof of Theorem 2.1.

We remark that the $(d+d')$-dimensional direct product process

$$(X_w(t), B^1(t), \dots, B^{d'}(t))$$

given by the pair of $X_w(t)$ and a d'-dimensional Brownian motion $(B^1(t), \dots, B^{d'}(t))$ independent of $X_w(t)$, is transient whenever $d' \geq 3$, because the marginal $(B^1(t), \dots, B^{d'}(t))$ is transient. To our best knowledge, the case $d' = 2$ is an open problem but we believe that it will be transient in view of the result in discrete cases. In fact, a similar problem was already concerned in discrete cases. In [2], the authors considered $d + d'$ independent walks on $\mathbb{Z}$, d of them performing Sinai's walk and d' of them performing simple symmetric random walk, and proved that the direct product is recurrent almost all environments if and only if $d' \leq 1$, or $d = 2$ and $d' = 0$.

Next, we consider specific laws of random environments. Precisely, we show some sufficient conditions in the cases that Gaussian random environments and random environments generated by Lévy processes. Such environments are concerned in [9] and [11], respectively. By giving some assumptions on laws of environments we are able to discuss clearer sufficient conditions for the recurrence as follows, while the sufficient condition given in Theorem 2.1 is somewhat abstract.

Let us consider a probability measure $\mathcal{Q}$ on $(\mathcal{W}, \mathscr{B}(\mathcal{W}))$ as a Gaussian measure. We assume that $\mathcal{Q}(w(0) = 0) = 1$ and $E^{\mathcal{Q}}[w(x)] = 0$ for $x \in \mathbb{R}^d$. Here $E^{\mathcal{Q}}$ stands for the expectation with respect to $\mathcal{Q}$. Let K be the covariance kernel of $\mathcal{Q}$, that is, $K(x,y) = E^{\mathcal{Q}}[w(x)w(y)]$ for $x, y \in \mathbb{R}^d$. Since $\mathcal{Q}$ is a probability measure on $(\mathcal{W}, \mathscr{B}(\mathcal{W}))$, K is a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$.

Theorem 2.2. *Assume that*

$$\sup_{1 \leq |x| \leq r^2} \int_{|y| \leq r^2} K(x,y)\mathrm{d}y - \inf_{|x| \leq 1} \int_{|y| \leq r^2} K(x,y)\mathrm{d}y < 2 \inf_{1 \leq |x| \leq r^2} \int_{|y| \leq r^2} K(x,y)\mathrm{d}y, \quad (2.4)$$

$$\lim_{n \to \infty} r^{-\alpha n} \sup_{x,y \in D_1} K(r^n x, y) = 0. \quad (2.5)$$

Then, the $(d+1)$-dimensional direct product of diffusion process $(X_w(t), B(t))$ given by the pair of $X_w(t)$ and a one-dimensional Brownian motion $B(t)$ independent of $X_w(t)$ is recurrent for almost all environments.

Note that the condition (2.4) above is stronger than those in Theorem 3.5 in [9], because we need the strength of the recurrence for $X_w(t)$ to show the recurrence of the direct product process.

As a direct consequence of Theorem 2.2, we have the following corollary.

Corollary 2.3. *The two-dimensional direct product of diffusion process $(Y_w(t), B(t))$ given by the pair of the Brox's diffusion process $Y_w(t)$ and a one-dimensional Brownian motion $B(t)$ independent of $Y_w(t)$ is recurrent for almost all environments.*

A similar problem was concerned in Proposition 3.1 in [11], but the absolute value of the Brownian environment was taken in the component of the diffusion process in random environment. The advantage of the method in the present paper is that we are able to show the recurrence without taking the absolute value. Proofs of Theorem 2.2 and Corollary 2.3 are given in Section 5.1.

Finally, we give the result on the case that random environments generated by Lévy processes. Let $\mathcal{W}$ be the space of functions w on $\mathbb{R}$ satisfying the following: $w(0) = 0$, w is right (resp. left) continuous with left (resp. right) limits on $[0, \infty)$ (resp. $(-\infty, 0)$). For $i = 1, 2, \ldots, d$, we set a probability measure $\mathcal{Q}_i$ on $(\mathcal{W}, \mathscr{B}(\mathcal{W}))$ such that $(w(x), x \in [0, \infty), \mathcal{Q}_i)$ and $(w(-x), x \in [0, \infty), \mathcal{Q}_i)$ are independent semi-selfsimilar Lévy processes with an exponent $\alpha_i \in (0, 2]$ (see Definitions 13.4 and 13.12 in [14]). Define the probability measure $\mathcal{Q}$ on $(\mathcal{W}^d, \mathscr{B}(\mathcal{W})^{\otimes d})$ by

$$\mathcal{Q} := \bigotimes_{i=1}^{d} \mathcal{Q}_i.$$

Denote the i th component of $w \in \mathcal{W}^d$ by w^i and write $w_+^i(t) := w^i(t)$ and $w_-^i(t) := w^i(-t)$ for $t \in [0, \infty)$.

Theorem 2.4. *Let $i = 1, 2, \ldots, d$. If $\alpha_i = 2$ or both w_+^i and w_-^i have positive jumps with positive probabilities, then the $(d+1)$-dimensional direct product process $(X_w(t), B(t))$ given by the pair of $X_w(t)$ and a one-dimensional Brownian motion $B(t)$ independent of $X_w(t)$ is recurrent for almost all environments.*

This result can be regarded as an extension of Theorem 1.2 (i) in [11]. Theorem 2.4 implies that diffusion processes in random environments generated by Lévy processes with positive jumps have very strong recurrence, and even taking a direct product of the diffusion process and a one-dimensional Brownian motion, the direct product process is still recurrent. Proof of Theorem 2.4 is given in Section 5.2.

The rest of the present paper is organized as follows. In Section 3, we give criteria for the recurrence of direct products of general symmetric Markov processes including a random time changed version, and prove some lemmas on diffusion processes in non-random environments. In Section 4, we give some sufficient conditions on the random environment for the recurrence of a multi-dimensional direct product process in an ergodic random environment. In Section 5, we consider concrete examples for the result obtained in Section 4 with Gaussian and stable Lévy environments. For notational convenience, we let $a \wedge b := \min\{a, b\}$ for any $a, b \in \mathbb{R}$.

3 Recurrence of products of Dirichlet forms and some lemmas on diffusion processes in non-random environments

3.1 Recurrence of products of Dirichlet forms

In this section, we give some analytic recurrence criteria for direct products of symmetric Dirichlet forms (or, of symmetric Markov processes). The result will be obtained by a simple obsevation for the recurrence of direct products of symmetric Markov processes due to [4] and [13].

For $i = 1, 2, \ldots, N$, let $E^{(i)}$ be a locally compact separable metric space and $m^{(i)}$ be a positive Radon measure on $E^{(i)}$ with full support. Let $(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$ be a symmetric regular Dirichlet form on $L^2(E^{(i)}, m^{(i)})$ possessing $\mathcal{C}^{(i)}$ as its core. It is well-known that $(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$ generates a strongly continuous Markovian semigroup $(T_t^{(i)})_{t \geq 0}$ of symmetric operators on $L^2(E^{(i)}, m^{(i)})$. Let $\mathbf{X}^{(i)} = (\Omega^{(i)}, \mathcal{M}^{(i)}, X^{(i)}(t), P_{x^{(i)}}^{(i)})$ be the $m^{(i)}$-symmetric Hunt process associated to $(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$. We say that $(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$ (or $\mathbf{X}^{(i)}$) is irreducible if any $(T_t^{(i)})_{t \geq 0}$-invariant set B satisfiess $m^{(i)}(B) = 0$ or $m^{(i)}(E^{(i)} \setminus B) = 0$.

