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A class of asymmetric regression models for left-censored data

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Abstract. A common assumption in the standard tobit model is the normality for the error distribution. However, asymmetry and bimodality may be present and alternative tobit models must be used in such cases. In this paper, we propose a tobit model based on the class of log-symmetric distributions, which includes as special cases heavy/light tailed distributions and bimodal distributions. We implement a likelihood-based approach for parameter estimation and consider a type of residual. We then discuss the problem of performing hypothesis tests within the proposed class by using the likelihood ratio and gradient statistics, which are particularly convenient for tobit models, as they do not require the information matrix. An elaborate Monte Carlo study is carried out for evaluating the performance of the maximum likelihood estimates, the likelihood ratio and gradient tests and the empirical distribution of the residuals. Finally, we illustrate the proposed methodology with the use of a real data set.

1 Introduction

After its introduction by Tobin (1958), the tobit model has been used extensively in several applied areas including economics, environmental sciences, engineering, biology, medicine and sociology; see, for example, Barros, Paula and Leiva (2008), Leiva et al. (2007), Villegas, Paula and Leiva (2011), Amemiya (1984), Thorarinsdottir and Gneiting (2010), Helsel (2011) and Martínez-Flores, Bolfarine and Gómez (2013a, 2013b). The tobit model is used to describe censored responses and gained its motivation based on a study intended to analyze the relationship between household expenditure on a durable good and household incomes. In that study, Tobin (1958) faced the existence of many cases wherein the expenditure was zero, which violated the linearity assumption of common regression approaches. Tobin (1958) introduced a regression model whose response was censored at a prefixed limiting value; see Amemiya (1984).

A strong assumption of the tobit model is that the error term is normally distributed, but it is not always the case in many applications; see, for example, Barros et al., Barros et al. (2010, 2018). The normality assumption may not be appropriate to describe the behavior of strictly positive data, as well as bimodal and/or light- and heavy-tailed data. The use of flexible distributions is very important as often real-world data are better modeled by non-normal distributions, especially with regard to the robustness of results. In the context of censored responses, some authors have emphasized the importance of using more flexible distributions; see, for example, Arellano et al. (2012), Martínez-Flores, Bolfarine and Gómez (2013a, 2013b), Garay et al. (2015), Massuia et al. (2015) and Barros et al. (2010, 2018). The reader is also referred to Yenilmez, Mert Kantar and Acitaş (2018) and the references therein for more work on robust and flexible models as well as nonparametric approaches in this regard.

Key words and phrases. Log-symmetric distributions, likelihood ratio test, gradient test, R software, tobit models.

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The log-symmetric distribution class, investigated by Jones (2008), arises when a random variable (RV) has the same distribution as its reciprocal, or when the distribution of the logarithm of the RV is symmetrical. This class is very useful for modeling strictly positive, asymmetric, bimodal and light- and heavy-tailed data. The class of log-symmetric distributions is a generalization of the log-normal distribution, which provides more flexible alternatives; see, for example, Vanegas and Paula (2016b). Vanegas and Paula (2015) proposed a semiparametric regression model by allowing both median and shape to be modeled, Vanegas and Paula (2016b) discussed some statistical properties of the log-symmetric class of distributions, Vanegas and Paula (2016a) proposed an extension of the log-symmetric regression models used by Vanegas and Paula (2015) by considering an arbitrary number of non-parametric additive components to describe the median and shape, Vanegas and Paula (2017) proposed log-symmetric regression models with non-informative left- or right-censored observations being allowed, while Medeiros and Ferrari (2017) considered hypothesis testing procedures in symmetric and log-symmetric linear regression models.

A prominent procedure for hypothesis testing in parametric models is the gradient (GR) test, which was proposed by Terrell (2002). This procedure is simple to compute and only involves the score vector and the maximum likelihood (ML) estimates of the parameter vector under the unrestricted and restricted models. Similar to the generalized likelihood ratio (LR) statistic discussed by Wilks (1938), the GR statistic is also attractive for censored samples, as is the case with tobit models, since no computation of the information matrix (neither observed nor expected) is required; see, for example, Lemonte and Ferrari (2011).

In this context, the primary objective of this paper is to propose a class of tobit models based on the log-symmetric distribution. The secondary objectives are: (i) to obtain the ML estimates of the model parameters; (ii) to deal with the issue of performing hypothesis tests concerning the parameters of the proposed tobit-log-symmetric model, for which LR and GR tests are developed; (iii) to carry out Monte Carlo simulations to evaluate the performance of the ML estimates and the LR and GR tests; and (iv) to discuss a real data application of the proposed methodology. The tobit-Birnbaum–Saunders model introduced recently by Desousa et al. (2018) is a special case of the proposed tobit-log-symmetric models. In general, the proposed models are expected to be more flexible options as compared to the normal-based tobit model. As the proposed tobit-log-symmetric models involve a logarithmic transformation, their main advantage lies in the possibility of obtaining symmetric models with greater flexibility in terms of bimodality and/or light- and heavy-tails. The present paper has three main differences as compared to the work of Vanegas and Paula (2017): (a) a new class of tobit models, namely, novel left-censored regression models is proposed here. On the other hand, the work of Vanegas and Paula (2017) assumes non-informative censoring, that is, the censoring times are statistically independent of the event times; (b) unlike is the work of Vanegas and Paula (2017), the proposed tobit-log-symmetric models focus on the problem of performing hypothesis testing based on the LR and GR tests, which is of great practical interest since they do not require the information matrix information; and (c) the present work does not consider nonparametric components and estimates the model parameters by using the Broyden–Fletcher–Goldfarb–Shanno (BFGS) quasi-Newton method, whereas the work of Vanegas and Paula (2017) applies a backfitting iterative process for obtaining the estimates in the nonparametric component and a Newton-Raphson algorithm for obtaining the estimates in the parametric component. The quasi-Newton class of methods class provides very efficient algorithms that eliminate the need for calculating second derivatives and also typically perform very well.

The rest of this paper proceeds as follows. In Section 2, we briefly describe the class of log-symmetric distributions and some properties. In Section 3, we formulate the tobit model based on the log-symmetric class, and then detail the associated estimation, inference and residual analysis based on the ML method. In Section 4, we carry out Monte Carlo simulations, and an illustration with a real data is done in Section 5. Finally, in Section 6, we discuss conclusions and some possible future research in this topic.

2 Log-symmetric distributions

Consider a continuous RV Y having a symmetric distribution with location parameter $\mu \in \mathbb{R}$, dispersion parameter $\phi > 0$, density generator $g(\cdot)$ and probability density function (PDF)

$$f_Y(y; \mu, \phi, g) = \frac{1}{\phi} g\left(\frac{(y-\mu)^2}{\phi^2}\right), \quad y \in \mathbb{R},$$

with $g(u) > 0$ for $u > 0$, and $\int_0^\infty u^{-1/2} g(u)\,\mathrm{d}u = 1$; see Fang, Kotz and Ng (1990). In this case, the notation $Y \sim \mathrm{S}(\mu, \phi^2, g)$ is used. The class of log-symmetric distributions arises when we set $T = \exp(Y)$, that is, we obtain a continuous and positive RV T such that the distribution of its logarithm belongs to the symmetric family. The PDF of T can be expressed as

$$f_T(t; \eta, \phi, g) = \frac{1}{\phi t} g(\widetilde{t}^2), \quad t > 0,$$

where $\widetilde{t} = \log([t/\eta]^{1/\phi})$ and $\eta = \exp(\mu) > 0$ is a scale parameter. In this case, we write $T \sim \mathrm{LS}(\eta, \phi^2, g)$. The density generator g may contain an extra parameter ξ (or an extra parameter vector $\boldsymbol{\xi}$). The cumulative distribution function (CDF) of T is given by

$$F_T(t; \eta, \phi, g) = F_Z(\widetilde{t}; 0, 1, g),$$

where $F_Z(\cdot)$ is the CDF of a symmetric random variable $Z = (Y - \mu)/\phi \sim \mathrm{S}(0, 1, g)$. Note that the density generator g leads to different log-symmetric distributions. Some members of log-symmetric distributions are the log-normal (Crow and Shimizu, 1988, Johnson, Kotz and Balakrishnan, 1994), log-logistic (Marshall and Olkin, 2007), log-Laplace (Johnson, Kotz and Balakrishnan, 1995), log-Cauchy (Marshall and Olkin, 2007), log-power-exponential (Vanegas and Paula, 2016b), log-Student-t (Vanegas and Paula, 2016b), log-power-exponential (Vanegas and Paula, 2016b), log-slash (Vanegas and Paula, 2016b), harmonic law (Podlaski, 2008), Birnbaum–Saunders (Birnbaum and Saunders, 1969, Rieck and Nedelman, 1991, Balakrishnan and Kundu, 2019), generalized Birnbaum–Saunders (Díaz-García and Leiva, 2005, 2007, Balakrishnan and Kundu, 2019), and F (Johnson, Kotz and Balakrishnan, 1995) distributions; see Table 1.

Let $T \sim \mathrm{LS}(\eta, \phi^2, g)$; we then readily have the following properties: (P1) $cT \sim \mathrm{LS}(c\eta, \phi^2, g)$, with $c > 0$; (P2) $T^c \sim \mathrm{LS}(\eta^c, c^2\phi^2, g)$, with $c \neq 0$; and (P3) the median of the distribution of T is η. The properties (P1) and (P2) say that the log-symmetric distribution holds the proportionality and reciprocation properties, respectively. Moreover, (P2) is

Table 1 *Density generator $g(u)$ for some log-symmetric distributions*

Distribution	$g(u)$
Log-normal(η, ϕ)	$\propto \exp(-\frac{1}{2}u)$
Log-Student-$t(\eta, \phi, \xi)$	$\propto (1 + \frac{u}{\xi})^{-\frac{\xi+1}{2}}$, $\xi > 0$
Log-power-exponential(η, ϕ, ξ)	$\propto \exp(-\frac{1}{2}u^{\frac{1}{1+\xi}})$, $-1 < \xi \leq 1$
Birnbaum–Saunders$(\eta, \phi = 2, \xi)$	$\propto \cosh(u^{1/2}) \exp(-\frac{2}{\xi^2}\sinh^2(u^{1/2}))$, $\xi > 0$
Birnbaum–Saunders-$t(\eta, \phi = 2, \boldsymbol{\xi} = (\xi_1, \xi_2)^\top)$	$\propto \cosh(u^{1/2})(\xi_2\xi_1^2 + 4\sinh^2(u^{1/2}))^{-\frac{\xi_2+1}{2}}$, $\xi_1, \xi_2 > 0$

useful to propose modified moment estimators; see Ng, Kundu and Balakrishnan (2003) for the Birnbaum–Saunders case. Finally, (P3) can be used to specify a dynamic point process model in terms of the conditional median; see Saulo et al. (2019).

