Exponential growth and continuous phase transitions for the contact process on trees

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Abstract

We study the supercritical contact process on Galton-Watson trees and periodic trees. We prove that if the contact process survives weakly then it dominates a supercritical Crump-Mode-Jagers branching process. Hence the number of infected sites grows exponentially fast. As a consequence we conclude that the contact process dies out at the critical value λ_1 for weak survival, and the survival probability $p(\lambda)$ is continuous with respect to the infection rate λ. Applying this fact, we show the contact process on a general periodic tree experiences two phase transitions in the sense that $\lambda_1 < \lambda_2$, which confirms a conjecture of Stacey's [12]. We also prove that if the contact process survives strongly at λ then it survives strongly at a $\lambda' < \lambda$, which implies that the process does not survive strongly at the critical value λ_2 for strong survival.

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1 Introduction

Harris [6] introduced the contact process on $\mathbb{Z}^d$ in 1974, which has been extensively studied since then. The contact process on a graph is usually viewed as a model that describes the spread of an infection. Vertices of the graph represent individuals and the states 0 and 1 indicate that an individual is healthy or infected. The contact process can be defined on any graph as follows: infected sites become healthy at rate 1, while healthy sites become infected at rate λ times the number of infected neighbors.

Pemantle [10] began the study of the contact process on trees and found that there exist two critical values

$$\lambda_1 = \inf\{\lambda : P(\xi_t^0 \neq \varnothing \text{ for all } t) > 0\}$$
$$\lambda_2 = \inf\{\lambda : \liminf_{t \to \infty} P(0 \in \xi_t^0) > 0\},$$

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where ξ_t^0 denotes the contact process on the tree starting from only the root infected. The contact process is said to survive *weakly* if the process survives but the root 0 is infected for finitely many times almost surely, and survive *strongly* if the root 0 is infected for infinitely many times with positive probability. It is natural to guess that the contact process dies out at λ_1 and does not survive strongly at λ_2. This is true for the contact process on a d-regular tree $\mathbb{T}^d$ where each vertex has degree $d+1$. (See Liggett [9] for most of the known results on regular trees.) However, proofs on regular trees rely heavily on translation invariance, and special functions such as $w_\rho(\xi_t) = \sum_{x \in \xi_t} \rho^{\ell(x)}$, where ρ is some positive constant and $\ell(x)$ is a function from $\mathbb{T}^d$ to $\mathbb{Z}$ so that for each $x \in \mathbb{T}^d$, $\ell(y) = \ell(x) - 1$ for exactly one neighbor y of x and $\ell(y) = \ell(x) + 1$ for the other d neighbors y of x. The expectation of $w_\rho(\xi_t)$ satisfies a submultiplicative relation

$$Ew_\rho(\xi_{t+s}) \le Ew_\rho(\xi_t)Ew_\rho(\xi_s)$$

which implies $Ew_\rho(\xi_t) \ge [\phi(\rho)]^t$ where $\phi(\rho) = \lim_{t \to \infty}[Ew_\rho(\xi_t)]^{1/t}$. Properties of $\phi(\rho)$ help us obtain a lot of detailed information about the behavior of the contact process on regular trees.

1.1 Galton-Watson trees

Turning to Galton-Watson trees, the analysis of the contact process becomes more complicated because of the spatial heterogeneity and randomness in the tree structure. Throughout the discussion we consider the contact process on Galton-Watson trees with an offspring distribution D that is a random variable on $\mathbb{N}$ that satisfies

$$P(D \ge 1) = 1 \text{ and } ED > 1. \tag{1.1}$$

A Galton-Watson tree is said to have a *subexponential* offspring distribution if its offspring distribution D satisfies

$$\limsup_{k \to \infty} (\log P(D = k))/k = 0.$$

In this case it is proved in Huang and Durrett [7] that

Theorem 1.1. *If the offspring distribution D for a Galton-Watson tree is subexponential and has mean $\mu > 1$ then $\lambda_1 = \lambda_2 = 0$.*

Clearly when the offspring distribution is subexponential the contact process dies out at both critical values. To give some intuition why $\lambda_2 = 0$ we discuss briefly the case where the offspring distribution satisfies $P(D \ge k) = \exp(-k^\alpha)$ for all $k \ge 0$ where $\alpha < 1$. Within distance k of the root there are roughly μ^k vertices where μ is the mean offspring number. Hence the vertex with the largest degree within distance k has degree $\approx (k \log \mu)^{1/\alpha}$. A vertex with degree n along with its neighbors is called a star graph with degree n. The contact process on a star graph with degree n can survive for time $\exp(C\lambda^2 n)$ for some $C > 0$ when n is sufficiently large (see [2]). Here and in what follows C is a positive constant whose value can change. While surviving on a star graph within distance k of the root, the contact process can try to push the infection back to the root with success probability $(\lambda/(1+\lambda))^k$ for each attempt. Since the process can survive for time $\exp(C\lambda^2(k \log \mu)^{1/\alpha})$ on a star graph with degree $(k \log \mu)^{1/\alpha}$, the probability that it fails to push the infection back to the root is (while omitting some details)

$$\le \left(1 - (\lambda/(1+\lambda))^k\right)^{\exp(C\lambda^2(k \log \mu)^{1/\alpha})} \le \exp\left(-\left(\frac{\lambda}{1+\lambda}\right)^k \exp(C\lambda^2(k \log \mu)^{1/\alpha})\right). \tag{1.2}$$

Since the first term in the exponent decays exponentially while the second grows superexponentially, the failure probability goes to 0 as $k \to \infty$ for any positive values of

λ. That is, the contact process can successfully push the infection back to the root and thus survive strongly for any $\lambda > 0$.

It turns out that Theorem 1.1 is sharp. The critical value is positive on Galton-Watson trees with tails thinner than the subexponential distribution. Recently Shankar Bhamidi, Danny Nam, Oanh Nguyen and Allan Sly [1] proved that

Theorem 1.2. *Consider the contact process on the Galton-Watson tree with offspring distribution D. If $E(\exp(cD)) < \infty$ for some $c > 0$, then $\lambda_1 > 0$.*

An offspring distribution with this property is said to have an *exponential tail*, which is thinner than the tail of the subexponential distribution. For example, suppose $P(D \geq k) = \exp(-ck)$ for $k \geq 0$ and some $c > 0$. Following the previous heuristics the vertex with the largest degree within distance k of the root has degree $\approx (k \log \mu)/c$. Again the contact process can survival for time $\exp(C\lambda^2 k)$ on a star graph with degree $(k \log \mu)/c$ for some $C > 0$, and can push the infection back to the root with failure probability

$$\left(1 - (\lambda/(1+\lambda))^k\right)^{\exp(C\lambda^2 k)} \approx \exp\left(-\left(\frac{\lambda}{1+\lambda}\right)^k \exp(C\lambda^2 k)\right) \qquad (1.3)$$

when λ is sufficiently small. As $\log(\lambda/(1+\lambda)) + C\lambda^2 < 0$ when λ is small, (1.3) does not go to 0 when k goes to infinity. This provides some intuition why λ_1 is strictly positive when the offspring distribution has an exponential tail. Comparing (1.2) and (1.3), we can see the difference between the subexpnential distribution and the exponential-tail distribution is that in the first case the largest degree within distance k (i.e., $(k \log \mu)^{1/\alpha}$) grows superlinearly in k and overwhelms the success probability for pushing the infection back to the root.

Since $\lambda_2 \geq \lambda_1 > 0$ on Galton-Watson trees with exponential tails, it is now interesting to (i) determine if $\lambda_1 < \lambda_2$ and (ii) understand the behavior of the contact process at the critical values λ_1 and λ_2. We have not been able to solve the first difficult problem but we have solved the second.

The first step is to construct a supercritical Crump-Mode-Jagers (CMJ) branching process that is dominated by the contact process. As a consequence, we prove that the number of infected sites in the contact process grows exponentially fast on the event of survival. Let ξ_t^0 be the contact process on $\mathbf{GW}(D)$ with only the root infected initially.

Theorem 1.3. *Suppose the offspring distribution D satisfies (1.1). When $\lambda > \lambda_1(\mathbf{GW}(D))$, there is a positive constant c so that*

$$\liminf_{t \to \infty} |\xi_t^0|/e^{ct} > 0 \quad P_\lambda\text{-a.s. on } \Omega_\infty \equiv \{\xi_t^0 \neq \varnothing \ \forall t \geq 0\}. \qquad (1.4)$$

Here P_λ is the annealed measure which is defined rigorously in Section 2. Suppose an individual in the comparison CMJ branching process gives birth only once and let τ denote that birth time. We can then look at the branching process in terms of time blocks of duration τ and roughly know the growth of this process by time τ. This suggests that a "block argument" should be a good strategy to prove the continuity of the phase transition at λ_1. Indeed, if the contact process ξ_t^λ with infection rate λ survives then it dominates a supercritical CMJ branching process. For a sufficiently small $\varepsilon > 0$, the contact process $\xi_t^{\lambda-\varepsilon}$ should behave very much the same as ξ_t^λ within a finite time block of length τ and thus also dominate a supercritical branching process. Hence we can show that

Theorem 1.4. *The contact process on $\mathbf{GW}(D)$ dies out at $\lambda_1(\mathbf{GW}(D))$ and the survival probability $p(\lambda)$ is continuous in $[0, \infty)$.*

Finally, we prove that

Theorem 1.5. *The contact process on $\mathbf{GW}(D)$ does not survive strongly at $\lambda_2(\mathbf{GW}(D))$.*

The proof of this result is more complicated than that of Theorem 1.4 but similar in spirit. We find the suitable "block events" and construct a comparison process based on the block events. If the contact process with infection rate λ survives strongly then it dominates a comparison process that guarantees strong survival, and hence the contact process survives strongly at $\lambda - \varepsilon$ for sufficiently small $\varepsilon > 0$.

1.2 Periodic trees

Let $\mathbb{T}_\kappa = (a_1, a_2, \ldots, a_\kappa)$ be a periodic tree in which the number of children in successive generations is $a_1, a_2, \ldots, a_\kappa$. We only assume $a_i \geq 1$ for $1 \leq i \leq \kappa$ and $a_1 \cdot a_2 \cdots a_k \neq 1$. In order for the graph to be quasi-transitive and thus simplify the computation, $\mathbb{T}_\kappa = (a_1, a_2, \ldots, a_\kappa)$ is arranged in the way that every vertex x in $\mathbb{T}_\kappa$ has one neighbor above it and $d(x)$ neighbors below it, where $d(x) \in \{a_1, \ldots, a_\kappa\}$ is the offspring number of x.

