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ROBUST ESTIMATION OF SUPERHEDGING PRICES

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We consider statistical estimation of superhedging prices using historical stock returns in a frictionless market with d traded assets. We introduce a plug-in estimator based on empirical measures and show it is consistent but lacks suitable robustness. To address this, we propose novel estimators which use a larger set of martingale measures defined through a tradeoff between the radius of Wasserstein balls around the empirical measure and the allowed norm of martingale densities. We then extend our study, in part, to estimation of risk measures, to the case of markets with traded options, to a multi-period setting and to settings with model uncertainty. We also study convergence rates of estimators and convergence of super-hedging strategies.

1. Introduction. Computation of risk associated to a given financial position is one of the fundamental operations market participants have to perform. For institutional players, like banks, it is regulated by the Basel Committee [39] which dictates rules and requirements for such risk assessments. A golden standard has long been given by Value-at-Risk (VaR), however, more recently this is being replaced by convex risk measures like Average VaR (Expected Shortfall) or more sophisticated approaches which include market modelling. Consequently, there is an abundant literature on VaR estimation and some more recent works related to statistical estimation of law-invariant risk measures; see [5, 11, 12, 28–30, 41]. All of these works consider a static situation with no trading involved.

In contrast, in this paper we consider estimation of risk for an agent who can trade in the market to offset her risk exposure. To put in evidence the novelty and relevance of our setting, we concentrate on one, simple but canonical, way to assess risk: the *superhedging price*. Consider a one-period frictionless market with prices (S_t, S_{t+1}) denominated in units of a fixed numeraire. The current stock prices S_t are known and the future prices S_{t+1} are modelled as random variables, say with return $r := S_{t+1}/S_t$ drawn from a distribution $\mathbb{P}$ on $\mathbb{R}^d_+$. For a payoff $g : \mathbb{R}^d_+ \to \mathbb{R}$, its superhedging price is given by

$$\pi^{\mathbb{P}}(g) := \inf\{x \in \mathbb{R} | \exists H \in \mathbb{R}^d \text{ s.t. } x + H(r-1) \geq g(r) \ \mathbb{P}\text{-a.s.}\}. \tag{1.1}$$

In this simple setting, an arbitrage strategy is $H \in \mathbb{R}^d$ such that $\mathbb{P}(H(r-1) \geq 0) = 1$ and $\mathbb{P}(H(r-1) > 0) > 0$ and if no such strategy exists we say that no-arbitrage NA($\mathbb{P}$) holds. By the fundamental theorem of asset pricing, absence of arbitrage is equivalent to existence of a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which S is a martingale, that is, $\mathbb{E}_{\mathbb{Q}}[r] = 1$. There might be more than one such measure and they can all be used for pricing. Taking the supremum over $\mathbb{E}_{\mathbb{Q}}[g]$ enables to compute the maximal feasible price for g and this, by the fundamental pricing-hedging duality, is the same as the superhedging price of g:

$$\pi^{\mathbb{P}}(g) = \sup_{\mathbb{Q} \sim \mathbb{P}, \mathbb{E}_{\mathbb{Q}}[r]=1} \mathbb{E}_{\mathbb{Q}}[g] \tag{1.2}$$

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for all Borel g; cf. [13], Theorem 1.31. Despite its theoretical importance and practical relevance, to the best of our knowledge, there has been no attempt to study statistical estimation of the superhedging price. Our paper fills this important gap. Instead of postulating a measure $\mathbb{P}$, we build estimators of $\pi^{\mathbb{P}}(g)$ directly from historical observations of returns $r_1, \dots, r_N$ and study their properties. Furthermore, we extend the estimators to take into account also the option price data. This is practically relevant and methodologically novel in that it allows a coherent and simultaneous use of historical time-series data with current option price data or, in mathematical finance jargon, the *physical measure* data and the *risk neutral measure* data.

In contrast, in existing approaches historical returns are, if at all, only used indirectly to compute $\pi^{\mathbb{P}}(g)$. In classical mathematical finance, one first postulates a family of plausible models $\{\mathbb{P}_\theta : \theta \in \Theta\}$. Such choice may be influenced by stylised features of historical returns; see [16] for a recent example, as well as by other considerations, for example, of computational tractability. Thereon, historical returns are not used and only the "future facing" options price data is exploited to select a candidate pricing measure $\mathbb{Q}_\theta$. More recently, pioneered by Mykland [35–37] in a continuous-time setting and pursued within the so-called robust approach to pricing and hedging, it was suggested to use historical returns to select a *prediction set*, that is, the set of paths on which the superhedging property is required, and then to compute the resulting cheapest superhedge which trades in stocks and options; see [1, 21]. Our approach inherits from that perspective but takes a statistical viewpoint and evolves it into a dynamic and asymptotically consistent methodology.

To describe our approach, suppose we observe d-dimensional historical returns $r_1 = S_1/S_0, \dots, r_N = S_N/S_{N-1}$ and for simplicity assume that these are nonnegative i.i.d. realisations of a distribution $\mathbb{P}$ which satisfies the no-arbitrage condition. We can equivalently represent the observations through their associated empirical measures

$$\hat{\mathbb{P}}_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{r_i},$$

which are well known to converge weakly to $\mathbb{P}$ as $N \to \infty$; see [47], Theorem 19.1, p. 266. This suggests a very natural way to approximate the superhedging price by simply using $\hat{\mathbb{P}}_N$ in place of $\mathbb{P}$. We show in Theorem 2.1 below that the resulting *plugin estimator* $\hat{\pi}_N(g) := \pi^{\hat{\mathbb{P}}_N}(g)$ is asymptotically consistent:

$$\lim_{N\to\infty} \hat{\pi}_N(g) = \pi^{\mathbb{P}}(g) \quad \mathbb{P}^\infty\text{-a.s.},$$

where $\mathbb{P}^\infty$ denotes the law of the process $(r_N)_{N\geq 1}$. However, we also show that $\hat{\pi}_N$ has serious shortcomings. First, it is not (statistically) robust: small perturbations of $\mathbb{P}$ can lead to large changes in the distribution of $\hat{\pi}_N$. We argue that the Lévy–Prokhorov metric used in the classical definition of statistical robustness, Definition 2.6, is not appropriate when looking at the financial context of derivatives pricing. We propose and study alternative metrics and ensuing notions of statistical robustness in Section 4.

Second, the plug-in estimator also lacks robustness from the financial point of view of risk management. In fact, $\hat{\pi}_N$ is monotone in N and converges from below so it is always a lower estimate of the risk: $\hat{\pi}_N \leq \pi^{\mathbb{P}}$. In Theorem 2.11, and in more detail in [38], Section B.8, we study the convergence rates for the plug-in estimators. This, in the one-dimensional case $d = 1$, could be exploited to build conservative estimates for the superhedging price $\pi^{\mathbb{P}}$.

A first intuition to improve the plug-in estimator could be to turn to estimators of the support of $\mathbb{P}$. Indeed, the superhedging price $\pi^{\mathbb{P}}(g)$, say for a continuous g, only depends on $\mathbb{P}$ via its support. We could thus replace the $\hat{\mathbb{P}}_N$-a.s. inequality in the plug-in estimator by an inequality on an estimator of the support of $\mathbb{P}$. Such estimators are well studied in statistics,

going back to [3, 8, 17, 18]; see also [6, 7, 20, 26, 33, 42, 44, 46]. Unfortunately, this approach does not seem to hold any ground. First, convergence of support estimators usually imposes strong conditions on $\mathbb{P}$, for example, compactness and convexity of the support and/or existence of a density. Second, for the convergence of the superhedging prices we would also need to impose some uniform continuity assumptions on g. However, under such conditions on $\mathbb{P}$ and g, we could directly improve the plug-in estimator and consider a suitable $\hat{\pi}_N + a_N$; see Section 2.4.

Instead, to address the shortcomings of the plug-in estimator, we propose novel estimators, which we introduce in Section 3. They exploit the dual formulation of the superhedging price in (1.2). In order to achieve financial robustness and to increase our point estimates, we need to consider a larger class of martingale measures. Thus we consider

$$\pi_{\mathcal{Q}_N}(g) = \sup_{\mathbb{Q}\in\mathcal{Q}_N} \mathbb{E}_{\mathbb{Q}}[g],$$

where $\mathcal{Q}_N$ is a subset of all martingale measures $\mathcal{M}$. The plug-in estimator corresponds to taking $\mathcal{Q}_N = \{\mathbb{Q} \in \mathcal{M} : \mathbb{Q} \sim \hat{\mathbb{P}}_N\}$ and it is natural to replace it with

$$\mathcal{Q}_N = \{\mathbb{Q} \in \mathcal{M} : \exists \tilde{\mathbb{P}} \in B_N(\hat{\mathbb{P}}_N) \text{ s.t. } \mathbb{Q} \sim \tilde{\mathbb{P}}\},$$

where $B_N(\hat{\mathbb{P}}_N)$ is some "ball" in the space of probability measures around the empirical measure $\hat{\mathbb{P}}_N$. We show that this can lead to a consistent estimator if we use a sufficiently strong metric, for example, the Wasserstein infinity metric $\mathcal{W}^\infty$. In general, however, such $\mathcal{Q}_N$ is too large. Instead, our main insight is to consider a tradeoff between the radius of the balls and the behaviour of martingale densities:

$$\hat{\mathcal{Q}}_N := \{\mathbb{Q} \in \mathcal{M} | \|d\mathbb{Q}/d\tilde{\mathbb{P}}\|_\infty \le k_N \text{ for some } \tilde{\mathbb{P}} \in B^p_{\varepsilon_N}(\hat{\mathbb{P}}_N)\},$$

where $B^p_{\varepsilon_N}(\hat{\mathbb{P}}_N)$ denotes the p-Wasserstein ball of radius ε_N around $\hat{\mathbb{P}}_N$ and $\varepsilon_N \to 0$ as well as $k_N \to \infty$. With a suitable choice of ε_N, k_N, we establish consistency of $\pi_{\hat{\mathcal{Q}}_N}(g)$ for a regular g; see Theorem 3.6, and also their financial robustness, see Corollary 3.7. This also allows us to study the cases when the estimator naturally extends to the setting of superhedging under model uncertainty about $\mathbb{P}$; see Corollary 3.8. The statistical robustness of $\pi_{\hat{\mathcal{Q}}_N}(g)$ is shown in Section 4; see Theorem 4.2. In Section 5, we extend our analysis to the case when risk is assessed not using the superhedging capital but rather via a generic risk measure ρ admitting a Kusuoka representation ([31], see (5.1) for a definition). We stress that this is substantially different to all the works recalled at the beginning of this Introduction since we consider an agent who can trade and optimises her position to offset the risk. We propose an estimator, inspired by $\pi_{\hat{\mathcal{Q}}_N}(g)$, and show its consistency.

Finally, we also propose another estimator:

$$\sup_{\mathbb{Q}\in\mathcal{M}} \left(\mathbb{E}_{\mathbb{Q}}[g] - C_N \left(\inf_{\hat{\mathbb{Q}}\sim\hat{\mathbb{P}}_N, \hat{\mathbb{Q}}\in\mathcal{M}} \left\| \frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}} \right\|_\infty - 1 \right) \right),$$

which is inspired by penalty methods used in risk measures and their representations as nonlinear expectations. Asymptotic consistency of this estimator is shown in Thereom 3.12 and holds for an arbitrary measurable bounded g.

The rest of the paper is organised as follows. In Section 2, we study the plug-in estimator $\hat{\pi}_N$: its consistency, convergence rates and robustness, both statistical and financial. In Section 3, we propose improved estimators and establish consistency for all of them, under different sets of assumptions. Subsequently, in Section 4, we discuss statistical robustness of all the estimators. We show in particular that no estimator can be robust in the classical sense of Tukey–Huber–Hampel, and suggest ways to amend the classical definition to make

it more appropriate to the superhedging price estimation. Appendix A contains proofs of the most important results from the main body of the paper. Other proofs, along with auxiliary results and supplementary discussion, are presented in [38], including convergence of superhedging strategies, extensions to a multi-period setting and arbitrage considerations.

Notation. We write $\mathcal{P}(A)$ for the set of probability measures on $A \subset \mathbb{R}^d$. $\mathbb{P}_n \Rightarrow \mathbb{P}$ denotes weak convergence of measures. $\mathbb{P} \in \mathcal{P}(\mathbb{R}_+^d)$ is a generic distribution for returns r so that $\mathbb{E}_{\mathbb{P}}[r] = \int_{\mathbb{R}_+^d} x\mathbb{P}(dx)$. We let $\mathcal{M} = \{\mathbb{Q} \in \mathcal{P}(\mathbb{R}_+^d) : \mathbb{E}_{\mathbb{Q}}[r] = 1\}$ denote the set of martingale measures for the stock prices (S_t, S_{t+1}), where $S_t > 0$ is fixed and $S_{t+1} = rS_t$. We write $\mathcal{M}_A$ for the set of martingale measures supported on A. We say that $\mathbb{P} \in \mathcal{P}(\mathbb{R}_+^d)$ does not admit arbitrage, or that NA($\mathbb{P}$) holds, if $\{\mathbb{Q} \in \mathcal{M} : \mathbb{Q} \sim \mathbb{P}\} \neq \varnothing$. Above, and throughout, H is a row vector, r is a column vector and 1 denotes either a scalar or a column vector $(1, \dots, 1)^{\mathrm{T}}$. We write span$(A)$ for the linear hull of A.

