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Analyticity for rapidly determined properties of Poisson Galton–Watson trees

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Abstract

Let T_λ be a Galton–Watson tree with Poisson(λ) offspring, and let A be a tree property. In this paper, we are concerned with the regularity of the function $\mathbb{P}_\lambda(A) := \mathbb{P}(T_\lambda \models A)$. We show that if a property A can be uniformly approximated by a sequence of properties $\{A_k\}$, depending only on the first k vertices in the breadth first exploration of the tree, with a bound in probability of $\mathbb{P}_\lambda(A \triangle A_k) \le Ce^{-ck}$ over an interval $I = (\lambda_0, \lambda_1)$, then $\mathbb{P}_\lambda(A)$ is real analytic in λ for $\lambda \in I$. We also present some applications of our results, particularly to properties that are not expressible in first order logic on trees.

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1 Introduction

Let $X_1, X_2, \dots$ be a sequence of independent Poisson random variables with parameter λ. Set $\widetilde{X} = (X_1, X_2, X_3, \dots)$ and construct a tree T_λ so that node i has X_i children, labelling the nodes from top to bottom and left to right, i.e., breadth first ordering (see Figure 1). We call the sequence $\widetilde{X}$ the seed of the Poisson Galton–Watson tree T_λ with parameter λ. Note that if the tree has a finite number n of vertices then the values X_j for $j > n$ are irrelevant.

Although the offspring distribution completely determines the law of T_λ, it does not provide an immediate sense of the tree's structure. A more transparent structural description of T_λ is provided by tree property probabilities, i.e., for a given tree property A, what is the probability that T_λ has this property? For convenience, we will identify this event $T_\lambda \models A$ with the property A itself, defining

$$f_A(\lambda) := \mathbb{P}_\lambda(T_\lambda \models A)\,, \tag{1.1}$$

where $\mathbb{P}_\lambda(\cdot)$ is the probability measure on trees induced by the Poisson(λ) Galton–Watson process. In this paper we are interested in the regularity of $f_\lambda(A)$ as a function of λ for certain choices of the tree property A.

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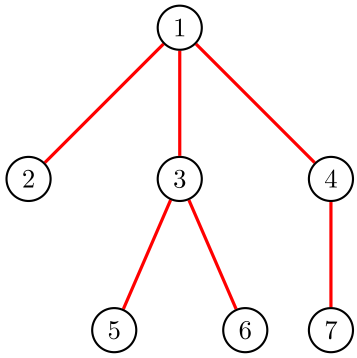

Figure 1: The node labelling convention. The seed used to define this tree is $(3, 0, 2, 1, 0, 0, 0, \dots)$.

In essence, this is a question about phase transitions: loss of regularity in $\mathbb{P}_\lambda(A)$ at a particular value of λ is interpreted as phase transition in structure of T_λ, as 'seen by' property A. We illustrate this idea as follows. Consider the two events

$$\begin{aligned} A_1 &= \{\text{The tree is infinite}\} = \{|T_\lambda| = \infty\}\,, \\ A_2 &= \{\text{The root has exactly one child}\} = \{X_1 = 1\}\,. \end{aligned} \tag{1.2}$$

As is well known (see, e.g., Prop. 5.4 in [3]), the probability $f_{A_1}(\lambda)$ that T_λ is infinite satisfies

$$f_{A_1}(\lambda) = 1 - \exp(-\lambda f_{A_1}(\lambda)), \tag{1.3}$$

or equivalently,

$$f_{A_1}(\lambda) = 1 + \frac{W_0(-\lambda e^{-\lambda})}{\lambda}, \tag{1.4}$$

where $W_0(x)$ is the principle branch of the Lambert W function studied in [2], the unique real solution to

$$W_0(x)e^{W_0(x)} = x, \quad W_0(x) \geq -1. \tag{1.5}$$

This function $f_{A_1}(\lambda) = \mathbb{P}_\lambda(A_1)$ is real analytic on $I_1 = (0, 1)$ and on $I_2 = (1, \infty)$, but has a branch cut singularity at $\lambda = 1$ and so is not real analytic on any interval containing this point: the interpretation is that the size of a Poisson Galton–Watson tree undergoes a phase transition at $\lambda = 1$. On the other hand, the probability that the root node has exactly one child is

$$f_{A_2}(\lambda) = \lambda e^{-\lambda}, \tag{1.6}$$

which is a real analytic function over the entire domain $I = (0, \infty)$. From the perspective of A_2, there is no phase transition.