To describe $\mathbb{T}_\kappa$ more rigorously we will introduce a function $\ell : \mathbb{T}_\kappa \to \mathbb{Z}$ that assigns a *level* to every vertex in $\mathbb{T}_k$. Without loss of generality, a distinguished vertex o is chosen to be the root where $d(o)$ can be any value in $\{a_1, a_2, \ldots, a_\kappa\}$. Now each vertex can be assigned a *level* according to their position relative to the root o. Specifically, we will define the function $\ell : \mathbb{T}_\kappa \to \mathbb{Z}$ so that (i) $\ell(o) = 0$ and (ii) for each $x \in \mathbb{T}_\kappa$, $\ell(y) = \ell(x) - 1$ for exactly one neighbor y of x, and $\ell(y) = \ell(x) + 1$ for the other $d(x)$ neighbors of x. For a vertex $x \in \mathbb{T}_\kappa$, $\ell(x)$ is said to be the level of x.

If we choose the root o to be a vertex with a_1 children, then the periodic tree $\mathbb{T}_\kappa = (a_1, a_2, \ldots, a_\kappa)$ is such that a vertex on level ℓ of $\mathbb{T}_\kappa$ has $a_{(\ell \mod \kappa)+1}$ children.

The treatment for the contact process on Galton-Watson trees can be extended to prove analogous conclusions on periodic trees.

Theorem 1.6. *Let ξ_t^o denote the contact process on $\mathbb{T}_\kappa$ starting with vertex o infected.*
(i) When $\lambda > \lambda_1(\mathbb{T}_\kappa)$, there is a positive constant c so that

$$\liminf_{t \to \infty} |\xi_t^o|/e^{ct} > 0 \quad P_\lambda\text{-a.s. on } \Omega_\infty \equiv \{\xi_t^o \neq \varnothing \ \forall t \geq 0\}. \tag{1.5}$$

(ii) The contact process ξ_t^o on the periodic tree T_κ dies out at $\lambda_1(\mathbb{T}_\kappa)$ and does not survive strongly at $\lambda_2(\mathbb{T}_\kappa)$.

We are also interested in determining if $\lambda_1 < \lambda_2$ on general periodic trees, which is believed to be true by Pemantle [10] and Stacey [12]. Stacey was able to prove $\lambda_1 < \lambda_2$ on what he called "isotropic block trees", which are to some extent similar to periodic trees but have more stringent assumptions on the structure. Intuitively, the assumptions require that there exist automorphisms on the tree that map distinguished vertices to one another. The rigorous definition of the isotropic block trees is a bit involved and hence will not be stated here. Curious readers are referred to page 1718 and 1719 in [12] for a full account.

To see the connection between isotropic block trees and periodic trees we will present two examples of the former.

Example 1. $G = (a_1, a_2, \ldots, a_\kappa)$ where $a_i = 1$ for $1 \leq i \leq \kappa - 1$ and $a_\kappa \geq 2$.

Example 2. $G = (a_1, a_2)$ where $a_1 a_2 \neq 1$, which is a general period-2 tree.

The above examples suggest that the isotropic block trees are, to some extent, similar to periodic trees. However, due to the somewhat stringent assumptions on the tree structure there are still many examples of periodic trees that fall outside this category. One of the simplest examples among them is the periodic (2,3,4) tree, where the the number of children in successive generations is 2,3,4.

The special structure of the isotropic blocks is a key ingredient that Stacey uses in his proof of

$$\exp(\alpha t) \leq E(|\xi_t^A|) \leq C \exp(\alpha t), \tag{1.6}$$

where C is a constant, α is a continuous function of λ and A is a set of vertices chosen specially, see Proposition 2.1 in [12]. For example, for a period-2 tree (a_1, a_2) the initially infected set A is chosen to be a finite tree of depth 2 where the first generation has a_1 offspring and each vertex in the second generation has a_2 offspring. The relation (1.6) implies that $\alpha = 0$ when $\lambda = \lambda_1$. Hence at λ_1 not only $E(|\xi_t^A|) \leq C$ for all $t \geq 0$ but also ξ_t^A dies out on an isotropic block tree. These facts are the key elements in Stacey's proof.

As we will see, the extinction at λ_1 is the real key to the proof of the existence of an intermediate phase. For this reason we can prove $\lambda_1 < \lambda_2$ for general periodic trees without trying to obtain the relation in (1.6), which is a nice relation but stronger than necessary for our purpose. We confirm Stacey's conjecture by showing

Theorem 1.7. *The contact process on the periodic tree T_κ has an intermediate phase in the sense that $\lambda_1 < \lambda_2$.*

The relation $\lambda_1 < \lambda_2$ is believed to hold for a much broader class of trees as discussed in [10] and [12]. It will be meaningful to prove

Conjecture 1. There exists an intermediate phase $\lambda_1 < \lambda_2$ on Galton-Watson trees whose offspring distribution has an exponential tail.

Part of the difficulty in proving this conjecture lies in the correlation between the state of a vertex and the geometry near this vertex. In an inhomogeneous graph such as a Galton-Watson tree, the fact that a vertex v is infected at a certain time would imply that it is more likely for v itself to have a large degree or to be connected to some vertices with large degrees. Our method requires a certain extent of independence between the state of a vertex and the geometry nearby, and hence does not extend to a highly inhomogeneous graph.

The outline of the rest of the paper is as follows. In Section 2, we will give an explicit construction of the contact process on Galton-Watson trees. Theorems 1.3, 1.4 and 1.5 will be proved in Section 3, 4 and 5, respectively. Periodic trees will be discussed in Section 6.

2 Model definition

The set of all realizations of the Galton-Watson trees with offspring distribution D is denoted by $\mathbf{GW}(D)$ while a certain realization is denoted by $\mathbb{T} \in \mathbf{GW}(D)$. Condition (1.1) guarantees that $\mathbb{T}$ is infinite almost surely. Let $P_{\mathbb{T},\lambda}$ be the law of the contact process with infection rate λ on the tree $\mathbb{T}$, and let μ be the probability measure over the realizations in $\mathbf{GW}(D)$. We define the annealed measure P_λ to be

$$P_\lambda(\ \cdot\ ) = E_{\mathbb{T}\sim\mu}(P_{\mathbb{T},\lambda}(\ \cdot\ )) = \sum_{\mathbb{T}\in\mathbf{GW}(D)} P_{\mathbb{T},\lambda}(\ \cdot\ )\mu(\mathbb{T}).$$

To simplify notation we will drop the index λ when the context is clear and use $\mathbf{GW}(D)$ to also represent a Galton-Watson tree with offspring distribution D.

Let ξ_t^0 denote the contact process on $\mathbf{GW}(D)$ with initially only the root infected. The critical value for weak survival for the contact process on $\mathbb{T}$ is defined as

$$\lambda_1(\mathbb{T}) = \inf\{\lambda : P_{\mathbb{T},\lambda}(\xi_t^0 \neq \varnothing \text{ for all } t \geq 0) > 0\}.$$

In fact, $\lambda_1(\mathbb{T})$ does not depend on the specific realization $\mathbb{T} \in \mathbf{GW}(D)$. That is, $\lambda_1(\mathbb{T})$ is a constant μ-almost surely. Pemantle proved a slightly different version of this result in Proposition 3.1 in [10]. We will follow his proof to show

Proposition 1. $\lambda_1(\mathbb{T})$ is a constant μ-almost surely.

Proof. Let $q(\mathbb{T}) = P_{\mathbb{T},\lambda}(\xi_t^0 \neq \varnothing \; \forall t \geq 0)$ and $p_1 = P_\lambda(q = 0)$. If the root has n children then there are n independent identically distributed subtrees that are themselves Galton-Watson trees with offspring distribution D, which we denote by $\mathbb{T}_1, \ldots, \mathbb{T}_n$. If $q(\mathbb{T}_i) > 0$ for some $1 \leq i \leq n$ then $q(\mathbb{T}) \geq \frac{\lambda}{1+\lambda} q(\mathbb{T}_i) > 0$, where the first term is the probability to push the infection to the root of $\mathbb{T}_i$. It follows that if the contact process dies out on $\mathbb{T}$ then it has to die out on all n subtrees $\mathbb{T}_1, \ldots, \mathbb{T}_n$.

Let f be the generating function of D and let $a_k = P(D = k)$. By the independence of the structures of the subtrees,

$$p_1 = P_\lambda(q = 0) \leq \sum_n a_n (p_1)^n = f(p_1) \tag{2.1}$$

Note that $f'(1) = ED > 1$ by (1.1). (2.1) combined with $f'(1) > 1$ and $f(0) = 0$ implies either $p_1 = 1$ or p_1 is bounded above by the unique fixed point of f in $(0,1)$, which then gives $p_1 = 0$. Hence $\lambda_1(\mathbb{T})$ is a constant μ-almost surely. $\square$

The critical value for strong survival is defined as

$$\lambda_2(\mathbb{T}) = \inf\{\lambda : P_{\mathbb{T},\lambda}(0 \in \xi_t^0 \;\; \text{infinitely often}) > 0\}.$$

It follows from the same reasoning as in the proof of Proposition 1 that

Proposition 2. $\lambda_2(\mathbb{T})$ is a constant μ-almost surely.

From now on we write $\lambda_1(\mathbf{GW}(D))$ and $\lambda_2(\mathbf{GW}(D))$ for the critical values for the contact process on a Galton Watson tree with offspring distribution D.

3 Exponential growth

In this section we prove Theorem 1.3 by showing that the contact process dominates a supercritical Crump-Mode-Jagers branching process. On an inhomogeneous structure such as the Galton-Watson tree, the infection state of a vertex x is correlated with the tree structure near x. To address this issue we observe that starting with only the root infected, when a vertex x becomes infected for the first time no information is known about the structure of the subtree rooted at x (i.e., the subtree made of x and all of its descendants). Hence we can "decouple" the infection state of x and the tree structure below it.

For a vertex $x \in \mathbb{T}$ we define $S(x)$ to be the *subtree* with root x, that is, $S(x)$ contains x together with all the descendants of x on $\mathbb{T}$. For a finite set $A \subset \mathbb{T}$, we define its *frontier* $F(A)$ to be the set of points $x \in A$ for which at least one of its children, say x', has $S(x') \cap A = \varnothing$. Let A' denote the set of x' such that x' is the child of some $x \in A$ and $S(x') \cap A = \varnothing$.