Let $\mathbf{X} = (\Omega, \mathcal{M}, X(t), P_x)$ be the process on E defined by the product of $\mathbf{X}^{(i)}$, where

$$\begin{aligned}
&\Omega = \Omega^{(1)} \times \Omega^{(2)} \times \cdots \times \Omega^{(N)}, \quad \mathcal{M} = \mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)} \otimes \cdots \otimes \mathcal{M}^{(N)}, \\
&E = E^{(1)} \times E^{(2)} \times \cdots \times E^{(N)}, \\
&P_x = P^{(1)}_{x^{(1)}} \otimes P^{(2)}_{x^{(2)}} \otimes \cdots P^{(N)}_{x^{(N)}}, \quad x = (x^{(1)}, x^{(2)}, \dots, x^{(N)}) \in E, \\
&X(t,\omega) = \left(X^{(1)}(t,\omega_1), X^{(2)}(t,\omega_2), \dots, X^{(N)}(t,\omega_N) \right), \quad \omega = (\omega_1, \omega_2, \dots, \omega_N) \in \Omega.
\end{aligned}$$

We note that the marginal processes $\{(X^{(i)}(t), t \geq 0), i = 1, 2, \dots, N\}$ are independent under P_x. Let m be the product measure of $\{m^{(i)}, i = 1, 2, \dots, N\}$. Assume that $\mathbf{X}^{(i)}$ is irreducible for any $i = 1, 2, \dots, N$. Then, $\mathbf{X}$ is also to be an m-symmetric irreducible Markov process on E ([4, Proposition 3.1], [13, Theorem 2.6]). Let $(\mathcal{E}, \mathcal{F})$ be the associated Dirichlet form of $\mathbf{X}$ on $L^2(E, m)$. Then $(\mathcal{E}, \mathcal{F})$ possesses the linear span of $\mathcal{C}^{(1)} \otimes \cdots \otimes \mathcal{C}^{(N)} := \{\phi^{(1)} \otimes \cdots \otimes \phi^{(N)} : \phi^{(i)} \in \mathcal{C}^{(i)}, i = 1, 2, \dots, N\}$ as its core, where $(\phi^{(1)} \otimes \cdots \otimes \phi^{(N)})(x) := \phi^{(1)}(x^{(1)})\phi^{(2)}(x^{(2)}) \dots \phi^{(N)}(x^{(N)})$. Thus the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is to be regular and also admits the following expressions: for $u^{(i)} \in \mathcal{F}^{(i)}$ $(i = 1, 2, \dots, N)$, $u := u^{(1)} \otimes \cdots \otimes u^{(N)} \in \mathcal{F}$ and

$$\mathcal{E}\left(u,u\right) = \sum_{i=1}^{N} \mathcal{E}^{(i)}\left(u^{(i)}, u^{(i)}\right) \prod_{j=1, j\neq i}^{N} \left(u^{(j)}, u^{(j)}\right)_{m^{(j)}} \tag{3.1}$$

where $(\cdot,\cdot)_{m^{(i)}}$ denotes the inner product on $L^2(E^{(i)}, m^{(i)})$ ([13, Theorems 1.3 and 1.4]).

Let X be a locally compact separable metric space and μ a positive Radon measure on X with full support. A regular Dirichlet form $(\mathcal{A}, \mathcal{V})$ on $L^2(X, \mu)$ (or the corresponding Markov process $\mathbf{M}$) is non-transient if and only if the following property holds:

(R) There exists a sequence $\{u_n\}_{n\geq 1} \subset \mathcal{V}$ such that $0 \leq u_n \leq 1$ μ-a.e., $\lim_{n\to\infty} u_n = 1$ μ-a.e. and $\mathcal{A}(u_n, u_n) \to 0$ as $n \to \infty$

(see [5, Theorem 1.6.3]). In particular, it is known that $(\mathcal{A}, \mathcal{V})$ (or $\mathbf{M}$) is to be recurrent if it is irreducible and possesses the property (R) ([5, Lemma 1.6.4]).

Now we give some simple criteria for the non-transience of $\mathbf{X}$ through the marginal processes $\{\mathbf{X}^{(i)}, i = 1, 2, \dots, N\}$ in an analytic way.

Proposition 3.1. *Let $\{(\mathcal{E}^{(i)}, \mathcal{F}^{(i)}), i = 1, 2, \cdots, N\}$ and $(\mathcal{E}, \mathcal{F})$ be as above. Assume that there exist sequences $\{u^{(i)}_n\}_{n\geq 1} \subset \mathcal{F}^{(i)}$ such that $0 \leq u^{(i)}_{n^{(i)}(k)} \leq 1$ $m^{(i)}$-a.e., $\lim_{k\to\infty} u^{(i)}_{n^{(i)}(k)} = 1$ $m^{(i)}$-a.e., $u^{(i)}_\infty := 1$ $m^{(i)}$-a.e. $(i = 1, 2, \dots, N)$ and*

$$\lim_{k\to\infty} \frac{1}{k} \prod_{i=1}^{N} \left(u^{(i)}_{n^{(i)}(k)}, u^{(i)}_{n^{(i)}(k)}\right)_{m^{(i)}} = 0 \tag{3.2}$$

for the index

$$n^{(i)}(k) := \inf\left\{ n \in \mathbb{N} : \mathcal{E}^{(i)}\left(u^{(i)}_n, u^{(i)}_n\right) \leq \frac{1}{k}\left(u^{(i)}_n, u^{(i)}_n\right)_{m^{(i)}} \right\}, \quad k \in \mathbb{N}. \tag{3.3}$$

Then the Dirichlet form $(\mathcal{E}, \mathcal{F})$ (or the direct product process $\mathbf{X}$ of $\{\mathbf{X}^{(i)}, i = 1, 2, \dots, N\}$) is non-transient.

Proof. Let

$$u_{n(k)} := u_{n^{(1,\dots,N)}(k)} = u^{(1)}_{n^{(1)}(k)} \otimes \cdots \otimes u^{(N)}_{n^{(N)}(k)}.$$

It is then easy to see that $0 \le u_{n(k)} \le 1$ m-a.e. and $u_{n(k)} \to 1$ as $k \to \infty$ m-a.e. Moreover, by (3.1) and the assumption (3.2), we have

$$\begin{aligned}\mathcal{E}\left(u_{n(k)}, u_{n(k)}\right) &= \sum_{i=1}^{N} \mathcal{E}^{(i)}\left(u^{(i)}_{n^{(i)}(k)}, u^{(i)}_{n^{(i)}(k)}\right) \prod_{j=1, j\neq i}^{N} \left(u^{(j)}_{n^{(j)}(k)}, u^{(j)}_{n^{(j)}(k)}\right)_{m^{(j)}} \\ &\le \frac{N}{k} \prod_{i=1}^{N} \left(u^{(i)}_{n^{(i)}(k)}, u^{(i)}_{n^{(i)}(k)}\right)_{m^{(i)}} \longrightarrow 0 \quad \text{as} \quad k \to \infty.\end{aligned}$$

Hence $(\mathcal{E}, \mathcal{F})$ (or $\mathbf{X}$) is non-transient. □

For a strictly positive continuous function $g^{(i)}$ on $E^{(i)}$, let $\{\mathbf{Y}^{(i)}, i = 1, 2, \ldots, N\}$ be the time changed processes of $\{\mathbf{X}^{(i)}, i = 1, 2, \ldots, N\}$ defined by

$$Y^{(i)}(t) := X^{(i)}\left(\tau^{(i)}_t\right),$$

where $\tau^{(i)}_t$ is the right continuous inverse of the positive continuous additive functional $A^{(i)}_t = \int_0^t g^{(i)}(X^{(i)}(s))\mathrm{d}s$ of $\mathbf{X}^{(i)}$, that is, $\tau^{(i)}_t = \inf\{s > 0 : A^{(i)}_s > t\}$ $(i = 1, 2, \ldots, N)$. Then, since the fine support of $g^{(i)}m^{(i)}$ equals to $E^{(i)}$, $\{\mathbf{Y}^{(i)}, i = 1, 2, \ldots, N\}$ are $g^{(i)}m^{(i)}$-symmetric Humt processes on $E^{(i)}$. Note that the irreducibility and non-transience are stable under time-changed transform (see [5, Theorem 6.2.3] and [15, Theorems 8.2 and 8.5]). Hence $\{\mathbf{Y}^{(i)}, i = 1, 2, \ldots, N\}$ are irreducible and non-transient if $\{\mathbf{X}^{(i)}, i = 1, 2, \ldots, N\}$ are so. Let $\{(\check{\mathcal{E}}^{(i)}, \check{\mathcal{F}}^{(i)}), i = 1, 2, \ldots N\}$ be the associated Dirichlet forms of $\{\mathbf{Y}^{(i)}, i = 1, 2, \ldots, N\}$ on $L^2(E^{(i)}, g^{(i)}m^{(i)})$. Then $(\check{\mathcal{E}}^{(i)}, \check{\mathcal{F}}^{(i)})$ is given by

