# Large deviations of radial SLE$_\infty$

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### Abstract

We derive the large deviation principle for radial Schramm-Loewner evolution (SLE) on the unit disk with parameter $\kappa \to \infty$. Restricting to the time interval $[0,1]$, the good rate function is finite only on a certain family of Loewner chains driven by absolutely continuous probability measures $\{\phi_t^2(\zeta)\,d\zeta\}_{t\in[0,1]}$ on the unit circle and equals $\int_0^1 \int_{S^1} |\phi_t'|^2/2\,d\zeta\,dt$. Our proof relies on the large deviation principle for the long-time average of the Brownian occupation measure by Donsker and Varadhan.

## 1 Introduction

The Schramm-Loewner evolution is a one parameter family of random fractal curves (denoted as $\mathrm{SLE}_\kappa$ with parameter $\kappa > 0$). It was introduced by Oded Schramm [16] and has been a central topic in the two dimensional random conformal geometry. A version of such curves starting from a fixed boundary point to a fixed interior point on some two-dimensional simply connected domain $D$ are called *radial* SLEs. Let us recall briefly the definition. The radial $\mathrm{SLE}_\kappa$ on the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ targeted at $0$ is the random curve associated to the radial Loewner chain, whose *driving function* $t \mapsto \zeta_t$ is given by a Brownian motion on the unit circle $S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ with variance $\kappa$. That is,

$$\zeta_t := B_t^\kappa := e^{iW_{\kappa t}}, \tag{1.1}$$

where $W_t$ is a standard linear Brownian motion. More precisely, we consider the Loewner ODE for all $z \in \mathbb{D}$

$$\partial_t g_t(z) = -g_t(z)\frac{g_t(z) + \zeta_t}{g_t(z) - \zeta_t}, \tag{1.2}$$

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or equivalently, the Loewner PDE satisfied by $f_t := g_t^{-1}$

$$\partial_t f_t(z) = z f_t'(z) \frac{z + \zeta_t}{z - \zeta_t}, \quad z \in \mathbb{D} \tag{1.3}$$

with the initial condition $f_0(z) = g_0(z) = z$. For a given $t > 0$, $f_t$ is a conformal map from $\mathbb{D}$ onto a simply connected domain $D_t \subset \mathbb{D}$ (and $s \mapsto g_s(z)$ is a well-defined solution of (1.2) up to time $t$ if and only if $z \in D_t$) such that $f_t(0) = 0$ and $f_t'(0) = e^{-t}$. The family of conformal maps $\{f_t\}_{t \geq 0}$ is called the *capacity parametrized radial Loewner chain* or *normalized subordination chain* driven by $t \mapsto \zeta_t$. SLE$_\kappa$ is the curve $t \mapsto \gamma_t$ which is defined as $\gamma_t := \lim_{r \to 1-} f_t(r\zeta_t)$, see [14]. In particular, the curve starts at $\gamma_0 = 1$. The radial SLE$_\kappa$ on an arbitrary simply connected domain $D$ is defined via the unique conformal map from $\mathbb{D}$ to $D$ respecting the starting and target points. It is well-known that SLE$_\kappa$ exhibits phase transitions as $\kappa$ varies. Larger values of $\kappa$ correspond in some sense to "wilder" SLE$_\kappa$ curves; in the $\kappa \geq 8$ regime the curve is space-filling.

In this work, we study the $\kappa \to \infty$ asymptotic behavior of radial SLE. To simplify notation we consider SLE$_\kappa$ run on the time interval $[0, 1]$ throughout the paper, but our results are easily generalized to arbitrarily bounded time intervals. Hence we denote by $\{\cdot\}$ the family $\{\cdot\}_{t \in [0,1]}$ to avoid repeating indices.

Our first result (Proposition 1.1) characterizes the limit as $\kappa \to \infty$ of the time-evolution of the SLE$_\kappa$ hulls. We argue heuristically as follows. We view the time-dependent vector field $\{-z(z + \zeta_t)/(z - \zeta_t)\}$ which generates the flow $\{g_t\}$ as $\{\int_{S^1} -z(z + \zeta)/(z - \zeta)\delta_{B_t^\kappa}(\zeta)\}$, where $\delta_{B_t^\kappa}$ is the Dirac mass at $B_t^\kappa$. During a short time interval where the flow is well-defined for the point $z$, we have $g_t(z) \approx g_{t+\Delta t}(z)$ and hence

$$\Delta g_t(z) \approx \int_t^{t+\Delta t} \int_{S^1} -g_t(z)(g_t(z) + \zeta)/(g_t(z) - \zeta)\delta_{B_s^\kappa}(\zeta)ds$$
$$= \int_{S^1} -g_t(z)(g_t(z) + \zeta)/(g_t(z) - \zeta)d(L_{t+\Delta t}^\kappa(\zeta) - L_t^\kappa(\zeta)),$$

where $L_t^\kappa$ is the *occupation measure* (or *local time*) on $S^1$ of $B^\kappa$ up to time $t$. We show that as $\kappa \to \infty$, the driving function oscillates so quickly that its local time in $[t, t + \Delta t]$ is almost uniform on $S^1$, so in the limit we get a *measure-driven Loewner chain* with driving measure uniform on $S^1$. That is,

$$\partial_t g_t(z) = \frac{1}{2\pi} \int_{S^1} -g_t(z) \frac{g_t(z) + \zeta}{g_t(z) - \zeta} d\zeta,$$

where $d\zeta$ denotes the Lebesgue measure. This implies $\partial_t g_t(z) = g_t(z)$, that is, $g_t(z) = e^t z$ or equivalently $f_t(z) = e^{-t}z$. See Section 2 for more details on the measure-driven Loewner chain. We show in Section 3.2:

**Proposition 1.1.** *As $\kappa \to \infty$, the Loewner chain $\{f_t\}$ driven by $\{\zeta_t\}$ (defined in (1.1)) converges to $\{z \mapsto e^{-t}z\}$ almost surely, with respect to the uniform Carathéodory topology.*

We shall mention that Loewner chains are also used in the study of the Hastings-Levitov model of randomly aggregating particles and similar small-particle limits have been studied, see [9] and references therein.

The heuristic argument above suggests that the large deviations of SLE$_\kappa$ boil down to the large deviations of the Brownian occupation measure, which we now describe.