Let $A_t = \cup_{s \leq t} \xi_s^0$ be the set of vertices ever infected by time t. We are interested in the set of vertices that have never been infected and are accessible to the infection ξ_t^0 at time t, i.e.,

$$B_t = \{x \in (A_t)' : \text{the parent of } x \text{ is in } \xi_t^0 \cap F(A_t)\}.$$

Let $\tau_k = \inf\{t \geq 0 : |\xi_t^0 \cap F(A_t)| \geq k\}$. The fact that each vertex in $|\xi_t^0 \cap F(A_t)|$ corresponds to at least one vertex in B_t gives $|B_{\tau_k}| \geq k$, i.e., there are at least k unexplored subtrees accessible to the contact process $\xi_{\tau_k}^0$. At time τ_k, ξ_t^0 can try to start infections on each

of the k subtrees, among which each infection could survive forever with a positive probability equal to $P_\lambda(\xi_t^0 \neq \varnothing \ \forall t \geq 0)$. When k is taken to be sufficiently large, at time τ_k the contact process ξ_t^0 gives birth to an expected number of more than one new process that lives forever, which make up an underlying branching process.

Recall that $\Omega_\infty = \{\xi_t^0 \neq \varnothing \ \forall t \geq 0\}$. We will begin the proof of Theorem 1.3 by showing

Lemma 3.1. *Suppose $\lambda > \lambda_1(\mathbf{GW}(D))$. If for every $k \in \mathbb{N}$, $\tau_k < \infty$ P_λ-almost surely on Ω_∞ then (1.4) holds.*

Proof. When $\lambda > \lambda_1(\mathbf{GW}(D))$ we have $\rho := P_\lambda(\xi_t^0 \neq \varnothing \ \forall t \geq 0) > 0$. Observe that with probability $\geq e^{-2}(1 - e^{-\lambda})$ an infected vertex $x \in \xi_t^0 \cap F(A_t)$ can infect its child $x' \in B_t$ within time 1 and x' will stay infected until time 1.

Since the subtree $S(x')$ is still unexplored when $x' \in B_t$ first becomes infected, we can first generate $S(x')$ according to the measure μ of $\mathbf{GW}(D)$ and then start a contact process on it. The probability that the process $\xi_t^{x'}$ restricted to $S(x')$ will survive is

$$E_{S(x') \sim \mu}(P_{S(x'),\lambda}(\xi_t^{x'} \neq t \text{ for all } t \geq 0)) = \rho.$$

If this infection on $S(x')$ survives, we say x' is a *particle with an infinite line of descendants*. Note that each particle with an infinite line of descendants will contribute at least size 1 to $|\xi_t^0|$ at all times. So it suffices to show the number of particles with an infinite line of descendants grows exponentially in time on the event Ω_∞.

We will construct a branching process Z_t that is dominated by the set of particles that has an infinite line of descendants. Let Z_0 contain only the root 0. Since $\{\tau_k < \infty\}$ almost surely on Ω_∞ we can choose M_1 large so that $P(\tau_k < M_1) \geq \rho/2$. If $\tau_k \geq M_1$ we discard the initial particle and there are no offspring. If $\tau_k < M_1$ then the initial particle gives birth to $X \sim Binomial(k, e^{-2}(1 - e^{-\lambda}))$ offspring at time τ_k. This is because the birth events of different vertices in $\xi_t^0 \cap F(A_t)$ are independent, since they involve vertices that are disjoint in space. If k is chosen large enough then the number of offspring $\bar{X}$ that will start a new process on its subtree with $\tau_k < M_1$ has expectation

$$E\bar{X} \geq ke^{-2}(1 - e^{-\lambda})\rho/2 > 1,$$

which implies that Z_t is supercritical.

The branching process is a Crump-Mode-Jagers (CMJ) branching process [3] because the initial particle gives birth at time τ_k. If $N(t)$ is the expected number of births by time t then the Malthusian parameter c is defined through

$$\int_0^\infty e^{-ct} dN(t) = 1.$$

Since our process has no births after time M_1 it is easy to see that it satisfies the conditions in Theorem 2 of [4], which gives

$$W(t) := \frac{Z_t}{e^{ct}} \to W \quad \text{a.s. as } t \to \infty,$$

where W is a non-negative random variable with $P(W > 0) > 0$. To improve this to the desired conclusion we note that for any $m < \infty$, $\tau_m < \infty$ almost surely on Ω_∞. At time τ_m, each vertex in $\xi_t^0 \cap F(A_t)$ can start an independent branching process construction with probability at least $e^{-2}(1 - e^{-\lambda})$ and each branching process will live forever with probability $P(W > 0)$. As long as one of the branching processes lives forever we will have the desired exponential growth of ξ_t^0. Hence

$$P\left(\liminf_{t \to \infty} |\xi_t^0|/e^{ct} > 0\right) \geq P(\Omega_\infty)P(Binomial(m, e^{-2}(1 - e^{-\lambda})P(W > 0)) \geq 1)$$

$$= P(\Omega_\infty)\left(1 - \left(1 - e^{-2}(1 - e^{-\lambda})P(W > 0)\right)^m\right)$$

for any $m \in \mathbb{N}$. Letting $m \to \infty$ and noting that the limit can only be positive on Ω_∞ proves the desired result. $\qquad\square$

Having established Lemma 3.1, to complete the proof of Theorem 1.3 it remains to show

Lemma 3.2. *For any $k \in \mathbb{N}$, $\tau_k < \infty$ almost surely on Ω_∞.*

Proof. In a contact process, an infected site is also referred to as a *particle*. The event that an infected site passes the infection to a susceptible site is also described as a particle *gives birth* to another particle. The recovery of an infected site is hence also referred to as the *death* of a particle. In this proof we adopt the narrative of the particles.

If some particle $x \in \xi_t^0 \cap F(A_t)$ gives birth to a particle $y \in B_t$, then at the moment of this birth y is added to $\xi_t^0 \cap F(A_t)$. In the worst case y is the last unexplored child of x, so x is removed from $\xi_t^0 \cap F(A_t)$ and the size of $\xi_t^0 \cap F(A_t)$ remains the same. Note that the size of $\xi_t^0 \cap F(A_t)$ will not decrease at a birth event.

When $y \in B_t$ becomes infected, we try to grow $|\xi_t^0 \cap F(A_t)|$ through consecutive birth events on the subtree $S(y)$. Since $S(y)$ has not been explored by the infection, every vertex in $S(y)$ still has all of its offspring unexplored. When a children of y (except the last one) is infected, the size of $\xi_t^0 \cap F(A_t)$ increases by 1. Hence if every particle gives birth onto a particle with offspring number at least 2, we need at most $2k$ consecutive birth events on $S(y)$ for $|\xi_t^0 \cap F(A_t)|$ to reach size k.

Taking account of the first birth event from x to y, we need $2k+1$ consecutive birth events before any death occurs and each of them is onto a vertex with offspring number at least 2. Since each vertex in $S(y) \cap \xi_t^0 \cap F(A_t)$ is connected to at least one vertex outside, the probability that a birth occurs before any death in $S(y) \cap \xi_t^0 \cap F(A_t)$ has probability $\geq \lambda/(1+\lambda)$.

We start the first trial at the first time when $|\xi_t^0 \cap F(A_t)| \geq 1$, i.e., when $t = 0$. A trial is said to have failed if a death occurs or a birth to a vertex with offspring number 1 occurs before we have $2k+1$ consecutive birth events to vertices with offspring number at least 2. Define the running time S_i of the i-th trial to be the first time either this trial fails or succeeds. S_i is dominated by the sum of $2k+1$ i.i.d. $Exp(1+\lambda)$ random variables and hence is finite almost surely. Let $T_0 = 0$ and define

$$T_i = \inf\{t \geq T_{i-1} + S_{i-1} : |\xi_t^0 \cap F(A_t)| \geq 1\}$$

to be the time when the i-th trial starts. If $T_i = \infty$ for some $i \geq 1$ then set $T_j = \infty$ for all $j \geq i$. Note that with an independent probability

$$\geq \left(\frac{\lambda}{1+\lambda} \cdot P(D \geq 2) \right)^{2k+1} \equiv \alpha_k$$

a trial will succeed, so it takes finitely many attempts to have one success, which then gives $|\xi_t^0 \cap F(A_t)| \geq k$. If the i-th trial fails, either we still have $|\xi_t^0 \cap F(A_t)| \geq 1$, which means $T_{i+1} = T_i + S_i$, or $|\xi_t^0 \cap F(A_t)| = 0$ and we have to wait until $|\xi_t^0 \cap F(A_t)|$ becomes 1 before starting the next trial. On the event that ξ_t^0 survives, infinitely many sites will be infected, i.e.,

$$\lim_{t\to\infty} |A_t| = \infty \quad \text{a.s. on } \Omega_\infty.$$

Since A_t cannot grow when $\xi_t^0 \cap F(A_t) = \varnothing$ we have $|\xi_t^0 \cap F(A_t)| \geq 1$ infinitely often on the event Ω_∞, meaning that for any $n \in \mathbb{N}$

$$\{T_n < \infty\} \quad \text{a.s. on } \Omega_\infty.$$

Hence we have infinitely many trials almost surely on the event of survival. Since the birth and death symbols involved in different trials come from regions disjoint in space and time, each trial succeeds with probability $\geq \alpha_k$ independent of the fates of previous trials. Therefore we will have a success in finite time almost surely on Ω_∞. The proof of Lemma 3.2 is complete and Theorem 1.3 has been established. $\square$

Remark 3.3. Our method is robust in the sense that it works on a broad class of trees. For example, in order to prove exponential growth for the contact process on periodic trees it suffices to extend the proof of Lemma 3.2 to periodic trees. The key element in the proof of Lemma 3.2 is the expansion of the infected frontier $\xi_t^0 \cap F(A_t)$ through consecutive birth events, which holds true for periodic trees too. It is easy to see that if $a_i \geq 2$ for all $1 \leq i \leq \kappa$ in the periodic tree $\mathbb{T}_\kappa = (a_1, \ldots, a_\kappa)$, then each birth event in the infected frontier will cause the size $|\xi_t^0 \cap F(A_t)|$ to increase by 1. If we only assume $a_1 a_2 \cdots a_\kappa \neq 1$, then we need κ consecutive birth events in the frontier to ensure that the size $|\xi_t^0 \cap F(A_t)|$ increases by 1.

4 Continuous survival probability

In the branching process Z_t in the proof of Lemma 3.1, a particle can only give birth within time M_1, which means we are only looking at a finite time "block" in the corresponding graphical representation. We drop the superscript 0 and let ξ_t^λ denote the contact process with infection rate λ and recovery rate 1 starting from the root infected. Within this "block" the process ξ_t^λ and $\xi_t^{\lambda'}$ should behave very much the same if λ' is sufficiently close to λ. If ξ_t^λ survives with positive probability, tuning down the infection rate λ slightly to λ' the resulting process $\xi_t^{\lambda'}$ should still behave similarly in the finite time "block", which then implies the survival of $\xi_t^{\lambda'}$. We will follow this idea to prove both results in this section. A superscript λ is used to denote the infection rate. For example, τ_k^λ is the corresponding hitting time of the event $\{|\xi_t^\lambda \cap F(A_t)| \geq k\}$.