3 The tobit-log-symmetric model

Consider a censored response variable to the left Y_i for case i, which is observable for values greater than γ and censored for values less than or equal to γ. Then, in the tobit formulation

$$Y_i = \begin{cases} \gamma, & Y_i^* \leq \gamma, i = 1, \ldots, m, \\ \boldsymbol{x}_i^\top \boldsymbol{\beta} + \varepsilon_i, & Y_i^* > \gamma, i = m+1, \ldots, n, \end{cases} \tag{3.1}$$

where γ is a known quantity, $Y_i^* = \boldsymbol{x}_i^\top \boldsymbol{\beta} + \varepsilon_i$, m is the number of cases censored to the left, n is the total number of cases, $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ip})^\top$ is an $n \times 1$ vector of covariates fixed and known, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^\top$ is a $p \times 1$ vector of regression coefficients, and $\{\varepsilon_i\}$ are independent and identically distributed (IID) RVs. The tobit-normal (tobit-NO) model is obtained from (3.1) when ε_i follows a normal distribution with mean zero and variance ς^2, that is, $\varepsilon_i \stackrel{\text{IID}}{\sim} \mathrm{N}(0, \varsigma^2)$.

Consider the log-symmetric regression model (Vanegas and Paula, 2015)

$$T_i = \eta_i \epsilon_i^{\phi_i}, \quad i = 1, \ldots, n, \tag{3.2}$$

where η_i is the median of T_i, ϕ_i is a shape parameter associated with the skewness or relative dispersion, and $\{\epsilon_i\}$ are standard log-symmetrically distributed IID RVs denoted by $\epsilon_i \stackrel{\text{IID}}{\sim} \mathrm{LS}(1, 1, g)$. Consequently, $T_i \stackrel{\text{IND}}{\sim} \mathrm{LS}(\eta_i, \phi_i^2, g)$. The structures for η_i and ϕ_i are expressed as

$$\eta_i = \exp(\boldsymbol{x}_i^\top \boldsymbol{\beta}), \quad i = 1, \ldots, n,$$
$$\log(\phi_i) = \boldsymbol{w}_i^\top \boldsymbol{\zeta}, \quad i = 1, \ldots, n,$$

where $\boldsymbol{x}_i$ and $\boldsymbol{\beta}$ are as in (3.1), $\boldsymbol{w}_i = (w_{ik}, \ldots, w_{ik})$ is an $n \times 1$ vector of covariates for ϕ_i, and $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_k)^\top$ is a $p \times 1$ parameter vector. For the sake of ease and simplicity, we shall hereafter assume that $\phi_i = \phi$, for $i = 1, \ldots, n$.

By taking logarithm in Equation (3.2), we obtain

$$\underbrace{\log(T_i)}_{Y_i} = \underbrace{\log(\eta_i)}_{\mu_i} + \phi \underbrace{\log(\epsilon_i)}_{\varepsilon_i}, \quad i = 1, \ldots, n, \tag{3.3}$$

where ε_i is a standard symmetrically distributed RV, $\varepsilon_i \stackrel{\text{IID}}{\sim} \mathrm{S}(0, 1, g)$, and $Y_i \stackrel{\text{IND}}{\sim} \mathrm{S}(\mu_i, \phi^2, g)$. Then, based on Equations (3.1) and (3.3), we propose a tobit model based on the log-symmetric distribution, denoted by tobit-LS, as

$$Y_i = \begin{cases} \gamma, & Y_i^* \leq \gamma, i = 1, \ldots, m, \\ \boldsymbol{x}_i^\top \boldsymbol{\beta} + \varepsilon_i, & Y_i^* > \gamma, i = m+1, \ldots, n, \end{cases} \tag{3.4}$$

where $Y_i^* = \log(T_i^*) = \boldsymbol{x}_i^\top \boldsymbol{\beta} + \varepsilon_i$, $\boldsymbol{\beta}$ and $\boldsymbol{x}_i$ are as in (3.1), and ε_i is as in (3.3).

Consider a sample of size n, $\boldsymbol{Y} = (Y_1, \ldots, Y_m, Y_{m+1}, \ldots, Y_n)^\top$ say, from a tobit-LS model that contains m left-censored data, that is, the values of Y less than a threshold point γ, and $n - m$ complete or uncensored data, namely, values of Y greater than γ. Then, the corresponding likelihood function for $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \phi)^\top$ is

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{m} F_Z(\zeta_i^{\mathrm{c}}; 0, 1, g) \prod_{i=m+1}^{n} \frac{1}{\phi} g(\zeta_i^2), \tag{3.5}$$

where F_Z is the CDF of the symmetric distribution and

$$\zeta_i^{\mathrm{c}} = \left(\frac{\gamma - \boldsymbol{x}_i^\top \boldsymbol{\beta}}{\phi}\right) \quad \text{and} \quad \zeta_i = \left(\frac{y_i - \boldsymbol{x}_i^\top \boldsymbol{\beta}}{\phi}\right).$$

By taking the logarithm in (3.5), we obtain the log-likelihood function for $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \phi)^\top$ as

$$\ell(\boldsymbol{\theta}) = \sum_i \ell_i(\boldsymbol{\theta}), \tag{3.6}$$

where

$$\ell_i(\boldsymbol{\theta}) = \begin{cases} \log\big(F_Z(\zeta_i^{\mathrm{c}}; 0, 1, g)\big), & i = 1, \ldots, m, \\ -\log(\phi) + \log\big(g(\zeta_i^2)\big), & i = m+1, \ldots, n. \end{cases}$$

The score vector for $\boldsymbol{\beta}$ and ϕ are given by

$$\dot{\boldsymbol{\ell}}(\boldsymbol{\theta}) = \frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \sum_{i=1}^n \dot{\boldsymbol{\ell}}_i(\boldsymbol{\theta}), \quad \text{where } \dot{\boldsymbol{\ell}}_i(\boldsymbol{\theta}) = \big(\dot{\boldsymbol{\ell}}_{i\boldsymbol{\beta}}^\top(\boldsymbol{\theta}), \dot{\ell}_{i\phi}(\boldsymbol{\theta})\big)^\top, \tag{3.7}$$

with

$$\dot{\boldsymbol{\ell}}_{i\boldsymbol{\beta}}(\boldsymbol{\theta}) = \begin{cases} -\dfrac{1}{\phi}\Omega_i \boldsymbol{x}_i, & i = 1, \ldots, m, \\ -\dfrac{2}{\phi} W_i \zeta_i \boldsymbol{x}_i, & i = m+1, \ldots, n, \end{cases}$$

$$\dot{\ell}_{i\phi}(\boldsymbol{\theta}) = \begin{cases} -\dfrac{1}{\phi}\Omega_i \zeta_i^{\mathrm{c}}, & i = 1, \ldots, m, \\ -\dfrac{1}{\phi} - \dfrac{2}{\phi} W_i \zeta_i^2, & i = m+1, \ldots, n, \end{cases}$$

with $\Omega_i = \frac{\mathrm{d}F_Z(u;0,1,g)/\mathrm{d}u|_{u=\zeta_i^{\mathrm{c}}}}{F_Z(\zeta_i^{\mathrm{c}};0,1,g)}$ and $W_i = \frac{\mathrm{d}g(u)/\mathrm{d}u|_{u=\zeta_i^2}}{g(\zeta_i^2)}$. To obtain the ML estimate of $\boldsymbol{\theta}$, it is necessary to maximize the expression in (3.6) by equating the score vector $\dot{\boldsymbol{\ell}}(\boldsymbol{\theta})$ in (3.7) to zero, providing the likelihood equations. They are solved using the BFGS quasi-Newton method; see Mittelhammer, Judge and Miller (2000), p. 199. The corresponding standard errors (SEs) can be approximated by computing the square roots of the diagonal elements of the inverse of the observed Fisher information matrix (Efron and Hinkley, 1978), which is obtained as $\mathcal{J}(\boldsymbol{\theta}) = -\ddot{\boldsymbol{\ell}}(\boldsymbol{\theta})$, where $\ddot{\boldsymbol{\ell}}(\boldsymbol{\theta})$ denotes the Hessian matrix, that is,

$$\ddot{\boldsymbol{\ell}}(\boldsymbol{\theta}) = \frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} = \sum_{i=1}^n \ddot{\boldsymbol{\ell}}_i(\boldsymbol{\theta}), \quad \text{where } \ddot{\boldsymbol{\ell}}_i(\boldsymbol{\theta}) = \begin{bmatrix} \ddot{\ell}_{i\boldsymbol{\beta}\boldsymbol{\beta}}(\boldsymbol{\theta}) & \ddot{\ell}_{i\boldsymbol{\beta}\phi}(\boldsymbol{\theta}) \\ \ddot{\ell}_{i\phi\boldsymbol{\beta}}(\boldsymbol{\theta}) & \ddot{\ell}_{i\phi\phi}(\boldsymbol{\theta}) \end{bmatrix},$$

with

$$\ddot{\boldsymbol{\ell}}_{i\boldsymbol{\beta}\boldsymbol{\beta}}(\boldsymbol{\theta}) = \begin{cases} -\dfrac{1}{\phi}\Omega_i' \boldsymbol{x}_i, & i = 1, \ldots, m, \\ -\dfrac{2}{\phi}\left[-W_i \dfrac{\boldsymbol{x}_i}{\phi} + W_i' \zeta_i\right]\boldsymbol{x}_i, & i = m+1, \ldots, n, \end{cases}$$

$$\ddot{\boldsymbol{\ell}}_{i\boldsymbol{\beta}\phi}(\boldsymbol{\theta}) = \ddot{\boldsymbol{\ell}}_{i\phi\boldsymbol{\beta}}(\boldsymbol{\theta}) = \begin{cases} -\left[\dfrac{1}{\phi}\Omega_i' - \dfrac{1}{\phi^2}\Omega_i\right]\boldsymbol{x}_i, & i = 1, \ldots, m, \\ -\dfrac{2\zeta_i}{\phi}\left\{\left[-\dfrac{W_i}{\phi} + W_i'\right] - \dfrac{1}{\phi} W_i\right\}\boldsymbol{x}_i, & i = m+1, \ldots, n, \end{cases}$$

$$\dot{\ell}_{i\phi\phi}(\boldsymbol{\theta}) = \begin{cases} \dfrac{1}{\phi^2}[2\Omega_i - \phi\Omega_i']\zeta_i^{\mathrm{c}}, & i = 1, \dots, m, \\ \dfrac{1}{\phi^2} + \dfrac{2}{\phi^2}[3W_i - \phi W_i']\zeta_i^2, & i = m+1, \dots, n. \end{cases}$$

Note that from (3.7), we can define the quantity

$$\nu_i = -2W_i, \tag{3.8}$$

which can be interpreted as a weight. In general, ν_i gives smaller weights for outlying observations under heavy-tailed error models; see Vanegas and Paula (2016b) and Medeiros and Ferrari (2017).