For any metric space $X$, let $\mathcal{M}_1(X)$ denote the set of Borel probability measures equipped with the Prokhorov topology (the topology of weak convergence). Let

$$\mathcal{N} = \{\rho \in \mathcal{M}_1(S^1 \times [0,1]) : \rho(S^1 \times I) = |I| \text{ for all intervals } I \subset [0,1]\}. \tag{1.4}$$

The condition imposed here allows us to write $\rho \in \mathcal{N}$ as a disintegration $\{\rho_t\}$ over the time interval $[0,1]$ (with $\rho_t \in \mathcal{M}_1(S^1)$ for a.e. $t$); see (2.1). We identify $\rho$ and the time-indexed family $\{\rho_t\}$. The second result we show is:

**Theorem 1.2.** *The process of measures $\{\delta_{B_t^\kappa}\} \in \mathcal{N}$ satisfies the large deviation principle with good rate function $\mathcal{E}(\rho) := \int_0^1 I(\rho_t)\,dt$ for $\rho \in \mathcal{N}$, where $I(\mu)$ is defined for each $\mu \in \mathcal{M}_1(S^1)$ as*

$$I(\mu) := \frac{1}{2} \int_{S^1} |\phi'(\zeta)|^2 d\zeta \tag{1.5}$$

*if $\mu(d\zeta) = \phi^2(\zeta)\,d\zeta$ and $\phi$ is absolutely continuous, and $I(\mu) := \infty$ otherwise. That is, for every closed set $C$ and open set $G$ of $\mathcal{N}$,*

$$\limsup_{\kappa \to \infty} \frac{1}{\kappa} \log \mathbb{P}\left[\{\delta_{B_t^\kappa}\} \in C\right] \leq -\inf_{\rho \in C} \mathcal{E}(\rho);$$

$$\liminf_{\kappa \to \infty} \frac{1}{\kappa} \log \mathbb{P}\left[\{\delta_{B_t^\kappa}\} \in G\right] \geq -\inf_{\rho \in G} \mathcal{E}(\rho);$$

*and the sub-level set $\{\rho \in \mathcal{N} : \mathcal{E}(\rho) \leq c\}$ is compact for all $c > 0$.*

Our proof is based on a result by Donsker and Varadhan [6] on large deviations of the Brownian occupation measure (see Sections 3.3–3.4). The $\kappa \to \infty$ large deviations of SLE then follows immediately from the continuity of the Loewner transform (Theorem 2.2) and the contraction principle [5, Theorem 4.2.1].

**Corollary 1.3.** *The family of $\mathrm{SLE}_\kappa$ satisfies the $\kappa \to \infty$ large deviation principle with the good rate function*

$$I_{\mathrm{SLE}_\infty}(\{K_t\}) := \mathcal{E}(\rho),$$

*where $\{\rho_t\}$ is the driving measure whose Loewner transform is $\{K_t\}$.*

Let us conclude the introduction with two comments.

The study of large deviations of SLE, while of inherent interest, is also motivated by problems from complex analysis and geometric function theory. In a forthcoming work [17], Viklund and the third author investigate the *duality* between the rate functions of $\mathrm{SLE}_{0+}$ (termed as the *Loewner energy* introduced in [18, 15]) and $\mathrm{SLE}_\infty$ that is reminiscent of the SLE duality [8, 20] which couples $\mathrm{SLE}_\kappa$ to the outer boundary of $\mathrm{SLE}_{16/\kappa}$ for $\kappa < 4$. Note that $\mathcal{E}(\rho)$ attains its minimum if and only if $\{D_t\}$ are concentric disks, and $\{\partial D_t\}$ are circles which also have the minimal Loewner energy.

It is also natural to consider the large deviations of *chordal* $\mathrm{SLE}_\infty$ (say, in $\mathbb{H}$ targeted at $\infty$). However, in contrast with the radial case, the family indexed by $\kappa$ of random measures $\{\delta_{W_{\kappa t}}\}$ on $\mathbb{R} \times [0,1]$ is not tight and the corresponding Loewner flow converges to the identity map for any fixed time $t$. To obtain a non-trivial limit, one needs to renormalize appropriately (see e.g., Beffara's thesis [1, Sec.5.2] for a non-conformal normalization) and consider generalized Stieltjes transformation of measures for the large deviations. Therefore, for simplicity we choose to study the radial case and suggest the large deviations of chordal $\mathrm{SLE}_\infty$ as an interesting question. We will show a simulation of large-$\kappa$ chordal SLEs and discuss some other questions at the end of the paper.

The paper is organized as follows: In Section 2, we explain the measure-driven radial Loewner evolution. In Section 3 we prove the main results of our paper. In Section 4 we present some comments, observations and questions.

## 2  Measure-driven radial Loewner evolution

In this section we collect some known facts on the measure-driven Loewner evolution (also known as Loewner-Kufarev evolution) that are essential to our proofs. Recall that

$$\mathcal{N} = \{\rho \in \mathcal{M}_1(S^1 \times [0,1]) : \rho(S^1 \times I) = |I| \text{ for all intervals } I \subset [0,1]\}$$

endowed with the Prokhorov topology. From the disintegration theorem (see e.g. [2, Theorem 33.3]), for each measure $\rho \in \mathcal{N}$ there exists a Borel measurable map $t \mapsto \rho_t$ (sending $[0,1] \to \mathcal{M}_1(S^1)$) such that for every measurable function $\varphi : S^1 \times [0,1] \to \mathbb{R}$ we have

$$\int_{S^1 \times [0,1]} \varphi(\zeta, t)\, d\rho = \int_0^1 \int_{S^1} \varphi(\zeta, t)\, \rho_t(d\zeta)\, dt. \tag{2.1}$$

We say $\{\rho_t\}$ is a *disintegration* of $\rho$; it is unique in the sense that any two disintegrations $\{\rho_t\}, \{\widetilde{\rho}_t\}$ of $\rho$ must satisfy $\rho_t = \widetilde{\rho}_t$ for a.e. $t$. We always denote by $\{\rho_t\}$ one such disintegration of $\rho \in \mathcal{N}$.

The Loewner chain driven by a measure $\rho \in \mathcal{N}$ is defined similarly to (1.2). For $z \in \mathbb{D}$, consider the *Loewner-Kufarev ODE*

$$\partial_t g_t(z) = -g_t(z) \int_{S^1} \frac{g_t(z) + \zeta}{g_t(z) - \zeta}\, \rho_t(d\zeta)$$

with the initial condition $g_0(z) = z$. Let $T_z$ be the supremum of all $t$ such that the solution is well-defined up to time $t$ with $g_t(z) \in \mathbb{D}$, and $D_t := \{z \in \mathbb{D} : T_z > t\}$ is a simply connected open set containing 0. We define the *hull* $K_t := \mathbb{D} \setminus D_t$ associated with the Loewner chain. Note that when $\kappa \geq 8$, the family $\{\gamma[0,t]\}$ of radial SLE$_\kappa$ is exactly the family of hulls $\{K_t\}$ driven by the measure $\{\delta_{B_t^\kappa}\}$.

The function $g_t$ defined above is the unique conformal map of $D_t$ onto $\mathbb{D}$ such that $g_t(0) = 0$ and $g_t'(0) > 0$; moreover $g_t'(0) = e^t$ (i.e. $D_t$ has conformal radius $e^{-t}$ seen from 0) since $\partial_t \log g_t'(0) = |\rho_t| = 1$ (see e.g. [10, Thm. 4.13]).