Lemma 4.1. *The contact process on $\mathbf{GW}(D)$ dies out at $\lambda_1(\mathbf{GW}(D))$.*

Proof. To prove the conclusion it suffices to show $P_\lambda(\xi_t^0 \neq \varnothing$ for all $t \geq 0) > 0$ implies $\lambda > \lambda_1$. Let $\rho := P_\lambda(\xi_t^0 \neq \varnothing$ for all $t \geq 0) > 0$. For reasons that will become clear later we choose k, M_1 so that

$$ke^{-2}(1 - e^{-\lambda})\rho/2 > 2 \quad \text{and} \quad P(\tau_k^\lambda < M_1) \geq \rho/2. \tag{4.1}$$

For a small $\delta > 0$ we have

$$P(\tau_k^{\lambda-\delta} < M_1) \geq P(\{\xi_t^\lambda = \xi_t^{\lambda-\delta} \text{ for all } t \leq \min\{\tau_k^\lambda, M_1\}\} \cap \{\tau_k^\lambda < M_1\}).$$

For any $\varepsilon > 0$, when δ is sufficiently small we have

$$P(\xi_t^\lambda \neq \xi_t^{\lambda-\delta} \text{ for some } t \leq \min\{\tau_k^\lambda, M_1\}) < \varepsilon.$$

Hence,

$$P(\tau_k^{\lambda-\delta} < M_1) \geq 1 - \varepsilon - (1 - \rho/2) = \rho/2 - \varepsilon.$$

Following the proof of Lemma 3.1 we can construct a branching process Z_t dominated by $\xi_t^{\lambda-\delta}$, where a particle gives birth to $X \sim Binomial(k, e^{-2}(1 - e^{-\lambda-\delta}))$ offspring if $\tau_k^{\lambda-\delta} < M_1$, and to no offspring if $\tau_k^{\lambda-\delta} \geq M_1$. When ε and δ are sufficiently small, by the choice of k in (4.1) the mean number of offspring is

$$EX \geq ke^{-2}(1 - e^{-\lambda-\delta})(\rho/2 - \varepsilon) > 1.$$

Hence Z_t is supercritical and survives with positive probability, which implies $\xi_t^{\lambda-\delta}$ survives with positive probability. So $\lambda > \lambda - \delta \geq \lambda_1$ and this completes the proof. $\square$

Let $p(\lambda) := P_\lambda(\xi_t^0 \neq \varnothing \ \forall t \geq 0)$ be a function of λ that represents the survival probability of the contact process ξ_t^λ.

Lemma 4.2. $p(\lambda)$ *is continuous on* $[0, \infty)$.

Proof. The right continuity of $p(\lambda)$ is immediate since

$$P_\lambda(\xi_t^0 \neq \varnothing) \downarrow p(\lambda)$$

as $t \uparrow \infty$ and $P_\lambda(\xi_t^0 \neq \varnothing)$ is increasing and continuous in λ for each t.

We have proved that $p(\lambda) \equiv 0$ on $[0, \lambda_1]$, so it remains to prove the left continuity for $p(\lambda)$ on (λ_1, ∞). Take $\lambda > \lambda_1$, which trivially gives $p(\lambda) > 0$. We will show for any $\varepsilon > 0$, there exists $\lambda' < \lambda$ such that $p(\lambda') > p(\lambda) - \varepsilon$.

Since $\lambda > \lambda_1$ there exists λ_0 such that $\lambda > \lambda_0 > \lambda_1$. It follows from the same argument as in the proof of Lemma 4.1 that we can construct a supercritical branching process Z_t that is dominated by $\xi_t^{\lambda_0}$. Moreover, there exists some constant $c > 0$ and a non-negative random variable W_0 with $P(W_0 > 0) > 0$ such that

$$\frac{Z_t}{e^{ct}} \to W_0 \quad \text{as } t \to \infty.$$

To simplify notation we will write $\sigma = e^{-2}(1 - e^{-\lambda_0})P(W_0 > 0)$. For a given $\delta > 0$ we will choose m so that $(1 - \sigma)^m < \delta$. Since $\tau_m^\lambda < \infty$ almost surely on $\Omega_\infty \equiv \{\xi_t^\lambda \neq \varnothing \ \forall t \geq 0\}$, we can choose M_2 sufficiently large so that

$$P(\tau_m^\lambda \leq M_2) \geq (1 - \delta)p(\lambda). \tag{4.2}$$

When $\lambda' < \lambda$ is sufficiently close to λ we also have

$$P(\xi_t^\lambda \neq \xi_t^{\lambda'} \text{ for some } t \leq M_2) < \delta. \tag{4.3}$$

Without loss of generality we can assume $\lambda' > \lambda_0$. Observe that each vertex in the frontier $\xi_t^{\lambda'} \cap F(A_t)$ can start an independent branching process with probability $\geq e^{-2}(1 - e^{-\lambda_0})$ and each branching process lives forever with probability at least $P(W_0 > 0)$ since $\lambda' > \lambda_0$. On the event $\{\xi_t^\lambda = \xi_t^{\lambda'} \text{ for } t \leq M_2\} \cap \{\tau_m^\lambda \leq M_2\}$, the process $\xi_t^{\lambda'}$ can give birth to $X \sim Binomial(m, \sigma)$ branching processes that survive. If $X \geq 1$ then the process $\xi_t^{\lambda'}$ survives since it dominates a surviving branching process. Hence

$$p(\lambda') \geq P\left(\{\xi_t^\lambda = \xi_t^{\lambda'} \text{ for } t \leq M_2\} \cap \{\tau_m^\lambda \leq M_2\}\right) P(Binomial(m, \sigma) \geq 1) \tag{4.4}$$

It follows from (4.2) and (4.3) that when λ' is sufficiently close to λ

$$P\left(\{\xi_t^\lambda = \xi_t^{\lambda'} \text{ for } t \leq M_2\} \cap \{\tau_m^\lambda \leq M_2\}\right) \geq (1 - \delta)p(\lambda) - \delta,$$

which then gives

$$p(\lambda') \geq ((1 - \delta)p(\lambda) - \delta)(1 - (1 - \sigma)^m) \geq ((1 - \delta)p(\lambda) - \delta)(1 - \delta).$$

Therefore, for any given $\varepsilon > 0$ there exists δ sufficiently small and λ' sufficiently close to λ so that $p(\lambda') > p(\lambda) - \varepsilon$. $\square$

Remark 4.3. The same construction can be carried out on periodic trees straightforwardly, thus proving the continuity of survival probability $p(\lambda)$ for the contact process on periodic trees. As a consequence, the contact process dies out at λ_1 on periodic trees.

5 No strong survival at λ_2

We start by introducing some new notation. Let $\mathbf{GW}^+(D)$ be the set of all realizations of the "root-added" Galton-Watson tree in which the root 0 has offspring number 1 and its only offspring is the root of a Galton-Watson tree with offspring distribution D. A realization in $\mathbf{GW}^+(D)$ is denoted by $\mathbb{T}^+$ and the measure over the realizations by ν.

Define the measure $P_{\lambda,+}$ to be

$$P_{\lambda,+}(\,\cdot\,) = E_{\mathbb{T}^+ \sim \nu}(P_{\mathbb{T}^+,\lambda}(\,\cdot\,)) = \sum_{\mathbb{T}^+ \in \mathbf{GW}^+(D)} P_{\mathbb{T}^+,\lambda}(\,\cdot\,)\nu(\mathbb{T}^+).$$

We will drop the index λ when the context is clear. The contact process on $\mathbf{GW}^+(D)$ with with root 0 initially infected will be denoted by η_t^0 while the contact process on $\mathbf{GW}(D)$ will still be denoted by ξ_t^0.

From now on we set $A_t = \cup_{0 \le s \le t}\ \eta_s^0$. Let $|v|$ denote the distance from vertex v to the root 0. For a vertex $x \in \eta_t^0 \cap F(A_t)$, define a *branch* of x to be

$$S^+(x) = \{x\} \cup S(x'),$$

where x' can be any child of x that has not been infected by time t. Observe that $S^+(x)$ follows the distribution of $\mathbf{GW}^+(D)$.

The strategy of proving Theorem 1.5 is to identify a tractable process embedded in the contact process that would infect the root infinitely often when the contact process survives strongly. To find this embedded process it helps to look at the the macroscopic behavior of the contact process when $\lambda > \lambda_2$.

Observe that for a sufficiently large j a lot of vertices on level j will be infected at a certain time jt_1 with probability δ. This observation is intuitive if we think about the contact process on regular trees and will be proved in Lemma 5.1. If there are sufficiently many of these vertices we should be able to use them to pass the infection back to the root with a sufficiently large probability q. Then we partition the Galton-Watson tree into blocks of depth j. The root of each block (since its first infection) will be infected after time jt_1 with probability at least δq. The nice thing about trees is that at distance jn there will be $O(e^{cn})$ such blocks so that the success probability of pushing the infection from distance jn to $(j-1)n$ is sufficiently large. For this reason we can use a recursive relation to show that in the embedded process the probability that the root is infected infinitely often is positive.

Having identified the embedded process (which is essentially a block construction), we will slightly tune down the infection rate λ and show the embedded process with a reduced infection rate $\lambda - \epsilon$ still can infect the root infinitely often. The continuity of phase transition at λ_2 is thus established by a classical argument of contradiction.

The first step is to establish the observation that a lot of vertices on level j will be infected at a certain time jt_1 with probability δ. This will come in handy when we estimate the probability of successfully pushing the infection back to the root. Specifically, we consider the infected vertices in the frontier $F(A_t)$ in order to obtain the independence between the subtree structure and the infection state of its root.

Despite the seemingly complicated statement, the proof of Lemma 5.1 is standard calculation of Binomial random variable.

Lemma 5.1. *If $\lambda > \lambda_1$, there exists $\alpha, \beta, \delta > 0$ and t_1 such that for all $j \ge 1$,*

$$P_{\lambda,+}(|\eta_{jt_1}^0 \cap F(A_{jt_1}) \cap \{v : j\alpha t_1 \le |v| \le j\beta t_1\}| \ge e^{jc_\lambda t_1}) \ge \delta,$$

where c_λ is a positive number which may depend on λ.