2. The plug-in estimator. Recall that we want to build an estimator for the superhedging price $\pi^{\mathbb{P}}(g)$. The easiest and possibly most natural way to do this is simply to replace the measure $\mathbb{P}$ with the empirical measures $\hat{\mathbb{P}}_N$. This yields the *plug-in estimator*:

$$\hat{\pi}_N(g) := \pi^{\hat{\mathbb{P}}_N}(g). \tag{2.1}$$

In this section, we develop the necessary tools to show asymptotic consistency of this estimator and understand its properties. The main proofs are reported in Appendix A with supplementary proofs in [38], Section B.1.

2.1. *Consistency.* We now state the main result of this section.

THEOREM 2.1. *Let $\mathbb{P}_1, \mathbb{P} \in \mathcal{P}(\mathbb{R}_+^d)$ and $g : \mathbb{R}_+^d \to \mathbb{R}$ be Borel measurable. Assume that $r_1, r_2, \dots$ are realisations of a time-homogeneous ergodic Markov chain with initial distribution $\mathbb{P}_1$ and unique invariant distribution $\mathbb{P}$ such that $\mathbb{P}_1 \ll \mathbb{P}$. Then*

$$\lim_{N\to\infty} \hat{\pi}_N(g) = \pi^{\mathbb{P}}(g) \quad \mathbb{P}^\infty\text{-a.s.}, \tag{2.2}$$

where $\mathbb{P}^\infty$ denotes the law of the Markov process started from $\mathbb{P}_1$.

REMARK 2.2. The assumptions in the above theorem are standard in econometric theory and cover a variety of models frequently used for modelling of financial returns data. We refer to Corollary B.3 in [38] for sufficient conditions for stationarity (with exponential decay rates) for various random coefficient autoregressive models, for example, linear and power GARCH and stochastic autoregressive volatility models, which are frequently used for option pricing. Nevertheless, we remark that this assumption rules out deterministic trends, structural breaks and seasonalities, which need to be treated separately.

The proof for a general g follows by Lusin's theorem from the case of a continuous claim g which in turn depends on the characterisation of the superhedging price using concave envelopes, which we now recall.

DEFINITION 2.3. Let $g : \mathbb{R}_+^d \to \mathbb{R}$ be Borel. For $A \subseteq \mathbb{R}_+^d$ and $x \in A$, we define the pointwise concave envelope

$$\hat{g}_A(x) = \inf\{u(x) | u : \mathbb{R}_+^d \to \mathbb{R} \text{ concave}, u \geq g \text{ on } A\}.$$

We define the $\mathbb{P}$-a.s. concave envelope as

$$\hat{g}_{\mathbb{P}}(x) = \inf\{u(x) | u : \mathbb{R}_+^d \to \mathbb{R} \text{ concave}, u \geq g \ \mathbb{P}\text{-a.s.}\}.$$

It is well known that in the definition of concave envelopes above we could take infimum over affine functions instead of concave functions. It follows from the definition of the superhedging price in (1.1) that we have

$$\pi^{\mathbb{P}}(g) = \hat{g}_{\mathbb{P}}(1) \quad \text{and} \quad \hat{\pi}_N(g) = \hat{g}_{\hat{\mathbb{P}}_N}(1) = \hat{g}_{\{r_1,\dots,r_N\}}(1). \tag{2.3}$$

Properties and computational methods for concave envelopes, or more generally for convex hulls of a set of discrete points, have been studied in many applied sciences and there are a number of efficient numerical routines available for their calculation. Naturally computational complexity increases with higher dimensions. Nevertheless, there exist algorithms determining approximative convex hulls, whose complexity is independent of the dimension; see, for instance, [45].

To establish a dual formulation for the plug-in estimator, assume now that $\mathbb{P}_1 \ll \mathbb{P}$ as well as no-$\mathbb{P}$-arbitrage, NA($\mathbb{P}$), holds and recall this implies the pricing-hedging duality; cf. (1.2). It turns out that since $\operatorname{supp}(\hat{\mathbb{P}}_N) \subseteq \operatorname{supp}(\mathbb{P})$ this already implies that NA($\hat{\mathbb{P}}_N$) holds for N large enough. More generally, we have the following.

PROPOSITION 2.4. *Let $\mathbb{P} \in \mathcal{P}(\mathbb{R}_+^d)$ and $(\mathbb{P}^N)_{N\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}_+^d$ such that $\mathbb{P}^N \Rightarrow \mathbb{P}$ and $\operatorname{supp}(\mathbb{P}^N) \subseteq \operatorname{supp}(\mathbb{P})$. Then*

$$\mathrm{NA}(\mathbb{P}) \Leftrightarrow \exists N_0 \in \mathbb{N} \text{ s.t. } \mathrm{NA}\big(\mathbb{P}^N\big) \quad \text{for all } N \geq N_0.$$

In particular, if NA($\mathbb{P}$) holds then in the setup of Theorem 2.1 we also have

$$\lim_{N\to\infty} \hat{\pi}_N(g) = \lim_{N\to\infty} \sup_{\mathbb{Q}\sim\hat{\mathbb{P}}^N, \mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[g] = \pi^{\mathbb{P}}(g) \quad \mathbb{P}^\infty\text{-a.s.} \tag{2.4}$$

We close this section considering an extended setup where in addition to the traded assets S, whose historical prices we observe, there also exist options in the market, which can be used for hedging g. If the market enlarged with those options does not allow for an arbitrage, the superhedging price of g in this market is again approximated by the plugin estimator, which now also allows for trading in the options. More precisely, we have the following.

COROLLARY 2.5. *Let $\mathbb{P} \in \mathcal{P}(\mathbb{R}_+^d)$ and $g : \mathbb{R}_+^d \to \mathbb{R}$ be Borel-measurable. In addition to the assets S, assume that there are $\tilde{d}$ traded options with continuous payoffs $f_1(r)$ and prices f_0 in the market. Define the evaluation map*

$$e(r) = \big(r^1, \dots, r^d, f_1^1(r)/f_0^1, \dots, f_1^{\tilde{d}}(r)/f_0^{\tilde{d}}\big)^{\mathrm{T}}$$

and $\tilde{\mathbb{P}} := \mathbb{P} \circ e^{-1}$. Finally, assume no arbitrage, NA($\tilde{\mathbb{P}}$), holds. Then, under the assumptions of Theorem 2.1, we have $\mathbb{P}^\infty$-a.s.,

$$\begin{aligned}
&\lim_{N\to\infty} \inf\big\{x \in \mathbb{R} | \exists H \in \mathbb{R}^{d+\tilde{d}} \text{ s.t. } x + H\big(e(r) - 1\big) \geq g(r) \ \forall r \in \{r_1, \dots, r_N\}\big\} \\
&\quad = \inf\big\{x \in \mathbb{R} | \exists H \in \mathbb{R}^{d+\tilde{d}} \text{ s.t. } x + H\big(e(r) - 1\big) \geq g(r) \ \mathbb{P}\text{-a.s.}\big\} \\
&\quad = \sup_{\mathbb{Q}\sim\mathbb{P}, \mathbb{Q}\in\mathcal{M}, \mathbb{E}_{\mathbb{Q}}[f_1]=f_0} \mathbb{E}_{\mathbb{Q}}[g].
\end{aligned}$$

It is worth stressing that in the classical approach to pricing and hedging, the historical returns are seen as *physical measure* inputs and might be used, for example, for extracting stylised features which models should exhibit. In contrast, option prices f_0 are *risk-neutral measure* inputs and would be used to calibrate the pricing measures. To the best of our knowledge consistent use of both in one estimator has not been achieved before.

2.2. *Statistical robustness*. Robustness of estimators is concerned with their sensitivity to perturbation of the sampling measure $\mathbb{P}$. To formalise this, suppose we have a sequence of estimators T_N which can be expressed as a fixed functional $T : \mathcal{P}(\mathbb{R}_+^d) \to \mathbb{R}$ evaluated on the sequence of empirical measures, that is, $T_N = T(\hat{\mathbb{P}}_N)$. This is clearly the case with the plug-in estimator of the superhedging price in (2.1). Hampel [19] proposed the following definition of statistical robustness.

DEFINITION 2.6 ([22], p. 42). Let $r_1, r_2, \dots$ be i.i.d. from $\mathbb{P} \in \mathcal{P}(\mathbb{R}_+^d)$. The sequence of estimators $T_N = T(\hat{\mathbb{P}}_N)$ is said to be robust at $\mathbb{P}$ if for every $\varepsilon > 0$ there is $\delta > 0$ and $N_0 \in \mathbb{N}$ such that for all $\tilde{\mathbb{P}} \in \mathcal{P}(\mathbb{R}_+^d)$ and $N \geq N_0$ we have

$$d_L(\mathbb{P}, \tilde{\mathbb{P}}) \leq \delta \quad \Longrightarrow \quad d_L\big(\mathcal{L}_{\mathbb{P}}(T_N), \mathcal{L}_{\tilde{\mathbb{P}}}(T_N)\big) \leq \varepsilon,$$

where d_L is the Lévy–Prokhorov metric

$$d_L(\mathbb{P}, \tilde{\mathbb{P}}) := \inf\big\{\delta > 0 | \mathbb{P}(B) \leq \tilde{\mathbb{P}}\big(B^\delta\big) + \delta \text{ for all } B \in \mathcal{B}\big(\mathbb{R}_+^d\big)\big\}. \tag{2.5}$$

We sometimes say that T_N is robust with respect to d_L to stress the dependency on the particular choice of the metric. A classical result of Hampel (see [22], Theorem 2.21) states that if T is asymptotically consistent, that is,

$$T_N = T(\hat{\mathbb{P}}_N) \longrightarrow T(\mathbb{P}) \quad \text{for all } \mathbb{P} \in \mathcal{P}\big(\mathbb{R}_+^d\big)$$

then T_N is robust at $\mathbb{P}$ if and only if $T(\cdot)$ is continuous at $\mathbb{P}$. The following theorem characterises weak continuity of the superhedging price, and hence also robustness of its estimators. In particular, it implies that even for i.i.d. returns $\hat{\pi}_N$ is robust only for special combinations of g and $\mathbb{P}$.

THEOREM 2.7. *Let g be continuous and $\mathbb{P} \in \mathcal{P}(\mathbb{R}_+^d)$. Then the functional $\tilde{\mathbb{P}} \mapsto \pi^{\tilde{\mathbb{P}}}(g)$ is lower semicontinuous at $\mathbb{P}$. It is continuous if and only if*

$$\pi^{\mathbb{P}}(g) = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[g]. \tag{2.6}$$

In consequence, any asymptotically consistent estimator T_N is robust at $\mathbb{P}$ only if the above equality holds true.

In particular, we see that, in general, the plug-in estimator $\hat{\pi}_N(g)$ is not robust w.r.t. d_L. The fact that this holds for any asymptotically consistent estimator suggests strongly that the classical definition of robustness is not adequate in the present context. The superhedging price $\pi^{\mathbb{P}}(g)$ is concerned with the support of $\mathbb{P}$ in the sense that for $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}(\mathbb{R}_+^d)$ with equal supports, and for a continuous g, we have $\pi^{\mathbb{P}_1}(g) = \pi^{\mathbb{P}_2}(g)$. In contrast, any δ-perturbation in the Lévy–Prokhorov sense allows for arbitrary changes to the support; see Lemma A.1. In particular, even if $\mathbb{P}$ satisfies no-arbitrage, measures in its neighbourhood may not and one may not employ (1.2) for these. To control the support, we can consider $d_H(\mathrm{supp}(\mathbb{P}), \mathrm{supp}(\tilde{\mathbb{P}}))$, for $\mathbb{P}, \tilde{\mathbb{P}} \in \mathcal{P}(\mathbb{R}_+^d)$ and where d_H denote the Hausdorff metric on closed subsets of $\mathbb{R}_+^d$.

PROPOSITION 2.8. *Let $g : \mathbb{R}_+^d \to \mathbb{R}$ be uniformly continuous and let $\mathbb{P} \in \mathcal{P}(\mathbb{R}_+^d)$ such that $NA(\mathbb{P})$ holds and* $\mathrm{span}(\mathrm{supp}(\mathbb{P})) = \mathbb{R}^d$. *Then the functional $\mathcal{P}(\mathbb{R}_+^d) \ni \tilde{\mathbb{P}} \to \pi^{\tilde{\mathbb{P}}}(g)$ is continuous w.r.t. the pseudo-metric $d_H(\mathrm{supp}(\mathbb{P}), \mathrm{supp}(\tilde{\mathbb{P}}))$.*

Alas, this does not allow us to recover statistical robustness of the plug-in estimator as the pseudometric above does not admit control over the tails of $\mathbb{P}$. Instead, in Section 4.2, we consider a stronger $\mathcal{W}^\infty$ metric which allows to obtain an analogue to Hampel's robustness result.

