

On the peaks of a stochastic heat equation on a sphere with a large radius

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Abstract

For every $R > 0$, consider the stochastic heat equation

$$\partial_t u_R(t, x) = \frac{1}{2} \Delta_{S_R^2} u_R(t, x) + \sigma(u_R(t, x)) \xi_R(t, x)$$

on S_R^2 , where $\xi_R = \dot{W}_R$ are centered Gaussian noises with the covariance structure given by $E[\dot{W}_R(t, x)\dot{W}_R(s, y)] = h_R(x, y)\delta_0(t - s)$, where h_R is symmetric and semi-positive definite and there exist some fixed constants $-2 < C_{h_{up}} < 2$ and $\frac{1}{2}C_{h_{up}} - 1 < C_{h_{down}} \leq C_{h_{up}}$ such that for all $R > 0$ and $x, y \in S_R^2$, $(\log R)^{C_{h_{down}}/2} = h_{down}(R) \leq h_R(x, y) \leq h_{up}(R) = (\log R)^{C_{h_{up}}/2}$, $\Delta_{S_R^2}$ denotes the Laplace-Beltrami operator defined on S_R^2 and $\sigma : \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz continuous, positive and uniformly bounded away from 0 and ∞ . Under the assumption that $u_{R,0}(x) = u_R(0, x)$ is a nonrandom continuous function on $x \in S_R^2$ and the initial condition that there exists a finite positive U such that $\sup_{R>0} \sup_{x \in S_R^2} |u_{R,0}(x)| \leq U$, we prove that for every finite positive t , there exist finite positive constants $C_{down}(t)$ and $C_{up}(t)$ which only depend on t such that as $R \rightarrow \infty$, $\sup_{x \in S_R^2} |u_R(t, x)|$ is asymptotically bounded below by $C_{down}(t)(\log R)^{1/4+C_{h_{down}}/4-C_{h_{up}}/8}$ and asymptotically bounded above by $C_{up}(t)(\log R)^{1/2+C_{h_{up}}/4}$ with high probability.

Keywords: stochastic partial differential equations; spheres.

AMS MSC 2010: Primary 35R60; 60H15, Secondary 60G15.

Submitted to EJP on January 10, 2019, final version accepted on January 10, 2020.

1 Introduction

Suppose $\{(\Omega_R, \mathcal{F}_R, P_R)\}_{R>0}$ is a collection of probability spaces. For each $R > 0$, let E_R denote the expectation with respect to P_R . For each $R > 0$, let ξ_R denote time-white space-colored noise on $S_R^2 \times [0, \infty)$, with S_R^2 being a sphere of radius R , defined on the probability space $(\Omega_R, \mathcal{F}_R, P_R)$. The covariance structure of $\xi_R = \dot{W}_R$ is given by

$$E_R \left[\dot{W}_R(t, x) \dot{W}_R(s, y) \right] = h_R(x, y) \delta_0(t - s), \quad (1.1)$$

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where h_R is a symmetric, positive semi-definite function on $S_R^2 \times S_R^2$ and there exist some fixed constants $-2 < C_{h_{up}} < 2$ and $\frac{1}{2}C_{h_{up}} - 1 < C_{h_{down}} \leq C_{h_{up}}$ such that for all $R > 0$ and $x, y \in S_R^2$,

$$(\log R)^{C_{h_{down}}/2} = h_{down}(R) \leq h_R(x, y) \leq h_{up}(R) = (\log R)^{C_{h_{up}}/2}. \tag{1.2}$$

For $0 < C_{\sigma_{down}} < C_{\sigma_{up}} < \infty$, let $\sigma : \mathbb{R} \mapsto [C_{\sigma_{down}}, C_{\sigma_{up}}]$ be Lipschitz continuous with the Lipschitz constant $0 < \mathbb{L}_\sigma < \infty$. Consider a collection of stochastic heat equations, each of which is defined on $[0, \infty) \times S_R^2 \times \Omega_R$,

$$\partial_t u_R(t, x) = \frac{1}{2} \Delta_{S_R^2} u_R(t, x) + \sigma(u_R(t, x)) \xi_R(t, x), \tag{1.3}$$

$0 \leq t < \infty, x \in S_R^2$, subject to the initial value condition,

$$u_R(0, x) = u_{R,0}(x) \quad \text{for all } x \in S_R^2,$$

where $\Delta_{S_R^2}$ is the Laplace-Beltrami operator on S_R^2 and the initial function $u_{R,0}(\cdot)$ is nonrandom and continuous. In this paper, we consider the following mild form of solution to (1.3).

Definition 1.1. A mild solution to (1.3) is defined to be a jointly measurable random field $u_R(\cdot, \cdot, \cdot) : [0, \infty) \times S_R^2 \times \Omega_R \mapsto \mathbb{R}$ which for each $0 \leq t < \infty, 0 < R < \infty, x \in S_R^2$, P_R -almost surely satisfies the equation

$$u_R(t, x) = \int_{S_R^2} p_R(t, x, y) u_{R,0}(y) dy + \int_0^t \int_{S_R^2} p_R(t-s, x, y) \sigma(u_R(s, y)) W_R(ds, dy), \tag{1.4}$$

where p_R is the heat kernel on S_R^2 and Ω_R is a probability space which depends on R .

Remark 1. We will show in Section 3 that $u_R(t, x)$ exists and is unique up to a P_R -modification independent of t and $x \in S_R^2$. For this reason, we also say $u_R(t, x)$ is the mild solution to (1.3).

Remark 2. Whenever it is clear from the context, we write Ω for Ω_R , P for P_R and E for E_R for brevity. For example, we can rewrite (1.1) as

$$E \left[\dot{W}_R(t, x) \dot{W}_R(s, y) \right] = h_R(x, y) \delta_0(t - s),$$

whenever there is no confusion.

The goal of this paper is to give an asymptotic estimate of $\sup_{x \in S_R^2} |u_R(t, x)|$ as $R \rightarrow \infty$. The following is the main theorem of this paper.

Theorem 1.1. If there exists a finite positive U such that $\sup_{R>0} \sup_{x \in S_R^2} |u_{R,0}(x)| \leq U$, then for any $0 < t < \infty$, there exist constants $0 < C_{down}(t) \leq C_{up}(t) < \infty$, which only depend on t , such that

$$\lim_{R \rightarrow \infty} P \left(C_{down}(t) (\log R)^{\alpha_l} \leq \sup_{x \in S_R^2} |u_R(t, x)| \leq C_{up}(t) (\log R)^{\alpha_u} \right) = 1, \tag{1.5}$$

where $\alpha_l = 1/4 + C_{h_{down}}/4 - C_{h_{up}}/8$ and $\alpha_u = 1/2 + C_{h_{up}}/4$.

Remark 3. The scaling property of the heat kernel (see (2.5) below) suggests the mild solution u_R for different R might also follow a similar scaling property. Suppose the noise $\xi_R = \dot{W}_R$ satisfies the condition $\xi_R(t, x) = \xi_1(t/R^2, x/R)$ for all $t \geq 0$ and $x \in S_R^2$. Then from (1.4) the mild solution u_R satisfies the scaling property $u_R(t, x) = u(t/R^2, x/R)$. The equation in Theorem 1.1 can then be rewritten as

$$\lim_{R \rightarrow \infty} P \left(C_{down}(t) (\log R)^{\alpha_l} \leq \sup_{x \in S^2} |u(t/R^2, x)| \leq C_{up}(t) (\log R)^{\alpha_u} \right) = 1, \tag{1.6}$$

where u denotes u_1 , provided that $\sup_{x \in S^2} |u(0, x)|$ is finite. Unfortunately, the scaling relation $\xi_R(t, x) = \xi_1(t/R^2, x/R)$ forces a strong growth rate condition on h_R , which is incompatible with (1.2).

The stochastic heat equation (1.3) provides a model of the heat flow on a large sphere. In this model, Theorem 1.1 gives an estimate of the highest temperature on a large heated sphere. The result of this paper offers a potential explanation for the existence of solar flares on a large-sized star and estimates the temperatures of the solar flares relative to the radius of the star. While a majority of papers in the theory of SPDE focus on SPDEs on Euclidean spaces, there are a smaller number of published works that study SPDEs on Riemannian manifolds. We find several papers related to SPDEs on Riemannian manifolds: Lang, Schwab [1], Elliott, Hairer, Scott [3], Dalang, Lévêque [6] [7], Gyöngy [12] [13], Peszat, Tindel [18], Funaki [20], Kazashi, Le Gia [22] and Brzeźniak, Goldys, Le Gia [23], [24]. These papers though focus on more general theories of SPDEs on spheres or Riemannian manifolds in general instead of investigating a specific quantitative property of a SPDE such as giving an asymptotic estimate of the peaks of a SPDE, which is the main goal of our paper.

The challenge in finding an accurate asymptotic estimate on the peaks, as given in Theorem 1.1, is to unveil the effect of the curvature of a sphere on the heat flow on its surface under a noisy environment modeled by (1.3). Unlike its Euclidean counterpart, the heat kernel of on a sphere does not have a compact form. The series expansion of the heat kernel on a Riemannian manifold is well-developed via the spectral theory of Laplace-Beltrami operator (See [19]). The technique to estimate of the maximal temperature of peaks, $\sup_{x \in S_R^2} |u_R(t, x)|$, relies on finding sufficiently-many “independent” points on a large sphere in the sense that heat flows originate from these points will not interact with each other in a short amount of time. This idea was introduced in [4]. While there always exist sufficiently-many “independent” points in a Euclidean space as done in [4], cleverly fitting in these “independent” points on a sphere is the key to achieving the goal of this paper. This fitting requirement poses strong restrictions on the choices of various variables used to define an underlying coupling process. Successful coordination on the choice of these variables makes everything fall into the right place. In addition to having to circumvent the “dependence” among points, we will need access to accurate estimations on the heat kernel on a sphere. Among various works on heat kernel estimations such as Li, Yau [17], Varadhan [21], and Molchanov [16], we will use Molchanov’s result to prove the main theorem of this paper. Molchanov [16] gives a uniform estimation on a compact subset of the sphere excluding the South pole.

Before moving to the more technical details and the long series of calculations, an outline of our paper is given. This paper is organized as follows. In Section 2, we recall the Laplacian-Beltrami operator [19] and Molchanov’s heat kernel estimates [16], and develop some preliminary estimates associated with the spherical heat kernels which will be frequently used throughout this paper. In Section 3, we show that the mild solution (1.4) exists uniquely and prove that it is jointly measurable. In Section 4, we show that the mild solution has spatial continuity. In Section 5, we follow the method in [4] to give an asymptotic upper bound of the supremum of the mild solution by noting that there exist sufficiently many “independent” points on a sphere of large radius. In Section 6, necessary tail probability estimates are developed which will be used to give an asymptotic lower bound of the supremum of the mild solution. In Section 7, we use a discretization technique as in [5] along with spatial continuity to give an asymptotic lower bound of the supremum of the mild solution, thus finishing the proof of the main Theorem 1.1 of the paper. In the appendix, we follow the argument in [14] to give the proof of the spherical version of Garsia’s Lemma that is used in Section 4.

Throughout this paper, the following notations will be used. Let S^2 denote S_1^2 the unit

sphere, as usual. For each $k, R > 0$, “ $\|\cdot\|_{k,R}$ ” denotes the $\|\cdot\|_{L^k(\Omega_R)}$ -norm. Denote x/R to be \tilde{x} for each $x \in S^2_R, R > 0$. When there is no confusion as to which probability space $(\Omega_R, \mathcal{F}_R, P_R)$ is involved, we write $\|\cdot\|_k$ instead of $\|\cdot\|_{k,R}$ for brevity. For real-valued functions f and g , which are defined on $[0, \infty)$, we write “ $f(t) \sim_t g(t)$ ” to mean that there exist a constant $0 < \epsilon_0 < 1$ such that $1 - \epsilon_0 \leq \liminf_{t \rightarrow 0} |f(t)/g(t)| \leq \limsup_{t \rightarrow 0} |f(t)/g(t)| \leq 1 + \epsilon_0$. For real-valued functions f and g , which are defined on $[M, \infty)$ for some finite positive M , we write “ $f(R) \asymp_R g(R)$ ” to mean that there exist constants $0 < C_1 \leq C_2 < \infty$ such that $C_1 \leq \liminf_{R \rightarrow \infty} |f(R)/g(R)| \leq \limsup_{R \rightarrow \infty} |f(R)/g(R)| \leq C_2$.

2 The heat kernels on spheres and some preliminary estimates

We use the following definition of the heat kernel. See [19] and [9] for references.

Definition 2.1. *The heat kernel on a Riemannian manifold M is a function $p(t, x, y) \in C^\infty(R^+ \times M \times M)$ such that*

1. *it satisfies the heat equation*

$$\partial_t p(t, x, y) = \frac{1}{2} \Delta_{M,x} p(t, x, y), \tag{2.1}$$

where $\Delta_{M,x}$ is the Laplace-Beltrami operator acting on x ,

2. *for every continuous function f with compact support in M and every $x \in M$,*

$$\lim_{t \rightarrow 0} \int_M p(t, x, y) f(y) dy = f(x). \tag{2.2}$$

It is well known that $\Delta_{S^2_R} = R^{-2} \Delta_{S^2}$ [19] and that the spherical harmonics $\{Y_{lm}\}_{l=0, \dots, \infty; -l \leq m \leq l}$ are eigenfunctions of Δ_{S^2} which form an orthonormal basis in $L^2(S^2)$ with the relations [19]

$$\Delta_{S^2} Y_{lm} = -l(l+1) Y_{lm},$$

for every $l \geq 0$ and $-l \leq m \leq l$. Define the collection of functions $Y_{lm;R}(\cdot) = Y_{lm}(\cdot/R)$ on S^2_R for every $l \geq 0$ and $-l \leq m \leq l$, then for all $x \in S^2_R$,

$$\Delta_{S^2_R} Y_{lm;R}(x) = -\frac{l(l+1)}{R^2} Y_{lm;R}(x).$$

The orthogonality of $\{Y_{lm;R}\}_{l \geq 0, -l \leq m \leq l}$ inherits from that of $\{Y_{lm}\}_{l \geq 0, -l \leq m \leq l}$ and that for every $l \geq 0$ and $-l \leq m \leq l$, every $R > 0$,

$$\frac{1}{R^2} \int_{S^2_R} |Y_{lm;R}(x)|^2 dx = 1.$$

Hence, for every $R > 0$, $\{R^{-1} Y_{lm;R}\}_{l \geq 0, -l \leq m \leq l}$ form an orthonormal basis of $L^2(S^2_R)$. By Proposition 3.1 in [19], and Proposition 3.29 in [15], for every $t, R > 0, x, y \in S^2_R$,

$$\begin{aligned} p_R(t, x, y) &= \frac{1}{R^2} \sum_{l=0}^{\infty} \sum_{m=-l}^l e^{-l(l+1)t/2R^2} Y_{lm;R}(x) \overline{Y_{lm;R}(y)} \\ &= \frac{1}{R^2} \sum_{l=0}^{\infty} \sum_{m=-l}^l e^{-l(l+1)t/2R^2} Y_{lm}(x/R) \overline{Y_{lm}(y/R)}, \end{aligned} \tag{2.3}$$

where (2.3) holds in the sense of pointwise convergence and $L^2(S^2(R))$ -convergence. By the well-known summation formula of spherical harmonics [15],

$$\sum_{m=-l}^l Y_{lm}(x) \overline{Y_{lm}(y)} = \frac{2l+1}{4\pi} P_l(x \cdot y) \quad \text{for each } l \geq 0 \text{ and any } x, y \in S^2, \tag{2.4}$$

where P_l denotes the $l - th$ Legendre polynomial and “ \cdot ” is the inner product for vectors, i.e., for every $x, y \in S^2$ whose Cartesian coordinates are given by $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ respectively, $x \cdot y = x_1y_1 + x_2y_2 + x_3y_3$. Denote $d : S_R^2 \times S_R^2 \mapsto [0, \infty)$ to be the geodesic distance on S_R^2 and $\theta(\cdot, \cdot) = d(\cdot, \cdot)/R$ the angle formed by two points on S_R^2 . Then, by (2.3) and (2.4),

$$\begin{aligned} p_R(t, x, y) &:= p_R(t, \theta(x, y)) \\ &= \sum_{l=0}^{\infty} \frac{(2l+1)e^{-l(l+1)t/2R^2}}{4\pi R^2} P_l(\cos \theta(x, y)) \\ &= \frac{1}{R^2} p_1(t/R^2, \theta(x, y)). \end{aligned} \tag{2.5}$$

It has been proved in [16] that for every $\theta_0 \in (0, \pi)$,

$$p_1(t, x, y) = p_1(t, \theta(x, y)) \sim_t \frac{e^{-\theta(x,y)^2/2t}}{2\pi t} \sqrt{\frac{\theta(x, y)}{\sin \theta(x, y)}}, \tag{2.6}$$

uniformly for all $0 \leq \theta(x, y) \leq \theta_0$. This together with the scaling property (2.5) gives the following.

Lemma 2.1. *For every $t > 0, 0 < \theta_0 < \pi, 0 < \epsilon_0 < 1$, there exists $0 < R_{mol}(t, \theta_0, \epsilon_0) < \infty$ such that for all $R > R_{mol}(t, \theta_0, \epsilon_0)$,*

$$p_R(t, \theta(x, y)) = C(t/R^2, \theta(x, y)) \frac{e^{-R^2\theta(x,y)^2/2t}}{2\pi t} \sqrt{\frac{\theta(x, y)}{\sin \theta(x, y)}}, \tag{2.7}$$

where $1 - \epsilon_0 \leq \inf_{0 \leq \theta \leq \theta_0} C(t/R^2, \theta) \leq \sup_{0 \leq \theta \leq \theta_0} C(t/R^2, \theta) \leq 1 + \epsilon_0$.

The fact that the heat kernel is a transition density function gives

Lemma 2.2. *For all $R, t > 0, x \in S_R^2$,*

$$\int_{S_R^2} p_R(t, \theta(x, y)) dy = 1. \tag{2.8}$$

The following three quantities will be useful in the upcoming chapters.

For every nonnegative $\alpha, \beta, t, R > 0$, let $B_R(x, \sqrt{\beta t})$ be the geodesic ball centered at x with radius $\sqrt{\beta t}$ on S_R^2 and define

$$f_e(\alpha, R, t) = \int_0^t ds \int_{S_R^2 \times S_R^2} e^{-2\alpha s} p_R(s, \theta(x, y_1)) p_R(s, \theta(x, y_2)) h_R(y_1, y_2) dy_1 dy_2, \tag{2.9}$$

and

$$\begin{aligned} f_{e,\beta}(\alpha, R, t) & \\ &= \int_0^t ds \int_{B_R(x, \sqrt{\beta t}) \times B_R(x, \sqrt{\beta t})} e^{-2\alpha s} p_R(s, \theta(x, y_1)) p_R(s, \theta(x, y_2)) h_R(y_1, y_2) dy_1 dy_2, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \tilde{f}_{e,\beta}(\alpha, R, t) & \\ &= \int_0^t ds \int_{S_R^2 \setminus B_R(x, \sqrt{\beta t}) \times S_R^2 \setminus B_R(x, \sqrt{\beta t})} e^{-2\alpha s} p_R(s, \theta(x, y_1)) p_R(s, \theta(x, y_2)) h_R(y_1, y_2) dy_1 dy_2. \end{aligned} \tag{2.11}$$

For notational convenience, denote $B_R(x, \sqrt{\beta t}) \times B_R(x, \sqrt{\beta t})$ by $T_1(\beta, x, R, t)$ and $S_R^2 \setminus B_R(x, \sqrt{\beta t}) \times S_R^2 \setminus B_R(x, \sqrt{\beta t})$ by $T_2(\beta, x, R, t)$. The following estimates will be used later.

Lemma 2.3. For every $0 < t, R, \alpha < \infty$, $f_e(\alpha, R, t) \leq (2\alpha)^{-1}h_{up}(R)$.

Proof. By (2.9) and Lemma 2.2, for every $0 < t, R, \alpha < \infty$,

$$\begin{aligned} f_e(\alpha, R, t) &\leq h_{up}(R) \int_0^t ds \int_{S_R^2 \times S_R^2} e^{-2\alpha s} p_R(s, \theta(x, y_1)) p_R(s, \theta(x, y_2)) dy_1 dy_2 \\ &= h_{up}(R) \int_0^t e^{-2\alpha s} ds \left(\int_{S_R^2} p_R(s, \theta(x, y)) dy \right)^2 \\ &\leq \frac{h_{up}(R)}{2\alpha}. \end{aligned} \quad \square$$

Lemma 2.4. For every $0 < t < \infty$, there exists a finite positive $R_{mol}(t)$ such that for $R \geq R_{mol}(t)$,

$$\tilde{f}_{e,\beta}(\alpha, R, t) \leq 2h_{up}(R)te^{-2\sqrt{\alpha\beta t}}, \tag{2.12}$$

provided that $\alpha \asymp_R \beta \asymp_R (\log R)^c$ where $0 < c < 1$ is a constant.

