

CONFORMAL GROWTH RATES AND SPECTRAL GEOMETRY ON DISTRIBUTIONAL LIMITS OF GRAPHS

BY JAMES R. LEE

Paul G. Allen School of Computer Science and Engineering, University of Washington, jrl@cs.washington.edu

For a unimodular random graph (G, ρ) , we consider deformations of its intrinsic path metric by a (random) weighting of its vertices. This leads to the notion of the *conformal growth exponent* of (G, ρ) , which is the best asymptotic degree of volume growth of balls that can be achieved by such a reweighting. Under moment conditions on the degree of the root, we show that the conformal growth exponent of a unimodular random graph bounds its almost sure spectral dimension. This has interesting consequences for many low-dimensional models.

The consequences in dimension two are particularly strong. It establishes that models like the uniform infinite planar triangulation (UIPT) and quadrangulation (UIPQ) almost surely have spectral dimension at most two. It also establishes a conjecture of Benjamini and Schramm (*Electron. J. Probab.* **6** (2001) no. 23) by extending their recurrence theorem from planar graphs to arbitrary families of H -minor-free graphs. More generally, it strengthens the work of Gurel-Gurevich and Nachmias (*Ann. of Math. (2)* **177** (2013) 761–781) who established recurrence for distributional limits of planar graphs when the degree of the root has exponential tails.

We further present a general method for proving subdiffusivity of the random walk on a large class of models, including UIPT and UIPQ, using only the volume growth profile of balls in the intrinsic metric.

CONTENTS

1. Introduction	2672
1.1. Discrete conformal metrics and the growth exponent	2673
1.2. Dimension two: Gauged quadratic growth and recurrence	2675
1.3. Estimates on the spectral measure and the heat kernel	2677
1.4. Intrinsic volume growth, Markov type and subdiffusivity	2679
1.5. Discussion and related work	2683
1.6. Preliminaries	2685
1.6.1. Unimodular random graphs and distributional limits	2686
2. Quadratic conformal growth and recurrence	2687
2.1. Comparing graph balls to conformal balls	2688
2.2. Bounding the effective resistance	2690
2.3. Region intersection graphs and energy-minimizing conformal weights	2692
3. Return probabilities and spectral geometry on finite graphs	2694
3.1. The normalized Laplacian spectrum	2694
3.2. Random partitions	2695
3.3. Eigenvalues and the degree distribution	2697
3.4. Return probabilities and spectral delocalization	2698
3.5. Constructing bump functions	2701
3.5.1. Many disjoint bumps	2701
3.5.2. Exhausting the stationary measure by capacitors	2704
4. Conformal growth rates and random walks	2706

Received July 2020.

MSC2020 subject classifications. 60-XX, 60D05, 60J10, 52C26.

Key words and phrases. Unimodular random graphs, uniform infinite planar triangulation, random walk, spectral dimension.

4.1. Invariant amenability and soficity	2706
4.2. Conformal growth exponent bounds the spectral dimension	2708
4.3. On-diagonal heat kernel bounds	2711
4.3.1. Fatter degree tails and transience	2714
4.4. Spectrally heterogeneous graphs	2714
4.4.1. Spectral measures on infinite graphs	2716
4.4.2. Spectrally significant vertices in unimodular random graphs	2718
5. Markov type and speed of the random walk	2720
5.1. Diffusive bounds in the conformal metric	2720
5.2. Weights from separators	2725
Appendix: Growth bounds for UIPT/UIPQ	2727
Acknowledgments	2728
References	2729

1. Introduction. Motivated by the study of random surfaces in quantum geometry [5], Benjamini and Schramm [18] sought to understand the behavior of random planar triangulations. Toward this end, they introduced the notion of the *distributional limit* of a sequence of finite graphs $\{G_n\}$. This limit is a random rooted infinite graph (G, ρ) with the property that the laws of neighborhoods of a randomly chosen vertex in G_n converge, as $n \rightarrow \infty$, to the laws of neighborhoods of ρ in G . When the limit exists, it is a *unimodular random graph* in the sense of Aldous and Lyons [3]. (See Section 1.6.1 for a discussion of the weak local topology and unimodular random graphs.)

An example of central importance is the *uniform infinite planar triangulation (UIPT)* of Angel and Schramm [10], which is obtained by taking the distributional limit of a uniform random triangulation of the 2-sphere with n vertices. A well-studied variant is the *uniform infinite planar quadrangulation (UIPQ)* constructed by Krikun [46]. More recently, Benjamini and Curien [14] sought to extend these studies to graphs that can be sphere-packed in \mathbb{R}^d for $d \geq 3$, but noted that the defining natural models in higher dimensions is a subtle issue.

In general, the goal of this line of work is to understand the almost sure geometric properties of the limit object, where often interesting phenomena emerge. For instance, Angel [7] has shown that almost surely balls of radius R in UIPT have volume $R^{4+o(1)}$, but such a ball can be separated from infinity by removing only $R^{1+o(1)}$ vertices. This reflects the fractal geometry of UIPT and leads one to suspect, for instance, that the random walk should be recurrent, and the speed of the walk should be subdiffusive.

Indeed, Benjamini and Curien [16] proved that the random walk in UIPQ is almost surely subdiffusive: The average distance from the starting point is at most $T^{1/3+o(1)}$ after T steps; the correct exponent is conjectured to be $1/4$, as predicted by the KPZ relations (see the discussion in [16]). More recently, Gurel-Gurevich and Nachmias [35] established that the random walk on UIPT and UIPQ is almost surely recurrent.

While the theory developed here applies to a wide range of distributional limits, our methods yield new proofs of these preceding results in substantially more general settings, and more detailed information even for the specific models of UIPT and UIPQ. For instance, we establish that the spectral dimension of UIPT/UIPQ is almost surely at most two, confirming a long-held belief. In fact, this holds for any distributional limit of finite planar graphs when the degree of the root has sufficiently nice tails. For UIPT (and a family of other planar models), a matching lower bound was recently established by Gwynne and Miller [37]. We are additionally able to strengthen the almost sure recurrence for UIPT/UIPQ to the conclusion that almost surely the number of returns to the root by time T grows asymptotically faster than $\log \log T$.

We show that the $T^{1/3+o(1)}$ speed bound for UIPT/UIPQ actually holds for any unimodular random planar graph with quartic volume growth, or alternately in any unimodular random

graph where there is a large enough discrepancy between the volume growth and isoperimetric profile. After initial circulation of this manuscript, this method has been used to establish that the speed in UIPT is $T^{1/4+o(1)}$ [36]; this estimate is sharp, as it matches the lower bound proved in [37]. Moreover, the method of conformal weights was used recently to show that random walk on the 2D incipient infinite cluster is almost surely subdiffusive in the chemical distance [33].

Previous work on distributional limits of planar graphs relies heavily on the analysis of circle packings, which can be thought of as ambient representations that conformally uniformize the geometry of the underlying graph. Here, we take an intrinsic approach, deforming the graph geometry directly using a family of discrete graph metrics. This makes our methods significantly more flexible and applicable to a much broader family of graphs. The connection between discrete uniformization and spectral geometry of graphs is present in earlier joint works with Biswal and Rao [19] and Kelner, Price and Teng [41], where we showed how such metrics can be used to control the spectrum of the Laplacian in bounded-degree graphs.

1.1. *Discrete conformal metrics and the growth exponent.* Consider a locally finite, connected graph G . A *conformal metric on G* is a map $\omega : V(G) \rightarrow \mathbb{R}_+$. The metric endows G with a graph distance as follows: Give to every edge $\{u, v\} \in E(G)$ a length $\text{len}_\omega(\{u, v\}) := \frac{1}{2}(\omega(u) + \omega(v))$. This prescribes to every path $\gamma = \{v_0, v_1, v_2, \dots\}$ in G the induced length

$$\text{len}_\omega(\gamma) := \sum_{k \geq 0} \text{len}_\omega(\{v_k, v_{k+1}\}).$$

Now for $u, v \in V(G)$, one defines the path metric $\text{dist}_\omega(u, v)$ as the infimum of the lengths of all u - v paths in G . Denote the closed ball

$$B_\omega(x, R) = \{y \in V(G) : \text{dist}_\omega(x, y) \leq R\}.$$

If (G, ρ) is a unimodular random graph, then a *conformal metric on (G, ρ)* is a (marked) unimodular random graph (G', ω, ρ') with $\omega : V(G) \rightarrow \mathbb{R}_+$ such that (G, ρ) and (G', ρ') have the same law. We say that the conformal weight is *normalized* if $\mathbb{E}[\omega(\rho)^2] = 1$. See Section 1.6.1 for precise definitions.

One thinks of such a metric $\omega : V(G) \rightarrow \mathbb{R}_+$ as deforming the geometry of the underlying graph. It will turn out that normalized conformal metrics with nice geometric properties form a powerful tool in understanding the structure of (G, ρ) . A basic property one might hope for is controlled volume growth of balls: $|B_\omega(\rho, R)| \leq O(R^d)$ for some fixed $d > 0$. As we will see, the best exponent d one can achieve controls the spectral dimension of G from above.

Spectral dimension vs. conformal growth exponent. Consider a unimodular random graph (G, ρ) . We define the *upper and lower conformal growth exponents of (G, ρ)* , respectively, by

$$\begin{aligned} \overline{\text{dim}}_{\text{cg}}(G, \rho) &:= \inf_{\omega} \limsup_{R \rightarrow \infty} \frac{\log \| \#B_\omega(\rho, R) \|_{L^\infty}}{\log R}, \\ \underline{\text{dim}}_{\text{cg}}(G, \rho) &:= \inf_{\omega} \liminf_{R \rightarrow \infty} \frac{\log \| \#B_\omega(\rho, R) \|_{L^\infty}}{\log R}, \end{aligned}$$

and the infimum is over all normalized conformal metrics on (G, ρ) , and we use

$$\|X\|_{L^\infty} := \sup\{\lambda > 0 : \mathbb{P}(X < \lambda) = 1\}$$

to denote the essential supremum of a random variable X , and $\#S$ to denote the cardinality of a finite set S .

When $\overline{\dim}_{\text{cg}}(G, \rho) = \underline{\dim}_{\text{cg}}(G, \rho)$, we define the *conformal growth exponent* by

$$\dim_{\text{cg}}(G, \rho) := \overline{\dim}_{\text{cg}}(G, \rho) = \underline{\dim}_{\text{cg}}(G, \rho).$$

Note that the quantities $\overline{\dim}_{\text{cg}}$, $\underline{\dim}_{\text{cg}}$, \dim_{cg} are functions of the law of (G, ρ) ; they are not defined on (fixed) rooted graphs.

As an indication that the conformal growth exponent can be bounded in interesting settings, let us state the next theorem, which is proved in the companion paper [49]. We use \Rightarrow to denote convergence in the distributional sense; see Section 1.6.1. Say that a graph G is *sphere-packed in \mathbb{R}^d* if G is the tangency graph of a collection of interior-disjoint Euclidean balls in \mathbb{R}^d .