2.3. *Financial robustness*. The plug-in estimator $\hat{\pi}_N$ not only lacks statistical robustness, as seen above, but is also not a financially robust estimate of risk. In fact, if $\mathbb{P}_1 \ll \mathbb{P}$, it converges to the superhedging price from below, that is, $\hat{\pi}_N \nearrow \pi^{\mathbb{P}}$. From a risk-management perspective one would like to find a consistent estimator for the $\mathbb{P}$-a.s. superhedging price converging from above. However, as we now show, this is not possible in general. As a direct consequence of the discontinuity of the superhedging functional with respect to the Lévy–Prokhorov metric d_L, the convergence from above at some confidence level cannot be achieved in practical applications.

PROPOSITION 2.9. *Let $\mathbb{P} \in \mathcal{P}(\mathbb{R}_+^d)$ satisfy* NA($\mathbb{P}$) *and g be bounded and Lipschitz continuous. Then there exists no consistent estimator T_N of $\pi^{\mathbb{P}}(g)$ such that for a confidence level $\alpha \in [0,1]$ there exists $N_0 \in \mathbb{N}$ and*

$$\mathbb{P}^\infty\Big(T_N \geq \sup_{\mathbb{Q}\in\mathcal{M},\mathbb{Q}\sim\mathbb{P}} \mathbb{E}_{\mathbb{Q}}[g] \text{ for all } N \geq N_0\Big) \geq \alpha \tag{2.7}$$

for all $\mathbb{P} \in \mathbb{P}(\mathbb{R}_+^d)$.

Thus, in order to achieve the above property (2.7) it is necessary to make additional regularity assumptions on $\mathbb{P}$ and g. We show that this is possible for suitably conservative estimators; see Section 3.2 below. In the case of the plug-in estimator, we can never achieve convergence from above but we can develop an understanding of the order of magnitude of the difference $\pi^{\mathbb{P}}(g) - \hat{\pi}_N(g)$. We first do this by studying the convergence rates; see also [38], Section B.8. Second, we achieve this via notions of statistical robustness suited for the plug-in estimator; see Section 4.2.

2.4. *Convergence rates*. We now investigate the convergence rate in (2.4). While motivated by financial considerations, the question is of independent interest. We focus on the one-dimensional case. We let $F_{\mathbb{P}}$ be the cumulative distribution function of $\mathbb{P} \in \mathcal{P}(\mathbb{R}_+)$ and $d_N = \sup_{r\in\mathbb{R}_+} |F_{\hat{\mathbb{P}}_N}(r) - F_{\mathbb{P}}(r)|$ denote the Kolmogorov–Smirnov distance between $\hat{\mathbb{P}}_N$ and $\mathbb{P}$.

DEFINITION 2.10. For $N \in \mathbb{N}$ and $k = 1, \dots, \lfloor 1/(3d_N)\rfloor$, we define the interquantile distance

$$\kappa_k^N = F_{\mathbb{P}}^{-1}(3kd_N) - F_{\mathbb{P}}^{-1}\big(3(k-1)d_N \vee 0+\big) \quad \text{for } k = 1, \dots \lfloor 1/(3d_N)\rfloor.$$

Furthermore, we set

$$\kappa_0^N = \begin{cases} F_{\mathbb{P}}^{-1}(1) - F_{\mathbb{P}}^{-1}(1-d_N) & \text{if } \mathbb{P} \text{ has bounded support,} \\ 0 & \text{otherwise,} \end{cases}$$

and let $\kappa^N = \sup_{k\in\{0,\dots,\lfloor 1/(3d_N)\rfloor\}} \kappa_k^N$.

We can now establish the speed of convergence for the plug-in estimator.

THEOREM 2.11. *In the setup of Theorem 2.1 assume $d = 1$,* NA($\mathbb{P}$) *holds and g is bounded and uniformly continuous with $|g(r) - g(\tilde{r})| \leq \delta(|r - \tilde{r}|)$ for some $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\delta(r) \to 0$ for $r \to 0$. Then, as $N \to \infty$,*

$$\begin{aligned} \pi^{\mathbb{P}}(g) - \hat{\pi}_N(g) &= \sup_{\mathbb{Q}\sim\mathbb{P},\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[g] - \sup_{\mathbb{Q}\sim\hat{\mathbb{P}}_N,\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[g] \\ &= \begin{cases} \mathcal{O}\big(\delta(\kappa^N)\big) & \text{if } \mathbb{P} \text{ has bounded support,} \\ \mathcal{O}\Big(\delta(\kappa^N) + \dfrac{1}{F_{\mathbb{P}}^{-1}(1-d_N)}\Big) & \text{otherwise.} \end{cases} \end{aligned}$$

REMARK 2.12. When the support of $\mathbb{P}$ is bounded, the above result holds for all continuous g. Furthermore, κ^N tends to 0 as $N \to \infty$.

LEMMA 2.13 (Dvoretzky–Kiefer–Wolfowitz, cf. [27], Theorem 11.6). *Suppose the returns $r_1, r_2, \ldots$ are i.i.d. samples from $\mathbb{P} \in \mathcal{P}(\mathbb{R}_+)$. Then for every $\varepsilon > 0$,*

$$\mathbb{P}^\infty(d_N > \varepsilon) \leq 2e^{-2N\varepsilon^2}.$$

Theorem 2.11 and Lemma 2.13 yield probabilistic bounds on the distribution of $\pi^{\mathbb{P}}(g) - \hat{\pi}_N(g)$. This is explored in [38], Section B.8, where we also prove Theorem 2.11 and provide extensions of Lemma 2.13.

3. Improved estimators for the $\mathbb{P}$-a.s. superhedging price. In the last section, we have seen that the plug-in estimator is asymptotically consistent but has important shortcomings from a statistical and financial point of view. To address these, we propose now new estimators and investigate their asymptotic behaviour as well as their robustness. To construct these, we consider "balls" around the empirical measure $\hat{\mathbb{P}}_N$ and we rely on recent convergence rate results of $\hat{\mathbb{P}}_N$ to $\mathbb{P}$ for the choice of the radii.

We start by considering balls in the Wasserstein$-\infty$ metric, which offers a very good control over the support but where we need to make strong assumptions on $\mathbb{P}$ to control the rate of convergence for $\hat{\mathbb{P}}_N$. Subsequently, in Section 3.2, we consider Wasserstein-p metrics, $p \geq 1$. The use of weaker metrics allows us to treat all measures admitting suitable finite moments but requires a penalisation over the dual (pricing) measures. In fact, our estimators rely on a suitable combination of results on convergence of empirical measures with insights into pricing and control over martingale densities. Similar to the spirit of Corollary 2.5 above, we combine the *physical measure*—and the *risk neutral measure*—arguments; see (3.4). We note that using Wasserstein metrics, as opposed to weaker metrics, allows us to control the first moment which is important for no-arbitrage reasons; see [38], Section B.7. Finally, in Section 3.3, we consider much larger balls, indeed all of $\mathcal{M}$, and let penalisation select the appropriate measures. Short proofs are given here, the main proofs are reported in Appendix A.2 while supplementary proofs are given in [38], Section B.2.

3.1. *Wasserstein $\mathcal{W}^\infty$ balls.* When considering robustness of the plug-in estimator we saw that to consider measures in a ball around $\hat{\mathbb{P}}_N$ we have to consider a notion of distance which, unlike the Lévy–Prokhorov metric, controls the supports. This is achieved by the Wasserstein-∞ distance

$$\begin{aligned}\mathcal{W}^\infty(\mathbb{P}, \tilde{\mathbb{P}}) &:= \inf_{\gamma \in \Pi(\mathbb{P}, \tilde{\mathbb{P}})} \gamma\text{-ess-sup } |x - y| \\ &= \inf\{\varepsilon > 0 | \mathbb{P}(B) \leq \tilde{\mathbb{P}}(B^\varepsilon), \tilde{\mathbb{P}}(B) \leq \mathbb{P}(B^\varepsilon)\ \forall B \in \mathcal{B}(\mathbb{R}_+^d)\},\end{aligned} \tag{3.1}$$

where $\Pi(\mathbb{P}, \tilde{\mathbb{P}})$ denotes the set of all probability measures γ with marginals $\mathbb{P}$ and $\tilde{\mathbb{P}}$ and where the equality between the definition and the second representation is a consequence of the Skorokhod representation theorem. A direct comparison of (3.1) with (2.5) reveals that $\mathcal{W}^\infty$ controls the support in a way that d_L does not. However, one immediate issue with considering $\mathcal{W}^\infty$ is that if $\mathbb{P}$ has unbounded support then $\mathcal{W}^\infty(\mathbb{P}, \hat{\mathbb{P}}_N) = \infty$ for all $N \in \mathbb{N}$ since $\hat{\mathbb{P}}_N$ are finitely supported. For this reason, and also to obtain appropriate confidence intervals, in order to build a good estimator using $\mathcal{W}^\infty$-balls we have to impose relatively strong assumptions on $\mathbb{P}$.

ASSUMPTION 3.1. The measure $\mathbb{P}$ is an element of $\mathcal{P}(A)$ for a connected, open and bounded set $A \in \mathcal{B}(\mathbb{R}^d_+)$ with a Lipschitz boundary. Furthermore, $\mathbb{P}$ admits a density $\rho : A \to (0, \infty)$ such that there exists $\alpha \geq 1$ for which $1/\alpha \leq \rho(r) \leq \alpha$ on A.

Under the above assumption, we have explicit bounds on $\mathcal{W}^\infty(\mathbb{P}, \hat{\mathbb{P}}_N)$. The case $d = 1$ follows from Kiefer–Wolfowitz bounds while the case $d \geq 2$ was established in [15], Theorem 1.1.

LEMMA 3.2. *Assume that* $\mathbb{P}$ *fulfils Assumption* 3.1 *and* NA($\mathbb{P}$) *holds. Furthermore, let* $r_1, r_2, \ldots$ *be i.i.d. samples from* $\mathbb{P}$. *If* $d = 1$, *then except on a set with probability* $O(\exp(-2\sqrt{N}))$, $\mathcal{W}^\infty(\mathbb{P}, \hat{\mathbb{P}}_N) \leq l_N(1, \alpha, A) := \alpha N^{-1/4}$. *If* $d \geq 2$, *then except on a set with probability* $O(N^{-2})$,

$$\mathcal{W}^\infty(\mathbb{P}, \hat{\mathbb{P}}_N) \leq l_N(d, \alpha, A) := C(\alpha, A) \begin{cases} \dfrac{\log(N)^{3/4}}{N^{1/2}} & \text{if } d = 2, \\ \dfrac{\log(N)^{1/d}}{N^{1/d}} & \text{if } d \geq 3. \end{cases}$$

We let $B^\infty_\varepsilon(\mathbb{P})$ denote a $\mathcal{W}^\infty$-ball of radius ε around $\mathbb{P}$. The above lemma allows to deduce consistency of the estimator based on such $\mathcal{W}^\infty$ balls.

PROPOSITION 3.3. *Consider* $\mathbb{P} \in \mathcal{P}(\mathbb{R}^d_+)$ *satisfying* NA($\mathbb{P}$) *and Assumption* 3.1, *and let* α, $l_N := l_N(d, \alpha, A)$ *be as in Lemma* 3.2. *Let* g *be a continuous function and* $r_1, r_2, \ldots$ *be i.i.d. samples from* $\mathbb{P}$. *Then*

$$\hat{\pi}^\infty_N(g) := \sup_{\tilde{\mathbb{P}} \in B^\infty_{l_N}(\hat{\mathbb{P}}_N)} \pi^{\tilde{\mathbb{P}}}(g) \searrow \pi^{\mathbb{P}}(g) \quad \text{as } N \to \infty, \mathbb{P}^\infty\text{-a.s.}$$

REMARK 3.4. The practical use of $\mathcal{W}^\infty$ estimators requires a good handle on l_N. Its dependence on the set A is mild; from [15], we see that if a bi-Lipschitz homeomorphism $\phi : \overline{A} \to [0, 1]^d$ exists, then the constant C depends on the domain A only via the Lipschitz constant of ϕ. However, the knowledge of α requires uniform *a priori* estimates on the density of $\mathbb{P}$ on A. This should be contrasted with $\mathcal{W}^p$ estimators below, which only require finiteness of certain moments.