Proof. By checking the details in [16], for every $0 < t < \infty$, $0 < \theta_0 < \pi$, there exist finite positive $\delta, c_0, R_{mol}(t, \delta, c_0, \theta_0)$ such that for all $R \geq R_{mol}(t, \delta, c_0, \theta_0)$, $0 \leq \theta \leq \theta_0$, and $0 < s < t$,

$$p_1(s/R^2, \theta) \geq \left(1 - e^{-R^2\delta/s}\right) (1 - c_0\sqrt{s}/R) \frac{e^{-R^2\theta^2/2s}}{2\pi s} \sqrt{\theta/\sin\theta}. \tag{2.13}$$

This together with (2.5) and the elementary inequality $\sqrt{\theta \sin \theta} \geq \theta \sqrt{1 - \theta^2/6}$ (for all $0 \leq \theta \leq \pi$) implies that for all finite positive t, β , there exists a finite positive $R_{mol}(t, \beta)$ such that for all finite positive α and $R \geq R_{mol}(t, \beta)$,

$$\begin{aligned} \tilde{f}_{e,\beta}(\alpha, R, t) &\leq h_{up}(R) \int_0^t e^{-2\alpha s} ds \left(1 - 2\pi \int_0^{\sqrt{\beta t}/R} \left(1 - e^{-R^2\delta/s}\right) (1 - c_0\sqrt{s}/R) \right. \\ &\quad \left. \times R^2\theta \frac{e^{-R^2\theta^2/2s}}{2\pi s} \sqrt{1 - \theta^2/6} d\theta \right)^2 \\ &\leq h_{up}(R) \int_0^t e^{-2\alpha s} \left(1 - \left(1 - e^{-\frac{R^2\delta}{t}}\right) \left(1 - \frac{c_0\sqrt{t}}{R}\right) \sqrt{1 - \frac{\beta t}{6R^2}} \left(1 - e^{-\frac{\beta t}{2s}}\right)\right) ds \\ &\leq h_{up}(R) \left(\int_0^t e^{-2\alpha s - \beta t/2s} ds + \int_0^t e^{-2\alpha s} \left(1 - \sqrt{1 - \frac{\beta t}{6R^2}}\right) ds \right. \\ &\quad \left. + \int_0^t e^{-2\alpha s} \left(e^{-R^2\delta/t} + c_0\sqrt{t}/R\right) ds \right) \\ &\leq h_{up}(R)te^{-2\sqrt{\alpha\beta t}} + \frac{h_{up}(R)\beta t}{12\alpha R^2} + \frac{h_{up}(R)}{2\alpha} \left(e^{-R^2\delta/t} + c_0\sqrt{t}/R\right). \end{aligned}$$

This implies for every $0 < t < \infty$, there exists a finite positive $R_{mol}(t)$ such that for $R \geq R_{mol}(t)$,

$$\tilde{f}_{e,\beta}(\alpha, R, t) \leq 2h_{up}(R)te^{-2\sqrt{\alpha\beta t}}, \tag{2.14}$$

provided that $\alpha \asymp_R \beta \asymp_R (\log R)^c$ where $0 < c < 1$ is a constant. □

Lemma 2.5. For every $0 < t, \beta < \infty$, $0 < \epsilon_0 < 1$, there exists a finite positive $R_{mol}(t, \pi/4, \epsilon_0)$ such that for all $R \geq \max\{R_{mol}(t, \pi/4, \epsilon_0), 4\sqrt{\beta t}/\pi\}$,

$$f_{e,\beta}(0, R, t) \geq 2\pi^2 t h_{down}(R) (1 - \epsilon_0)^2 \left(1 - e^{-\beta/2}\right)^2. \tag{2.15}$$

Proof. By (2.10) and Lemma 2.1, for every $0 < t, \beta < \infty, 0 < \epsilon_0 < 1$, there exists a finite positive $R_{mol}(t, \pi/4, \epsilon_0)$ such that for all $R \geq \max\{R_{mol}(t, \pi/4, \epsilon_0), 4\sqrt{\beta t}/\pi\}$,

$$\begin{aligned} f_{e,\beta}(0, R, t) &\geq h_{down}(R)(1 - \epsilon_0)^2 \int_0^t s^{-2} ds \left(\int_{B_R(x, \sqrt{\beta t})} e^{-R^2 \theta(x, y_1)^2 / 2s} \sqrt{\frac{\theta(x, y_1)}{\sin \theta(x, y_1)}} dy_1 \right)^2 \\ &= 4\pi^2 h_{down}(R)(1 - \epsilon_0)^2 R^4 \int_0^t s^{-2} ds \left(\int_0^{\sqrt{\beta t}/R} e^{-R^2 \theta^2 / 2s} \sqrt{\theta \sin \theta} d\theta \right)^2 \\ &\geq 2\pi^2 h_{down}(R)(1 - \epsilon_0)^2 R^4 \int_0^t s^{-2} ds \left(\int_0^{\sqrt{\beta t}/R} \theta e^{-R^2 \theta^2 / 2s} d\theta \right)^2 \\ &= 2\pi^2 t h_{down}(R)(1 - \epsilon_0)^2 \left(1 - e^{-\frac{\beta}{2}} \right)^2, \end{aligned}$$

where in the second inequality, the assumption $R \geq 4\sqrt{\beta t}/\pi$ comes into play. It implies $\sqrt{\beta t}/R \leq \pi/4$ and hence $\sqrt{\theta \sin \theta} \geq \sqrt{2}\theta/2$ for all $0 < \theta \leq \sqrt{\beta t}/R$. \square

3 Existence, uniqueness, and measurability

Following the development in the Section 1, we establish in this section the existence and uniqueness of the mild solution.

Remark 4. The existence and uniqueness theorem follows from the contraction mapping principle. See Section 7.1 of [11] for details on the proof of the existence and uniqueness theorem on a general class of stochastic differential equations. We present the proof here for the completeness of this paper.

Definition 3.1. For every $0 < T, R < \infty$, and each $0 \leq t \leq T, x \in S_R^2$, define the sequence of approximating random field $\{u_R^{(n)}(t, x)\}_{n \geq 0}$ to be

$$u_R^{(0)}(t, x) = u_R(0, x) = u_{R,0}(x), \tag{3.1}$$

and

$$u_R^{(n+1)}(t, x) = \int_{S_R^2} p_R(t, x, y) u_R^{(n)}(0, y) dy + \int_0^t \int_{S_R^2} p_R(t-s, x, y) \sigma(u_R^{(n)}(s, y)) W(ds, dy). \tag{3.2}$$

Theorem 3.1. For every $0 < T, R < \infty$, the sequence of approximating random field $\{u_R^{(n)}(t, x)\}_{n \geq 0}$ defined above has a P_R -limit $u_R(t, x)$ for every $0 \leq t \leq T, x \in S_R^2$ and is unique up to a P_R -modification independent of t, x .

Remark 5. We have not proved that the P_R -limit $u_R(t, x)$ is the mild solution since the jointly measurability has not been established yet.

Proof. Let $t \rightarrow 0$ in (2.2) to get

$$u_R^{(n+1)}(0, x) = \lim_{t \rightarrow 0} \int_{S_R^2} p_R(t, x, y) u_R^{(n)}(0, y) dy = u_R^{(n)}(0, x).$$

By induction,

$$u_R^{(n)}(0, x) = u_{R,0}(x) \quad \text{for all } n. \tag{3.3}$$

From (3.2),

$$\begin{aligned} u_R^{(n+1)}(t, x) - u_R^{(n)}(t, x) &= \int_0^t \int_{S_R^2} p_R(t-s, x, y) \left(\sigma(u_R^{(n)}(s, y)) - \sigma(u_R^{(n-1)}(s, y)) \right) W(ds, dy). \end{aligned} \tag{3.4}$$

For notational brevity, denote for all $0 < s < t < \infty$, $0 < R < \infty$, positive integer n , $x, y_1, y_2 \in S_R^2$,

$$V_1(t, s, R, n, x, y_1, y_2) = p_R(t - s, x, y_1)p_R(t - s, x, y_2) \left(\sigma(u_R^{(n)}(s, y_1)) - \sigma(u_R^{(n-1)}(s, y_1)) \right) \cdot \left(\sigma(u_R^{(n)}(s, y_2)) - \sigma(u_R^{(n-1)}(s, y_2)) \right),$$

and

$$V_2(t, s, R, n, x, y_1, y_2) = p_R(t - s, x, y_1)p_R(t - s, x, y_2) \cdot e^{-2\alpha s} \left| u_R^{(n)}(s, y_1) - u_R^{(n-1)}(s, y_1) \right| \cdot \left| u_R^{(n)}(s, y_2) - u_R^{(n-1)}(s, y_2) \right|.$$

By Carlen-Kr ee’s bound [2] for Burkholder-Gundy-Davis inequality, and a similar argument in [10], and (3.3), we have for any $k \geq 2$, $0 \leq t \leq T < \infty$, $0 < \alpha, R < \infty$ and $x \in S_R^2$ that

$$\begin{aligned} & e^{-\alpha t} \left\| u_R^{(n+1)}(t, x) - u_R^{(n)}(t, x) \right\|_k \tag{3.5} \\ &= \left\| e^{-\alpha t} \int_0^t \int_{S_R^2} p_R(t - s, x, y) \left(\sigma(u_R^{(n)}(s, y)) - \sigma(u_R^{(n-1)}(s, y)) \right) W(ds, dy) \right\|_k \\ &\leq 2\sqrt{k} \left\| e^{-\alpha t} \sqrt{\int_{[0, t] \times S_R^2 \times S_R^2} h_R(y_1, y_2) V_1(t, s, R, n, x, y_1, y_2) ds dy_1 dy_2} \right\|_k \\ &\leq 2\mathbb{L}_\sigma \sqrt{k} \left\| \sqrt{\int_{[0, t] \times S_R^2 \times S_R^2} h_R(y_1, y_2) e^{-2\alpha(t-s)} V_2(t, s, R, n, x, y_1, y_2) ds dy_1 dy_2} \right\|_k \\ &\leq 2\mathbb{L}_\sigma \sqrt{k} f_e(\alpha, R, t) \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \left\| u_R^{(n)}(t, x) - u_R^{(n-1)}(t, x) \right\|_k. \end{aligned}$$

Along Lemma 2.3, this implies for any $k \geq 2$, $0 \leq T < \infty$, $0 < \alpha, R < \infty$ and $x \in S_R^2$ that

$$\begin{aligned} & \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \left\| u_R^{(n+1)}(t, x) - u_R^{(n)}(t, x) \right\|_k \tag{3.6} \\ & \leq \frac{\mathbb{L}_\sigma \sqrt{2h_{up}(R)k}}{\sqrt{\alpha}} \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \left\| u_R^{(n)}(t, x) - u_R^{(n-1)}(t, x) \right\|_k. \end{aligned}$$

Define the norm $\| \cdot \|_{\alpha, k}$ for the collection of random fields on $[0, T] \times S_R^2 \times \Omega_R$ by

$$\|X\|_{\alpha, k} = \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \left\| X(t, x, \omega) \right\|_k \tag{3.7}$$

where $X : [0, T] \times S_R^2 \times \Omega_R \mapsto \mathbb{R}$.

Choose $\alpha = k^2 > \max\{4, 2L_\sigma^2 h_{up}(R)\}$ then $\frac{\mathbb{L}_\sigma \sqrt{2h_{up}(R)k}}{\sqrt{\alpha}} = \frac{\mathbb{L}_\sigma \sqrt{2h_{up}(R)}}{k} < 1$.

The contraction mapping principle implies that $u_R^{(n)}(\cdot, \cdot)$ converges to a unique limit in $\| \cdot \|_{k^2, k}$ -norm for $k > \max\{2, L_\sigma \sqrt{2h_{up}(R)}\}$. We denote this limit by $u_R(\cdot, \cdot)$. By Markov’s inequality, for each $0 \leq t \leq T$, $x \in S_R^2$, $u_R(t, x)$ is the P_R -limit of $u_R^{(n)}(t, x)$ and is hence P_R -measurable. $u_R(t, x)$ is unique up to a modification independent of t, x since if $\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} \|u_R(t, x) - \tilde{u}_R(t, x)\|_{k^2, k} = 0$ then almost surely $u_R(t, x) = \tilde{u}_R(t, x)$ for all $0 \leq t \leq T$, $x \in S_R^2$, for every $0 < T, R < \infty$. \square

Next, we want to show the joint measurability of the P_R -limit $u_R(t, x)$. To do this we develop three lemmas, which state the P_R -limit $u_R(t, x)$ is spatial-continuous and time-continuous in $L^k(\Omega)$ for each $k \geq 2$ and is a uniform limit in probability of its Picard iterations, independent of space and time.

Lemma 3.2. For every $0 < T, R < \infty$, let $u_R(t, x)$ be the limit random field defined in Theorem 3.1. Then $u_R(t, x)$ is spatial-continuous in the L^k sense. More precisely, for every $k \geq 2, 0 < t < T, 0 < \epsilon_0 < 1$ there exists a finite positive $R_{mol}(t, \epsilon_0)$ such that for all $R \geq R_{mol}(t, \epsilon_0)$, and every $x, x' \in S_R^2$ such that $\theta(x, x') < t^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$,

$$\|u_R(t, x) - u_R(t, x')\|_k^k \leq \left(4\sqrt{2}C_{\sigma_{up}}\sqrt{kh_{up}(R)}(1 + \epsilon_0)^{1/3}R^{4/3}\theta(x, x')^{1/3}\right)^k. \quad (3.8)$$

Proof. Assume throughout the proof that $k \geq 2$. Denote for every positive integer $n, 0 < s, R < \infty, x, x', y_1, y_2 \in S_R^2$,

$$Q_{n,\sigma}(s, R, x, x', y_1, y_2) = [p_R(s, \theta(x, y_1))\sigma(u^{(n)}(s, y_1)) - p_R(s, \theta(x', y_1))\sigma(u^{(n)}(s, y_1))] \\ \times [p_R(s, \theta(x, y_2))\sigma(u^{(n)}(s, y_2)) - p_R(s, \theta(x', y_2))\sigma(u^{(n)}(s, y_2))],$$

and

$$Q(s, R, x, x', y_1, y_2) = |p_R(s, \theta(x, y_1)) - p_R(s, \theta(x', y_1))| \cdot |p_R(s, \theta(x, y_2)) - p_R(s, \theta(x', y_2))|,$$

for notational brevity. By Carlen-Kr ee’s optimal bound ([2]) on the Burkholder-Gundy-Davis inequality, for every $0 < t, R < \infty$ and $x, x' \in S_R^2$,

$$\|u_R^{(n+1)}(t, x) - u_R^{(n+1)}(t, x')\|_k^k \quad (3.9) \\ \leq (2\sqrt{k})^k \left\| \sqrt{\int_0^t ds \int_{S_R^2 \times S_R^2} dy_1 dy_2 h_R(y_1, y_2) Q_{n,\sigma}(s, R, x, x', y_1, y_2)} \right\|_k^k \\ \leq (2\sqrt{k})^k \left(h_{up}(R) C_{\sigma_{up}}^2 \int_0^t ds \int_{S_R^2 \times S_R^2} dy_1 dy_2 Q(s, R, x, x', y_1, y_2) \right)^{k/2}.$$

For every $0 < \delta < t, 0 < R < \infty, x, x' \in S_R^2$,

$$\int_0^t \int_{S_R^2 \times S_R^2} Q(s, R, x, x', y_1, y_2) ds dy_1 dy_2 \quad (3.10) \\ = \int_0^\delta \int_{S_R^2 \times S_R^2} Q(s, R, x, x', y_1, y_2) ds dy_1 dy_2 + \int_\delta^t \int_{S_R^2 \times S_R^2} Q(s, R, x, x', y_1, y_2) ds dy_1 dy_2.$$

Since p_R is a transition density function, for every $0 < \delta < t, 0 < R < \infty, x, x' \in S_R^2$,

$$\int_0^\delta \int_{S_R^2 \times S_R^2} Q(s, R, x, x', y_1, y_2) ds dy_1 dy_2 \quad (3.11) \\ \leq \int_0^\delta \int_{S_R^2 \times S_R^2} (p_R(s, \theta(x, y_1)) + p_R(s, \theta(x', y_1))) (p_R(s, \theta(x, y_2)) + p_R(s, \theta(x', y_2))) ds dy_1 dy_2 \\ = 4\delta.$$

Denote for any $0 < \delta < t < \infty, 0 < R < \infty, x, x', y \in S_R^2$,

$$S(\delta, t, R, x, x', y) = \sum_{l=1}^\infty \frac{2l+1}{2\pi l(l+1)} \left(e^{-l(l+1)\delta/2R^2} - e^{-l(l+1)t/2R^2} \right) \\ \times [P_l(\cos \theta(x, y)) - P_l(\cos \theta(x', y))],$$

for notational brevity. Then for any $0 < \delta < t, 0 < R < \infty, x, x' \in S_R^2$,

$$\int_\delta^t \int_{S_R^2 \times S_R^2} Q(s, R, x, x', y_1, y_2) ds dy_1 dy_2 \\ = \int_{S_R^2 \times S_R^2} S(\delta, t, R, x, x', y_1) S(\delta, t, R, x, x', y_2) dy_1 dy_2. \quad (3.12)$$

By uniform convergence and that $\sup_{-1 \leq a \leq 1} |P'_l(a)| \leq l(l+1)/2$, we have

$$\left| \sum_{l=1}^{\infty} \frac{2l+1}{2\pi l(l+1)} \left(e^{-\frac{l(l+1)\delta}{2R^2}} - e^{-\frac{l(l+1)t}{2R^2}} \right) (P_l(a) - P_l(b)) \right| \leq \left(\sum_{l=1}^{\infty} \frac{2l+1}{4\pi} e^{-\frac{l(l+1)\delta}{2R^2}} \right) |a-b|, \tag{3.13}$$

for any $0 < \delta < t$ and any $-1 \leq a, b \leq 1$.

