Electron. Commun. Probab. **25** (2020), no. 34, 1–13.
https://doi.org/10.1214/20-ECP313
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Anchored isoperimetric profile of the infinite cluster in supercritical bond percolation is Lipschitz continuous*

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Abstract

We consider the standard model of i.i.d. first passage percolation on $\mathbb{Z}^d$ given a distribution G on $\mathbb{R}_+$. We consider a cube oriented in the direction $\overrightarrow{v}$ whose sides have length n. We study the maximal flow from the top half to the bottom half of the boundary of this cube. We already know that the maximal flow renormalized by n^{d-1} converges towards the flow constant $\nu_G(\overrightarrow{v})$. We prove here that the map $p \mapsto \nu_{p\delta_1+(1-p)\delta_0}$ is Lipschitz continuous on all intervals $[p_0, p_1] \subset (p_c(d), 1)$ where $p_c(d)$ denotes the critical parameter for i.i.d. bond percolation on $\mathbb{Z}^d$. For $p > p_c(d)$, we know that there exists almost surely a unique infinite open cluster $\mathcal{C}_p$ [8]. We are interested in the regularity properties in p of the anchored isoperimetric profile of the infinite cluster $\mathcal{C}_p$. For $d \geq 2$, using the result on the regularity of the flow constant, we prove here that the anchored isoperimetric profile defined in [4] is Lipschitz continuous on all intervals $[p_0, p_1] \subset (p_c(d), 1)$.

Keywords: regularity; flow constant; percolation; isoperimetric constant; first passage percolation.
AMS MSC 2010: Primary 60K35, Secondary 82B43.
Submitted to ECP on January 7, 2019, final version accepted on April 8, 2020.

1 Introduction

1.1 Flow constant

The model of first passage percolation was first introduced by Hammersley and Welsh [9] in 1965 as a model for the spread of a fluid in a porous medium. In this model, mathematicians studied intensively geodesics, *i.e.*, fastest paths between two points in the grid. The study of maximal flows in first passage percolation started later in 1984 in dimension 2 with an article of Grimmett and Kesten [7]. In 1987, Kesten studied maximal flows in dimension 3 in [10]. The study of maximal flows is associated with the study of random cutsets that can be seen as $(d-1)$-dimensional surfaces. Their study presents more technical difficulties than the study of geodesics. Thus, the interpretation of first passage percolation in terms of maximal flows has been less studied.

Let us consider a large box in $\mathbb{Z}^d$ oriented according to a direction $\overrightarrow{v}$, to each edge we assign a random i.i.d. capacity with distribution G. We interpret this capacity as a rate of flow, *i.e.*, it corresponds to the maximal amount of water that can cross the

*Research was partially supported by the ANR project PPPP (ANR-16-CE40-0016)

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edge per second. Next, we consider the top half and bottom half of the boundary of the box. We are interested in the maximal flow that can cross the box from its top half to its bottom half per second. A first issue is to understand if the maximal flow in the box properly renormalized converges when the size of the box grows to infinity. This question was addressed in [10], [11] and [16] where one can find laws of large numbers and large deviations estimates for this maximal flow when the dimensions of the box grow to infinity under some moments assumptions on the capacities. The maximal flow properly renormalized converges towards the so-called flow constant $\nu_G(\overrightarrow{v})$. In [12], Rossignol and Théret proved the same results without any moment assumption on G, they even allow the capacities to take infinite value as long as $G(\{+\infty\}) < p_c(d)$ where $p_c(d)$ denotes the critical parameter of i.i.d. bond percolation on $\mathbb{Z}^d$. Moreover, the two authors have shown that the flow constant is continuous with regard to the distribution of the capacities. Let us denote $\beta_p = \nu_{p\delta_1+(1-p)\delta_0}$ for $p > p_c(d)$. Thanks to the result of Zhang in [15], we know that β_p is a norm. This norm will be properly defined in section 2. In this paper, we prove that the map $p \mapsto \beta_p$ is Lipschitz continuous on every compact interval included in $(p_c, 1)$.

Theorem 1 (Regularity of the flow constant)**.** *Let $p_c(d) < p_0 < p_1 < 1$. There exists a positive constant κ depending only on d, p_0 and p_1, such that*

$$\forall p, q \in [p_0, p_1] \qquad \sup_{x\in\mathbb{S}^{d-1}} |\beta_p(x) - \beta_q(x)| \leq \kappa |q-p| \,.$$

The proof of this theorem will strongly rely on an adaptation of the proof of Theorem 1 in [16].

1.2 Anchored isoperimetric profile

The study of isoperimetric problems in the discrete setting is more recent than in the continuous setting. In the continuous setting, we study the perimeter to volume ratio; in the context of graphs, the analogous problem is the study of the size of edge boundary to volume ratio. This can be encoded by the Cheeger constant. For a finite graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$, we define the edge boundary $\partial_{\mathcal{G}} A$ of a subset A of $V(\mathcal{G})$ as

$$\partial_{\mathcal{G}} A = \left\{ e = \langle x, y \rangle \in E(\mathcal{G}) : x \in A, y \notin A \right\}.$$

We denote by $|B|$ the cardinality of the finite set B. The isoperimetric constant of $\mathcal{G}$, also called Cheeger constant, is defined as

$$\varphi_{\mathcal{G}} = \min \left\{ \frac{|\partial_{\mathcal{G}} A|}{|A|} \,:\, A \subset V(\mathcal{G}), 0 < |A| \leq \frac{|V(\mathcal{G})|}{2} \right\} .$$

This constant was introduced by Cheeger in his thesis [2] in order to obtain a lower bound for the smallest eigenvalue of the Laplacian. The isoperimetric constant of a graph gives information on its geometry.

Let $d \geq 2$. We consider an i.i.d. supercritical bond percolation on the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ having for vertices $\mathbb{Z}^d$ and for edges $\mathbb{E}^d$ the set of pairs of nearest neighbors in $\mathbb{Z}^d$ for the Euclidean norm. Every edge $e \in \mathbb{E}^d$ is open with probability $p > p_c(d)$. We know that there exists almost surely a unique infinite open cluster $\mathcal{C}_p$ [8]. In this paper, we want to study how the geometry of $\mathcal{C}_p$ varies with p through its Cheeger constant. However, if we minimize the isoperimetric ratio over all possible subgraphs of $\mathcal{C}_p$ without any constraint on the size, one can prove that $\varphi_{\mathcal{C}_p} = 0$ almost surely. For that reason, we shall minimize the isoperimetric ratio over all possible subgraphs of $\mathcal{C}_p$ given a constraint on the size. There are several ways to do it. We can for instance study the Cheeger constant of the graph $\mathcal{C}_n = \mathcal{C}_p \cap [-n,n]^d$ or of the largest connected component $\widetilde{\mathcal{C}}_n$ of $\mathcal{C}_n$ for $n \geq 1$. Since we have $\varphi_{\mathcal{C}_p} = 0$ almost surely, the isoperimetric constants $\varphi_{\mathcal{C}_n}$ and $\varphi_{\widetilde{\mathcal{C}}_n}$ go to 0 when n goes to infinity. Roughly speaking, by analogy with the full lattice, we expect that subgraphs of $\widetilde{\mathcal{C}}_n$ that minimize the isoperimetic ratio have edge boundary size of order n^{d-1} and size of order n^d with high probability.