Recently, Podder and Spencer [7, 8] studied this question in the context of first order properties on trees. Informally speaking, a first order property can be expressed as a sentence in first order logic, which contains an infinite number of variables, the equality "$=$" relation, the binary parent relation $\pi(x, y)$ which is true if y is the parent of x, the root symbol R, universal and existential quantifiers and the usual Boolean connectives.

In [8], Podder and Spencer used the Ehrenfeucht game for rooted trees and a contraction mapping theorem to prove the following:

Theorem 1.1. *Let A be a first order property. Then $f_A(\lambda)$ is a $C^\infty(0, \infty)$ function.*

Our main result, Theorem 1.5, is an extension of this to a larger class of properties. We also improve the smoothness. Before stating our result, we introduce some notation and definitions.

The **k-truncated seed** $\widetilde{X}^{(k)} = (X_1, \dots, X_k)$ is given by the first k elements of the seed $\widetilde{X}$.

Definition 1.2. *An event A is called* k-tautologically determined *if there exists a set $B \subseteq \mathbb{N}^k$ such that*

$$A = \left\{ \widetilde{X}^{(k)} \in B \right\}. \tag{1.7}$$

Definition 1.3. *An event A is called* rapidly determined *over an open interval $I \subseteq (0, \infty)$, if for every $\lambda \in I$, there exist positive constants c and C, $k_0 \in \mathbb{N}$, and a sequence of k-tautologically determined events A_k such that for all $k \geq k_0$,*

$$\mathbb{P}_\lambda(A \triangle A_k) \leq Ce^{-ck}. \tag{1.8}$$

Theorem 1.4 ([8, Theorem 6.6])**.** *Every first order property is rapidly determined over $(0, \infty)$.*

We can now state our main result:

Theorem 1.5. *Suppose that a tree property A is rapidly determined over an open interval $I \subseteq (0, \infty)$. Then $f_A(\lambda)$ is a real analytic function on I.*

The conclusion of the theorem means that for every $\lambda \in I$, there exists $\delta > 0$ so that the function $f_A(\lambda)$ can be extended to a complex analytic function $f_A(z)$ on the disc $D_\delta(\lambda) = \{z : |z - \lambda| \leq \delta\}$. Theorem 1.5 improves on Theorem 1.1 in two ways. Firstly, we broaden the scope of applicability to the larger class of rapidly determined properties, and secondly, we improve the regularity from C^∞ to real analytic. The collection of first order properties is countable, since every first order property is specified by a finite sequence from a countable alphabet. On the other hand, Proposition 3.5 in Section 3 describes uncountably many rapidly determined properties.

Corollary 1.6. *Let A be a first order property. Then $f_A(\lambda)$ is a real analytic function of $\lambda \in (0, \infty)$.*

Proposition 1.7. *Let*

$$A = \{\textit{there exists a node on an even level with exactly } n_A \textit{ children}\}$$
$$B = \{\textit{there exists a node on a prime level with exactly } n_B \textit{ children}\}.$$

Then A and $A \cup B$ are both rapidly determined on $(0, \infty)$.

Remark 1.8. We note that neither A nor $A \cup B$ are first order properties. This follows from a simple modification of [9, Theorems 2.1.3 and 2.3.3].

Unlike Podder and Spencer, our methods are not model theoretic in nature. Instead, we take a more direct, complex analytic approach. It is similar in spirit to the route taken in [5, 6], where the regularity of Lyapunov exponents for products of discrete random matrices were studied.

2 Analyticity for rapidly determined properties

In this section we prove Theorem 1.5. We begin with a preliminary result.

Lemma 2.1. *Let $k \in \mathbb{N}$ and let A be a k-tautologically determined event. Then $f_A(\lambda)$ may be analytically continued to an entire function $f_A(z)$.*