Note that $\sum_{l=1}^{\infty} \frac{2l+1}{4\pi} e^{-\frac{l(l+1)\delta}{2R^2}} = p_1\left(\frac{\delta}{R^2}, z, z\right)$ for any $0 < \delta, R < \infty$ and any $z \in S_R^2$. By Molchanov's heat kernel estimate (Lemma 2.1), for every $0 < \delta < t, 0 < \epsilon_0 < 1$ there exists $0 < R_{mol}(t, \epsilon_0) < \infty$ such that for any $z \in S^2$ and any $R \geq R_{mol}(t, \epsilon_0)$,

$$p_1(\delta/R^2, z, z) \leq \frac{(1 + \epsilon_0)R^2}{2\pi\delta}.$$

Hence, for any $-1 \leq a, b \leq 1, 0 < \delta < t, 0 < \epsilon_0 < 1$ and $R \geq R_{mol}(t, \epsilon_0)$,

$$\left| \sum_{l=1}^{\infty} \frac{2l+1}{2\pi l(l+1)} \left(e^{-\frac{l(l+1)\delta}{2R^2}} - e^{-\frac{l(l+1)t}{2R^2}} \right) (P_l(a) - P_l(b)) \right| \leq \frac{(1 + \epsilon_0)R^2}{2\pi\delta} |a-b|. \tag{3.14}$$

Use (3.14), the triangle inequality and the trigonometric inequality $|\cos \alpha - \cos \beta| \leq |\alpha - \beta|$ in (3.12) to get for any $0 < \delta < t, 0 < \epsilon_0 < 1$ and $R > R_{mol}(t, \epsilon_0)$, and any $x, x' \in S_R^2$,

$$\int_{\delta}^t \int_{S_R^2 \times S_R^2} Q(s, R, x, x', y_1, y_2) ds dy_1 dy_2 \leq \frac{4(1 + \epsilon_0)^2 R^8}{\delta^2} \theta(x, x')^2. \tag{3.15}$$

Use (3.11) and (3.15) in (3.10) to get, for any $0 < \delta < t, 0 < \epsilon_0 < 1$ and $R > R_{mol}(t, \epsilon_0)$, and any $x, x' \in S_R^2$,

$$\int_0^t \int_{S_R^2 \times S_R^2} Q(s, R, x, x', y_1, y_2) ds dy_1 dy_2 \leq 4\delta + \frac{4(1 + \epsilon_0)^2 R^8}{\delta^2} \theta(x, x')^2. \tag{3.16}$$

Take

$$\delta = (1 + \epsilon_0)^{2/3} R^{8/3} \theta(x, x')^{2/3}. \tag{3.17}$$

Then for any $0 < t < \infty, 0 < \epsilon_0 < 1$ and $R > R_{mol}(t, \epsilon_0)$, and any $x, x' \in S_R^2$ such that

$$\theta(x, x') < \frac{t^{3/2}}{(1 + \epsilon_0)R^4}, \tag{3.18}$$

we have $\delta < t$. (3.14) and hence (3.16) can be applied to give that for any $0 < t < \infty, 0 < \epsilon_0 < 1$ and $R > R_{mol}(t, \epsilon_0)$, and any $x, x' \in S_R^2$ such that $\theta(x, x') < t^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$,

$$\int_0^t \int_{S_R^2 \times S_R^2} Q(s, R, x, x', y_1, y_2) ds dy_1 dy_2 \leq 8(1 + \epsilon_0)^{2/3} R^{8/3} \theta(x, x')^{2/3}. \tag{3.19}$$

Use (3.19) in (3.9) to get for any $0 < t < \infty, 0 < \epsilon_0 < 1, R > R_{mol}(t, \epsilon_0)$, and $x, x' \in S_R^2$ such that $\theta(x, x') < t^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$,

$$\left\| u_R^{(n+1)}(t, x) - u_R^{(n+1)}(t, x') \right\|_k^k \leq \left(4\sqrt{2}C_{\sigma_{up}} \sqrt{kh_{up}(R)} (1 + \epsilon_0)^{1/3} R^{4/3} \theta(x, x')^{1/3} \right)^k. \tag{3.20}$$

Let $n \rightarrow \infty$ to get for any $0 < t < \infty, 0 < \epsilon_0 < 1$ and $R > R_{mol}(t, \epsilon_0)$, and any $x, x' \in S_R^2$, such that $\theta(x, x') < t^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$,

$$\left\| u_R(t, x) - u_R(t, x') \right\|_k^k \leq \left(4\sqrt{2}C_{\sigma_{up}} \sqrt{kh_{up}(R)} (1 + \epsilon_0)^{1/3} R^{4/3} \theta(x, x')^{1/3} \right)^k. \tag{3.21}$$

□

Lemma 3.3. For every $0 < T, R < \infty$, let $u_R(t, x)$ be the limit random field defined in Theorem 3.1. Then $u_R(t, x)$ is time-continuous in L^k sense. More precisely, for every $k \geq 2, 0 < t_1 < t_2 \leq T, R > 0$,

$$\sup_{x \in S_R^2} \left\| u_R(t_1, x) - u_R(t_2, x) \right\|_k^k \leq (2\sqrt{k})^k \left(h_{up}(R) C_{\sigma_{up}}^2 (t_2 - t_1) \right)^{k/2}. \tag{3.22}$$

Proof. By Carlen’s optimal bound ([2]) on Burkholder-Gundy-Davis inequality and Lemma 2.2, for every $0 < t_1 < t_2 < \infty, x \in S_R^2$,

$$\begin{aligned} & \left\| u_R^{(n)}(t_1, x) - u_R^{(n)}(t_2, x) \right\|_k^k & (3.23) \\ &= \left\| \int_{t_1}^{t_2} p_R(s, \theta(x, y)) \sigma \left(u_R^{(n)}(s, y) \right) \right\|_k^k \\ &\leq (2\sqrt{k})^k \left(h_{up}(R) C_{\sigma_{up}}^2 \int_{t_1}^{t_2} \int_{S_R^2 \times S_R^2} p_R(s, \theta(x, y_1)) p_R(s, \theta(x, y_2)) ds dy_1 dy_2 \right)^{k/2} \\ &\leq (2\sqrt{k})^k \left(h_{up}(R) C_{\sigma_{up}}^2 (t_2 - t_1) \right)^{k/2}. \end{aligned}$$

Let $n \rightarrow \infty$ to finish. □

Lemma 3.4. For every $k \geq 2, 0 < T < \infty$, there exists a finite positive $R_{mol}(T)$ such that for all $R \geq R_{mol}(T)$, there exists a full probability space $\Omega_{T,R}$ on which $\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} |u_R^{(n)}(t, x) - u_R(t, x)|$ is P_R -measurable. Moreover, for all nonnegative integer n and $\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} |u_R^{(n)}(t, x) - u_R(t, x)|$ converges to zero almost surely as $n \rightarrow \infty$.

Proof. For each positive integer n , define

$$T_n = \{T \cdot 2^{-n}, 2T \cdot 2^{-n}, 3T \cdot 2^{-n}, \dots, T\}, \tag{3.24}$$

and

$$\begin{aligned} G_{R,n} = \{ & x \in S_R^2 : x = (R \sin(i_1 \pi 4^{-n}) \cos(2i_2 \pi 4^{-(n+1)}), R \sin(i_1 \pi 4^{-n}) \sin(2i_2 \pi 4^{-(n+1)}), \\ & R \cos(i_1 \pi 4^{-n})) \text{ for some } i_1, i_2 \in \mathbb{Z} \}. \end{aligned} \tag{3.25}$$

By Doob’s separability theory, Theorem 2.4 in [8] specifically (since $[0, T] \times S_R^2$ can be parametrized by t, θ, ϕ each of which is linear), for each n there exists a version of $u_R^{(n)}(t, x) - u_R(t, x)$ such that there exists a countable subset of $[0, T] \times S_R^2$, denoted by $D_n(T, R)$ such that $\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} |u_R^{(n)}(t, x) - u_R(t, x)| = \sup_{(t,x) \in D_n(T,R)} |u_R^{(n)}(t, x) - u_R(t, x)|$ and hence $\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} |u_R^{(n)}(t, x) - u_R(t, x)|$ is measurable with respect to P_R .

By throwing away the bad sets for each n where $\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} |u_R^{(n)}(t, x) - u_R(t, x)|$ is non-measurable with respect to P_R , we get a full probability subset $\Omega_{T,R}$ of Ω on which $\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} |u_R^{(n)}(t, x) - u_R(t, x)|$ is P_R -measurable for each n . For the rest of the proof, we redefine for all $0 \leq t \leq T, R > 0$ and $x \in S_R^2$,

$$u_R(t, x, \omega) = u_R(t, x, \omega) \mathbb{1}_{\{\omega \in \Omega_{T,R}\}}, \tag{3.26}$$

and for each nonnegative n

$$u_R^{(n)}(t, x, \omega) = u_R^{(n)}(t, x, \omega) \mathbb{1}_{\{\omega \in \Omega_{T,R}\}}. \tag{3.27}$$

For every $\epsilon > 0, 0 < \epsilon_0 < 1, k \geq 2, 0 < T, R < \infty$, positive integer n ,

$$\begin{aligned} & \mathbb{P} \left(\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} |u_R(t, x) - u_R^{(n)}(t, x)| > \epsilon \right) \\ & \leq \epsilon^{-k} \mathbb{E} \left(\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} |u_R^{(n)}(t, x) - u_R(t, x)|^k \right) \\ & \leq \epsilon^{-k} 2^n 4^{2n+1} 5^{k-1} \sup_{t \in T_n} \sup_{|t'-t| \leq T \cdot 2^{-n}} \sup_{x \in G_{R,n}} \sup_{\theta(x', x) \leq \pi \cdot 4^{-n}} (Mo_1 + Mo_2 + Mo_3 + Mo_4 + Mo_5), \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} Mo_1 &= \mathbb{E} \left(|u_R^{(n)}(t, x) - u_R^{(n)}(t, x')|^k \right), Mo_2 = \mathbb{E} \left(|u_R^{(n)}(t, x') - u_R^{(n)}(t', x')|^k \right), \\ Mo_3 &= \mathbb{E} \left(|u_R^{(n)}(t', x') - u_R(t', x')|^k \right), Mo_4 = \mathbb{E} \left(|u_R(t', x') - u_R(t, x')|^k \right), \\ Mo_5 &= \mathbb{E} \left(|u_R(t, x') - u_R(t, x)|^k \right). \end{aligned}$$

Similar to (3.6) we will get for any $0 < \alpha, T, R < \infty, k \geq 2$ and $m > n$,

$$\begin{aligned} & \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \|u_R^{(m+1)}(t, x) - u_R^{(n)}(t, x)\|_k \\ & \leq \left[\left(\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} \right)^m + \dots + \left(\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} \right)^n \right] \\ & \quad \times \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \|u_R^{(1)}(t, x) - u_R^{(0)}(t, x)\|_k. \end{aligned}$$

Let $m \rightarrow \infty$ to get for any $0 < \alpha < \infty$,

$$\begin{aligned} & \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \|u_R(t, x) - u_R^{(n)}(t, x)\|_k \\ & \leq \frac{\left(\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} \right)^n}{1 - \left(\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} \right)} \sup_{0 < t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \|u_R^{(1)}(t, x) - u_R^{(0)}(t, x)\|_k. \end{aligned}$$

Choose $\alpha = 8\mathbb{L}_\sigma^2 h_{up}(R)k$ to get for every $0 \leq t \leq T < \infty, 0 < R < \infty, k \geq 2$ and $x \in S_R^2$,

$$\|u_R(t, x) - u_R^{(n)}(t, x)\|_k^k \leq e^{8\mathbb{L}_\sigma^2 h_{up}(R)k^2 T} 2^{-(n-1)k} \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} \|u_R^{(1)}(t, x) - u_R^{(0)}(t, x)\|_k^k. \tag{3.29}$$

By taking the supremum on the left, we have for $0 < R < \infty, k \geq 2$ and $x \in S_R^2$,

$$\begin{aligned} \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} \|u_R(t, x) - u_R^{(n)}(t, x)\|_k^k & \leq e^{8\mathbb{L}_\sigma^2 h_{up}(R)k^2 T} 2^{-(n-1)k} \sup_{0 \leq t \leq T} \\ & \sup_{x \in S_R^2} \|u_R^{(1)}(t, x) - u_R^{(0)}(t, x)\|_k^k. \end{aligned}$$

Use the above upper bound, Lemma 3.2 and (3.20), Lemma 3.3 and (3.23) in (3.28) to get for any fixed $k \geq 2$, and every $0 < T < \infty, 0 < \epsilon_0 < 1$, there exists a finite positive $R_{mol}(T, \epsilon_0)$ such that for all $R \geq R_{mol}(T, \epsilon_0)$ and any n such that

$\pi 4^{-n} < (2^{-n}T)^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$ (so Lemma 3.2 and (3.20) can be applied),

$$\begin{aligned} & \mathbb{P} \left(\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} \left| u_R(t, x) - u_R^{(n)}(t, x) \right| > \epsilon \right) \\ & \leq \epsilon^{-k} 2^{5n+2} 5^{k-1} \left(e^{8L_\sigma^2 h_{up}(R)k^2 T} 2^{-(n-1)k} \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} \left\| u_R^{(1)}(t, x) - u_R^{(0)}(t, x) \right\|_k^k \right. \\ & \quad \left. + 2 \left(4\sqrt{2} C_{\sigma_{up}} \sqrt{kh_{up}(R)} (1 + \epsilon_0)^{1/3} R^{4/3} (\pi \cdot 4^{-n})^{1/3} \right)^k + 2(2\sqrt{k})^k \left(h_{up}(R) C_{\sigma_{up}}^2 2^{-n} T \right)^{k/2} \right). \end{aligned}$$

Choose $\epsilon = 2^{-n/4}$ to get for any fixed $k \geq 2$, $0 < \epsilon_0 < 1$, and every $0 < T < \infty$, there exists a finite positive $R_{mol}(T) = R_{mol}(T, \epsilon_0)$ such that for all $R \geq R_{mol}(T)$ and any n such that $\pi 4^{-n} < (2^{-n}T)^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$,

$$\begin{aligned} & \mathbb{P} \left(\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} \left| u_R(t, x) - u_R^{(n)}(t, x) \right| > 2^{-n/4} \right) \\ & \leq 2^{(5+k/4)n+2} 5^{k-1} \left(e^{8L_\sigma^2 h_{up}(R)k^2 T} 2^{-(n-1)k} \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} \left\| u_R^{(1)}(t, x) - u_R^{(0)}(t, x) \right\|_k^k \right. \\ & \quad \left. + 2 \left(4\sqrt{2} C_{\sigma_{up}} \sqrt{kh_{up}(R)} (1 + \epsilon_0)^{1/3} R^{4/3} (\pi \cdot 4^{-n})^{1/3} \right)^k + 2(2\sqrt{k})^k \left(h_{up}(R) C_{\sigma_{up}}^2 2^{-n} T \right)^{k/2} \right). \end{aligned}$$

Choose any $k > 20$ so $2^{(5+k/4)n+2} \cdot 2^{-(n-1)k} = 2^{(5-\frac{3k}{4})n+k+2}$ and $2^{(5+k/4)n+2} \cdot 2^{-2nk/3} = 2^{(5-\frac{5k}{12})n+2}$ and $2^{(5+k/4)n+2} \cdot 2^{-nk/2} = 2^{(5-\frac{k}{4})n+2}$ all decay exponentially fast as $n \rightarrow \infty$. Hence,

$$\sum_{n=1}^{\infty} \mathbb{P} \left(\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} \left| u_R(t, x) - u_R^{(n)}(t, x) \right| > 2^{-n/4} \right) < \infty.$$

Borel-Cantelli's lemma implies there almost surely exists a finite $N(\omega)$ such that for $n \geq N(\omega)$,

$$\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} \left| u_R(t, x) - u_R^{(n)}(t, x) \right| \leq 2^{-n/4}. \tag{3.30}$$

We are now ready to show that the P_R -limit $u_R(t, x)$ is jointly measurable.

Theorem 3.5. *For every $0 < T < \infty$, there exists a finite positive $R_{mol}(T)$ such that for every $R \geq R_{mol}(T)$, there is a version of the P_R -limit $u_R(t, x)$ such that $u_R(\cdot, \cdot, \cdot) : [0, T] \times S_R^2 \times \Omega \mapsto \mathbb{R}$ is measurable.*

Proof. First, make the modification that for all $0 \leq t \leq T$ and $x \in S_R^2$,

$$u_R(t, x, \omega) = u_R(t, x, \omega) \mathbb{1}_{\{\omega \in \Omega_{T,R}\}},$$

and for each nonnegative n ,

$$u_R^{(n)}(t, x, \omega) = u_R^{(n)}(t, x, \omega) \mathbb{1}_{\{\omega \in \Omega_{T,R}\}},$$

where $\Omega_{T,R}$ is given in Lemma 3.4. For notational brevity, define for each real number α random sets

$$\begin{aligned} M_1(\alpha) &= \left\{ t \in [0, T], x \in S_R^2, \omega \in \Omega \mid u_R(t, x, \omega) \geq \alpha \right\} \\ & \quad \cap \left\{ t \in [0, T], x \in S_R^2, \omega \in \Omega \mid u_R(t, x, \omega) = \limsup_{n \rightarrow \infty} u_R^{(n)}(t, x, \omega) \right\}, \end{aligned}$$

and

$$M_2(\alpha) = \left\{ t \in [0, T], x \in S_R^2, \omega \in \Omega \mid u_R(t, x, \omega) \geq \alpha \right\} \\ \cap \left\{ t \in [0, T], x \in S_R^2, \omega \in \Omega \mid u_R(t, x, \omega) \neq \limsup_{n \rightarrow \infty} u_R^{(n)}(t, x, \omega) \right\}.$$

Then

$$M_1(\alpha) \cap M_2(\alpha) = \left\{ t \in [0, T], x \in S_R^2, \omega \in \Omega \mid u_R(t, x, \omega) \geq \alpha \right\}.$$

It suffices to show for each real number α , $M_1(\alpha)$ is measurable with respect to the product measure in the measurable space $[0, T] \times S_R^2 \times \Omega$ since $M_2(\alpha)$ has product measure zero by Lemma 3.4. Note that

$$M_1(\alpha) = \left\{ t \in [0, T], x \in S_R^2, \omega \in \Omega \mid \limsup_{n \rightarrow \infty} u_R^{(n)}(t, x, \omega) \geq \alpha \right\} \\ \cap \left\{ t \in [0, T], x \in S_R^2, \omega \in \Omega \mid u_R(t, x, \omega) = \limsup_{n \rightarrow \infty} u_R^{(n)}(t, x, \omega) \right\}.$$

The set

$$\left\{ t \in [0, T], x \in S_R^2, \omega \in \Omega \mid u_R(t, x, \omega) = \limsup_{n \rightarrow \infty} u_R^{(n)}(t, x, \omega) \right\}$$

has full product measure. Moreover,

$$\left\{ t \in [0, T], x \in S_R^2, \omega \in \Omega \mid \limsup_{n \rightarrow \infty} u_R^{(n)}(t, x, \omega) \geq \alpha \right\} \\ = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} \left\{ t \in [0, T], x \in S_R^2, \omega \in \Omega \mid u_R^{(n)}(t, x, \omega) \geq \alpha - \frac{1}{k} \right\}$$

is jointly measurable since each of $u_R^{(n)}$ is by iteration. This finishes the proof. \square

We now have enough recipes to prove the uniqueness and existence theorem of the mild solution.

Theorem 3.6. *For every $0 < T < \infty$, there exists a finite positive $R_{mol}(T)$ such that for every $R \geq R_{mol}(T)$, the mild solution $u_R(t, x)$ exists for every $0 \leq t \leq T$ and $x \in S_R^2$ and is unique up to a P_R -modification independent of t, x .*

Proof. By Theorem 3.5, for every $0 < T < \infty$, there exists a finite positive $R_{mol}(T)$ such that for every $R \geq R_{mol}(T)$, the P_R -limit $u_R(t, x)$ of the sequence of approximating random field is jointly measurable up to a P_R -modification independent of t, x . Let n go to infinity in (3.2), we see that $u_R(t, x)$ satisfies (1.3) and is unique up to a P_R -modification independent of t, x . \square

4 Spatial continuity

In this section, we apply a version of Kolmogorov's continuity theorem to show the mild solution is spatial-continuous almost surely. Time continuity can also be obtained by a similar method but we do not prove it in the paper since time continuity is not used to prove our main results.

We follow the developments in [14] to prove a spherical version of Kolmogorov's continuity theorem by setting up the Garsia's theorem. Since we are working on spheres, some necessary arguments for the spherical versions of Garsia's theorems and

Kolmogorov's continuity theorem will be given, which will be similar to the arguments in [14]. Further details are given in the appendix.

We begin by setting up some necessary notations and terminologies. Suppose $\{\mu_k\}_{k \geq 2}$ is a sequence of subadditive measure. Fix $k \geq 2$ and $r_0(k) = 1$, define iteratively,

$$r_{n+1}(k) = \sup \left\{ r > 0 : \mu_k(r) = \frac{1}{2} \mu_k(r_n(k)) \right\}. \tag{4.1}$$

Define for every $x \in S_R^2$,

$$\bar{f}_{n,k}(x) = \frac{1}{|B_R(x, r_n(k))|} \int_{B_R(x, r_n(k))} f(z) dz, \tag{4.2}$$

and

$$C_{\mu_k} = \sup_{r > 0} \frac{\mu_k(2r)}{\mu_k(r)}, \tag{4.3}$$

where $B_R(x, r)$ is the geodesic ball centered at x with radius r in S_R^2 and $|\cdot|$ denotes the surface measure. For notational convenience, denote

$$B_R(r) = B_R(N, r), \tag{4.4}$$

where N is the North Pole of S_R^2 .

Define the operator $\tilde{\cdot}$ on spheres by assigning for any $x, z \in S_R^2$ the point $x\tilde{+}z$ to be the isotropic image of z by rotating S_R^2 along the great circle which contains x, z, N from x to N if $x \neq N$ and $x\tilde{+}z = z$ if $x = N$. We can rewrite $\bar{f}_{n,k}(x)$ as

$$\bar{f}_{n,k}(x) = \frac{1}{|B_R(r_n(k))|} \int_{B_R(r_n(k))} f(x\tilde{+}z) dz. \tag{4.5}$$

Define Garsia's integral

$$I_k = \int_{S_R^2} dx \int_{S_R^2} dy \left| \frac{f(x) - f(y)}{\mu_k(R\theta(x, y))} \right|^k. \tag{4.6}$$

The spherical version of Kolmogorov's continuity theorem is based on the following lemma and theorem of Garsia's theory. The proofs on the results of Garsia's theory are omitted in this section and are given in the appendix.

Lemma 4.1. *Suppose $f \in L^1(S_R^2)$, $\bar{f}_{n,k}$ is defined as in (4.2), I_k is defined as in (4.6) and there exists $1 \leq k < \infty$ such that*

1. $I_k < \infty$ and
2. $\int_0^1 |B_R(r)|^{-2/k} d\mu_k(r) < \infty$.