PROOF OF PROPOSITION 3.3. From Lemma 3.2, an application of Borel–Cantelli shows that, $\mathbb{P}^\infty$-a.s., for N large enough $\mathbb{P} \in B^\infty_{l_N}(\hat{\mathbb{P}}_N)$ and, in particular, $\hat{\pi}^\infty_N \geq \pi^{\mathbb{P}}$. Further, as $d_H(\text{supp}(\mathbb{P}), \text{supp}(\tilde{\mathbb{P}})) \leq \mathcal{W}^\infty(\mathbb{P}, \tilde{\mathbb{P}})$ by (3.1), for all $\tilde{\mathbb{P}} \in B^\infty_{l_N}(\hat{\mathbb{P}}_N)$ we have $\text{supp}(\tilde{\mathbb{P}}) \subseteq \overline{\text{supp}(\mathbb{P})^{2l_N}}$, a compact on which g is uniformly continuous. For measures supported on this compact, Proposition 2.8 yields continuity of $\tilde{\mathbb{P}} \mapsto \pi^{\tilde{\mathbb{P}}}$ with respect to $\mathcal{W}^\infty$ at $\mathbb{P}$, which in turn implies consistency of $\hat{\pi}^\infty_N$ and concludes the proof. □

Thus $\hat{\pi}^\infty_N$ is not only consistent but also financially robust. We shall see in Corollary 4.6 below, that it is also statistically robust with respect to $\mathcal{W}^\infty$. However, these results only hold for measures $\mathbb{P}$ which satisfy Assumption 3.1. In the next section, we introduce a family of estimators which exhibit similar desirable properties for a much larger class of measures $\mathbb{P}$.

3.2. *Wasserstein $\mathcal{W}^p$ balls and martingale densities.* We assume no-arbitrage NA($\mathbb{P}$) holds and exploit (1.2) to consider estimators of the form

$$\pi_{\mathcal{Q}_N}(g) := \sup_{\mathbb{Q}\in\mathcal{Q}_N} \mathbb{E}_{\mathbb{Q}}[g] \tag{3.2}$$

for different specifications of the sets of martingale measures $\mathcal{Q}_N$ based on "balls" around $\hat{\mathbb{P}}_N$. In order to guarantee asymptotic consistency, we have to ascertain that the true measure $\mathbb{P}$ is contained in these balls and that we have some control over the tails of the martingale measures in $\mathcal{Q}_N$. Our crucial insight, following recent work of [34], is to work with Wasserstein metrics defined, for $p \geq 1$ and $\mathbb{P}, \tilde{\mathbb{P}} \in \mathcal{P}(\mathbb{R}^d_+)$ with a finite p^{th} moment, by

$$\mathcal{W}^p(\mathbb{P}, \tilde{\mathbb{P}}) = \left(\inf\left\{\int_{\mathbb{R}^d_+\times\mathbb{R}^d_+} |r-s|^p \gamma(dr, ds) \middle| \gamma \in \Pi(\mathbb{P}, \tilde{\mathbb{P}})\right\}\right)^{1/p},$$

where $\Pi(\mathbb{P}, \tilde{\mathbb{P}})$ is the set of probability measures on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$ with marginals $\mathbb{P}$ and $\tilde{\mathbb{P}}$. In case $p = 1$, [25] showed that Kantorovitch–Rubinstein duality (see [9], Theorem 11.8.2, p. 421) has a particularly nice expression:

$$\mathcal{W}^1(\mathbb{P}, \tilde{\mathbb{P}}) = \sup_{f\in\mathcal{L}_1} \left| \int_{\mathbb{R}^d_+} f(y)\, d\mu(y) - \int_{\mathbb{R}^d_+} f(y)\, d\nu(y) \right|, \tag{3.3}$$

where $\mathcal{L}_1$ denotes the 1-Lipschitz continuous functions $f : \mathbb{R}^d_+ \to \mathbb{R}$.[1] A Wasserstein ball around $\mathbb{P}$ is denoted

$$B^p_\varepsilon(\mathbb{P}) = \{\tilde{\mathbb{P}} \in \mathcal{P}(\mathbb{R}^d_+) | \mathcal{W}^p(\mathbb{P}, \tilde{\mathbb{P}}) \leq \varepsilon\}.$$

For a given $\varepsilon \geq 0$ and $k \in (0, \infty]$, let

$$D^p_{\varepsilon,k}(\mathbb{P}) := \left\{\mathbb{Q} \in \mathcal{M} \middle| \left\| \frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}} \right\|_\infty \leq k \text{ for some } \tilde{\mathbb{P}} \in B^p_\varepsilon(\mathbb{P})\right\}. \tag{3.4}$$

One's first intuition might be to use $\mathcal{Q}_N = D^p_{\varepsilon_N,\infty}(\hat{\mathbb{P}}_N)$ in (3.2). Interestingly, this does not work as the balls are too large. Indeed, Wasserstein distance metrises weak convergence and Lemma A.1 shows that any ball around $\hat{\mathbb{P}}_N$ includes measures with full support. As it turns out, to obtain a consistent estimator a subtle interplay is required between ε and k in (3.4).

ASSUMPTION 3.5.

1. $r_1, r_2, \ldots$ are realisations of a time-homogeneous ergodic Markov chain with initial distribution $\mathbb{P}_1$ and unique invariant distribution $\mathbb{P}$ such that $\mathbb{P}_1 \ll \mathbb{P}$ and $\|d\mathbb{P}_1/d\mathbb{P}\|_{L^s(\mathbb{P})} < \infty$ for some $s > 3$. Furthermore, $\mathbb{E}_{\mathbb{P}}[|r|^q] < \infty$ for some $q > 2ps/(s-2)$ and there exists a sequence $(\rho_N)_{N\in\mathbb{N}}$ with $\sum_{N\in\mathbb{N}} \rho_N < \infty$ such that if $r_1 \sim \mathbb{P}$,

$$\mathbb{E}\big[\mathbb{E}\big[f(r_N) - m(f)|r_1\big]^2\big] \leq \rho_N^2, \tag{3.5}$$

holds for all $\|f\|_\infty \leq 1$, all $N \in \mathbb{N}$, where $m(f) = \mathbb{E}[f(r_1)]$.

2. $r_1, r_2, \ldots$ are i.i.d. samples of $\mathbb{P}$ and there exist $a, c > 0$ such that $\mathbb{E}_{\mathbb{P}}[\exp(c|r|^a)] < \infty$.

[1] We develop the theory for all $p \geq 1$. In practice, the choice of p has to be made by the statistician. From Theorem 4.2 and the equation above Corollary 3.7, it is apparent that, for robustness, one wants to take p as large as possible. This however makes moment assumptions more restrictive. The cases $p = 1$ and $p = 2$ are the most popular in literature, given in particular the nice duality for $p = 1$ and the fact that L^2 is a Hilbert space.

We again refer to [38], Corollary B.3, for examples of processes, which satisfy Assumption 3.5. Clearly, Assumption 3.5.2 implies 3.5.1. Under this assumption [14], see also [34], used concentration of measure techniques to obtain rates of the decay for $\mathbb{P}^\infty(\mathcal{W}(\mathbb{P},\hat{\mathbb{P}}_N)\geq\varepsilon)$; see Lemma A.2 and [38], Lemma B.4. This allows to compute explicitly a function $\varepsilon_N:(0,1)\to\mathbb{R}_+$ with $\varepsilon_N(\beta)\searrow 0$ as $N\to\infty$, such that

$$\mathbb{P}^\infty\big(\mathcal{W}^p(\mathbb{P},\hat{\mathbb{P}}_N)\geq\varepsilon_N(\beta)\big)\leq\beta,\quad N\geq 1.$$

We say that Assumption 3.5 holds if either Assumption 3.5.1 holds and then ε_N is given in (B.3) or Assumption 3.5.2 holds and ε_N is then given in (A.1). We state now the main result in this section.

THEOREM 3.6. *Let g be either Lipschitz continuous and bounded from below or continuous and bounded, $p\geq 1$ and $\mathbb{P}\in\mathcal{P}(\mathbb{R}_+^d)$ satisfying* NA($\mathbb{P}$). *Suppose Assumption 3.5 holds and $\beta_N\in(0,1)$ satisfy $\lim_{N\to\infty}\beta_N=0$ and $\lim_{N\to\infty}\varepsilon_N(\beta_N)=0$. Pick a sequence $k_N=o(1/\varepsilon_N(\beta_N))$. Then the limit in $\mathbb{P}^\infty$-probability*

$$\lim_{N\to\infty}\pi_{\hat{\mathcal{Q}}_N}(g)=\pi^{\mathbb{P}}(g), \tag{3.6}$$

holds, where $\hat{\mathcal{Q}}_N:=D^p_{\varepsilon_N(\beta_N),k_N}(\hat{\mathbb{P}}_N)$. Furthermore, if $(\beta_N)_{N\in\mathbb{N}}$ satisfies $\sum_{N=1}^\infty\beta_N<\infty$ then the limit (3.6) also holds $\mathbb{P}^\infty$-almost surely.

The above result shows that $\pi_{\hat{\mathcal{Q}}_N}$ is an asymptotically consistent estimator of $\pi^{\mathbb{P}}$. Note that we assume no arbitrage NA($\mathbb{P}$) so that, using (1.2), the convergence above is equivalent to

$$\lim_{N\to\infty}\sup_{\mathbb{Q}\in\hat{\mathcal{Q}}_N}\mathbb{E}_{\mathbb{Q}}[g]=\sup_{\mathbb{Q}\sim\mathbb{P},\mathbb{Q}\in\mathcal{M}}\mathbb{E}_{\mathbb{Q}}[g].$$

We write $\hat{\mathcal{Q}}_N^p=\hat{\mathcal{Q}}_N$ when we want to stress the dependence on p. As mentioned above, the consistency depends crucially on the choice of $\hat{\mathcal{Q}}_N$. We discuss this further and motivate the above choice in [38], Section B.7. For $p>1$, $D^p_{\varepsilon,k}(\mathbb{P})$ are weakly compact but $D^1_{\varepsilon,k}(\mathbb{P})$ is not even weakly closed in general; see Lemma A.3. In case of $p=1$, taking weak closure of $\hat{\mathcal{Q}}_N^1$ could destroy the asymptotic consistency of the estimator, for example, taking $g(r)=(r-1)$ in the example in the proof of Lemma A.3.

For the particular choice of $\beta_N=\exp(-\sqrt{N})$ under Assumption 3.5.2, an explicit computation yields that for N large enough we have

$$\varepsilon_N(\beta_N)=\left(\frac{\log(c_1\exp(\sqrt{N}))}{c_2N}\right)^{1/\min(\max(d/p,2),a/(2p))}\sim\frac{1}{N^{1/\min(\max(2d/p,4),a/p)}}.$$

However, many other choices of β_N are feasible. The essential point is that for summable (β_N), a Borel–Cantelli argument implies that for N large enough the true distribution $\mathbb{P}$ is within an $\varepsilon_N(\beta_N)$-ball around $\hat{\mathbb{P}}_N$. This allows us to deduce a sufficient condition for financial robustness of our estimator.

COROLLARY 3.7. *In the setup of Theorem 3.6 with $\sum_{N=1}^\infty\beta_N<\infty$, let g be such that*

$$\exists C\in\mathbb{R}_+\quad\text{s.t.}\quad\sup_{\mathbb{Q}\in\mathcal{M},\|\frac{d\mathbb{Q}}{d\mathbb{P}}\|_\infty\leq C}\mathbb{E}_{\mathbb{Q}}[g]=\sup_{\mathbb{Q}\sim\mathbb{P},\mathbb{Q}\in\mathcal{M}}\mathbb{E}_{\mathbb{Q}}[g]=\pi^{\mathbb{P}}(g). \tag{3.7}$$

Then $\pi_{\hat{\mathcal{Q}}_N}(g)\geq\pi^{\mathbb{P}}(g)$ for N large enough so that the estimator is asymptotically consistent and converges from above.

The condition (3.7) is motivated by an approximation result; see Lemma A.4. It allows us also to consider the case when we are unsure about the *true* measure $\mathbb{P}$ and instead prefer to superhedge under all measures in its small neighbourhood.

COROLLARY 3.8. *In the setup of Theorem 3.6, fix $C > 0$ and assume there exists $\mathbb{Q} \in \mathcal{M}$ such that $\|d\mathbb{Q}/d\mathbb{P}\|_\infty < C$. Consider $C_N \to C$ and a fixed $\varepsilon > 0$. Then*

$$\lim_{N\to\infty} \sup_{\mathbb{Q}\in D^p_{\varepsilon+\varepsilon_N, C_N}(\hat{\mathbb{P}}_N)} \mathbb{E}_{\mathbb{Q}}[g] = \sup_{\mathbb{Q}\in D^p_{\varepsilon,C}(\mathbb{P})} \mathbb{E}_{\mathbb{Q}}[g]$$

holds in $\mathbb{P}^\infty$-probability and $\mathbb{P}^\infty$-a.s. whenever $\sum_{N=1}^\infty \beta_N < \infty$.