If $g_t$ is the solution of a Loewner-Kufarev ODE then its inverse $f_t = g_t^{-1}$ satisfies the *Loewner-Kufarev PDE*:

$$\partial_t f_t(z) = z f_t'(z) \int_{S^1} \frac{z + \zeta}{z - \zeta} \rho_t(d\zeta),$$

and vice versa. Note that $f_t(0) = 0$, $f_t'(0) = e^{-t}$, and $f_t(\mathbb{D}) = D_t \subset f_s(\mathbb{D})$ for $s \leq t$. Such a time-indexed family $\{f_t\}$ is called *a normalized chain of subordinations*. We write $\mathcal{S}$ for the set of normalized chains of subordinations $\{f_t\}$ on $[0,1]$. An element of $\mathcal{S}$ can be equivalently represented by either $\{f_t\}$ or the process of hulls $\{K_t\}$. The map $\mathcal{L} : \rho \mapsto \{f_t\}$ (or interchangeably $\mathcal{L} : \rho \mapsto \{K_t\}$) is called the *Loewner transform*. In fact, $\mathcal{L}$ is a bijection:

**Theorem 2.1** (Bijectivity of the Loewner transform [13, Satz 4]). *The family $(f_t)_{t \in [0,1]}$ is a normalized chain of subordination over $[0,1]$ if and only if*

- *$t \mapsto f_t(z)$ is absolutely continuous in $[0,1]$ and for all $r < 1$, there is $K(r) > 0$ such that $|f_t(z) - f_s(z)| \leq K(r)|t - s|$ for all $z \in r\mathbb{D}$;*

- *and there is a ($t$-a.e. unique) function $h(z,t)$ that is analytic in $z$, measurable in $t$ with $h(0,t) = 1$ and $\operatorname{Re} h(z,t) > 0$, so that for $t$-a.e. we have*

$$\partial_t f_t(z) = -z f_t'(z) h(z,t).$$

From the Herglotz representation of $h(\cdot, t)$, there exists a unique $\rho_t \in \mathcal{M}_1(S^1)$ such that

$$h(z,t) = \int_{S^1} \frac{\zeta + z}{\zeta - z} \rho_t(d\zeta), \quad \forall z \in \mathbb{D}.$$

Therefore $\{f_t\}$ satisfies the Loewner PDE driven by the (a.e. uniquely determined) measurable function $t \mapsto \rho_t$.

We now equip $\mathcal{S}$ with a topology. View $\mathcal{S}$ as the set of normalized chains of subordinations $\{f_t\}$ on $[0,1]$, and change notation by writing $f(z,t) = f_t(z)$. We endow $\mathcal{S}$ with the topology of uniform convergence of $f$ on compact subsets of $\mathbb{D} \times [0,1]$. (Equivalently, if we view $\mathcal{S}$ as the set of processes of hulls $\{K_t\}$, this is the topology of *uniform Carathéodory convergence*.) The continuity of $\mathcal{L}$ has been, e.g., derived in [12, Proposition 6.1] (see also [9]).

**Theorem 2.2** (Continuity). *The Loewner transform $\mathcal{L} : \mathcal{N} \to \mathcal{S}$ is a homeomorphism.*

## 3 Proofs of the main results

In this section, we study the random measure $\{\delta_{B_t^\kappa}\} \in \mathcal{N}$. In Section 3.1 we approximate $\mathcal{N}$ by spaces of time-averaged measures. In Section 3.2 we verify that $\{\delta_{B_t^\kappa}\} \in \mathcal{N}$ converges almost surely as $\kappa \to \infty$ to the uniform measure on $S^1 \times [0,1]$; this yields Proposition 1.1. In Section 3.3, we review the large deviation principle for the circular Brownian motion occupation measure, which is a special case of seminal work of Donsker and Varadhan [6]. Finally, in Section 3.4 we prove Theorem 1.2, the large deviation principle for $\{\delta_{B_t^\kappa}\} \in \mathcal{N}$.

### 3.1 Time-discretized approximations of measures

We emphasize that the results of this section are wholly deterministic.

For $n \geq 0$, let $\mathcal{I}_n := \{0, 1, 2, \cdots, 2^n - 1\}$ be an index set, and define $\mathcal{Y}_n := \left(\mathcal{M}_1(S^1)\right)^{\mathcal{I}_n}$. We note that $\mathcal{Y}_n$ is endowed with the product topology. For each $i \in \mathcal{I}_n$ we define a function $P_n^i : \mathcal{N} \to \mathcal{M}_1(S^1)$ via

$$P_n^i(\rho) := 2^n \int_{i/2^n}^{(i+1)/2^n} \rho_t \, dt, \tag{3.1}$$

where here $\{\rho_t\}$ is a disintegration of $\rho$ with respect to $t$, as in (2.1). We define also the map $P_n : \mathcal{N} \to \mathcal{Y}_n$ via $P_n = (P_n^i)_{i \in \mathcal{I}_n}$. That is, $P_n$ averages $\rho$ along each $2^{-n}$-time interval, and outputs the $2^n$-tuple of these $2^n$ time-averages.

We consider $\mathcal{Y}_n$ to be the space of time-discretized approximations of $\mathcal{N}$, in the following sense. Define a map $F_n : \mathcal{Y}_n \to \mathcal{N}$ via

$$F_n \left((\mu_i)_{i \in \mathcal{I}_n}\right) := \sum_{i \in \mathcal{I}_n} \mu_i \otimes \mathrm{Leb}_{[i/2^n, (i+1)/2^n]}.$$

Then one can view $F_n(P_n(\rho))$ as a "level-$n$ approximation" of $\rho$ (see Lemma 3.1).

We have provided a way of projecting an element of $\mathcal{N}$ to the space of level-$n$ approximations $\mathcal{Y}_n$. Now we write down a map $P_{n,n+1} : \mathcal{Y}_{n+1} \to \mathcal{Y}_n$ which takes in a finer approximation and outputs a coarser approximation:

$$P_{n,n+1} \left((\mu_i)_{i \in \mathcal{I}_{n+1}}\right) := \left(\frac{\mu_0 + \mu_1}{2}, \ldots, \frac{\mu_{2^{n+1}-2} + \mu_{2^{n+1}-1}}{2}\right).$$

That is, we average pairs of components of $\mathcal{Y}_{n+1}$. It is clear that

$$P_n = P_{n,n+1} \circ P_{n+1}. \tag{3.2}$$

The convergence of $P_n(\rho_j) \xrightarrow{j \to \infty} P_n(\rho)$ in $\mathcal{Y}_n$ is equivalent to the convergence $\rho_j(f) \xrightarrow{j \to \infty} \rho(f)$ for the functions $f$ which are piecewise constant in time for each time interval $(i/2^n, (i+1)/2^n)$. For each fixed $n$, this is a coarser topology than that of $\mathcal{N}$. The following lemma shows that the $n \to \infty$ topology agrees with that of $\mathcal{N}$.

**Lemma 3.1.** *We have $\mathcal{N} = \varprojlim \mathcal{Y}_n$. That is, as topological spaces, $\mathcal{N}$ is the projective (inverse) limit of $\mathcal{Y}_n$ as $n \to \infty$.*

*Proof.* Let $\mathcal{Y} = \varprojlim \mathcal{Y}_n$; this is the subset of $\prod_{j=0}^{\infty} \mathcal{Y}_j$ comprising elements $(y^0, y^1, \dots)$ such that $P_{n,n+1}(y^{n+1}) = y^n$ for all $n \geq 0$. The topology on $\mathcal{Y}$ is inherited from $\prod_{j=0}^{\infty} \mathcal{Y}_j$. Because of the coherence relation (3.2), we can define a map $P : \mathcal{N} \to \mathcal{Y}$ by $P(\rho) := (P_j(\rho))_{j \geq 0}$. We now show that $P$ is a homeomorphism.