Then $\bar{f}_k = \lim_{n \rightarrow \infty} \bar{f}_{n,k}$ exists and for each integer $l \geq 0$,

$$\sup_{x \in S_R^2} |\bar{f}_k(x) - \bar{f}_{l,k}(x)| \leq 4C_{\mu_k} I_k^{1/k} \int_0^{r_{l+1}(k)} |B_R(r)|^{-2/k} d\mu_k(r). \tag{4.7}$$

Moreover, $\bar{f}_k = f$ a.e.

Theorem 4.2. *Suppose $f \in L^1(S_R^2)$ and \bar{f}_k is defined for some $1 \leq k < \infty$ as in Lemma 4.1. Then for all $x, x' \in S_R^2$ such that $R\theta(x, x') \leq 1$,*

$$|\bar{f}_k(x) - \bar{f}_k(x')| \leq 4C_{\mu_k} (2 + C_{\mu_k}) I_k^{1/k} \int_0^{R\theta(x, x')} |B_R(r)|^{-2/k} d\mu_k(r). \tag{4.8}$$

Our spherical version of Kolmogorov’s theorem will be a consequence of the following two theorems. They will be of use again later when we give an asymptotic upper bound of $\sup_{x \in S_R^2} |u_R(t, x)|$ as $R \rightarrow \infty$.

Theorem 4.3. For every $0 < t < \infty$, $0 < \epsilon_0 < 1$, $0 < a < 2$, $0 < q < 1/3$, there exist finite positive $K(a, q)$ and $R_{mol}(t, \epsilon_0)$ such that for all $k \geq \max\{2, K(a, q)\}$, $R \geq \max\left\{ (3(2\pi)^{-1}(1 + \epsilon_0)^{-1}t^{3/2})^{1/4}, R_{mol}(t, \epsilon_0) \right\}$ and n such that $\pi R 2^{-n} \leq 1$ and $\pi 2^{-n} < t^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$,

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 < \theta(x, x') \leq \pi 2^{-n}} \left| \frac{u_R(t, x) - u_R(t, x')}{(R\theta(x, x'))^q} \right|^k \right] \\ & \leq \pi^{a-4} \left(12288 \sqrt{2h_{up}(R)} C_{\sigma_{up}} (2-a)^{-1} (1 + \epsilon_0)^{1/3} \pi^{13/3-a-q} 2^q R^{1/3-q} \right)^k k^{k/2}. \end{aligned} \tag{4.9}$$

Theorem 4.4. For every $0 < t < \infty$, $0 < \epsilon_0 < 1$, $0 < a < 2$, $0 < q < 1/3$, there exist finite positive $K(a, q)$ and $R_{mol}(t, \epsilon_0)$ such that for all $R \geq \max\left\{ \left(\frac{3t^{3/2}}{2\pi(1+\epsilon_0)} \right)^{1/4}, R_{mol}(t, \epsilon_0) \right\}$, $n \geq \max\left\{ 2, K(a, q), \log_2(\pi R) \right\}$, $\lfloor \log_2(\pi(1 + \epsilon_0)R^4) - 3(\log_2 t)/2 \rfloor + 1$, $0 < \gamma < \infty$,

$$\begin{aligned} & \mathbb{P} \left(\sup_{\theta(x, x') \leq \pi 2^{-n}} |u_R(t, x) - u_R(t, x')| \geq \pi R 2^{-n\gamma} \right) \\ & \leq \pi^{a-4} \left(12288 \sqrt{2h_{up}(R)} C_{\sigma_{up}} (2-a)^{-1} (1 + \epsilon_0)^{1/3} \pi^{10/3-a} 2^q R^{-2/3} \right)^n n^{n/2} 2^{(\gamma-q)n^2}. \end{aligned} \tag{4.10}$$

We postpone the proofs of the above two theorems but state and prove the spatial continuity theorem.

Theorem 4.5. For every $0 < t < \infty$, there exists a finite positive $R(t)$ such that for all $R \geq R(t)$ and $0 < \gamma < 1/3$, there exists a finite positive $n(R, t, \gamma) > \max\left\{ 2, K(\gamma), \log_2(\pi R), \log_2(3\pi R^4/2) - 3(\log_2 t)/2 \right\}$ where $K(\gamma)$ is a finite positive number such that for all positive integer $n \geq n(R, t, \gamma)$,

$$\sup_{\theta(x, x') \leq \pi 2^{-n}} |u_R(t, x) - u_R(t, x')| \leq \pi R 2^{-\gamma n}. \tag{4.11}$$

Moreover, $n(R, t, \gamma)$ is increasing in R .

Proof of Theorem 4.5. By Theorem 4.4, for every $0 < t < \infty$, $0 < \gamma < \frac{1}{3}$, there exist finite positive $K(1, \frac{\gamma}{2} + \frac{1}{6})$ and $R_{mol}(t, 1/2)$ such that for all $R \geq \max\left\{ (t^{3/2}/\pi)^{1/4}, R_{mol}(t, 1/2) \right\}$, $n \geq n_0 := \max\left\{ 2, K(1, \frac{\gamma}{2} + \frac{1}{6}), \log_2(\pi R), \lfloor \log_2(3\pi R^4/2) - \frac{3}{2} \log_2 t \rfloor + 1 \right\}$,

$$\begin{aligned} & \sum_{n=n_0}^{\infty} \mathbb{P} \left(\sup_{\theta(x, x') \leq \pi 2^{-n}} |u_R(t, x) - u_R(t, x')| > \pi R 2^{-\gamma n} \right) \\ & \leq \sum_{n=n_0}^{\infty} \pi^{-3} \left(12288 \sqrt{2h_{up}(R)} C_{\sigma_{up}} (3/2)^{1/3} \pi^{7/3} 2^{\frac{\gamma}{2} + \frac{1}{6}} R^{-2/3} \right)^n n^{n/2} 2^{(\frac{\gamma}{2} - \frac{1}{6})n^2}. \end{aligned} \tag{4.12}$$

By the Borel-Cantelli Lemma, almost surely there exists a random finite N , $N \geq n_0$ such that for $n \geq N$,

$$\sup_{\theta(x, x') \leq \pi 2^{-n}} |u_R(t, x) - u_R(t, x')| \leq \pi R 2^{-\gamma n}. \tag{4.13}$$

□

We finish this section by demonstrating the proof of Theorem 4.3 and Theorem 4.4.

Proof of Theorem 4.3. Lemma 3.2 implies that for every $k \geq 2$, $0 < t < \infty$, $0 < \epsilon_0 < 1$, there exists $0 < R_{mol}(t, \epsilon_0) < \infty$ such that for all $R > R_{mol}(t, \epsilon_0)$ and $x, x' \in S_R^2$ such that $\theta(x, x') < \frac{t^{3/2}}{(1+\epsilon_0)R^4}$,

$$\|u_R(t, x) - u_R(t, x')\|_k^k \leq \left(4\sqrt{2}C_{\sigma_{up}}\sqrt{kh_{up}(R)}(1 + \epsilon_0)^{1/3}R^{4/3}\theta(x, x')^{1/3}\right)^k. \quad (4.14)$$

By (4.14) and Fubini's theorem we have local integrability for $u_R(t, \cdot)$ so for a fixed $k \geq 2$, we can define

$$\bar{u}_R(t, x) = \liminf_{n \rightarrow \infty} \frac{1}{|B_R(x, r_n(k))|} \int_{B_R(x, r_n(k))} u_R(t, y) dy, \quad (4.15)$$

where $r_n(k)$ is such that $\mu_k(r_{n+1}(k)) = \mu_k(r_n(k))/2$.

Define

$$I_k = \int_{S_R^2} dx \int_{S_R^2} dx' \left| \frac{u_R(t, x) - u_R(t, x')}{\mu_k(R\theta(x, x'))} \right|^k. \quad (4.16)$$

By (4.14), for any $0 < a < 2$, $k \geq 2$, $0 < t < \infty$, $0 < \epsilon_0 < 1$, $R > R_{mol}(t, \epsilon_0)$ and $x, x' \in S_R^2$ such that $\theta(x, x') < t^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$,

$$\begin{aligned} EI_k &\leq \int_{S_R^2} dx \int_{S_R^2} dx' \frac{\left(4\sqrt{2}C_{\sigma_{up}}\sqrt{kh_{up}(R)}(1 + \epsilon_0)^{1/3}R^{4/3}\theta(x, x')^{1/3}\right)^k}{(R\theta(x, x'))^{a+\frac{k}{3}}} \\ &= \left(4\sqrt{2}C_{\sigma_{up}}\sqrt{kh_{up}(R)}(1 + \epsilon_0)^{1/3}\right)^k R^{4-a+k} \int_{S^2} \left(2\pi \int_0^\pi \theta^{-a} \sin \theta d\theta\right) dx_1 \\ &\leq \left(4\sqrt{2}C_{\sigma_{up}}\sqrt{kh_{up}(R)}(1 + \epsilon_0)^{1/3}\right)^k R^{4-a+k} \left(8\pi^2 \int_0^\pi \theta^{1-a} d\theta\right) \\ &= \left((2 - a)^{-1}\pi^{4-a}32\sqrt{2}C_{\sigma_{up}}\sqrt{kh_{up}(R)}(1 + \epsilon_0)^{1/3}\right)^k R^{4-a+k}. \end{aligned} \quad (4.17)$$

The identity $1 - \cos \theta = 2 \sin^2(\theta/2)$ gives us

$$\begin{aligned} \int_0^{\pi R} |B_R(x, r)|^{-2/k} dr &= \int_0^{\pi R} \left(2\pi \int_0^{r/R} R^2 \sin \theta d\theta\right)^{-2/k} dr \\ &= \frac{1}{(4\pi R^2)^{2/k}} \int_0^{\pi R} \frac{1}{(\sin(r/2R))^{4/k}} dr. \end{aligned} \quad (4.18)$$

For $k > 4$, (4.18) is finite.

Apply Lemma 4.1 to $u_R(t, x)$ to get for any $k > 4$, $0 < t, R < \infty$ and almost all $x \in S_R^2$,

$$\bar{u}_R(t, x) = \lim_{n \rightarrow \infty} \frac{1}{|B_R(x, r_n(k))|} \int_{B_R(x, r_n(k))} u_R(t, y) dy, \quad (4.19)$$

and is spatial-continuous.

By Fatou's lemma, Fubini's theorem, and (4.14), for every $m > 0$, $k > 4$, $0 < \epsilon_0 < 1$,

$0 < t < \infty, R > R_{mol}(t, \epsilon_0),$

$$\begin{aligned} & \mathbb{P}(|\bar{u}_R(t, x) - u_R(t, x)| > 2^{-m}) \\ & \leq 2^m \mathbb{E} \left(\liminf_{n \rightarrow \infty} \frac{\int_{B_R(x, r_n(k))} |u_R(t, y) - u_R(t, x)| dy}{|B_R(x, r_n(k))|} \right) \\ & \leq 2^m \liminf_{n \rightarrow \infty} \left(\frac{\int_{B_R(x, r_n(k))} \mathbb{E}[|u_R(t, y) - u_R(t, x)|^k] dy}{|B_R(x, r_n(k))|} \right) \\ & \leq 2^m \lim_{n \rightarrow \infty} \left(4\sqrt{2} C_{\sigma_{up}} \sqrt{kh_{up}(R)(1 + \epsilon_0)^{1/3} R^{4/3} r_n(k)^{1/3}} \right)^k \\ & = 0. \end{aligned}$$

Let $m \rightarrow \infty$, then for any $x \in S_R^2$, $\bar{u}_R(t, x) = u_R(t, x)$ on a full probability subset Ω_x of Ω . By Doob's separability theory, $\sup_{x \in S_R^2} |\bar{u}_R(t, x) - u_R(t, x)| = \sup_{x \in D_R} |\bar{u}_R(t, x) - u_R(t, x)|$ on a full probability subset Ω_0 of Ω where D_R is a countable subset of S_R^2 . Then on the full probability subset $\Omega_0 \cup (\cup_{x \in D_R} \Omega_x)$, $\sup_{x \in S_R^2} |\bar{u}_R(t, x) - u_R(t, x)| = 0$. This shows for each $0 < t < \infty$, $\bar{u}_R(t, \cdot)$ is an a.s.-continuous modification of $u_R(t, \cdot)$, independent of the spatial variable. For any fixed $0 < a < 2, k \geq 2$, take

$$\mu_k(r) = r^{\frac{1}{3} + \frac{a}{k}}. \tag{4.20}$$

Then

$$C_{\mu_k} = 2^{\frac{1}{3} + \frac{a}{k}}. \tag{4.21}$$

By Theorem 4.2, for every $0 < t < \infty, k \geq 2$ and $x, x' \in S_R^2$ such that $R\theta(x, x') \leq 1$,

$$\begin{aligned} & |\bar{u}_R(t, x) - \bar{u}_R(t, x')| \tag{4.22} \\ & \leq 4 \cdot 2^{1/3+a/k} \left(2 + 2^{1/3+a/k} \right) I_k^{1/k} \int_0^{R\theta(x, x')} |B_R(r)|^{-2/k} d(r^{1/3+a/k}). \end{aligned}$$

Using the identity $\cos \theta = 1 - 2 \sin^2(\theta/2)$, we have for k such that $\frac{3a+k}{3a-12+k} (4/\pi)^{2/k} \leq 2$, every $0 < t < \infty$, every $0 < \epsilon_0 < 1$, and $R \geq \max \{ (3(2\pi)^{-1}(1 + \epsilon_0)^{-1}t^{3/2})^{1/4}, R_{mol}(t, \epsilon_0) \}$ and $x, x' \in S_R^2$ such that $\theta(x, x') \leq t^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$,

$$\begin{aligned} & \int_0^{R\theta(x, x')} |B_R(r)|^{-2/k} d(r^{1/3+a/k}) \tag{4.23} \\ & = \left(\frac{1}{3} + \frac{a}{k} \right) (4\pi)^{-2/k} R^{(a-4)/k+1/3} \int_0^{\theta(x, x')} u^{a/k-2/3} (\sin(u/2))^{-4/k} du \\ & \leq \left(\frac{1}{3} + \frac{a}{k} \right) (4\pi)^{-2/k} R^{(a-4)/k+1/3} \int_0^{\theta(x, x')} u^{a/k-2/3} (u/4)^{-4/k} du \\ & \leq 2R^{(a-4)/k+1/3} \theta(x, x')^{(a-4)/k+1/3}. \end{aligned}$$

Along these lines, it is required that $\theta(x, x') \leq 2\pi/3$ in order to imply that $\sin(u/2) \geq u/4$ for all $0 \leq u \leq \theta(x, x')$. That $\theta(x, x') \leq \theta(x, x') \leq t^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$ and $R \geq (3(2\pi)^{-1}(1 + \epsilon_0)^{-1}t^{3/2})^{1/4}$ will suffice for this purpose.

Define $K_0 = \inf \{ k \geq 0 : \frac{3a+k}{3a-12+k} (4/\pi)^{2/k} \leq 2 \}$ then by (4.22) for every $0 < t < \infty$, every $0 < \epsilon_0 < 1, k \geq K_0, R \geq \max \{ (3(2\pi)^{-1}(1 + \epsilon_0)^{-1}t^{3/2})^{1/4}, R_{mol}(t, \epsilon_0) \}$ and $x, x' \in S_R^2$ such that $\theta(x, x') \leq \theta(x, x') \leq t^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$ and $R\theta(x, x') \leq 1$,

$$|\bar{u}_R(t, x) - \bar{u}_R(t, x')| \leq 8 \cdot 2^{1/3+a/k} \left(2 + 2^{1/3+a/k} \right) I_k^{1/k} R^{(a-4)/k+1/3} \theta(x, x')^{(a-4)/k+1/3}. \tag{4.24}$$

This implies (using the estimate $2^{1/3+a/k} < 6$) for every $0 < t < \infty$, $0 < \epsilon_0 < 1$, $0 < \epsilon \leq 1$, $0 < a < 2$, $k \geq \max\{2, K_0\}$, $R \geq \max\{(3(2\pi)^{-1}(1 + \epsilon_0)^{-1}t^{3/2})^{1/4}, R_{mol}(t, \epsilon_0)\}$ and $x, x' \in S_R^2$ such that $\theta(x, x') \leq \theta(x, x') \leq t^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$ and $R\theta(x, x') \leq \epsilon$,

$$|\bar{u}_R(t, x) - \bar{u}_R(t, x')|^k \leq 384^k I_k \epsilon^{k/3+a-4}. \tag{4.25}$$

By (4.17), (4.25), for every $0 < t < \infty$, $0 < \epsilon_0 < 1$, $0 < a < 2$, $k \geq \max\{2, K_0\}$, $R \geq \max\{(3(2\pi)^{-1}(1 + \epsilon_0)^{-1}t^{3/2})^{1/4}, R_{mol}(t, \epsilon_0)\}$, n such that $\pi R 2^{-n} \leq 1$ and $\pi 2^{-n} < \theta(x, x') \leq t^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$, and $0 < q < \frac{1}{3}$,

$$\begin{aligned} & \mathbb{E} \left[\sup_{\pi 2^{-(n+1)} \leq \theta(x, x') \leq \pi 2^{-n}} \left| \frac{\bar{u}_R(t, x) - \bar{u}_R(t, x')}{(R\theta(x, x'))^q} \right|^k \right] \tag{4.26} \\ & \leq \mathbb{E} \left[\sup_{\theta(x, x') \leq \pi 2^{-n}} \left| \frac{\bar{u}_R(t, x) - \bar{u}_R(t, x')}{(\pi R 2^{-(n+1)})^q} \right|^k \right] \\ & \leq 384^k \mathbb{E} I_k (\pi R 2^{-n})^{k/3+a-4} \frac{1}{(\pi R 2^{-(n+1)})^{qk}} \\ & \leq \pi^{a-4} \left(12288 \sqrt{2h_{up}(R)} C_{\sigma_{up}} (2-a)^{-1} (1 + \epsilon_0)^{1/3} \pi^{13/3-a-q} 2^q R^{1/3-q} \right)^k k^{k/2} \\ & \quad \times 2^{-n((\frac{1}{3}-q)k+a-4)}. \end{aligned}$$

Define $K_1 = \inf\{k \geq 0 : 2^{-((1/3-q)k+a-4)} < 1\}$. Summing from n to ∞ in (4.26) to get for every $0 < t < \infty$, $0 < \epsilon_0 < 1$, $0 < a < 2$, $0 < q < \frac{1}{3}$, $k \geq \max\{2, K_0, K_1\}$, $R \geq \max\{(3(2\pi)^{-1}(1 + \epsilon_0)^{-1}t^{3/2})^{1/4}, R_{mol}(t, \epsilon_0)\}$, n such that $\pi R 2^{-n} \leq 1$ and $\pi 2^{-n} < t^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$,

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 < \theta(x, x') \leq \pi 2^{-n}} \left| \frac{\bar{u}_R(t, x) - \bar{u}_R(t, x')}{(R\theta(x, x'))^q} \right|^k \right] \tag{4.27} \\ & \leq \pi^{a-4} \left(12288 \sqrt{2h_{up}(R)} C_{\sigma_{up}} (2-a)^{-1} (1 + \epsilon_0)^{1/3} \pi^{13/3-a-q} 2^q R^{1/3-q} \right)^k k^{k/2} \\ & \quad \times \frac{2^{-n((\frac{1}{3}-q)k+a-4)}}{1 - 2^{-(k(\frac{1}{3}-q)+a-4)}} \\ & \leq \pi^{a-4} \left(12288 \sqrt{2h_{up}(R)} C_{\sigma_{up}} (2-a)^{-1} (1 + \epsilon_0)^{1/3} \pi^{13/3-a-q} 2^q R^{1/3-q} \right)^k k^{k/2}. \end{aligned}$$

This finishes the proof since $\bar{u}_R(t, \cdot)$ is a version of $u_R(t, \cdot)$ with the modification uniform in the spatial variable. \square

Proof of Theorem 4.4. By Markov's inequality, and Theorem 4.3, for every $0 < t, \gamma < \infty$, $0 < \epsilon_0 < 1$, $0 < a < 2$, $0 < q < \frac{1}{3}$, there exist finite positive $K(a, q)$ and $R_{mol}(t, \epsilon_0)$ such that for all $k \geq \max\{2, K(a, q)\}$, $R \geq \max\{(3(2\pi)^{-1}(1 + \epsilon_0)^{-1}t^{3/2})^{1/4}, R_{mol}(t, \epsilon_0)\}$ and n such that $\pi R 2^{-n} \leq 1$ and $\pi 2^{-n} < t^{3/2}(1 + \epsilon_0)^{-1}R^{-4}$,

$$\begin{aligned} & \mathbb{P} \left(\sup_{\theta(x, x') \leq \pi 2^{-n}} |u_R(t, x) - u_R(t, x')| > \pi R 2^{-\gamma n} \right) \tag{4.28} \\ & \leq \mathbb{E} \left[\sup_{\theta(x, x') \leq \pi 2^{-n}} |u_R(t, x) - u_R(t, x')|^k \right] \frac{1}{(\pi R 2^{-\gamma n})^k} \\ & \leq \pi^{a-4} \left(12288 \sqrt{2h_{up}(R)} C_{\sigma_{up}} (2-a)^{-1} (1 + \epsilon_0)^{1/3} \pi^{10/3-a} 2^q R^{-2/3} \right)^k k^{k/2} 2^{(\gamma-q)nk}. \end{aligned}$$

Choose $k = n$ to finish the proof. □

Remark 6. It is worth mentioning that the spatial continuity could possibly be proved by the method in [25] which favors a semi-group approach on Banach spaces. Since it is not the major concern of this paper, we will not give further arguments on this alternative approach in proving spatial continuity.