THEOREM 1.1. *If $\{G_n\}$ are finite graphs such that each G_n is sphere-packed in \mathbb{R}^d , and $\{G_n\} \Rightarrow (G, \rho)$, then $\overline{\dim}_{\text{cg}}(G, \rho) \leq d$.*

Theorem 1.1 is proved in somewhat greater generality: One can replace \mathbb{R}^d by any Ahlfors d -regular metric measure space and relax the notion of “packing” to allow bounded-multiplicity overlap of balls. We refer to [49] for details.

For a locally finite, connected graph G , denote the discrete-time heat kernel

$$p_T^G(x, y) := \mathbb{P}[X_T = y \mid X_0 = x], \quad x, y \in V(G),$$

where $\{X_n\}$ is the standard random walk on G . We recall the *spectral dimension* of G :

$$\dim_{\text{sp}}(G) := \lim_{n \rightarrow \infty} \frac{-2 \log p_{2n}^G(x, x)}{\log n},$$

whenever the limit exists. If the limit does exist, then it is the same for all $x \in V(G)$.

The spectral dimension is considered an important quantity in the study of quantum gravity, since it can be defined in a reparameterization-invariant way [4, 6]. It has long been conjectured that the spectral dimension of 2D quantum gravity is equal to two. Our results confirm that the almost sure spectral dimension is at most two for these models.

Define also the *upper and lower spectral dimension* of G , respectively,

$$\begin{aligned} \overline{\dim}_{\text{sp}}(G) &:= \limsup_{n \rightarrow \infty} \frac{-2 \log p_{2n}^G(x, x)}{\log n}, \\ \underline{\dim}_{\text{sp}}(G) &:= \liminf_{n \rightarrow \infty} \frac{-2 \log p_{2n}^G(x, x)}{\log n}. \end{aligned}$$

It turns out that conformal growth exponent bounds the spectral dimension in somewhat general settings.

Say that a real-valued random variable X has *negligible tails* if its tails decay faster than any inverse polynomial:

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{\log n}{|\log \mathbb{P}[|X| > n]|} = 0,$$

where we take $\log(0) = -\infty$ in the preceding definition (in the case that X is essentially bounded).

For the sake of clarity in the next statement, we use (G, ρ) to denote the law μ of (G, ρ) , and $(\mathbf{G}, \boldsymbol{\rho})$ to denote the random variable with law μ .

THEOREM 1.2. *If (G, ρ) is a unimodular random graph and $\text{deg}_G(\rho)$ has negligible tails, then almost surely:*

$$\begin{aligned} \overline{\dim}_{\text{sp}}(\mathbf{G}) &\leq \overline{\dim}_{\text{cg}}(G, \rho), \\ \underline{\dim}_{\text{sp}}(\mathbf{G}) &\leq \underline{\dim}_{\text{cg}}(G, \rho). \end{aligned}$$

In conjunction with Theorem 1.1, this shows that if (G, ρ) is the distributional limit of finite \mathbb{R}^d -packable graphs (and $\text{deg}_G(\rho)$ has negligible tails), then almost surely:

$$p_{2T}^G(\rho, \rho) \geq T^{-d/2-o(1)} \quad \text{as } T \rightarrow \infty.$$

In particular, this establishes that the vast majority of random planar maps in the literature have almost sure spectral dimension at most two. This was unknown even for UIPT and UIPQ. For UIPQ, it was recently established by Gwynne and Miller [37] that this is tight: the spectral dimension is almost surely two. For UIPT, a matching lower bound remains open.

An overview of the proof of Theorem 1.2 in the special case of 2-dimensional growth is given in Section 1.3. The inequalities in Theorem 1.2 imply that the conformal growth rate provides a lower bound on the return probabilities up to a $T^{o(1)}$ correction factor. We remark that if one makes the stronger assumption that $\text{deg}_G(\rho)$ has exponential tails, then the implied correction factors are only polylogarithmic; see the discussion in Section 4.2.

Finally, we note that the inequalities in Theorem 1.2 cannot be reversed in general: There is a unimodular random graph (G, ρ) with uniformly bounded degrees such that $\dim_{\text{sp}}(G, \rho)$ is finite but $\dim_{\text{cg}}(G, \rho)$ is infinite; see Section 4.4 where we review an example due to [9].

Uniformization and intrinsic dimension. Certainly circle packings of planar graphs are a powerful, elegant and “conformally natural” [60] tool. Still, it is enlightening to think of situations where ambient representations do a poor job of emphasizing the intrinsic geometry of the underlying graph. In general, this is the case when the dimension of the graph differs from that of the ambient space.

A basic example is the planar graph $G = (V, E)$ which is the product of a triangle and a bi-infinite path: $V = \{0, 1, 2\} \times \mathbb{Z}$ and $\{x, y\} \in E$ if and only if $\|x - y\|_1 = 1$. This graph is quasi-isometric to \mathbb{Z} , and thus manifestly one-dimensional. The appropriate uniformizing discrete conformal metric (by transitivity) is $\omega \equiv 1$. The circle packing in \mathbb{R}^2 (which is unique up to Möbius transformations) has an accumulation point in \mathbb{R}^2 , and the radii of the circles grow with geometrically increasing radii from the accumulation point to infinity.

Consider another example: the incipient infinite cluster (IIC) of critical percolation $(G_d^{\text{IIC}}, 0)$ on \mathbb{Z}^d , for d sufficiently large. In their solution to the Alexander–Orbach conjecture in high dimensions, Kozma and Nachmias [45] show that for $d \geq 11$,¹ almost surely $\dim_{\text{sp}}(G_d^{\text{IIC}}) = 4/3$. Theorem 1.2 implies that $\dim_{\text{cg}}(G_d^{\text{IIC}}) \geq 4/3$. Moreover, one can show that this is tight. For instance, G_d^{IIC} is spectrally homogeneous in the sense of (4.21) in Section 4.4 with $d = 4/3$ [57]. In this case, the inequality in Theorem 1.2 can be reversed, and $\dim_{\text{cg}}(G_d^{\text{IIC}}) = 4/3$.

1.2. Dimension two: Gauged quadratic growth and recurrence. The conformal growth exponent is not precise enough to study recurrence (which depends on lower-order factors in the heat kernel $p_T^G(\rho, \rho)$). Say that a unimodular random graph (G, ρ) is (C, R) -quadratic for $C > 0$ and $R \geq 1$ if

$$(1.2) \quad \inf_{\omega} \|\#B_{\omega}(\rho, R)\|_{L^{\infty}} \leq CR^2,$$

¹One needs to use [32] to obtain $d \geq 11$; the original reference proves it for $d \geq 19$.

where the infimum is over all normalized conformal metrics on (G, ρ) . We say that (G, ρ) has *gauged quadratic conformal growth* (gQCG) if there is a constant $C > 0$ such that (G, ρ) is (C, R) -quadratic for all $R \geq 1$. Note that we allow a different conformal weight ω for every choice of R , and this is necessary for distributional limits of finite planar graphs to have gQCG (see Lemma 2.16).

THEOREM 1.3. *If (G, ρ) is a unimodular random graph with uniformly bounded degrees and gauged quadratic conformal growth, then G is almost surely recurrent.*

In Section 2.3, we argue that distributional limits of planar graphs, H -minor free graphs, string graphs, and other families have gauged quadratic conformal growth. Thus Theorem 1.3 generalizes the Benjamini–Schramm recurrence theorem [18] to H -minor-free graphs, confirming a conjecture stated there. After initial dissemination of a draft of this manuscript, we learned that Angel and Szegedy (personal communication) had previously discovered a proof of the H -minor-free case using a detailed analysis of the Robertson–Seymour classification [59].²

REMARK 1.4 (String graphs). By the Koebe–Andreev–Thurston circle packing theorem, planar graphs are precisely the tangency graphs of interior-disjoint disks in the plane. *String graphs* are a significant generalization: They are the intersection graphs of a collection of arbitrary path-connected regions in the plane (with no assumption on disjointness). Such graphs can be dense, but string graphs with uniformly bounded degrees have quadratic conformal growth (see Section 2.3).

Unbounded degrees. Recently, Gurel-Gurevich and Nachmias [35] resolved a central open problem by showing that the uniform infinite planar triangulation (UIPT) and quadrangulation (UIPQ) are almost surely recurrent. They achieved this by extending the recurrence theorem of Benjamini and Schramm in a different direction: In every distributional limit of finite planar graphs where the degree of the root has exponential tails, the limit is almost surely recurrent. It was previously known that both UIPQ and UIPT satisfy this hypothesis.

Let μ denote the law of (G, ρ) , and define $\bar{d}_\mu : [0, 1] \rightarrow \mathbb{R}_+$ by

$$\bar{d}_\mu(\varepsilon) := \sup\{\mathbb{E}[\deg_G(\rho) \mid \mathcal{E}] : \mathbb{P}(\mathcal{E}) \geq \varepsilon\},$$

where the supremum is over all measurable sets \mathcal{E} with $\mathbb{P}(\mathcal{E}) \geq \varepsilon$. That $\deg_G(\rho)$ has exponential tails is equivalent to $\bar{d}_\mu(1/t) \leq O(\log t)$ as $t \rightarrow \infty$. Negligible tails as defined in (1.1) is equivalent to the assumption that

$$(1.3) \quad \bar{d}_\mu(1/t) \leq t^{o(1)} \quad \text{as } t \rightarrow \infty.$$

ASSUMPTION 1.5. Suppose (G, ρ) is a unimodular random graph with law μ satisfying the following:

1. (G, ρ) has gauged quadratic conformal growth.
2. (G, ρ) is uniformly decomposable (cf. Section 3.2).
3. $\mathbb{E}[\deg_G(\rho)^2] < \infty$.

²We remark that establishing quadratic conformal growth for H -minor-free graphs does not require the Robertson–Seymour theory.

THEOREM 1.6. *Under Assumption 1.5, if additionally*

$$\sum_{t \geq 1} \frac{1}{t \bar{d}_\mu(1/t)} = \infty,$$

then G is almost surely recurrent.

It was previously known (see Section 3.2) that many families of finite graphs—planar graphs, H -minor-free graphs, and string graphs—are uniformly decomposable. This property passes to distributional limits, hence Theorem 1.6 generalizes the result of [35]. Note that we allow slightly heavier tails: For instance, $\bar{d}_\mu(1/t) \leq O(\log t \log \log t)$ is still enough to guarantee recurrence.

Moreover, Theorem 1.6 is tight in the following sense: For any monotonically nondecreasing sequence $\{d_t : t = 1, 2, \dots\}$ such that $\sum_{t \geq 1} \frac{1}{td_t} < \infty$, there is a unimodular random planar graph satisfying Assumption 1.5 that is almost surely transient, and such that $\bar{d}_\mu(1/t) \leq d_t$ for all t sufficiently large; see Section 4.3.1.

1.3. *Estimates on the spectral measure and the heat kernel.* Let us now describe some of the elements of the proof of Theorem 1.6, along with more detailed information about the random walk. In Section 4.1, we argue that $\underline{\dim}_{\text{cg}}(G, \rho) < \infty$ implies that (G, ρ) is invariantly amenable, and thus it is a distributional limit of finite graphs: $\{G_n\} \Rightarrow (G, \rho)$.