Proof. Let $\theta = P_+(\eta_t^0 \neq \varnothing$ for all $t \geq 0)$. It follows from the proof of Lemma 3.1 that there exists $c > 0$ such that when t_1 is sufficiently large

$$P_+(|\eta_{t_1}^0 \cap F(A_{t_1})| \geq e^{ct_1}) \geq \frac{3}{4}\theta.$$

For reasons that will become clear later, we choose t_1 large enough so that the following also holds

$$(\theta e^{ct_1})/8 > 1 \tag{5.1}$$

Given the choice of t_1 we can then choose α small and β large so that

$$P_+(|\eta_{t_1}^0 \cap F(A_{t_1}) \cap \{v : \alpha t_1 \leq |v| \leq \beta t_1\}| \geq e^{ct_1}) \geq \frac{1}{2}\theta \equiv \delta. \tag{5.2}$$

A vertex x is said to be *good* if the event in (5.2) with probability δ occurs for the contact process on $S^+(x)$ with x initially infected. If the root 0 is good, we say the root is a good vertex at level 0. The good vertices in $\eta_{jt_1}^0 \cap F(A_{jt_1}) \cap \{v : j\alpha t_1 \leq |v| \leq j\beta t_1\}$ are called the good vertices at level j. Let N_j be the number of good vertices at level j. Trivially we have

$$|\eta_{jt_1}^0 \cap F(A_{jt_1}) \cap \{v : j\alpha t_1 \leq |v| \leq j\beta t_1\}| \geq N_j.$$

Hence it suffices to estimate N_j. Note that the good vertices in $\eta_{t_1}^0 \cap F(A_{t_1}) \cap \{v : \alpha t_1 \leq |v| \leq \beta t_1\}$ correspond to disjoint branches, so N_1 dominates a $Binomial(e^{ct_1}, \delta)$ random variable if the root 0 is good.

For a random variable $X \sim Binomial(n, p)$, Chebyshev's inequality gives

$$P\left(X \leq \frac{np}{2}\right) \leq \frac{Var(X)}{(np/2)^2} \leq \frac{4}{np}. \tag{5.3}$$

By (5.3) with $n = e^{ct_1}$ and $p = \delta$,

$$P\left(N_1 \leq \frac{\delta e^{ct_1}}{2}\,\Big|\,N_0 = 1\right) \leq P\left(Binomial(e^{ct_1}, \delta) \leq \frac{\delta e^{ct_1}}{2}\right)$$
$$\leq (4/\delta)e^{-ct_1},$$

where $(4/\delta)e^{-ct_1} < 1$ by (5.1). The branches of good vertices at level 1 are disjoint, so the random variable N_2 dominates $Binomial(N_1 e^{ct_1}, \delta)$, which gives

$$P\left(N_2 \leq \left(\frac{\delta e^{ct_1}}{2}\right)^2\,\Big|\,N_1 \geq \frac{\delta e^{ct_1}}{2}\right) \leq P\left(Binomial\left(\frac{\delta e^{2ct_1}}{2}, \delta\right) \leq \left(\frac{\delta e^{ct_1}}{2}\right)^2\right)$$
$$\leq 2\left(\frac{2e^{-ct_1}}{\delta}\right)^2 \leq \left((4/\delta)e^{-ct_1}\right)^2.$$

It follows from the Markov property that N_{i+1} depends only on N_i for $i \geq 0$. Hence, for any $j \geq 1$

$$P\left(N_j \geq \left(\frac{\delta e^{ct_1}}{2}\right)^j\right) \geq \delta P\left(N_1 \geq \frac{\delta e^{ct_1}}{2}\right)\prod_{i=1}^{j-1} P\left(N_{i+1} \geq \left(\frac{\delta e^{ct_1}}{2}\right)^{i+1}\,\Big|\,N_i \geq \left(\frac{\delta e^{ct_1}}{2}\right)^i\right)$$
$$\geq \delta \prod_{i=1}^{j}\left(1 - \left((4/\delta)e^{-ct_1}\right)^i\right) \geq \delta\sigma$$

where $\sigma \equiv \prod_{i=1}^{\infty}\left(1 - ((4/\delta)e^{-ct_1})^i\right) > 0$. Since $(\theta e^{ct_1})/8 > 1$ there exists $0 < c_\lambda < c$ such that

$$e^{c_\lambda t_1} < 2 < \frac{\theta e^{ct_1}}{4} = \frac{\delta e^{ct_1}}{2}.$$

The last equality follows from the definition $\delta = \theta/2$. Hence for any $j \geq 1$, $P(N_j \geq e^{jc_\lambda t_1}) \geq \delta\sigma > 0$. □

We are now ready to prove Theorem 1.5, i.e., the contact process on $\mathbf{GW}(D)$ does not survive strongly at $\lambda_2(\mathbf{GW}(D))$.

Proof of Theorem 1.5. Since $P_{\lambda_2}(0 \in \xi_t^0 \text{ i.o.}) = 0$ if and only if $P_{\lambda_2,+}(0 \in \eta_t^0 \text{ i.o.}) = 0$, it suffices to show the latter. Our goal is to show that $P_{\lambda,+}(0 \in \eta_t^0 \text{ i.o.}) > 0$ implies $\lambda > \lambda_2$. Now suppose $P_{\lambda,+}(0 \in \eta_t^0 \text{ i.o.}) > 0$. Theorem 4.1 implies that $\lambda > \lambda_1$, so there exists λ_0 such that $\lambda > \lambda_0 > \lambda_1$. Applying Lemma 5.1 to the contact process with infection rate λ_0, there exists positive c_{λ_0} so that for all $j \geq 1$,

$$P_{\lambda_0,+}(|\eta_{jt_1}^0 \cap F(A_{jt_1}) \cap \{v : j\alpha t_1 \leq |v| \leq j\beta t_1\}| \geq e^{jc_{\lambda_0}t_1}) \geq \delta \qquad (5.4)$$

for some choice of α, β, δ and t_1.

Let $B_j = \eta_{jt_1}^0 \cap F(A_{jt_1}) \cap \{v : j\alpha t_1 \leq |v| \leq j\beta t_1\}$ and let x be a vertex in B_j. We use the notation $(x,0) \to (0,t)$ to denote the event that running the contact process on the shortest path connecting x and 0 with x initially infected, 0 will be infected at time t. Trivially, if every particle in the direct path from $x \in B_j$ to 0 lives for at least $2/\alpha$ and pushes the infection toward the root 0 between time $[1/\alpha, 2/\alpha]$ then

$$P_{\lambda_0}((x,0) \to (0,t) \text{ for some } t \in [jt_1, (2\beta/\alpha)jt_1]) \geq \left(e^{-2/\alpha}(e^{-\lambda_0/\alpha} - e^{-2\lambda_0/\alpha})\right)^{j\beta t_1} \equiv (q_0)^{j\beta t_1}.$$

Fix $\varepsilon = (c_{\lambda_0}/2)$ and $\rho = \frac{1}{4}P_{\lambda,+}(0 \in \eta_t^0 \text{ i.o.})$. For reasons that will become clear later we choose j_0 such that

$$C_{j_0} \equiv \delta \rho e^{(c_{\lambda_0} - \varepsilon)j_0 t_1} > 1. \qquad (5.5)$$

From now on we write $t_0 \equiv j_0 t_1$. We want to use the vertices in B_{j_0} to push the infection back to the root 0. For some vertex $x \in B_{j_0}$, let η_t^x denote the contact process restricted to the branch $S^+(x)$ starting with x infected. If $x \in \eta_t^x$ for many times, then the infection can try many times to push back to the root. Hence the probability of successfully pushing the infection from x back to 0 is increased if $x \in \eta_t^x$ for many times. To estimate the success probability let $\sigma_0^x = 0$ and define

$$\sigma_{i+1}^x = \inf\{t > \sigma_i + (2\beta/\alpha)t_0 : x \in \eta_t^x\}.$$

Choose m so that

$$1 - (1 - (q_0)^{\beta t_0})^m \geq e^{-\varepsilon t_0}, \qquad (5.6)$$

which is a lower bound on the probability that x successfully pushes the infection to the root 0 within m independent trials. If the process η_t^x restricted to $S^+(x)$ reinfects x for infinitely many times then $\sigma_m^x < \infty$, i.e., $P_{\lambda,+}(\sigma_m^x < \infty) \geq P_{\lambda,+}(0 \in \eta_t^0 \text{ i.o.})$. So we can choose M_2 such that

$$P_{\lambda,+}(\sigma_m^x \leq M_2) \geq \frac{1}{2}P_{\lambda,+}(0 \in \eta_t^0 \text{ i.o.}) = 2\rho. \qquad (5.7)$$

For any $\epsilon < \lambda - \lambda_0$, the estimates in (5.4) through (5.6) are still true for the contact process with infection rate $\lambda - \epsilon$ since $\lambda - \epsilon > \lambda_0$. By (5.7) there exists a sufficiently small $\epsilon > 0$ with $\epsilon < \lambda - \lambda_0$ that satisfies

$$P_{\lambda-\epsilon,+}(\sigma_m^x \leq M_2) \geq \rho.$$

A vertex x is said to be $(m, \lambda - \epsilon)$-*recurrent* if $\{\sigma_m^x \leq M_2\}$ occurs for the contact process η_t^x with infection rate $\lambda - \epsilon$. If x is $(m, \lambda - \epsilon)$-recurrent then with probability $\geq e^{-\varepsilon t_0}$ it can reinfect the root after time t_0. Each vertex in B_{j_0} is $(m, \lambda - \epsilon)$-recurrent independently with probability at least ρ.