$$\begin{cases} \check{\mathcal{F}}^{(i)} = \left\{\varphi \in L^2\left(E^{(i)}, g^{(i)}m^{(i)}\right) : \text{there exists } f \in \mathcal{F}^{(i)}_e \text{ such that } \varphi = f \ g^{(i)}m^{(i)}\text{-a.e.}\right\} \\ \check{\mathcal{E}}^{(i)}(\varphi, \varphi) = \mathcal{E}^{(i)}(f, f) \quad \text{for } \varphi \in \check{\mathcal{F}}^{(i)} \text{ and } f \in \mathcal{F}^{(i)}_e \text{ with } \varphi = f \ g^{(i)}m^{(i)}\text{-a.e.,} \end{cases}$$

where $\mathcal{F}^{(i)}_e$ is the extended Dirichlet space of $\mathcal{F}^{(i)}$ (see [5] for the definition). Then we can obtain the following corollary as a consequence of Proposition 3.1.

Corollary 3.2. *For $i = 1, 2 \ldots, N$, let $g^{(i)}$ be a strictly positive continuous function on $E^{(i)}$. Assume that the marginal processes $\{\mathbf{X}^{(i)}, i = 1, 2, \ldots, N\}$ are irreducible and non-transient. If there exist sequences $\{u^{(i)}_n\}_{n\ge1} \subset \mathcal{F}^{(i)}_e \cap L^2(E^{(i)}, g^{(i)}m^{(i)})$ such that $0 \le u^{(i)}_{n^{(i)}(k)} \le 1$ $g^{(i)}m^{(i)}$-a.e., $u^{(i)}_{n^{(i)}(k)} \to 1$ as $k \to \infty$ $g^{(i)}m^{(i)}$-a.e. $(i = 1, 2, \ldots, N)$ and*

$$\lim_{k\to\infty} \frac{1}{k} \prod_{i=1}^{N} \left(u^{(i)}_{n^{(i)}(k)}, u^{(i)}_{n^{(i)}(k)}\right)_{g^{(i)}m^{(i)}} = 0, \tag{3.4}$$

for the index

$$n^{(i)}(k) := \inf\left\{n \in \mathbb{N} : \mathcal{E}^{(i)}\left(u^{(i)}_n, u^{(i)}_n\right) \le \frac{1}{k}\left(u^{(i)}_n, u^{(i)}_n\right)_{g^{(i)}m^{(i)}}\right\}, \tag{3.5}$$

then the direct product process $\mathbf{Y}$ of $\{\mathbf{Y}^{(i)}, i = 1, 2, \ldots, N\}$ is recurrent.

3.2 Some lemmas on diffusion processes in non-random environments

Let w be a locally bounded and Borel measurable function on $\mathbb{R}^d$. Consider the strongly local Dirichlet form $(\mathcal{E}^w, \mathcal{F}^w)$ defined by (2.1). Denote $C_0^\infty(\mathbb{R}^d)$ by the set of all smooth functions with compact support in $\mathbb{R}^d$. Note that the local boundedness of w implies that $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{F}^w$, in particular $(\mathcal{E}^w, \mathcal{F}^w)$ is regular. Let $\mathbf{X}^w = (X_w(t), P^w_x)$ be the diffusion process associated with $(\mathcal{E}^w, \mathcal{F}^w)$. The d-dimensional Brownian motion is associated to $(\mathcal{E}^0, \mathcal{F}^0)$, the Dirichlet form $(\mathcal{E}^w, \mathcal{F}^w)$ with $w \equiv 0$.

For $r \in (1,\infty)$, let $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \varphi(x) \le 1$ on $\mathbb{R}^d$, $\varphi(x) = 1$ on $|x| \le 1$, and $\varphi(x) = 0$ on $|x| \ge r$. For fixed r and φ, define the sequence $\{u_n\} \subset \mathcal{F}^w$ by

$$u_n(x) := \varphi\left(r^{-n}x\right), \quad x \in \mathbb{R}^d,\ n \in \mathbb{N} \cup \{0\}. \tag{3.6}$$

It is clear that $\lim_{n\to\infty} u_n(x) = 1$ for $x \in \mathbb{R}^d$. We let

$$\overline{w}(a,b) := \sup_{a \le |x| \le b} w(x), \quad \underline{w}(a,b) := \inf_{a \le |x| \le b} w(x)$$

for $a, b \ge 0$ as in Theorem 2.1. Then, it is easy to see by the definition of u_n and the assumption on φ that for $n \in \mathbb{N}$,

$$C_1 r^{dn} \exp\left(-\overline{w}(0, r^{n+1})\right) \le \int_{\mathbb{R}^d} |u_n(x)|^2 e^{-w(x)} \mathrm{d}x \le C_1 r^{dn} \exp\left(-\underline{w}(0, r^{n+1})\right), \tag{3.7}$$

where $C_1 := C_1(d, r, \varphi) = \int_{|x| \le r} \varphi(x)^2 \, \mathrm{d}x$. Moreover, since

$$\int_{\mathbb{R}^d} |u_n(x)|^2 e^{-w(x)} \mathrm{d}x \ge r^{dn} \int_{|x| \le r^{-\ell}} e^{-w(r^n x)} \mathrm{d}x$$

for any $\ell \in \mathbb{N}$, it also follows that for $n \in \mathbb{N}$

$$\int_{\mathbb{R}^d} |u_n(x)|^2 e^{-w(x)} \mathrm{d}x \ge V_d \, r^{d(n-\ell)} \exp\left(-\overline{w}(0, r^{n-\ell})\right). \tag{3.8}$$

In particular

$$\int_{\mathbb{R}^d} |u_n(x)|^2 e^{-w(x)} \mathrm{d}x \ge V_d \exp\left(-\overline{w}(0,1)\right), \tag{3.9}$$

where V_d denotes the volume of the unit ball in $\mathbb{R}^d$. On the other hand, the relation

$$\int_{\mathbb{R}^d} |\nabla u_n(x)|^2 e^{-w(x)} \mathrm{d}x = r^{(d-2)n} \int_{1 \le |x| \le r} |\nabla \varphi(x)|^2 e^{-w(r^n x)} \mathrm{d}x$$

implies that for $n \in \mathbb{N}$,

$$C_2 r^{(d-2)n} \exp\left(-\overline{w}(r^n, r^{n+1})\right) \le \int_{\mathbb{R}^d} |\nabla u_n(x)|^2 e^{-w(x)} \mathrm{d}x \le C_2 r^{(d-2)n} \exp\left(-\underline{w}(r^n, r^{n+1})\right), \tag{3.10}$$

where $C_2 := C_2(d, r, \varphi) = \int_{1 \le |x| \le r} |\nabla \varphi(x)|^2 \mathrm{d}x$.