Thus for simplicity, let us consider a finite planar graph G_n and a root $\rho_n \in V(G_n)$ chosen uniformly at random. Without loss, we may assume that $n = |V(G_n)|$. Define

$$\Delta_{G_n}(k) := \max_{S \subseteq V(G_n): |S| \leq k} \sum_{x \in S} \deg_{G_n}(x)$$

to be the sum of the k largest vertex degrees in G_n . In Section 3, we establish the bound

$$(1.4) \quad \lambda_k(G_n) \leq c \frac{\Delta_{G_n}(k)}{n},$$

where c is a universal constant and $\{1 - \lambda_k(G_n) : k = 0, 1, \dots, n - 1\}$ are the eigenvalues of the random walk operator on G_n . In [41], a weaker bound was proved, with $k \cdot \Delta_{G_n}(1)$ in place of $\Delta_{G_n}(k)$.

Such a bound provides average estimates for the diagonal of the heat kernel: Let P denote the random walk operator on G_n . Then for an integer $T \geq 0$,

$$(1.5) \quad \begin{aligned} \mathbb{E}[p_T^{G_n}(\rho_n, \rho_n)] &= \frac{1}{n} \sum_{x \in V(G_n)} \langle \mathbb{1}_x, P^T \mathbb{1}_x \rangle = \frac{\text{tr}(P^T)}{n} \\ &= \frac{1}{n} \sum_{k=0}^{n-1} (1 - \lambda_k(G_n))^T \geq \frac{\#\{k : \lambda_k(G_n) \leq 1/T\}}{4n}, \end{aligned}$$

where the last inequality holds for $T \geq 2$.

Thus if the vertex degrees are uniformly bounded along the sequence $\{G_n\}$, then we have $\Delta_{G_n}(k) \leq O(k)$, and combining (1.4) and (1.5) yields

$$\mathbb{E}[p_T^G(\rho, \rho)] \geq \liminf_{n \rightarrow \infty} \mathbb{E}[p_T^{G_n}(\rho_n, \rho_n)] \gtrsim \frac{1}{T}.$$

If $\{G_n\} \Rightarrow (G, \rho)$ and we impose only the weaker assumption that $\deg_G(\rho)$ has exponential tails, then it must hold that for n sufficiently large, $\Delta_{G_n}(\frac{n}{T}) \leq O(\frac{n}{T} \log T)$, and one obtains

$$(1.6) \quad \mathbb{E}[p_T^G(\rho, \rho)] \geq \liminf_{n \rightarrow \infty} \mathbb{E}[p_T^{G_n}(\rho_n, \rho_n)] \gtrsim \frac{1}{T \log T}.$$

In this way, the degree-modified two-dimensional Weyl law in (1.4) predicts recurrence when the degree of the root has exponential tails, since $\sum_{T \geq 1} \frac{1}{T \log T} = \infty$. But a significant obstacle is that the annealed estimate (1.6) does not necessarily imply anything for the distributional limit. The issue is that the lower bound in (1.6) could come entirely from a small set of vertices (and such small sets could be negligible in the distributional limit). Indeed, one could add to G_n only εn isolated vertices to achieve $\frac{1}{n} \sum_{x \in V(G_n)} p_T^{G_n}(x, x) \geq \varepsilon$. (See Section 4.4 for a family of connected examples where the contribution of the expected return probabilities come from a small measure of roots.) Thus even to obtain almost sure recurrence, we need an estimate stronger than (1.5). We state now the following strengthening of Theorem 1.6.

THEOREM 1.7. *Under Assumption 1.5, the following holds. There is a constant $C = C(\mu)$ such that for every $\delta > 0$ and all $T \geq C/\delta^2$,*

$$\mathbb{P} \left[p_{2T}^G(\rho, \rho) < \frac{\delta}{T \bar{d}_\mu(1/T^3)} \right] \leq C\delta^{1/17}.$$

Note that even for the special case of UIPT/UIPQ, this estimate yields $\mathbb{E} p_{2T}^G(\rho, \rho) \gtrsim \frac{1}{T \log T}$ for every $T \geq 2$, improving over the bound $\mathbb{E} p_{2T}^G(\rho, \rho) \gtrsim \frac{1}{T^{4/3}(\log T)^{o(1)}}$ from [16]. As a consequence, one obtains a bound on the rate of divergence of the Green function: Define

$$g_\mu(T) := \sum_{t=1}^T \frac{1}{t \bar{d}_\mu(1/t)}.$$

For instance, for UIPT/UIPQ, one has $g_\mu(T) \asymp \log \log T$.

THEOREM 1.8. *Under Assumption 1.5, the following holds. If $g_\mu(T) \rightarrow \infty$, then G is almost surely recurrent. Moreover, almost surely*

$$\limsup_{T \rightarrow \infty} \frac{\sum_{t=1}^T p_t^G(\rho, \rho)}{g_\mu(T)} > 0.$$

Exhaustion by capacitors. For a finite graph G and $\varphi : V(G) \rightarrow \mathbb{R}$, define the normalized Dirichlet energy

$$\mathcal{E}_G(\varphi) := \frac{1}{|E(G)|} \sum_{\{x,y\} \in E(G)} |\varphi(x) - \varphi(y)|^2.$$

Let us call a pair (A, Ω) of subsets $A \subseteq \Omega \subseteq V(G)$ a *capacitor*, and define the *capacity* of (A, Ω) by

$$\text{cap}_\Omega^G(A) := \inf_{\varphi: V(G) \rightarrow [0,1]} \{ \mathcal{E}_G(\varphi) : \varphi|_A \equiv 1, \text{supp } \varphi \subseteq \Omega \},$$

where $\text{supp } \varphi := \{x \in V(G) : \varphi(x) \neq 0\}$. We remark that, by the Dirichlet principle, the capacity of (A, Ω) is precisely the inverse of the effective resistance $R_{\text{eff}}^G(A \leftrightarrow (V(G) \setminus \Omega))$; see Section 2.2.

Theorem 1.2, Theorem 1.7 and Theorem 1.8 are proved as follows. One uses an appropriate conformal weight on a finite graph G to locate capacitors $(A_1, \Omega_1), \dots, (A_k, \Omega_k)$ such that the sets $\{\Omega_i\}$ are pairwise disjoint. Using reversibility of the random walk, such a collection of capacitors yields a lower bound on typical return probabilities (see Theorem 3.7): For every $T \geq 1$ and $\varepsilon > 0$,

$$\pi \left(\left\{ x \in V(G) : p_{2T}^G(x, x) \geq \frac{\varepsilon^2}{4M} \right\} \right) \geq -3\varepsilon \bar{d}_G(\varepsilon) + \sum_{i=1}^k \pi(A_i) - 2T \sum_{i=1}^k \text{cap}_{\Omega_i}^G(A_i),$$

where $M = \max\{|\Omega_i| : i = 1, \dots, k\}$, and $\bar{d}_G(\varepsilon) := \frac{\Delta_G(\varepsilon n)}{\varepsilon n}$ (this is equal to $\bar{d}_\mu(\varepsilon)$ when μ is the law of (G, ρ) with G a fixed finite graph and $\rho \in V(G)$ chosen uniformly at random), and π is the stationary measure on G .

1.4. *Intrinsic volume growth, Markov type and subdiffusivity.* Consider a connected, infinite, locally finite graph G . For $x \in V(G)$ and $r \geq 0$, let

$$B_G(x, r) = \{y \in V(G) : \text{dist}_G(x, y) \leq r\},$$

where dist_G denotes the (unweighted) graph distance in G . Suppose that G has nearly uniform d -dimensional volume growth in the sense that for r sufficiently large,

$$(1.7) \quad r^{d-o(1)} \leq |B_G(x, r)| \leq r^{d+o(1)}$$

holds uniformly for all $x \in V(G)$.

When G is planar and $d > 2$, one suspects that the structure of G should be somewhat degenerate. Indeed, Itai Benjamini has put forth a number of conjectures to this effect. For instance, in [17] it is conjectured that if G is planar and (1.7) is satisfied, then the random walk on G should be *subdiffusive* with the natural speed estimate:

$$(1.8) \quad \mathbb{E}[\text{dist}_G(X_0, X_T)] \leq T^{1/d+o(1)}.$$

Recent examples show this to be false.

THEOREM 1.9 ([30], Theorem 1.3). *For every rational $d > 3$ and $\varepsilon > 0$, there is a constant $c(\varepsilon) > 0$ and a unimodular random planar graph (G, ρ) such that G almost surely has uniform polynomial growth of degree d , and*

$$\mathbb{E}[\text{d}_G(X_0, X_T) \mid X_0 = \rho] \geq c(\varepsilon)T^{1/(d-1+\varepsilon)} \quad \forall T \geq 1.$$

Nevertheless, we will see that (1.8) holds if d is replaced by $d - 1$. Subdiffusivity was confirmed specifically for UIPQ: In [16], it is shown that

$$(1.9) \quad \mathbb{E}[\text{dist}_G(X_0, X_T) \mid X_0 = \rho] \leq T^{1/3}(\log T)^{O(1)}.$$

For UIPT [7] and UIPQ [16], an almost sure asymptotic variant of (1.7) is satisfied with $d = 4$. Thus the estimate (1.9), while nontrivial, does not meet the conjectured exponent of $1/4$.

In establishing (1.9), the authors undertook a detailed study of the geometry of UIPQ. We show that the phenomenon is somewhat more general: To obtain subdiffusivity for a unimodular random planar graph, one need only assume asymptotic d -dimensional volume growth for some $d > 3$.

THEOREM 1.10. *Suppose that (G, ρ) is a unimodular random planar graph and for some $d > 3$, there is a function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $h(r) \leq r^{o(1)}$ and such that almost surely,*

$$(1.10) \quad \frac{r^d}{h(r)} \leq |B_G(\rho, r)| \leq h(r)r^d \quad \forall r \geq 1.$$

Then the random walk on (G, ρ) is strictly subdiffusive. More specifically,

$$(1.11) \quad \mathbb{E}[\text{dist}_G(X_0, X_T)^{d-1} \mid X_0 = \rho] \leq T^{1+o(1)} \quad \text{as } T \rightarrow \infty.$$

REMARK 1.11 (Sharpness of the assumption $d > 3$). After initial circulation of this manuscript, Omer Angel and Asaf Nachmias constructed, for every $\varepsilon > 0$ sufficiently small, a unimodular random planar graph (G, ρ) with uniformly bounded degrees and such that almost surely the random walk is diffusive and yet

$$(1.12) \quad \lim_{r \rightarrow \infty} \frac{\log |B_G(\rho, r)|}{\log r} = 3 - \varepsilon.$$

This suggests that the assumption $d > 3$ in Theorem 1.10 may be tight, except possibly at the critical value $d = 3$. Note that Theorem 1.10 does not apply directly because (1.12) only entails an almost sure asymptotic bound, while (1.10) demands a uniform bound over the choice of (G, ρ) .