Now we are ready to show $\lambda > \lambda_2$. The following argument is inspired by the proof of Proposition 4.57 in Liggett [9]. Let $r_i = P_{\lambda-\epsilon,+}(\exists \ t \geq 2it_0 \text{ such that } 0 \in \eta_t^0)$. To estimate

r_{i+1} we first run the contact process up to time t_0. According to (5.4) at time t_0 the set of infected frontier B_{j_0} has size $\geq L_0 \equiv \exp(c_{\lambda_0} t_0)$ with probability δ. Recall that η_t^x is the contact process restricted to the branch $S^+(x)$ starting with x infected. Let

$$R_x = \{\exists \ t \geq 2it_0 \text{ such that } x \in \eta_t^x\} \quad \text{and} \quad G_x = \{x \text{ is } (m, \lambda - \epsilon)\text{-recurrent}\}.$$

If for some $x \in B_{j_0}$ the event $R_x \cap G_x$ occurs then x is reinfected after $(2i + 1)t_0$ and can push the infection back to the root after $2(i + 1)t_0$ with probability at least $e^{-\varepsilon t_0}$. It follows that

$$
\begin{aligned}
r_{i+1} &\geq P_{\lambda-\epsilon,+} \left(\exists x \in B_{j_0} \text{ such that } R_x \cap G_x \text{ occurs}\right) \cdot e^{-\varepsilon t_0} \\
&\geq P_{\lambda_0,+} \left(Binomial(|B_{j_0}|, P_{\lambda_0,+}(R_x \cap G_x)) \geq 1\right) \cdot e^{-\varepsilon t_0} \qquad (\text{since } \lambda_0 < \lambda - \epsilon). \quad (5.8)
\end{aligned}
$$

On a given tree $\mathbb{T}^+$, R_x and G_x are positively correlated, as they are both increasing functions of the infection events and decreasing functions of the recovery events (see Liggett [9] for an account of positive correlation). Thus

$$P_+(R_x \cap G_x) = E_{\mathbb{T}^+\sim\nu}(P_{\mathbb{T}^+}(R_x \cap G_x)) \geq E_{\mathbb{T}^+\sim\nu}(P_{\mathbb{T}^+}(R_x) \cdot P_{\mathbb{T}^+}(G_x)).$$

In order to estimate $E_{\mathbb{T}^+\sim\nu}(P_{\mathbb{T}^+}(R_x) \cdot P_{\mathbb{T}^+}(G_x))$ we will adopt a normal positive correlation argument. The reader can skip this part and jump straight to (5.9).

Now observe that the root-added Galton-Watson tree $\mathbb{T}^+$ can be represented by a collection of i.i.d. random variables. Let the random variables $Z_0 = 1$ and $Z_1 \sim D$ be the number of offspring of the added root and the root, respectively. Then we sample a series of i.i.d. random variables $Z_{1,1}, \ldots, Z_{1,Z_1} \sim D$ to represent the number of offspring for each individual in the first generation. The offspring of each individual in the second generation is then given by the collection $\{Z_{1,i,1}, \ldots, Z_{1,i,Z_{1,i}}\}_{i=1}^{Z_1}$, and so on.

To simplify notation we write the index $(i_1, i_2, \ldots, i_n)$ as $\mathbf{i} = (i_1, i_2, \ldots, i_n, 0, \ldots) \in \mathbb{N}^\mathbb{N}$. Hence, given a collection $\vec{Z} = \{Z_\mathbf{i} : \mathbf{i} \in \mathbb{N}^\mathbb{N}\}$ of i.i.d. random variables we can construct a Galton-Watson tree accordingly. Note that different $\vec{Z}$'s can be mapped to the same $\mathbb{T}^+ \in \mathbf{GW}^+(D)$. For example, if $Z_1 = 1$ then we only use the element $Z_{1,1}$ to construct the next generation, ignoring the elements $Z_{1,2}, Z_{1,3}$ and so on. Let μ^* denote the measure over the realizations of $\vec{Z}$. A function h of $\vec{Z}$ is said to be increasing if for a pair $\vec{Z}^1, \vec{Z}^2$ with $Z_\mathbf{i}^1 \leq Z_\mathbf{i}^2$ for all $\mathbf{i}$ we have $h(\vec{Z}^1) \leq h(\vec{Z}^2)$. It follows from the Harris inequality (see Section 2.2 in [5]) that μ^* has positive correlations, that is, $E_{\mu^*} fg \geq E_{\mu^*} f \cdot E_{\mu^*} g$ for all increasing functions f and g with finite second moments. Note that the measure ν on $\mathbf{GW}^+(D)$ can be obtained from μ^* by collapsing different $\vec{Z}$'s mapping to the same $\mathbb{T}^+$, which means, for example,

$$E_{\mathbb{T}^+\sim\nu}(P_{\mathbb{T}^+}(R_x)) = E_{\vec{Z}\sim\mu^*}(P_{\vec{Z}}(R_x))$$

Both $P_{\vec{Z}}(R_x)$ and $P_{\vec{Z}}(G_x)$ are increasing with respect to $\vec{Z}$, so it follows from the positive correlations of μ^* that

$$
\begin{aligned}
E_{\mathbb{T}^+\sim\nu} \left(P_{\mathbb{T}^+}(R_x) \cdot P_{\mathbb{T}^+}(G_x)\right) &= E_{\vec{Z}\sim\mu^*}(P_{\vec{Z}}(R_x) \cdot P_{\vec{Z}}(G_x)) \geq E_{\vec{Z}\sim\mu^*}(P_{\vec{Z}}(R_x)) \cdot E_{\vec{Z}\sim\mu^*}(P_{\vec{Z}}(G_x)) \\
&= E_{\mathbb{T}^+\sim\nu}(P_{\mathbb{T}^+}(R_x)) E_{\mathbb{T}^+\sim\nu}(P_{\mathbb{T}^+}(G_x)) = P_+(R_x) P_+(G_x)
\end{aligned}
$$

That is,

$$P_+(R_x \cap G_x) \geq P_+(R_x) P_+(G_x) \geq r_i \rho. \qquad (5.9)$$

Now we can finish the calculation in (5.8)

$$r_{i+1} \geq P(|B_{j_0}| \geq L_0) P(Binomial(L_0, r_i\rho) \geq 1) e^{-\varepsilon t_0} \geq \delta e^{-\varepsilon t_0} (1 - (1 - r_i\rho)^{L_0})$$

where the term δ comes from (5.4). Set $f(r) = \delta e^{-\varepsilon t_0}(1 - (1-r\rho)^{L_0})$. Then $f(0) = 0$ and $f(1) < 1$. Now the reason for our choice of j_0 in (5.5) has become clear. Under this choice of j_0 we have

$$f'(0) = \delta \rho e^{-\varepsilon t_0} L_0 = C_{j_0} > 1.$$

It follows that f has a fixed point $r^* \in (0,1)$, i.e., $f(r^*) = r^*$. Now we prove $r_i \geq r^*$ inductively for $i \geq 0$. First we have $r_0 = 1$. Suppose $r_i \geq r^*$, then the monotonicity of f implies that

$$r_{i+1} \geq f(r_i) \geq f(r^*) = r^*.$$

Thus we have proved $r_i \geq r^*$ for all $i \geq 0$, which implies that $P_{\lambda-\epsilon,+}(0 \in \eta_t^0 \text{ i.o.}) > 0$ and thus $\lambda - \epsilon \geq \lambda_2$. Therefore we have shown $\lambda > \lambda_2$ and this completes the proof. $\qquad\square$

6 The weak survival phase on periodic trees

In this section we consider the contact process on a general periodic tree $\mathbb{T}_\kappa$. We will prove (i) $\lambda_1 < \lambda_2$ (see Theorem 1.7) and (ii) the contact process does not survive strongly at λ_2 (see Theorem 1.6, (ii)). The rest of Theorem 1.6 can be proved by similar arguments to that of the Galton-Watson trees and hence is omitted here. The reader is referred to the remarks in previous sections for ideas of the proofs.

Recall that the periodic tree $\mathbb{T}_\kappa = (a_1, a_2, \ldots, a_\kappa)$ is arranged in the way that every vertex x in $\mathbb{T}_\kappa$ has one neighbor above it and $d(x)$ neighbors below it, where $d(x) \in \{a_1, \ldots, a_\kappa\}$ is the offspring number of x. To simplify notation we will let $\gamma = \prod_{i=1}^\kappa a_i$. Without loss of generality, a distinguished vertex o is chosen to be the root and is said to be on level 0. Recall that the function $\ell : \mathbb{T}_\kappa \to \mathbb{Z}$ assigns a *level* to each vertex in $\mathbb{T}_\kappa$, which is defined as follows: (i) $\ell(o) = 0$ and (ii) for each $x \in \mathbb{T}_\kappa$, $\ell(y) = \ell(x) - 1$ for exactly one neighbor y of x, and $\ell(y) = \ell(x) + 1$ for the other $d(x)$ neighbors of x.

6.1 The growth profile

We now follow Liggett [8] and Stacey [12] to define the weight of a vertex x by

$$w_\rho(x) = \rho^{\ell(x)},$$

where $\rho > 0$ is a constant to be specified later. The weight of a set A of vertices is defined to be

$$w_\rho(A) = \sum_{x \in A} \rho^{\ell(x)}.$$

Take $\{e_n : n \in \mathbb{Z}\}$ in $\mathbb{T}_\kappa$ such that $\ell(e_n) = n$ and $|e_n - e_{n+1}| = 1$, where e_0 is what we called the root o. As many of the things we will discuss are analogous to results in Liggett [9] for the homogeneous tree $\mathbb{T}^d$, we will use the same notation as in [9]. Throughout the discussion we will use ξ_t to denote the contact process on $\mathbb{T}_\kappa$ starting with only the root o infected.

Define

$$u(n) = P(e_n \in \xi_t \text{ for some } t).$$

In order for the infection ξ_t to reach $e_{(n+m)\kappa}$ it has to first reach $e_{n\kappa}$. Thus if we restart the contact process with only $e_{n\kappa}$ infected at the moment ξ_t reaches $e_{n\kappa}$, by the strong Markov property

$$u((m+n)\kappa) \geq u(m\kappa)u(n\kappa). \tag{6.1}$$

See (4.47) in Liggett [9] for a detailed argument. Hence

$$\beta(\lambda) := \lim_{n \to \infty} [u(n\kappa)]^{1/n}$$

exists and satisfies

$$\beta(\lambda) = \sup_n [u(n\kappa)]^{1/n}.$$

It is easy to show that $\beta(\lambda)$ is independent of the type of vertex that is initially infected.

Let $S(x)$ denote the subtree that contains vertex x and all of its descendants. We will use $\bar{\xi}_t$ to denote the contact process restricted to $S(o)$, starting with only the root o infected. The following can be derived from the proof of Lemma 4.53 in [9]. Hence we only give a sketch of proof here.

Lemma 6.1. *For any λ,*

$$\lim_{n\to\infty} \left(\sup_t P(e_{n\kappa} \in \bar{\xi}_t)\right)^{\frac{1}{n}} = \beta(\lambda).$$

Sketch of proof. Let $B(x,n) = \{y \in \mathbb{T}_\kappa : |y - x| \le n\}$. Let $v_r(m)$ be the probability that ξ_t^o restricted to $B(o,r)$ infects e_m for some $t \le r$. Then

$$\lim_{r\to\infty} v_r(m) = u(m).$$

Now take positive integers j, k, m, n satisfying

$$jm + r \le n.$$

Let $x_0 = e_{(n-jm)\kappa}, x_1 = e_{(n-(j-1)m)\kappa}, \ldots, x_j = e_{n\kappa}$. In order to have an infection path from $(o,0)$ to $(e_{n\kappa}, t)$ for some $t \le nr$, we consider the intersection of following events:

the infection starting from o reaches x_0 within time 1 without exiting $S(o)$,

the infection from x_0 reaches x_1 within time r without exiting $B(x_1, r)$,

$\ldots$

the infection from x_{j-1} reaches x_j within time r without exiting $B(x_j, r)$.