**Showing that $P$ is continuous.** Since the topology on $\mathcal{Y}$ is inherited from the product topology on $\prod_{n=0}^{\infty} \mathcal{Y}_j$, it suffices to show that the map $P : \mathcal{N} \to \prod_{n=0}^{\infty} \mathcal{Y}_n$ is continuous, i.e. $P_n : \mathcal{N} \to \mathcal{Y}_n$ is continuous for each $n$. But this is clear: if two measures in $\mathcal{N}$ are close in the Prokhorov topology, then so is the time-average of these measures on a time interval.

**Showing that $P$ is a bijection.** Fix $f \in C(S^1 \times [0,1])$. We claim that for any $\varepsilon > 0$, there exists $n_0 = n_0(f, \varepsilon)$ such that for all $m, n \geq n_0$ and $y = (y^0, y^1, \dots) \in \mathcal{Y}$ we have

$$|(F_m(y^m))(f) - (F_n(y^n))(f)| < \varepsilon. \tag{3.3}$$

To that end we note that $f$ is uniformly continuous; we can choose $\delta > 0$ so that $|f(\zeta, t) - f(\zeta, t')| < \varepsilon$ whenever $|t - t'| < \delta$. Choosing $n_0$ such that $2^{-n_0} < \delta$, we obtain (3.3).

Now we show that $P$ is a bijection. By (3.3), for each $y = (y^0, y^1, \dots) \in \mathcal{Y}$ we can define a bounded linear functional $T_y : C(S^1 \times [0,1]) \to \mathbb{R}$ via

$$T_y(f) = \lim_{n \to \infty} (F_n(y^n))(f) \quad \text{for } y = (y^0, y^1, \dots).$$

Clearly $T_y$ maps nonnegative functions to nonnegative reals, so the Riesz-Markov-Kakutani representation theorem tells us there is a unique[1] Borel measure $\rho$ on $S^1 \times [0,1]$ such that $\rho(f) = T_y(f)$ for all $f \in C^1(S^1 \times [0,1])$; it is easy to check that $\rho \in \mathcal{N}$. Thus, for each $y \in \mathcal{Y}$ the equation $P(\rho) = y$ has a unique solution in $\mathcal{N}$, so $P$ is a bijection.

**Concluding that $P$ is a homeomorphism.** We note that $\mathcal{N}$ is compact (since $S^1 \times [0,1]$ is compact) and $\mathcal{Y}$ is compact (since each $\mathcal{Y}_n$ is compact and Hausdorff). Since $P$ is a continuous bijection of compact sets, it is a homeomorphism. $\qquad\square$

### 3.2 Almost sure limit of SLE driving measures

Consider a Brownian motion $B_t^\kappa$ on the unit circle $S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ started at $1$ with variance $\kappa$ as (1.1). Define the *occupation measure* of $B_t^\kappa$:

$$L_t^\kappa(A) = \int_0^t \mathbf{1}\{B_s^\kappa \in A\} \, ds \quad \text{for Borel sets } A \subset S^1.$$

Let $\overline{L}_t^\kappa = t^{-1} L_t^\kappa$ be the *average occupation measure* of $B^\kappa$ at time $t$ (its normalization gives $\overline{L}_t^\kappa \in \mathcal{M}_1(S^1)$). An easy consequence of the ergodic theorem is the following almost sure $t \to \infty$ limit of $\overline{L}_t^1$; we include the proof for completeness.

**Lemma 3.2.** *Almost surely, as $t \to \infty$ we have $\overline{L}_t^1 \to (2\pi)^{-1} \operatorname{Leb}_{S^1}$ in $\mathcal{M}(S^1)$.*

*Proof.* It suffices to show that for any continuous function $f : S^1 \to \mathbb{R}$ we have almost surely

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(B_s^1) \, ds = \frac{1}{2\pi} \int_{S^1} f(\zeta) \, d\zeta. \tag{3.4}$$

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[1]The representation theorem yields a unique *regular* Borel measure, but since $S^1 \times [0,1]$ is compact, *all* Borel measures on $S^1 \times [0,1]$ are regular.

Given this, by choosing a suitable countable collection of functions, we obtain the lemma.

Let $(\Omega, \mathbb{P})$ be a Wiener space so that $\mathbb{P}$ is the law of $B_t^1$. Consider the expanded probability space given by $(\Omega \times S^1, \mathbb{P} \otimes (2\pi)^{-1} \operatorname{Leb}_{S^1})$, and let $(\omega, e^{i\theta})$ correspond to the random path $e^{i\theta} B_t^1(\omega)$. That is, after sampling an instance of Brownian motion $B_t^1(\omega)$ started at $1$, we apply an independent uniform rotation to the circle so the Brownian motion starts at $e^{i\theta}$ instead. A consequence of Birkhoff's ergodic theorem is that for a.e. $(\omega, e^{i\theta}) \in \Omega \times S^1$, we have

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(e^{i\theta} B_s^1(\omega)) \, ds = \frac{1}{2\pi} \int_{S^1} f(\zeta) \, d\zeta. \tag{3.5}$$

Equivalently, for a.e. $e^{i\theta} \in S^1$, we have (3.5) for a.e. $\omega$. Taking a sequence of $e^{i\theta}$ converging to $1$ and using the uniform continuity of $f$, we obtain (3.4). This concludes the proof of Lemma 3.2. $\qquad\square$

Now we justify the heuristic argument in the introduction, which said that as $\kappa \to \infty$, the Brownian motion $B_t^\kappa$ moves so quickly that the driving measure converges to $(2\pi)^{-1} \operatorname{Leb}_{S^1} \otimes \operatorname{Leb}_{[0,1]}$.

**Lemma 3.3.** As $\kappa \to \infty$, $\{\delta_{B_t^\kappa}\}$ converges almost surely to $(2\pi)^{-1} \operatorname{Leb}_{S^1} \otimes \operatorname{Leb}_{[0,1]}$ in $\mathcal{N}$.