In [1], Biskup, Louidor, Procaccia and Rosenthal defined a modified Cheeger constant $\widetilde{\varphi}_{\mathcal{C}_n}$ and proved that $n\widetilde{\varphi}_{\mathcal{C}_n}$ converges towards a deterministic constant in dimension 2. In [6], Gold proved the same result in dimension $d \geq 3$. Instead of considering the open edge boundary of subgraphs within $\mathcal{C}_n$, they considered the open edge boundary within the whole infinite cluster $\mathcal{C}_p$, this is more natural because $\mathcal{C}_n$ has been artificially created by restricting $\mathcal{C}_p$ to the box $[-n,n]^d$. They also added a stronger constraint on the size of subgraphs of $\mathcal{C}_n$ to ensure that minimizers do not touch the boundary of the box $[-n,n]^d$. Moreover, they proved that the subgraphs achieving the minimum, properly rescaled, converge towards a deterministic shape that is the Wulff crystal. Namely, it is the shape solving the continuous anisotropic isoperimetric problem associated with the norm β_p corresponding to the surface tension in the percolation setting. The quantity $n\widetilde{\varphi}_{\mathcal{C}_n}$ converges towards the solution of a continuous isoperimetric problem.

This modified Cheeger constant was inspired by the anchored isoperimetric profile $\varphi_n(p)$. This is another way to define the Cheeger constant of $\mathcal{C}_p$, that is more natural in the sense that we do not restrict minimizers to remain in the box $[-n,n]^d$. It is defined as follows:

$$\varphi_n(p) = \min\left\{ \frac{|\partial_{\mathcal{C}_p} H|}{|H|} : 0 \in H \subset \mathcal{C}_p,\ \text{H connected, } 0 < |H| \leq n^d \right\},$$

where we condition on the event $\{0 \in \mathcal{C}_p\}$. We say that H is a valid subgraph if $0 \in H \subset \mathcal{C}_p$, H is connected and $|H| \leq n^d$.

We need to introduce some definitions to be able to define properly a limit shape in dimension $d \geq 2$. In order to build a continuous limit shape, we shall define a continuous analogue of the cardinality of the open edge boundary. In fact, the cardinality of the open edge boundary may be interpreted in terms of a surface energy associate with the norm β_p. Given a subset E of $\mathbb{R}^d$ having a regular boundary, we define $\mathcal{I}_p$ as

$$\mathcal{I}_p(E) = \int_{\partial E} \beta_p(n_E(x))\mathcal{H}^{d-1}(dx)\,,$$

where $\mathcal{H}^{d-1}$ denotes the Hausdorff measure in dimension $d-1$ and $n_E(x)$ is the normal unit exterior vector of E at x. The quantity $\mathcal{I}_p(E)$ represents the surface energy of E for the norm β_p. At the point x, the tension has intensity $\beta_p(n_E(x))$ in the direction of $n_E(x)$. To understand the link between β_p and the open edge boundary, we refer to sections 3 in [6] or [4]. We denote by $\mathcal{L}^d$ the d-dimensional Lebesgue measure. We can associate with the norm β_p the following isoperimetric problem:

$$\text{minimize } \frac{\mathcal{I}_p(E)}{\mathcal{L}^d(E)} \text{ subject to } \mathcal{L}^d(E) \leq 1\,.$$

We use the Wulff construction to build a minimizer for this anisotropic isoperimetric problem (see [14]). We define the set $\widehat{W}_p$ as

$$\widehat{W}_p = \bigcap_{v \in \mathbb{S}^{d-1}} \left\{ x \in \mathbb{R}^d : x \cdot v \leq \beta_p(v) \right\},$$

where $\cdot$ denotes the standard scalar product and $\mathbb{S}^{d-1}$ is the unit sphere of $\mathbb{R}^d$. Taylor proved in [13] that the set $\widehat{W}_p$ properly rescaled is the unique minimizer, up to translations and modifications on a null set, of the associated isoperimetric problem.

In [4], Dembin proves the existence of the limit of $n\varphi_n(p)$ and that it converges towards the solution of the continuous isoperimetric problem associated with the norm β_p.

Proposition 1. *Let $d \geq 2$, $p > p_c(d)$ and let β_p be the norm that will be properly defined in section 2. Let W_p be a dilate of the Wulff crystal $\widehat{W}_p$ for the norm β_p such that $\mathcal{L}^d(W_p) = 1/\theta_p$ where $\theta_p = \mathbb{P}(0 \in \mathcal{C}_p)$. Then, conditionally on the event $\{0 \in \mathcal{C}_p\}$,*

$$\lim_{n\to\infty} n\varphi_n(p) = \frac{\mathcal{I}_p(W_p)}{\theta_p \mathcal{L}^d(W_p)} = \mathcal{I}_p(W_p) \text{ a.s.}.$$

In this paper, we aim to study the regularity properties of the anchored isoperimetric profile. This was first studied by Garet, Marchand, Procaccia, Théret in [5], they proved that the modified Cheeger constant in dimension 2 is continuous on $(p_c(2), 1]$. We aim here to prove the two following theorems. Theorem 2 asserts that the anchored isoperimetric profile is Lipschitz continuous on every compact interval $[p_0, p_1] \subset (p_c(d), 1)$.

Theorem 2 (Regularity of the anchored isoperimetric profile)**.** *Let $d \geq 2$. Let $p_c(d) < p_0 < p_1 < 1$. There exits a positive constant ν depending only on d, p_0 and p_1, such that*

$$\forall p, q \in [p_0, p_1] \qquad |\mathcal{I}_p(W_p) - \mathcal{I}_q(W_q)| \leq \nu |q - p| \,.$$

Remark 1.1. Actually, the map $p \mapsto \mathcal{I}_p(W_p)$ is also continuous at 1, this is not a consequence of theorem 2 but it comes from the fact that the map $p \to \beta_p$ is continuous on $(p_c(d), 1]$. This result is a corollary of theorem 2.6. in [12].

Theorem 3 studies the Hausdorff distance between two Wulff crystals associated with norms β_p and β_q.

Theorem 3 (Regularity of the anchored isoperimetric profile)**.** *Let $d \geq 3$. Let $p_c(d) < p_0 < p_1 < 1$. There exits a positive constant ν' depending only on d, p_0 and p_1, such that*

$$\forall p, q \in [p_0, p_1] \qquad d_{\mathcal{H}}(\widehat{W}_p, \widehat{W}_q) \leq \nu' |q - p| \,,$$

where $d_{\mathcal{H}}$ is the Hausdorff distance between non empty compact sets of $\mathbb{R}^d$.

Theorem 1 is the key element to prove these two theorems.