The extra parameter ξ is estimated by using the profile log-likelihood. The motivation to leave the extra parameter fixed in the estimation process comes from the work of Lucas (1997), in which it has been shown that robustness to outlying observations under Student-t models holds only if the degree of freedom is fixed, rather than directly estimated in the ML method. Moreover, some difficulties in computing the extra parameter in the power-exponential model have also been reported by Kano, Berkane and Bentler (1993); see Vanegas and Paula (2016b) for more details. The following two steps are necessary to obtain the estimates of the model parameters:

(1) Let $\xi_k = k$ and for each $k = a, \dots, b$, where a and b are predefined limits, compute the ML estimates of $\boldsymbol{\theta}$ by using the above procedures; compute also the log-likelihood function;
(2) The final estimate of ξ is the one that maximizes the log-likelihood function and the associated estimate of $\boldsymbol{\theta}$ is then the final one.

3.1 Statistical tests

We consider here the LR and GR statistical tests for the tobit-log-symmetric regression model. We choose these tests because they do not require the information matrix, as mentioned earlier. Let $\boldsymbol{\theta}$ be a p-vector of parameters that index a tobit-log-symmetric model. Suppose our interest lies in testing the hypothesis $\mathcal{H}_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ against $\mathcal{H}_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$, where $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^\top, \boldsymbol{\theta}_2^\top)^\top$, $\boldsymbol{\theta}_1$ is an $r \times 1$ vector of parameters of interest and $\boldsymbol{\theta}_2$ is a $(p-r) \times 1$ vector of nuisance parameters.

Two popular methods for testing these linear hypotheses are by the use of LR and GR test statistics, which are given by

$$\Lambda_{\mathrm{LR}} = 2\{\ell(\widehat{\boldsymbol{\theta}}) - \ell(\widetilde{\boldsymbol{\theta}})\},$$

$$\Lambda_{\mathrm{GR}} = \dot{\boldsymbol{\ell}}^\top(\widetilde{\boldsymbol{\theta}})(\widehat{\boldsymbol{\theta}} - \widetilde{\boldsymbol{\theta}}),$$

where $\ell(\cdot)$ is the log-likelihood function defined in (3.6), and $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\theta}}_1^\top, \widehat{\boldsymbol{\theta}}_2^\top)^\top$ and $\widetilde{\boldsymbol{\theta}} = (\boldsymbol{\theta}_1^{(0)\top}, \widetilde{\boldsymbol{\theta}}_2^\top)^\top$ are unrestricted and restricted ML estimators of $\boldsymbol{\theta}$, respectively. Moreover, $\dot{\boldsymbol{\ell}}(\cdot)$ is the score vector defined in (3.7). In regular cases, we have, under $\mathcal{H}_0$ and as $n \to \infty$, both statistical tests converging in distribution to χ_r^2. Then, $\mathcal{H}_0$ is rejected at nominal level δ if the test statistic is larger than $\chi^2_{1-\delta,r}$, the upper δ quantile of the χ_r^2 distribution.

3.2 Model checking

Residuals analysis are frequently used to evaluate the validity of the assumptions of the model, presence of outliers and also as tools for model selection. In the context of regression models, usually Pearson and studentized residuals are often used. Nevertheless, in a tobit scenario, these two types of residuals, even under normality, are not suitable; see, for example, Barros et al. (2010). Moreover, Saulo et al. (2020) have studied, in the log-symmetric context, the empirical distribution of the generalized Cox-Snell (GCS) and quantile (Dunn and Smyth, 1996) residuals under misspecification. These authors found that the former is more sensitive to misspecification and also to be suitable for assessing the adjustment. For this reason, we opt to work only with the generalized Cox-Snell residuals. In the log-symmetric tobit case, the GCS residual is given by

$$r_i^{\mathrm{GCS}} = -\log\big(\widehat{S}_Y(y_i; \widehat{\mu}_i, \widehat{\phi}^2, g)\big) = -\log\big(1 - \widehat{F}_Y(y_i; \widehat{\mu}_i, \widehat{\phi}^2, g)\big),$$

where $\widehat{S}_Y$ denotes survival function fitted to the data. The GCS residual is asymptotically standard exponential, EXP(1) in short, if the model is correctly specified whatever the specification of the model is.

4 Monte Carlo simulation studies

Three Monte Carlo simulation studies are carried out to evaluate the performances of the ML estimates, the statistical tests and the empirical distribution of the residuals. The `R` software has been used to do all numerical calculations; see R-Team (2016).

4.1 ML estimates

A Monte Carlo simulation study is carried out to evaluate the performance of the ML estimates. We focus on three tobit-log-symmetric models: tobit-log-normal (tobit-LN), tobit-log-Student-t (tobit-Lt) and tobit-log-power-exponential (tobit-LPE). The study considers simulated data generated from each one of the above-mentioned models according to

$$Y_i = \begin{cases} \gamma, & Y_i^* \le \gamma, i = 1, \ldots, m, \\ Y_i^* = \beta_0 + \beta_1 x_i + \varepsilon_i, & Y_i^* > \gamma, i = m+1, \ldots, n, \end{cases}$$

where ε_i is as in (3.4), x_i is a covariate obtained from a uniform distribution in the interval (0,1), and the true parameter values are taken as $\beta_0 = 0.2$ $\beta_1 = 0.5$. Moreover, the simulation scenario considers sample size $n \in \{50, 100, 300, 500\}$, scale parameter $\phi \in \{1.00, 3.00, 5.00\}$, extra parameter $\xi_1 = 0.5$ (tobit-LPE), $\xi_1 = 4$ (tobit-Lt), censoring proportion $\varrho = m/n \in \{0.20, 0.50\}$, with 5000 Monte Carlo replications for each combination of above given parameters, censoring proportion and sample size. The γ value is defined as the ϱ quantile of the generated values of the dependent variable, where ϱ is the censoring proportion.

The ML estimation results for the considered tobit-log-symmetric models are presented in Tables 2–4. The empirical bias and mean squared error (MSE) are reported. A look at the results in Tables 2–4 allows us to conclude that, for $\phi \in \{1.00, 3.00, 5.00\}$ and $\varrho \in \{0.20, 0.50\}$, as the sample size increases, the empirical bias and MSE both decrease, as expected. Moreover, we note that, as the value of the parameter ϕ increases, the performance of the estimate of this parameter deteriorates. In general, the performances of the estimates decrease when the censoring proportion increases.

4.2 Statistical tests

We now present Monte Carlo simulation studies to evaluate the performance of the LR and GR tests. We consider again the following models: tobit-LN, tobit-Lt and tobit-LPE. The simulation scenario considers the following setting: sample size $n \in \{50, 100, 300, 500\}$, scale parameter $\phi = 3.00$, extra parameter $\xi_1 = 0.5$ (tobit-LPE), $\xi_1 = 5$ (tobit-Lt), censoring proportion $\varrho = m/n \in \{0.3, 0.5\}$, with 5000 Monte Carlo replications for each combination of parameters, censoring proportion and sample size. The value γ is as defined in Section 4.1.

Table 2 *Empirical bias and MSE (in parentheses) from simulated data for the indicated ML estimates of the tobit-LN model parameters, n and* ϱ

n	ϕ	$\varrho = 0.20$			$\varrho = 0.50$		
		$\widehat{\phi}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$	$\widehat{\phi}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$
50	1.00	−0.0099 (0.0141)	−0.0147 (0.0918)	0.0086 (0.2715)	−0.0132 (0.0249)	−0.0212 (0.1218)	0.0087 (0.3263)
	3.00	−0.0297 (0.1269)	−0.0367 (0.8167)	0.0114 (2.4409)	−0.0394 (0.2250)	−0.0589 (1.0393)	0.0169 (2.9163)
	5.00	−0.0491 (0.3526)	−0.0589 (2.2629)	0.0144 (6.7631)	−0.0652 (0.6263)	−0.0951 (2.8684)	0.0210 (8.1019)
100	1.00	−0.0045 (0.0068)	−0.0092 (0.0441)	0.0072 (0.1308)	−0.0084 (0.0125)	−0.0094 (0.0573)	0.0060 (0.1518)
	3.00	−0.0138 (0.0610)	−0.0239 (0.3924)	0.0148 (1.1741)	−0.0257 (0.1122)	−0.0267 (0.4887)	0.0158 (1.3546)
	5.00	−0.0228 (0.1695)	−0.0370 (1.0831)	0.0191 (3.2483)	−0.0426 (0.3112)	−0.0428 (1.3439)	0.0225 (3.7660)
300	1.00	−0.0006 (0.0023)	−0.0048 (0.0147)	0.0043 (0.0428)	−0.0014 (0.0040)	−0.0053 (0.0187)	0.0040 (0.0498)
	3.00	−0.0017 (0.0209)	−0.0137 (0.1302)	0.0110 (0.3834)	−0.0041 (0.0365)	−0.0166 (0.1602)	0.0123 (0.4428)
	5.00	−0.0028 (0.0578)	−0.0222 (0.3609)	0.0171 (1.0650)	−0.0068 (0.1007)	−0.0283 (0.4382)	0.0215 (1.2277)
500	1.00	−0.0003 (0.0014)	−0.0025 (0.0088)	0.0028 (0.0258)	−0.0007 (0.0024)	−0.0016 (0.0113)	0.0003 (0.0309)
	3.00	−0.0006 (0.0127)	−0.0064 (0.0778)	0.0061 (0.2309)	−0.0011 (0.0221)	−0.0064 (0.0981)	0.0018 (0.2770)
	5.00	−0.0011 (0.0354)	−0.0105 (0.2159)	0.0104 (0.6406)	−0.0012 (0.0617)	−0.0108 (0.2697)	0.0023 (0.7681)