We close this section with two examples illustrating that the assumptions on regularity of g in Theorem 3.6 can not be easily relaxed.

EXAMPLE 3.9 (g unbounded, not Lipschitz). Set $g(r) = (r-1)^2$ and consider $(r_N)_{N\geq 1}$ i.i.d. from $\mathbb{P} = \delta_1$. For $r_N \geq 2$, consider the measures

$$\nu_N = \frac{\varepsilon_N(\beta_N)}{2}\delta_0 + \left(1 - \frac{r_N\varepsilon_N(\beta_N)}{2(r_N-1)}\right)\delta_1 + \frac{\varepsilon_N(\beta_N)}{2(r_N-1)}\delta_{r_N}$$

and

$$\mathbb{Q}_N = \frac{\varepsilon_N(\beta_N)}{2\sqrt{\varepsilon_N(\beta_N)}}\delta_0 + \left(1 - \frac{r_N\varepsilon_N(\beta_N)}{2(r_N-1)\sqrt{\varepsilon_N(\beta_N)}}\right)\delta_1 + \frac{\varepsilon_N(\beta_N)}{2(r_N-1)\sqrt{\varepsilon_N(\beta_N)}}\delta_{r_N}.$$

Then $\mathcal{W}^1(\nu_N, \delta_1) \leq \varepsilon_N(\beta_N)$,

$$\left\|\frac{d\mathbb{Q}_N}{d\nu_N}\right\|_\infty \leq \frac{1}{\sqrt{\varepsilon_N(\beta_N)}}$$

and choosing $r_N = 1/\varepsilon_N(\beta_N)$ we find

$$\mathbb{E}_{\mathbb{Q}_N}[g] \geq \frac{\sqrt{\varepsilon_N(\beta_N)}}{2(1/\varepsilon_N(\beta_N)-1)}\big(1/\varepsilon_N(\beta_N)-1\big)^2 \to \infty \quad \text{as } N\to\infty.$$

EXAMPLE 3.10 (g bounded, discontinuous). Set $g(r) = \mathbb{1}_{\{r\neq 1\}}$ and consider $(r_N)_{N\geq 1}$ i.i.d. from $\mathbb{P} = \delta_1$. Let

$$\nu_N = \frac{1}{2}\delta_{1-\varepsilon_N(\beta_N)/2} + \frac{1}{2}\delta_{1+\varepsilon_N(\beta_N)/2},$$

then $\mathcal{W}^1(\nu_N, \delta_1) = \varepsilon_N(\beta_N)/2$. We conclude

$$\lim_{N\to\infty} \sup_{\mathbb{Q}\in\hat{\mathcal{Q}}_N} \mathbb{E}_{\mathbb{Q}}[g] \geq \lim_{N\to\infty} \sup_{\mathbb{Q}\ll\nu_N, \mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[g] = 1 \neq 0 = \sup_{\mathbb{Q}\sim\mathbb{P}, \mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[g].$$

Let us remark that $\pi_{\hat{\mathcal{Q}}_N}(g)$ acting on infinite dimensional spaces is bounded by a more sophisticated version of the plug-in estimator. To see this, define the average value at risk of g at level $1/k$, for $k \geq 1$, by

$$AV@R^{\mathbb{P}}_{1/k}(g) = \max_{\tilde{\mathbb{P}}\sim\mathbb{P}, \|d\tilde{\mathbb{P}}/d\mathbb{P}\|_\infty \leq k} \mathbb{E}_{\tilde{\mathbb{P}}}[g].$$

In dimension one, $d = 1$, it can be re-expressed (see [13], Theorem 4.47), as

$$AV@R^{\mathbb{P}}_{1/k}(g) := k\int_{1-1/k}^1 F^{-1}_{\mathbb{P}\circ g^{-1}}(x)\,dx,$$

which makes the link with the classical value-at-risk apparent. If we now include the ability to trade and optimise the final position, by the translation-invariance of $AV@R^{\mathbb{P}}_{1/k}(\cdot)$ we can write

$$\begin{aligned}&\inf_{H\in\mathbb{R}^d} AV@R^{\mathbb{P}}_{1/k}\big(g(r)-H(r-1)\big)\\&\quad=\inf\big\{x\in\mathbb{R}|\exists H\in\mathbb{R}^d \text{ s.t. } AV@R^{\mathbb{P}}_{1/k}\big(g(r)-H(r-1)-x\big)\le 0\big\},\end{aligned}$$

which is a superhedging price, where the acceptance cone is given by an $AV@R$ constraint. An analogous representation and bounds for $\pi_{\hat{\mathcal{Q}}_N}$ follow.

COROLLARY 3.11. *In the setup of Theorem 3.6, let g be 1-Lipschitz and $\mathbb{P}\in\mathcal{P}(\mathbb{R}^d_+)$ satisfying* NA($\mathbb{P}$). *Then there exists $N_0\in\mathbb{N}$ such that for all $N\ge N_0$,*

$$\begin{aligned}&\inf_{H\in\mathbb{R}^d} AV@R^{\hat{\mathbb{P}}_N}_{1/k_N}\big(g(r)-H(r-1)\big)\\&\quad\le \inf_{H\in\mathbb{R}^d}\sup_{\tilde{\mathbb{P}}\in B^p_{\varepsilon_N(\beta_N)}(\hat{\mathbb{P}}_N)} AV@R^{\tilde{\mathbb{P}}}_{1/k_N}\big(g(r)-H(r-1)\big)=\pi_{\hat{\mathcal{Q}}_N}(g)\\&\quad=\inf\Big\{x\in\mathbb{R}|\exists H\in\mathbb{R}^d \text{ s.t.}\\&\qquad\sup_{\tilde{\mathbb{P}}\in B^p_{\varepsilon_N(\beta_N)}(\hat{\mathbb{P}}_N)} AV@R^{\tilde{\mathbb{P}}}_{1/k_N}\big(g(r)-H(r-1)-x\big)\le 0\Big\}\qquad(3.8)\\&\quad\le \inf_{|H|\le 1} AV@R^{\hat{\mathbb{P}}_N}_{1/k_N}\big(g(r)-H(r-1)\big)+2k_N\varepsilon_N(\beta_N)\end{aligned}$$

on a set of probability greater or equal than $1-\beta_N$.

Note that

$$\pi_{\hat{\mathcal{Q}}_N}(g)=\sup_{\tilde{\mathbb{P}}\in B^p_{\varepsilon_N(\beta_N)}(\hat{\mathbb{P}}_N)}\ \sup_{\|d\nu/d\tilde{\mathbb{P}}\|_\infty\le k_N}\ \inf_{H\in\mathbb{R}^d}\mathbb{E}_\nu\big[g-H(r-1)\big].$$

The proof proceeds by using a min-max argument to interchange the two suprema and the infimum above and uses continuity of $\tilde{\mathbb{P}}\mapsto AV@R^{\tilde{\mathbb{P}}}_{1/k_N}(g-H(r-1))$ w.r.t. to $\mathcal{W}^1$; see [41] and [38], Section B.2, for details.

We provide a method for the direct calculation of the Wasserstein estimator implemented in TensorFlow,[2] which is based on recent duality results obtained in [10]. As this approximation is computationally quite costly when a large sample size is used, we opt to compute the upper bound of $\pi_{\hat{\mathcal{Q}}_N}(g)$ given in (3.8) instead. This is shown in Figure 1.

3.3. *A penalty approach*: *Estimator for discontinuous payoffs*. In the previous section, we introduced the estimator $\pi_{\hat{\mathcal{Q}}_N}$ where $\hat{\mathcal{Q}}_N$ were based on Wasserstein balls around $\hat{\mathbb{P}}_N$. This estimator allowed us to address fundamental shortcomings of the plug-in estimator but, as the counterexamples demonstrated, it is only asymptotically consistent under suitable regularity assumptions on g and/or further assumptions on $\mathbb{P}$. To construct an estimator which would be consistent also for discontinuous payoffs while preserving some of the desirable robustness properties of $\pi_{\hat{\mathcal{Q}}_N}$, it is natural to turn to penalty methods used in risk measures and their representations as nonlinear expectations. Namely, we use the maximum norm of the Radon–Nikodym derivative, rather than the Wasserstein distance, in the penalisation term.

[2]Our Python implementation for all of the numerical examples in the paper can be found at https://github.com/johanneswiesel/Stats-for-superhedging.

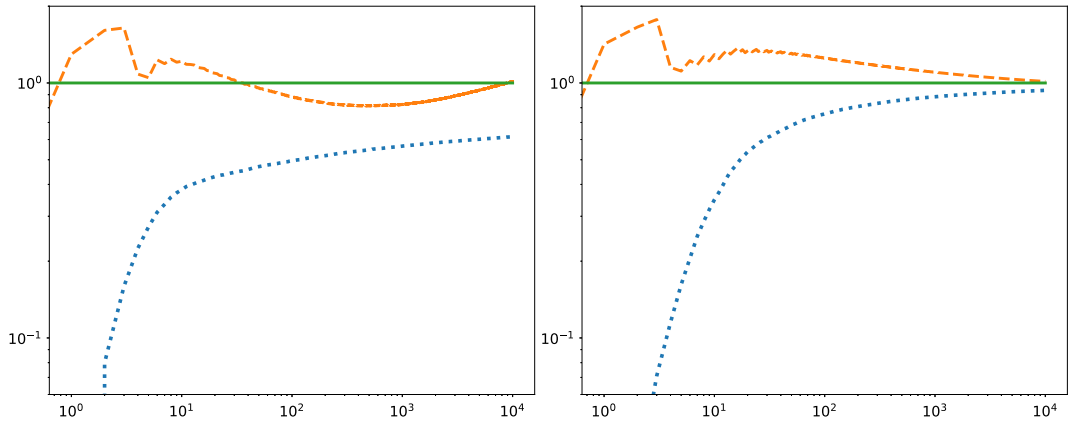

FIG. 1. *Convergence of the plug-in estimator* $\hat{\pi}_N$ *(dotted) and the Wasserstein estimator* $\pi_{\hat{\mathcal{Q}}_N}$ *(dashed) to the true value (solid) as* $N \to \infty$ *for* $g(r) = (1-r)\mathbb{1}_{\{r\leq 1\}} - \sqrt{r-1}\mathbb{1}_{\{r>1\}}$, $\mathbb{P} = \mathrm{Exp}(1)$ *(left) and* $g(r) = (r-2)^+$, $\mathbb{P} = \exp(\mathcal{N}(0,1))$ *(right). Results averaged over* 10^3 *runs.*

THEOREM 3.12. *In the setting of Theorem 2.1, let* NA($\mathbb{P}$) *hold and let* $g : \mathbb{R}^d_+ \to \mathbb{R}_+$ *be Borel-measurable and bounded by some constant* $C > 0$. *Then for any* $C_N \overset{N\to\infty}{\longrightarrow} C$ *we have*

$$\lim_{N\to\infty} \sup_{\mathbb{Q}\in\mathcal{M}} \left(\mathbb{E}_{\mathbb{Q}}[g] - C_N \left(\inf_{\hat{\mathbb{Q}}\sim\hat{\mathbb{P}}_N, \hat{\mathbb{Q}}\in\mathcal{M}} \left\| \frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}} \right\|_\infty - 1 \right) \right) = \sup_{\mathbb{Q}\sim\mathbb{P}, \mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[g] \tag{3.9}$$

$\mathbb{P}^\infty$*-a.s., where for two probability measures* $\mathbb{Q}, \hat{\mathbb{Q}}$ *the expression* $\|\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}}\|_\infty = \infty$ *if* $\hat{\mathbb{Q}}$ *is not absolutely continuous w.r.t.* $\mathbb{Q}$.

The direct implementation of (3.9) proves numerically expensive and unstable due to the fraction $\|d\hat{\mathbb{Q}}/d\mathbb{Q}\|_\infty$ appearing in the penalisation term. Thus, in Figure 2, we show an upper bound on the penalty estimator derived in the proof of Theorem 3.12 in [38], Section B.2. We focus on more tractable properties of the plug-in and Wasserstein estimator for the rest of the paper.