Remark 1.2. In this paper, we choose to work on the anchored isoperimetric profile instead of the modified Cheeger constant because the norm we use is the same for all dimensions $d \geq 2$. The existence of the modified Cheeger constant in dimension 2 uses another norm specific to this dimension (see [1]). In [6], Gold proved the existence of the modified Cheeger constant for $d \geq 3$ with the same norm β_p. Actually, we believe that his proof also holds in dimension 2 up to using similar combinatorial arguments as in [4]. Therefore, the theorem 2 may be shown for the modified Cheeger constant in dimension $d \geq 2$ using the same ingredients as in this paper.

Here is the structure of the paper. In section 2, we define the norm β_p. We prove that the map $p \mapsto \beta_p$ is Lipschitz continuous in section 3. We prove the main results on the regularity of the anchored isoperimetric profile (theorems 2 and 3) in section 4. Finally, we write an adaptation of the proof of Zhang [16] in section 5 that is necessary to prove theorem 1.

2 Definition of the norm β_p

We introduce now many notations used for instance in [11] concerning flows through cylinders. Let A be a non-degenerate hyperrectangle, that is to say a rectangle of dimension $d-1$ in $\mathbb{R}^d$. Let $\overrightarrow{v}$ be one of the two unit vectors normal to A. Let $h>0$, we denote by $\mathrm{cyl}(A,h)$ the cylinder with base A and height $2h$ defined by

$$\mathrm{cyl}(A,h)=\{x+t\overrightarrow{v}\,:\,x\in A,\,t\in[-h,h]\}\,.$$

The set $\mathrm{cyl}(A,h)\setminus A$ has two connected components, denoted by $C_1(A,h)$ and $C_2(A,h)$. For $i=1,2$, we denote by $C_i'(A,h)$ the discrete boundary of $C_i(A,h)$ defined by

$$C_i'(A,h)=\left\{x\in\mathbb{Z}^d\cap C_i(A,h)\,:\,\exists y\notin\mathrm{cyl}(A,h),\,\langle x,y\rangle\in\mathbb{E}^d\right\}\,.$$

We say that the set of edges E cuts $C_1'(A,h)$ from $C_2'(A,h)$ in $\mathrm{cyl}(A,h)$ if any path γ from $C_1'(A,h)$ to $C_2'(A,h)$ in $\mathrm{cyl}(A,h)$ contains at least one edge of E. We call such a set a cutset. For any cutset E, let $|E|_{o,p}$ denote the number of p-open edges in E. We shall call it the p-capacity of E. Define

$$\tau_p(A,h)=\min\left\{|E|_{o,p}:\;\;E\text{ cuts }C_1'(A,h)\text{ from }C_2'(A,h)\text{ in }\mathrm{cyl}(A,h)\right\}\,.$$

Note that it is a random quantity as $|E|_{o,p}$ is random, and that the cutsets in this definition are anchored at the border of A. This quantity is related to the fact that graphs that achieve the infimum in the definition of $\varphi_n(p)$ try to minimize their open edge boundary. To build a norm upon this quantity, we use the fact that the quantity $\tau_p(A,h)$ properly renormalized converges towards a deterministic constant when the size of the cylinder goes to infinity. The following proposition is a corollary of proposition 3.5 in [11].

Proposition 2 (Definition of the norm β_p)**.** *Let $d\geq 2$, $p>p_c(d)$, A be a non-degenerate hyperrectangle and $\overrightarrow{v}$ one of the two unit vectors normal to A. Let h be a height function such that $\lim_{n\to\infty}h(n)=\infty$. The limit*

$$\beta_p(\overrightarrow{v})=\lim_{n\to\infty}\frac{\mathbb{E}[\tau_p(nA,h(n))]}{\mathcal{H}^{d-1}(nA)}$$

exists and is finite. Moreover, the limit is independent of A and h and the homogeneous extension of β_p to $\mathbb{R}^d$ is a norm.

As the limit does not depend on A and h, in what follows for simplicity, we will take $h(n)=n$ and $A=S(\overrightarrow{v})$ where $S(\overrightarrow{v})$ is an hyper-square centered at 0, isometric to $[-1,1]^{d-1}\times\{0\}$ and normal to $\overrightarrow{v}$. We will denote by $B(n,\overrightarrow{v})$ the cube $\mathrm{cyl}(nS(\overrightarrow{v}),n)$ and by $\tau_p(n,\overrightarrow{v})$ the quantity $\tau_p(nS(\overrightarrow{v}),n)$.

3 Regularity of the map $p\mapsto\beta_p$

Let $p_0>p_c(d)$ and let $q>p\geq p_0$. Our strategy is the following, we easily get that $\beta_p\leq\beta_q$ by properly coupling the percolations of parameters $p_c(d)<p<q$. The second inequality requires more work. We denote by $E_{n,p}$ the random cutset of minimal size that achieves the minimum in the definition of $\tau_p(n,\overrightarrow{v})$. By definition, as $E_{n,p}$ is a cutset, we can bound $\tau_q(n,\overrightarrow{v})$ from above by the number of edges in $E_{n,p}$ that are q-open, which we expect to be at most $\tau_p(n,\overrightarrow{v})+C(q-p)|E_{n,p}|$ where C is a constant. We next need to get a control of $|E_{n,p}|$ which is uniform in $p\in[p_0,1]$ of the kind c_0n^{d-1} where c_0 depends only on p_0 and d. In [16], Zhang obtained a control on the size of the smallest minimal cutset that separates the top from the bottom of a cylinder in the general first passage percolation model, but his control depends on the distribution G of the passage times. We only consider probability measures $G_p=p\delta_1+(1-p)\delta_0$ for $p>p_c(d)$, but we need to adapt Zhang's proof in this particular case to obtain a control that does not depend on p anymore and a control for cutsets that separates the bottom half from the top half of the boundary of the cylinder. More precisely, let us denote by $\mathcal{N}_{n,p}$ the total number of edges in $E_{n,p}$. We have the following control on $\mathcal{N}_{n,p}$.

Theorem 4 (Adaptation of theorem 2 in [16]). *Let $p_0 > p_c(d)$. There exist constants C_1, C_2 and α that depend only on d and p_0 such that*

$$\forall p \in [p_0, 1] \quad \forall n \geq 1 \qquad \mathbb{P}_p\left(\mathcal{N}_{n,p} > \alpha n^{d-1}\right) \leq C_1 \exp(-C_2 n^{d-1}) .$$

We postpone the proof of theorem 4 to section 5. We have now the key ingredients to prove that the map $p \mapsto \beta_p$ is Lipschitz continuous.