Table 3 *Empirical bias and MSE (in parentheses) from simulated data for the indicated ML estimates of the tobit-Lt model parameters, n and ϱ*

		$\varrho = 0.20$			$\varrho = 0.50$		
n	ϕ	$\widehat{\phi}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$	$\widehat{\phi}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$
50	1.00	−0.0038 (0.0190)	−0.0058 (0.1148)	0.0085 (0.3473)	0.0056 (0.0347)	−0.0217 (0.1471)	−0.0181 (0.4173)
	3.00	−0.0118 (0.1706)	−0.0113 (1.0352)	0.0098 (3.1385)	0.0149 (0.3147)	−0.0464 (1.2505)	−0.0961 (3.6955)
	5.00	−0.0202 (0.4739)	−0.0196 (2.8732)	0.0287 (8.7144)	0.0252 (0.8749)	−0.0715 (3.4329)	−0.0308 (9.2639)
100	1.00	0.0011 (0.0102)	−0.0051 (0.0560)	0.0050 (0.1668)	0.0031 (0.0185)	−0.0154 (0.0719)	0.0115 (0.1958)
	3.00	0.0023 (0.0911)	−0.0111 (0.5021)	0.0079 (1.4986)	0.0107 (0.1682)	−0.0409 (0.6101)	0.0218 (1.7335)
	5.00	0.0037 (0.2529)	−0.0173 (1.3919)	0.0197 (4.1527)	0.0176 (0.4673)	−0.0640 (1.6827)	0.0286 (4.8195)
300	1.00	−0.0005 (0.0033)	−0.0027 (0.0181)	0.0047 (0.0547)	0.0003 (0.0059)	−0.0069 (0.0229)	0.0083 (0.0636)
	3.00	−0.0018 (0.0294)	−0.0073 (0.1622)	0.0123 (0.4910)	0.0007 (0.0536)	−0.0168 (0.1941)	0.0180 (0.5621)
	5.00	−0.0027 (0.0814)	−0.0117 (0.4497)	0.0109 (1.3623)	0.0022 (0.1494)	−0.0285 (0.5374)	0.0292 (1.5663)
500	1.00	−0.0004 (0.0019)	−0.0010 (0.0108)	0.0011 (0.0331)	0.0001 (0.0035)	−0.0024 (0.0134)	0.0013 (0.0379)
	3.00	−0.0010 (0.0173)	−0.0023 (0.0972)	0.0019 (0.2977)	0.0005 (0.0316)	−0.0037 (0.1155)	0.0021 (0.3394)
	5.00	−0.0018 (0.0482)	−0.0038 (0.2699)	0.0031 (0.8267)	0.0002 (0.0882)	−0.0067 (0.3167)	0.0037 (0.9394)

Table 4 *Empirical bias and MSE (in parentheses) from simulated data for the indicated ML estimates of the tobit-LPE model parameters, n and* ϱ

n	ϕ	$\varrho = 0.20$			$\varrho = 0.50$		
		$\widehat{\phi}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$	$\widehat{\phi}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$
50	1.00	−0.0090 (0.0194)	−0.0064 (0.1903)	0.0087 (0.5807)	−0.0109 (0.0314)	−0.0282 (0.2277)	0.0228 (0.6642)
	3.00	−0.0272 (0.1742)	−0.0151 (1.7096)	0.0187 (5.2207)	−0.0312 (0.2793)	−0.0653 (1.9717)	0.0269 (5.9557)
	5.00	−0.0456 (0.4837)	−0.0237 (4.7496)	0.0285 (9.5076)	−0.0728 (1.5164)	−0.1393 (5.5873)	0.0391 (9.4605)
100	1.00	−0.0047 (0.0099)	−0.0028 (0.0910)	0.0002 (0.2815)	−0.0037 (0.0159)	−0.0149 (0.1077)	0.0106 (0.3170)
	3.00	−0.0061 (0.0888)	−0.0081 (0.8171)	0.0142 (2.5290)	−0.0122 (0.1427)	−0.0339 (0.9188)	0.0174 (2.7931)
	5.00	−0.0102 (0.2469)	−0.0135 (2.2718)	0.0048 (7.0339)	−0.0205 (0.3969)	−0.0537 (2.5311)	0.0170 (7.7414)
300	1.00	−0.0021 (0.0032)	−0.0019 (0.0295)	0.0002 (0.0865)	−0.0010 (0.0052)	−0.0063 (0.0341)	0.0043 (0.0956)
	3.00	−0.0026 (0.0287)	−0.0051 (0.2653)	0.0030 (0.7783)	−0.0069 (0.0468)	−0.0172 (0.2901)	0.0130 (0.8346)
	5.00	−0.0047 (0.0798)	−0.0078 (0.7366)	0.0021 (2.1619)	−0.0097 (0.1299)	−0.0253 (0.7998)	0.0047 (2.3190)
500	1.00	−0.0008 (0.0019)	−0.0017 (0.0177)	0.0001 (0.0525)	−0.0003 (0.0031)	−0.0005 (0.0204)	0.0033 (0.0574)
	3.00	−0.0021 (0.0176)	−0.0047 (0.1593)	0.0018 (0.4733)	−0.0017 (0.0280)	−0.0033 (0.1730)	0.0030 (0.5050)
	5.00	−0.0011 (0.0176)	−0.0051 (0.1593)	0.0014 (0.4733)	−0.0079 (0.0775)	−0.0162 (0.4754)	0.0031 (1.3952)

This simulation study assesses the performance of the LR and GR tests under two data generating models.

4.2.1 *Model* 1. The first data generating model has one covariate and is given by

$$Y_i = \begin{cases} \gamma, & Y_i^* \leq \gamma, i = 1, \dots, m, \\ Y_i^* = \beta_0 + \beta_1 x_i + \varepsilon_i, & Y_i^* > \gamma, i = m+1, \dots, n, \end{cases}$$

where ε_i is as in (3.4), with $\beta_0 = 1.0$ and $\beta_1 \in \{-1.00, -0.75, -0.25, 0.00, 0.25, 0.75, 1.00\}$. The covariate values were obtained as U(0, 1) random draws. The interest lies in testing $\mathcal{H}_0 : \beta_1 = 0$ against $\mathcal{H}_1 : \beta_1 \neq 0$.

Tables 5–7 present the simulation results of powers of the LR and GR tests, namely, their capacity to identify a false null hypothesis. We also consider the case when the null hypothesis is true ($\beta_1 = 0.00$ in the data generation). From Tables 5–7, we observe that the power associated with the LR and GR tests increases as a function of the sample size, as expected. We also observe that the power of the tests increase when the true parameter deviates from the value 0. Finally, we note that the power of the tests decrease when the censoring proportion increases. In general, the results show that both tests have similar power.

Table 5 *Power study* (%) *for different values of* β_1 *and models* (*nominal level* = 1%)

		tobit-LN				tobit-L*t*				tobit-LPE			
		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$	
n	β_1	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR
50	−1.00	3.50	2.74	2.92	2.28	3.80	2.90	3.10	2.48	4.38	3.22	3.58	2.86
	−0.75	2.32	1.86	2.04	1.64	2.60	2.10	2.22	1.86	2.96	2.20	2.56	2.20
	−0.25	1.34	1.08	1.16	0.94	1.52	1.20	1.52	1.18	1.58	1.28	1.52	1.30
	0.00	1.20	0.98	1.14	0.88	1.30	0.98	1.40	1.08	1.54	1.12	1.38	1.32
	0.25	1.38	1.00	1.28	0.96	1.28	1.04	1.26	1.08	1.56	1.16	1.40	1.30
	0.75	2.00	1.70	1.88	1.40	2.18	1.58	2.14	1.64	2.54	2.04	2.20	2.04
	1.00	3.12	2.42	2.74	2.02	3.56	2.72	3.14	2.56	3.88	2.92	3.30	2.88
100	−1.00	5.86	5.22	4.80	4.38	6.08	5.60	5.18	4.78	6.70	5.88	5.50	5.24
	−0.75	3.58	3.04	3.22	2.92	3.66	3.24	3.58	3.06	4.28	3.62	3.62	3.50
	−0.25	1.12	1.04	1.42	1.26	1.32	1.18	1.64	1.32	1.58	1.24	1.68	1.54
	0.00	1.10	0.98	1.06	0.80	1.10	1.00	1.12	1.00	1.28	1.14	1.26	1.24
	0.25	1.26	1.10	1.40	1.20	1.60	1.30	1.48	1.36	1.62	1.36	1.54	1.46
	0.75	3.44	3.24	3.02	2.72	4.24	3.94	3.52	3.32	4.68	4.00	4.02	3.72
	1.00	5.92	5.28	4.98	4.54	6.42	5.90	5.74	5.18	6.88	6.14	6.24	6.04
300	−1.00	17.82	17.44	14.90	14.26	18.26	18.02	14.86	14.52	18.50	18.06	15.56	15.40
	−0.75	9.80	9.52	7.94	7.76	9.72	9.44	8.16	8.00	10.68	10.02	8.88	8.66
	−0.25	2.00	1.94	1.96	1.86	2.26	2.18	2.24	2.12	2.58	2.44	2.30	2.36
	0.00	1.16	1.06	1.22	1.12	1.30	1.26	1.34	1.26	1.72	1.56	1.62	1.52
	0.25	1.76	1.64	1.64	1.60	1.96	1.86	1.82	1.76	2.18	2.08	2.00	2.00
	0.75	9.28	8.98	7.50	7.22	9.82	9.72	8.00	7.72	10.42	9.84	8.30	8.20
	1.00	17.88	17.68	14.34	14.02	18.24	17.84	14.50	14.24	18.36	18.20	15.50	15.52
500	−1.00	32.58	31.90	27.24	26.92	32.48	32.28	27.36	27.04	32.82	32.74	27.80	27.68
	−0.75	16.22	16.00	14.08	13.76	17.10	16.92	14.44	14.28	17.50	17.22	14.94	14.66
	−0.25	2.22	2.14	2.44	2.40	2.72	2.70	2.88	2.80	3.28	3.16	2.98	3.02
	0.00	1.22	1.22	0.94	0.94	1.16	1.16	1.18	1.18	1.50	1.42	1.42	1.34
	0.25	1.94	1.86	1.74	1.70	2.30	2.28	2.06	2.04	2.78	2.66	2.24	2.22
	0.75	15.38	15.22	12.84	12.58	16.52	16.30	13.68	13.50	17.06	16.70	14.38	14.30
	1.00	32.16	31.90	26.62	26.38	31.80	31.42	27.44	27.04	32.48	32.14	27.94	27.94