Define a number $n(k) \in \mathbb{N} \cup \{\infty\}$ by

$$n(k) := \inf\left\{ n \in \mathbb{N} : \mathcal{E}^w(u_n, u_n) \le \frac{1}{k} \int_{\mathbb{R}^d} |u_n(x)|^2 e^{-w(x)} \mathrm{d}x \right\}, \quad k \in \mathbb{N}.$$

Lemma 3.3. *Let $k \in \mathbb{N}$ such that $n(k) < \infty$ and $\mathcal{E}^w(u_{n(k)}, u_{n(k)}) \ne 0$. Then we have*

$$\begin{aligned}\int_{\mathbb{R}^d} |u_{n(k)}(x)|^2 e^{-w(x)} \mathrm{d}x < \frac{C_2}{2} k r^{(d-2)(n(k)-1)} \exp\left(-\underline{w}(r^{n(k)-1}, r^{n(k)})\right) \\ \times \left(1 + \frac{(r^d - 1) r^{dn(k)} \exp\left(-\underline{w}(r^{n(k)-1}, r^{n(k)+1})\right)}{\exp\left(-\overline{w}(0,1)\right)}\right).\end{aligned}$$

Proof. We note that the choice of $k \in \mathbb{N}$ and the definition of $n(k)$ imply

$$\int_{\mathbb{R}^d} |u_{n(k)-1}(x)|^2 e^{-w(x)} \mathrm{d}x < \frac{k}{2} \int_{\mathbb{R}^d} |\nabla u_{n(k)-1}(x)|^2 e^{-w(x)} \mathrm{d}x.$$

In view of (3.7), (3.9) and (3.10), we then have

$$\begin{aligned}
&\int_{\mathbb{R}^d} |u_{n(k)}(x)|^2 e^{-w(x)} \mathrm{d}x = \frac{\int_{\mathbb{R}^d} |u_{n(k)}(x)|^2 e^{-w(x)} \mathrm{d}x}{\int_{\mathbb{R}^d} |u_{n(k)-1}(x)|^2 e^{-w(x)} \mathrm{d}x} \int_{\mathbb{R}^d} |u_{n(k)-1}(x)|^2 e^{-w(x)} \mathrm{d}x \\
&\quad < \frac{1}{2} \left(1 + \frac{\int_{\mathbb{R}^d} \left(|u_{n(k)}|^2 - |u_{n(k)-1}|^2 \right) e^{-w(x)} \mathrm{d}x}{\int_{\mathbb{R}^d} |u_{n(k)-1}|^2 e^{-w(x)} \mathrm{d}x} \right) k \int_{\mathbb{R}^d} |\nabla u_{n(k)-1}(x)|^2 e^{-w(x)} \mathrm{d}x \\
&\quad \le \frac{C_2}{2} \left(1 + \frac{\int_{r^{n(k)-1} \le |x| \le r^{n(k)+1}} e^{-w(x)} \mathrm{d}x}{V_d \exp\left(-\overline{w}(0,1)\right)} \right) k r^{(d-2)(n(k)-1)} \exp\left(-\underline{w}(r^{n(k)-1}, r^{n(k)}) \right) \\
&\quad \le \frac{C_2}{2} \left(1 + \frac{(r^d - 1) r^{dn(k)} \exp\left(-\underline{w}(r^{n(k)-1}, r^{n(k)+1}) \right)}{\exp\left(-\overline{w}(0,1)\right)} \right) \\
&\qquad\qquad\qquad\qquad \times k r^{(d-2)(n(k)-1)} \exp\left(-\underline{w}(r^{n(k)-1}, r^{n(k)}) \right). \quad \square
\end{aligned}$$

Lemma 3.4. *Let $k \in \mathbb{N}$ such that $n(k) < \infty$ and $\mathcal{E}^w(u_{n(k)}, u_{n(k)}) \neq 0$. Then we have*

$$\begin{aligned}
&\int_{\mathbb{R}^d} |u_{n(k)}(x)|^2 e^{-w(x)} \mathrm{d}x \\
&\quad < \widetilde{C} k^{d/2} \exp\left(-\underline{w}(0, r^{n(k)+1}) - \frac{d}{2}\, \underline{w}(r^{n(k)-1}, r^{n(k)}) + \frac{d}{2}\, \overline{w}(0, r^{n(k)-2}) \right),
\end{aligned}$$

where $\widetilde{C} := (2V_d)^{-d/2} C_1 C_2^{d/2} r^{d(d+2)/2}$.

Proof. In view of (3.8) for $\ell = 1$ and (3.10), we see that for $n \in \mathbb{N}$

$$\frac{\int_{\mathbb{R}^d} |\nabla u_n(x)|^2 e^{-w(x)} \mathrm{d}x}{\int_{\mathbb{R}^d} |u_n(x)|^2 e^{-w(x)} \mathrm{d}x} \le \frac{C_2}{V_d} r^{-2n+d} \exp\left(-\underline{w}(r^n, r^{n+1}) + \overline{w}(0, r^{n-1}) \right).$$

From this inequality and the definition of $n(k)$, we have

$$\frac{1}{k} < \frac{C_2}{2V_d} r^{-2n(k)+d+2} \exp\left(-\underline{w}(r^{n(k)-1}, r^{n(k)}) + \overline{w}(0, r^{n(k)-2}) \right)$$

for $k \in \mathbb{N}$ such that $n(k) < \infty$ and $\mathcal{E}^w(u_{n(k)}, u_{n(k)}) \neq 0$. Hence

$$r^{dn(k)} = \left(r^{2n(k)} \right)^{d/2} < \left(\frac{r^{d+2} C_2}{2V_d} \right)^{d/2} k^{d/2} \exp\left(-\frac{d}{2}\, \underline{w}(r^{n(k)-1}, r^{n(k)}) + \frac{d}{2}\, \overline{w}(0, r^{n(k)-2}) \right).$$

Applying this inequality to the upper estimate in (3.7), we can obtain the assertion. $\square$

The condition $n(k) < \infty$ for any $k \in \mathbb{N}$ is guaranteed in the case of d-dimensional Brownian motion (or the Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$). Therefore, by virtue of Lemma 3.4, we have the following fact.

Corollary 3.5. *For any $k \in \mathbb{N}$, it holds that*

$$k^{-d/2} \int_{\mathbb{R}^d} |u_{n(k)}(x)|^2 \mathrm{d}x < \widetilde{C},$$

where $\widetilde{C}$ is the constant which appeared in Lemma 3.4.

4 Recurrence of diffusion processes in random environments

Let $\mathcal{W}$ be the space of locally bounded and Borel measurable functions on $\mathbb{R}^d$ with the topology generated by the uniform convergence on compact sets. Let $\mathscr{B}(\mathcal{W})$ be the Borel σ-field of $\mathcal{W}$ and $\mathcal{Q}$ be a probability measure on $(\mathcal{W}, \mathscr{B}(\mathcal{W}))$. We call an element $w \in \mathcal{W}$ an environment and assume that $\mathcal{Q}(w(0) = 0) = 1$. For given $w \in \mathcal{W}$, we define the Dirichlet form $(\mathcal{E}^w, \mathcal{F}^w)$ by (2.1) and let $\mathbf{X}^w = (X_w(t), P_x^w)$ be the associated diffusion process of $(\mathcal{E}^w, \mathcal{F}^w)$.

For $r > 1$ and $\alpha > 0$, let T be a mapping from $\mathcal{W}$ to $\mathcal{W}$ defined by $Tw(x) = r^{-\alpha} w(rx)$ for $x \in \mathbb{R}^d$. We assume the α-semi-selfsimilarity of $(\mathcal{W}, \mathscr{B}(\mathcal{W}), \mathcal{Q})$ by (2.2). We say that a mapping T is weakly mixing if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \mathcal{Q}\left(T^k A_1 \cap A_2\right) - \mathcal{Q}(A_1)\mathcal{Q}(A_2) \right| = 0$$

for $A_1, A_2 \in \mathscr{B}(\mathcal{W})$.

As in the proof of Theorem 2.2 in [9], we can prove the following lemma.

Lemma 4.1. *Assume that T is weakly mixing. If $A \in \mathscr{B}(\mathcal{W})$ satisfies $\mathcal{Q}(A) > 0$, then, for $\mathcal{Q}$-almost every $w \in \mathcal{W}$, $\{n \in \mathbb{N} : T^n w \in A\}$ is an infinite set.*

Let φ and $\{u_n\}$ be the functions defined as in Section 3. For given $\mathbb{N}$-valued increasing sequence $\{n_\ell\}_{\ell \geq 1}$, define $\ell(k) \in \mathbb{N} \cup \{\infty\}$ by

$$\ell(k) := \ell(k, w) = \inf\left\{\ell \in \mathbb{N} : \mathcal{E}^w(u_{n_\ell}, u_{n_\ell}) \leq \frac{1}{k} \int_{\mathbb{R}^d} |u_{n_\ell}(x)|^2 e^{-w(x)} \mathrm{d}x \right\}, \quad k \in \mathbb{N}.$$

By using Lemma 4.1 above, we have the estimate as follows.