Proof of Theorem 1. Let $p_c < p_0 < p_1 < 1, \overrightarrow{v} \in \mathbb{S}^{d-1}$, and p, q such that $p_0 \leq p < q \leq p_1$. First, we fix a cube $B(n, \overrightarrow{v})$ and we couple the percolations of parameters p and q in the standard way, *i.e.*, we consider the i.i.d. family $(U(e))_{e \in \mathbb{E}^d}$ distributed according to the uniform law on $[0, 1]$ and we say that an edge e is p-open (resp. q-open) if $U(e) \geq p$ (resp. $U(e) \geq q$). Thanks to this coupling, we easily obtain that $\tau_p(\overrightarrow{v}, n) \leq \tau_q(\overrightarrow{v}, n)$ and by dividing by $\mathcal{H}^{d-1}(nS(\overrightarrow{v})) = (2n)^{d-1}$, taking the expectation and letting n go to infinity we conclude that

$$\beta_p(\overrightarrow{v}) \leq \beta_q(\overrightarrow{v}) . \tag{3.1}$$

Let $E_{n,p}$ be a random cutset of minimal size that achieves the minimum in the definition of $\tau_p(n, \overrightarrow{v})$. We consider now another coupling. The idea is to introduce a coupling of the percolations of parameter p and q such that if an edge is p-open then it is q-open and $E_{n,p}$ is independent of the q-state of any edge. Unfortunately, we cannot find such a coupling but we can introduce a coupling that almost has this property. To do so, for each edge $e \in \mathbb{E}^d$, we consider two independent Bernoulli random variables $U(e)$ and $V(e)$ of parameters p and $(q - p)/(1 - p)$. We say that an edge e is p-open if $U(e) = 1$ and that it is q-open if $U(e) = 1$ or $V(e) = 1$. Indeed,

$$\mathbb{P}\left(U(e) = 1, V(e) = 1\right) = p + (1 - p)\frac{q - p}{1 - p} = q .$$

Let $\delta > 0$. We have,

$$\begin{aligned}
\mathbb{P}&\left(\tau_q(n, \overrightarrow{v}) > \tau_p(n, \overrightarrow{v}) + \left(\frac{q-p}{1-p} + \delta\right)\alpha n^{d-1}, \mathcal{N}_{n,p} < \alpha n^{d-1}\right) \\
&\leq \mathbb{P}\left(\tau_q(n, \overrightarrow{v}) - \tau_p(n, \overrightarrow{v}) > \left(\frac{q-p}{1-p} + \delta\right)|E_{n,p}|\right) \\
&\leq \sum_{\mathfrak{E}} \mathbb{P}\left(E_{n,p} = \mathfrak{E}, |\{e \in \mathfrak{E} : (U(e), V(e)) = (0,1)\}| > \left(\frac{q-p}{1-p} + \delta\right)|\mathfrak{E}|\right) \\
&\leq \sum_{\mathfrak{E}} \mathbb{P}\left(E_{n,p} = \mathfrak{E}, |\{e \in \mathfrak{E} : V(e) = 1\}| > \left(\frac{q-p}{1-p} + \delta\right)|\mathfrak{E}|\right) \\
&\leq \sum_{\mathfrak{E}} \mathbb{P}(E_{n,p} = \mathfrak{E})\, \mathbb{P}\left(|\{e \in \mathfrak{E} : V(e) = 1\}| > \left(\frac{q-p}{1-p} + \delta\right)|\mathfrak{E}|\right) \\
&\leq \exp(-2\delta^2 n^{d-1})
\end{aligned} \tag{3.2}$$

where the sum is over sets $\mathfrak{E}$ that cut $C_1'(nS(\overrightarrow{v}), n)$ from $C_2'(nS(\overrightarrow{v}), n)$ in $B(n, \overrightarrow{v})$ and where we use in the last inequality Chernoff bound and the fact that $|E_{n,p}| \geq n^{d-1}$ (uniformly in $\overrightarrow{v}$). Finally, using inequality (3.2) and theorem 4, we get

$$\begin{aligned}
\mathbb{E}[\tau_q(n, \overrightarrow{v})] &\leq \mathbb{E}[\tau_q(n, \overrightarrow{v})\mathbb{1}_{\mathcal{N}_{n,p} < \alpha n^{d-1}}] + \mathbb{E}[\tau_q(n, \overrightarrow{v})\mathbb{1}_{\mathcal{N}_{n,p} \geq \alpha n^{d-1}}] \\
&\leq \mathbb{E}[\tau_p(n, \overrightarrow{v})] + \left(\frac{q-p}{1-p} + \delta\right)\alpha n^{d-1} + |B(n, \overrightarrow{v})|\left(\mathrm{e}^{-2\delta^2 n^{d-1}} + C_1\,\mathrm{e}^{-C_2 n^{d-1}}\right) \\
&\leq \mathbb{E}[\tau_p(n, \overrightarrow{v})] + \left(\frac{q-p}{1-p} + \delta\right)\alpha n^{d-1} + C_d(2n)^d\left(\mathrm{e}^{-2\delta^2 n^{d-1}} + C_1\,\mathrm{e}^{-C_2 n^{d-1}}\right) ,
\end{aligned}$$

where C_d is a constant depending only on d. Dividing by $(2n)^{d-1}$ and by letting n go to infinity, we obtain

$$\beta_q(\overrightarrow{v}) \leq \beta_p(\overrightarrow{v}) + \left(\frac{q-p}{1-p} + \delta\right)\frac{\alpha}{2^{d-1}} \tag{3.3}$$

and by letting δ go to 0,

$$\beta_q(\overrightarrow{v}) \leq \beta_p(\overrightarrow{v}) + \kappa(q-p) \tag{3.4}$$

where $\kappa = \alpha/(2^{d-1}(1-p_1))$. Combining inequalities (3.1) and (3.4), we obtain that

$$\sup_{\overrightarrow{v}\in\mathbb{S}^{d-1}} |\beta_q(\overrightarrow{v}) - \beta_p(\overrightarrow{v})| \leq \kappa|q-p|\,. \qquad \square$$

4 Proof of theorems 2 and 3

Proof of theorem 2. Let $p_c < p_0 < p_1 < 1$. We recall that, for $p > p_c(d)$, W_p denotes the Wulff crystal for the norm β_p such that $\mathcal{L}^d(W_p) = 1/\theta_p$. In this section we aim to prove that the map $p \mapsto \mathcal{I}_p(W_p)$ is Lipschitz continuous on $[p_0, p_1]$. Notice that as the map $p \mapsto \theta_p$ is non-decreasing, we have

$$\forall p, q \in (p_c(d), 1] \qquad p < q \implies \mathcal{L}^d(W_p) \geq \mathcal{L}^d(W_q)$$

and using the fact that W_q is a minimizer for $\mathcal{I}_q$ for sets of equal volume, it follows that

$$\forall p, q \in (p_c(d), 1] \qquad p < q \implies \mathcal{I}_q(W_p) \geq \mathcal{I}_q(W_q)\,. \tag{4.1}$$

Moreover, the map $p \mapsto \theta_p$ is infinitely differentiable on $[p_0, p_1]$, see for instance theorem 8.92 in [8]. Therefore, there exists a constant L depending on p_0, p_1 and d such that

$$\forall p, q \in [p_0, p_1] \qquad |\theta_p - \theta_q| \leq L|q-p|\,. \tag{4.2}$$