Their examples *do show* that the assumption $d > 3$ in Theorem 1.17 below is sharp, as they satisfy Assumption (A) with $d = 3 - \varepsilon$ for every $q > 1$ and $\alpha > 0$.

The key geometric fact about planar graphs used to prove Theorem 1.10 is due to Benjamini and Papasoglu [17]. They show that in any planar graph satisfying (1.7), for any $x \in V(G)$ and r sufficiently large, there is a set of vertices of size at most $r^{1+o(1)}$ that separates $B_G(x, r)$ from $V(G) \setminus B_G(x, 2r)$ (see Figure 1(a)). In fact, the existence of such separators together with (1.10) is enough to obtain the estimate (1.11), without requiring that the graph be planar.

For a connected, locally-finite graph G , a node $x \in V(G)$, and two radii $r' > r > 0$, let $\kappa_G(x; r, r')$ denote the cardinality of the smallest set

$$U \subseteq B_G(x, r') \setminus B_G(x, r)$$

such that U separates x and $V(G) \setminus B_G(x, r')$ in G . The next lemma follows easily from the argument in [17]; we simply note the quantitative dependence on the growth rate.

LEMMA 1.12 ([17]). *If (G, ρ) is a unimodular random planar graph satisfying (1.7), then there is a constant $c > 1$ such that*

$$\kappa_G(\rho; r, 2r) \leq cr \frac{h(4r)}{h(r/4)} \quad \forall r \geq 1.$$

Using Lemma 1.12, the next theorem generalizes Theorem 1.10.

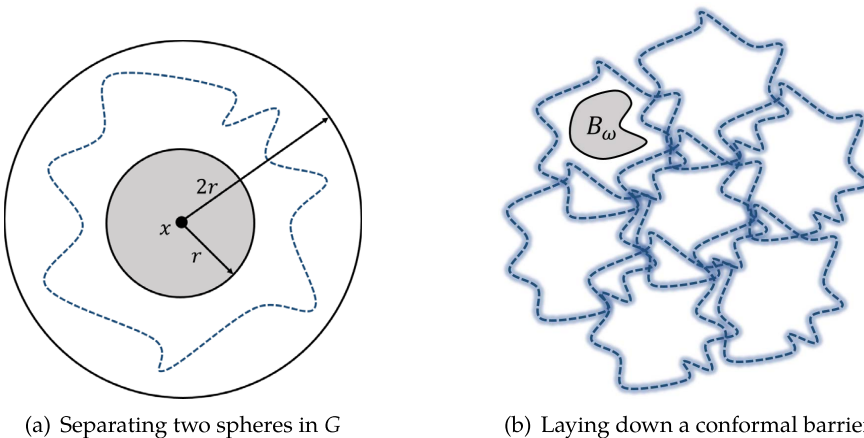


FIG. 1. Establishing subdiffusivity.

THEOREM 1.13. *Suppose that (G, ρ) is a unimodular random graph and for some numbers $d \geq k + 1 \geq 2$, there is a function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $h(r) \leq r^{o(1)}$ and such that almost surely (1.10) holds and,*

$$(1.13) \quad \kappa_G(\rho; r, h(r)r) \leq h(r)r^{k-1} \quad \forall r \geq 1.$$

Then one has the estimate

$$\mathbb{E}[\text{dist}_G(X_0, X_T)^{d-k+1} \mid X_0 = \rho] \leq T^{1+o(1)} \quad \text{as } T \rightarrow \infty.$$

In particular, when $d > k + 1$, the random walk is strictly subdiffusive.

One can view this result as saying that when a graph has d -dimensional volume growth, but k -dimensional isoperimetry and $d > k + 1$, then this discrepancy necessitates subdiffusivity. We now elaborate on the proof of Theorem 1.13 in the special case when G is almost surely planar. In particular, one will find in the argument below the relatively simple construction of a useful discrete conformal metric.

Markov type of normalized conformal metrics. A central tool is K. Ball’s notion of Markov type [11].

DEFINITION 1.14 (Markov type). A metric space (X, d) is said to have *Markov type* $p \in [1, \infty)$ if there is a constant $M > 0$ such that for every $n \in \mathbb{N}$, the following holds. For every reversible Markov chain $\{Z_t\}_{t=0}^\infty$ on $\{1, \dots, n\}$, every mapping $f : \{1, \dots, n\} \rightarrow X$, and every time $t \in \mathbb{N}$,

$$(1.14) \quad \mathbb{E}[d(f(Z_t), f(Z_0))^p] \leq M^p t \mathbb{E}[d(f(Z_0), f(Z_1))^p],$$

where Z_0 is distributed according to the stationary measure of the chain. One denotes by $M_p(X, d)$ the infimal constant M such that the inequality holds.

In Section 5.1, basic Markov type theory is used to prove the following.

THEOREM 1.15. *Suppose that (G, ρ) is a unimodular random graph that almost surely satisfies*

$$|B_G(\rho, r)| \leq Cr^q \quad \forall r \geq 1$$

for some numbers $C, q \geq 1$. Then for any normalized conformal metric ω on (G, ρ) , the following holds: For any $q' \geq 1$ and $T \geq 2$,

$$\mathbb{E}[T^{q'} \wedge \text{dist}_\omega(X_0, X_T)^2 \mid X_0 = \rho] \leq C'T(\log T)^2,$$

where $C' = C'(C, q, q')$.

We remark that the truncation by $T^{q'}$ is a technical matter that will not play a significant role in the application of Theorem 1.15 to speed in the graph metric. We will compare $\text{dist}_\omega(X_0, X_T)$ to $\text{dist}_G(X_0, X_T)$ and, since $\text{dist}_G(X_0, X_T) \leq T$ always holds, the truncation will be irrelevant after passing to the graph distance; see the argument below.

Thus in order to prove a subdiffusive estimate for the speed in dist_G , it suffices to construct a normalized conformal metric on (G, ρ) with a suitable relationship between dist_ω and dist_G .

Weights from separators. First, let us fix a radius $s \geq 1$. By iteratively “cutting out” separators guaranteed by (1.13) in a unimodular way (see Figure 1(b)), a generalization of the following fact is established in Section 5.2 (see Lemma 5.8).

LEMMA 1.16. *Suppose the assumptions of Theorem 1.13 hold. For every $s > 0$, there is a triple (G, ρ, W_s) that is unimodular as a marked network, and such that the following holds:*

1. $\mathbb{P}[\rho \in W_s] \leq s^{k-1-d+o(1)}$,
2. *Almost surely, every connected component of $G[V(G) \setminus W_s]$ has diameter at most s (in the metric dist_G).*

Heuristically, these parameters make sense: The separator occupies $s^{k-1+o(1)}$ nodes out of the $s^{d+o(1)}$ nodes in the s -ball. Now we define, for every $j \in \mathbb{N}$, the normalized conformal weight:

$$\omega_j := \frac{\mathbb{1}_{W_{2^j}}}{\sqrt{\mathbb{P}[\rho \in W_{2^j}]}}$$

and we note that from Lemma 1.16(1), we have

$$\omega_j \geq 2^{j(d-k+1)/2-o(1)} \mathbb{1}_{W_{2^j}}.$$

In particular, combined with Lemma 1.16(2), this implies that

$$(1.15) \quad \text{dist}_G(x, y) > 2^j \implies \text{dist}_{\omega_j}(x, y) \geq 2^{j(d-k+1)/2-o(1)}.$$

Finally, consider the normalized metric

$$\hat{\omega}_T := \sqrt{\frac{1}{\lceil \log_2 T \rceil} \sum_{j=0}^{\lceil \log_2 T \rceil} \omega_j^2}.$$

From (1.15), we have almost surely for every $x, y \in V(G)$ with $\text{dist}_G(x, y) \leq T$:

$$\text{dist}_G(x, y) \leq \text{dist}_{\hat{\omega}_T}(x, y)^{2/(d-k+1)} T^{o(1)} \quad \text{as } T \rightarrow \infty.$$

In particular, for every $T \geq 1$:

$$\begin{aligned} \mathbb{E}[\text{dist}_G(X_0, X_T)^{d-k+1}] &= \mathbb{E}[T^{d-k+1} \wedge \text{dist}_G(X_0, X_T)^{d-k+1}] \\ &\leq T^{o(1)} \mathbb{E}[T^{d-k+1} \wedge \text{dist}_{\hat{\omega}_T}(X_0, X_T)^2] \\ &\leq T^{1+o(1)}, \end{aligned}$$

where the final inequality follows from Theorem 1.15. This yields Theorem 1.13.

Theorem 1.10 is not strong enough to reproduce (1.9) because even the vertex degrees in UIPQ are unbounded, and thus no uniform estimate of the form (1.10) can hold. The next result remedies this. It shows that if we have a $(1 + \delta)$ -moment bound on the size of balls and a stretched exponential lower tail, then our methods can still be applied.

THEOREM 1.17. *Suppose that (G, ρ) is a unimodular random graph that satisfies the following conditions for some numbers $C > 0$ and $d > 3$:*

1. *For every $r \geq 2$: $\mathbb{E}|B_G(\rho, r)| \leq r^C$.*
2. *The degree of the root has exponential tails:*

$$\mathbb{P}[\text{deg}_G(\rho) > \lambda] \leq e^{-\lambda/C} \quad \forall \lambda \geq 1.$$

Assume, additionally, that one of the following two conditions holds for some $\alpha > 0$:

(A) Planar; volume statistics.

- (i) G is almost surely planar, and
- (ii) For some $q > 1$ and for every $r \geq 2$:

$$\begin{aligned} \mathbb{E}|B_G(\rho, 6r)|^q &\leq Cr^d, \\ \mathbb{P}\left[|B_G(\rho, r)| < \frac{\varepsilon}{C}r^d\right] &\leq \exp(-(1/\varepsilon)^{2/\alpha}) \quad \forall \varepsilon > 0. \end{aligned}$$

(B) Unimodular shattering. For every $r \geq 2$, there is a random subset $W_r \subseteq V(G)$ such that (G, ρ, W_r) is unimodular (as a marked network), and:

- (i) $\mathbb{P}[\rho \in W_r] \leq Cr^{1-d}(\log r)^\alpha$
- (ii) Almost surely, every component of $G[V(G) \setminus W_r]$ has diameter at most r in dist_G .

Then the random walk on (G, ρ) is almost surely subdiffusive: There is a constant $C' \geq 1$ such that for all $T \geq 2$,

$$\mathbb{E}[\text{dist}_G(X_0, X_T)^{d-1} \mid X_0 = \rho] \leq C'T(\log T)^{\alpha(d-1)+5}.$$

Application to UIPT and UIPQ. It is well known that UIPT and UIPQ satisfy conditions (1) and (2). See [10] for UIPT and [16] and for UIPQ. The fact that (A) is satisfied with $\alpha = 4$ for UIPT/UIPQ is somewhat more delicate, and is discussed in the Appendix. In Section 5.2, it is shown that (A) \implies (B), but we include (A) to demonstrate that for unimodular random planar graphs, sufficient volume statistics are enough to yield subdiffusive behavior of the random walk.