Table 6 *Power study (%) for different values of β_1 and models (nominal level = 5%)*

		tobit-LN				tobit-L*t*				tobit-LPE			
		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$	
n	β_1	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR
50	−1.00	12.44	11.64	11.10	10.18	12.70	12.06	11.72	10.94	13.50	12.44	12.52	12.02
	−0.75	9.24	8.58	8.46	7.90	9.84	9.16	8.78	8.16	10.54	9.60	9.50	9.12
	−0.25	5.64	5.22	5.34	4.78	5.76	5.26	5.76	5.18	6.90	6.00	6.20	5.90
	0.00	4.90	4.50	4.90	4.36	5.38	4.80	5.26	4.88	6.12	5.52	5.78	5.28
	0.25	5.20	4.70	5.36	4.84	5.48	5.24	5.54	5.28	6.48	5.60	6.06	5.62
	0.75	8.70	8.02	7.62	7.04	9.22	8.46	8.18	7.78	9.78	8.80	9.34	8.54
	1.00	11.94	11.12	10.02	9.20	12.50	11.62	10.80	9.88	13.98	12.02	11.72	11.14
100	−1.00	17.60	17.02	15.84	15.26	18.42	17.88	16.40	15.86	19.20	18.20	17.10	16.64
	−0.75	11.90	11.60	10.82	10.24	12.36	12.04	11.44	11.18	13.28	12.72	11.86	11.72
	−0.25	5.84	5.50	5.84	5.62	6.44	6.00	6.44	6.18	6.90	6.50	6.88	6.74
	0.00	5.08	4.96	5.26	5.00	5.54	5.10	5.68	5.50	6.66	6.24	6.30	6.08
	0.25	5.60	5.24	5.92	5.60	6.60	6.26	6.30	6.14	7.28	6.84	7.04	6.84
	0.75	11.54	11.10	10.72	10.48	12.28	11.92	11.40	11.10	13.26	12.90	11.98	11.94
	1.00	17.02	16.72	15.18	14.48	17.56	17.02	16.20	15.82	18.56	17.80	17.08	16.90
300	−1.00	38.22	37.96	32.64	32.46	37.96	37.72	33.26	33.08	38.82	38.34	34.04	33.96
	−0.75	23.40	23.14	21.30	21.12	23.86	23.64	21.48	21.30	24.80	24.42	21.80	21.50
	−0.25	7.28	7.14	7.26	7.18	8.02	7.82	7.42	7.40	8.54	8.34	8.16	8.06
	0.00	5.44	5.40	5.72	5.60	6.00	5.94	6.12	6.04	7.12	7.00	6.48	6.48
	0.25	7.90	7.88	7.06	6.96	8.56	8.36	7.36	7.24	8.92	5.52	7.88	7.82
	0.75	23.96	23.88	20.50	20.24	24.06	23.84	20.96	20.78	24.54	24.32	21.58	21.42
	1.00	37.74	37.56	32.80	32.60	38.28	38.04	33.36	33.14	38.10	37.84	33.72	33.60
500	−1.00	56.50	56.32	49.84	49.68	56.40	56.20	50.48	50.40	55.96	55.80	50.48	50.20
	−0.75	35.52	35.42	31.84	31.60	35.60	35.52	31.78	31.72	36.00	35.66	32.50	32.26
	−0.25	9.12	9.04	8.24	8.20	9.68	9.60	8.80	8.76	10.22	10.16	9.58	9.40
	0.00	4.96	4.90	5.28	5.20	5.66	5.66	6.04	6.02	6.72	6.56	6.60	6.52
	0.25	8.06	8.06	7.30	7.24	8.44	8.32	7.94	7.80	9.36	9.34	8.50	8.36
	0.75	35.40	35.18	30.98	30.78	35.56	35.36	31.42	31.26	35.78	35.48	31.84	31.76
	1.00	56.28	56.26	49.86	49.64	56.18	55.96	50.04	49.94	55.84	55.58	50.16	49.98

4.2.2 *Model* 2. We consider as data generating process the following four-covariate model:

$$Y_i = \begin{cases} \gamma, & Y_i^* \leq \gamma, i = 1, \ldots, m, \\ Y_i^* = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \varepsilon_i, & Y_i^* > \gamma, i = m+1, \ldots, n, \end{cases}$$

where ε_i is as in (3.4), with $\beta_0 = 1.0$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\beta_3 = 0.8$ and $\beta_4 \in \{-1.00, -0.75, -0.25, 0.00, 0.25, 0.75, 1.00\}$. The covariate values were taken as random draws from the U(0, 1) distribution. The interest lies in testing $\mathcal{H}_0 : \beta_4 = 0$ against $\mathcal{H}_1 : \beta_4 \neq 0$.

Tables 8–10 present the simulation results of powers of the tests, which also include the case when the null hypothesis is true ($\beta_4 = 0.00$ in the data generation). From these tables, we observe that the power of the LR and GR tests increases as a function of the sample size, that is, the nonnull rejection rates of the tests converge to 100% as the sample size increases, as expected. We also observe that the power of the tests decrease (increase) when the censoring proportion increases (the true parameter deviates from the value 0). The results also show that both tests have similar power.

4.3 Empirical distribution of the residuals

We now present a Monte Carlo simulation study for evaluating the performance of GCS residuals. The sample generation, as well as the simulation scenario, are almost the same as

Table 7 *Power study (%) for different values of β_1 and models (nominal level = 10%)*

		tobit-LN				tobit-Lt				tobit-LPE			
		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$	
n	β_1	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR
50	−1.00	20.16	19.46	19.30	18.50	20.48	19.84	19.46	18.74	21.48	20.42	20.88	19.86
	−0.75	16.14	15.62	15.20	14.46	16.34	15.78	15.90	15.20	17.18	16.20	16.90	16.14
	−0.25	11.00	10.32	11.04	10.48	11.78	11.18	11.18	10.66	12.86	11.96	12.38	11.96
	0.00	9.98	9.60	9.84	9.28	11.02	10.42	10.46	10.00	12.34	11.48	11.80	11.18
	0.25	10.48	9.92	10.26	9.62	11.38	10.74	11.02	10.64	12.38	11.50	12.54	12.02
	0.75	15.60	14.90	13.92	13.40	16.54	15.90	14.96	14.52	17.52	16.54	16.12	15.60
	1.00	19.82	19.18	17.56	16.74	20.52	19.90	18.44	17.76	21.66	20.44	19.96	19.28
100	−1.00	27.52	27.18	25.28	24.80	28.08	27.78	25.52	25.08	28.34	27.68	26.86	26.62
	−0.75	20.10	19.68	18.80	18.42	20.82	20.36	19.16	18.90	21.72	21.08	20.14	19.78
	−0.25	11.36	11.12	11.20	11.06	12.02	11.80	11.96	11.80	13.58	13.22	13.00	12.36
	0.00	10.38	10.02	10.12	9.84	11.34	11.00	11.20	11.08	12.14	11.54	11.98	11.78
	0.25	11.36	11.10	11.32	11.12	11.80	11.52	11.94	11.76	13.34	12.70	12.88	12.54
	0.75	19.68	19.40	18.32	18.02	19.94	19.68	18.66	18.40	20.96	20.16	19.88	19.62
	1.00	26.76	26.30	24.54	24.14	26.60	26.24	24.46	24.14	27.38	26.54	25.62	25.48
300	−1.00	50.70	50.66	45.36	45.24	50.12	50.00	45.18	45.10	50.28	50.04	45.94	45.64
	−0.75	34.68	34.50	30.78	30.68	35.08	34.98	31.58	31.36	35.28	35.10	31.80	31.74
	−0.25	13.56	13.46	12.74	12.66	14.02	13.96	13.48	13.42	15.04	14.76	14.16	13.92
	0.00	10.82	10.78	10.82	10.70	11.76	11.64	10.80	10.76	12.36	12.12	11.78	11.64
	0.25	13.52	13.46	12.68	12.56	14.32	14.16	13.42	13.38	15.28	15.14	14.34	14.38
	0.75	34.84	34.66	30.68	30.50	34.80	34.68	31.34	31.26	35.04	34.74	31.64	31.68
	1.00	49.92	49.78	44.98	44.86	50.52	50.36	45.36	45.22	50.18	50.22	45.28	45.14
500	−1.00	68.34	68.30	62.70	62.56	68.28	68.20	62.32	62.26	67.56	67.42	62.28	62.16
	−0.75	47.74	47.66	43.04	42.94	47.98	47.86	44.18	44.04	48.30	48.14	43.88	43.84
	−0.25	14.94	14.90	14.44	14.44	15.68	15.60	14.84	14.76	16.80	16.66	15.62	15.28
	0.00	10.56	10.50	10.14	10.10	11.26	11.16	10.84	10.82	12.56	12.36	11.90	11.80
	0.25	14.30	14.26	13.52	13.48	15.42	15.34	14.58	14.56	16.74	16.54	15.60	15.56
	0.75	47.60	47.52	14.88	42.78	47.56	47.50	42.88	42.84	47.22	47.28	43.20	42.88
	1.00	68.60	68.52	62.18	62.04	67.68	67.60	62.20	62.10	66.88	66.92	61.84	61.74

in Section 4.1 with the difference that in this case we use $\varrho = 0.2$. In addition to the following tobit-log-symmetric models, tobit-LN, tobit-Lt and tobit-LPE, we further consider the tobit-Birnbaum–Saunders (tobit-BS) and tobit-Birnbaum–Saunders-t (tobit-BS-t) models.

Table 11 presents the empirical mean, standard deviation (SD), coefficient of skewness (CS) and coefficient of (excess) kurtosis (CK), whose values are expected to be 1, 1, 2 and 6, respectively, for the GCS residuals. From this table, note that, as the sample size increases, the values of the empirical mean, SD, CS and CK approach these values of the reference EXP(1) distribution. Therefore, the considered residuals conform well with the reference distribution.