Let us compute now some useful inequalities. For any set $E \subset \mathbb{R}^d$ with Lipschitz boundary, by theorem 1, we have for any $p, q \in [p_0, p_1]$

$$\begin{aligned} |\mathcal{I}_p(E) - \mathcal{I}_q(E)| &= \left| \int_{\partial E} (\beta_p(n_E(x)) - \beta_q(n_E(x)))\, \mathcal{H}^{d-1}(dx) \right| \\ &\leq \int_{\partial E} |\beta_p(n_E(x)) - \beta_q(n_E(x))|\, \mathcal{H}^{d-1}(dx) \leq \kappa|q-p|\mathcal{H}^{d-1}(\partial E) \end{aligned} \tag{4.3}$$

where κ is the constant associated with p_0 and p_1 in the statement of theorem 1. We recall that the map $p \to \beta_p$ is uniformly continuous on $[p_0, p_1]$. We denote by β^{min} and β^{max} its minimal and maximal value, i.e., we have

$$\forall \overrightarrow{v} \in \mathbb{S}^{d-1} \quad \forall p \in [p_0, p_1] \qquad \beta^{min} \leq \beta_p(\overrightarrow{v}) \leq \beta^{max}\,.$$

Together with inequality (4.1) and the fact that the Wulff crystal is a minimizer for an isoperimetric problem, we get for $p \in [p_0, p_1]$

$$\mathcal{I}_p(W_p) \leq \mathcal{I}_p(W_{p_0}) = \int_{\partial W_{p_0}} \beta_p(n_{W_{p_0}}(x))\mathcal{H}^{d-1}(dx) \leq \beta^{max}\mathcal{H}^{d-1}(\partial W_{p_0})\,. \tag{4.4}$$

We also have

$$\mathcal{H}^{d-1}(\partial W_p) = \int_{\partial W_p} \mathcal{H}^{d-1}(dx) \leq \int_{\partial W_p} \frac{\beta_p(n_{W_p}(x))}{\beta^{min}}\mathcal{H}^{d-1}(dx) \leq \frac{\mathcal{I}_p(W_p)}{\beta^{min}}$$

and so together with inequality (4.4), we get

$$\forall p \in [p_0, p_1] \qquad \mathcal{H}^{d-1}(\partial W_p) \le \mathcal{H}^{d-1}(\partial W_{p_0}) \frac{\beta^{max}}{\beta^{min}} \,. \tag{4.5}$$

Finally, we obtain combining inequalities (4.1), (4.3) and (4.5),

$$\mathcal{I}_p(W_p) \ge \mathcal{I}_q(W_p) - \kappa|q-p|\mathcal{H}^{d-1}(\partial W_p) \ge \mathcal{I}_q(W_q) - \kappa|q-p|\mathcal{H}^{d-1}(\partial W_{p_0}) \frac{\beta^{max}}{\beta^{min}} \,. \tag{4.6}$$

As $\mathcal{L}^d(W_p) = \mathcal{L}^d(W_q) \cdot \theta_q/\theta_p = \mathcal{L}^d(W_q(\theta_q/\theta_p)^{1/d})$ and as W_p is the minimizer for the isoperimetric problem associated with the norm β_p, we have

$$\mathcal{I}_p(W_p) \le \mathcal{I}_p\left(\left(\frac{\theta_q}{\theta_p}\right)^{1/d} W_q\right) \le \left(\frac{\theta_q}{\theta_p}\right)^{(d-1)/d} \mathcal{I}_p(W_q) \le \frac{\theta_q}{\theta_p}\mathcal{I}_p(W_q)$$

and so using inequalities (4.2), (4.3), (4.4) and (4.5)

$$\begin{aligned} \mathcal{I}_p(W_p) &\le \frac{\theta_q}{\theta_p}\big(\mathcal{I}_q(W_q) + \kappa|q-p|\mathcal{H}^{d-1}(\partial W_q)\big) \\ &\le \left(1 + \frac{L}{\theta_{p_0}}|q-p|\right)\left(\mathcal{I}_q(W_q) + \kappa|q-p|\mathcal{H}^{d-1}(\partial W_{p_0})\frac{\beta^{max}}{\beta^{min}}\right) \\ &\le \mathcal{I}_q(W_q) + \beta^{max}\mathcal{H}^{d-1}(\partial W_{p_0})\left(\frac{L}{\theta_{p_0}} + \frac{\kappa}{\beta^{min}}\left(1 + \frac{L}{\theta_{p_0}}\right)\right)|q-p| \,. \end{aligned} \tag{4.7}$$

Thus combining inequalities (4.6) and (4.7) together with Theorem 1, we get

$$|\mathcal{I}_p(W_p) - \mathcal{I}_q(W_q)| \le \nu|q-p| \tag{4.8}$$

where we set

$$\nu = \beta^{max}\mathcal{H}^{d-1}(\partial W_{p_0})\left(\frac{L}{\theta_{p_0}} + \frac{\kappa}{\beta^{min}}\left(1 + \frac{L}{\theta_{p_0}}\right)\right) \,. \qquad \square$$

Proof of theorem 3. Let $p_c < p_0 < p_1 < 1$ and $p, q \in [p_0, p_1]$. We consider β_p^* the dual norm of β_p, defined by

$$\forall x \in \mathbb{R}^d,\ \beta_p^*(x) = \sup\{x \cdot z \ :\ \beta_p(z) \le 1\} \,.$$

Then β_p^* is a norm. The Wulff crystal $\widehat{W}_p$ associated with β_p is in fact the unit ball associated with β_p^*. Note that the supremum in the definition of β_p^* is always achieved for a z such that $\beta_p(z) = 1$. Let $x \in \mathbb{S}^{d-1}$. Let $y \in \mathbb{S}^{d-1}$ be the direction that achieves the supremum for $\beta_p^*(x)$, thus we have

$$\beta_p^*(x) = x \cdot \frac{y}{\beta_p(y)}$$

and so using theorem 1,

$$\beta_p^*(x) - \beta_q^*(x) \le x \cdot \frac{y}{\beta_p(y)} - x \cdot \frac{y}{\beta_q(y)} \le \frac{\|x\|_2\|y\|_2}{\beta_p(y)\beta_q(y)}|\beta_p(y) - \beta_q(y)| \le \frac{\kappa}{(\beta^{min})^2}|q-p|$$

where β^{min} was defined in the proof of theorem 2. We proceed similarly for $\beta_q^*(x) - \beta_p^*(x)$. Finally, we obtain

$$\sup_{x \in \mathbb{S}^{d-1}} |\beta_p^*(x) - \beta_q^*(x)| \le \frac{\kappa}{(\beta^{min})^2}|q-p| \,. \tag{4.9}$$