5 An asymptotic lower bound of the supremum of the mild solution

Following the idea of [4], we show in this section that for some fixed positive constant $C(t)$ which depends on a fixed finite positive t , $\sup_{x \in S_R^2} |u_R(t, x)| \geq C(t)\sqrt{\log R}$ asymptotically as $R \rightarrow \infty$ with high probability. The goal of this section is to prove the following main theorem.

Theorem 5.1. *Assume $\sup_{R>0} \sup_{x \in S_R^2} |u_R(t, x)| \leq U < \infty$. For every $0 < t < \infty$, $0 < \epsilon_0 < 1$, there exists a finite positive constant $C(t, \epsilon_0)$ such that*

$$\lim_{R \rightarrow \infty} P \left(\exists x \in S_R^2 : |u_R(t, x)| \geq C(t, \epsilon_0) (\log R)^{1/4 + C_{h_{down}}/4 - C_{h_{up}}/8} \right) = 1. \tag{5.1}$$

Moreover, for all $0 < t < \infty$, $0 < \epsilon_0 < 1$, finite positive constant C , there exist finite positive constants $C(t, \epsilon_0)$ and $R(t, \epsilon_0, C)$ such that for $R \geq R(t, \epsilon_0, C)$,

$$P \left(\exists x \in S_R^2 : |u_R(t, x)| \geq C(t, \epsilon_0) (\log R)^{1/4 + C_{h_{down}}/4 - C_{h_{up}}/8} \right) \geq 1 - R^{-C\pi e^{-2(1/2 - C_{h_{up}}/4)}}. \tag{5.2}$$

We begin with some important definitions and lemmas that lead to the proof of Theorem 5.1.

Definition 5.1. *Define the “space-truncated” coupling process by*

$$U_{t,R}^{(\beta)}(x) = \int_{S_R^2} p_R(t, \theta(x, y)) u_{R,0}(y) dy + \int_{(0,t) \times B_R(x, \sqrt{\beta t})} p_R(t-s, \theta(x, y)) \sigma(U_{t,R}^{(\beta)}(y)) W(ds, dy). \tag{5.3}$$

Definition 5.2. *Define the n -th step Picard iteration of the “space-truncated” coupling process by*

$$U_{t,R}^{(\beta,0)}(x) = u_{R,0}(x)$$

and

$$U_{t,R}^{(\beta,n)}(x) = \int_{S_R^2} p_R(t, \theta(x, y)) U_{0,R}^{(\beta,(n-1))}(y) dy + \int_{(0,t) \times B_R(x, \sqrt{\beta t})} p_R(t-s, \theta(x, y)) \sigma(U_{t,R}^{(\beta,(n-1))}(y)) W(ds, dy).$$

As in Section 3, we can show the mild solution of (5.3) exists as the unique P-limit of its Picard iterations, is jointly measurable in time, space and probability and has spatial continuity up to a modification by Doob’s separability theory. By the same argument as in [4], we have the following independence result.

Lemma 5.2. *For every $0 < \beta, t, R, n < \infty$, and $x_1, x_2, \dots \in S_R^2$ such that $d(x_i, x_j) > 2n\sqrt{\beta t}$ whenever $i \neq j$, $\{U_{t,R}^{(\beta,n)}(x_j)\}_{j=1,2,\dots}$ is a collection of i.i.d random variables.*

As a first step, we find upper and lower bounds of the moments of $U_{t,R}^{(\beta,n)}(x)$ to give a tail probability estimate of $U_{t,R}^{(\beta,n)}(x)$. The following gives lower bounds of the moments of $U_{t,R}^{(\beta,n)}(x)$.

Lemma 5.3. *For every $0 < t, \beta < \infty$, $0 < \epsilon_0 < 1$, there exists a finite positive $R_{mol}(t, \pi/4, \epsilon_0)$ such that for every positive integers k , and every R such that $R \geq \max \{R_{mol}(t, \pi/4, \epsilon_0), 4\sqrt{\beta t}/\pi\}$,*

$$E \left[U_{t,R}^{(\beta,n)}(x)^{2k} \right] \geq \frac{2\sqrt{\pi}}{e} \left(\frac{4\pi^2 h_{lo}(R) t C_{\sigma_{down}}^2 (1 - \epsilon_0)^2 (1 - e^{-\beta/2})^{2k}}{e} \right)^k. \tag{5.4}$$

Proof. Take $t = 0$ in Definition 5.2 to get for every $0 < \beta, R < \infty$, positive integer n , $x \in S_R^2$,

$$U_{0,R}^{(\beta,n)}(x) = \int_{S_R^2} p_R(0, \theta(x, y)) U_{0,R}^{(\beta,n-1)}(y) dy$$

As in Section 3, we get by induction

$$U_{0,R}^{(\beta,n)}(x) = u_{R,0}(x), \tag{5.5}$$

and

$$U_{t,R}^{(\beta,n)}(x) = \int_{S_R^2} p_R(t, \theta(x, y)) u_{R,0}(y) dy \tag{5.6}$$

$$+ \int_{[0,t] \times B_R(x, \sqrt{\beta t})} p_R(t-s, \theta(x, y)) \sigma(U_{t,R}^{(\beta,n-1)}(y)) W(ds, dy).$$

Define a martingale $\{M(u)\}_{0 \leq u \leq t}$ by

$$M(u) = \int_{S_R^2} p_R(t, \theta(x, y)) u_{R,0}(y) dy \tag{5.7}$$

$$+ \int_{[0,u] \times B_R(x, \sqrt{\beta t})} p_R(t-s, \theta(x, y)) \sigma(U_{s,R}^{(\beta,n-1)}(y)) W(ds, dy).$$

By Ito's formula, for all $k \geq 1$,

$$M(u)^{2k} = \left(\int_{S_R^2} p_R(t, \theta(x, y)) u_{R,0}(y) dy \right)^{2k} \tag{5.8}$$

$$+ 2k \int_0^u M(s)^{(2k-1)} dM(s) + \frac{2k(2k-1)}{2} \int_0^u M(s)^{(2k-2)} d \langle M, M \rangle_s.$$

Let $u = t$ and take expectation to get

$$E[M(t)^{2k}] = \left(\int_{S_R^2} p_R(t, \theta(x, y)) u_{R,0}(y) dy \right)^{2k} + \frac{2k(2k-1)}{2} E \left[\int_0^t M(s)^{(2k-2)} d \langle M, M \rangle_s \right]. \tag{5.9}$$

For notational brevity, denote

$$g(t, R, n, x, y_1, y_2) = p_R(t-s, \theta(x, y_1)) p_R(t-s, \theta(x, y_2)) \sigma(U_{t,R}^{(\beta,n-1)}(y_1)) \sigma(U_{t,R}^{(\beta,n-1)}(y_2)), \tag{5.10}$$

and

$$I(t, R, \beta) = \int_{B_R(x, \sqrt{\beta t}) \times B_R(x, \sqrt{\beta t})} p_R(t-s, \theta(x, y_1)) p_R(t-s, \theta(x, y_2)) dy_1 dy_2. \tag{5.11}$$

Then

$$\begin{aligned} \mathbb{E} \left[\int_0^t M(s)^{(2k-2)} d \langle M, M \rangle_s \right] & \tag{5.12} \\ &= \mathbb{E} \left[\int_0^t M(s)^{(2k-2)} ds \int_{B_R(x, \sqrt{\beta t}) \times B_R(x, \sqrt{\beta t})} g(t, R, n, x, y_1, y_2) h_R(y_1, y_2) dy_1 dy_2 \right] \\ &\geq h_{lo}(R) C_{\sigma_{down}}^2 \int_0^t \mathbb{E} \left[M(s)^{(2k-2)} \right] ds I(t, R, \beta). \end{aligned}$$

Define

$$\mu_{R,\beta}(t, ds) := h_{lo}(R) ds \int_{B_R(x, \sqrt{\beta t}) \times B_R(x, \sqrt{\beta t})} p_R(t-s, \theta(x, y_1)) p_R(t-s, \theta(x, y_2)) dy_1 dy_2 \tag{5.13}$$

then (5.12) can be written as

$$\mathbb{E} \left[\int_0^t M(s)^{(2k-2)} d \langle M, M \rangle_s \right] \geq C_{\sigma_{down}}^2 \int_0^t \mathbb{E} \left[M(s)^{(2k-2)} \right] \mu_{R,\beta}(t, ds). \tag{5.14}$$

Use (5.14) in (5.9) to get

$$\mathbb{E}[M(t)^{2k}] \geq \frac{2k(2k-1)C_{\sigma_{down}}^2}{2} \int_0^t \mathbb{E} \left[M(s)^{(2k-2)} \right] \mu_{R,\beta}(t, ds). \tag{5.15}$$

By induction,

$$\begin{aligned} \mathbb{E}[M(t)^{2k}] &\geq \frac{(2k)!C_{\sigma_{down}}^{2k}}{2^k} \int_0^t \mu_{R,\beta}(t, ds_1) \int_0^{s_1} \mu_{R,\beta}(s_1, ds_2) \cdots \int_0^{s_{k-1}} \mu_{R,\beta}(s_{k-1}, ds_k) \tag{5.16} \\ &= \frac{(2k)!C_{\sigma_{down}}^{2k}}{2^k k!} \left(\int_0^t \mu_{R,\beta}(t, ds) \right)^k \\ &= \frac{(2k)!C_{\sigma_{down}}^{2k}}{2^k k!} (f_{e,\beta}(0, R, t))^k. \end{aligned}$$

By Stirling’s approximation that for all positive integer n ,

$$\sqrt{2\pi n}^{n+\frac{1}{2}} e^{-n} \leq n! \leq en^{n+\frac{1}{2}} e^{-n}. \tag{5.17}$$

This together with Lemma 2.5 implies that for every $0 < t, \beta < \infty$, $0 < \epsilon_0 < 1$, there exists a finite positive $R_{mol}(t, \pi/4, \epsilon_0)$ such that for all positive integers k, n , and all $R \geq \max \{R_{mol}(t, \pi/4, \epsilon_0), 4\sqrt{\beta t/\pi}\}$,

$$\mathbb{E} \left[U_{t,R}^{(\beta,n)}(x)^{2k} \right] \geq \frac{2\sqrt{\pi}}{e} \left(\frac{4\pi^2 h_{down}(R) t C_{\sigma_{down}}^2 (1-\epsilon_0)^2 (1-e^{-\beta/2})^2 k}{e} \right)^k. \tag{5.18} \quad \square$$

The next lemma gives upper bounds of the moments of $U_{t,R}^{(\beta,n)}(x)$.

Lemma 5.4. Assume for each finite positive R , $\sup_{x \in S_R^2} |u_{R,0}(x)| \leq U_R < \infty$. Then for every $0 < T, \alpha, R < \infty$, $0 < \beta < \pi^2 R^2/T$, k such that $\mathbb{L}_\sigma \sqrt{2k h_{up}(R)}/\alpha < 1$, and every positive integer n ,

$$\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \|U_{t,R}^{(\beta,n)}(x)\|_k \leq \frac{U_R + |\sigma(0)| \sqrt{2k h_{up}(R)}/\alpha}{1 - \mathbb{L}_\sigma \sqrt{2k h_{up}(R)}/\alpha}. \tag{5.19}$$

Proof. As in the proof of Theorem 3.1 we can apply the Carlen-Krée bound [2] on Burkholder-Gundy-Davis inequality, Lemma 2.3 and a similar argument in [10], to get for $0 \leq t \leq T < \infty$, $R > 0$, $0 < \beta < \pi^2 R^2/T$, $\alpha > 0$ and k such that $\mathbb{L}_\sigma \sqrt{2kh_{up}(R)}/\alpha < 1$,

$$\begin{aligned}
 e^{-\alpha t} \|U_{t,R}^{(\beta,n)}(x)\|_k &\leq e^{-\alpha t} \left| \int_{S_R^2} p_R(t, \theta(x, y)) u_{R,0}(y) dy \right| \\
 &+ 2\sqrt{k f_e(\alpha, R, t)} \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} \left\| e^{-\alpha t} \left(|\sigma(0)| + \mathbb{L}_\sigma \left| U_{t,R}^{(\beta,n-1)}(x) \right| \right) \right\|_k \\
 &\leq U_R + \sqrt{2kh_{up}(R)/\alpha} \left(|\sigma(0)| + \mathbb{L}_\sigma \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \|U_{t,R}^{(\beta,n-1)}(x)\|_k \right).
 \end{aligned}
 \tag{5.20}$$

By induction and a little algebra, we have

$$\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \|U_{t,R}^{(\beta,n)}(x)\|_k \leq \frac{U_R + |\sigma(0)| \sqrt{2kh_{up}(R)/\alpha}}{1 - \mathbb{L}_\sigma \sqrt{2kh_{up}(R)/\alpha}}. \tag{5.21}$$

□

With $U_{t,R}^{(\beta)}(x)$ replacing the role of $U_{t,R}^{(\beta,n)}(x)$ and $U_{t,R}^{(\beta,n-1)}(x)$ in the proof of Lemma 5.4, we will get

Lemma 5.5. Assume for each finite positive R , $\sup_{x \in S_R^2} |u_{R,0}(x)| \leq U_R < \infty$. Then for every $0 < T, \alpha, R < \infty$, $0 < \beta < \pi^2 R^2/T$, k such that $\mathbb{L}_\sigma \sqrt{2kh_{up}(R)}/\alpha < 1$,

$$\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \|U_{t,R}^{(\beta)}(x)\|_k \leq \frac{U_R + |\sigma(0)| \sqrt{2kh_{up}(R)/\alpha}}{1 - \mathbb{L}_\sigma \sqrt{2kh_{up}(R)/\alpha}}. \tag{5.22}$$

The following lemma gives a tail probability estimate based on the previous lemmas.

Lemma 5.6. Assume for each finite positive R , $\sup_{x \in S_R^2} |u_{R,0}(x)| \leq U_R < \infty$. For every $0 < t, \alpha, \beta < \infty$, $0 < \epsilon_0 < 1$, there exists a finite positive $R_{mol}(t, \pi/4, \epsilon_0)$ such that for all positive integer n , and all $R \geq \max \{R_{mol}(t, \pi/4, \epsilon_0), 4\sqrt{\beta t}/\pi\}$, positive integer k such that $\mathbb{L}_\sigma \sqrt{2kh_{up}(R)}/\alpha < 1$, all λ such that $0 < \lambda < \pi \sqrt{h_{down}(R)t/e} C_{\sigma_{down}} (1 - \epsilon_0)(1 - e^{-\beta/2}) \sqrt{k}$,

$$\begin{aligned}
 P \left(|U_{t,R}^{(\beta,n)}(x)| \geq \lambda \right) &\geq \pi e^{-(4\alpha t + 2)k - 2} \left(4\pi^2 h_{down}(R) t C_{\sigma_{down}}^2 (1 - \epsilon_0)^2 (1 - e^{-\beta/2})^2 k \right)^{2k} \\
 &\times \left(\frac{U_R + 2|\sigma(0)| \sqrt{\frac{2kh_{up}(R)}{\alpha}}}{1 - 2\mathbb{L}_\sigma \sqrt{\frac{2kh_{up}(R)}{\alpha}}} \right)^{-4k}.
 \end{aligned}$$

Proof. By the Paley-Zygmund inequality, Lemma 5.3 and Lemma 5.4, we have for every $0 < t, \alpha, \beta < \infty$, $0 < \epsilon_0 < 1$, there exists a finite positive $R_{mol}(t, \pi/4, \epsilon_0)$ such that for all positive integer $n \geq 2$, and all $R \geq \max \{R_{mol}(t, \pi/4, \epsilon_0), 4\sqrt{\beta t}/\pi\}$, positive integer k such that $\mathbb{L}_\sigma \sqrt{2kh_{up}(R)}/\alpha < 1$, all λ such that $0 < \lambda < \pi \sqrt{h_{down}(R)t/e} C_{\sigma_{down}} (1 - \epsilon_0)(1 -$

$$e^{-\beta/2})\sqrt{k},$$

$$\begin{aligned} \mathbb{P}\left(|U_{t,R}^{(\beta,n)}(x)| \geq \lambda\right) &\geq \mathbb{P}\left(|U_{t,R}^{(\beta,n)}(x)| \geq \frac{1}{2}\|U_{t,R}^{(\beta,n)}(x)\|_k\right) \\ &\geq \frac{\left(\mathbb{E}\left[|U_{t,R}^{(\beta,n-1)}(x)|^{2k}\right]\right)^2}{4\mathbb{E}\left[|U_{t,R}^{(\beta,n)}(x)|^{4k}\right]} \\ &\geq \pi e^{-(4\alpha t+2)k-2}\left(4\pi^2 h_{down}(R)tC_{\sigma_{down}}^2(1-\epsilon_0)^2(1-e^{-\beta/2})^2k\right)^{2k} \\ &\quad \times \left(\frac{U_R+2|\sigma(0)|\sqrt{\frac{2kh_{up}(R)}{\alpha}}}{1-2\mathbb{L}_\sigma\sqrt{\frac{2kh_{up}(R)}{\alpha}}}\right)^{-4k}. \quad \square \end{aligned}$$

Now we have obtained a tail probability estimate of $U_{t,R}^{(\beta,n)}(x)$. Based on the approximation to $u_{t,R}(x)$ by $U_{t,R}^{(\beta,n)}(x)$, we can achieve the goal of finding a tail probability estimate of $u_{t,R}(x)$. The accuracy of the approximation is described in the following three lemmas.

Lemma 5.7. Assume for each finite positive R , $\sup_{x \in S_R^2} |u_{R,0}(x)| \leq U_R < \infty$ and that $\alpha \asymp_R \beta \asymp_R (\log R)^c$ where $0 < c < 1$ is a constant. Then for every $0 < t < \infty$, $R > R_{mol}(t)$ where $R_{mol}(t)$ is as in Lemma 2.4 and all $k \geq 2$ such that $\mathbb{L}_\sigma \sqrt{2kh_{up}(R)}/\alpha < 1$,

$$\sup_{x \in S_R^2} e^{-\alpha t} \left\| u_{R,t}(x) - U_{t,R}^{(\beta)}(x) \right\|_k \leq \frac{2\sqrt{2}t^{1/2} \sqrt{kh_{up}(R)} e^{-\sqrt{\alpha}\beta t} \left(|\sigma(0)| + \mathbb{L}_\sigma \frac{U_R + |\sigma(0)|\sqrt{\frac{2kh_{up}(R)}{\alpha}}}{1 - \mathbb{L}_\sigma \sqrt{\frac{2kh_{up}(R)}{\alpha}}} \right)}{1 - \mathbb{L}_\sigma \sqrt{2h_{up}(R)k}/\alpha}.$$

Proof. Recall from Definition 5.1 that

$$\begin{aligned} U_{t,R}^{(\beta)}(x) &= \int_{S_R^2} p_R(t, \theta(x, y)) u_{R,0}(y) dy \\ &\quad + \int_{(0,t) \times B_R(x, \sqrt{\beta t})} p_R(t-s, \theta(x, y)) \sigma(U_{t,R}^{(\beta)}(y)) W(ds, dy). \end{aligned}$$

Define a coupling process by

$$\begin{aligned} V_{t,R}(x) &= \int_{S_R^2} p_R(t, \theta(x, y)) u_{R,0}(y) dy \\ &\quad + \int_{(0,t) \times S_R^2} p_R(t-s, \theta(x, y)) \sigma(U_{t,R}^{(\beta)}(y)) W(ds, dy). \end{aligned}$$

Denote

$$S_{t,R,\beta} = [0, t] \times S_R^2 \setminus B_R(x, \sqrt{\beta t}) \times S_R^2 \setminus B_R(x, \sqrt{\beta t}), \tag{5.23}$$

and

$$g_1(s, R, \beta, x, y_1, y_2) = e^{-2\alpha(t-s)} p_R(t-s, \theta(x, y_1)) p_R(t-s, \theta(x, y_2)) \sigma(U_{s,R}^{(\beta)}(y_1)) \sigma(U_{s,R}^{(\beta)}(y_2)), \tag{5.24}$$

and

$$\begin{aligned} g_2(s, R, \beta, x, y_1, y_2) &= e^{-2\alpha(t-s)} p_R(t-s, \theta(x, y_1)) p_R(t-s, \theta(x, y_2)) \\ &\quad \cdot \left| \sigma(u_{s,R}(y_1)) - \sigma(U_{s,R}^{(\beta)}(y_1)) \right| \cdot \left| \sigma(u_{s,R}(y_2)) - \sigma(U_{s,R}^{(\beta)}(y_2)) \right| \end{aligned} \tag{5.25}$$

for notational brevity.