Therefore

$$P(e_{n\kappa} \in \bar{\xi}_t \text{ for some } t \le nr) \ge P(e_{(n-jm)\kappa} \in \bar{\xi}_t \text{ for some } t \le 1)[v_r(m\kappa)]^j$$

Observe that

$$P(e_{n\kappa} \in \bar{\xi}_{i+1}) \ge e^{-1} P(e_{n\kappa} \in \bar{\xi}_t \text{ for some } t \in [i, i+1]).$$

Hence

$$\begin{aligned}
\sup_t P(e_{n\kappa} \in \bar{\xi}_t) &\ge \max_{0 \le i < nr} P(e_{n\kappa} \in \bar{\xi}_{i+1}) \\
&\ge e^{-1} \max_{0 \le i < nr} P(e_{n\kappa} \in \bar{\xi}_t \text{ for some } t \in [i, i+1]) \\
&\ge \frac{e^{-1}}{nr} P(e_{n\kappa} \in \bar{\xi}_t \text{ for some } t \le nr) \\
&\ge \frac{e^{-1}}{nr} P(e_{(n-jm)\kappa} \in \bar{\xi}_t \text{ for some } t \le 1)[v_r(m\kappa)]^j.
\end{aligned}$$

Therefore

$$\liminf_{n\to\infty} \left(\sup_t P(e_{n\kappa} \in \bar{\xi}_t)\right)^{\frac{1}{n}} \ge [v_r(m\kappa)]^{1/m}.$$

Taking $r \to \infty$ and then $m \to \infty$ gives the result. $\qquad\square$

To simplify notation, in this section we will write λ_1 for $\lambda_1(\mathbb{T}_\kappa)$ and λ_2 for $\lambda_2(\mathbb{T}_\kappa)$. The structure of the periodic trees are similar in a way to the homogeneous trees, which allows us to obtain many useful properties on $\beta(\lambda)$. Here we only list one property to be used later:

Lemma 6.2. $\left(\frac{\lambda}{1+\lambda}\right)^{\kappa} \leq \beta(\lambda_1) \leq 1/\gamma.$

Proof. When the vertex e_i is infected, the probability that it infects e_{i+1} before the infection recovers is $\lambda/(1+\lambda)$. If this event occurs for every vertex on the path from e_0 to $e_{n\kappa}$ then $e_{n\kappa} \in \xi_t^o$ for some t. That is,

$$u(n\kappa) = P(e_n \in \xi_t \text{ for some } t) \geq \left(\frac{\lambda}{1+\lambda}\right)^{n\kappa},$$

which implies that $\beta(\lambda) \geq \left(\frac{\lambda}{1+\lambda}\right)^{\kappa}.$

Suppose $\beta(\lambda) > 1/\gamma$. There exists some constant a so that $\beta(\lambda) > a > 1/\gamma$. By Lemma 6.1 there exists t_0 and n_0 such that

$$P(e_{n_0\kappa} \in \bar{\xi}_{t_0})^{1/n_0} \geq a.$$

For $n \in \mathbb{Z}$, we will use

$$\mathcal{L}_n = \{x \in \mathbb{T}_\kappa : \ell(x) = n\}$$

to denote the set of vertices on level n. Let $B_1 = \{x \in \bar{\xi}_{t_0} : x \in \mathcal{L}_{n_0\kappa}\}$. Then $E|B_1| \geq (a\gamma)^{n_0} > 1$. At time t_0, at each $x \in B_1$ we can start an independent contact process restricted to $S(x)$ and run them for time t_0. Continuing this construction gives a supercritical branching process. Therefore $\beta(\lambda) > 1/\gamma$ implies $\lambda > \lambda_1$, i.e., $\beta(\lambda_1) \leq 1/\gamma$. $\qquad\square$

6.2 Proof of Theorem 1.7

The above discussion completes our preparation for the proof. Now we will proceed to show the existence of an intermediate phase for the contact process on periodic trees.

There are κ types of vertices in the periodic tree T_κ categorized according to their offspring numbers and relative positions in the tree. A vertex that corresponds to a_{i+1} in the tree $T_\kappa = (a_1, a_2, \ldots, a_\kappa)$ is said to be of type i, where $i \in \{0, 1, \ldots, \kappa - 1\}$. Let $\xi_t^{i,\lambda}$ be the contact process on T_κ with infection rate λ, starting with an infected type i vertex that we denote by x_i. We will also set x_i to be the root on level 0 and define the function $\ell : T_\kappa \to \mathbb{Z}$ accordingly.

If we only look at levels $\{\mathcal{L}_{n\kappa}\}_{n\in\mathbb{Z}}$ of the tree $\mathbb{T}_\kappa$ then the structure resembles a homogeneous tree $\mathbb{T}^\gamma$. Here we state a Lemma in Liggett [9] that will be useful in our proof.

Lemma 4.26 (See [9] Part I). *Let T^d denote the homogeneous tree with degree $d+1$ and root o. Let*

$$\alpha_n(\rho) = \sum_{|x-o|=n} \rho^{\ell(x)}.$$

Then $\alpha_0(\rho) = 1$ and for $n \geq 1$

$$\alpha_n(\rho) = \begin{cases} \frac{d^{n-1}\rho^n(d^2\rho^2-1)+\rho^{-n}(\rho^2-1)}{d\rho^2-1}, & d\rho^2 \neq 1 \\ \rho^{-n}[(n+1)-(n-1)\rho^2], & d\rho^2 = 1. \end{cases}$$

It is known that for the contact process $\xi_t^{\lambda_1}$ on homogeneous trees, the expectation of its weight $Ew_\rho(\xi_t^{\lambda_1}) \to 0$ as $t \to \infty$, see [12]. This is a key ingredient in Stacey's proof in [12]. For a given $\varepsilon > 0$ and a sufficiently large t_0 we have $Ew_\rho(\xi_{t_0}^{\lambda_1}) \leq \varepsilon$. Hence for some λ slightly larger than λ_1 we have

$$Ew_\rho(\xi_{t_0}^{\lambda}) \leq 2\varepsilon.$$

Using this fact Stacey constructed a supermartingale $M_n := \frac{w_\rho(\xi^\lambda_{nt_0})}{(2\varepsilon)^n}$ and showed the contact process does not survive strongly at this λ.

We will carry out an analogous argument for the periodic trees. In Lemma 6.3 we estimate the weight function at λ_1 for the contact process on periodic trees and then in the proof of Theorem 1.7 the supermartingale is constructed. The new ingredient in our argument is the extinction of the contact process on periodic trees at λ_1, which is used in the proof of Lemma 6.3.

Lemma 6.3. *For any $\varepsilon > 0$, there exists $t_0 > 0$ so that*

$$\max_{i \in \{0,1,\dots,\kappa-1\}} E w_\rho(\xi^{i,\lambda_1}_{t_0}) \leq \varepsilon.$$

Proof. We will consider the contact process ξ^{i,λ_1}_t starting with a type i vertex x_i infected and set x_i to be the root of T_κ. Recall that $\mathcal{L}_n = \{x \in \mathbb{T}_\kappa : \ell(x) = n\}$ denotes the set of vertices on level n and $S(x)$ denotes the subtree that contains vertex x and all of its descendants. For a vertex $x \in \cup_{n \in \mathbb{Z}} \mathcal{L}_{n\kappa}$ we shall define a block with root x to be

$$S^x_\kappa = \{y \in S(x) : |y - x| \leq \kappa - 1\}.$$

Note that every vertex in $\cup_{n \in \mathbb{Z}} \mathcal{L}_{n\kappa}$ has type i. Define $\mathcal{D}_n(x_i) = \{x \in T_\kappa : |x - x_i| = n\}$ and

$$\mathcal{B}_n(x_i) = \{x \in T_\kappa : |x - x_i| = n\kappa, \ell(x) = 0 \pmod{\kappa}\}.$$

The tree T_κ can be partitioned into a disjoint union of blocks

$$T_\kappa = \cup_{x \in \cup_{m \in \mathbb{Z}} \mathcal{L}_{m\kappa}} S^x_\kappa.$$

The weight $E w_\rho(\xi^{i,\lambda_1}_t)$ can be split into two parts:

$$E w_\rho(\xi^{i,\lambda_1}_t) \leq \sum_{n=0}^{\kappa N - 1} \sum_{x \in \mathcal{D}_n(x_i)} w_\rho(x) P(x \in \xi^{i,\lambda_1}_t) + \sum_{n=\kappa N}^{\infty} \sum_{x \in \mathcal{D}_n(x_i)} w_\rho(x) P(x \in \xi^{i,\lambda_1}_t) \quad (6.2)$$

To obtain an upper bound on the second term in (6.2), we count the weight of the whole block S^x_κ if some vertex in this block is infected at time t. Hence

$$\sum_{n=\kappa N}^{\infty} \sum_{x \in \mathcal{D}_n(x_i)} w_\rho(x) P(x \in \xi^{i,\lambda_1}_t) \leq \sum_{n \geq N} \sum_{x \in \mathcal{B}_n(x_i)} w_\rho(S^x_\kappa) P(S^x_\kappa \cap \xi^{i,\lambda_1}_t \neq \varnothing) \quad (6.3)$$

Set $a_0 = 1$ and it is straightforward to compute

$$w_\rho(S^x_\kappa) = \sum_{m=0}^{\kappa-1} \left(\rho^{\ell(x)+m} \cdot \prod_{j=0}^{m} a_j \right) \leq \gamma \rho^{\ell(x)} \left(\sum_{m=0}^{\kappa-1} \rho^m \right) \leq \frac{\gamma}{1-\rho} \cdot \rho^{\ell(x)}. \quad (6.4)$$

Let $\Gamma_{x_i,x}$ denote the path on T_κ from x_i to x. If $\Gamma_{x_i,x} \cap S^x_\kappa \neq \{x\}$, then the first vertex in S^x_κ that will be reached by the infection is not x. In this case, let $x^* \in \Gamma_{x_i,x}$ denote the vertex that satisfies $|x - x^*| = \kappa$. That is, the infection starting at x_i would spread to S^x_κ through vertex $x^* \in S^x_\kappa$. Then we know

$$P(S^x_\kappa \cap \xi^{i,\lambda_1}_t \neq \varnothing) \leq \begin{cases} P(x \in \xi^{i,\lambda_1}_t \text{ for some } t) & \text{if } \Gamma_{x_i,x} \cap S^x_\kappa = \{x\}, \\ P(x^* \in \xi^{i,\lambda_1}_t \text{ for some } t) & \text{otherwise.} \end{cases}$$