Proposition 4.2. *Assume that T is a weakly mixing and*

$$\mathcal{Q}\left(\underline{w}(1, r^2) > a,\ \overline{w}(1, r^2) - \underline{w}(0,1) < b\right) > 0 \tag{4.1}$$

for $0 < a < b < \infty$. Then, for $\mathcal{Q}$-almost every $w \in \mathcal{W}$, there exists an $\mathbb{N}$-valued increasing sequence $\{n_\ell^w\}_{\ell \geq 1}$ such that

$$\frac{1}{k^\gamma} \int_{\mathbb{R}^d} \left| u_{n^w_{\ell(k)}}(x) \right|^2 e^{-w(x)} \mathrm{d}x = o\left(\exp\left((b(1-\gamma) - a + \varepsilon) r^{\alpha(n^w_{\ell(k)} - 1)}\right)\right) \quad (k \to \infty)$$

for any $\varepsilon > 0$ and $\gamma \in [0,1]$.

Remark 4.3. The assumption (4.1) implies that the probability that w looks like in Figure 1 is positive.

Proof. Set

$$A := \left\{ w \in \mathcal{W} : \underline{w}(r^{-1}, r) > a r^{-\alpha},\ \overline{w}(1, r) - \underline{w}(0,1) < b r^{-\alpha} \right\}.$$

Then we see

$$\begin{aligned} \mathcal{Q}(A) &= \mathcal{Q}\left(\underline{w}(1, r^2) > a,\ \overline{w}(r, r^2) - \underline{w}(0, r) < b\right) \\ &\geq \mathcal{Q}\left(\underline{w}(1, r^2) > a,\ \overline{w}(1, r^2) - \underline{w}(0, 1) < b\right) > 0 \end{aligned}$$

in view of (2.2). In view of Lemma 4.1, there exists $\mathcal{N} \in \mathscr{B}(\mathcal{W})$ such that $\mathcal{Q}(\mathcal{N}) = 0$, and for $w \in \mathcal{W} \setminus \mathcal{N}$, $w(0) = 0$ and $\{n \in \mathbb{N} : T^n w \in A\}$ is an infinite set. For $w \in \mathcal{W} \setminus \mathcal{N}$, let $\{n_\ell^w : \ell \in \mathbb{N}\}$ be a strictly increasing sequence in $\{n \in \mathbb{N} : T^n w \in A\}$. Then we have

$$\begin{cases} \underline{w}\left(r^{n_\ell^w - 1}, r^{n_\ell^w}\right) \geq \underline{w}\left(r^{n_\ell^w - 1}, r^{n_\ell^w + 1}\right) > a r^{\alpha(n_\ell^w - 1)} \\ \overline{w}\left(r^{n_\ell^w}, r^{n_\ell^w + 1}\right) - \underline{w}\left(0, r^{n_\ell^w}\right) < b r^{\alpha(n_\ell^w - 1)} \\ \underline{w}(0, r^{n_\ell}) = \underline{w}(0, r^{n_\ell + 1}). \end{cases} \tag{4.2}$$

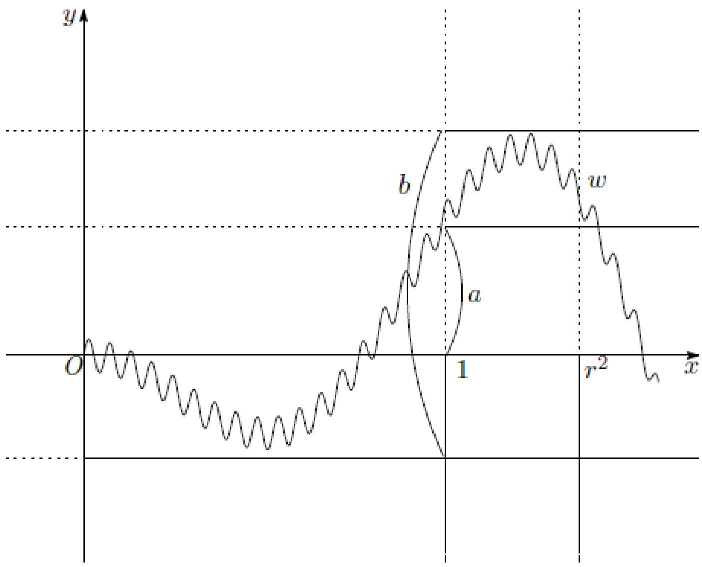

Figure 1: For Remark 4.3

For the last equality above, we used the fact that $w(0) = 0$ implies

$$\begin{aligned}\underline{w}(0, r^{n_\ell}) &= \min\left\{\underline{w}(0, r^{n_\ell}), ar^{\alpha(n_\ell - 1)}\right\} \\ &\le \min\left\{\underline{w}(0, r^{n_\ell}), \underline{w}(r^{n_\ell - 1}, r^{n_\ell + 1})\right\} \\ &= \underline{w}(0, r^{n_\ell + 1}).\end{aligned}$$

On the other hand, by applying the first inequality in (4.2) to Lemma 3.3, it holds that

$$\begin{aligned}\int_{\mathbb{R}^d} \left|u_{n^w_{\ell(k)}}(x)\right|^2 e^{-w(x)}\mathrm{d}x &\le C_2\, kr^{(d-2)(n^w_{\ell(k)}-1)} \exp\left(-ar^{\alpha(n^w_{\ell(k)}-1)}\right) \\ &\qquad \times \left(1 + \frac{(r^d - 1)r^{dn^w_{\ell(k)}} \exp\left(-ar^{\alpha(n^w_{\ell(k)}-1)}\right)}{\exp\left(-\overline{w}(0,1)\right)}\right)\end{aligned}$$

for $k \in \mathbb{N}$. From this, one can get for sufficiently large $k \in \mathbb{N}$ that

$$\int_{\mathbb{R}^d} \left|u_{n^w_{\ell(k)}}(x)\right|^2 e^{-w(x)}\mathrm{d}x \le Ckr^{(d-2)n^w_{\ell(k)}} \exp\left(-ar^{\alpha(n^w_{\ell(k)}-1)} + \overline{w}(0,1)\right), \tag{4.3}$$

where C is a constant depending on d, r, α, φ and a. Moreover, since

$$\mathcal{E}^w\left(u_{n^w_{\ell(k)}}, u_{n^w_{\ell(k)}}\right) \le \frac{1}{k}\int_{\mathbb{R}^d} \left|u_{n^w_{\ell(k)}}(x)\right|^2 e^{-w(x)}\mathrm{d}x, \quad k \in \mathbb{N},$$

we have by (3.7) and (3.10) that for $k \in \mathbb{N}$

$$C_2 r^{-2n^w_{\ell(k)}} \exp\left(-\overline{w}\left(r^{n^w_{\ell(k)}}, r^{n^w_{n_{\ell(x)}}+1}\right)\right) \le \frac{2}{k} C_1 \exp\left(-\underline{w}\left(0, r^{n^w_{n_{\ell(x)}}+1}\right)\right). \tag{4.4}$$

Then, by virtue of the second and third relations in (4.2), and (4.4)

$$\begin{aligned} r^{-2n^w_{\ell(k)}} \exp\left(-br^{\alpha(n^w_{\ell(k)}-1)}\right) &\le r^{-2n^w_{\ell(k)}} \exp\left(\underline{w}\left(0, r^{n^w_{n_{\ell(x)}}}\right) - \overline{w}\left(r^{n^w_{\ell(k)}}, r^{n^w_{n_{\ell(x)}}+1}\right)\right) \\ &= r^{-2n^w_{\ell(k)}} \exp\left(\underline{w}\left(0, r^{n^w_{n_{\ell(x)}}+1}\right) - \overline{w}\left(r^{n^w_{\ell(k)}}, r^{n^w_{n_{\ell(x)}}+1}\right)\right) \\ &\le \frac{2C_1}{kC_2}, \end{aligned}$$

hence, for $\gamma \in [0,1]$ and $k \in \mathbb{N}$

$$r^{-2(1-\gamma)n^w_{\ell(k)}} \exp\left(-b(1-\gamma)r^{\alpha(n^w_{\ell(k)}-1)}\right) \le \left(\frac{2C_1}{kC_2}\right)^{1-\gamma}.$$

From this inequality, we see that the right-hand side of (4.3) is dominated by

$$C\left(\frac{2C_1}{C_2}\right)^{1-\gamma} k^\gamma r^{(d-2\gamma)n^w_{\ell(k)}} \exp\left((b(1-\gamma)-a)r^{\alpha(n^w_{\ell(k)}-1)} + \overline{w}(0,1)\right).$$

The proof is completed. □

Now we prove Theorem 2.1 by applying Propositions 3.1 and 4.2.