1.5. *Discussion and related work.* Our use of “conformal metric” as a vertex-weighting $\omega : V(G) \rightarrow \mathbb{R}_+$ is inspired by the Riemannian setting and, in particular, the classical argument of Hersch [39], which relies on the fact that the Laplacian is conformally invariant in dimension two. In this analogy, $\mathbb{E}[\omega(\rho)^2]$ plays the role of the area. See, for instance, Theorem 3.14. When the degrees are unbounded, the families of vertex- and edge-weighted metrics on a graph G can be substantially different (see [48], where this plays a fundamental role in the main theorem).

Suppose G is a finite planar graph represented as the tangency graph of interior-disjoint circles $\{C_v : v \in V(G)\}$ with radii $\{r_v > 0 : v \in V(G)\}$. Considering only the weight given by $\omega(v) := r_v$, the topology of the Euclidean plane is removed; we are left only with an “area measure” on G , and the topology of the graph G itself. While it may seem we have abstracted out too much, many “conformally-flavored” properties remain.

For instance, if G is a triangulation with degree at most d , the classical ring lemma asserts the existence of a constant $C = C(d)$ such that $r_u \leq Cr_v$ for $\{u, v\} \in E(G)$. In Lemma 2.6, we show that for any normalized weight $\omega : V(G) \rightarrow \mathbb{R}_+$, one can choose a related normalized weight $\hat{\omega}$ that satisfies an analogous property. Similarly, in Section 5, we see that if (G, ρ, ω) is a reversible conformal random planar graph with a normalized conformal weight $\omega : V(G) \rightarrow \mathbb{R}_+$, then

$$\mathbb{E}[\text{dist}_\omega(X_0, X_T)^2 \mid X_0 = \rho] \leq O(T) \mathbb{E}[\omega(\rho)^2] \quad \forall T \geq 1.$$

In other words, the random walk has at most diffusive speed in the metric dist_ω , inheriting a property of Brownian motion in the Euclidean plane.

On the other hand, it does not hold in general that $(V(G), \text{dist}_\omega)$ is quasisymmetric to $(V(G), \text{dist}_G)$ (see [38] for background on quasisymmetric and quasiconformal maps), in the

same way that if $\{C_v : v \in V(G)\}$ is a circle packing of G in the Euclidean plane and $x_v \in \mathbb{R}^2$ is the center of C_v , then the map $v \mapsto x_v$ is not necessarily a quasisymmetry. Indeed, a circle packing of a planar graph can only be considered as a “snapshot” of a possible quasisymmetric mapping at a particular scale. This can be made precise by considering sequences of combinatorial approximations of metric spaces homeomorphic to the Euclidean sphere, as in the works of Cannon [26] and Bonk and Kleiner [22].

Anomalous diffusion on fractals. The topics studied here draw from a number of areas. We are interested largely in examples where various notions of dimension fail to coincide, for example, the topological dimension, exponent of volume growth (in the intrinsic metric), speed exponent of the random walk (cf. (1.8)), and the exponent of the isoperimetric profile (cf. Theorem 1.13). This phenomenon is characteristic of “anomalous diffusion” on fractals. We refer to [12] for a discussion of such exponents and their possible relationships.

While there are conditions under which these exponents are related in a natural way (see, e.g., [13]), those settings generally involve a detailed relationship between the graph distance and the effective resistance metric. In contrast, we are only able to obtain such estimates with respect to the conformal metric dist_ω (and, even then, only for most choices of the root). In this sense, the approach taken here is related to that of [8], where the authors obtain similar estimates in a planar graph under the metric pulled back from a circle packing (under the assumption of uniformly bounded degrees).

Geometric analysis on manifolds. The eigenvalue estimate (1.4) is closely related to one established much earlier by Korevaar [44] in addressing a question of S. T. Yau on the spectrum of the Laplace–Beltrami operator on orientable surfaces. There it is shown that if Ω is a subdomain in a complete d -dimensional Riemannian manifold (M, g_0) with nonnegative Ricci curvature and $(M, \varphi g_0)$ is a finite-volume conformal metric, then

$$\lambda_k \leq C_d \left(\frac{k}{\text{vol}(\Omega, \varphi g_0)} \right)^{d/2}, \quad k = 1, 2, \dots,$$

where C_d is a constant depending only on d , $\{\lambda_k\}$ are the Neumann eigenvalues of the Laplacian, and $\text{vol}(\cdot)$ is the Riemannian volume. For background, we refer to the lecture notes [61].

An immediate consequence of the uniformization theorem is the estimate

$$\lambda_k \leq C \frac{k}{\text{vol}(\mathbb{S}^2, g)}, \quad k = 1, 2, \dots$$

for any Riemannian metric g on \mathbb{S}^2 .

Korevaar’s approach (see also [34] where it is generalized and expounded upon at length) also proceeds by the construction of a family of disjoint capacitors (as we do Section 3.5). But at a technical level, his argument is substantially different from the one taken here: He constructs capacitors in the metric measure space $(\mathbb{S}^2, d_{\mathbb{S}^2}, \mu)$, where $d_{\mathbb{S}^2}$ is the standard (constant curvature) geodesic metric on the sphere, and μ is the pushforward of the volume measure of (\mathbb{S}^2, g) under a conformal map. In particular, one knows little about μ except that it is nonatomic. On the other hand, our conformal metric dist_ω is chosen to interact nicely with the counting measure on the graph G .

Quasisymmetric uniformization. The conformal growth exponent is—at least philosophically—related to Pansu’s notion of *conformal dimension* [58]. For a metric space X , this is the infimal Hausdorff dimension of all metric spaces that are quasisymmetrically equivalent to X . A closely related notion occurs implicitly in a paper of Bourdon and Pajot [24] and is defined by explicitly by Bonk and Kleiner [23]: The *Ahlfors regular conformal dimension*

of X is the infimal d such that X is quasimetric to an Ahlfors d -regular metric measure space.

The quasimetric uniformization problem asks when a metric space can be quasimetrically parameterized by a class of model spaces. We refer to the ICM lectures of Bonk [21] and Kleiner [43] for a survey of this area.

1.6. *Preliminaries.* We use the notation $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$. We also employ the asymptotic notation $A \lesssim B$ and $A \leq O(B)$ to denote that $A \leq c \cdot B$ where $c > 0$ is a positive constant that is independent of other parameters. Since the symbol \lesssim appears only finitely many times in this paper, one could in fact take $c > 0$ to be a fixed universal constant.

We sometimes write $[n] = \{1, 2, \dots, n\}$. When X is a finite set and $f : X \rightarrow \mathbb{R}$, we use the notation

$$\|f\|_{\ell^2(X)} := \sqrt{\sum_{x \in X} f(x)^2},$$

$$\|f\|_{L^2(X)} := \sqrt{\frac{1}{|X|} \sum_{x \in X} f(x)^2}.$$

All graphs appearing in this paper are undirected and locally finite and without loops or multiple edges. If G is such a graph, we use $V(G)$ and $E(G)$ to denote the vertex and edge set of G , respectively. If $S \subseteq V(G)$, we use $G[S]$ for the induced subgraph on S . For $A, B \subseteq V(G)$, we write $E_G(A, B)$ for the set of edges with one endpoint in A and the other in B . We write dist_G for the unweighted path metric on $V(G)$, and $B_G(x, r) = \{y \in V(G) : \text{dist}_G(x, y) \leq r\}$ to denote the closed r -ball around $x \in V(G)$. Also let $\text{deg}_G(x)$ denote the degree of a vertex $x \in V(G)$, and $d_{\max}(G) = \sup_{x \in V(G)} \text{deg}_G(x)$. Write $G_1 \cong G_2$ to denote that G_1 and G_2 are isomorphic as graphs. If (G_1, ρ_1) and (G_2, ρ_2) are rooted graphs, we write $(G_1, \rho_1) \cong_\rho (G_2, \rho_2)$ to denote the existence of a rooted isomorphism.

Consider a pseudometric space (X, d) (i.e., we allow for the possibility that $d(x, y) = 0$ when $x \neq y$). Throughout the paper, we will deal only with complete, separable, pseudometric spaces. For $x \in X$ and two subsets $S, T \subseteq X$, we use the notation $d(S, T) := \inf_{x \in S, y \in T} d(x, y)$ and $d(x, S) := d(\{x\}, S)$. Define $\text{diam}(S, d) := \sup_{x, y \in S} d(x, y)$ and for $R \geq 0$, define the closed balls

$$B_{(X, d)}(x, R) = \{y \in X : d(x, y) \leq R\}.$$

We omit the subscript (X, d) if the underlying metric space is clear from context.

Graph minors and region intersection graphs. If H and G are finite graphs, one says that H is a *minor* of G if H can be obtained from G by a sequence of edge deletions, vertex deletions and edge contractions. If G is infinite, say that H is a minor of G if there is a finite subgraph G' of G that contains an H minor. Recall Kuratowski's theorem: Planar graphs are precisely the graphs that do not contain $K_{3,3}$ or K_5 as a minor.

A graph G is a *region intersection graph over G_0* if the vertices of G correspond to connected subsets of G_0 and there is an edge between two vertices of G precisely when those subsets intersect. More formally, there is a family of connected subsets $\{R_u \subseteq V_0 : u \in V\}$ such that $\{u, v\} \in E \iff R_u \cap R_v \neq \emptyset$. We use $\text{rig}(G_0)$ to denote the family of all *finite* region intersection graphs over G_0 .

A prototypical family of region intersection graphs is the set of *string graphs*; these are the intersection graphs of continuous arcs in the plane. It is not difficult to see that $\text{rig}(\mathbb{Z}^2)$ is precisely the family of all finite string graphs (see [48], Lemma 1.4).

1.6.1. *Unimodular random graphs and distributional limits.* We begin with a discussion of unimodular random graphs and distributional limits. One may consult the extensive reference of Aldous and Lyons [3]. The paper [18] offers a concise introduction to distributional limits of finite planar graphs. We briefly review some relevant points.

Let \mathcal{G} denote the set of isomorphism classes of connected, locally finite graphs; let \mathcal{G}_\bullet denote the set of *rooted* isomorphism classes of *rooted*, connected, locally finite graphs. Define a metric on \mathcal{G}_\bullet as follows: $\mathbb{d}_{\text{loc}}((G_1, \rho_1), (G_2, \rho_2)) = 1/(1 + \alpha)$, where

$$\alpha = \sup\{r > 0 : B_{G_1}(\rho_1, r) \cong_\rho B_{G_2}(\rho_2, r)\}.$$

$(\mathcal{G}_\bullet, \mathbb{d}_{\text{loc}})$ is a separable, complete metric space. For probability measures $\{\mu_n\}$, μ on \mathcal{G}_\bullet , write $\{\mu_n\} \Rightarrow \mu$ when μ_n converges weakly to μ with respect to \mathbb{d}_{loc} .