We recall the following definition of the Hausdorff distance between two subsets E and F of $\mathbb{R}^d$:

$$d_{\mathcal{H}}(E,F)=\inf\left\{r\in\mathbb{R}^+ : E\subset F^r \text{ and } F\subset E^r\right\}$$

where $E^r=\{y:\exists x\in E \quad \|y-x\|_2\leq r\}$. Thus, we have

$$d_{\mathcal{H}}(\widehat{W}_p,\widehat{W}_q)\leq \sup_{y\in\mathbb{S}^{d-1}}\left\|\frac{y}{\beta_p^*(y)}-\frac{y}{\beta_q^*(y)}\right\|_2 .$$

Note that $y/\beta_p^*(y)$ (resp. $y/\beta_q^*(y)$) is in the unit sphere for the norm β_p^* (resp. β_q^*). Let $x\in\mathbb{S}^{d-1}$. Using the definition of β^*, we obtain

$$\frac{1}{\beta^{max}}\leq x\cdot\frac{x}{\beta_p(x)}\leq\beta_p^*(x) .$$

Finally, using inequality (4.9), we obtain

$$\begin{aligned} d_{\mathcal{H}}(\widehat{W}_p,\widehat{W}_q)&\leq \sup_{y\in\mathbb{S}^{d-1}}\left|\frac{1}{\beta_p^*(y)}-\frac{1}{\beta_q^*(y)}\right|\\ &\leq \sup_{y\in\mathbb{S}^{d-1}}\frac{1}{\beta_q^*(y)\beta_p^*(y)}\left|\beta_p^*(y)-\beta_q^*(y)\right|\\ &\leq \sup_{y\in\mathbb{S}^{d-1}}(\beta^{max})^2\left|\beta_p^*(y)-\beta_q^*(y)\right|\leq\frac{\kappa(\beta^{max})^2}{(\beta^{min})^2}|q-p| . \end{aligned} \tag{4.10}$$

The result follows. □

5 Proof of theorem 4

The proof of theorem 4 is going to be simpler than the proof of theorem 2 in [16], because passage times in our context can take only values 0 or 1, *i.e.*, to each edge we associate an i.i.d random variable of distribution $G_p=p\delta_1+(1-p)\delta_0$ whereas Zhang considers in [16] more general distributions. Our setting is equivalent to bond percolation of parameter p by saying that an edge is closed if its passage time is 0, and open if its passage time is 1. Let us briefly explain the idea behind that theorem. Let $p\geq p_0$. We work on bond percolation of parameter p (equivalently on first passage percolation with distribution $G_p=p\delta_1+(1-p)\delta_0$). We aim at bounding the size of the smallest minimal cutset that cuts the set $C_1'(nS(\overrightarrow{v}),n)$ from $C_2'(nS(\overrightarrow{v}),n)$ in $B(n,\overrightarrow{v})$. To do so we do a renormalization at a scale t in order to build a “smooth” minimal cutset. For $u\in\mathbb{Z}^d$, we define $B_t(u)=[0,t]^d+tu$ and $\bar{B}_t(u)=\bigcup_{v:\|v-u\|_\infty\leq 1}B_t(v)$. We say that the cubes $B_t(u)$ and $B_t(v)$ are $*$-neighbors if $\|u-v\|_\infty=1$. The $3t$-cube $\bar{B}_t(u)$ is the union of the cube $B_t(u)$ and its $*$-neighbors.

Let us now introduce some useful definitions. A connected cluster C is said to be p-crossing for a box B, if for all d directions, there is a p-open path in $C\cap B$ connecting the two opposite faces of B. We define the diameter of a finite cluster $\mathcal{C}$ as

$$\mathrm{Diam}(\mathcal{C}):=\max_{\substack{i=1,\dots,d\\ x,y\in\mathcal{C}}}|x_i-y_i| .$$

Let $T_{m,t}(p)$ be the event that B_t has a p-crossing cluster and contains some other p-open cluster D having diameter at least m. We say that $B_t(u)$ has a p-disjoint property if there exist two disconnected p-open clusters in $\bar{B}_t(u)$, both with vertices in $B_t(u)$ and in the boundary of $\bar{B}_t(u)$. We say that $B_t(u)$ has a p-blocked property if there is a p-open cluster C in $\bar{B}_t(u)$ with vertices in $B_t(u)$ and in the boundary of $\bar{B}_t(u)$, but without vertices in

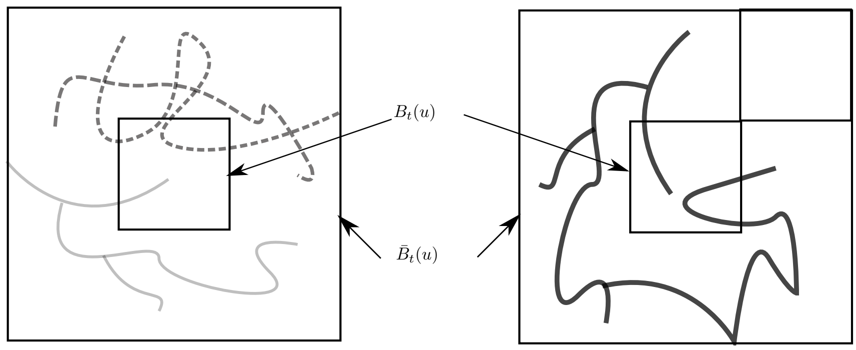

Figure 1: On the left a box $B_t(u)$ with a disjoint property, on the right a box with a blocked property

a t-cube of $\bar{B}_t(u)$. We say that a p-atypical event occurs in $B_t(u)$ if it has a p-blocked property or a p-disjoint property (see Figure 1).

As the original proof is very technical, the adaptation of the proof is also technical. There are two points that need to be adapted from Zhang's proof. First, Zhang controls the size of a minimal cutset from the top to the bottom of a box in theorem 2 but here we need to control the size of a minimal cutset from the top half $C_1'(nS(\overrightarrow{v}), n)$ to the bottom half $C_2'(nS(\overrightarrow{v}), n)$ of a box $B(n, \overrightarrow{v})$. The second point is that Zhang has a control on the size that holds for a fixed p, but we need here to have a uniform control of the size for $p \in [p_0, 1]$.

Adaptation of the proof of theorem 1 *in [16] to get theorem 4.* We keep the same notations as in [16]. The following adaptation is not self-contained. Let $p_0 > p_c(d)$ and $\overrightarrow{v} \in \mathbb{S}^{d-1}$. In [16], the author bounds the size of the smallest minimal cutset that cuts a given cylinder $B(k, m)$ from infinity. However, his construction of a linear cutset in section 2 of [16] is not specific to the set $B(k, m)$ and can be defined in the same way for any set of vertices. In particular we can replace $B(k, m)$ by $C_1'(nS(\overrightarrow{v}), n)$ and ∞ by $C_2'(nS(\overrightarrow{v}), n)$ (as it is done by Zhang in Theorem 2 in [16] with the top and the bottom of a cylinder). Note that given the configuration of passage times, the construction of Zhang's is totally deterministic. As we only focus on edges inside $B(n, \overrightarrow{v})$, we can assume that all edges outside $B(n, \overrightarrow{v})$ are closed.