Proof. By the assumption on A there exists $B \subseteq \mathbb{N}^k$ such that $A = \{\widetilde{X}_\lambda^{(k)} \in B\}$. Therefore we have

$$\begin{aligned}\mathbb{P}_\lambda(A) = \mathbb{P}\left(\widetilde{X}_\lambda^{(k)} \in B\right) &= \sum_{(m_1,\dots,m_k)\in B} \prod_{i=1}^{k} e^{-\lambda}\frac{\lambda^{m_i}}{m_i!} \\ &= e^{-k\lambda} \sum_{(m_1,\dots,m_k)\in B} \frac{\lambda^{m_1+\cdots+m_k}}{m_1!\dots m_k!} = e^{-k\lambda}\sum_{n=0}^{\infty} a_n \frac{\lambda^n}{n!},\end{aligned} \tag{2.1}$$

where

$$0 \le a_n = \sum_{\substack{(m_1,\dots,m_k)\in B \\ m_1+\cdots+m_k=n}} \binom{n}{m_1,\dots,m_k} \le \sum_{\substack{(m_1,\dots,m_k)\in \mathbb{N}^k \\ m_1+\cdots+m_k=n}} \binom{n}{m_1,\dots,m_k} = k^n. \tag{2.2}$$

Since

$$\lim_{n\to\infty} \left|\frac{a_n}{n!}\right|^{\frac{1}{n}} \le \lim_{n\to\infty} \left|\frac{k^n}{n!}\right|^{\frac{1}{n}} = 0, \tag{2.3}$$

it follows that $\mathbb{P}_\lambda(A)$ may be analytically continued to an entire function $\mathbb{P}_z(A)$. □

Proof of Theorem 1.5. Let $\lambda \in I$. Since A is rapidly determined over the interval I, there exists constants c and C, $k_0 \in \mathbb{N}$ and a sequence of k-tautologically determined events (A_k) such that for all $k \ge k_0$

$$\mathbb{P}_\lambda(A \triangle A_k) \le Ce^{-ck}. \tag{2.4}$$

From this it then follows that

$$\mathbb{P}_\lambda(A) = \lim_{k\to\infty} \mathbb{P}_\lambda(A_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{P}_\lambda(A_k \setminus A_{k-1}). \tag{2.5}$$

From Lemma 2.1 we get that $f_{A_k}(\lambda) = \mathbb{P}_\lambda(A_k)$ can be extended to a complex analytic function over $\mathbb{C}$ that we denote as $f_{A_k}(z)$. In order to establish that $f_A(\lambda)$ can also be extended to an analytic function in some neighbourhood of $\lambda \in I$, it suffices to show that for every $\lambda \in I$ there exist positive constants c_1 and c_2 and $\delta > 0$ such that for all $z \in D_\delta(\lambda) = \{z \in \mathbb{C} : |z-\lambda| \le \delta\}$, we have

$$|f_{A_k\setminus A_{k-1}}(z)| \le c_1 e^{-c_2 k}. \tag{2.6}$$

Indeed, this will then imply that $f_{A_n}(z)$ converges uniformly to a function denoted $f_A(z)$, which will also be analytic on $D_\delta(\lambda)$.

Since the event $A_k \setminus A_{k-1}$ is k-tautologically determined, we have a set $B \subseteq \mathbb{N}^k$ be such that

$$A_k \setminus A_{k-1} = \{\widetilde{X}^{(k)} \in B\}, \tag{2.7}$$

and can therefore write the analytic continuation of $f_{A_k\setminus A_{k-1}}(\lambda)$ as

$$f_{A_k\setminus A_{k-1}}(z) = \sum_{\ell=0}^{\infty} \sum_{\substack{(m_1,\dots,m_k)\in B \\ \sum_{i\le k} m_i=\ell}} \prod_{i=1}^{k} e^{-z}\frac{z^{m_i}}{m_i!}. \tag{2.8}$$

We then bound the modulus of $f_{A_k\setminus A_{k-1}}(z)$ as follows. First, we use that for all $\varepsilon > 0$, $r \ge 0$, and $z \in D_\delta(\lambda)$ with $\delta \le \min\left\{\frac{\lambda\epsilon}{2}, \log\frac{1+\epsilon}{1+\epsilon/2}\right\}$, we have

$$\left|\frac{z^r e^{-z}}{r!}\right| \le \left|1+\frac{\delta}{\lambda}\right|^r e^{\delta}\frac{\lambda^r e^{-\lambda}}{r!} \le (1+\epsilon)^r\,\frac{\lambda^r e^{-\lambda}}{r!}, \tag{2.9}$$

and apply this for all k terms of the product appearing in (2.8). This gives the bound

$$|f_{A_k\setminus A_{k-1}}(z)| \le \sum_{\ell=0}^{\infty}(1+\epsilon)^{\ell} \sum_{\substack{(m_1,\dots,m_k)\in B \\ \sum_{i\le k} m_i=\ell}} \prod_{i=1}^{k} e^{-\lambda}\frac{\lambda^{m_i}}{m_i!}. \tag{2.10}$$