Proof of Theorem 2.1. We note that

$$\begin{aligned} &\mathcal{Q}\left(\overline{w}(1,r^2) - \underline{w}(0,1) < 2\underline{w}(1,r^2)\right) \\ &= \mathcal{Q}\left(\bigcup_{\widetilde{a}\in(0,\infty)\cap\mathbb{Q}} \left\{\underline{w}(1,r^2) > \widetilde{a},\ \overline{w}(1,r^2) - \underline{w}(0,1) < 2\widetilde{a}\right\}\right) \\ &\le \sum_{\widetilde{a}\in(0,\infty)\cap\mathbb{Q}} \mathcal{Q}\left(\underline{w}(1,r^2) > \widetilde{a},\ \overline{w}(1,r^2) - \underline{w}(0,1) < 2\widetilde{a}\right). \end{aligned}$$

In view of this fact, the assumption (2.3) implies that there exists $a > 0$ such that

$$\mathcal{Q}\left(\underline{w}(1,r^2) > a,\ \overline{w}(1,r^2) - \underline{w}(0,1) < 2a - 4\varepsilon\right) > 0 \tag{4.5}$$

for a sufficiently small $\varepsilon > 0$. Thus, by applying Proposition 4.2 with $\gamma = 1/2$ and $b = 2a - \varepsilon$, we see that

$$k^{-1/2} \int_{\mathbb{R}^d} \left|u_{n^w_{\ell(k)}}(x)\right|^2 e^{-w(x)} \mathrm{d}x \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{4.6}$$

Then, by Corollary 3.5 and (4.6),

$$\begin{aligned} &\frac{2}{k}\left(\int_{\mathbb{R}^d} \left|u_{n^w_{\ell(k)}}(x)\right|^2 e^{-w(x)}\mathrm{d}x\right)\left(\int_{\mathbb{R}^1} \left|u_{n(k)}(x)\right|^2 \mathrm{d}x\right) \\ &\le \frac{2\widetilde{C}}{k^{1/2}} \int_{\mathbb{R}^d} \left|u_{n^w_{\ell(k)}}(x)\right|^2 e^{-w(x)}\mathrm{d}x \longrightarrow 0 \end{aligned}$$

as $k \to \infty$. Moreover, (4.5) implies $\mathcal{Q}(\underline{w}(1,r) > a) > 0$ and therefore, $X_w(t)$ is recurrent for almost all environments in view of [9, Theorem 2.2]. Hence, by virtue of Proposition 3.1, we can conclude that $(X_w(t), B(t))$ is recurrent for almost all environments. □

5 Applications to explicit random environments

In this section, as applications of a random environment appeared in Section 4, we consider the recurrence of the product of diffusion processes in semi-selfsimilar Gaussian and Lévy random environments, and show Theorems 2.2 and 2.4, and Corollary 2.3.

5.1 Gaussian random environments

Let $\mathcal{W}$ be the space of locally bounded Borel measurable functions w on $\mathbb{R}^d$, with the topology generated by the uniform convergence on compact sets. We define a probability measure $\mathcal{Q}$ on $(\mathcal{W}, \mathscr{B}(\mathcal{W}))$ by a Gaussian measure, that is, $(w(x_1), w(x_2), \dots w(x_n))$ has an n-dimensional Gaussian distribution under $\mathcal{Q}$, where $x_1, x_2, \dots, x_n \in \mathbb{R}^d$ for $n \in \mathbb{N}$. We assume that $\mathcal{Q}(w(0) = 0) = 1$ and $E^{\mathcal{Q}}[w(x)] = 0$ for $x \in \mathbb{R}^d$. Here $E^{\mathcal{Q}}$ stands for the expectation with respect to $\mathcal{Q}$. Let K be the covariance kernel of $\mathcal{Q}$, that is, $K(x,y) = E^{\mathcal{Q}}[w(x)w(y)]$ for $x, y \in \mathbb{R}^d$. Since $\mathcal{Q}$ is a probability measure on $(\mathcal{W}, \mathscr{B}(\mathcal{W}))$, K is a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$.

It is well-known that the law of a Gaussian measure is determined by the mean and the covariance kernel. First, we are going to consider a sufficient condition for (2.3) in Theorem 2.1.

Lemma 5.1. *Assume that for* $r > 1$

$$\sup_{1 \le |x| \le r^2} \int_{|y| \le r^2} K(x,y)\mathrm{d}y - \inf_{|x| \le 1} \int_{|y| \le r^2} K(x,y)\mathrm{d}y < 2 \inf_{1 \le |x| \le r^2} \int_{|y| \le r^2} K(x,y)\mathrm{d}y.$$

Then the assumption (2.3) *holds.*

Proof. The proof is similar to that of [9, Lemma 3.1]. So, we omit the detail and see only the sketch of the proof. By the general theory of the Gaussian system, for $f \in L^2(\mathbb{R}^d, \mathrm{d}x)$ with compact support, $\int_{\mathbb{R}^d} K(\cdot, y) f(y)\mathrm{d}y$ is in the Cameron-Martin space $\mathcal{H}$ associated to $\mathcal{Q}$ on $(\mathcal{W}, \mathscr{B}(\mathcal{W}))$. In particular,

$$\int_{|y| \le r^2} K(\cdot, y)\, \mathrm{d}y \in \mathcal{H}. \tag{5.1}$$

On the other hand, since $\mathcal{H}$ is dense in the support of $\mathcal{Q}$, for any $g \in \mathcal{H}$

$$\mathcal{Q}\left(\sup_{x \in \mathbb{R}^d} |w(x) - g(x)| < \varepsilon \right) > 0 \quad \text{for any} \quad \varepsilon > 0.$$

This inequality and (5.1) imply

$$\mathcal{Q}\left(\sup_{x \in \mathbb{R}^d} \left| w(x) - \int_{|y| \le r^2} K(x,y)\, \mathrm{d}y \right| < \varepsilon \right) > 0. \tag{5.2}$$

Let

$$\delta := 2 \inf_{1 \le |x| \le r^2} \int_{|y| \le r^2} K(x,y)\mathrm{d}y - \left(\sup_{1 \le |x| \le r^2} \int_{|y| \le r^2} K(x,y)\mathrm{d}y - \inf_{|x| \le 1} \int_{|y| \le r^2} K(x,y)\mathrm{d}y \right)$$

and choose $\varepsilon \in (0, \delta/4)$. Then, if

$$\sup_{x \in \mathbb{R}^d} \left| w(x) - \int_{|y| \le r^2} K(x,y)\mathrm{d}y \right| < \varepsilon,$$

we have

$$\sup_{1 \le |x| \le r^2} w(x) - \inf_{|x| \le 1} w(x) < 2 \inf_{1 \le |x| \le r^2} w(x) + 4\varepsilon - \delta < 2 \inf_{1 \le |x| \le r^2} w(x).$$

Therefore, by (5.2) we have the assertion. □

Next we consider a sufficient condition for the mixing condition. In the sequel, let $r > 1$, $\alpha > 0$ and T be a mapping from $\mathcal{W}$ to $\mathcal{W}$ defined by $Tw(x) = r^{-\alpha}w(rx)$ for $x \in \mathbb{R}^d$ satisfying (2.2). We say that T is strongly mixing if

$$\lim_{n\to\infty} \mathcal{Q}(T^n A_1 \cap A_2) = \mathcal{Q}(A_1)\mathcal{Q}(A_2), \quad A_1, A_2 \in \mathscr{B}(\mathcal{W}).$$

It is well known that every strongly mixing transformation is weakly mixing, hence is ergodic (see [20]). Set $D_1 := \{x \in \mathbb{R}^d; 1 < |x| < r\}$. Then, we see the following.