We denote by $\mathcal{C}(n)$ the set that corresponds to $C(k, m)$ defined in Lemma 1 in [16]:

$$\mathcal{C}(n) = \{v \in \mathbb{Z}^d : \ v \text{ is connected to } C_1'(nS(\overrightarrow{v}), n) \text{ by an open path } \} .$$

We denote by $\mathcal{G}(n)$ the event that $\mathcal{C}(n) \cap C_2'(nS(\overrightarrow{v}), n) = \emptyset$ (it corresponds to $\mathcal{G}(k, m)$ in [16]). On this event, the exterior edge boundary $\Delta_e \mathcal{C}(n)$ of $\mathcal{C}(n)$ is a closed cutset that cuts $C_1'(nS(\overrightarrow{v}), n)$ from $C_2'(nS(\overrightarrow{v}), n)$. The problem is that the cutset $\Delta_e \mathcal{C}(n)$ may be very entangled. We use renormalization to be able to build a smooth closed cutset upon $\Delta_e \mathcal{C}(n)$. We denote by $\underline{A}$ the set of t-cubes that intersect $\Delta_e \mathcal{C}(n)$. By Zhang's construction, we can extract from $\underline{A}$ a set of cubes Γ_t such that Γ_t is $*$-connected and the union $\bar{\Gamma}_t$ of the $3t$-cubes in Γ_t (the cubes in Γ_t and their $*$-neighbors) contains a closed cutset that separates the set $C_1'(nS(\overrightarrow{v}), n)$ from $C_2'(nS(\overrightarrow{v}), n)$. Moreover, each cube in Γ_t has a $*$-neighbor where a p-atypical event occurs.

The set $E = \{\langle x, y\rangle \in B(n, \overrightarrow{v}) \ : \ x \in C_1'(nS(\overrightarrow{v}), n)\}$ cuts the set $C_1'(nS(\overrightarrow{v}), n)$ from the set $C_2'(nS(\overrightarrow{v}), n)$ in $B(n, \overrightarrow{v})$ and there exists a constant c_d depending only on d but not on $\overrightarrow{v}$ such that $|E| \leq c_d n^{d-1}$. Thus, we obtain that

$$\tau_p(n, \overrightarrow{v}) \leq |E| \leq c_d n^{d-1} .$$

We denote by $E_{n,p}$ the cutset that achieves the infimum in $\tau_p(n,\overrightarrow{v})$ and such that $|E_{n,p}| = \mathcal{N}_{n,p}$ ($E_{n,p}$ corresponds to $W(k,m)$ the minimal cutset between the top and the bottom of $B(k,m)$ in [16]). For a configuration ω, we denote by $e_1,\dots,e_{J(\omega)}$ the p-open edges in $E_{n,p}$. We have $J(\omega)=\tau_p(n,\overrightarrow{v})(\omega)\leq c_d n^{d-1}$. We denote by $\sigma(\omega)$ the configuration which coincides with ω except in edges $e_1,\dots,e_{J(\omega)}$ that are closed for $\sigma(\omega)$. Thus, the set $E_{n,p}(\sigma(\omega))$ is a p-closed (for the configuration $\sigma(\omega)$) cutset that cuts $C'_1(nS(\overrightarrow{v}),n)$ from $C'_2(nS(\overrightarrow{v}),n)$ in $B(n,\overrightarrow{v})$. Note that the set of edges $E_{n,p}(\sigma(\omega))$ is determined by the configuration ω whereas we consider the state of its edges is given by the configuration $\sigma(\omega)$. We recall that all the edges outside $B(n,\overrightarrow{v})$ are closed so that the event $\mathcal{G}(n)$ occurs in the configuration $\sigma(\omega)$ and we can use the construction of section 2 in [16] for the configuration $\sigma(\omega)$: there exists a set of cube Γ_t such that $\bar{\Gamma}_t$ contains a p-closed (for $\sigma(\omega)$) cutset Γ that cuts $C'_1(nS(\overrightarrow{v}),n)$ from $C'_2(nS(\overrightarrow{v}),n)$ (see Lemma 4 in [16]). The set $\Gamma\cap B(n,\overrightarrow{v})$ is a closed cutset that separates $C'_1(nS(\overrightarrow{v}),n)$ from $C'_2(nS(\overrightarrow{v}),n)$ in $B(n,\overrightarrow{v})$.

We now change $\sigma(\omega)$ back to ω. For $i\in\{1,\dots,J(\omega)\}$, the state of the edge e_i changes from closed to open. We write $\Gamma(\omega)$ when we consider the edge set Γ (the edge set Γ is determined by $\sigma(\omega)$ with its edges capacities determined by the configuration ω. The set $\Gamma(\omega)$ exists as an edge set, it is still a cutset but it is no longer closed, all edges in $\Gamma(\omega)\setminus\{e_1,\dots,e_{J(\omega)}\}$ are closed. Therefore, $|\Gamma(\omega)|_{o,p}\leq J(\omega)$, but by definition of $E_{n,p}$, we have $J(\omega)=|E_{n,p}(\omega)|_{o,p}\leq|\Gamma(\omega)|_{o,p}\leq J(\omega)$ and so $|\Gamma(\omega)|_{o,p}=J(\omega)$ and $\{e_1,\dots,e_{J(\omega)}\}\subset\Gamma$. Moreover, for each ω, by definition of $\mathcal{N}_{n,p}(\omega)$, we get that $|\Gamma(\omega)|\geq\mathcal{N}_{n,p}(\omega)$.

Note that for the t-cubes $B_t(u)\in\Gamma_t$ such that $\bar{B}_t(u)$ intersects the boundary of $B(n,\overrightarrow{v})$, we cannot be sure that there exists a t-cube in $\bar{B}_t(u)$ where a p-atypical event occurs. Thus, we need to obtain a control of the numbers of such cubes. Since edges outside $B(n,\overrightarrow{v})$ are closed, the set $\Delta_e\mathcal{C}(n)\setminus B(n,\overrightarrow{v})$ is included in the exterior edge boundary $\Delta_e B(n,\overrightarrow{v})$ of $B(n,\overrightarrow{v})$. Therefore, the cubes $B_t(u)$ in Γ_t such that $\bar{B}_t(u)$ is not contained in the strict interior of $B(n,\overrightarrow{v})$ satisfy $\bar{B}_t(u)\cap\Delta_e B(n,\overrightarrow{v})\neq\emptyset$. We deduce that there are at most $C_{d,t}n^{d-1}$ such cubes in Γ_t where $C_{d,t}$ is a constant depending only on the dimension d and t. Thus, if the number of t-cubes in Γ_t is greater than βn^{d-1}, then the number of t-cubes $B_t(u)$ in Γ_t that do not intersect the boundary of $B(n,\overrightarrow{v})$ and such that $\overline{B}_t(u)$ do not contain any edge among $e_1,\dots,e_J$ is greater than $(\beta-C_{d,t}-3^d c_d)n^{d-1}$. All these t-cubes have at least one $*$-neighbor with a blocked or disjoint property for the configuration ω.