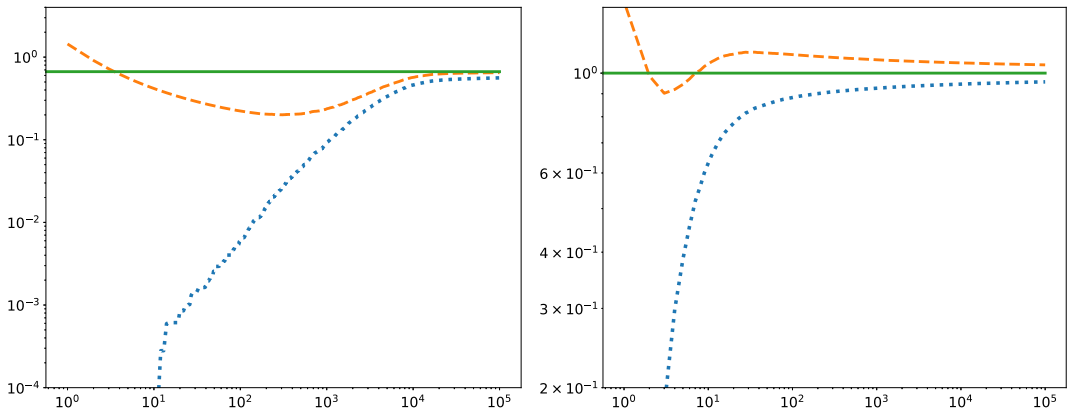

FIG. 2. *Convergence of the penalty estimator (dashed) in Theorem 3.12 and the plugin estimator* $\hat{\pi}_N$ *(dotted) to the true value (solid) as* $N \to \infty$ *for* $g(r) = \mathbb{1}_{\{r\leq 0.5\}}$, $\mathbb{P} = \mathbb{P}^{10}$ *from* [38], *Example* B.23, *(left) and* $g(r) = \mathbb{1}_{\{r\leq 0.5\}}$, $\mathbb{P} = \mathrm{Exp}(1)$ *(right). Results averaged over* 10^3 *runs.*

4. Statistical robustness of superhedging price estimators. Recall that in Theorem 2.7 we showed that classical robustness in the sense of Hampel cannot hold unless the superhedging price is trivial in that (2.6) holds. This is closely related to properties of Lévy–Prokhorov metric (see Lemma A.1), and we are naturally led to consider stronger metrics than d_L, which offer a better control on the support. Below, we investigate the use of Wasserstein distances. First, we consider $\mathcal{W}^p$ for $p \geq 1$ which is sufficient to establish robustness of the estimator $\pi_{\hat{\mathcal{Q}}_N}$ from Section 3.2. Then we turn to an even stronger metric $\mathcal{W}^\infty$ which is needed to study the plug-in estimator.

4.1. *Robustness with respect to the Wasserstein–Hausdorff metric.* Following [32], we consider Wasserstein–Hausdorff distance, that is, a Hausdorff distance between subsets of $\mathcal{P}(\mathbb{R}^d_+)$ equipped with $\mathcal{W}^p$.

DEFINITION 4.1. Let $\mathfrak{P}, \tilde{\mathfrak{P}} \subseteq \mathcal{P}(\mathbb{R}^d_+)$. The p-Wasserstein–Hausdorff metric between sets $\mathfrak{P}$ and $\tilde{\mathfrak{P}}$ is given by

$$\mathcal{W}^p(\mathfrak{P}, \tilde{\mathfrak{P}}) := \max\Big(\sup_{\mathbb{P}\in\mathfrak{P}} \inf_{\tilde{\mathbb{P}}\in\tilde{\mathfrak{P}}} \mathcal{W}^p(\mathbb{P}, \tilde{\mathbb{P}}), \sup_{\tilde{\mathbb{P}}\in\tilde{\mathfrak{P}}} \inf_{\mathbb{P}\in\mathfrak{P}} \mathcal{W}^p(\mathbb{P}, \tilde{\mathbb{P}}) \Big).$$

In this generality, $\mathcal{W}^p(\mathfrak{P}, \tilde{\mathfrak{P}})$ can take the value infinity. Properties of this quantity are discussed in [32] assuming compactness and uniform integrability of $\mathfrak{P}$ and $\tilde{\mathfrak{P}}$. We apply this distance to the sets of the form $\hat{\mathcal{Q}}_N = D^p_{\varepsilon_N(\beta_N),k_N}(\hat{\mathbb{P}}_N)$ (see (3.4)), and we note that $D^1_{\varepsilon_N(\beta_N),k_N}(\hat{\mathbb{P}}_N)$ is neither compact nor uniformly integrable; see Lemma A.3. We used these sets in Section 3.2 to define consistent estimators $\pi_{\hat{\mathcal{Q}}_N}$; see (3.2) and Theorem 3.6. The following establishes their robustness.

THEOREM 4.2. *Fix $p \geq 1$. The estimator $\pi_{\hat{\mathcal{Q}}_N}$ studied in Theorem 3.6 is robust with respect to the p-Wasserstein–Hausdorff metric in the sense that*

$$\sup_{g\in\mathcal{L}_1} |\pi_{\hat{\mathcal{Q}}^1_N}(g) - \pi_{\hat{\mathcal{Q}}^2_N}(g)| \leq \mathcal{W}^p(\hat{\mathcal{Q}}^1_N, \hat{\mathcal{Q}}^2_N),$$

where $\hat{\mathcal{Q}}^i_N = D^p_{\varepsilon_N(\beta_N),k_N}(\hat{\mathbb{P}}^i_N)$ for $\mathbb{P}^i \in \mathcal{P}(\mathbb{R}^d_+)$, $i = 1, 2$.

PROOF. Note that for all $g \in \mathcal{L}_1$ and $\mathbb{Q}^i \in \hat{\mathcal{Q}}^i_N$, $i = 1, 2$, we have

$$\mathbb{E}_{\mathbb{Q}^1}[g] - \mathbb{E}_{\mathbb{Q}^2}[g] = \int_{\mathbb{R}^d_+\times\mathbb{R}^d_+} g(r) - g(s)\, d\gamma(r, s) \leq \int_{\mathbb{R}^d_+\times\mathbb{R}^d_+} |r - s|\, d\gamma(r, s),$$

where $\gamma \in \Pi(\mathbb{Q}^1, \mathbb{Q}^2)$ is a probability measure on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$ with marginals $\mathbb{Q}^1$ and $\mathbb{Q}^2$. Taking the infimum over all these probability measures γ yields

$$\mathbb{E}_{\mathbb{Q}^1}[g] - \mathbb{E}_{\mathbb{Q}^2}[g] \leq \mathcal{W}^p(\mathbb{Q}^1, \mathbb{Q}^2)$$

for all $p \geq 1$. The claim follows. □

REMARK 4.3. It follows in particular that if $\mathbb{P}^1, \mathbb{P}^2$ admit no arbitrage then $\lim_{N\to\infty} \mathcal{W}^p(\hat{\mathcal{Q}}^1_N, \hat{\mathcal{Q}}^2_N) = 0$ implies $\operatorname{supp}(\mathbb{P}^1) = \operatorname{supp}(\mathbb{P}^2)$. Indeed, otherwise there exists a Lipschitz continuous function g such that

$$\sup_{\mathbb{Q}\sim\mathbb{P}^1, \mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[g] \neq \sup_{\mathbb{Q}\sim\mathbb{P}^2, \mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[g],$$

so, by consistency, $\lim_{N\to\infty} \mathcal{W}^p(\hat{\mathcal{Q}}^1_N, \hat{\mathcal{Q}}^2_N) > 0$, $\mathbb{P}^\infty$-a.s.

4.2. *Robustness with respect to $\mathcal{W}^\infty$ and perturbations of the support.* We reconsider now robustness of the plug-in estimator from Section 2. In analogy to the previous section, it seems natural to simply consider the Hausdorff distance between the supports of $\hat{\mathbb{P}}_N^1$ and $\hat{\mathbb{P}}_N^2$. In Proposition 2.8, we established continuity of $\mathcal{P}(\mathbb{R}_+^d) \ni \tilde{\mathbb{P}} \to \pi^{\tilde{\mathbb{P}}}(g)$ in the pseudometric $d_H(\mathrm{supp}(\mathbb{P}), \mathrm{supp}(\tilde{\mathbb{P}}))$ but noted that it was not sufficient for a robustness result. Recalling the Lévy metric in (2.5), if $d_L(\mathbb{P}, \tilde{\mathbb{P}}) \le \varepsilon$ then $\tilde{\mathbb{P}}$ can be obtained from $\mathbb{P}$ by redistributing ε mass to arbitrary points on $\mathbb{R}_+^d$, while $(1-\varepsilon)$ mass can only be moved in an ε-neighbourhood (in the Euclidean distance) of where $\mathbb{P}$ allocated mass. As we have observed before, the former operation causes problems, as it changes the null sets of the measure uncontrollably. This is no longer possible under our pseudo-metric. However, to obtain robustness, we have to restrict redistribution of mass to an ε-neighbourhood for all sets and not only for the whole support. This is achieved by the $\mathcal{W}^\infty$ metric as is clear from the second representation in (3.1). This leads to the following extended notion of robustness.

DEFINITION 4.4. Let $\mathfrak{P} \subseteq \mathcal{P}(\mathbb{R}_+^d)$ and $r_1, r_2, \ldots$ be i.i.d. with distribution $\mathbb{P} \in \mathfrak{P}$. The sequence of estimators $T_N = T(\hat{\mathbb{P}}_N)$ is said to be robust at $\mathbb{P} \in \mathfrak{P}$ w.r.t. $\mathcal{W}^\infty$ on $\mathfrak{P}$, if for all $\varepsilon > 0$ there exist $\delta > 0$ and $N_0 \in \mathbb{N}$ such that for all $N \ge N_0$ and all $\tilde{\mathbb{P}} \in \mathfrak{P}$,

$$\mathcal{W}^\infty(\tilde{\mathbb{P}}, \mathbb{P}) \le \delta \quad \Longrightarrow \quad d_L\big(\mathcal{L}_{\tilde{\mathbb{P}}}(T_N), \mathcal{L}_{\mathbb{P}}(T_N)\big) \le \varepsilon.$$

The following asserts robustness of the plug-in estimator in the above sense and is the main result in this section.

THEOREM 4.5. *Let $\mathbb{P} \in \mathcal{P}(\mathbb{R}_+^d)$ such that* NA($\mathbb{P}$) *holds and* span(supp($\mathbb{P}$)) $= \mathbb{R}^d$. *Then, for a uniformly continuous g, the plug-in estimator $\hat{\pi}_N(g)$ is robust at $\mathbb{P}$ w.r.t. $\mathcal{W}^\infty$ on $\mathcal{P}(\mathbb{R}_+^d)$.*

The proof of Theorem 4.5 is reported in [38], Section B.3. There are ways to weaken the continuity assumption on g and obtain robustness on some $\mathfrak{P} \subseteq \mathcal{P}(\mathbb{R}_+^d)$; see [38], Corollary B.7. We close this section with a result on robustness of $\hat{\pi}_N^\infty(g)$ from Proposition 3.3.

COROLLARY 4.6. *Let g be a continuous function and $\mathbb{P} \in \mathcal{P}(\mathbb{R}_+^d)$ satisfying* NA($\mathbb{P}$) *and Assumption 3.1. Then the estimator $\hat{\pi}_N^\infty(g)$ from Proposition 3.3 is robust at $\mathbb{P}$ w.r.t. $\mathcal{W}^\infty$ on $\mathcal{P}(\mathbb{R}_+^d)$.*

5. Risk measurement estimation. The $\mathbb{P}$-a.s. superhedging price $\pi^{\mathbb{P}}(g)$ is a very conservative assessment of risk of a short position in a liability with payoff g. Instead, we could use a risk measure $\rho_{\mathbb{P}}$ for such an assessment, as proposed by [2], leading to

$$\pi^{\rho_{\mathbb{P}}}(g) := \inf\big\{x \in \mathbb{R} | \exists H \in \mathbb{R}^d \text{ such that } \rho_{\mathbb{P}}\big(g - x - H(r-1)\big) \le 0\big\}.$$

Note that we include above the ability to trade in the market in order to (optimally) reduce the risk of g. We consider $\rho_{\mathbb{P}}$ with Kusuoka's representation

$$\rho_{\mathbb{P}}(g) = \sup_{\mu \in \mathfrak{P}} \int_0^1 \mathrm{AV@R}_\alpha^{\mathbb{P}}(g)\, d\mu(\alpha) \tag{5.1}$$

for a set $\mathfrak{P}$ of probability measures on $[0,1]$. This is not very restrictive since this representation, first obtained in [31], holds for any law invariant coherent risk measure; see [23]. Importantly, it enables us to think of $\rho_{\mathbb{P}}(g)$ as a function of the underlying measure $\mathbb{P}$. Much

like we did for $\pi^{\mathbb{P}}(g)$, we would like to estimate $\pi^{\rho_{\mathbb{P}}}(g)$ directly from the observed stock returns. To this end, we introduce the following estimator:

$$\pi^{\rho}_{B^p_{\varepsilon_N(\beta_N)}(\hat{\mathbb{P}}_N)}(g) := \inf\Big\{x \in \mathbb{R} \big| \exists H \in \mathbb{R}^d \text{ such that } \sup_{\tilde{\mathbb{P}} \in B^p_{\varepsilon_N(\beta_N)}(\hat{\mathbb{P}}_N)} \rho_{\tilde{\mathbb{P}}}\big(g - x - H(r-1)\big) \le 0\Big\}$$

as the natural equivalent to $\pi_{\hat{\mathcal{Q}}_N}(g)$. In particular, if $\mathfrak{P} = \{\delta_\alpha\}$ where $\alpha \in [0,1]$, we simply have $\rho_{\mathbb{P}}(g) = \mathrm{AV@R}^{\mathbb{P}}_\alpha(g)$ and the corresponding estimator $\pi^{\rho}_{B^p_{\varepsilon_N(\beta_N)}(\hat{\mathbb{P}}_N)}(g)$ resembles the Wasserstein $\mathcal{W}^p$ estimator for fixed level $1/k_N := \alpha$. We have the following consistency result.