5 Application to real data

Tobit-log-symmetric models are now used to analyze a data set from a case-study of measles vaccine, corresponding to antibody concentration levels (response variable, T_i) collected from 330 children at 12 months of age; see Moulton and Halsey (1995). A natural assumption that can be made is that this response variable follows a log-normal distribution. According to Ott (1990), the log-normal distribution has been used to model concentrations of several substances. In other words, concentration data are usually modeled by the log-normal distribution. As mentioned earlier, the class of log-symmetric distributions is a generalization of

Table 8 *Power study (%) for different values of β_4 and models (nominal level = 1%)*

		tobit-LN				tobit-Lt				tobit-LPE			
		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$	
n	β_4	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR
50	−1.00	4.04	3.18	3.38	2.62	4.00	3.16	3.54	2.92	4.26	3.38	3.66	3.18
	−0.75	2.90	2.38	2.56	1.94	2.88	2.48	2.82	2.28	3.24	2.48	2.82	2.52
	−0.25	1.86	1.34	1.58	1.20	2.28	1.60	2.12	1.62	2.44	1.72	2.28	1.92
	0.00	1.60	1.24	1.46	1.16	1.88	1.40	1.86	1.48	2.04	1.62	2.08	1.60
	0.25	1.76	1.42	1.74	1.24	2.02	1.62	1.92	1.50	2.36	1.82	2.20	1.78
	0.75	2.74	2.06	2.56	2.06	3.00	2.14	2.58	2.28	3.16	2.44	2.84	2.38
	1.00	3.48	2.80	3.36	2.78	3.62	3.06	3.40	2.96	3.98	3.10	4.06	3.20
100	−1.00	5.60	5.08	4.84	4.34	5.82	5.36	5.28	4.80	6.50	5.60	5.60	5.08
	−0.75	3.52	3.30	3.28	2.96	3.62	3.30	3.58	3.28	4.58	3.84	3.84	3.50
	−0.25	1.42	1.24	1.44	1.26	1.76	1.54	1.72	1.62	1.96	1.70	1.98	1.84
	0.00	1.12	1.02	1.34	1.14	1.38	1.26	1.54	1.44	1.68	1.42	1.78	1.64
	0.25	1.40	1.24	1.46	1.36	1.82	1.50	1.72	1.54	1.98	1.56	1.84	1.64
	0.75	2.90	2.60	2.84	2.50	3.58	3.16	3.14	2.84	3.82	3.40	3.40	3.30
	1.00	4.72	4.18	4.20	3.82	4.98	4.62	4.62	4.36	5.80	5.26	4.98	4.62
300	−1.00	15.52	15.20	12.90	12.52	16.54	16.20	13.78	13.66	17.34	16.90	14.62	14.50
	−0.75	7.90	7.60	6.56	6.44	8.48	8.22	7.24	7.00	9.62	9.18	7.84	7.62
	−0.25	1.64	1.60	1.52	1.42	1.84	1.76	1.82	1.72	2.16	2.04	1.84	1.84
	0.00	1.10	0.98	1.18	1.12	1.44	1.38	1.41	1.40	1.70	1.62	1.80	1.64
	0.25	1.78	1.74	1.80	1.78	1.88	1.74	2.08	1.98	2.14	2.02	2.20	2.18
	0.75	8.54	8.14	7.42	7.24	9.00	8.70	8.12	7.84	9.84	9.32	8.32	7.92
	1.00	16.34	16.06	13.68	13.14	16.88	16.46	14.08	13.78	17.54	16.96	14.74	14.30
500	−1.00	29.58	29.12	23.40	23.04	29.34	29.16	23.74	23.46	29.90	29.50	24.28	24.04
	−0.75	13.72	13.56	11.30	11.12	14.08	13.74	11.90	11.72	15.20	14.94	12.74	12.42
	−0.25	2.00	1.94	1.94	1.86	2.44	2.40	2.34	2.32	3.00	3.02	2.62	2.60
	0.00	1.08	1.08	1.04	1.02	1.30	1.22	1.16	1.10	1.86	1.80	1.64	1.62
	0.25	2.02	1.98	1.90	1.82	2.46	2.36	2.24	2.22	2.80	2.60	2.38	2.40
	0.75	15.24	15.16	12.82	12.52	15.44	15.32	12.84	12.80	16.38	16.04	13.46	13.20
	1.00	29.84	29.52	24.56	24.20	30.26	29.98	25.12	24.88	30.90	30.16	25.12	25.00

the log-normal model, and so it is natural to extend the range of distributions that the response variable, that is, the antibody concentration level, can follow.

In the measurement of antibody concentration by quantitative assays, there is always a concentration value, γ say, below which an exact measurement cannot be made, independently of the employed technique. Then, this value γ can be used to substitute a value for the censored observation. In the measles vaccine data, the value of γ was 0.1 international units (IU) or −2.306 in the logarithmic scale. It was verified that 86 (26.1%) of the observations fell below γ and then were recorded as 0.1. The covariates considered in the study were: x_{i1} is the type of vaccine used (0 if Schwarz and 1 if Edmonston-Zagreb); x_{i2} is the level of the dosage (0 if medium and 1 if high); and x_{i3} is the gender where 0 is male and 1 is female.

Table 12 reports descriptive statistics of the observed antibody concentration levels, including the median (MD), mean ($\bar{t}$), SD, coefficient of variation (CV), CS and CK (excess), and minimum ($t_{(1)}$) and maximum ($t_{(n)}$) values. From this table, we observe a skewed and high kurtosis features in the data.

Figure 1 presents the histogram and boxplots for the measles vaccine data. Note that the skewness observed in Table 12 is confirmed by the histogram presented in Figure 1(a). The adjusted boxplot for the measles vaccine data indicates that some potential outliers identified by the usual boxplot are not outliers; see Figure 1(b). The adjusted boxplot is used when the data is skew distributed; see Hubert and Vandervieren (2008).

Table 9 *Power study (%) for different values of β_4 and models (nominal level = 5%)*

		tobit-LN				tobit-Lt				tobit-LPE			
		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$	
n	β_4	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR
50	−1.00	11.58	10.76	10.98	10.06	12.82	11.96	11.76	10.92	13.66	12.46	12.40	11.52
	−0.75	9.84	9.14	9.22	8.54	10.52	9.94	9.92	9.30	11.50	10.24	10.58	9.82
	−0.25	7.58	6.84	7.06	6.48	7.74	7.12	7.90	6.92	8.56	7.60	8.38	7.86
	0.00	7.08	6.66	6.78	6.00	7.60	6.74	7.12	6.44	8.04	7.16	7.82	7.14
	0.25	7.42	6.76	6.74	6.16	7.72	7.22	7.22	6.80	8.48	7.42	8.16	7.30
	0.75	8.90	8.26	8.60	7.82	9.60	8.84	9.04	8.48	10.94	9.66	10.08	9.30
	1.00	10.66	9.92	9.90	9.30	11.50	10.42	10.76	10.04	12.28	11.44	11.72	10.88
100	−1.00	16.30	15.76	14.32	14.02	16.88	16.30	15.06	14.68	17.78	17.28	15.44	15.18
	−0.75	11.36	11.04	10.94	10.64	12.44	11.82	11.52	11.12	13.34	12.54	12.36	11.86
	−0.25	6.72	6.66	6.64	6.28	7.22	6.74	6.92	6.58	8.02	7.56	7.44	7.20
	0.00	5.36	5.14	5.56	5.30	6.30	5.90	6.30	5.98	7.36	6.68	6.26	6.18
	0.25	5.70	5.42	5.78	5.54	6.50	6.26	6.72	6.32	7.44	7.00	7.10	6.70
	0.75	10.94	10.54	10.06	9.60	11.84	11.28	10.72	10.48	12.64	11.82	11.70	11.30
	1.00	15.42	14.86	13.46	13.02	16.32	15.64	14.14	13.78	16.82	15.92	14.98	14.54
300	−1.00	34.22	33.94	30.08	29.80	35.06	34.84	30.84	30.64	34.98	34.70	31.30	31.30
	−0.75	21.94	21.78	19.16	18.90	22.56	22.40	20.20	20.12	23.38	23.14	20.68	20.72
	−0.25	6.46	6.22	6.74	6.66	7.40	7.28	7.18	7.12	8.62	8.50	7.80	7.70
	0.00	5.30	5.28	5.28	5.22	5.62	5.54	5.68	5.58	6.18	6.10	6.18	6.12
	0.25	6.78	6.70	7.06	6.92	7.60	7.40	7.68	7.54	8.60	8.50	8.08	8.00
	0.75	22.56	22.46	19.96	19.62	22.80	22.58	20.12	19.90	24.04	23.70	20.80	20.42
	1.00	35.04	34.60	31.22	30.92	35.38	35.22	31.24	31.02	36.12	36.08	32.18	32.08
500	−1.00	53.32	53.18	47.88	47.68	52.94	52.74	47.68	47.60	52.50	52.36	47.38	47.38
	−0.75	33.12	32.96	28.62	28.50	33.08	32.96	28.70	28.46	33.64	33.38	29.34	29.14
	−0.25	7.82	7.78	7.36	7.30	8.64	8.54	8.24	8.14	9.46	9.42	8.58	8.60
	0.00	5.40	5.40	5.22	5.10	5.72	5.64	5.72	5.62	6.68	6.50	6.26	6.22
	0.25	8.16	8.12	7.98	7.78	8.90	8.86	8.52	8.42	9.54	9.36	9.16	9.04
	0.75	33.86	33.78	29.16	29.02	34.42	34.26	30.10	29.94	34.48	34.54	29.90	29.88
	1.00	53.10	53.06	46.98	46.78	52.96	52.94	47.24	47.08	52.70	52.58	47.44	47.26

Figure 2 shows normal QQ plots for each covariate combination of the observed antibody concentration levels. Note that for the three covariates each at 0 or 1, we have $2^3 = 8$ possibilities. From this figure, we observe that a normal or symmetric distribution is not a reasonable assumption, since we have to accommodate skewness. We then analyze the measles vaccine data using the tobit-log-symmetric model, expressed as

$$Y_i = \begin{cases} -2.306, & Y_i^* \leq -2.306, i = 1, \ldots, 85, \\ Y_i^* = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{2i} + \beta_3 x_{3i} + \varepsilon_i, & Y_i^* > -2.306, i = 86, \ldots, 330, \end{cases}$$

where $\varepsilon_i \overset{\text{IID}}{\sim} \text{S}(0, 1, g)$.

Table 13 reports the ML estimates, computed by the BFGS quasi-Newton method, SEs and Akaike (AIC) and Bayesian information (BIC) criteria values. For comparison, the results of the classical tobit-NO model (Tobin, 1958) showed in Equation (3.1), are presented as well. The two-step procedure presented in Section 3 is used to get the estimates of ξ_1 in the tobit-Lt and tobit-LPE models, and the estimate of ξ_2 in the tobit-BS-t model. The extra parameter ξ_1 in the tobit-BS and tobit-BS-t models is the shape parameter, and it is estimated directly.