Lemma 5.2 (Lemma 3.3 in [9]). *If*

$$\lim_{n\to\infty} r^{-\alpha n} \sup_{x,y\in D_1} K(r^n x, y) = 0,$$

then T is strongly mixing.

Now we prove Theorem 2.2 and Corollary 2.3.

Proof of Theorem 2.2. For $w \in \mathcal{W}$, let $\mathbf{X}^w = (X_w(t), P_x^w)$ be the d-dimensional diffusion process associated to $(\mathcal{E}^w, \mathcal{F}^w)$ defined by (2.1). Note that the condition (2.3) implies $\mathcal{Q}(\underline{w}(1, r) > a_0) > 0$ for some $a_0 > 0$. Thus, if T on $\mathcal{W}$ is weakly mixing, $\mathbf{X}^w$ is recurrent for almost all environment ([9, Theorem 2.2]). From this fact with Theorem 2.1, Lemmas 5.1 and 5.2, we have Theorem 2.2. □

Proof of Corollary 2.3. Let w be the two-sided Brownian motion on $\mathbb{R}$ under $\mathcal{Q}$. In this case, w and $r^{-1/2}w(r\,\cdot)$ have the same law. Furthermore, the covariance kernel $K(x, y)$ is given by

$$K(x, y) = (|x| \wedge |y|)\mathbf{1}_{(0,\infty)}(xy), \quad x, y \in \mathbb{R}.$$

Choose r satisfying $1 < r < \sqrt{2 + \sqrt{2}}$. Then, since

$$\sup_{1\le|x|\le r^2} \int_{|y|\le r^2} K(x, y)\mathrm{d}y = \frac{1}{2}r^4, \quad \inf_{|x|\le 1} \int_{|y|\le r^2} K(x, y)\mathrm{d}y = 0,$$
$$\inf_{1\le|x|\le r^2} \int_{|y|\le r^2} K(x, y)\mathrm{d}y = r^2 - \frac{1}{2},$$

(2.4) is satisfied. Moreover, it is easy to see that (2.5) is satisfied. On the other hand, we note that the Brox's diffusion process $Y_w(t)$ is a time changed process of the one-dimensional diffusion process $X_w(t)$ by the positive continuous additive functional $\int_0^\cdot e^{-w(X_w(s))}\mathrm{d}s$. Then the Dirichlet form $(\check{\mathcal{E}}^w, \check{\mathcal{F}}^w)$ corresponding to $Y_w(t)$ is given by (2.1) replacing the underlying measure $e^{-w(x)}\mathrm{d}x$ with $e^{-2w(x)}\mathrm{d}x$. It is well-known that $X_w(t)$ is recurrent (see [19]). Hence, by Corollary 3.2 and Theorem 2.2, we obtain the assertion. □

5.2 Products of environments generated by Lévy processes

Let $\mathcal{W}$ be the space of functions w on $\mathbb{R}$ satisfying the following: $w(0) = 0$, w is right (resp. left) continuous with left (resp. right) limits on $[0, \infty)$ (resp. $(-\infty, 0)$). For $i = 1, 2, \ldots, d$ we set a probability measure $\mathcal{Q}_i$ on $(\mathcal{W}, \mathscr{B}(\mathcal{W}))$ such that $(w(x), x \in [0, \infty), \mathcal{Q}_i)$ and $(w(-x), x \in [0, \infty), \mathcal{Q}_i)$ are independent semi-selfsimilar Lévy processes with an exponent $\alpha_i \in (0, 2]$ (see Definitions 13.4 and 13.12 in [14]). Define the probability measure $\mathcal{Q}$ on $(\mathcal{W}^d, \mathscr{B}(\mathcal{W})^{\otimes d})$ by

$$\mathcal{Q} := \bigotimes_{i=1}^{d} \mathcal{Q}_i.$$

Denote the ith component of $w \in \mathcal{W}^d$ by w^i and denote $w^i_+(t) := w^i(t)$ and $w^i_-(t) := w^i(-t)$ for $t \in [0, \infty)$.

By a similar argument to the proof of Proposition 2.1 in [10], we have the following.

Lemma 5.3. *Let $i = 1, 2, \dots, d$. If $\alpha_i = 2$ or both w_+^i and w_-^i have positive jumps with positive probabilities, then there exists a positive constant M such that for any $a > 0$*

$$\mathcal{Q}\left(\left\{\underline{w}^i(0,1) > -M\right\} \cap \left\{\left(w^i(-1) \wedge w^i(1)\right) > a\right\}\right) > 0.$$

Proof. First we prove that there exists $M > 0$ such that for any $a > 0$

$$\mathcal{Q}\left(\left\{\inf_{t\in[0,1]} w_+^i(t) > -M\right\} \cap \left\{w_+^i(1) > a\right\}\right) > 0, \quad i = 1, 2, \dots, d. \tag{5.3}$$

If $\alpha_i = 2$, then w_+^i is a Brownian motion and hence (5.3) holds. Assume that $\alpha_i \in (0, 2)$. In this case, we note that the Lévy measure ν of w_+^i is not trivial and its Gaussian part is to be 0. Since w_+^i has positive jumps with a positive probability, we can choose $\varepsilon \in (0, 1]$ such that $\nu((\varepsilon, \infty)) > 0$. For $i = 1, 2, \dots, d$, let v_1^i, v_2^i and v_3^i be independent Lévy processes associated to the triplets $(0, \nu(\cdot \cap (\varepsilon, \infty)), 0)$, $(0, \nu(\cdot \cap [-1, \varepsilon]), 0)$ and $(0, \nu(\cdot \cap (-\infty, -1)), 0)$, respectively. Then, the equality in law

$$\left(w_+^i(t), t \in [0, \infty)\right) \stackrel{\text{law}}{=} \left(v_1^i(t) + v_2^i(t) + v_3^i(t) + ct, t \in [0, \infty)\right) \tag{5.4}$$

holds for a constant $c \in \mathbb{R}$. Note that v_2^i is right-continuous with left limits almost surely. So there exists $M > 0$ such that

$$\mathcal{Q}\left(\sup_{t\in[0,1]} \left|v_2^i(t)\right| < M - |c|\right) > 0. \tag{5.5}$$

Also, since $\nu((\varepsilon, \infty)) > 0$, $v_1^i \geq 0$ almost surely and we have

$$\mathcal{Q}\left(v_1^i(1) > a + M + |c|\right) > 0 \tag{5.6}$$

for any $a > 0$. On the other hand, the definition of the Lévy measure implies $\nu((-\infty, -1)) < \infty$, and hence

$$\mathcal{Q}\left(v_3^i(t) = 0 \text{ for } t \in [0, 1]\right) > 0. \tag{5.7}$$

From (5.4), (5.5), (5.7), (5.6) and the independence of v_1^i, v_2^i and v_3^i, we then obtain

$$\begin{aligned}
&\mathcal{Q}\left(\left\{\inf_{t\in[0,1]} w_+^i(t) > -M\right\} \cap \left\{w_+^i(1) > a\right\}\right) \\
&\quad \geq \mathcal{Q}\left(\left\{v_1^i(1) > a + M + |c|\right\} \cap \left\{\inf_{t\in[0,1]} v_2^i(t) > -M + |c|\right\} \cap \left\{v_3^i(t) = 0 \text{ for } t \in [0,1]\right\}\right) \\
&\quad = \mathcal{Q}\left(v_1^i(1) > a + M + |c|\right) \mathcal{Q}\left(\inf_{t\in[0,1]} v_2^i(t) > -M + |c|\right) \mathcal{Q}\left(v_3^i(t) = 0 \text{ for } t \in [0,1]\right) > 0.
\end{aligned}$$

Thus, we obtain (5.3). Similarly to above, we also have

$$\mathcal{Q}\left(\left\{\inf_{t\in[0,1]} w_-^i(t) > -M\right\} \cap \left\{w_-^i(1) > a\right\}\right) > 0. \tag{5.8}$$

Now, on account of (5.3) and (5.8)

$$\begin{aligned}
&\mathcal{Q}\left(\left\{\underline{w}^i(0,1) > -M\right\} \cap \left\{w^i(-1) \wedge w^i(1) > a\right\}\right) \\
&\quad = \mathcal{Q}\left(\left\{\inf_{t\in[0,1]} w_+^i(x) > -M\right\} \cap \left\{w_+^i(1) > a\right\}\right) \mathcal{Q}\left(\left\{\inf_{t\in[0,1]} w_-^i(t) > -M\right\} \cap \left\{w_-^i(1) > a\right\}\right) \\
&\quad > 0
\end{aligned}$$

for any $a > 0$ and $i = 1, 2, \dots, d$. $\square$

From this lemma, we have the following.