*Proof.* Lemma 3.1 states that $\mathcal{N}$ is the projective limit of the spaces $\mathcal{Y}_n$ defined in Section 3.1, with projection map from $\mathcal{N}$ to $\mathcal{Y}_n$ given by $(P_n^i)_{i \in \mathcal{I}_n}$. It thus suffices to show that as $\kappa \to \infty$, the random measure $P_n^i(\{\delta_{B_t^\kappa}\})$ converges almost surely to $P_n^i((2\pi)^{-1} \operatorname{Leb}_{S^1} \otimes \operatorname{Leb}_{[0,1]}) = (2\pi)^{-1} \operatorname{Leb}_{S^1}$ in the Prokhorov topology, namely,

$$\lim_{\kappa \to \infty} 2^n \int_{i/2^n}^{(i+1)/2^n} \delta_{B_t^\kappa} \, dt = \frac{1}{2\pi} \operatorname{Leb}_{S^1}.$$

This is true since Lemma 3.2 tells us that almost surely

$$\lim_{\kappa \to \infty} \frac{2^n}{i} \int_0^{i/2^n} \delta_{B_t^\kappa} \, dt = \lim_{\kappa \to \infty} \frac{2^n}{i+1} \int_0^{(i+1)/2^n} \delta_{B_t^\kappa} \, dt = \frac{1}{2\pi} \operatorname{Leb}_{S^1}.$$

Hence Lemma 3.3 holds. $\qquad\square$

*Proof of Proposition 1.1.* It follows immediately from Theorem 2.2 and Lemma 3.3. $\quad\square$

### 3.3 Large deviation principle of occupation measures

In this section, we discuss the large deviation principle of Brownian motion occupation measures on $S^1$ as $\kappa \to \infty$.

Recall that $\overline{L}_t^\kappa = t^{-1} L_t^\kappa$ is the average occupation measure of $B^\kappa$ at time $t$. By Brownian scaling we have that (recall that the upper index is diffusivity and the lower index is time) $\overline{L}_{\kappa t}^1$ and $\overline{L}_t^\kappa$ equal in law, so it suffices to understand the large deviation principle for $\overline{L}_t^1$ as $t \to \infty$. This follows from a more general result of Donsker and Varadhan; we state the result for Brownian motion on $S^1$.

**Theorem 3.4** ([6, Theorem 3]). *Define $\tilde{I} : \mathcal{M}_1(S^1) \to \mathbb{R}_{\geq 0}$ by*

$$\tilde{I}(\mu) := -\inf_{u > 0, \, u \in C^2(S^1)} \int_{S^1} \frac{u''}{2u}(\zeta) \mu(d\zeta) = -\inf_{u > 0, \, u \in C^2(S^1)} \int_{S^1} \frac{L(u)}{u}(\zeta) \mu(d\zeta), \tag{3.6}$$

*where $L(u) = u''/2$ is the infinitesimal generator of the Brownian motion on $S^1$. The average occupation measure $\overline{L}_1^\kappa$ admits a large deviation principle as $\kappa \to \infty$, with rate*

function $\tilde{I}$. That is, for any closed set $C \subset \mathcal{M}_1(S^1)$,

$$\limsup_{\kappa \to \infty} \frac{1}{\kappa} \log \mathbb{P}[\overline{L}_1^\kappa \in C] \le - \inf_{\mu \in C} \tilde{I}(\mu), \tag{3.7}$$

and for any open set $G \in \mathcal{M}_1(S^1)$,

$$\liminf_{\kappa \to \infty} \frac{1}{\kappa} \log \mathbb{P}[\overline{L}_1^\kappa \in G] \ge - \inf_{\mu \in G} \tilde{I}(\mu). \tag{3.8}$$

Moreover, $\tilde{I}$ is good, lower-semicontinuous, and convex.

Note that the lower-semicontinuity (hence the goodness, since $\mathcal{M}_1(S^1)$ is compact) and convexity follow directly from the expression of $\tilde{I}$. For the convenience of those readers who may not be so familiar with the statement of Theorem 3.4, let us provide an outline of the proof of the upper bound (3.7) in order to explain where this rate function comes from.

Let $P_\zeta$ denote the law of a Brownian motion $B^1$ on $S^1$ starting from $\zeta \in S^1$ and $Q_{\zeta,t}$ the law of the average occupation measure $\overline{L}_t^1$ under $P_\zeta$. Fix a small number $h > 0$, and let $\pi_h(\zeta, d\xi)$ be the law of $B_h$ under $P_\zeta$. We consider the Markov chain $X_n := B_{nh}$, so that $\pi_h$ is the transition kernel of $X$. We write $\mathbb{E}$ for the expectation with respect to $P_1$.

Now let $u \in C^2(S^1)$ such that $u > 0$. From the Markov property, we inductively get

$$\mathbb{E}\left[ \frac{u(X_0) \cdots u(X_{n-1})}{\pi_h u(X_0) \cdots \pi_h u(X_{n-1})} u(X_n) \right] = \mathbb{E}[u(X_0)] = u(1).$$

Since the Brownian motion is a Feller process with infinitesimal generator $L$, we have

$$\log \frac{u(\zeta)}{\pi_h u(\zeta)} = \log\left( 1 - h\frac{Lu(\zeta)}{u(\zeta)} + o(h) \right) = -h\frac{Lu(\zeta)}{u(\zeta)} + o(h).$$

Therefore,

$$\begin{aligned}
u(1) &= \mathbb{E}\left[ \exp\left( -\sum_{i=0}^n h\frac{Lu(X_i)}{u(X_i)} + o(h) \right) \pi_h u(X_n) \right] \\
&= \mathbb{E}\left[ \exp\left( -\int_0^t \frac{Lu(B_s)}{u(B_s)} ds \right) \pi_h u(B_t) + o(1) \right],
\end{aligned}$$

where $n$ is chosen to be the integer part of $t/h$. Hence,

$$\mathbb{E}^{Q_{1,t}}\left[ \exp\left( -t\int_{S^1} \frac{Lu(\zeta)}{u(\zeta)} \overline{L}_t^1(d\zeta) \right) \right] \le \frac{u(1)}{\inf_{\xi \in S^1} \pi_h u(\xi)} \le \frac{u(1)}{\inf_{\xi \in S^1} u(\xi)} \le M(u)$$

for some $M(u) < \infty$ depending only on the function $u > 0$. For any measurable set $C \subset \mathcal{M}_1(S^1)$, since

$$\begin{aligned}
M(u) &\ge \mathbb{E}^{Q_{1,t}}\left[ \exp\left( -t\int_{S^1} \frac{Lu(\zeta)}{u(\zeta)} \overline{L}_t^1(d\zeta) \right) \right] \\
&\ge Q_{1,t}(C) \exp\left( -t \sup_{\mu \in C} \int_{S^1} \frac{Lu}{u}(\zeta)\, \mu(d\zeta) \right)
\end{aligned}$$

for arbitrary $u$, we have

$$\limsup_{t \to \infty} \frac{1}{t} \log Q_{1,t}(C) \le \inf_{u > 0, u \in C^2} \sup_{\mu \in C} \int_{S^1} \frac{Lu}{u}(\zeta)\, \mu(d\zeta).$$

When $C$ is closed (hence compact), some topological considerations allow us to swap the inf and sup in the above expression, and we obtain

$$\inf_{u>0, u \in C^2} \sup_{\mu \in C} \int_{S^1} \frac{Lu}{u}(\zeta)\mu(d\zeta) \leq \sup_{\mu \in C} \inf_{u>0, u \in C^2} \int_{S^1} \frac{Lu}{u}(\zeta)\mu(d\zeta) = -\inf_{\mu \in C} \tilde{I}(\mu),$$

which is the upper bound (3.7). As it is often the case in the derivation of large deviation principles, the lower bound turns out to be trickier, and uses approximation by discrete time Markov chains and a change of measure argument. We refer to the original paper [6] for more details.