The mass-transport principle. Let $\mathcal{G}_{\bullet\bullet}$ denote the set of doubly-rooted isomorphism classes of doubly-rooted, connected, locally finite graphs. A probability measure μ on \mathcal{G}_\bullet is *unimodular* if it obeys the following *mass-transport principle*: For all Borel-measurable $F : \mathcal{G}_{\bullet\bullet} \rightarrow [0, \infty]$,

$$(1.16) \quad \int \sum_{x \in V(G)} F(G, \rho, x) d\mu((G, \rho)) = \int \sum_{x \in V(G)} F(G, x, \rho) d\mu((G, \rho)).$$

If (G, ρ) is a random rooted graph with law μ , and μ is unimodular, we say that (G, ρ) is a *unimodular random graph*.

Distributional limits of finite graphs. As observed by Benjamini and Schramm [18], distributional limits of finite graphs are unimodular random graphs. Consider a (possibly random) sequence $\{G_n\} \subseteq \mathcal{G}$ of finite graphs, and let ρ_n denote a uniformly random element of $V(G_n)$. Then $\{(G_n, \rho_n)\}$ is a sequence of \mathcal{G}_\bullet -valued random variables, and one has the following.

LEMMA 1.18. *If $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$, then (G, ρ) is unimodular.*

If $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$, we say that (G, ρ) is the *distributional limit* of the sequence $\{(G_n, \rho_n)\}$. When $\{G_n\}$ is a (possibly random) sequence of finite graphs, we write $\{G_n\} \Rightarrow (G, \rho)$ for $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$ where $\rho_n \in V(G_n)$ is chosen uniformly at random.

Unimodular random conformal graphs. A *conformal graph* is a pair (G, ω) , where G is a connected, locally finite graph and $\omega : V(G) \rightarrow \mathbb{R}_+$. Let \mathcal{G}^* and \mathcal{G}_\bullet^* denote the collections of isomorphism classes of conformal graphs and conformal rooted graphs, respectively. As in Section 1.6.1, one can define a metric on \mathcal{G}_\bullet^* as follows: $\mathbb{d}_{\text{loc}}^*((G_1, \omega_1, \rho_1), (G_2, \omega_2, \rho_2)) = 1/(\alpha + 1)$, where

$$\alpha = \sup\left\{r > 0 : B_{G_1}(\rho_1, r) \cong_\rho B_{G_2}(\rho_2, r) \text{ and } \mathbb{d}(\omega_1|_{B_{G_1}(\rho_1, r)}, \omega_2|_{B_{G_2}(\rho_2, r)}) \leq \frac{1}{r}\right\},$$

where for two weights $\omega_1 : V(H_1) \rightarrow \mathbb{R}_+$ and $\omega_2 : V(H_2) \rightarrow \mathbb{R}_+$ on rooted-isomorphic graphs (H_1, ρ_1) and (H_2, ρ_2) , we write

$$\mathbb{d}(\omega_1, \omega_2) := \inf_{\psi : V(H_1) \rightarrow V(H_2)} \|\omega_2 \circ \psi - \omega_1\|_{\ell^\infty},$$

where the infimum is over all graph isomorphisms from H_1 to H_2 satisfying $\psi(\rho_1) = \rho_2$.

If $\{\mu_n\}$ and μ are probability measures on \mathcal{G}_\bullet^* , we abuse notation and write $\{\mu_n\} \Rightarrow \mu$ to denote weak convergence with respect to $\mathbb{d}_{\text{loc}}^*$. One defines unimodularity of a random rooted conformal graph (G, ω, ρ) analogously to (1.16): It should now hold that for all Borel-measurable $F : \mathcal{G}_{\bullet\bullet}^* \rightarrow [0, \infty]$,

$$\int \sum_{x \in V(G)} F(G, \omega, \rho, x) d\mu((G, \omega, \rho)) = \int \sum_{x \in V(G)} F(G, \omega, x, \rho) d\mu((G, \omega, \rho)).$$

Indeed, such decorated graphs are a special case of the marked networks considered in [3], and again it holds that every distributional limit of finite unimodular random conformal graphs is a unimodular random conformal graph.

Given a random conformal graph (G, ω, ρ) , we define

$$\|\omega\|_{L^2} := \sqrt{\mathbb{E} \omega(\rho)^2}.$$

Say that ω is *normalized* if $\|\omega\|_{L^2} = 1$.

Suppose that (G, ρ) is a unimodular random graph. A *conformal weight* on (G, ρ) is a unimodular random conformal graph (G', ω, ρ') such that (G, ρ) and (G', ρ') have the same law. We will speak simply of a “conformal metric ω on (G, ρ) .” Only such unimodular metrics are considered in this work.

Convergence of infinite sums of metrics. The following construction will occasionally be useful. Consider a family $\{\omega_j : j \geq 1\}$ of normalized conformal weights on (G, ρ) . Recall that formally this is a sequence $\{(G_j, \omega_j, \rho_j) : j \geq 1\}$ such that (G_j, ω_j, ρ_j) is a unimodular random conformal graph and (G_j, ρ_j) has the same law as (G, ρ) for every $j \geq 1$. Thus we may consider a joint coupling $(G, \{\omega_j : j \geq 1\}, \rho)$.

Now we would like to define a unimodular random conformal graph (G, ω, ρ) where

$$\omega := \sqrt{\sum_{j \geq 1} \frac{\omega_j^2}{j^2}}.$$

A priori, it is not clear that almost surely this sum converges for every $x \in V(G)$. So let us momentarily allow $\omega : V(G) \rightarrow [0, +\infty]$ to take extended real values. Fix $r \geq 1$ and define the transport $F(G, \omega, x, y) := \mathbb{1}_{\{\text{dist}_G(x,y) \leq r\}} \mathbb{1}_{\{\omega(y) = +\infty\}}$. Then by the mass-transport principle:

$$\begin{aligned} \mathbb{P}\left[\max_{x \in B_G(\rho, r)} \omega(x) = +\infty\right] &= \mathbb{E}\left[\sum_{x \in V(G)} F(G, \omega, \rho, x)\right] \\ &= \mathbb{E}\left[\sum_{x \in V(G)} F(G, \omega, x, \rho)\right] = \mathbb{E}[\mathbb{1}_{B_G(\rho, r)} \mathbb{1}_{\{\omega(\rho) = +\infty\}}] = 0, \end{aligned}$$

where the latter equality follows from the fact that each ω_j is normalized, hence $\mathbb{E}[\omega(\rho)^2] < \infty$. Since this holds for every $r \geq 1$, we conclude that almost surely, $\sup_{x \in V(G)} \omega(x) < \infty$.

2. Quadratic conformal growth and recurrence. The assumption $\dim_{\text{cg}}(G, \rho) \leq 2$ is not sufficient to ensure almost sure recurrence of G . Instead, we need a more delicate way to measure quadratic growth using a family of metrics. Recall that a unimodular random graph (G, ρ) is (C, R) -quadratic for $C > 0$ and $R \geq 1$ if

$$(2.1) \quad \inf_{\omega} \|\#B_{\omega}(\rho, R)\|_{L^\infty} \leq CR^2,$$

where the infimum is over all normalized conformal metrics on (G, ρ) . Say that (G, ρ) has *gauged quadratic conformal growth* ($gQCG$) if there is a constant $C > 0$ such that (G, ρ) is (C, R) -quadratic for all $R \geq 1$. Note that we allow a different conformal weight ω for every choice of R .

A sequence $\{(G_n, \rho_n)\}$ of unimodular random graphs has *uniform gQCG* if there is a constant $C > 0$ such that for (G_n, ρ_n) is (C, R) -quadratic for all $R \geq 1$ and $n \geq 1$. A family $\mathcal{F} \subseteq \mathcal{G}$ of finite graphs has *uniform gQCG* if the family of unimodular random graphs $\{(G, \rho) : G \in \mathcal{F}\}$ has uniform gQCG, where $\rho \in V(G)$ is chosen uniformly at random.

Finally, we say that (G, ρ) has *asymptotic gQCG* if there is a constant $C > 0$ and a sequence of radii $\{R_n\}$ with $R_n \rightarrow \infty$ such that (G, ρ) is (C, R_n) -quadratic for all $n \geq 1$. We can now state the main theorem of this section. The proof appears in Section 2.2.

THEOREM 2.1. *If (G, ρ) has asymptotic gQCG and $\|\text{deg}_G(\rho)\|_{L^\infty} < \infty$, then G is almost surely recurrent.*

REMARK 2.2. Note that $(\mathbb{Z}^2, 0)$ has gQCG and, moreover, one can consider a single conformal weight $\omega \equiv 1$ for every $R \geq 1$. On the other hand, the infinite ternary tree does not have gQCG.

REMARK 2.3. Note that if (G, ρ) has gQCG, then $\dim_{\text{cg}}(G, \rho) \leq 2$. To see this, consider the corresponding family of normalized conformal weights $\{\omega_{2^k} : k \geq 1\}$ arising from applying (2.1) with $R = 2^k$, and let

$$\omega := \sqrt{\frac{6}{\pi^2} \sum_{k \geq 1} \frac{\omega_{2^k}^2}{k^2}}.$$

Then $\|\omega\|_{L^2} = 1$ and, moreover, $\omega \geq \frac{\sqrt{6}}{k\pi} \omega_{2^k}$ for every $k \geq 1$, hence

$$\left\| \#B_\omega\left(\rho, \frac{\sqrt{6}}{k\pi} 2^k\right) \right\|_{L^\infty} \leq \left\| \#B_{\omega_{2^k}}(\rho, 2^k) \right\|_{L^\infty} \leq C4^k,$$

implying that $\dim_{\text{cg}}(G, \rho) \leq 2$.

REMARK 2.4. In Section 2.3, we will show that the family of all finite planar graphs has uniform gQCG. But for this to hold, it must be that we allow $\omega = \omega_R$ to depend on the scale R in (2.1). Indeed, let T_n denote the complete binary tree of height n , then for some constant $c > 0$, and any normalized conformal metric $\omega : V(T_n) \rightarrow \mathbb{R}_+$,

$$(2.2) \quad \max_{R \geq 0} \frac{|B_\omega(x, R)|}{R^2} \geq c\sqrt{n}.$$

This is proved in Lemma 2.16. Let (T, ρ) denote the distributional limit of $\{T_n\}$ (this is known as the ‘‘canopy tree’’ from [2]). Then (2.2) implies that no single normalized conformal metric ω on (T, ρ) can have quadratic growth.

2.1. Comparing graph balls to conformal balls. In order to use a conformal weight $\omega : V(G) \rightarrow \mathbb{R}_+$ to establish recurrence, we will need a way of comparing the conformal metric dist_ω to the graph metric dist_G . Say that a conformal graph (G, ω) is *C-regulated* if it satisfies the following properties:

1. $\omega(x) \geq 1/2$ for all $x \in V(G)$.
2. If $\{u, v\} \in E(G)$, then $\omega(u) \leq C\omega(v)$.

This definition allows us to compare balls in the metrics dist_G and dist_ω .