In the proof of theorem 1 in [16], Zhang sums over all possible sets Γ_t. To do so, he needs to find at least one cube $B_t(v)$ that belongs to Γ_t and then he will be able to sum over all possible $*$-connected sets that contained $B_t(v)$ of a given size. In our case, any cube $B_t(u)$ that intersects the boundary $\Delta_e C'_1(nS(\overrightarrow{v}),n)\setminus B(n,\overrightarrow{v})$ belongs to $\underline{A}$ as it also intersects $\Delta_e\mathcal{C}(n)$ and by Zhang construction, we can prove that the cube $B_t(u)$ also belongs to Γ_t. Thanks to this remark, we avoid the part of Zhang's proof where he tries to find a vertex z in the intersection between the cutset $W(k,m)$ and a line L in order to find a cube that is in Γ_t. Thus, the term $\exp(\beta^{-1}n)$ in (6.19) in [16] is not necessary in our case. This leads to small modifications of constants in the proof of [16]. The remainder of the proof is the same except that we need a uniform decay for $p\in[p_0,1]$ of the probability of a p-atypical event in B_t instead of using the control in [16]. We need to prove the following lemma:

Lemma 1 (Uniform decay of the probability an atypical event occurs)**.** *Let $p_0>p_c(d)$. There exist positive constants $C_1(p_0)$ and $C_2(p_0)$ depending only on p_0 and d such that*

$$\forall p\geq p_0\quad\forall t\geq 1\qquad\mathbb{P}\left(\text{a } p\text{-atypical event occurs in } B_t\right)\leq C_1(p_0)\exp(-C_2(p_0)t)\,.\tag{5.1}$$

We would like to highlight the fact that in lemmas 6 and 7 in [16], Zhang proves the same result but with constants C_1 and C_2 depending on p. Obtaining a decay that is uniform for $p \in [p_0, 1]$ is the key element to adapt the proof of Zhang and show that the constant α in the statement of the theorem 4 does depend only on p_0 and d. $\square$

Let us now prove lemma 1. We need to adapt some existing proofs in order to obtain a decay which is uniform in p.

Proof of lemma 1. First, note that if B_t has a p-disjoint property and $\bar{B}_t$ has a p-crossing cluster, then one of the two disjoint clusters is different from the p-crossing cluster. Therefore, there is a p-open cluster of diameter greater than t different from the p-crossing cluster, so the event $T_{t,3t}(p)$ occurs in the box $\bar{B}_t$. Similarly, let us assume that B_t has a p-blocked property and $\bar{B}_t$ and all of its sub-boxes (*i.e*, boxes $B_t(v)$ such that $B_t(v) \subset \bar{B}_t$) have a p-crossing cluster. We denote by C the p-open cluster in the definition of the p-blocked property. Thus, there is at least one cluster among C and the p-crossing clusters of the sub-boxes that are disjoint from the p-crossing cluster of $\bar{B}_t$ and so the event $T_{t,3t}(p)$ occurs in the box $\bar{B}_t$. Thus,

$$\begin{aligned} \mathbb{P}(\text{a } p\text{-atypical event occurs in } B_t \ ) \leq \mathbb{P}(&\bar{B}_t \text{ does not have a } p\text{-crossing cluster}) \\ &+ 3^d \mathbb{P}(B_t \text{ does not have a } p\text{-crossing cluster}]) + \mathbb{P}\left(T_{t,3t}(p)\right) \end{aligned} \quad (5.2)$$

As the event $\{B_t \text{ doesn't have a } p\text{-crossing cluster}\}$ is non-increasing in p, we have

$$\mathbb{P}(B_t \text{ doesn't have a } p\text{-crossing cluster} \leq \mathbb{P}(B_t \text{ doesn't have a } p_0\text{-crossing cluster})\,.$$

The probability for a box B_t not to have a p_0-crossing cluster is decaying exponentially fast with t^{d-1}, see for instance theorem 7.68 in [8]. Therefore, there exist positive constants $c_1(p_0)$ and $c_2(p_0)$ such that

$$\mathbb{P}(B_t \text{ does not have a } p\text{-crossing cluster}) \leq c_1(p_0)\exp(-c_2(p_0)t^{d-1})\,. \quad (5.3)$$

It remains to prove that there exist positive constants $\kappa(p_0)$ and $\mu(p_0)$ depending only on p_0 such that for all $p \geq p_0$, for all positive integers m and N

$$\mathbb{P}(T_{m,N}(p)) \leq \kappa N^{2d} \exp(-\mu m)\,. \quad (5.4)$$

In dimension $d \geq 3$, we refer to the proof of lemma 7.104 in [8]. The proof of lemma 7.104 requires the proof of lemma 7.78. The probability controlled in lemma 7.78 is clearly non decreasing in the parameter p. Thus, if we choose $\delta(p_0)$ and $L(p_0)$ as in the proof of lemma 7.78 for $p_0 > p_c(d)$, then these parameters can be kept unchanged for some $p \geq p_0$. Thanks to lemma 7.104, we obtain

$$\begin{aligned} \forall p \geq p_0 \quad \mathbb{P}(T_{m,N}(p)) &\leq d(2N+1)^{2d} \exp\left(\left(\frac{m}{L(p_0)+1} - 1\right)\log(1-\delta(p_0))\right) \\ &\leq \frac{d.3^d}{1-\delta(p_0)} N^{2d} \exp\left(-\frac{-\log(1-\delta(p_0))}{L(p_0)+1} m\right)\,. \end{aligned}$$

We get the result with

$$\kappa = \frac{d.3^d}{1-\delta(p_0)} \quad \text{and} \quad \mu = \frac{-\log(1-\delta(p_0))}{L(p_0)+1} > 0\,.$$

In dimension 2, the result is obtained by Couronné and Messikh in the more general setting of FK-percolation, see theorem 9 in [3]. We proceed similarly as in dimension $d \geq 3$, the constant appearing in this theorem first appeared in proposition 6. The probability of the event considered in this proposition is clearly increasing in the parameter of the underlying percolation which have parameter $1-p$, it is an event for the subcritical regime of the Bernoulli percolation. Let us fix a $p_0 > p_c(2) = 1/2$, then $1 - p_0 < p_c(2)$ and we can choose the parameter $c(1 - p_0)$ and keep it unchanged for some $1 - p \leq 1 - p_0$. In theorem 9, we get the expected result with $c(1 - p_0)$ for a $p \geq p_0$ and $g(n) = n$. Finally, combining inequalities (5.2), (5.3) and (5.4), we get

$$\begin{aligned}&\mathbb{P}(\text{a } p\text{-atypical event occurs in } B_t)\\&\quad\leq c_1(p_0)\exp(-c_2(p_0)(3t)^{d-1}) + 3^d c_1(p_0)\exp(-c_2(p_0)t^{d-1}) + \kappa(p_0)(3t)^{2d}\exp(-\mu(p_0)t)\,.\end{aligned}$$

The result follows. □

Acknowledgments. The author would like to thank an anonymous referee for his valuable suggestions on the structure of the paper.

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