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SECOND ERRATA TO "DISTANCE COVARIANCE IN METRIC SPACES"

BY RUSSELL LYONS

Department of Mathematics, Indiana University, rdlyons@indiana.edu

There is a slight gap and error in Remark 3.4 of *Ann. Probab.* **41**, no. 5 (2013), 3284–3305, that was not noticed before the first errata were published (*Ann. Probab.* **46**, no. 4 (2018), 2400–2405). We take this opportunity to provide some additional updates as well.

(i) Proposition 2.6, whose proof was noted in the first errata to be incorrect without stronger hypotheses, has been given a correct proof by Janson (2019).

(ii) Remark 3.4 is slightly incorrect and has a gap in its proof. It should read as follows.

If $\mathscr{X}$ is a metric space of strong negative type, then $\alpha : \mu \mapsto a_\mu$ is injective on $\mu \in M_1(\mathscr{X})$ with finite first moment, as stated in Theorem 4.1 of Klebanov (2005) and later in Theorem 3.6 of Nickolas and Wolf (2009). Conversely, if $\mathscr{X}$ is a metric space of negative type and α is injective on $M_1(\mathscr{X})$ with finite first moment, then there is, at most, one pair $\mu, \mu' \in M_1(\mathscr{X})$ with $D(\mu - \mu') = 0$ and $\|\mu - \mu'\| = 2$ (i.e., μ and μ' are mutually singular). Note that if there are no such pairs, then $\mathscr{X}$ has strong negative type.

The first part follows from the fact that if $a_\mu = a_{\mu'}$, then strong negative type guarantees that $D(\mu - \mu') = \int (a_\mu - a_{\mu'}) d(\mu - \mu') = 0$, whence $\mu = \mu'$. (It suffices here to assume only that $a_\mu = a_{\mu'}$ on the support of $|\mu - \mu'|$.) For the converse, suppose that α is injective on measures in $M_1(\mathscr{X})$ with finite first moment. Let ϕ be an embedding of $\mathscr{X}$. Then

$$a_\mu(x) = \|\phi(x)\|^2 - 2\langle \phi(x), \beta(\mu)\rangle + \int \|\phi(x')\|^2 \, d\mu(x'),$$

whence if $D(\mu - \mu') = 0$ (which is the same as $\beta(\mu) = \beta(\mu')$),

$$a_\mu(x) - a_{\mu'}(x) = \int \|\phi(x')\|^2 \, d(\mu - \mu')(x') =: V(\mu - \mu')$$

does not depend on x and is not 0 if $\mu \neq \mu'$ by injectivity of α. Suppose that for $i = 1, 2$, there are $\mu_i, \mu_i' \in M_1(\mathscr{X})$ such that $D(\mu_i - \mu_i') = 0$ and $\|\mu_i - \mu_i'\| = 2$. Define

$$\nu := \frac{\mu_1 - \mu_1'}{V(\mu_1 - \mu_1')} - \frac{\mu_2 - \mu_2'}{V(\mu_2 - \mu_2')}.$$

Then $a_\nu(x) = 1 - 1 = 0$ for all x. Since $\nu(\mathscr{X}) = 0$, injectivity of α yields $\nu = 0$. It follows that $\mu_1 - \mu_1' = \mu_2 - \mu_2'$, as desired.

There are metric spaces $\mathscr{X}$ of negative type with injective α but $\mathscr{X}$ not of strong negative type; for example, use e_k, f_k and v_k as in the corrected Remark 3.3, but rather than w_k, use $u_0 := f_1$ and $u_k := -f_k + f_{k+1}/3$ for $k \geq 1$. Again, the collection $\{v_k, u_k; k \geq 0\}$ has no obtuse angles and is affinely independent, yet $(1/3)(v_0 + \sum_{k \geq 1} v_k/2^{k-1}) = \mathbf{0} = (2/5)(w_0 + \sum_{k \geq 1} u_k/3^{k-1})$ exhibits the only pair of mutually singular probability measures with the same barycenter. At the same time, $(1/3)(\|v_0\|^2 + \sum_{k \geq 1} \|v_k/2^{k-1}\|^2) \neq (2/5)(\|w_0\|^2 + \sum_{k \geq 1} \|u_k/3^{k-1}\|^2)$, so α is injective.

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Of course, if $\mathscr{X}$ has a sequence of selfisometries ι_n such that $\lim_{n\to\infty} d(x, \iota_n(x)) = \infty$, then there cannot be a unique pair $\mu, \mu' \in M_1(\mathscr{X})$ with $D(\mu - \mu') = 0$ and $\|\mu - \mu'\| = 2$, whence, in such a case, the conjunction of negative type and injectivity of α is equivalent to strong negative type. For example, this holds if $\mathscr{X}$ is a Banach space.

On the other hand, there are metric spaces not of negative type for which α is injective on the probability measures: for example, take a finite metric space for which the distance matrix is nonsingular. The map α is injective also for all separable L^p spaces $(1 < p < \infty)$; see Linde (1986b) or Gorin and Koldobskiĭ (1987).

(iii) Our paper dealt explicitly with metric spaces only, except for a hint in Remark 3.6 that the triangle inequality is not crucial for negative type. Schoenberg (1938) assumed only that d is symmetric, is nonnegative and vanishes on the diagonal and that $\mathscr{X}$ is separable in the topology given by this premetric. With the additional assumption that d vanishes *only* on the diagonal, that is, that $(\mathscr{X}, d)$ is a semimetric, such conditions are sufficient for all we do in Section 3, except when we want examples specifically of metric spaces. Note that when $(\mathscr{X}, d)$ is a premetric space of negative type, then the embedding in a Hilbert space and the parallelogram law there give a replacement for the triangle inequality, to wit, $d(x, z) \le 2(d(x, y) + d(y, z))$.

Allowing semimetrics in Remark 3.19 gives additional results. For example, Remark 3.19 gives that $(H, \|\cdot\|^{2r})$ has strong negative type for $0 < r < 1$, which was later proved as Theorem 4.2 by Dehling et al. (2020) and then as Theorem 6.6 in Janson (2019).

More generally, separable spaces $(L^p, \|\cdot\|^{pr})$ have strong negative type for $0 < p \le 2$ and $0 < r < 1$. This is because $(L^p, \|\cdot\|^p)$ has negative type, as shown by Schoenberg (1938).

(iv) It is also the case that the proof in Remark 3.19 may be slightly clearer if expressed as follows, where we now include the case of semimetrics for convenience.

We claim that if $(\mathscr{X}, d)$ is a separable, semimetric space of negative type, then $(\mathscr{X}, d^r)$ has strong negative type when $0 < r < 1$. When $\mathscr{X}$ is finite and so strong negative type is the same as strict negative type; this result is due to (Li and Weston (2010), Theorem 5.4). To prove our claim, we use the result of Linde (1986a) that the map $\alpha : \mu \mapsto a_\mu$ of Remark 3.4 is injective on $M_1^1(H, \|\cdot\|^p)$ for all $p \in \mathbb{R}^+ \setminus 2\mathbb{N}$. Since $(H, \|\cdot\|^{2r})$ has negative type by Schoenberg (1938), it follows that $(H, \|\cdot\|^{2r})$ has strong negative type by Remark 3.4. Let $\phi : (\mathscr{X}, d^{1/2}) \to (H, \|\cdot\|)$ be an isometric embedding. Then, ϕ also provides an isometry from $(\mathscr{X}, d^r)$ to $(H, \|\cdot\|^{2r})$, whence the claim follows.

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