LEMMA 2.5. *If (G, ω) is C-regulated for some $C \geq 2$, then it holds that for every $x \in V(G)$ and $r \geq 0$,*

$$B_G\left(x, \frac{\log \frac{r}{2\omega(x)}}{\log C}\right) \subseteq B_\omega(x, r) \subseteq B_G(x, 2r).$$

PROOF. The latter inclusion is straightforward from property (1) of *C-regulated*. The proof of the former inclusion is by induction. Trivially, $B_G(x, 0) \subseteq B_\omega(x, r)$. Suppose that

$B_G(x, k - 1) \subseteq B_\omega(x, r)$ and $v \in B_G(x, k)$. Property (2) of C -regulated yields $\omega(v) \leq \omega(x)C^k$, which implies inductively that

$$\text{dist}_\omega(x, v) \leq \omega(x) \sum_{j=0}^k C^j \leq 2\omega(x)C^k \leq r,$$

as long as $k \leq \frac{\log \frac{r}{2\omega(x)}}{\log C}$. \square

Now let us see that when the degrees are uniformly bounded, one can convert any conformal weight into a C -regulated weight where C is a constant depending only on the maximum degree.

LEMMA 2.6. *Let (G, ω, ρ) be a normalized unimodular random conformal graph, and suppose that $d := \|\text{deg}_G(\rho)\|_{L^\infty}$. Then there exists a normalized, $\sqrt{2d}$ -regulated unimodular random conformal graph $(G, \hat{\omega}, \rho)$ such that $\hat{\omega} \geq \frac{1}{2}\omega$.*

PROOF. For a conformal pair (G, ω) , we define

$$\omega_0(x) = \sqrt{\sum_{y \in V(G)} \omega(y)^2 (2d)^{-\text{dist}_G(x,y)}}.$$

Notice that $\omega_0 \geq \omega$ pointwise and, moreover, for $\{u, v\} \in E(G)$, we have

$$(2.3) \quad \omega_0(u)^2 \leq 2d\omega_0(v)^2$$

by construction.

In order to analyze $\|\omega_0\|_{L^2}$, we define a mass transportation: For $x, y \in V(G)$,

$$F(G, \omega, x, y) = \omega(x)^2 (2d)^{-\text{dist}_G(x,y)}.$$

Note that the total flow out of x is bounded by

$$\omega(x)^2 \sum_{y \in V(G)} (2d)^{-\text{dist}_G(x,y)} \leq \omega(x)^2 \sum_{k \geq 0} 2^{-k} \leq 2\omega(x)^2.$$

Therefore, by the mass-transport principle, it holds that

$$\begin{aligned} 2\mathbb{E}[\omega(\rho)^2] &\geq \mathbb{E}\left[\sum_{x \in V(G)} F(G, \omega, \rho, x)\right] \\ &= \mathbb{E}\left[\sum_{x \in V(G)} F(G, \omega, x, \rho)\right] \\ &= \mathbb{E}[\omega_0(\rho)^2], \end{aligned}$$

where the last equality follows from the definition of ω_0 and F . In particular, we conclude that $\mathbb{E}[\omega_0(\rho)^2] \leq 2\mathbb{E}[\omega(\rho)^2] = 2$.

Now define the normalized weight

$$\hat{\omega} := \frac{\sqrt{\frac{1}{4}\mathbb{1} + \frac{3}{8}\omega_0^2}}{\sqrt{\frac{1}{4} + \frac{3}{8}\mathbb{E}[\omega_0^2]}}.$$

It satisfies property (1) of C -regulated by construction, and also $\hat{\omega} \geq \frac{1}{2}\omega_0 \geq \frac{1}{2}\omega$ pointwise. Furthermore, property (2) of C -regulated is a consequence of (2.3) with $C = \sqrt{2d}$. \square

2.2. *Bounding the effective resistance.* In the present section, we will use the notion of the *effective resistance* $R_{\text{eff}}^G(S \leftrightarrow T)$ between two subsets $S, T \subseteq V(G)$ in a graph. For completeness, we present one definition that aligns with our use of the quantity; for more background, we refer the reader to [53], Chapters 2 and 9. For $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, the Dirichlet principle asserts that

$$R_{\text{eff}}^G(S \leftrightarrow T) = \left(\inf_{f \in \mathcal{F}_{S,T}} \mathcal{E}(f) \right)^{-1},$$

where $\mathcal{F}_{S,T} = \{f : V(G) \rightarrow \mathbb{R} \mid f|_S = 0, f|_T = 1\}$, and we recall the (unnormalized) Dirichlet energy functional

$$\mathcal{E}(f) := \sum_{\{x,y\} \in E(G)} |f(x) - f(y)|^2.$$

Let us define $R_{\text{eff}}^G(S \leftrightarrow T) = 0$ when $S \cap T \neq \emptyset$. We will soon prove Theorem 2.1 using the following well-known characterization; see, for example, [53], Lemma 9.22.

THEOREM 2.7. *A graph G is recurrent if and only if there is some vertex $x \in V(G)$ and constant $c > 0$ such that for all $R \geq 0$, there is a finite set $S_R \subseteq V(G)$ such that*

$$R_{\text{eff}}^G(B(x, R) \leftrightarrow V(G) \setminus S_R) \geq c.$$

First, we will need a lemma about the expected area of balls. Let us define

$$\mathcal{A}_\omega(x, R) := \sum_{y \in B_\omega(x, R)} \omega(y)^2.$$

LEMMA 2.8. *Let (G, ω, ρ) be a unimodular random conformal graph with $\mathbb{E} \omega(\rho)^2 = 1$. Then for every $R \geq 1$,*

$$\mathbb{E}[\mathcal{A}_\omega(\rho, R)] \leq \|\#B_\omega(\rho, R)\|_{L^\infty}.$$

PROOF. We employ the mass-transport principle: For a conformal graph (G, ω) and $x, y \in V(G)$, define the flow

$$F(G, \omega, x, y) = \omega(x)^2 \mathbb{1}_{\{\text{dist}_\omega(x,y) \leq R\}}.$$

Then

$$\begin{aligned} \mathbb{E}[\mathcal{A}_\omega(\rho, R)] &= \mathbb{E} \left[\sum_{x \in V(G)} F(G, \omega, x, \rho) \right] = \mathbb{E} \left[\sum_{x \in V(G)} F(G, \omega, \rho, x) \right] \\ &= \mathbb{E}[\omega(\rho)^2 |B_\omega(\rho, R)|] \leq \|\#B_\omega(\rho, R)\|_{L^\infty}. \quad \square \end{aligned}$$

In order to apply Theorem 2.7, we use a conformal weight to construct a test function of small energy.

LEMMA 2.9. *Consider a graph G , vertex $x \in V(G)$, and scale $R \geq 0$. Then for any C -regulated conformal weight $\omega : V(G) \rightarrow \mathbb{R}_+$, it holds that*

$$R_{\text{eff}} \left(B_G \left(x, \frac{\log \frac{R}{4\omega(x)}}{\log C} \right) \leftrightarrow V(G) \setminus B_G(x, 2R) \right) \geq \frac{1}{4(1+C)^2 d_{\max}(G)} \cdot \frac{R^2}{\mathcal{A}_\omega(x, R)}.$$

PROOF. Define $f : V(G) \rightarrow \mathbb{R}$ by

$$f(x) = \frac{2}{R} \min\left(\frac{R}{2}, \max\left(0, \text{dist}_\omega(x, v) - \frac{R}{2}\right)\right).$$

Note that $v \in B_\omega(x, \frac{R}{2}) \implies f(v) = 0$ and $v \notin B_\omega(x, R) \implies f(v) = 1$.

Therefore, by the Dirichlet principle,

$$R_{\text{eff}}\left(B_\omega\left(x, \frac{R}{2}\right) \leftrightarrow V(G) \setminus B_\omega(x, R)\right) \geq \frac{1}{\mathcal{E}_G(f)},$$

and

$$\begin{aligned} \mathcal{E}(f) &\leq \frac{4d_{\max}(G)}{R^2} (1 + C)^2 \sum_{v \in B_\omega(x, R)} \omega(v)^2 \\ &= 4d_{\max}(G)(1 + C)^2 \frac{\mathcal{A}_\omega(x, R)}{R^2}, \end{aligned}$$

where we have used the fact that f is $2/R$ -Lipschitz, and the fact that ω is C -regulated which, in particular, asserts that for $\{u, v\} \in E(G)$, one has

$$\text{dist}_\omega(u, v)^2 \leq \omega_k(u)^2 + \omega_k(v)^2 \leq (1 + C)^2 \omega_k(v)^2.$$

Finally, we use Lemma 2.5 to arrive at the desired conclusion, replacing dist_ω balls by dist_G balls. \square

PROOF OF THEOREM 2.1. Consider a radius $R \geq 1$ and a normalized, C -regulated conformal weight $\omega : V(G) \rightarrow \mathbb{R}_+$ satisfying $\| \#B_\omega(\rho, R) \|_{L^\infty} \leq cR^2$ for some constant $c > 0$. From Lemma 2.8, we have $\mathbb{E}[\mathcal{A}_\omega(\rho, R)] \leq cR^2$, hence employing Markov’s inequality,

$$\mathbb{P}\left[\omega(\rho)^2 < R \text{ and } \mathcal{A}_\omega(\rho, R) < \frac{c}{\varepsilon} R^2\right] \geq 1 - \varepsilon - \frac{1}{R}.$$

Combining this with Lemma 2.9, we see that

$$(2.4) \quad \mathbb{P}\left[R_{\text{eff}}\left(B_G\left(\rho, \frac{\log(R/4)}{2 \log C}\right) \leftrightarrow V(G) \setminus B_G(\rho, 2R)\right) \geq c'\varepsilon\right] \geq 1 - \varepsilon - \frac{1}{R},$$

where c' is a constant depending only on C and $\|\text{deg}_G(\rho)\|_{L^\infty}$.

By assumption, (G, ρ) has asymptotic gQCG and $\|\text{deg}_G(\rho)\|_{L^\infty} < \infty$. Combining the definition of asymptotic gQCG with Lemma 2.6 (to derive a C -regulated conformal metric) shows that (2.4) holds for $R = R_n$, where $\{R_n\}$ is a sequence of radii with $R_n \rightarrow \infty$.

In particular, Fatou’s lemma tells us that

$$\mathbb{P}\left[\limsup_{n \rightarrow \infty} R_{\text{eff}}\left(B_G\left(\rho, \frac{\log(R_n/4)}{2 \log C}\right) \leftrightarrow V(G) \setminus B_G(\rho, 2R_n)\right) \geq c'\varepsilon\right] \geq 1 - \varepsilon.$$

Since $\{B_G(\rho, 2R_n)\}$ is a sequence of finite sets, Theorem 2.7 yields

$$\mathbb{P}[G \text{ recurrent}] \geq 1 - \varepsilon.$$

Sending $\varepsilon \rightarrow 0$ completes the proof. \square

2.3. *Region intersection graphs and energy-minimizing conformal weights.* The next two theorems essentially follow from prior work. Note that for the special case of planar graphs, an alternate proof of the next result based on circle packings appears in [49]. It has the advantage that it extends suitably to graphs that are sphere-packed in any Euclidean space.