PROPOSITION 5.1. *Assume g satisfies $|g(r) - g(\tilde{r})| \le L|r - \tilde{r}|$ for some $L \in \mathbb{R}$ and that $\sup_{\mu\in\mathfrak{P}} \int_0^1 \mu(d\alpha)/\alpha^{1/p} < \infty$. Then for any $\mathbb{P}$ satisfying* NA($\mathbb{P}$) *and Assumption 3.5 the limit in $\mathbb{P}^\infty$-probability*

$$\lim_{n\to\infty} \pi^{\rho}_{B^p_{\varepsilon_N(\beta_N)}(\hat{\mathbb{P}}_N)}(g) = \pi^{\rho_{\mathbb{P}}}(g)$$

holds. If Assumption 3.5.2 is satisfied, then the limit also holds $\mathbb{P}^\infty$-a.s.

PROOF. The "$\ge$"-inequality follows in the proof of Theorem 3.6. We now prove the opposite inequality using [41], Corollary 11, p. 538. Fix $\varepsilon > 0$. Note that there exists $H \in \mathbb{R}^d$ such that $\rho_{\mathbb{P}}(g - \pi^{\rho_{\mathbb{P}}}(g) - \varepsilon - H(r-1)) \le 0$. Then for all $\tilde{\mathbb{P}} \in B^p_{\varepsilon_N(\beta_N)}(\hat{\mathbb{P}}_N)$,

$$\begin{aligned}
\pi^{\rho_{\tilde{\mathbb{P}}}}(g) &\le \rho_{\tilde{\mathbb{P}}}\big(g - H(r-1)\big) = \rho_{\tilde{\mathbb{P}}}\big(g - \pi^{\rho_{\mathbb{P}}}(g) - \varepsilon - H(r-1)\big) + \pi^{\rho_{\mathbb{P}}}(g) + \varepsilon \\
&= \big[\rho_{\tilde{\mathbb{P}}}\big(g - \pi^{\rho_{\mathbb{P}}}(g) - \varepsilon - H(r-1)\big) - \rho_{\mathbb{P}}\big(g - \pi^{\rho_{\mathbb{P}}}(g) - \varepsilon - H(r-1)\big)\big] \\
&\quad + \rho_{\mathbb{P}}\big(g - \pi^{\rho_{\mathbb{P}}}(g) - \varepsilon - H(r-1)\big) + \pi^{\rho_{\mathbb{P}}}(g) + \varepsilon \\
&\le \tilde{L}\mathcal{W}^p(\tilde{\mathbb{P}}, \mathbb{P}) \sup_{\mu\in\mathfrak{P}} \int_0^1 \frac{\mu(d\alpha)}{\alpha^{1/p}} + \pi^{\rho_{\mathbb{P}}}(g) + \varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, the claim follows. □

We note that, in analogy to the above, $\sup_{\tilde{\mathbb{P}} \in B^p_{\varepsilon_N(\beta_N)}(\hat{\mathbb{P}}_N)} \rho_{\tilde{\mathbb{P}}}(g)$ offers a natural consistent estimator for $\rho_{\mathbb{P}}(g)$.

Finally, we present a simple empirical test of the performance of our estimators. We simulate weekly returns according to a GARCH(1, 1) model:

$$r_n = \sqrt{\frac{\mu - 2}{\mu}}\eta_n\sqrt{h_n}, \quad h_n = \omega + \beta h_{n-1} + \alpha r^2_{n-1},$$

where $\omega = 0.02$, $\beta = 0.1$, $\alpha = 0.8$ and η_n is standard student-t distributed with $\mu = 5$ degrees of freedom. We take 1000 samples from the above $\mathbb{P} \sim$ GARCH(1, 1) and calculate the plug-in estimator $\pi^{\mathrm{AV@R}_{0.95}}_{\hat{\mathbb{P}}_N}((r-1)^+)$ and the Wasserstein estimator $\pi^{\mathrm{AV@R}_{0.95}}_{B^p_{\varepsilon_N(\beta_N)}(\hat{\mathbb{P}}_N)}((r-1)^+)$. We compare this to a parametric estimator of $\pi^{\mathrm{AV@R}^{\mathbb{P}}_{0.95}}((r-1)^+)$, where we first estimate the parameters of the GARCH(1, 1) model and then compute $\pi^{\mathrm{AV@R}^{\tilde{\mathbb{P}}}_{0.95}}((r-1)^+)$ given the estimated model $\tilde{\mathbb{P}}$. For each of these estimates, we use a running window of 50 weeks, which is in line with the Basel III regulations for calculating the 10-day AV@R (see [40],

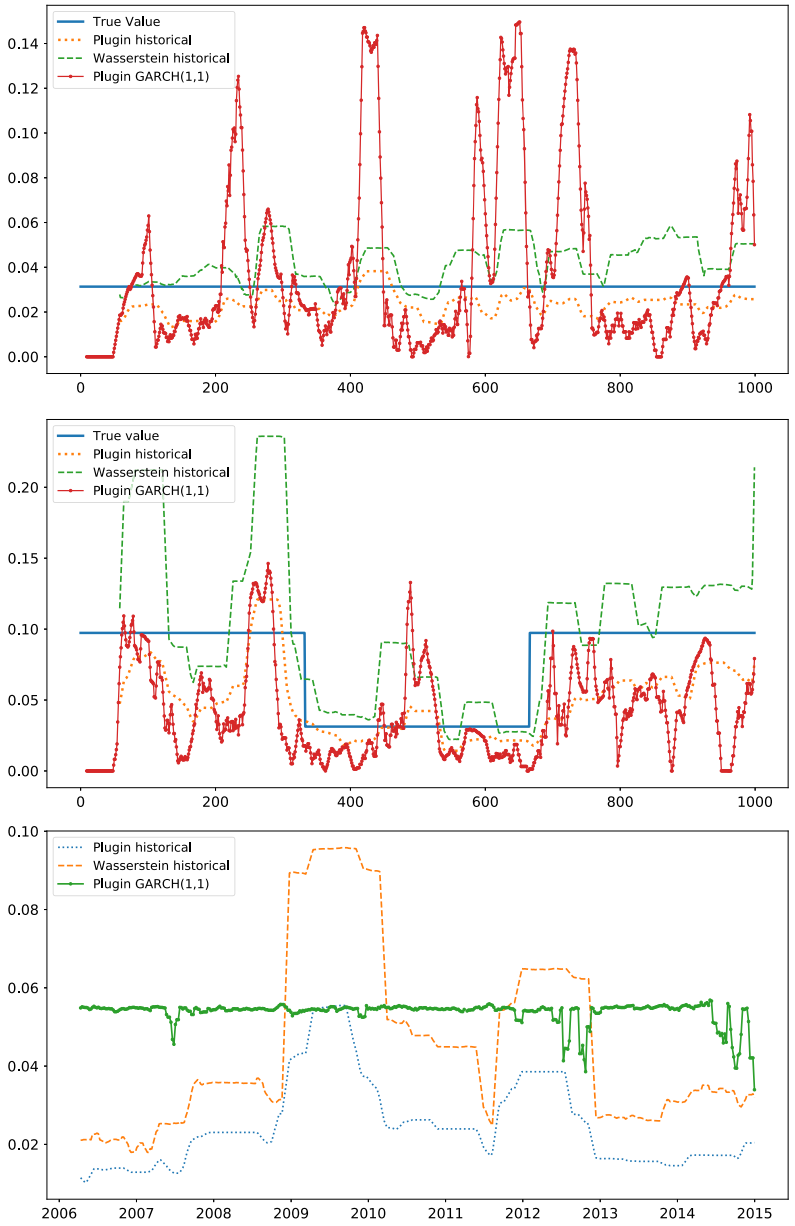

FIG. 3. *Comparison of estimates for* $\pi^{\mathrm{AV@R}^{\tilde{\mathbb{P}}}_{0.95}}((r-1)^+)$. *Estimates use a rolling window of* 50 *data points and we plot the average of the last* 10 (*first two panes*) *or* 5 (*last pane*) *estimates. The data is from* $\mathbb{P} \sim \mathrm{GARCH}(1,1)$ (*first pane*) *and its variant with a change in the parameters for the middle third of the time series. The last pane uses S&P*500 *returns.*

MAR33, p. 89), which set the minimum length of the historical observation period to be one year. We also consider the case when the parameters of the model change for observations 330–670. The behaviour of the three estimators is shown in Figure 3. Both the Wasserstein and plug-in estimator approximate the true value reasonably well—the Wasserstein estimator being the most conservative estimator. The parametric estimator exhibits the most erratic behaviour which is due to the unstable estimation of the GARCH(1, 1) parameters with only 50 data points. This shows advantages of our proposed estimators when compared with a parametric approach, even when the model is correctly selected. The last pane in Figure 3 uses S&P500 weekly returns data from 01/01/2006–01/01/2015 with a moving window of 50

weeks. The GARCH(1, 1) estimator, for which the model is mis-specified, does not pick up any markets movements, while both the plug-in and Wasserstein estimator detect the financial crisis and its aftermath. For similar plots but with GARCH(1, 1) using log-returns, we refer to [38], Section B.4. While preliminary, we believe that this simple empirical study points to clear advantages of our approach. In particular, it is encouraging to see that in the last pane, the Wasserstein estimators clearly picks up the financial crisis and the Eurozone debt crisis periods. A further in-depth study of comparative performance of different estimators is clearly needed and is left for future research.

APPENDIX A: ADDITIONAL RESULTS AND PROOFS

A.1. Additional results and proofs for Section 2. PROOF OF THEOREM 2.1. First, note that if A_n is a nondecreasing sequence of sets with $A = \lim_n A_n = \bigcup_n A_n$ then $\hat{g}_A = \lim_n \hat{g}_{A_n}$. The "$\geq$" inequality is obvious and the reverse follows since $\hat{g}_{A_n}$ is a nondecreasing sequence of concave functions thus its limit is a concave function dominating g on A.

Using Lusin's theorem (see [4], Theorem 7.4.3, p. 227), we can find an increasing sequence K_n of compact subsets of $\operatorname{supp}(\mathbb{P})$ such that $\mathbb{P}(\mathbb{R}^d_+ \setminus K_n) \leq 1/n$ and $g|_{K_n}$ is continuous. Continuity of g on K_n implies that $\hat{g}_{K_n} = \hat{g}_{\mathbb{P}|_{K_n}} \leq \hat{g}_{\mathbb{P}}$. On the other hand, by the argument above, $\lim_n \hat{g}_{K_n} = \hat{g}_{\bigcup_n K_n} \geq \hat{g}_{\mathbb{P}}$ since $\mathbb{P}(\bigcup_n K_n) = 1$. We conclude that $\lim_n \hat{g}_{K_n} = \hat{g}_{\mathbb{P}}$. Further, by Birkhoff's ergodic theorem (see [24], Theorem 9.6, p. 159) and $\mathbb{P}_1 \ll \mathbb{P}$ we have

$$\bigcup_N \operatorname{supp}(\hat{\mathbb{P}}^N) = \{r_1, r_2, \ldots\} \quad \text{is a.s. dense in } \operatorname{supp}(\mathbb{P})$$

and hence $\hat{g}_{K_n \cap \{r_1, r_2, \ldots\}} = \hat{g}_{K_n}$ a.s. By the argument above, we thus have

$$\lim_{N\to\infty} \hat{g}_{\hat{\mathbb{P}}_N} = \hat{g}_{\{r_1,\ldots\}} = \hat{g}_{\bigcup_n K_n \cap \{r_1,\ldots\}} = \lim_{n\to\infty} \hat{g}_{K_n} = \hat{g}_{\mathbb{P}} \quad \mathbb{P}^\infty\text{-a.s.},$$

where the second equality follows since the inclusion $\{r_1, r_2, \ldots\} \subset \bigcup_n K_n$ holds $\mathbb{P}^\infty$-a.s. We conclude using (2.3). $\square$

LEMMA A.1. *Let $\mathbb{P} \in \mathcal{P}(\mathbb{R}^d_+)$ with finite first moment and $\varepsilon > 0$. Then for all $x \in \mathbb{R}^d_+$ the ball $B^1_\varepsilon(\mathbb{P})$ in the 1-Wasserstein metric contains $\lambda\delta_x + (1-\lambda)\mathbb{P}$ for some $\lambda \in (0, 1)$. In particular, any $\mathbb{P} \in \mathcal{P}(\mathbb{R}^d_+)$ can be written as a weak limit of probability measures $\mathbb{P}^N$ with $\operatorname{supp}(\mathbb{P}^N) = \mathbb{R}^d_+$.*