Therefore, for $x \in \mathcal{B}^i_n$,

$$P(S^x_\kappa \cap \xi^{i,\lambda_1}_t \neq \varnothing) \leq \beta(\lambda_1)^{|x_i - x| - 1} = \beta(\lambda_1)^{n-1}. \quad (6.5)$$

Putting (6.4) and (6.5) together we have

$$(6.3) \leq \sum_{n \geq N} \sum_{x \in \mathcal{B}_n(x_i)} \frac{\gamma}{1-\rho} \cdot \rho^{\ell(x)} \beta(\lambda_1)^{n-1}. \tag{6.6}$$

Define $\alpha_n(\rho) = \sum_{x \in \mathcal{B}_n(x_i)} \rho^{\ell(x)}$. As a simple corollary of Lemma 4.26 we have $\alpha_0(\rho) = 1$ and

$$\alpha_n(\rho) = \rho^{-n\kappa}[(n+1) - (n-1)\rho^{2\kappa}] \quad \text{when } \gamma \rho^{2\kappa} = 1. \tag{6.7}$$

It follows from (6.7) and $\beta(\lambda_1) \leq 1/\gamma$ (see Lemma 6.2) that we can choose N sufficiently large so that

$$\sum_{n \geq N} \sum_{x \in \mathcal{B}_n(x_i)} \frac{\gamma}{1-\rho} \cdot \rho^{\ell(x)} \beta(\lambda_1)^{n-1} \leq \frac{\gamma}{\beta(\lambda_1)(1-\rho)} \left(\sum_{n \geq N} \gamma^{-n/2}(n+1) \right) \leq \varepsilon/2. \tag{6.8}$$

Notice that this choice of N is independent of the type of vertex that is initially infected as well as the time t. When N is fixed, since at λ_1 the contact process dies out

$$\lim_{t \to \infty} \sum_{n=0}^{\kappa N - 1} \sum_{x \in \mathcal{D}_n^i} w_\rho(x) P(x \in \xi_t^{i,\lambda_1}) = 0$$

for any $i \in \{0, 1, \ldots, \kappa - 1\}$. That is, there exists $M_i > 0$ so that when $t \geq M_i$,

$$\sum_{n=0}^{\kappa N - 1} \sum_{x \in \mathcal{D}_n^i} w_\rho(x) P(x \in \xi_t^{i,\lambda_1}) \leq \varepsilon/2. \tag{6.9}$$

Taking $t_0 = \max\{M_0, M_1, \ldots, M_{\kappa-1}\}$ and adding up (6.8) and (6.9) proves the desired result. $\qquad\square$

Proof of Theorem 1.7. Given $\varepsilon < 1/4$, by Lemma 6.3 we can choose t_0 so that

$$\max_{i \in \{0,1,\ldots,\kappa-1\}} E_\lambda w_\rho(\xi_{t_0}^{i,\lambda_1}) \leq \varepsilon.$$

Since $\max_{i \in \{0,1,\ldots,\kappa-1\}} E_\lambda w_\rho(\xi_{t_0}^{i,\lambda})$ is a continuous function with respect to λ, there exists $\lambda > \lambda_1$ so that

$$\max_{i \in \{0,1,\ldots,\kappa-1\}} E_\lambda w_\rho(\xi_{t_0}^{i,\lambda}) < 2\varepsilon \equiv \delta.$$

Without loss of generality we consider the contact process $\xi_t^{0,\lambda}$ that starts with a type 0 vertex infected. Let $\mathcal{F}_t$ be the σ-algebra generated by $\xi_t^{0,\lambda}$ up to time t. We can show that

$$E[w_\rho(\xi_{(n+1)t_0}^{0,\lambda})|\mathcal{F}_{nt_0}] \leq \sum_{x \in \xi_{nt_0}^{0,\lambda}} \rho^{\ell(x)} E \sum_{y \in \xi_{t_0}^{x,\lambda}} \rho^{\ell(y) - \ell(x)}$$

$$\leq \left(\sum_{x \in \xi_{nt_0}^{0,\lambda}} \rho^{\ell(x)} \right) \cdot \max_{i \in \{0,1,\ldots,\kappa-1\}} E w_\rho(\xi_{t_0}^{i,\lambda}) \leq \delta \cdot w_\rho(\xi_{nt_0}^{0,\lambda}).$$

It follows that

$$M_n := \frac{w_\rho(\xi_{nt_0}^{0,\lambda})}{\delta^n}$$

is a non-negative supermartingale, which converges almost surely as n goes to infinity. Since $\delta = 2\varepsilon < 1/2$,

$$w_\rho(\xi_{nt_0}^{0,\lambda}) \to 0 \qquad \text{as } n \to \infty.$$

Intuitively this implies that the process $\xi_t^{0,\lambda}$ (where $\lambda > \lambda_1$) does not survive strongly. See Proposition 1.0 in [12] for a complete argument. Therefore $\lambda_1 < \lambda \leq \lambda_2$. $\qquad\square$

6.3 Weak survival at λ_2

In this section we complete the proof of Theorem 1.6 by showing the contact process on $\mathbb{T}_\kappa$ does not survive strongly at λ_2. Since $\lambda_1(\mathbb{T}_\kappa) < \lambda_2(\mathbb{T}_\kappa)$ we know the process survives weakly at λ_2. The proof on periodic trees is very much similar to that of Galton-Watson trees. In this case, the estimations are simpler due to the more regular structure that comes from periodicity.

Without loss of generality, a vertex with offspring number a_1 is chosen to be the root o. Let $\bar{\xi}_t$ denote the contact process restricted to the subtree $S(o)$ starting with the root o infected.

Proof of Theorem 1.6 (ii). Our goal is to show that $\rho \equiv P(o \in \bar{\xi}_t^\lambda \text{ i.o.}) > 0$ implies $\lambda > \lambda_2$.

Suppose when the infection rate is λ we have $\rho > 0$. First observe that

$$\rho = P(o \in \bar{\xi}_t^\lambda \text{ for a sequence of times } t \uparrow \infty) \leq u(n)$$

for all n and $\lambda > 0$. (The reader is referred to Theorem 4.65 in [9] for more details.) Hence $\beta(\lambda) = \lim_{n\to\infty} u(n)^{1/n} \geq \lim_{n\to\infty} \rho^{1/n} = 1$. Lemma 6.1 then implies that for any $\delta > 0$ there exists n_0 and t_0 so that

$$P(e_{n_0\kappa} \in \bar{\xi}_{t_0}^\lambda) \geq (1-\delta)^{n_0}.$$

As $P(e_{n_0\kappa} \in \bar{\xi}_{t_0}^\lambda)$ is continuous with respect to λ, there exists $\varepsilon > 0$ so that

$$P(e_{n_0\kappa} \in \bar{\xi}_{t_0}^{\lambda-\varepsilon}) \geq (1-2\delta)^{n_0}.$$

Now we will estimate the size of the infection by comparing with a branching process. Define $\mathcal{L}_n = \{x \in S(o) : |x-o| = n\kappa\}$. Let $B_0 = \{o\}$ and $B_1 = \{x \in \bar{\xi}_{t_0}^{\lambda-\varepsilon} : x \in \mathcal{L}_{n_0}\}$. Then $E|B_1| \geq (\gamma(1-2\delta))^{n_0} > 1$ when we choose δ to be sufficiently small. Specifically we will choose δ so that $\gamma(1-2\delta)^2 > 1$ for reasons that will become clear later.

At time t_0, at each $x \in B_1$ we can start an independent contact process restricted to $S(x)$ and run it for time t_0. Continuing this construction gives a supercritical branching process $|B_m|$ with mean $E|B_1|$. It is clear from the construction that $B_m \subset \bar{\xi}_{mt_0}^{\lambda-\varepsilon} \cap \mathcal{L}_{mn_0}$. Since $|B_m|$ is a supercritical branching process,

$$\lim_{m\to\infty} \frac{|B_m|}{(E|B_1|)^m}$$

exists almost surely and is not trivially zero. Therefore, there exists a $\epsilon > 0$ such that

$$P(|B_m| \geq \epsilon(E|B_1|)^m) \geq \epsilon \tag{6.10}$$

for all sufficiently large m. For reasons that will become clear later we choose m_0 sufficiently large so that

$$P(|\bar{\xi}_{m_0t_0}^{\lambda-\varepsilon} \cap \mathcal{L}_{m_0n_0}| \geq \epsilon(\gamma(1-2\delta))^{m_0n_0}) \geq P(|B_{m_0}| \geq \epsilon(\gamma(1-2\delta))^{m_0n_0}) \geq \epsilon$$

and

$$\epsilon^2((1-2\delta)^2\gamma)^{m_0n_0} > 1.$$

Let $r_i = P(o \in \bar{\xi}_{2im_0t_0}^{\lambda-\varepsilon})$ and $M \equiv \epsilon(\gamma(1-2\delta))^{m_0}$. We can obtain the following recursive relation:

$$\begin{aligned}
r_{i+1} &\geq P(Binomial(|\bar{\xi}_{m_0t_0}^{\lambda-\varepsilon} \cap \mathcal{L}_{m_0n_0}|, r_i) \geq 1)P(e_{m_0n_0\kappa} \in \bar{\xi}_{m_0t_0}^{\lambda-\varepsilon}) \\
&\geq P(|\bar{\xi}_{m_0t_0}^{\lambda-\varepsilon} \cap \mathcal{L}_{m_0n_0}| \geq M)(1-(1-r_i)^M)P(e_{n_0\kappa} \in \bar{\xi}_{t_0}^{\lambda-\varepsilon})^{m_0} \tag{6.11} \\
&\geq \epsilon(1-2\delta)^{m_0n_0}(1-(1-r_i)^M). \tag{6.12}
\end{aligned}$$

Setting $f(r) = \epsilon(1 - 2\delta)^{m_0 n_0}(1 - (1 - r)^M)$. By our choices of δ and m_0,

$$f'(0) = \epsilon^2((1 - 2\delta)^2 \gamma)^{m_0 n_0} > 1.$$

Following the same argument as in the proof of Theorem 1.5 we can show there exists $r^* > 0$ such that $r_i \geq r^*$ for all $i \geq 0$. That is, $\bar{\xi}_t^{\lambda - \varepsilon}$ survives strongly, which implies $\lambda > \lambda - \varepsilon \geq \lambda_2$. $\qquad\qquad\square$

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