Lemma 5.4. *Let $i = 1, 2, \ldots, d$. If $\alpha_i = 2$ or both w^i_+ and w^i_- have positive jumps with positive probabilities, then there exists $\widehat{a} > 0$ such that for any $\varepsilon > 0$*

$$\mathcal{Q}\left(\underline{w}^i(1, r^2) > \widehat{a},\ \overline{w}^i(1, r^2) - \underline{w}^i(0, 1) < \widehat{a}(1+\varepsilon)\right) > 0.$$

Proof. It is sufficient to show that there exists $\widehat{a} > 0$ such that for any $\varepsilon > 0$ and $i = 1, 2, \ldots, d$,

$$\mathcal{Q}\left(\inf_{t\in[1,r^2]} w^i_+(t) > \widehat{a},\ \sup_{t\in[1,r^2]} w^i_+(x) - \inf_{t\in[0,1]} w^i_+(x) < \widehat{a}(1+\varepsilon)\right) > 0, \tag{5.9}$$

because the proof of (5.9) for $(w^i_-(x), x \in (-\infty, 0))$ is almost same. Since w^i_+ is a Lévy process, $(w^i_+(t) - w^i_+(1), t \in [1, \infty))$ and $(w^i_+(t), t \in [0, 1])$ are independent. Then, for any $\widehat{a} > 0$ and $i = 1, 2, \ldots, d$, we have

$$\begin{aligned}
&\mathcal{Q}\left(\inf_{t\in[1,r^2]} w^i_+(t) > \widehat{a},\ \sup_{t\in[1,r^2]} w^i_+(t) - \inf_{t\in[0,1]} w^i_+(t) < \widehat{a}(1+\varepsilon)\right) \\
&= \mathcal{Q}\left(\widehat{a} - w^i_+(1) < w^i_+(t) - w^i_+(1) < \widehat{a}(1+\varepsilon) - w^i_+(1) + \inf_{t\in[0,1]} w^i_+(t) \quad \text{for } t \in [1, r^2]\right) \\
&\geq \mathcal{Q}\left(\left\{-\frac{\varepsilon\widehat{a}}{4} < w^i_+(t) - w^i_+(1) < \frac{\varepsilon\widehat{a}}{4} \quad \text{for } t \in [1, r^2]\right\}\right. \\
&\qquad\qquad \left.\cap \left\{\inf_{t\in[0,1]} w^i_+(t) > -\frac{\varepsilon\widehat{a}}{4}\right\} \cap \left\{\frac{(4+\varepsilon)\widehat{a}}{4} < w^i_+(1) < \frac{(2+\varepsilon)\widehat{a}}{2}\right\}\right) \\
&= \mathcal{Q}\left(-\frac{\varepsilon\widehat{a}}{4} < w^i_+(t) - w^i_+(1) < \frac{\varepsilon\widehat{a}}{4} \quad \text{for } t \in [1, r^2]\right) \\
&\qquad \times \mathcal{Q}\left(\left\{\inf_{t\in[0,1]} w^i_+(t) > -\frac{\varepsilon\widehat{a}}{4}\right\} \cap \left\{\frac{(4+\varepsilon)\widehat{a}}{4} < w^i_+(1) < \frac{(2+\varepsilon)\widehat{a}}{2}\right\}\right) \\
&\geq \mathcal{Q}\left(\sup_{t\in[1,r^2]} |w^i_+(t) - w^i_+(1)| < \frac{\varepsilon\widehat{a}}{4}\right) \\
&\qquad \times \mathcal{Q}\left(\left\{\inf_{t\in[0,1]} w^i_+(t) > -\frac{\varepsilon\widehat{a}}{4}\right\} \cap \left\{\frac{(4+\varepsilon)\widehat{a}}{4} < w^i_+(1) < \frac{(2+\varepsilon)\widehat{a}}{2}\right\}\right).
\end{aligned}$$

Let M be the constant appeared in Lemma 5.3 and let $\widetilde{M} > 0$ be a constant satisfying

$$\mathcal{Q}\left(\sup_{t\in[1,r^2]} \left|w^i_+(t) - w^i_+(1)\right| < \widetilde{M}\right) > 0.$$

Then, by taking $\widehat{a} \in [4(M \vee \widetilde{M})/\varepsilon, \infty)$, we have

$$\mathcal{Q}\left(\sup_{t\in[1,r^2]} \left|w^i_+(t) - w^i_+(1)\right| < \frac{\varepsilon\widehat{a}}{4}\right) > 0. \tag{5.10}$$

On the other hand, by noting the fact that (5.3) holds for any $a > 0$, we can take $\widehat{a} > 0$ such that

$$\begin{aligned}
&\mathcal{Q}\left(\left\{\inf_{t\in[0,1]} w^i_+(t) > -\frac{\varepsilon\widehat{a}}{4}\right\} \cap \left\{\frac{(4+\varepsilon)\widehat{a}}{4} < w^i_+(1) < \frac{(2+\varepsilon)\widehat{a}}{2}\right\}\right) \\
&\qquad \geq \mathcal{Q}\left(\left\{\inf_{t\in[0,1]} w^i_+(t) > -M\right\} \cap \left\{\frac{(4+\varepsilon)\widehat{a}}{4} < w^i_+(1) < \frac{(2+\varepsilon)\widehat{a}}{2}\right\}\right) > 0.
\end{aligned} \tag{5.11}$$

Therefore, we can conclude the assertion for $\widehat{a} > 0$ satisfying both (5.10) and (5.11). □

Now we prove Theorem 2.4 by applying Propositions 3.1 and 4.2, and Lemma 5.4.

Proof of Theorem 2.4. Let define a random function w by

$$w(x) := \sum_{i=1}^{d} w^i \left(x^{(i)}\right), \quad x = (x^{(1)}, x^{(2)}, \dots, x^{(d)}) \in \mathbb{R}^d.$$

For this w, let $\mathbf{X}^w = (X_w(t), P_x^w)$ be the diffusion process associated to the Dirichlet form $(\mathcal{E}^w, \mathcal{F}^w)$ given by (2.1). Then, it is the d-dimensional direct products of diffusion processes in products of random environments generated by one-dimensional semi-selfsimilar Lévy processes $\{(w^i(x), x \in \mathbb{R}), i = 1, 2, \dots, d\}$, that is,

$$X_w(t) = \left(X^1_{w^1}(t), X^2_{w^2}(t), \dots, X^d_{w^d}(t)\right).$$

We remark that the components of $X_t^{(w)}$ are independent for each environment w, because

$$e^{-w(x)}\mathrm{d}x = \prod_{i=1}^{d} e^{-w^i(x^{(i)})}\mathrm{d}x^{(i)}$$

for $x = (x^{(1)}, x^{(2)}, \dots, x^{(d)}) \in \mathbb{R}^d$. In view of Propositions 3.1 and 4.2, and Lemma 5.4 we obtain the assertion. □

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