Table 10 *Power study* (%) *for different values of* β_4 *and models* (*nominal level* = 10%)

		tobit-LN				tobit-Lt				tobit-LPE			
		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$	
n	β_4	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR
50	−1.00	18.92	18.28	18.06	17.54	20.06	19.46	19.52	18.78	21.02	19.86	20.00	19.64
	−0.75	16.50	16.00	16.24	15.52	17.46	16.76	17.40	16.68	18.40	17.44	17.98	17.30
	−0.25	13.42	12.94	12.98	12.36	14.54	13.94	13.54	12.96	15.76	14.40	14.68	13.92
	0.00	12.68	12.20	12.58	11.92	13.66	13.20	13.22	12.36	15.20	14.14	14.26	13.46
	0.25	12.88	12.42	12.68	12.02	14.00	13.22	13.38	12.84	14.88	14.28	14.42	13.54
	0.75	15.54	14.88	14.48	13.76	16.50	15.68	15.72	15.18	17.40	16.18	16.76	16.20
	1.00	18.06	17.44	16.66	16.02	19.18	18.56	17.92	17.24	20.20	18.90	18.60	17.88
100	−1.00	24.26	23.70	22.84	22.54	25.08	24.90	23.10	22.86	25.98	25.12	23.62	23.40
	−0.75	19.06	18.82	17.86	17.28	20.10	19.80	18.50	18.32	20.80	20.24	18.74	18.76
	−0.25	11.98	11.80	11.90	11.62	13.00	12.74	12.68	12.48	14.46	13.76	13.84	13.32
	0.00	11.22	10.90	11.50	11.20	11.70	11.52	12.14	12.00	12.88	12.62	12.82	12.58
	0.25	11.80	11.50	11.88	11.52	12.40	12.16	12.66	12.36	14.10	13.42	13.78	13.58
	0.75	18.36	17.94	17.56	17.04	19.94	19.48	18.06	17.90	20.94	20.18	18.82	18.52
	1.00	24.56	24.02	22.48	21.92	25.38	25.00	23.06	22.64	26.48	26.00	23.70	23.20
300	−1.00	46.70	46.66	42.04	41.90	46.56	46.48	42.02	41.98	46.52	46.50	42.78	42.64
	−0.75	32.12	31.98	29.26	29.06	32.90	32.74	30.24	30.16	33.34	32.94	30.60	30.52
	−0.25	12.76	12.68	11.78	11.76	13.80	13.66	13.68	13.62	15.06	14.66	14.62	14.54
	0.00	9.58	9.54	10.34	10.26	10.74	10.68	11.28	11.20	12.62	12.62	12.82	12.58
	0.25	13.26	13.22	12.68	12.58	13.76	13.62	13.28	13.26	14.60	14.48	14.48	14.22
	0.75	32.60	32.46	29.94	29.74	33.40	33.30	30.38	30.18	34.14	33.68	31.04	30.70
	1.00	47.20	47.08	43.34	43.04	47.40	47.28	43.36	43.24	47.16	47.04	43.86	43.54
500	−1.00	64.84	64.78	59.90	59.82	65.06	65.02	59.90	59.78	64.18	63.96	59.92	59.88
	−0.75	45.82	45.74	41.26	41.12	45.40	45.30	41.22	41.14	46.02	45.92	41.28	41.20
	−0.25	13.62	13.56	13.04	12.92	14.26	14.24	14.14	13.98	15.78	15.58	14.92	14.76
	0.00	10.66	10.04	10.76	10.68	10.84	10.84	10.97	10.90	12.32	12.22	12.04	11.90
	0.25	14.80	14.68	14.12	14.04	15.34	15.30	15.12	15.08	16.66	16.46	15.70	15.70
	0.75	45.98	45.90	41.46	41.40	45.46	45.36	41.16	41.08	45.82	45.86	41.84	41.84
	1.00	65.68	65.66	60.16	60.12	65.58	65.54	60.14	60.00	65.64	65.24	59.74	59.86

From Table 13, we observe that all the tobit-log-symmetric models provide better adjustments compared to the tobit-NO model based on the values of AIC and BIC. Particularly, the tobit-LN has the lowest AIC and BIC values.

Figure 3 displays the quantile versus quantile (QQ) plots with simulated envelope of the GCS residuals for the tobit-NO, tobit-LN, tobit-Lt, tobit-LPE, tobit-BS and tobit-BS-t models. This figure indicates that the GCS residuals in the tobit-log-symmetric models show better agreements with the expected EXP(1) distribution. In particular, we observe quite a good agreement in the tobit-BS case and a poor agreement in the tobit-NO case. The results of these plots are corroborated by the histograms of the GCS residuals for the tobit-NO, tobit-LN, tobit-Lt, tobit-LPE, tobit-BS and tobit-BS-t models, superimposed with the true EXP(1) density; see Figure 4.

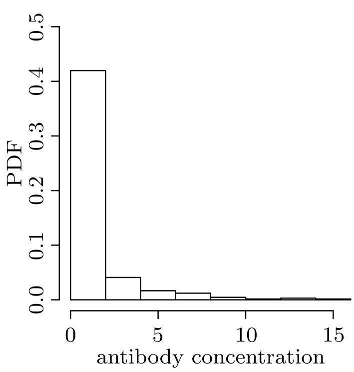
Figure 5(a)–(e) plots the quantity v_i defined in (3.8) for the tobit-log-symmetric models. Note that for the tobit-Lt and tobit-LPE models, v_i gives smaller weights for observations with larger values. In the tobit-BS-t case, Figure 5(e), we initially observe a fall followed by a rise in weights, which later, after a certain point, starts to fall again. To confirm that the weight drop behavior is sustainable, we plot v_i by considering a simulated sequence, Figure 5(f).

Table 11 *Summary statistics for the GCS residuals*

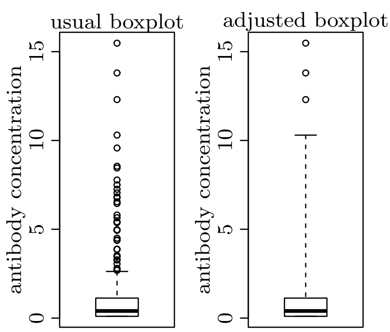
Model	n	Mean	SD	CV	CK
tobit-LN	50	1.1734	0.8648	2.0313	4.5596
	100	1.1662	0.8749	2.2664	6.2262
	300	1.1620	0.8822	2.4677	7.9836
	500	1.1601	0.8842	2.5111	8.4318
tobit-Lt	50	1.1063	0.9039	1.8481	3.9135
	100	1.1047	0.9143	2.0597	5.3346
	300	1.1048	0.9218	2.2355	6.7793
	500	1.1040	0.9227	2.2765	7.1839
tobit-LPE	50	1.1913	0.8569	2.0821	4.8415
	100	1.1822	0.8642	2.2964	6.3818
	300	1.1767	0.8730	2.5060	8.2183
	500	1.1747	0.8750	2.5427	8.5601
tobit-BS	50	1.0692	0.9291	1.7526	3.3923
	100	1.0655	0.9402	1.9567	4.7599
	300	1.0630	0.9480	2.1356	6.1950
	500	1.0613	0.9503	2.1728	6.5470
tobit-BS-t	50	1.7853	2.3409	1.9194	3.8142
	100	1.7691	2.3438	2.0925	4.9781
	300	1.7607	2.3527	2.2418	6.1481
	500	1.7583	2.3567	2.2720	6.4136

Table 12 *Summary statistics for the measles vaccine data*

$\bar{t}$	MD	SD	CV	CS	CK	$t_{(1)}$	$t_{(n)}$	n
1.20	0.40	2.10	174.74%	3.46	14.37	0.10	15.47	330



(a) Histogram



(b) Boxplots

Figure 1 *Histogram and boxplots for the measles vaccine data.*

From this figure, we observe the decay in the weights, indicating that increasingly large observations get penalized with increasingly small weights. Therefore, the tobit-Lt, tobit-LPE and tobit-BS-t models tend to produce robust estimates against outlying observations, which is an important advantage as compared to models based on the normal distribution.

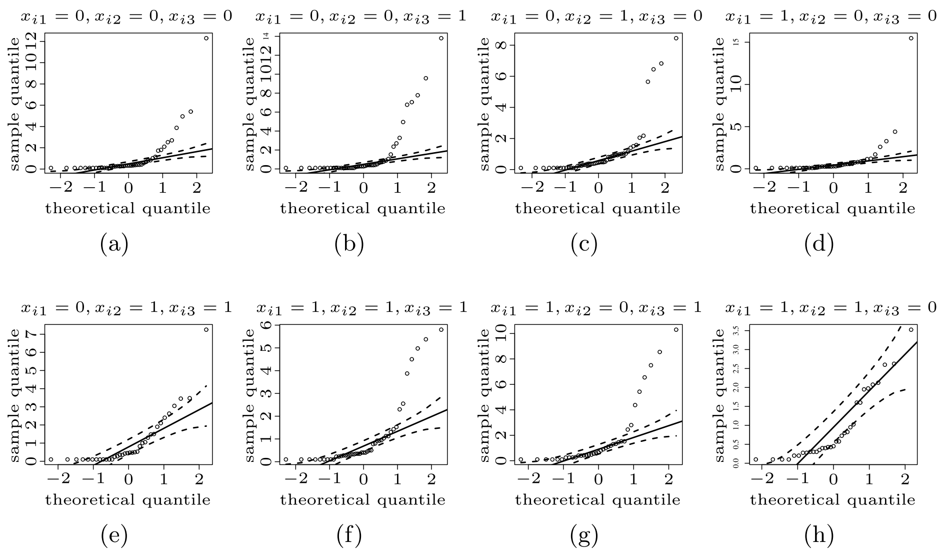


Figure 2 *Normal QQ plots for each covariate combination of the observed antibody concentration levels.*

Table 13 *ML estimates (with SE in parentheses) and AIC and BIC values for the indicated models for the measles vaccine data*