Then, as in the proof of Theorem 3.1 we can apply Carlen-Krée’s bound on Burkholder-Gundy-Davis inequality [2], Lemma 2.4, a similar argument in [10], and Lemma 5.5 to get under the assumption that $\alpha \asymp_R \beta \asymp_R (\log R)^c$ where $0 < c < 1$ is a constant, for every $0 < t < \infty$ there exists a finite positive $R_{mol}(t)$ such that for all $R > R_{mol}(t)$, $k \geq 2$ such that $\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} < 1$,

$$\begin{aligned}
 e^{-\alpha t} \left\| U_{t,R}^{(\beta)}(x) - V_{t,R}(x) \right\|_k & \tag{5.26} \\
 & \leq 2\sqrt{k} \left\| \sqrt{\int_{S_{t,R,\beta}} h_R(y_1, y_2) e^{-2\alpha s} g_1(s, R, \beta, x, y_1, y_2) ds dy_1 dy_2} \right\|_k \\
 & \leq 2\sqrt{k} \tilde{f}_{e,\beta}(\alpha, R, t) \sup_{t \geq 0} \sup_{x \in S_R^2} e^{-\alpha t} \left(|\sigma(0)| + \mathbb{L}_\sigma \left\| U_{t,R}^{(\beta)}(x) \right\|_k \right) \\
 & \leq 2\sqrt{2}t^{1/2} \sqrt{kh_{up}(R)} e^{-\sqrt{\alpha}\beta t} \left(|\sigma(0)| + \mathbb{L}_\sigma \frac{U_R + |\sigma(0)|\sqrt{2kh_{up}(R)/\alpha}}{1 - \mathbb{L}_\sigma \sqrt{2kh_{up}(R)/\alpha}} \right).
 \end{aligned}$$

By Lemma 2.3, for every $0 < t, R, \alpha < \infty$, $0 < \beta < \pi^2 R^2/t$, $k \geq 2$,

$$\begin{aligned}
 e^{-\alpha t} \left\| u_R(t, x) - V_{t,R}(x) \right\|_k & \tag{5.27} \\
 & \leq 2\sqrt{k} \left\| \sqrt{\int_{S_{t,R,\beta}} h_{up}(R) e^{-2\alpha s} g_2(s, R, \beta, x, y_1, y_2) ds dy_1 dy_2} \right\|_k \\
 & \leq 2\mathbb{L}_\sigma \sqrt{kf_e(\alpha, R, t)} \sup_{x \in S_R^2} e^{-\alpha t} \left\| u_R(t, x) - U_{t,R}^{(\beta)}(x) \right\|_k \\
 & \leq \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} \sup_{x \in S_R^2} e^{-\alpha t} \left\| u_R(t, x) - U_{t,R}^{(\beta)}(x) \right\|_k.
 \end{aligned}$$

From (5.26) and (5.27), we get for every $0 < t < \infty$, $R > R_{mol}(t)$, and $0 < \beta < \pi^2 R^2/16t$, $k \geq 2$ and $\alpha > 0$ such that $\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} < 1$,

$$\begin{aligned}
 \sup_{x \in S_R^2} e^{-\alpha t} \left\| u_R(t, x) - U_{t,R}^{(\beta)}(x) \right\|_k & \tag{5.28} \\
 & \leq 2\sqrt{2}t^{1/2} \sqrt{kh_{up}(R)} e^{-\sqrt{\alpha}\beta t} \left(|\sigma(0)| + \mathbb{L}_\sigma \frac{U_R + |\sigma(0)|\sqrt{2h_{up}(R)k/\alpha}}{1 - \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha}} \right) \\
 & \quad + \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} \sup_{x \in S_R^2} e^{-\alpha t} \left\| u_R(t, x) - U_{t,R}^{(\beta)}(x) \right\|_k.
 \end{aligned}$$

By the same argument, we can also get for every $0 < t < \infty$, $R > R_{mol}(t)$, and $0 < \beta < \pi^2 R^2/16t$, $k \geq 2$ and $\alpha > 0$ such that $\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} < 1$, and every positive integer n ,

$$\begin{aligned}
 \sup_{x \in S_R^2} e^{-\alpha t} \left\| u_{t,R}^{(n)}(x) - U_{t,R}^{(\beta,n)}(x) \right\|_k & \tag{5.29} \\
 & \leq 2\sqrt{2}t^{1/2} \sqrt{kh_{up}(R)} e^{-\sqrt{\alpha}\beta t} \left(|\sigma(0)| + \mathbb{L}_\sigma \frac{U_R + |\sigma(0)|\sqrt{2kh_{up}(R)/\alpha}}{1 - \mathbb{L}_\sigma \sqrt{2kh_{up}(R)/\alpha}} \right) \\
 & \quad + \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} \sup_{x \in S_R^2} e^{-\alpha t} \left\| u_{t,R}^{(n-1)}(x) - U_{t,R}^{(\beta,n-1)}(x) \right\|_k.
 \end{aligned}$$

This rules out the possibility of

$$\sup_{x \in S_R^2} e^{-\alpha t} \left\| u_R(t, x) - U_{t,R}^{(\beta)}(x) \right\|_k = \infty, \tag{5.30}$$

since by assumption

$$\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} < 1. \tag{5.31}$$

After a little algebra in (5.28), we arrive at the inequality

$$\begin{aligned} \sup_{x \in S_R^2} e^{-\alpha t} \left\| u_R(t, x) - U_{t,R}^{(\beta)}(x) \right\|_k & \tag{5.32} \\ & \leq \frac{2\sqrt{2}t^{1/2} \sqrt{k h_{up}(R)} e^{-\sqrt{\alpha}\beta t} \left(|\sigma(0)| + \mathbb{L}_\sigma \frac{U_R + |\sigma(0)| \sqrt{2k h_{up}(R)/\alpha}}{1 - \mathbb{L}_\sigma \sqrt{2k h_{up}(R)/\alpha}} \right)}{1 - \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha}}. \quad \square \end{aligned}$$

Lemma 5.8. Assume for each finite positive R , $\sup_{x \in S_R^2} |u_{R,0}(x)| \leq U_R < \infty$. Then for every $0 < T, R < \infty$, $0 < \beta < \pi^2 R^2/T$, $k \geq 2$, $\alpha > 0$ such that $\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} < 1$ and $\sqrt{2h_{up}(R)k/\alpha} < 1$ and every positive integer n ,

$$\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \left\| U_{t,R}^{(\beta)}(x) - U_{t,R}^{(\beta,n)}(x) \right\|_k \leq (C_{\sigma_{up}} + 2U_R) \frac{\left(\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} \right)^n}{U_R - \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha}}.$$

Proof. As in the proof of Theorem 3.1 we can apply Carlen-Kr ee’s bound on Burkholder-Gundy-Davis inequality [2], Lemma 2.3, a similar argument in [10], and Lemma 5.5 to get for every $0 < T, R < \infty$, $0 < \beta < \pi^2 R^2/T$, $k \geq 2$, $\alpha > 0$ such that $\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} < 1$ and $\sqrt{2h_{up}(R)k/\alpha} < 1$ and every positive integer n ,

$$\begin{aligned} \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \left\| U_{t,R}^{(\beta,n+1)}(x) - U_{t,R}^{(\beta,n)}(x) \right\|_k & \\ & \leq 2\mathbb{L}_\sigma \sqrt{k f_e(\alpha, R, t)} \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \left\| U_{t,R}^{(\beta,n)}(x) - U_{t,R}^{(\beta,n-1)}(x) \right\|_k \\ & \leq \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \left\| U_{t,R}^{(\beta,n)}(x) - U_{t,R}^{(\beta,n-1)}(x) \right\|_k. \end{aligned}$$

By induction, for $m > n$,

$$\begin{aligned} \sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \left\| U_{t,R}^{(\beta,m)}(x) - U_{t,R}^{(\beta,n)}(x) \right\|_k & \tag{5.33} \\ & \leq \frac{\left(\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} \right)^n}{1 - \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha}} \sup_{t \geq 0} \sup_{x \in S_R^2} e^{-\alpha t} \left\| U_{t,R}^{(\beta,1)}(x) - U_{t,R}^{(\beta,0)}(x) \right\|_k. \end{aligned}$$

Note that

$$\begin{aligned} U_{t,R}^{(\beta,1)}(x) - U_{t,R}^{(\beta,0)}(x) & = \int_{S_R^2} p_R(t, \theta(x, y)) u_{R,0}(y) dy - u_{R,0}(x) \\ & \quad + \int_{[0,t] \times B_R(x, \sqrt{\beta t})} p_R(t-s, \theta(x, y)) \sigma(u_{R,0}(y)) W(ds, dy). \end{aligned}$$

For every $0 < t, R < \infty$,

$$\begin{aligned} e^{-\alpha t} \left| \int_{S_R^2} p_R(t, \theta(x, y)) u_{R,0}(y) dy - u_{R,0}(x) \right| & \\ & \leq e^{-\alpha t} \sup_{x \in S_R^2} |u_{R,0}(x)| \left(1 + \int_{S_R^2} p_R(t, \theta(x, y)) dy \right) \\ & \leq 2U_R. \end{aligned}$$

Since $\sqrt{2h_{up}(R)k/\alpha} < 1$, by Carlen-Krée’s bound on the Burkholder-Gundy-Davis inequality,

$$e^{-\alpha t} \left\| \int_{[0,t] \times B_R(x, \sqrt{\beta t})} p_R(t-s, \theta(x, y)) \sigma(u_{R,0}(y)) W(ds, dy) \right\|_k \leq C_{\sigma_{up}}.$$

Let $m \rightarrow \infty$ in (5.33) to get

$$\sup_{0 \leq t \leq T} \sup_{x \in S_R^2} e^{-\alpha t} \left\| U_{t,R}^{(\beta)}(x) - U_{t,R}^{(\beta,n)}(x) \right\|_k \leq (C_{\sigma_{up}} + 2U_R) \frac{(\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha})^n}{1 - \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha}}. \quad \square$$

Lemma 5.9. Assume for each finite positive R , $\sup_{x \in S_R^2} |u_{R,0}(x)| \leq U_R < \infty$ and that $\alpha \asymp_R \beta \asymp_R (\log R)^c$ where $0 < c < 1$ is a constant. Then for every $0 < t < \infty$, $R > R_{mol}(t)$ where $R_{mol}(t)$ is as in Lemma 2.4, $k \geq 2$ such that $\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} < 1$ and $\sqrt{2h_{up}(R)k/\alpha} < 1$, every positive integer n , $\lambda > 0$, $N > 1$ points $x_1, \dots, x_N \in S_R^2$,

$$\begin{aligned} & \mathbb{P} \left(\max_{1 \leq j \leq N} |U_{t,R}^{(\beta,n)}(x_j) - u_{t,R}(x_j)| > \lambda \right) \\ & \leq \frac{N}{2} (\lambda/2)^{-k} \left(\frac{2\sqrt{2}t^{1/2} \sqrt{kh_{up}(R)} e^{\alpha t - \sqrt{\alpha\beta}t} \left(|\sigma(0)| + \mathbb{L}_\sigma \frac{U_R + |\sigma(0)| \sqrt{2kh_{up}(R)/\alpha}}{1 - \mathbb{L}_\sigma \sqrt{2kh_{up}(R)/\alpha}} \right)}{1 - \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha}} \right)^k \\ & \quad + \frac{N}{2} (\lambda/2)^{-k} \left((C_{\sigma_{up}} + 2U_R) \frac{(\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha})^n}{1 - \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha}} \right)^k. \end{aligned}$$

Proof. By Lemma 5.7, Lemma 5.8, Markov’s inequality and Jensen’s inequality,

$$\begin{aligned} & \mathbb{P} \left(\max_{1 \leq j \leq N} |U_{t,R}^{(\beta,n)}(x_j) - u_{t,R}(x_j)| > \lambda \right) \\ & \leq N \lambda^{-k} \sup_{t \geq 0} \sup_{x \in S_R^2} \mathbb{E} \left(\left[U_{t,R}^{(\beta,n)}(x) - u_{t,R}(t, x) \right]^k \right) \\ & \leq N (\lambda/2)^{-k} \cdot \frac{1}{2} \sup_{t \geq 0} \sup_{x \in S_R^2} \left(\left\| U_{t,R}^{(\beta)}(x) - U_{t,R}^{(\beta,n)}(x) \right\|_k^k + \left\| u_{t,R}(t, x) - U_{t,R}^{(\beta)}(x) \right\|_k^k \right) \\ & \leq \frac{N}{2} (\lambda/2)^{-k} \left(\frac{2\sqrt{2}t^{1/2} \sqrt{kh_{up}(R)} e^{\alpha t - \sqrt{\alpha\beta}t} \left(|\sigma(0)| + \mathbb{L}_\sigma \frac{U_R + |\sigma(0)| \sqrt{2kh_{up}(R)/\alpha}}{1 - \mathbb{L}_\sigma \sqrt{2kh_{up}(R)/\alpha}} \right)}{1 - \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha}} \right)^k \\ & \quad + \frac{N}{2} (\lambda/2)^{-k} \left((C_{\sigma_{up}} + 2U_R) \frac{(\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha})^n}{1 - \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha}} \right)^k. \quad \square \end{aligned}$$

We are now ready to prove the main theorem of this section that gives the asymptotic lower bound of $\sup_{x \in S_R^2} |u_R(t, x)|$.

Proof of Theorem 5.1. Assume throughout the proof, $\alpha \asymp_R \beta \asymp_R (\log R)^{1/2 + C_{h_{up}}/4}$, $n \asymp_R \log R$, $k \asymp_R (\log R)^{1/2 - C_{h_{up}}/4}$, $\lambda \asymp_R (\log R)^{1/4 + C_{h_{down}}/4 - C_{h_{up}}/8}$, and

$$\sup_{R > 0} U_R \leq U < \infty, \tag{5.34}$$

and

$$\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} < 1, \tag{5.35}$$

and

$$\sqrt{2h_{up}(R)k/\alpha} < 1, \tag{5.36}$$

and

$$0 < \lambda < \pi \sqrt{h_{down}(R)t/e} C_{\sigma_{down}} (1 - \epsilon_0) (1 - e^{-\beta/2}) \sqrt{k}/2. \tag{5.37}$$

Whenever a statement/an equality/an inequality involves the above variables, it is assumed the involved variables are subjected to the above estimates. More accurate estimations will be given along the proof. By Lemma 5.2, for all $0 < t, R < \infty$ and every positive integer n, N such that

$$2n\sqrt{\beta}tN < 2\pi R, \tag{5.38}$$

there exist N points x_1, \dots, x_N such that $U_{t,R}^{(\beta,n)}(x_1), \dots, U_{t,R}^{(\beta,n)}(x_N)$ are i.i.d. random variables. By Lemma 5.6 and Lemma 5.9, for every $0 < t < \infty$ and $0 < \epsilon_0 < 1$, there exists a finite positive $R(t, \epsilon_0)$ such that for all $R \geq R(t, \epsilon_0)$,

$$\begin{aligned} & \mathbb{P} \left(\max_{1 \leq j \leq N} |u_R(t, x_j)| < \lambda \right) \tag{5.39} \\ & \leq \mathbb{P} \left(\max_{1 \leq j \leq N} |U_{t,R}^{(\beta,n)}(x_j)| < 2\lambda \right) + \mathbb{P} \left(\max_{1 \leq j \leq N} |U_{t,R}^{(\beta,n)}(x_j) - u_R(t, x_j)| \geq \lambda \right) \\ & = \prod_{j=1}^N \left[1 - \mathbb{P} \left(|U_{t,R}^{(\beta,n)}(x_j)| \geq 2\lambda \right) \right] + \mathbb{P} \left(\max_{1 \leq j \leq N} |U_{t,R}^{(\beta,n)}(x_j) - u_R(t, x_j)| \geq \lambda \right) \\ & \leq \left(1 - \pi e^{-(4\alpha+2)k-2} \left(M(t, C_{\sigma_{down}}, \epsilon_0, \beta) h_{down}(R) k \right)^{2k} \right. \\ & \quad \times \left. \left(\frac{U + 2|\sigma(0)|\sqrt{2kh_{up}(R)/\alpha}}{1 - 2\mathbb{L}_\sigma \sqrt{2kh_{up}(R)/\alpha}} \right)^{-4k} \right)^N \\ & \quad + \frac{N}{2} (\lambda/2)^{-k} \left(\frac{2\sqrt{2}t^{1/2} \sqrt{kh_{up}(R)} e^{\alpha t - \sqrt{\alpha}\beta t} \left(|\sigma(0)| + \mathbb{L}_\sigma \frac{U + |\sigma(0)|\sqrt{2kh_{up}(R)/\alpha}}{1 - \mathbb{L}_\sigma \sqrt{2kh_{up}(R)/\alpha}} \right)}{1 - \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha}} \right)^k \\ & \quad + \frac{N}{2} (\lambda/2)^{-k} \left((C_{\sigma_{up}} + 2U) \frac{\left(\mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha} \right)^n}{1 - \mathbb{L}_\sigma \sqrt{2h_{up}(R)k/\alpha}} \right)^k, \end{aligned}$$

where $M(t, C_{\sigma_{down}}, \epsilon_0, \beta) = 4\pi^2 t C_{\sigma_{down}}^2 (1 - \epsilon_0)^2 (1 - e^{-\beta/2})^2$. Take

$$N = \lfloor k^{C_k} N(k) \rfloor + 1, \tag{5.40}$$

for some finite positive constant $C_k < 2$, where

$$N(k) = \left(\frac{e^{\alpha t + 1/2}}{\sqrt{M(t, C_{\sigma_{down}}, \epsilon_0, \beta) h_{down}(R) k}} \frac{U + 2|\sigma(0)|\sqrt{2kh_{up}(R)/\alpha}}{1 - 2\mathbb{L}_\sigma \sqrt{2kh_{up}(R)/\alpha}} \right)^{4k}. \tag{5.41}$$

Then

$$\begin{aligned} & \left(1 - \pi e^{-(4\alpha+2)k-2} \left(M(t, C_{\sigma_{down}}, \epsilon_0, \beta) h_{down}(R) k \right)^{2k} \left(\frac{U + 2|\sigma(0)|\sqrt{2kh_{up}(R)/\alpha}}{1 - 2\mathbb{L}_\sigma \sqrt{2kh_{up}(R)/\alpha}} \right)^{-4k} \right)^N \\ & \leq e^{-\pi e^{-2} k^{C_k}}. \tag{5.42} \end{aligned}$$

Take

$$\alpha = 8\pi^2 h_{up}(R) (\max\{1, \mathbb{L}_\sigma\})^2 k, \tag{5.43}$$

(so (5.35) and (5.36) are satisfied) and

$$\beta = 4\alpha t. \tag{5.44}$$

Since $\sqrt{2kh_{up}(R)/\alpha} \leq (2\pi L_\sigma)^{-1}$, we have

$$\begin{aligned} & \frac{2\sqrt{2}t^{1/2}\sqrt{kh_{up}(R)}e^{\alpha t - \sqrt{\alpha\beta}t} \left(|\sigma(0)| + \mathbb{L}_\sigma \frac{U_R + |\sigma(0)|\sqrt{2kh_{up}(R)/\alpha}}{1 - \mathbb{L}_\sigma\sqrt{2kh_{up}(R)/\alpha}} \right)}{1 - \mathbb{L}_\sigma\sqrt{2h_{up}(R)k/\alpha}} \\ & \leq \frac{8\sqrt{2}\pi^2 (|\sigma(0)| + \mathbb{L}_\sigma U) t^{1/2} \sqrt{kh_{up}(R)}}{(2\pi - 1)^2} e^{-8\pi^2 h_{up}(R)t(\max\{1, \mathbb{L}_\sigma\})^2 k}, \end{aligned} \tag{5.45}$$

and

$$(C_{\sigma_{up}} + 2U) \frac{(\mathbb{L}_\sigma\sqrt{2h_{up}(R)k/\alpha})^n}{1 - \mathbb{L}_\sigma\sqrt{2h_{up}(R)k/\alpha}} \leq \frac{2\pi}{2\pi - 1} \cdot (C_{\sigma_{up}} + 2U)(2\pi)^{-n}. \tag{5.46}$$

Take

$$n = \log R. \tag{5.47}$$

Then for every $0 < t < \infty$, there exists a finite positive $R_n(t)$ such that for all $R \geq R_n(t)$,

$$\begin{aligned} \frac{2\pi}{2\pi - 1} \cdot (C_{\sigma_{up}} + 2U)(2\pi)^{-n} & \leq \frac{8\sqrt{2}\pi^2 (|\sigma(0)| + \mathbb{L}_\sigma U) t^{1/2} \sqrt{kh_{up}(R)}}{(2\pi - 1)^2} \\ & \times e^{-8\pi^2 h_{up}(R)t(\max\{1, \mathbb{L}_\sigma\})^2 k}. \end{aligned} \tag{5.48}$$

Use (5.42), (5.48), (5.45), (5.46) in (5.39) to get under the constraints of (5.38), for all $0 < t < \infty$, $0 < \epsilon_0 < 1$ and $R \geq \max\{R_n(t), R(\epsilon_0, t)\}$,

$$\begin{aligned} & \mathbb{P} \left(\max_{1 \leq j \leq N} |u_R(t, x_j)| < \lambda \right) \leq e^{-\pi e^{-2} k^{C_k}} \\ & + N(\lambda/2)^{-k} \left(\frac{8\sqrt{2}\pi^2 (|\sigma(0)| + \mathbb{L}_\sigma U) t^{1/2} \sqrt{kh_{up}(R)}}{(2\pi - 1)^2} e^{-8\pi^2 h_{up}(R)t(\max\{1, \mathbb{L}_\sigma\})^2 k} \right)^k. \end{aligned} \tag{5.49}$$

Take

$$\lambda = \sqrt{h_{down}(R)t/eC_{\sigma_{down}}(1 - \epsilon_0)(1 - e^{-\beta/2})\sqrt{k}}, \tag{5.50}$$

then (5.37) is satisfied.