We now split this sum, and bound each term as

$$\begin{aligned} |f_{A_k\setminus A_{k-1}}(z)| \le \sum_{\ell\le[3k\lambda]}(1+\epsilon)^{\ell} &\sum_{\substack{(m_1,\dots,m_k)\in B \\ \sum_{i\le k} m_i=\ell}} \prod_{i=1}^{k} e^{-\lambda}\frac{\lambda^{m_i}}{m_i!} \\ &+ \sum_{\ell>[3k\lambda]}(1+\epsilon)^{\ell} \sum_{\substack{(m_1,\dots,m_k)\in B \\ \sum_{i\le k} m_i=\ell}} \prod_{i=1}^{k} e^{-\lambda}\frac{\lambda^{m_i}}{m_i!} \\ &\le (1+\epsilon)^{3k\lambda}\left[\sum_{\ell=0}^{\infty} \sum_{\substack{(m_1,\dots,m_k)\in B \\ \sum_{i\le k} m_i=\ell}} \prod_{i=1}^{k} e^{-\lambda}\frac{\lambda^{m_i}}{m_i!}\right] \\ &+ \sum_{\ell>[3k\lambda]}(1+\epsilon)^{\ell}\left[\sum_{\substack{(m_1,\dots,m_k)\in \mathbb{N}^k \\ \sum_{i\le k} m_i=\ell}} \prod_{i=1}^{k} e^{-\lambda}\frac{\lambda^{m_i}}{m_i!}\right] \\ &= (1+\epsilon)^{3k\lambda}\cdot \mathbb{P}_\lambda(A_k\setminus A_{k-1}) + \sum_{\ell>[3k\lambda]}(1+\epsilon)^{\ell}\cdot \mathbb{P}_\lambda\left(\sum_{i=1}^{k} X_i=\ell\right). \end{aligned} \tag{2.11}$$

Since $\sum_{i=1}^{k} X_i$ has the Poisson distribution with parameter $k\lambda$, it follows that there exists a positive constant c_1 so that for $\ell>[3k\lambda]$,

$$\mathbb{P}_\lambda\left(\sum_{i=1}^{k} X_i=\ell\right) \le e^{-c_1\ell}. \tag{2.12}$$

From (2.4) we get that there exist positive constants c_2 and c_3 so that for all k,

$$\mathbb{P}_\lambda(A_k\setminus A_{k-1}) \le c_2 e^{-c_3 k}. \tag{2.13}$$

Taking ϵ sufficiently small and using the two bounds above in (2.11) we obtain for positive constants c_4 and c_5,

$$|f_{A_k\setminus A_{k-1}}(z)| \le c_4 e^{-c_5 k}, \tag{2.14}$$

and this concludes the proof of (2.6) and hence the theorem. □

3 Examples of rapidly determined properties

In this section we provide some examples of rapidly determined properties to demonstrate the applicability of Theorem 1.5.

We start by showing that when the tree is subcritical, every property is rapidly determined. For a tree T, we write $|T|$ for the total number of vertices in T.

Proposition 3.1. *Let $0\le\lambda_0<\lambda_1\le 1$. Then every property A is rapidly determined on the interval $I=(\lambda_0,\lambda_1)$.*

Proof. Let $A_k = A \cap \{|T_\lambda| < k\}$. Since A_k is a k-tautologically determined event, it suffices to show that for every $\lambda < 1$ there exist positive constants c and C so that for all k

$$\mathbb{P}_\lambda(A \triangle A_k) \leq Ce^{-ck}. \tag{3.1}$$

We now have

$$\mathbb{P}_\lambda(A \triangle A_k) = \mathbb{P}_\lambda(A \setminus A_k) \leq \mathbb{P}_\lambda(|T_\lambda| \geq k) \leq \mathbb{P}_\lambda\left(\sum_{i=1}^{k} X_i \geq k\right). \tag{3.2}$$

Using that $\sum_{i=1}^{k} X_i$ has the Poisson distribution with parameter $k\lambda$ and $\lambda < 1$ proves (3.1) (See, e.g., Appendix A in [1]), and this concludes the proof. □

Remark 3.2. One interpretation of the proposition above is that Poisson Galton–Watson trees do not exhibit a phase transition in *any* property over the interval $I = (0, 1)$.