The rate function $\tilde{I}$ of Theorem 3.4 is somewhat unwieldy but can be simplified for Brownian motion as noted in [6]. We provide here an alternative elementary proof.

**Theorem 3.5** ([6, Theorem 5]). *For $\mu \in \mathcal{M}_1(S^1)$, the rate function $\tilde{I}(\mu)$ is finite if and only if $\mu = \phi^2(\zeta)d\zeta$ for some function $\phi \in W^{1,2}$. In this case, we have $\tilde{I}(\mu) = I(\mu)$, where*

$$I(\mu) = \frac{1}{2} \int_{S^1} |\phi'(\zeta)|^2 \, d\zeta.$$

*Proof.* First assume that $\mu = \phi^2 d\zeta$ for some $\phi \in W^{1,2}$ and that $I(\mu)$ is finite, we will show that $\tilde{I}(\mu) = I(\mu)$. For this, take a sequence $\phi_n \in C^2$ with $\phi_n > 0$ converging to $\phi$ almost everywhere such that $\int_{S^1} (\phi_n')^2 \, d\zeta \to \int_{S^1} (\phi')^2 \, d\zeta$. For any $u \in C^2$ and any $\varepsilon > 0$, we have for sufficiently large $n$ that

$$\int_{S^1} \frac{u''}{2u} \phi^2 \, d\zeta + \varepsilon \geq \int_{S^1} \frac{(v\phi_n)''}{2(v\phi_n)} \phi_n^2 \, d\zeta = \int_{S^1} \frac{\phi_n'' \phi_n}{2} \, d\zeta + \int_{S^1} \frac{(\phi_n^2 v')'}{2v} \, d\zeta,$$

where $v := u/\phi_n \in C^2$. From integration by parts, this latter expression equals to

$$-\frac{1}{2} \int_{S^1} |\phi_n'|^2 \, d\zeta + \frac{1}{2} \int_{S^1} \frac{\phi_n^2 v'^2}{v^2} \, d\zeta \geq -\frac{1}{2} \int_{S^1} |\phi_n'|^2 \, d\zeta \geq -I(\mu) - \varepsilon$$

by taking $n$ larger if necessary. Since $\varepsilon$ is arbitrary, we obtain $-\int_{S^1} \frac{u''}{2u} \phi^2 \, d\zeta \leq I(\mu)$, and thus $\tilde{I}(\mu) \leq I(\mu)$ by taking supremum over $u$. The opposite inequality can be shown by taking $u = \phi_n$ (i.e. $v = 1$) and sending $n \to \infty$. Therefore $\tilde{I}(\mu) = I(\mu)$ when $I(\mu) < \infty$.

It remains to prove that if $\tilde{I}(\mu) < \infty$ then $I(\mu) < \infty$, so consider $\mu$ such that $\tilde{I}(\mu)$ is finite. Let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be a family of nonnegative smooth functions on $S^1$ with $\int_{S^1} \eta_\varepsilon \, d\zeta = 1$ and converging weakly to the Dirac delta function at $1$ as $\varepsilon \to 0$. Writing $\mu^\xi$ for $\mu$ rotated by $\xi \in S^1$, we define $\mu_\varepsilon = \int_{S^1} \eta_\varepsilon(\xi)\mu^\xi d\xi$ as a weighted average of probability measures so that $\mu_\varepsilon$ converges weakly to $\mu$. Observe that $\tilde{I}$ is rotation invariant and convex. Therefore by Jensen's inequality,

$$\tilde{I}(\mu_\varepsilon) = \tilde{I} \left( \int_{S^1} \eta_\varepsilon(\xi)\mu^\xi \, d\xi \right) \leq \int_{S^1} \eta_\varepsilon(\xi)\tilde{I}(\mu^\xi) \, d\xi = \tilde{I}(\mu).$$

Write $\phi_\varepsilon := \sqrt{\eta_\varepsilon * \mu}$, so that $\mu_\varepsilon = \phi_\varepsilon^2(\zeta)d\zeta$. We will show that

$$\int_{S^1} (\phi_\varepsilon')^2 \, d\zeta \leq 2\tilde{I}(\mu_\varepsilon). \tag{3.9}$$

Given (3.9), we see that $\int_{S^1} (\phi_\varepsilon')^2 \, d\zeta$ is uniformly bounded above (by $2\tilde{I}(\mu)$); letting $\varepsilon \to 0$ implies $\mu$ is an absolutely continuous measure, and furthermore $\sqrt{\mu(d\zeta)/d\zeta} \in W^{1,2}$ so $I(\mu) < \infty$, concluding the proof of the theorem.

We turn to the proof of (3.9), which follows the argument of [7, Lemma 2.4]. In the definition (3.6), take $u = e^{\lambda h}$ where $h$ is smooth and $\lambda$ is a real number. This gives

$$\lambda^2 \int_{S^1} h'^2 \phi_\varepsilon^2 \, d\zeta + \lambda \int_{S^1} h'' \phi_\varepsilon^2 \, d\zeta = \lambda^2 \int_{S^1} h'^2 \phi_\varepsilon^2 \, d\zeta - 2\lambda \int_{S^1} h' \phi_\varepsilon' \phi_\varepsilon \, d\zeta \geq -2\tilde{I}(\mu_\varepsilon)$$

which holds for any real number $\lambda$. By choosing $\lambda$ so that the quadratic function takes the minimum, we have

$$\left( \int_{S^1} h' \phi'_\varepsilon \phi_\varepsilon \, d\zeta \right)^2 \leq 2\tilde{I}(\mu_\varepsilon) \int_{S^1} h'^2 \phi_\varepsilon^2 \, d\zeta. \tag{3.10}$$

For $n \in \mathbb{N}$, consider an auxiliary function $\nu_n$ on positive real numbers defined as

$$\nu_n(x) := \begin{cases} 0 & \text{if } 0 < x \leq 1/2n \\ 1/x & \text{if } x \geq 1/n \end{cases}$$

and extended on $[1/2n, 1/n]$ so that $0 \leq \nu_n(x) \leq 1/x$ and $\nu_n$ is smooth for all $x$. And define $V_n(x) = \int_0^x \nu_n(y) \, dy$. Plugging $h = V_n(\phi_\varepsilon)$ to (3.10) gives

$$\left( \int_{S^1} \nu_n(\phi_\varepsilon) \phi'^2_\varepsilon \phi_\varepsilon \, d\zeta \right)^2 \leq 2\tilde{I}(\mu_\varepsilon) \int_{S^1} \nu_n^2(\phi_\varepsilon) \phi'^2_\varepsilon \phi_\varepsilon^2 \, d\zeta \leq 2\tilde{I}(\mu_\varepsilon) \int_{S^1} \nu_n(\phi_\varepsilon) \phi'^2_\varepsilon \phi_\varepsilon \, d\zeta$$

where $\nu_n(\phi_\varepsilon) \leq 1/\phi_\varepsilon$ was used and the common terms on both sides cancel out. As $n \to \infty$, Fatou's lemma implies (3.9) as desired. $\square$

### 3.4 Large deviations for $\{\delta_{B_t^\kappa}\}$

In this section, we prove Theorem 1.2. That is, we establish the large deviation principle for the Brownian trajectory measure $\{\delta_{B_t^\kappa}\}$. We use the notation of Section 3.1.