Use (5.50) in (5.49) to get under the constraints of (5.38), for all $0 < t < \infty$, $0 < \epsilon_0 < 1$ and $R \geq \max\{R_n(t), R(\epsilon_0, t)\}$,

$$\begin{aligned} & \mathbb{P} \left(\max_{1 \leq j \leq N} |u_R(t, x_j)| < \lambda \right) \\ & \leq \exp(-\pi e^{-2} k^{C_k}) \\ & + N \left(\frac{h_{up}(R)}{h_{down}(R)} \right)^{k/2} M_2(U, C_{\sigma_{down}}, \epsilon_0, \beta)^k \exp \left(-8\pi^2 h_{up}(R)t(\max\{1, \mathbb{L}_\sigma\})^2 k^2 \right), \end{aligned} \tag{5.51}$$

where $M_2(U, C_{\sigma_{down}}, \epsilon_0, \beta) = \frac{16\sqrt{2}\pi^2 \sqrt{e(|\sigma(0)| + \mathbb{L}_\sigma U)}}{(2\pi-1)^2 C_{\sigma_{down}} (1-\epsilon_0)(1-e^{-\beta/2})}$. By (5.40) and (5.41), for every $0 < t < \infty$, there exists a finite positive $R_N(t)$ such that for all $R \geq R_N(t)$,

$$N \leq 2k^{(4C_k-2)k} h_{down}(R)^{-2k} e^{(4\alpha t+2)k}. \tag{5.52}$$

Hence, by (5.43), under the constraint of (5.38), for all $0 < t < \infty$, $0 < \epsilon_0 < 1$ and $R \geq \max\{R_n(t), R_N(t), R(\epsilon_0, t)\}$,

$$\begin{aligned} P\left(\max_{1 \leq j \leq N} |u_R(t, x_j)| < \lambda\right) &\leq \exp(-\pi e^{-2} k^{C_k}) \\ &+ 2\left(M_2(t, C_{\sigma_{down}}, \epsilon_0, \beta)^k k^{(4C_k-2)k} h_{down}(R)^{-5k/2} h_{up}(R)^{k/2}\right. \\ &\quad \left. \times R^{-2k} \exp\left(-8\pi^2 h_{up}(R)t (\max\{1, \mathbb{L}_\sigma\})^2 k^2 + 2k\right)\right). \end{aligned} \tag{5.53}$$

(5.44), (5.47), (5.52) imply that it suffices to have

$$4t (\log R) \sqrt{\alpha} k^{(4C_k-2)k} h_{down}(R)^{-2k} e^{(4\alpha t+2)k} < \pi R, \tag{5.54}$$

in order for (5.38) to hold. Take

$$k = \left\lceil \frac{(\log R)^{1/2-C_{h_{up}}/4}}{2\sqrt{2}\pi \sqrt{(4+\epsilon_\alpha)t (\max\{1, \mathbb{L}_\sigma\})}} \right\rceil, \tag{5.55}$$

where ϵ_α is a finite positive constant. Then

$$e^{8\pi^2(4+\epsilon_\alpha)th_{up}(R)(\max\{1, \mathbb{L}_\sigma\})^2 k^2} < R. \tag{5.56}$$

Note that (5.56) implies (5.54) by (5.43). Hence, by choosing a larger $R_N(t)$ if necessary, for all $0 < t < \infty$, $0 < \epsilon_0 < 1$ and $R \geq \max\{R_n(t), R_N(t), R(\epsilon_0, t)\}$, (5.38) is satisfied. Since $C_k < 2$, by choosing a larger $R_N(t)$ (hence a larger k) if necessary, we have for all $0 < t < \infty$, $0 < \epsilon_0 < 1$ and $R \geq \max\{R_n(t), R_N(t), R(\epsilon_0, t)\}$,

$$\begin{aligned} &2\left(M_2(t, C_{\sigma_{down}}, \epsilon_0, \beta)^k k^{(4C_k-2)k} h_{down}(R)^{-5k/2} h_{up}(R)^{k/2} R^{-2k}\right. \\ &\quad \left. \times \exp\left(-8\pi^2 h_{up}(R)t (\max\{1, \mathbb{L}_\sigma\})^2 k^2 + 2k\right)\right) \leq \exp(-\pi e^{-2} k^{C_k}). \end{aligned} \tag{5.57}$$

By (5.50), (5.53) and (5.57), for all $0 < t < \infty$, $0 < \epsilon_0 < 1$, finite positive constant $C_k < 2$, and $R \geq \max\{R_n(t), R_N(t), R(\epsilon_0, t)\}$,

$$P\left(\max_{1 \leq j \leq N} |u_R(t, x_j)| < \sqrt{h_{down}(R)t/e} C_{\sigma_{down}} (1-\epsilon_0)(1-e^{-\beta/2})\sqrt{k}\right) \leq 2 \exp(-\pi e^{-2} k^{C_k}). \tag{5.58}$$

By (5.44) and (5.55), for every $0 < t < \infty$, $0 < \epsilon_0 < 1$, there exists a finite positive constant $C(t, \epsilon_0)$ such that

$$\lim_{R \rightarrow \infty} P\left(\sup_{x \in S_R^2} |u_R(t, x)| \geq C(t, \epsilon_0) (\log R)^{1/4+C_{h_{down}}/4-C_{h_{up}}/8}\right) = 1. \tag{5.59}$$

By Theorem 4.5, the results can be restated as every all $0 < t < \infty$, $0 < \epsilon_0 < 1$, there exists a finite positive constant $C(t, \epsilon_0)$ such that

$$\lim_{R \rightarrow \infty} P\left(\exists x \in S_R^2 : |u_R(t, x)| \geq C(t, \epsilon_0) (\log R)^{1/4+C_{h_{down}}/4-C_{h_{up}}/8}\right) = 1. \tag{5.60}$$

Moreover, by (5.58), for every $0 < t < \infty$, $0 < \epsilon_0 < 1$, finite positive constant C , there exist finite positive constants $C(t, \epsilon_0)$ and $R(t, \epsilon_0, C)$ such that for $R \geq R(t, \epsilon_0, C)$,

$$P\left(\exists x \in S_R^2 : |u_R(t, x)| \geq C(t, \epsilon_0) (\log R)^{1/4+C_{h_{down}}/4-C_{h_{up}}/8}\right) \geq 1 - R^{-C\pi e^{-2}(1/2-C_{h_{up}}/4)}. \tag{5.61}$$

□

6 Tail probability estimates

Assume throughout this section that $u_{R,0}(x) = 0$. Then $u_R(t, x)$ has mean zero and is subgaussian. In this section, we give a tail probability estimate of $u_R(t, x)$, which will be useful when we derive an asymptotic upper bound for $\sup_{x \in S_R^2} |u_R(t, x)|$ in the next section.

Lemma 6.1. *Then for any $0 < t, R < \infty, x \in S_R^2$,*

$$\text{Var}(u_R(t, x)) \leq h_{up}(R)tC_{\sigma_{up}}^2. \tag{6.1}$$

Proof. By Lemma 2.2,

$$\begin{aligned} \text{Var}(u_R(t, x)) &\leq C_{\sigma_{up}}^2 h_{up}(R) \int_0^t ds \int_{S_R^2 \times S_R^2} p_R(s, \theta(x, y_1)) p_R(s, \theta(x, y_2)) dy_1 dy_2 \\ &= h_{up}(R)tC_{\sigma_{up}}^2. \end{aligned} \tag{6.2}$$

□

Lemma 6.2. *For any $0 < t, R < \infty, x \in S_R^2$, any integer $k \geq 2$,*

$$\mathbb{E}[|u_R(t, x)|^k] \leq \left(2C_{\sigma_{up}}\sqrt{h_{up}(R)t}\right)^k k^{k/2}. \tag{6.3}$$

Proof. By Lemma 2.2, Carlen-Kr ee’s optimal bound on the Burkholder-Gundy-Davis inequality [2],

$$\mathbb{E}[|u_R(t, x)|^k] \leq (2\sqrt{k})^k \|\sqrt{\text{Var}(u_R(t, x))}\|_k^k \leq \left(2C_{\sigma_{up}}\sqrt{h_{up}(R)t}\right)^k k^{k/2}. \tag{6.4}$$

□

Lemma 6.3. *For all $0 < t, R, \lambda < \infty, x \in S_R^2$,*

$$\mathbb{E}[\exp(\lambda u_R(t, x))] < \left(1 + 2\lambda C_{\sigma_{up}}\sqrt{h_{up}(R)t}\right) \exp\left(4C_{\sigma_{up}}^2 h_{up}(R)t\lambda^2\right). \tag{6.5}$$

Proof. By Lemma 6.2,

$$\mathbb{E}[\exp(\lambda u_R(t, x))] = \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{\lambda^k (u_R(t, x))^k}{k!}\right] \leq 1 + \sum_{k=2}^{\infty} \frac{\left(2\lambda C_{\sigma_{up}}\sqrt{h_{up}(R)t}\right)^k k^{k/2}}{k!}. \tag{6.6}$$

By Sterling’s estimation,

$$\sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k} \leq k!. \tag{6.7}$$

This implies

$$k^{k/2} \leq \frac{e^{k/2}\sqrt{k!}k^{-1/4}}{(2\pi)^{1/4}}. \tag{6.8}$$

So now

$$\begin{aligned} \mathbb{E}[\exp(\lambda u_R(t, x))] &\leq 1 + \sum_{k=2}^{\infty} \frac{\left(2\lambda C_{\sigma_{up}}\sqrt{h_{up}(R)t}\right)^k e^{k/2}k^{-1/4}}{(2\pi)^{1/4}\sqrt{k!}} \\ &< 1 + \sum_{k=2}^{\infty} \frac{\left(2\lambda C_{\sigma_{up}}\sqrt{h_{up}(R)t}\right)^k}{\sqrt{k!}} \\ &< \left(1 + \left(2\lambda C_{\sigma_{up}}\sqrt{h_{up}(R)t}\right)\right) \exp\left(4C_{\sigma_{up}}^2 h_{up}(R)t\lambda^2\right). \end{aligned} \tag{6.9}$$

□

Theorem 6.4. For any $0 < t, R < \infty$, $x \in S_{R'}^2$, $M > 4C_{\sigma_{up}}\sqrt{h_{up}(R)te}$,

$$\sup_{x \in S_R^2} P(|u_R(t, x)| > M) < \left(\frac{M}{2C_{\sigma_{up}}\sqrt{h_{up}(R)te}} \right) \exp \left(-\frac{M^2}{16C_{\sigma_{up}}^2 h_{up}(R)te} \right). \quad (6.10)$$

Proof. By Markov's inequality and Lemma 6.3, for all $0 < t, R, \lambda < \infty$ and $x \in S_R^2$,

$$\begin{aligned} P(|u_R(t, x)| > M) &\leq e^{-M\lambda} E[\exp(\lambda u_R(t, x))] \\ &< \left(1 + \left(2C_{\sigma_{up}}\sqrt{h_{up}(R)te}\lambda \right) \right) \exp \left(4C_{\sigma_{up}}^2 h_{up}(R)te\lambda^2 - M\lambda \right). \end{aligned} \quad (6.11)$$

Take $\lambda = M(8C_{\sigma_{up}}^2 h_{up}(R)te)^{-1}$ to get for $M > 4C_{\sigma_{up}}\sqrt{h_{up}(R)te}$,

$$\begin{aligned} P(|u_R(t, x)| > M) &< \left(1 + \frac{M}{4C_{\sigma_{up}}\sqrt{h_{up}(R)te}} \right) \exp \left(-\frac{M^2}{16C_{\sigma_{up}}^2 h_{up}(R)te} \right) \\ &< \left(\frac{M}{2C_{\sigma_{up}}\sqrt{h_{up}(R)te}} \right) \exp \left(-\frac{M^2}{16C_{\sigma_{up}}^2 h_{up}(R)te} \right). \end{aligned} \quad (6.12)$$

Take the supremum on the left-hand side to finish the proof. \square

7 An asymptotic upper bound of the supremum of the mild solution and the proof of the main theorem

In this section, we prove that for any fixed $t > 0$, there exists a constant $C > 0$ the probability of the event $\{\sup_{x \in S_R^2} |u_R(t, x)| > C\sqrt{\log R}\}$ is small as R tends to ∞ , hence obtaining an asymptotic upper bound of $\sup_{x \in S_R^2} |u_R(t, x)|$. The main result of this section is the following.

Theorem 7.1. For every $0 < t < \infty$, there exists a positive constant C such that

$$\lim_{R \rightarrow \infty} P\left(\exists x \in S_R^2 \text{ such that } |u_R(t, x)| \geq C(\log R)^{1/2+C_{h_{up}}/4}\right) = 0. \quad (7.1)$$

Moreover, for every $2 - 2^{1/4} < r < 1$, $\log_2(2 - r) < \gamma < 1/3$, $0 < t < \infty$, there exists a finite positive $R(t, r, \gamma)$ such that for $R \geq R(t, r, \gamma)$ and $32C_{\sigma_{up}}\sqrt{te}\log_{2-r}(2) < C < \infty$,

$$\begin{aligned} &P\left(\exists x \in S_R^2 \text{ such that } |u_R(t, x)| \geq C(\log R)^{1/2+C_{h_{up}}/4}\right) \\ &\leq \frac{C(\log(2 - r))^{1/2}}{C_{\sigma_{up}}\sqrt{te}} (\log_{2-r}(R))^{1/2} \exp \left((\log_{2-r}(R)) \left(\log(16) - \frac{C^2 \log(2 - r)}{256C_{\sigma_{up}}^2 te} \right) \right). \end{aligned} \quad (7.2)$$

As a result, we have obtained:

Proof of Theorem 1.1. Theorem 1.1 is obtained by combining Theorem 5.1 and Theorem 7.1. \square

We begin by establishing some notations. Suppose $C > 0$ is a constant. For each $t > 0$, $R > 0$, $\alpha > 0$, $\gamma > 0$, $0 < r < 1$ and positive integer $k > 1$, positive integer n , define sets as follows.

Definition 7.1.

$$A_{t,R} = \left\{ \exists x \in S_R^2 \text{ such that } |u_R(t, x)| \geq C(\log R)^{1/2+C_{h_{up}}/4} \right\},$$

$$A_{t,R,n,\alpha} = \left\{ \exists x \in S_R^2 \text{ such that } |u_R(t, x)| > C (\log R)^{1/2+C_{hup}/4} - 2^{-\alpha n} \right\},$$

$$G_{R,n} = \left\{ x \in S_R^2 : x = (R \sin(i_1 \pi 4^{-n}) \cos(2i_2 \pi 4^{-(n+1)}), R \sin(i_1 \pi 4^{-n}) \sin(2i_2 \pi 4^{-(n+1)}), R \cos(i_1 \pi 4^{-n})) \text{ for some } i_1, i_2 \in \mathbb{Z} \right\},$$

$$L_{t,R,n,\alpha} = \left\{ \exists x \in S_R^2 \text{ such that } |u_R(t, x)| \geq C (\log R)^{1/2+C_{hup}/4} \text{ and for all } x \in G_{R,n}, |u_R(t, x)| \leq C (\log R)^{1/2+C_{hup}/4} - 2^{-\alpha n} \right\},$$

$$K_{t,R,n,\alpha} = \left\{ \exists x \in G_{R,n} \text{ such that } |u_R(t, x)| > C (\log R)^{1/2+C_{hup}/4} - 2^{-\alpha n} \right\},$$

$$C_{r,t,n,\gamma} = \left\{ \sup_{\theta(x,x') \leq \pi 2^{-n}} |u_{(2-r)^n}(t, x) - u_{(2-r)^n}(t, x')| \leq \pi \left(\frac{2-r}{2^\gamma} \right)^n \right\}.$$

We record the following result which will be used in the proof of the main theorem of this section.

Lemma 7.2. For every $0 < t < \infty$, $0 < q < \frac{1}{3}$, $2 - 2^{1/4} < r < 1$, there exists a finite positive $n(t, q, r)$ such that for all $n \geq n(t, q, r)$, and $0 < \gamma < \infty$,

$$\begin{aligned} P \left(\sup_{\theta(x,x') \leq \pi 2^{-n}} |u_{(2-r)^n}(t, x) - u_{(2-r)^n}(t, x')| \geq \pi (2-r)^n 2^{-n\gamma} \right) & \quad (7.3) \\ & \leq \pi^{-3} \left(12288 \sqrt{2} (\log(2-r))^{C_{hup}/4} C_{\sigma_{up}} (3/2)^{1/3} \pi^{7/3} 2^q \right)^n \\ & \quad \times n^{(1/2+C_{hup}/4)n} \left(\frac{2^\gamma - q}{(2-r)^{2/3}} \right)^{n^2}. \end{aligned}$$

Proof. Choose $\epsilon_0 = 1/2$, $a = 1$, $R = (2-r)^n$ in Theorem 4.4. □

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. Assume throughout the proof that $2 - 2^{1/4} < r < 1$, and $\log_2(2-r) < \gamma < 1/3$.