Lemma 3.3. *Let E_k be the set of nodes amongst the first k which lie on an even level. Then for every $\lambda \in (0, \infty)$ there exists a positive constant c so that*

$$\mathbb{P}_\lambda\left(|E_k| \leq \frac{k}{2\lambda+1}, |T_\lambda| \geq k\right) \leq e^{-ck}. \tag{3.3}$$

In the proof of Lemma 3.3, we will need the following version of the Azuma–Hoeffding inequality for supermartingales with bounded exponential moments:

Lemma 3.4 ([4, Lemma 4.3])**.** *Let $(M_k)_{k\in\mathbb{N}_0}$ be a supermartingale with respect to the filtration $\mathcal{F}_k$ with $M_0 = 0$. Suppose that the increments $Y_k = M_k - M_{k-1}$ satisfy $\mathbb{E}\left[e^{Y_k}|\mathcal{F}_{k-1}\right] \leq C$. Then for all $k > 0$ and real $\alpha \in [0, 2C]$, we have*

$$\mathbb{P}(M_k \geq \alpha k) \leq e^{-\frac{\alpha^2 k}{4C}}. \tag{3.4}$$

Proof of Lemma 3.3. Let T_λ be a Poisson(λ) Galton–Watson tree with associated seed $\widetilde{X} = (X_1, X_2, \dots)$, and let E_k (resp. O_k) be the set of nodes amongst the first k which lie on an even (resp. odd) level, with the convention that if $k > |T_\lambda|$ then $E_k = E_{|T_\lambda|}$ and $O_k = O_{|T_\lambda|}$. Denote the set of children of E_k by $C(E_k)$.

On the event $\{|T_\lambda| \geq k\}$, the first k nodes exist and we have

$$|E_k| + |O_k| = k. \tag{3.5}$$

Noting that $O_k \subseteq C(E_k)$, we see that

$$|C(E_k)| \geq k - |E_k|, \tag{3.6}$$

and hence on the event $\{|E_k| \leq \frac{k}{2\lambda+1}\} \cap \{|T_\lambda| \geq k\}$, we have

$$|C(E_k)| \geq \frac{2\lambda k}{2\lambda+1}. \tag{3.7}$$

Now consider the sequence of random variables $(M_k)_{k\in\mathbb{N}_0}$ with

$$M_k = |C(E_k)| - \lambda|E_k|. \tag{3.8}$$

By (3.7), on the event $\{|E_k| \leq \frac{k}{2\lambda+1}\} \cap \{|T_\lambda| \geq k\}$, we have

$$M_k \geq \frac{\lambda k}{2\lambda+1} \tag{3.9}$$

almost surely, and hence

$$\begin{aligned}\mathbb{P}_\lambda\left(|E_k| \le \frac{k}{2\lambda+1}, |T_\lambda| \ge k\right) &= \mathbb{P}_\lambda\left(M_k \ge \frac{\lambda k}{2\lambda+1}, |E_k| \le \frac{k}{2\lambda+1}, |T_\lambda| \ge k\right) \\ &\le \mathbb{P}_\lambda\left(M_k \ge \frac{\lambda k}{2\lambda+1}\right).\end{aligned} \tag{3.10}$$

We now use the fact that $(M_k)_{k\in\mathbb{N}_0}$ is a martingale with respect to the filtration generated by the truncated seed $\widetilde{X}^{(k)} = (X_1, \dots, X_k)$, and satisfies the conditions of Lemma 3.4. To see this, note that M_k can alternatively be written as

$$M_k = \sum_{i=0}^{k} 1_{i\in E_i}(X_i - \lambda), \tag{3.11}$$

and hence has increments $Y_i = M_i - M_{i-1} = 1_{i\in E_i}(X_i - \lambda)$. These may be bounded as

$$\mathbb{E}\left[e^{Y_i}\middle|\widetilde{X}^{(i-1)}\right] \le \mathbb{E}\left[e^{X_i}\right] = e^{\lambda(e-1)}. \tag{3.12}$$

Clearly, $\frac{\lambda}{2\lambda+1} \le 2e^{\lambda(e-1)}$ for all $\lambda > 0$, and hence by Lemma 3.4,

$$\mathbb{P}_\lambda\left(|E_k| \le \frac{k}{2\lambda+1}, |T_\lambda| \ge k\right) \le \mathbb{P}_\lambda\left(M_k \ge \frac{\lambda k}{2\lambda+1}\right) \le e^{-ck}, \tag{3.13}$$

with $c = \frac{\lambda^2}{4(2\lambda+1)^2 e^{\lambda(e-1)}}$. □

Next we prove a more general statement than the one given in Proposition 1.7. As noted in the introduction, this statement implies that there are uncountably many rapidly determined properties.