PROOF OF THEOREM 2.7. Fix $\mathbb{P} \in \mathcal{P}(\mathbb{R}^d_+)$ and a sequence $\mathbb{P}^N$ converging to $\mathbb{P}$. Let $\{r_1, r_2, \ldots\}$ be dense in $\operatorname{supp}(\mathbb{P})$. Fix $n \geq 1$ and note that, for any $i \geq 1$, weak convergence implies that $\mathbb{P}^N(B_{1/n}(r_i)) > 0$ for all N large enough. In particular, there exists $r_i^n \in B_{1/n}(r_i)$ such that $\hat{g}_{\mathbb{P}^N}(r_i^n) \geq g(r_i^n)$. Thus, by the same reasoning as in the proof of Theorem 2.1 above,

$$\liminf_{\mathbb{P}^N \Rightarrow \mathbb{P}} \pi^{\mathbb{P}^N}(g) = \liminf_{N\to\infty} \hat{g}_{\mathbb{P}^N}(1) \geq \lim_{n\to\infty} \lim_{k\to\infty} \hat{g}_{\{r_1^n, r_2^n, \ldots, r_k^n\}}(1) = \pi^{\mathbb{P}}(g).$$

We conclude using Lemma A.1 since for a sequence with $\operatorname{supp}(\mathbb{P}^N) = \mathbb{R}^d_+$, by continuity of g, we have, for all $N \geq 1$,

$$\pi^{\mathbb{P}^N}(g) = \inf\{x \in \mathbb{R} | \exists H \in \mathbb{R}^d \text{ s.t. } x + H(r-1) \geq g \text{ on } \mathbb{R}^d_+\} = \sup_{\mathbb{Q}\in\mathcal{M}_{\mathbb{R}^d_+}} \mathbb{E}_{\mathbb{Q}}[g].$$

For the second part of the theorem, assume $\mathbb{P} \in \mathcal{P}(\mathbb{R}^d_+)$ is such that

$$\pi^{\mathbb{P}}(g) < \sup_{\mathbb{Q}\in\mathcal{M}_{\mathbb{R}^d_+}} \mathbb{E}_{\mathbb{Q}}[g] = \inf\{x \in \mathbb{R} | \exists H \in \mathbb{R}^d \text{ s.t. } x + H(r-1) \geq g \text{ on } \mathbb{R}^d_+\}.$$

Take a sequence $(\mathbb{P}^N)_{N\in\mathbb{N}}$, as above, with $\operatorname{supp}(\mathbb{P}^N) = \mathbb{R}^d_+$ and $\mathbb{P}^N \Rightarrow \mathbb{P}$. Fix $\varepsilon > 0$ such that

$$2\varepsilon < \pi^{\mathbb{P}^N}(g) - \pi^{\mathbb{P}}(g).$$

For every $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have $d_L(\mathbb{P}^N, \mathbb{P}) \leq \delta$. Let T_N be an asymptotically consistent estimator of $\pi^{\mathbb{P}}(g)$. Then, for all N large enough,

$$\begin{aligned} d_L\big(\mathcal{L}_{\mathbb{P}^{N_0}}(T_N), \mathcal{L}_{\mathbb{P}}(T_N)\big) &\geq d_L\big(\delta_{\pi^{\mathbb{P}}(g)}, \delta_{\pi^{\mathbb{P}^{N_0}}(g)}\big) - d_L\big(\mathcal{L}_{\mathbb{P}^{N_0}}(T_N), \delta_{\pi^{\mathbb{P}^{N_0}}(g)}\big) \\ &\quad - d_L\big(\delta_{\pi^{\mathbb{P}}(g)}, \mathcal{L}_{\mathbb{P}}(T_N)\big) \\ &\geq \varepsilon. \end{aligned}$$

Thus T_N is not robust at $\mathbb{P}$, which shows the claim. □

A.2. Additional results and proofs for Section 3. We now prove Theorem 3.6 under Assumption 3.5.2. We adopt the notation of Section 3.

LEMMA A.2 ([14], Theorem 2). *Under Assumption 3.5.2, we have*

$$\mathbb{P}^\infty\big(\mathcal{W}^p(\mathbb{P}, \hat{\mathbb{P}}_N) \geq \varepsilon\big) \leq \begin{cases} c_1 \exp\big(-c_2 N \varepsilon^{\min(\max(d/p,2),a/(2p))}\big) & \text{if } \varepsilon \leq 1, \\ c_1 \exp\big(-c_2 N \varepsilon^{a/(2p)}\big) & \text{if } \varepsilon > 1 \end{cases}$$

for $N \geq 1$, $d \neq 2p$ and $\varepsilon > 0$, where c_1, c_2 are positive constants that only depend on p, d, a, c and $\mathbb{E}_{\mathbb{P}}[\exp(c|r|^a)]$. Thus for some confidence level $\beta \in (0,1)$ we can choose

$$\varepsilon_N(\beta) := \begin{cases} \left(\dfrac{\log(c_1\beta^{-1})}{c_2 N}\right)^{1/\min(\max(d/p,2),a/(2p))} & \text{if } N \geq \dfrac{\log(c_1\beta^{-1})}{c_2}, \\ \left(\dfrac{\log(c_1\beta^{-1})}{c_2 N}\right)^{(2p)/a} & \text{if } N < \dfrac{\log(c_1\beta^{-1})}{c_2} \end{cases} \tag{A.1}$$

which yields $\mathbb{P}^\infty(\mathcal{W}^p(\mathbb{P}, \hat{\mathbb{P}}_N) \geq \varepsilon_N(\beta)) \leq \beta$.

LEMMA A.3. *Fix $N \in \mathbb{N}$ and $p \geq 1$. Let $\mathbb{Q}_n \in D^p_{\varepsilon_N(\beta_N),k_N}(\mathbb{P})$ such that $\mathbb{Q}_n \Rightarrow \mathbb{Q} \in \mathcal{P}(\mathbb{R}^d_+)$ for $n \to \infty$. Then $|E_{\mathbb{Q}}[r-1]| \leq K k_N \varepsilon_N(\beta_N)$ for some $K > 0$. $D^p_{\varepsilon_N(\beta_N),k_N}(\mathbb{P})$ is weakly compact for $p > 1$. In general, $D^1_{\varepsilon_N(\beta_N),k_N}(\mathbb{P})$ is not weakly closed.*

Taking the closure of $D^1_{\varepsilon_N(\beta_N),k_N}(\mathbb{P})$ would ensure compactness, but consistency of the estimator $\sup_{\mathbb{Q}\in\hat{\mathcal{Q}}_N} \mathbb{E}_{\mathbb{Q}}[g]$ in Theorem 3.6 would be lost in general since the closure might include nonmartingale measures. To see this, take for instance $g(r) = (r-1)$ in the example in the proof of Lemma A.3.

LEMMA A.4 ([43], Cor. 3.3). *For a measurable function g bounded from below*

$$\sup_{\|\frac{d\mathbb{Q}}{d\mathbb{P}}\|_\infty < \infty, \mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[g] = \sup_{\mathbb{Q}\sim\mathbb{P}, \mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[g].$$

PROOF OF THEOREM 3.6. Let us assume that Assumption 3.5.2 is satisfied. Note that the "≥"-inequality follows from Lemma A.4. Indeed, Lemma A.4 implies that for all $N \in \mathbb{N}$ there exists a martingale measure $\mathbb{Q}_N \sim \mathbb{P}$ with $\|d\mathbb{Q}_N/d\mathbb{P}\|_\infty \leq k_N$ such that

$$\sup_{\mathbb{Q}\sim\mathbb{P}, \mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[g] \leq \mathbb{E}_{\mathbb{Q}_N}[g] + a_N,$$

with $a_N \to 0$ as $N \to \infty$. With $\mathbb{P}^\infty$-probability $(1-\beta_N)$ we have $\mathbb{P} \in B^p_{\varepsilon_N(\beta_N)}(\hat{\mathbb{P}}_N)$, and hence $\mathbb{Q}_N \in \hat{\mathcal{Q}}_N$. This gives

$$\mathbb{P}^\infty\Big(\sup_{\mathbb{Q}\sim\mathbb{P},\mathbb{Q}\in\mathcal{M}} \mathbb{E}_\mathbb{Q}[g] - a_N \le \sup_{\mathbb{Q}\in\hat{\mathcal{Q}}_N} \mathbb{E}_\mathbb{Q}[g]\Big) \ge 1-\beta_N, \quad N \ge 1.$$

We recall that NA($\mathbb{P}$) gives $\pi^\mathbb{P}(g) = \sup_{\mathbb{Q}\sim\mathbb{P},\mathbb{Q}\in\mathcal{M}} \mathbb{E}_\mathbb{Q}[g]$ and hence, for any $\varepsilon > 0$ and N large enough, $\mathbb{P}^\infty(\pi_{\hat{\mathcal{Q}}_N}(g) - \pi^\mathbb{P}(g) \le -\varepsilon) \le \beta_N \to 0$ as $N \to \infty$.

For the "$\le$"-inequality, we assume g is Lipschitz continuous bounded from below and we take $H \in \mathbb{R}^d$ such that

$$\pi^\mathbb{P}(g) + H(r-1) \ge g \quad \mathbb{P}\text{-a.s.}$$

Take a sequence $\mathbb{Q}_N \in \hat{\mathcal{Q}}_N$ with $\mathbb{E}_{\mathbb{Q}_N}[g] \ge \pi_{\hat{\mathcal{Q}}_N}(g) - a_N$. By definition, there exist $\nu_N \in B^p_{\varepsilon_N(\beta_N)}(\hat{\mathbb{P}}_N)$ such that $\|d\mathbb{Q}_N/d\nu_N\|_\infty \le k_N$. In particular, with $\mathbb{P}^\infty$-probability $(1-\beta_N)$ we have $\nu_N \in B^p_{2\varepsilon_N(\beta_N)}(\mathbb{P})$. Let us define

$$A = \{\pi^\mathbb{P}(g) + H(r-1) - g \ge 0\}.$$

Then

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_N}[g] &\le \mathbb{E}_{\mathbb{Q}_N}\big[(\pi^\mathbb{P}(g) + H(r-1))\mathbb{1}_A + g\mathbb{1}_{A^c}\big] \\ &= \pi^\mathbb{P}(g) + \mathbb{E}_{\mathbb{Q}_N}\big[H(r-1)\big] + \mathbb{E}_{\mathbb{Q}_N}\big[(g - H(r-1) - \pi^\mathbb{P}(g))\mathbb{1}_{A^c}\big]\end{aligned} \tag{A.2}$$

and the second term on the RHS vanishes since $\mathbb{Q}_N$ is a martingale measure. To treat the last term on the RHS consider the function

$$\tilde{g} := (g - H(r-1) - \pi^\mathbb{P}(g)) \vee 0$$

which is nonnegative, C-Lipschitz for some $C > 0$ and $\{\tilde{g} > 0\} = A^c$. Since $\mathbb{P}(A^c) = 0$, we have in particular

$$\left|\int_{A^c} \tilde{g}\, d\nu_N\right| = \left|\int_{A^c} \tilde{g}\, d\nu_N - \int_{A^c} \tilde{g} d\mathbb{P}\right| = \left|\int \tilde{g}\, d\nu_N - \int \tilde{g}\, d\mathbb{P}\right|$$

which by the Kantorovitch–Rubinstein duality (3.3) is dominated by $C\mathcal{W}^1(\nu_N, \mathbb{P}) \le C\mathcal{W}^p(\nu_N, \mathbb{P})$. We conclude that, for any $\varepsilon > 0$,

$$\mathbb{P}^\infty\big(\pi_{\hat{\mathcal{Q}}_N}(g) - \pi^\mathbb{P}(g) \ge \varepsilon\big) \le \mathbb{P}^\infty\big(a_N + Ck_N\mathcal{W}^p(\nu_N, \mathbb{P}) \ge \varepsilon\big) \le \beta_N$$

for N large enough since $\varepsilon_N k_N \to 0$. This establishes the convergence of $\pi_{\hat{\mathcal{Q}}_N}(g)$ to $\pi^\mathbb{P}(g)$ in $\mathbb{P}^\infty$-probability. Further, whenever $\sum_{N=1}^\infty \beta_N < \infty$, a simple application of Borel–Cantelli lemma, similarly as in [34], Lemma 3.7, shows that the convergence holds $\mathbb{P}^\infty$-a.s. This concludes the proof in the case of Lipschitz continuous g bounded from below and under Assumption 3.5.2. The remaining arguments, in particular the case when g is bounded and continuous, are given in [38], Section B.2. □

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SUPPLEMENTARY MATERIAL

Supplement to "Robust estimation of superhedging prices" (DOI: 10.1214/20-AOS1966SUPP; .pdf). Contains proofs of some results in the main paper as well as supplementary technical results.

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