The first step is the large deviation principle for $P_n(\{\delta_{B_t^\kappa}\})$, which follows easily from Theorem 3.4. Recall that $P_n(\{\delta_{B_t^\kappa}\})$ is a $2^n$-tuple of elements of $\mathcal{M}_1(S^1)$, the $i$th element being the average of $\{\delta_{B_t^\kappa}\}$ on the time interval $[i/2^n, (i+1)/2^n]$.

**Lemma 3.6.** *Fix $n \geq 1$. The random variable $P_n(\{\delta_{B_t^\kappa}\}) \in \mathcal{Y}_n$ satisfies the large deviation principle as $\kappa \to \infty$, with good rate function $I_n : \mathcal{Y}_n \to \mathbb{R}$ defined by*

$$I_n((\mu^i)_{i \in \mathcal{I}_n}) := \frac{1}{2^n} \sum_{i \in \mathcal{I}_n} I(\mu_i), \tag{3.11}$$

*where $I : \mathcal{M}_1(S^1) \to \mathbb{R}$ is the good rate function defined in (1.5).*

*Proof.* Since the large deviation rate function $I$ is rotation invariant, the same rate function is applicable to the setting of the occupation measure of Brownian motion started at any $\zeta \in S^1$. Furthermore, the Markov property of Brownian motion tells us that conditioned on the value $B_{j/2^n}^\kappa$, the process $(B_t^\kappa)_{[j/2^n,(j+1)/2^n]}$ is independent of $(B_t^\kappa)_{[0,j/2^n]}$. These observations, together with Theorem 3.4, yield the lemma. $\square$

Since $\mathcal{N} = \varprojlim \mathcal{Y}_n$, we can deduce the large deviation principle for $\{\delta_{B_t^\kappa}\}$.

**Proposition 3.7.** *The random measure $\{\delta_{B_t^\kappa}\} \in \mathcal{N}$ has the large deviation principle with good rate function*

$$\sup_{n \geq 0} I_n(P_n(\rho)) \quad \text{for } \rho \in \mathcal{N},$$

*where $I_n : \mathcal{Y}_n \to \mathbb{R}$ is defined in (3.11).*

*Proof.* This follows from the Dawson-Gärtner theorem [4] (or [5, Thm 4.6.1]), the fact that $\mathcal{N} = \varprojlim \mathcal{Y}_n$ by Lemma 3.1, and the large deviation principle for $P_n(\{\delta_{B_t^\kappa}\})$ (Lemma 3.6). $\square$

Finally, we can simplify the rate function $\sup_{n \geq 0} I_n(P_n(\rho))$.

**Lemma 3.8.** *Define* $\mathcal{E} : \mathcal{N} \to \mathbb{R}$ *by* $\mathcal{E}(\rho) := \int_0^1 I(\rho_t)\, dt$, *where* $\{\rho_t\}$ *is any disintegration of* $\rho$ *with respect to* $t$ *(see* (2.1)). *Then, with* $I_n : \mathcal{Y}_n \to \mathbb{R}$ *defined as in* (3.11), *we have*

$$\mathcal{E}(\rho) = \sup_{n \geq 0} I_n(P_n(\rho)).$$

*Proof.* By definition we have $P_n^i(\rho) = \frac{1}{2}(P_{n+1}^{2i}(\rho) + P_{n+1}^{2i+1}(\rho))$, so Jensen's inequality applied to the convex function $I$ yields $I_n(P_n(\rho)) \leq I_{n+1}(P_{n+1}(\rho))$, and hence

$$\sup_{n \geq 0} I_n(P_n(\rho)) = \lim_{n \to \infty} I_n(P_n(\rho)).$$

Next, we check that $\mathcal{E}(\rho) \geq \lim_{n \to \infty} I_n(P_n(\rho))$. This again follows from Jensen's inequality:

$$I_n(\rho) = \frac{1}{2^n} \sum_{i \in \mathcal{I}_n} I\left( 2^n \int_{i/2^n}^{(i+1)/2^n} \rho_t\, dt \right) \leq \sum_{i \in \mathcal{I}_n} \int_{i/2^n}^{(i+1)/2^n} I(\rho_t)\, dt = \mathcal{E}(\rho).$$

Thus, we are done once we prove the reverse inequality $\mathcal{E}(\rho) \leq \lim_{n \to \infty} I_n(P_n(\rho))$.

Consider the probability space given by $[0,1]$ endowed with its Borel $\sigma$-algebra $\mathcal{F}_\infty$, and let $\mu$ be the $\mathcal{M}_1(S^1)$-valued random variable defined by sampling $t \sim \mathrm{Leb}_{[0,1]}$ then setting $\mu := \rho_t$. Let $\mathcal{F}_n$ be the $\sigma$-algebra generated by sets of the form $[i/2^n, (i+1)/2^n]$ for $i \in \mathcal{I}_n$; note that $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$. Define $\mu_n := \mathbb{E}\left[\mu \big| \mathcal{F}_n\right]$. For any continuous function $f \in C(S^1)$, the bounded real-valued Doob martingale $\mu_n(f)$ converges a.s. to $\mu(f)$. Taking a suitable countable collection of $f$, we conclude that a.s. $\mu_n$ converges to $\mu$ in the Prokhorov topology. By Fatou's lemma and the lower-semicontinuity of $I$, we have

$$\liminf_{n \to \infty} \mathbb{E}[I(\mu_n)] \geq \mathbb{E}[\liminf_{n \to \infty} I(\mu_n)] \geq \mathbb{E}[I(\mu)]. \tag{3.12}$$

We can write $\mu_n$ explicitly as $\mu_n = 2^n \int_{i/2^n}^{(i+1)/2^n} \rho_t\, dt$ where $i \in \mathcal{I}_n$ is the index for which $t \in [i/2^n, (i+1)/2^n]$, so $\mathbb{E}[I(\mu_n)] = I_n(P_n(\rho))$. We also have $\mathbb{E}[I(\mu)] = \int_0^1 I(\rho_t)\, dt = \mathcal{E}(\rho)$. Combining these with (3.12), we conclude that $\lim_{n \to \infty} I_n(P_n(\rho)) \geq \mathcal{E}(\rho)$. $\qquad\square$

*Proof of Theorem 1.2.* Proposition 3.7 says that $\{\delta_{B_t^\kappa}\} \in \mathcal{N}$ has a large deviation principle with good rate function given by $\sup_{n \geq 0} I_n(P_n(\cdot))$, and Lemma 3.8 shows that this good rate function can alternatively be expressed as $\mathcal{E}$. $\qquad\square$

## 4  Comments

Let us make further comments and list a few questions in addition to those in the introduction.