Model	AIC	BIC	ϕ	ξ_1	ξ_2	β_0	β_1	β_2	β_3
tobit-NO	1299.27	1318.27	0.945			0.597	0.225	−0.228	0.271
			(0.047)			(0.288)	(0.297)	(0.295)	(0.296)
tobit-LN	1122.28	1141.28	1.666			−1.239	0.315	0.138	0.087
			(0.080)			(0.184)	(0.190)	(0.189)	(0.189)
tobit-L*t*	1130.68	1153.47	1.474	5		−1.207	0.319	0.208	0.077
			(0.081)			(0.183)	(0.189)	(0.188)	(0.189)
tobit-LPE	1123.67	1146.47	1.311	0.3		−1.182	0.260	0.178	0.070
			(0.070)			(0.173)	(0.180)	(0.175)	(0.181)
tobit-BS	1168.38	1187.37		1.545		−0.910	0.178	0.073	0.121
				(0.081)		(0.105)	(0.127)	(0.126)	(0.126)
tobit-BS-*t*	1126.16	1148.96		1.662	4	−1.241	0.305	0.086	0.113
				(0.102)		(0.186)	(0.191)	(0.190)	(0.190)

Next, we test the null hypotheses (a) $\mathcal{H}_0 : \beta_1 = 0$, (b) $\mathcal{H}_0 : \beta_2 = 0$ and (c) $\mathcal{H}_0 : \beta_3 = 0$, using the LR and GR tests. For illustrative purposes, we consider only the tobit-LN model as it has produced the lowest AIC and BIC values in the previously described results. The corresponding p-values of LR and GR tests are: (a) 0.0971 (LR) and 0.0975 (GR); (b) 0.4657 (LR) and 0.4658 (GR); (c) 0.4657 (LR) and 0.5926 (GR). Thus, these results indicate that in the tobit-LN model, only the EZ (type of vaccine) covariate should be used. Thus, the reduced predictive tobit-LN model is given by

$$\widehat{Y}_i = \begin{cases} -2.306, & \widehat{Y}_i^* \leq -2.306, i = 1, \ldots, 85, \\ \widehat{Y}_i^* = \underset{(0.1325)}{-1.1315} + \underset{(0.1898)}{0.3242} x_{i1}, & \widehat{Y}_i^* > -2.306, i = 86, \ldots, 330. \end{cases}$$

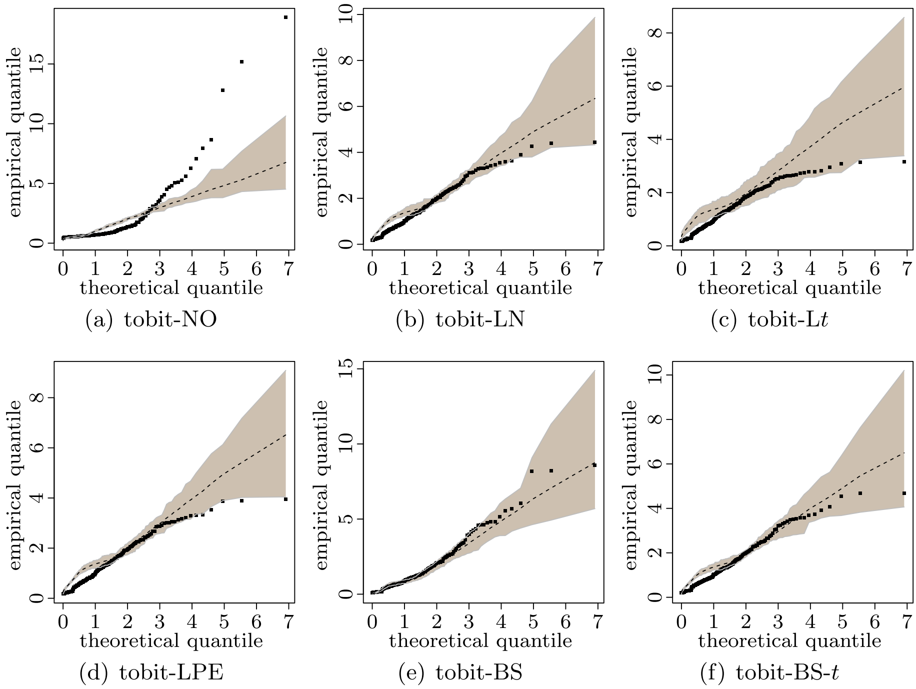


Figure 3 *QQ plot and its envelope for the GCS residuals for the tobit-log-symmetric models for measles vaccine data.*

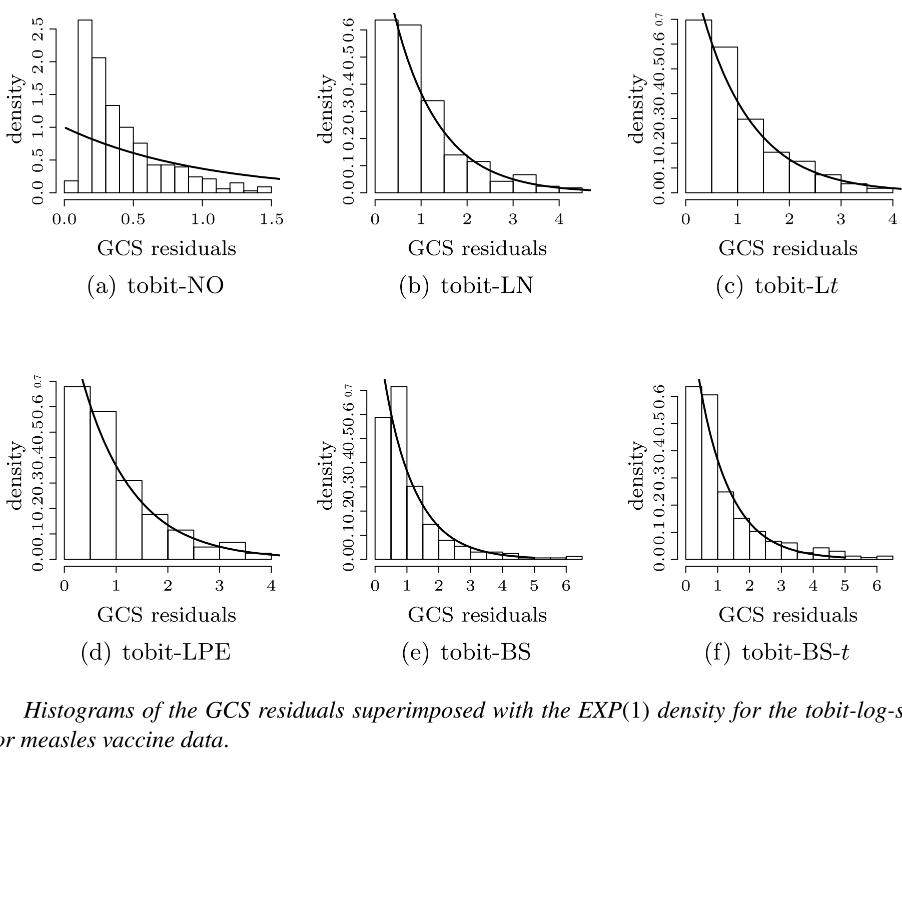


Figure 4 *Histograms of the GCS residuals superimposed with the EXP*(1) *density for the tobit-log-symmetric models for measles vaccine data.*

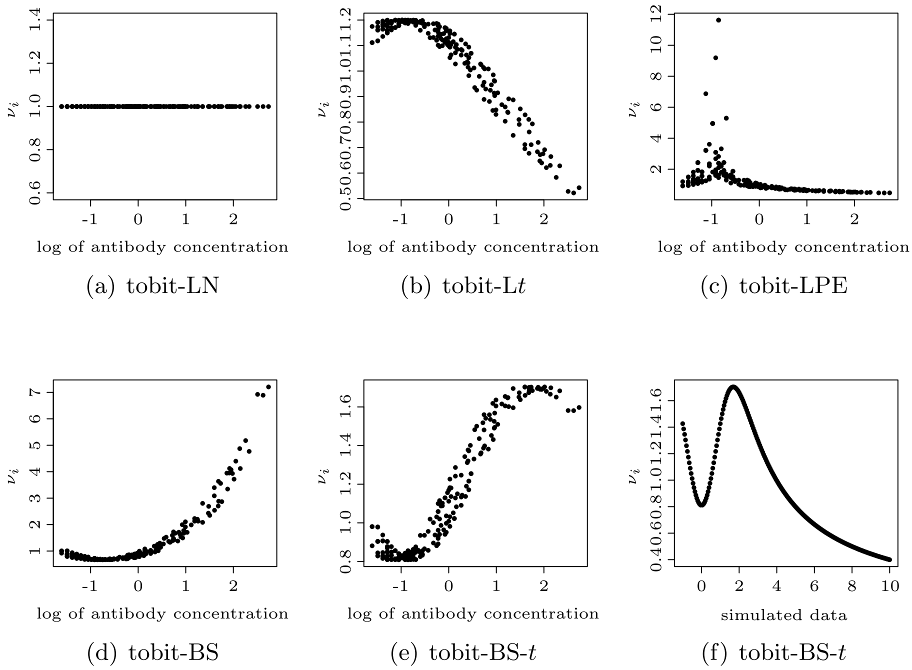


Figure 5 *Weights ν_i for the indicated models with the measles vaccine data* (*a*)–(*e*) *and with simulated data* (*f*).

6 Concluding remarks

We have proposed and analyzed a new class of tobit models for left-censored data. We have considered a likelihood-based approach for parameter estimation. We have addressed hypothesis testing within the proposed class of tobit models by using the likelihood ratio and gradient statistics. Monte Carlo simulations were carried out to evaluate the behaviour of the maximum likelihood estimates, the likelihood ratio and gradient tests and the empirical distribution of the residuals. The simulation results (a) have shown good performaces of the maximum likelihood estimates; (b) indicated that the likelihood ratio and gradient tests have similar powers; and (c) indicated that the considered residuals conform well with the reference distribution. The result (c) demonstrates similar results with other models and scenarios; see Lemonte (2016), p. 29. We have applied the proposed models to a real data set on measles vaccine in Haiti. The application has favored the use of tobit-log-symmetric models over the classical tobit-normal model. As part of future research, it will be of interest to apply Bartlett and Bartlett-type corrections (Medeiros and Ferrari, 2017) to attenuate the size distortion of the likelihood ratio and gradient tests. In addition, Monte Carlo simulations can evaluate the behavior of the maximum likelihood estimates as well as residuals under misspecification. Furthermore, influence diagnostic tools can be investigated and also multivariate models can be studied. Finally, Monte Carlo simulations can be applied to evaluate the accuracy of how some popular information criteria correctly choose log-symmetric regressions models; see Ventura et al. (2019). Work on these problems is currently in progress and we hope to report these findings in future papers.

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