For every $0 < t, \alpha < \infty$ and positive integer, on $L_{t,(2-r)^n,n,\alpha}$, there exists $x \in S_{(2-r)^n}^2$ such that $|u_{(2-r)^n}(t, x)| \geq C (n \log(2-r))^{1/2+C_{hup}/4}$ and for all $y \in G_{(2-r)^n,n}$, $|u_{t,(2-r)^n}(y)| \leq C (n \log(2-r))^{1/2+C_{hup}/4} - 2^{-\alpha n}$. For all positive integer n , there exists $y \in G_{(2-r)^n,n}$ such that $\theta(x, y) < \pi(2-r)^n 4^{-n}$. Hence for every $0 < t, \alpha < \infty$, positive integer n , on $L_{t,(2-r)^n,n,\alpha} \cap C_{r,t,n,\gamma}$, there exist $x \in S_{(2-r)^n}^2$ and $y \in G_{(2-r)^n,n}$, such that

$$2^{-\alpha n} \leq |u_{t,(2-r)^n}(x)| - |u_{t,(2-r)^n}(y)| \leq \pi \left(\frac{2-r}{2^\gamma} \right)^n. \quad (7.4)$$

Choose any fixed $0 < \alpha < \gamma - \log_2(2-r)$. Then for $n \geq \lceil (\gamma - \log_2(2-r) - \alpha)(\log_2 \pi)^{-1} \rceil + 1$,

$$2^{-\alpha n} > \pi \left(\frac{2-r}{2^\gamma} \right)^n. \quad (7.5)$$

This implies for every $0 < t < \infty$, $0 < \alpha < \gamma - \log_2(2 - r)$, positive integer $n \geq \lfloor (\gamma - \log_2(2 - r) - \alpha)(\log_2 \pi)^{-1} \rfloor + 1$,

$$P(L_{t,(2-r)^n,n,\alpha} \cap C_{r,t,n,\alpha}) = P\left(2^{-\alpha n} \leq \pi \left(\frac{2-r}{2^\gamma}\right)^n\right) = 0. \tag{7.6}$$

On $K_{t,(2-r)^n,n,\alpha}$ there exists $x \in G_{(2-r)^n,n}$ such that $|u_{t,(2-r)^n}(x)| \geq C(n \log(2 - r))^{1/2+C_{h_{up}}/4} - 2^{-\alpha n}$. By Theorem 6.4, for every $0 < t, \alpha < \infty$ and positive integer $n > \max \left\{ \inf \left\{ n \in \mathbb{Z} : C(n \log(2 - r))^{1/2+C_{h_{up}}/4} \geq 4U \right\}, \inf \left\{ n \in \mathbb{Z} : 2^{1-\alpha n} < C(n \log(2 - r))^{1/2+C_{h_{up}}/4} \right\} \right\}$,

$$\begin{aligned} &P(K_{t,(2-r)^n,n,\alpha} \cap C_{r,t,n,\gamma}) \\ &\leq 2^{4n+2} \sup_{x \in S_{(2-r)^n}^2} P\left(|u_{(2-r)^n}(t,x)| > C(n \log(2 - r))^{1/2+C_{h_{up}}/4} - 2^{-\alpha n}\right) \\ &\leq 2^{4n+2} \sup_{x \in S_{(2-r)^n}^2} P\left(\left|u_{(2-r)^n}(t,x) - u_{(2-r)^n,0}(x)\right| > \frac{C(n \log(2 - r))^{1/2+C_{h_{up}}/4}}{4}\right) \\ &\quad + 2^{4n+2} \sup_{x \in S_{(2-r)^n}^2} P\left(\left|u_{(2-r)^n,0}(x)\right| > \frac{C(n \log(2 - r))^{1/2+C_{h_{up}}/4}}{4}\right) \\ &= \frac{C(\log(2 - r))^{\frac{1}{2}}}{2C_{\sigma_{up}}\sqrt{te}} n^{1/2} \exp\left(n\left(\log(16) - \frac{C^2 \log(2 - r)}{256C_{\sigma_{up}}^2 te}\right)\right). \end{aligned} \tag{7.7}$$

By (7.6), (7.7) and Lemma 7.2, for every $0 < t < \infty$, $0 < q < 1/3$, $0 < \alpha < \gamma - \log_2(2 - r)$, positive integer $n > \max \left\{ \inf \left\{ n \in \mathbb{Z} : C(n \log(2 - r))^{1/2+C_{h_{up}}/4} \geq 4U \right\}, \inf \left\{ n \in \mathbb{Z} : 2^{1-\alpha n} < C(n \log(2 - r))^{1/2+C_{h_{up}}/4} \right\}, \lfloor (\gamma - \log_2(2 - r) - \alpha)(\log_2 \pi)^{-1} \rfloor + 1 \right\}$,

$$\begin{aligned} &P(A_{t,(2-r)^n} \neq \emptyset) \\ &\leq P(K_{t,(2-r)^n,n,\alpha} \cap C_{r,t,n,\gamma}) + P(L_{t,(2-r)^n,n,\alpha} \cap C_{r,t,n,\gamma}) + P(C_{r,t,n,\gamma}^c) \\ &\leq P(K_{t,(2-r)^n,n,\alpha} \cap C_{r,t,n,\gamma}) + P(C_{r,t,n,\gamma}^c) \\ &\leq \frac{C(\log(2 - r))^{\frac{1}{2}}}{2C_{\sigma_{up}}\sqrt{te}} n^{1/2} \exp\left(n\left(\log(16) - \frac{C^2 \log(2 - r)}{256C_{\sigma_{up}}^2 te}\right)\right) \\ &\quad + \pi^{-3} \left(12288\sqrt{2} (\log(2 - r))^{C_{h_{up}}/4} C_{\sigma_{up}} (3/2)^{1/3} \pi^{7/3} 2^q\right)^n n^{(1/2+C_{h_{up}}/4)n} \left(\frac{2^{\gamma-q}}{(2 - r)^{2/3}}\right)^{n^2}. \end{aligned}$$

Choose and fix $q \in (\gamma, 1/3)$, $0 < \alpha < \gamma - \log_2(2 - r)$. Then for every $0 < t < \infty$, there exists a finite positive $n(t)$ such that for all positive integer $n \geq n(t)$,

$$P(A_{t,(2-r)^n} \neq \emptyset) \leq \frac{C(\log(2 - r))^{\frac{1}{2}}}{C_{\sigma_{up}}\sqrt{te}} n^{1/2} \exp\left(n\left(\log(16) - \frac{C^2 \log(2 - r)}{256C_{\sigma_{up}}^2 te}\right)\right). \tag{7.8}$$

In terms of R , the above can be restated as: for every $0 < t < \infty$, there exists a finite

positive $R(t)$ such that for $R \geq R(t)$,

$$P(A_{t,R} \neq \emptyset) \leq \frac{C(\log(2-r))^{\frac{1}{2}}}{C_{\sigma_{up}}\sqrt{te}} (\log_{2-r}(R))^{1/2} \times \exp\left((\log_{2-r}(R))\left(\log(16) - \frac{C^2 \log(2-r)}{256C_{\sigma_{up}}^2 te}\right)\right).$$

This implies that for every $0 < t < \infty$, $32C_{\sigma_{up}}\sqrt{te} \log_{2-r}(2) < C < \infty$,

$$\lim_{R \rightarrow \infty} P\left(\exists x \in S_R^2 \text{ such that } |u_R(t, x)| \geq C(\log R)^{1/2+C_{h_{up}}/4}\right) = 0. \tag{7.9}$$

A Garsia’s theorem

We follow the arguments in [14] to give the proof of the Garsia’s theorem in the spherical context. Relevant notations and symbols are defined in Section 4.

Proof of Lemma 4.1. By Jensen’s inequality,

$$\begin{aligned} &|\bar{f}_{l+1,k}(x) - \bar{f}_{l,k}(x)|^k \\ &= \left| \frac{1}{|B_R(r_{l+1}(k))|} \int_{B_R(r_{l+1}(k))} f(x\tilde{z})dz - \frac{1}{|B_R(r_l(k))|} \int_{B_R(r_l(k))} f(x\tilde{z})dz \right|^k \\ &= \left| \frac{1}{|B_R(r_{l+1}(k))| \cdot |B_R(r_l(k))|} \int_{B_R(r_{l+1}(k))} dz \int_{B_R(r_l(k))} dy (f(x\tilde{z}) - f(x\tilde{y})) \right|^k \\ &\leq \frac{1}{|B_R(r_{l+1}(k))|^2} \int_{B_R(r_{l+1}(k))} dz \int_{B_R(r_l(k))} dy |f(x\tilde{z}) - f(x\tilde{y})|^k. \end{aligned} \tag{A.1}$$

For $\alpha > \sup_{z \in B_R(r_{l+1}(k))} \sup_{y \in B_R(r_l(k))} \mu_k(R\theta(z, y))$,

$$\begin{aligned} &\int_{B_R(r_{l+1}(k))} dz \int_{B_R(r_l(k))} dy |f(x\tilde{z}) - f(x\tilde{y})|^k \\ &\leq \alpha^k \int_{B_R(r_{l+1}(k))} dz \int_{B_R(r_l(k))} dy \frac{|f(x\tilde{z}) - f(x\tilde{y})|^k}{|\mu_k(R\theta(z, y))|^k} \\ &\leq \alpha^k I_k. \end{aligned} \tag{A.2}$$

Hence,

$$|\bar{f}_{l+1,k}(x) - \bar{f}_{l,k}(x)|^k \leq \frac{\alpha^k I_k}{|B_R(r_{l+1}(k))|^2}. \tag{A.3}$$

Let α converges to $\sup_{z \in B_R(r_{l+1}(k))} \sup_{y \in B_R(r_l(k))} \mu_k(R\theta(z, y))$. Then

$$|\bar{f}_{l+1,k}(x) - \bar{f}_{l,k}(x)| \leq \frac{\left(\sup_{z \in B_R(r_{l+1}(k))} \sup_{y \in B_R(r_l(k))} \mu_k(R\theta(z, y))\right) I_k^{1/k}}{|B_R(r_{l+1}(k))|^{2/k}}. \tag{A.4}$$

Note that

$$\begin{aligned} \sup_{z \in B_R(r_{l+1}(k))} \sup_{y \in B_R(r_l(k))} \mu_k(R\theta(z, y)) &\leq \sup_{z \in B_R(r_{l+1}(k))} \sup_{y \in B_R(r_l(k))} \mu_k(R\theta(z, N) + R\theta(y, N)) \\ &\leq \mu_k(r_{l+1} + r_l) \\ &\leq \mu_k(2r_l). \end{aligned} \tag{A.5}$$

So now

$$|\bar{f}_{l+1,k}(x) - \bar{f}_{l,k}(x)| \leq \frac{\mu_k(2r_l)I_k^{1/k}}{|B_R(r_{l+1}(k))|^{2/k}}. \tag{A.6}$$

For any positive integer L ,

$$\begin{aligned} |\bar{f}_{l+L,k}(x) - \bar{f}_{l,k}(x)| &\leq \sum_{n=l}^{l+L-1} |\bar{f}_{n+1,k}(x) - \bar{f}_{n,k}(x)| \\ &\leq I_k^{1/k} \sum_{n=l}^{\infty} \frac{\mu_k(2r_n)}{|B_R(r_{n+1}(k))|^{2/k}} \\ &\leq C_{\mu_k} I_k^{1/k} \sum_{n=l}^{\infty} \frac{\mu_k(r_n)}{|B_R(r_{n+1}(k))|^{2/k}}. \end{aligned} \tag{A.7}$$

Since

$$\mu_k(r_n) = 2(\mu_k(r_n) - \mu_k(r_{n+1})) = 4(\mu_k(r_{n+1}) - \mu_k(r_{n+2})), \tag{A.8}$$

we can continue to get

$$\begin{aligned} |\bar{f}_{l+L,k}(x) - \bar{f}_{l,k}(x)| &\leq 4C_{\mu_k} I_k^{1/k} \sum_{n=l}^{\infty} \frac{\mu_k(r_{n+1}) - \mu_k(r_{n+2})}{|B_R(r_{n+1}(k))|^{2/k}} \\ &\leq 4C_{\mu_k} I_k^{1/k} \sum_{n=l}^{\infty} \int_{r_{n+2}(k)}^{r_{n+1}(k)} \frac{d\mu_k(r)}{|B_R(r_{n+1}(k))|^{2/k}} \\ &\leq 4C_{\mu_k} I_k^{1/k} \int_0^{r_{l+1}(k)} \frac{d\mu_k(r)}{|B_R(r)|^{2/k}}. \end{aligned} \tag{A.9}$$

By letting $L \rightarrow \infty$, we have that

$$\bar{f}_k = \lim_{n \rightarrow \infty} \bar{f}_{n,k} \tag{A.10}$$

exists and for each integer $l \geq 0$,

$$\sup_{x \in S_R^2} |\bar{f}_k(x) - \bar{f}_{l,k}(x)| \leq 4C_{\mu_k} I_k^{1/k} \int_0^{r_{l+1}(k)} |B_R(r)|^{-2/k} d\mu_k(r). \tag{A.11}$$

To prove the last statement, let $\phi : S_R^2 \rightarrow \mathbb{R}$ be a continuous function. By (4.7),

$$\begin{aligned} \int_{S_R^2} \phi(x) \bar{f}_k(x) dx &= \lim_{n \rightarrow \infty} \int_{S_R^2} \phi(x) \bar{f}_{n,k}(x) dx \\ &= \lim_{n \rightarrow \infty} \int_{S_R^2} f(x) \bar{\phi}_{n,k}(x) dx \\ &= \int_{S_R^2} f(x) \phi(x) dx. \end{aligned} \tag{A.12}$$

This implies $f = \bar{f}_k$ a.e. □

Proof of Theorem 4.2. Suppose $r_{l+1}(k) \leq R\theta(x, x') \leq r_l(k)$ for some nonnegative integer l . Then by the triangle inequality and Lemma 4.1,

$$\begin{aligned} |\bar{f}_k(x) - \bar{f}_k(x')| &\leq 2 \sup_{z \in S_R^2} |\bar{f}_k(z) - \bar{f}_{l,k}(z)| + |\bar{f}_{l,k}(x) - \bar{f}_{l,k}(x')| \\ &\leq 8C_{\mu_k} I_k^{1/k} \int_0^{R\theta(x, x')} |B_R(r)|^{-2/k} d\mu_k(r) + |\bar{f}_{l,k}(x) - \bar{f}_{l,k}(x')|. \end{aligned} \tag{A.13}$$

We can use a similar argument as in the proof of the previous lemma to estimate the last term and get

$$\begin{aligned}
 |\bar{f}_{l,k}(x) - \bar{f}_{l,k}(x')|^k &\leq \frac{\alpha^k}{|B_R(r_l(k))|^2} \int_{B_R(r_l(k))} dz \int_{B_R(r_l(k))} dy \frac{|f(x\tilde{+}z) - f(x'\tilde{+}y)|^k}{|\mu(R\theta(x\tilde{+}z, x'\tilde{+}y))|^k} \\
 &= \frac{\alpha^k I_k}{|B_R(r_l(k))|^2},
 \end{aligned}
 \tag{A.14}$$

for any $\alpha \geq \sup_{z \in B_R(r_l(k))} \sup_{y \in B_R(r_l(k))} \mu_k(R\theta(x\tilde{+}z, x'\tilde{+}y))$.

Let α converges to $\sup_{z \in B_R(r_l(k))} \sup_{y \in B_R(r_l(k))} \mu_k(R\theta(x\tilde{+}z, x'\tilde{+}y))$ from above to get

$$|\bar{f}_{l,k}(x) - \bar{f}_{l,k}(x')| \leq \left(\sup_{z \in B_R(r_l(k))} \sup_{y \in B_R(r_l(k))} \mu_k(R\theta(x\tilde{+}z, x'\tilde{+}y)) \right) I_k^{1/k} |B_R(r_l(k))|^{-2/k}.
 \tag{A.15}$$

For any $z, y \in B_R(r_l(k))$,

$$\begin{aligned}
 \theta(x\tilde{+}z, x'\tilde{+}y) &\leq \theta(x\tilde{+}z, x) + \theta(x, x') + \theta(x', x'\tilde{+}y) \\
 &\leq 3r_l(k)/R \\
 &< 4r_l(k)/R.
 \end{aligned}
 \tag{A.16}$$

Hence,

$$\begin{aligned}
 |\bar{f}_{l,k}(x) - \bar{f}_{l,k}(x')| &\leq C_{\mu_k}^2 \mu_{r_l(k)} I_k^{1/k} |B_R(r_l(k))|^{-2/k} \\
 &\leq 4C_{\mu_k}^2 (\mu_{r_{l+1}(k)} - \mu_{r_{l+2}(k)}) I_k^{1/k} |B_R(r_{l+1}(k))|^{-2/k} \\
 &\leq 4C_{\mu_k}^2 I_k^{1/k} \int_0^{R\theta(x, x')} \frac{d\mu(r)}{|B_R(r)|^{2/k}}.
 \end{aligned}
 \tag{A.17}$$

Use (A.17) in (A.13) to get

$$|\bar{f}_k(x) - \bar{f}_k(x')| \leq 4C_{\mu_k} (2 + C_{\mu_k}) I_k^{1/k} \int_0^{R\theta(x, x')} |B_R(r)|^{-2/k} d\mu_k(r).
 \tag{A.18}$$

□

References

- [1] Annika Lang, Christoph Schwab, Isotropic Gaussian random fields on the sphere: Regularity, fast simulation and stochastic partial differential equations, *Ann. Appl. Probab.*, 25, (No. 6, 2015), 3047–3094. MR-3404631
- [2] Carlen, Eric, and Krée, Paul. L^p Estimates on iterated stochastic integrals, *Ann. Probab.* 19(1) (1991) 354–368. MR-1085341
- [3] C. M. Elliott, M. Hairer, M. R. Scott, Stochastic Partial Differential Equations on Evolving Surfaces and Evolving Riemannian Manifolds. ArXiv:1208.5958v1 [math.AP] 29 Aug 2012.
- [4] Conus, Daniel, Joseph, Mathew and Khoshnevisan, Davar. On the chaotic character of the stochastic heat equation, before the onset of intermittency, *Ann. Probab.* 41(3B) (2013) 2225–2260. MR-3098071
- [5] Dalang, Robert C., Mueller, Carl and Zambotti, Lorenzo. Hitting properties of parabolic S.P.D.E.'s with reflection, *Ann. Probab.* 34(4) (2006) 1423–1450. MR-2257651
- [6] Dalang, Robert C., Lévêque, Olivier. Second-order linear hyperbolic SPDEs driven by isotropic Gaussian noise on a sphere, *Ann. Probab.*, 32, (No. 1B, 2004), 1068–1099. MR-2044674
- [7] Dalang, Robert C., Lévêque, Olivier. Second-order hyperbolic SPDEs driven by Homogeneous Gaussian noise on a hyperplane, *Trancations of the American Mathematical Society*, 358, (No. 5, 2006), 2123–2159. MR-2197451

- [8] Doob, J. L., *Stochastic Processes*, Wiley Classics Library, 1990. MR-1038526
- [9] Elton, P. Hsu. *Stochastic Analysis on Manifolds*, American Math. Society, Graduate Studies in Mathematics 38 Providence RI. MR-1882015
- [10] Foondun, Mohammad, and Khoshnevisan, Davar. Intermittence and nonlinear stochastic partial differential equations *Electronic J. Probab.* Vol. 14, Paper no. 21 (2009) 548–568. MR-2480553
- [11] G. D. Prato, J. Zabczyk. *Stochastic Equations in Infinite Dimensions*, 2nd Edition, Cambridge University Press, Cambridge, 2014 MR-3236753
- [12] I. Gyöngy, Stochastic partial differential equations on manifolds I. *Potential Analysis*, 2(2):101–113, 1993. ISSN 0926-2601. MR-1246744
- [13] I. Gyöngy, Stochastic partial differential equations on manifolds II. *Nonlinear Filtering*, 6(1):39–56, 1997. ISSN 0926-2601. MR-1436821
- [14] Khoshnevisan, Davar. *Analysis of Stochastic Partial Differential Equations*, American Math. Society, CBMS Regional Conference Series in Mathematics 119 Providence RI, 2014 (116 pp). MR-3222416
- [15] Marinucci, Domenico and Peccati, Giovanni. *Random Fields on the Sphere: Representation, Limit Theorems and Cosmological Applications*, Cambridge University Press, 2011. MR-2840154
- [16] Molchanov, S. A., Diffusion processes and Riemannian geometry, *Russian Math. Surveys*, 30, (No. 1 1975), 1–63. MR-0413289
- [17] Peter Li and Shing Tung Yau, On the parabolic kernel of the Schrödinger operator. *Acta Math.*, Volume 156 (1986) 153–201. MR-0834612
- [18] S. Peszat, S. Tindel, Stochastic heat and wave equations on a Lie group. *Stoch. Anal. Appl.*, 28 (2010), no. 4, 662–695 MR-2739599
- [19] Steven Rosenberg. *The Laplacian on a Riemannian Manifold*, Cambridge University Press, 1997. MR-1462892
- [20] T. Funaki, A stochastic partial differential equations with values in a manifold. *Journal of functional analysis*, 109(2):257–288, 1992. ISSN 0022-1236. MR-1186323
- [21] Varadhan, S. R. S., On the behaviour of the fundamental solution of the heat equation with variable coefficients. *Comm. Pure Appl. Math.*, 20 (1967) 657–685. MR-0208191
- [22] Y. Kazashi, Q. T. Le Gia, A non-uniform discretization of stochastic heat equations with multiplicative noise on the unit sphere, *Journal of Complexity*, vol. 50, pp. 43–65 MR-3907363
- [23] Z. Brzeźniak, B. Goldys, Q. T. Le Gia, Random attractors for the stochastic Navier-Stokes equations on the 2D unit sphere. *J. Math. Fluid Mech.* 20 (2018), no. 1, 227–253. MR-3767663
- [24] Z. Brzeźniak, B. Goldys, Q. T. Le Gia, Random dynamical systems generated by stochastic Navier–Stokes equations on a rotating sphere. *J. Math. Anal. Appl.* 426 (2015), no. 1, 505–545. MR-3306386
- [25] Z. Brzeźniak, On stochastic convolution in Banach spaces and applications. *Stochastics Stochastics Rep.* 61 (1997), no. 3–4, 245–295. MR-1488138

Acknowledgments. This research is partly supported by the NSF grant 1608575. The author would also like to thank Davar Khoshnevisan for many motivating discussions and his careful reading of the manuscript, Thomas Alberts for his helpful advice on editing of the manuscript and the anonymous referees for suggestive comments on the first version of this paper.