1. As we have discussed in the introduction, one may wonder what the limit and large deviations of chordal SLE$_\infty$ are. Figure 1 shows two chordal SLE$_\kappa$ curves on $[-1,1]^2$ from a boundary point $-i$ to another boundary point $i$, for several large values of $\kappa$. We see that the interfaces stretch out to the target point and are close to horizontal lines after we map the square to $\mathbb{H}$ and the target point $i$ to $\infty$.

2. Corollary 1.3 shows that SLE$_\infty$ concentrates around the family of Loewner chains driven by an absolutely continuous measure $\rho$ with $\mathcal{E}(\rho) < \infty$. In [17] we geometrically characterize the Loewner chains driven by such measures. Note that the answer to the same question for the large deviation rate function of SLE$_{0+}$, namely the family of Jordan curves of finite Loewner energy, is well-understood. That family has been shown to be exactly the family of Weil-Petersson quasicircles [19], which has far-reaching connections to geometric function theory and Teichmüller theory.
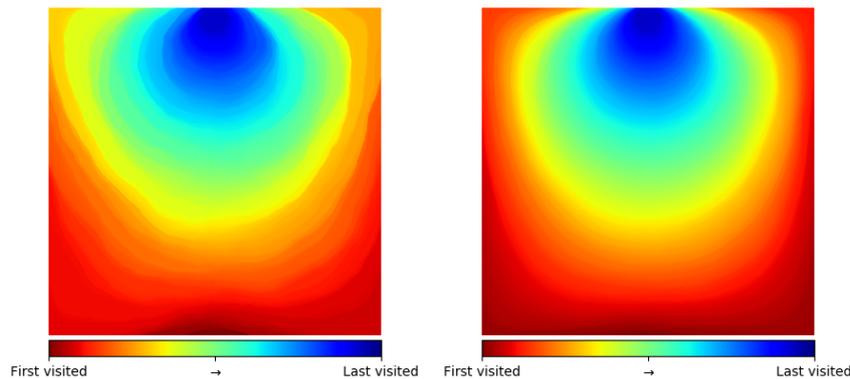
Figure 1: An instance of chordal SLE$_{128}$ and SLE$_{1000}$ on $[-1,1]^2$ from $-i$ to $i$. The simulation of these *counterflow lines* is done by imaginary geometry as described in [11], and are approximated via linear interpolation of an $800 \times 800$ discrete Gaussian free field with suitable boundary conditions. The color represents the time (capacity) parametrization of the SLE curve.

3. The rate function (1.5) for the Brownian occupation measure coincides with the rate function of the *square* of the Brownian bridge (or Gaussian free field) on $S^1$. Is there a profound reason or is this merely a coincidence? One could attempt to relate the large deviations of the Brownian occupation measure to the large deviations of the occupation measure of a *Brownian loop soup* on $S^1$.

4. The fluctuations of the circular Brownian occupation measure $L_t^\kappa$ were studied by Bolthausen. We express this result in terms of the *local time* $\ell_t : S^1 \to [0, \infty)$, defined via $L_t^1 = \ell_t(\zeta)d\zeta$. Note that $\ell_t$ is a.s. a random continuous function.

**Theorem 4.1** ([3]). *Identify each $\zeta = e^{i\theta} \in S^1$ with $\theta \in [0, 2\pi)$. As $t \to \infty$, the stochastic process $\sqrt{t}(\frac{\ell_t(\theta)}{t} - \frac{1}{2\pi})_{\theta \in [0, 2\pi)}$ converges in distribution to $(2b_\theta - \frac{1}{\pi}\int_0^{2\pi} b_\tau \, d\tau)_{\theta \in [0, 2\pi)}$, where $b$ is a Brownian bridge on the interval $[0, 2\pi]$ with endpoints pinned at $b_0 = b_{2\pi} = 0$.*

We wonder whether there are interesting consequences to the fluctuations of SLE$_\infty$.

# References

[1] Vincent Beffara, *Mouvement brownien plan, SLE, invariance conforme et dimensions fractales*, Ph.D. thesis, Paris 11, 2003.

[2] Patrick Billingsley, *Probability and measure*, third ed., Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1995, A Wiley-Interscience Publication. MR-1324786

[3] Erwin Bolthausen, *On the global asymptotic behavior of brownian local time on the circle*, Transactions of the American Mathematical Society **253** (1979), 317–328. MR-0536950

[4] Donald A. Dawson and Jürgen Gärtner, *Large deviations from the McKean-Vlasov limit for weakly interacting diffusions*, Stochastics: An International Journal of Probability and Stochastic Processes **20** (1987), no. 4, 247–308. MR-0885876

[5] Amir Dembo and Ofer Zeitouni, *Large deviations techniques and applications*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2010. MR-2571413

[6] Monroe D. Donsker and S.R. Srinivasa Varadhan, *Asymptotic evaluation of certain Markov process expectations for large time, I*, Communications on Pure and Applied Mathematics **28** (1975), no. 1, 1–47. MR-0386024

[7] Monroe D. Donsker and S.R. Srinivasa Varadhan, *A law of the iterated logarithm for total occupation times of transient brownian motion*, Communications on Pure and Applied Mathematics **33** (1980), no. 3, 365–393. MR-0562740

[8] Julien Dubédat, *Duality of Schramm-Loewner evolutions*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 5, 697–724. MR-2571956

[9] Fredrik Johansson Viklund, Alan Sola, and Amanda Turner, *Scaling limits of anisotropic Hastings–Levitov clusters*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques **48** (2012), no. 1, 235–257. MR-2919205

[10] Gregory F. Lawler, *Conformally invariant processes in the plane*, Mathematical Surveys and Monographs, vol. 114, American Mathematical Society, Providence, RI, 2005. MR-2129588

[11] Jason Miller and Scott Sheffield, *Imaginary geometry I: interacting SLEs*, Probab. Theory Related Fields **164** (2016), no. 3-4, 553–705. MR-3477777

[12] Jason Miller and Scott Sheffield, *Quantum Loewner evolution*, Duke Mathematical Journal **165** (2016), no. 17, 3241–3378. MR-3572845

[13] Christian Pommerenke, *Über die subordination analytischer funktionen.*, Journal für die reine und angewandte Mathematik **218** (1965), 159–173. MR-0180669

[14] Steffen Rohde and Oded Schramm, *Basic properties of SLE*, Ann. of Math. (2) **161** (2005), no. 2, 883–924. MR-2153402

[15] Steffen Rohde and Yilin Wang, *The Loewner energy of loops and regularity of driving functions*, Int. Math. Res. Not. (IMRN) (2019).

[16] Oded Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math. **118** (2000), 221–288. MR-1776084

[17] Fredrik Viklund and Yilin Wang, *Duality of Loewner-Kufarev and Dirichlet energies via foliations by Weil-Petersson quasicircles*, In preparation.

[18] Yilin Wang, *The energy of a deterministic Loewner chain: reversibility and interpretation via* SLE$_{0+}$, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 7, 1915–1941. MR-3959854

[19] Yilin Wang, *Equivalent descriptions of the Loewner energy*, Invent. Math. **218** (2019), no. 2, 573–621. MR-4011706

[20] Dapeng Zhan, *Duality of chordal SLE*, Invent. Math. **174** (2008), no. 2, 309–353. MR-2439609

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