In the following, if $F \subseteq \mathbb{N}$ is a set of levels, we say that a node lies on an F*-level* if the level of the node is contained in F.

Proposition 3.5. *The event*

$$A = \{\text{there exists a node on an even level with exactly } n_A \text{ children}\}$$

is rapidly determined on the interval $I = (\lambda_0, \lambda_1)$ for any $0 \le \lambda_0 < \lambda_1 \le \infty$. Moreover, if $F \subseteq \mathbb{N}$ is any set of levels, and B is the event

$$B = \{\text{there exists a node on an } F\text{-level with exactly } n_B \text{ children}\},$$

then $A \cup B$ is a rapidly determined event.

Proof. Let T_λ be a Poisson(λ) Galton–Watson tree with seed $\widetilde{X} = (X_1, X_2, \dots)$, and let E_k and F_k be the sets of nodes among the first k which lie on an even/F-level, respectively. We now define the event A_k (resp. B_k) that in E_k (resp. F_k) there exists a node with exactly n_A (resp. n_B) children, i.e,

$$A_k = \bigcup_{i\in E_k} \{X_i = n_A\}, \qquad B_k = \bigcup_{i\in F_k} \{X_i = n_B\}, \tag{3.14}$$

with the convention that if $|T_\lambda| < k$ then $E_k = E_{|T_\lambda|}$ and $F_k = F_{|T_\lambda|}$. As $A_k \subseteq A$ and $B_k \subseteq B$, we have $(A\cup B)\triangle(A_k \cup B_k) = (A\cup B)\cap(A_k^c \cap B_k^c)$, and on this event we must have $|T_\lambda| \ge k$. Hence,

$$\begin{aligned}\mathbb{P}_\lambda((A\cup B)\triangle(A_k\cup B_k)) &= \mathbb{P}_\lambda((A\cup B)\cap(A_k^c\cap B_k^c), |T_\lambda| \ge k) \\ &\le \mathbb{P}_\lambda(A_k^c, |T_\lambda| \ge k)\end{aligned} \tag{3.15}$$

Splitting this over $\{|E_k| \le \frac{k}{2\lambda+1}\} \cup \{|E_k| > \frac{k}{2\lambda+1}\}$,

$$\mathbb{P}_\lambda(A_k^c, |T_\lambda| \ge k) = \mathbb{P}_\lambda\left(A_k^c, |T_\lambda| \ge k, |E_k| \le \frac{k}{2\lambda+1}\right) + \mathbb{P}_\lambda\left(A_k^c, |T_\lambda| \ge k, |E_k| > \frac{k}{2\lambda+1}\right), \tag{3.16}$$

we may bound the first term using Lemma 3.4 as

$$\mathbb{P}_\lambda\left(A_k^c, |T_\lambda| \ge k, |E_k| \le \frac{k}{2\lambda+1}\right) \le \mathbb{P}_\lambda\left(|T_\lambda| \ge k, |E_k| \le \frac{k}{2\lambda+1}\right) \le e^{-ck}. \tag{3.17}$$

For the second term, we again use a martingale argument. Consider the martingale $(M_k)_{k\in\mathbb{N}_0}$ with respect to the filtration generated by the truncated seed $\widetilde{X}^{(k)}$ with

$$M_k = \sum_{i=0}^{k} 1_{i\in E_i}(p - 1_{X_i = n_A}), \tag{3.18}$$

where $p = \mathbb{P}_\lambda(X_i = n_A)$. On the event $A_k^c = \bigcap_{i\in E_k}\{X_i \ne 1\}$, we have $M_k = p|E_k|$ and hence, on $A_k^c \cap \{|E_k| > \frac{k}{2\lambda+1}\}$ we have $M_k \ge \frac{pk}{2\lambda+1}$. Noting that $|M_k - M_{k-1}| \le 1$, we may therefore bound the second term in equation (3.16) with the usual Azuma–Hoeffding inequality

$$\mathbb{P}_\lambda\left(A_k^c, |T_\lambda| \ge k, |E_k| > \frac{k}{2\lambda+1}\right) \le \mathbb{P}_\lambda\left(M_k \ge \frac{kp}{2\lambda+1}\right) \le e^{-c'k} \tag{3.19}$$

with $c' = \frac{p^2}{2(2\lambda+1)^2}$, and this concludes the proof. □

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