THEOREM 2.10 (Uniform gQCG for H -minor-free graphs [41]). *For every fixed graph H , the family of finite graphs excluding H as a minor has uniform gQCG. In particular, if $H = K_h$ for some $h \geq 2$, then every such graph is (κ, R) -quadratic for all $R \geq 1$, where $\kappa \leq O(h^2 \log h)$.*

THEOREM 2.11 (Uniform gQCG for region intersection graphs [48]). *For every $\lambda > 0$ and fixed graph H , the family of finite region intersection graphs G over an H -minor-free graph with $d_{\max}(G) \leq \lambda$ has uniform gQCG. In particular, if $H = K_h$ for some $h \geq 2$, then every such graph is (κ, R) -quadratic for all $R \geq 1$, where $\kappa \leq O(\lambda h^2 \log h)$.*

Let us remark on the proof of these results. Fix a finite graph $G = (V, E)$. Let $n = |V|$ and consider a positive number $k \leq n$. Define the set $P_k(G) \subseteq \ell^2(V)$ by

$$P_k(G) := \left\{ \omega \geq 0 : \frac{1}{k^2} \sum_{x,y \in S} \text{dist}_\omega(x, y) \geq 1 \ \forall S \subseteq V, |S| \geq k \right\}.$$

While it may not be immediately apparent, the set $P_k(G)$ is a polytope because one can replace every such inequality indexed by a subset $S \subseteq V$ with the family of inequalities:

$$\frac{1}{k^2} \sum_{x,y \in S} \text{len}_\omega(\gamma_{xy}) \geq 1 \quad \forall \{\gamma_{xy} : x, y \in S\},$$

where in the latter quantifier, γ_{xy} ranges over all simple x - y paths in G . Since $\text{len}_\omega(\gamma)$ is a linear function in the values $\{\omega(z) : z \in V\}$, the claim follows.

LEMMA 2.12. *For any $z \in V$ and $\omega \in P_k(G)$, it holds that $|B_\omega(z, 1/2)| < k$.*

PROOF. Denote $S := B_\omega(z, 1/2)$. If $|S| \geq k$, let $S' \subseteq S$ be a subset with $|S'| = k$. Since $\omega \in P_k(G)$, we have

$$\frac{1}{|S'|^2} \sum_{x,y \in S'} \text{dist}_\omega(x, y) \geq 1,$$

but this is a contradiction since $\text{diam}_\omega(S') \leq 1$. \square

Consider now the optimization problem

$$(2.5) \quad \theta_k(G) := \min\{\|\omega\|_{L^2(V)} : \omega \in P_k(G)\}.$$

We claim that the values $\{\theta_k(G) : k = 1, 2, \dots, n\}$ dictate the quadratic conformal growth: G is (κ, R) -quadratic for some $\kappa \geq 1$ and every $R \geq 0$ if and only if $\theta_k(G) \leq C/k^2$ for some $C \geq 1$ and every $k \leq n$.

CLAIM 2.13. *For every n -vertex graph $G = (V, E)$, the following holds:*

1. For every $k \leq n$, G is (κ, R) -quadratic with $\kappa = 4k\theta_k(G)^2$ and $R = 1/(2\theta_k(G))$.
2. For every $\kappa, R > 0$, it holds that if G is (κ, R) -quadratic, then

$$\theta_k(G) \leq \frac{2}{R} \quad \text{for all } k \geq 2\kappa R^2.$$

PROOF. Let us first prove (1). Consider $\omega \in P_k(G)$ with $\theta := \|\omega\|_{L^2(V)}$ and define $\hat{\omega} := \omega/\theta$. Then by Lemma 2.12, for any $z \in V$,

$$|B_\omega(z, 1/(2\theta))| = |B_{\hat{\omega}}(z, 1/2)| < k,$$

implying that G is $(4\theta^2k, 1/(2\theta))$ -quadratic.

To prove (2), consider $\omega : V \rightarrow \mathbb{R}_+$ such that $\|\omega\|_{L^2(V)} = 1$ and $|B_\omega(x, R)| \leq \kappa R^2$ for all $x \in V$. Consider any $S \subseteq V$ with $|S| \geq 2\kappa R^2$. Then for every $x \in S$, it holds that

$$|B_\omega(x, R) \cap S| \leq \frac{1}{2}|S|.$$

Hence,

$$(2.6) \quad \frac{1}{|S|^2} \sum_{x,y \in S} \text{dist}_\omega(x, y) \geq \frac{1}{|S|^2} \sum_{x \in S} \frac{|S|}{2} R \geq \frac{R}{2}.$$

If we now set $\hat{\omega} := (2/R)\omega$, then (2.6) gives $\hat{\omega} \in P_k(G)$ for any $k \geq 2\kappa R^2$; hence, $\theta_k(G) \leq 2/R$ for all such k . \square

Theorem 2.10 and Theorem 2.11 are proved by analyzing the optimization (2.5). It entails minimizing a strongly convex function over a polytope, thus (2.5) has a unique optimal solution. The authors of [19] developed a “flow crossing” theory for understanding the dual optimization problem, and that was expanded upon in the works [41, 48]. The following is a consequence of [49], Theorem 1.13.

LEMMA 2.14. *If $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$ and $\{(G_n, \rho_n)\}$ has uniform gQCG, then (G, ρ) has gQCG. In particular, if (G, ρ) is a distributional limit of finite H -minor-free graphs, then (G, ρ) has gQCG.*

In particular, combining Lemma 2.14 with the preceding two theorems and Theorem 2.1 yields the following corollary. We recall that $\text{rig}(\mathcal{F}(H))$ is the set of all finite graphs that are region intersection graphs over some graph G_0 that excludes the graph H as a minor.

COROLLARY 2.15. *For every fixed graph H , if $\{G_n\} \subseteq \text{rig}(\mathcal{F}(H))$ is a sequence of graphs with uniformly bounded degrees and $\{G_n\} \Rightarrow (G, \rho)$, then G is almost surely recurrent.*

We end this section by observing that one cannot equip every finite planar graph with a single normalized metric that achieves uniform quadratic volume growth at all scales simultaneously (thus justifying the necessity for multiple conformal weights in the definition of gQCG).

LEMMA 2.16. *There is a constant $C > 0$ such that the following holds. Let T_n be the complete binary tree of height n , and consider any normalized conformal weight ω on T_n . Then*

$$\max_{x \in V(T_n)} \max_{R \geq 0} \frac{|B_\omega(x, R)|}{R^2} \geq C\sqrt{n}.$$

PROOF. Let $\omega : V(T_n) \rightarrow \mathbb{R}_+$ be a conformal weight satisfying $|B_\omega(x, R)| \leq R^2$ for all $x \in V(T_n)$ and $R \geq 1$. Consider the family \mathcal{P} of $\binom{2^n}{2}$ paths in T_n between all the leaves of T_n . There must exist a constant $c > 0$ and a subset $\mathcal{P}_n \subseteq \mathcal{P}$ of paths going through the root of T_n

with $|\mathcal{P}_n| \geq c2^{2n}$ and such that every path in \mathcal{P}_n has ω -length at least $c2^{n/2}$. (Otherwise, there would be some leaf that could reach $c2^n$ other leaves using paths of length $\ll 2^{n/2}$, contradicting the quadratic volume assumption.) Similarly, there exist disjoint subsets $\mathcal{P}_{n-1}^{(0)}, \mathcal{P}_{n-1}^{(1)} \subseteq \mathcal{P}$ of paths in the left and right subtrees, each containing $c2^{2(n-1)}$ paths of ω -length at least $c2^{(n-1)/2}$, and so on.

Let $\mathcal{P}_k = \bigcup_{j \in \{0,1\}^{n-k}} \mathcal{P}_k^{(j)}$ be the set of such ‘‘long’’ paths in subtrees of height k . Observe that this union is disjoint by construction. For a vertex $v \in V(T_n)$, define

$$\alpha(v) = \sum_{k=1}^n 2^{-3k/2} \#\{\gamma \in \mathcal{P}_k : v \in \gamma\}.$$

Then we have

$$\begin{aligned} \sum_{k=1}^n c^2 2^{-3k/2} 2^{n-k} 2^{2k} 2^{k/2} &\leq \sum_{k=1}^n 2^{-3k/2} |\mathcal{P}_k| \min_{\gamma \in \mathcal{P}_k} \text{len}_\omega(\gamma) \leq \sum_{v \in V(T_n)} \alpha(v) \omega(v) \\ &\leq \|\alpha\|_{\ell^2(V(T_n))} \|\omega\|_{\ell^2(V(T_n))}, \end{aligned}$$

where the last inequality is Cauchy–Schwarz. The left-hand side is $c^2 n 2^n$.

Now a simple calculation yields

$$\sum_{v \in V(T_n)} \alpha(v)^2 \leq \sum_{k=1}^n 2^{n-k} 2^{4k} 2^{-3k} = n 2^n.$$

We conclude that

$$\|\omega\|_{\ell^2(V(T_n))} \geq c^2 \sqrt{n 2^n},$$

implying that $\|\omega\|_{L^2} = 2^{-n/2} \|\omega\|_{\ell^2(V(T_n))} \geq c^2 \sqrt{n}$, and completing the argument. \square

3. Return probabilities and spectral geometry on finite graphs. We now turn to heat kernel estimates on finite graphs.

3.1. *The normalized Laplacian spectrum.* Let $G = (V, E)$ be a connected, finite graph with $n = |V|$. Let $\pi(x) = \frac{\deg_G(x)}{2|E|}$ denote the stationary measure. We will use $L^2(\pi)$ for the Hilbert space of functions $f : V \rightarrow \mathbb{R}$ equipped with the inner product

$$\langle f, g \rangle_\pi = \sum_{x \in V} \pi(x) f(x) g(x),$$

and denote by

$$\langle f, g \rangle = \sum_{x \in V} f(x) g(x)$$

the inner product on $\ell^2(V)$. We use $\|\cdot\| := \|\cdot\|_{\ell^2(V)}$ and $\|f\|_\pi = \sqrt{\langle f, f \rangle_\pi}$.

Define the operators $A, D, P, L, \mathcal{L} : \ell^2(V) \rightarrow \ell^2(V)$ as follows:

$$Af(x) := \sum_{y: \{x,y\} \in E} f(y),$$

$$Df(x) := \deg_G(x) f(x),$$

$$P := D^{-1}A,$$

$$L := I - P,$$

$$\mathcal{L} := I - D^{-1/2}AD^{-1/2}.$$

The *normalized Laplacian* \mathcal{L} is symmetric and positive semidefinite. We denote its eigenvalues by

$$0 = \lambda_0(G) \leq \lambda_1(G) \leq \dots \leq \lambda_{n-1}(G).$$

We use $\lambda_k := \lambda_k(G)$ if the graph G is clear from context. Note that $P = I - D^{-1/2}\mathcal{L}D^{1/2}$, so if $\mathcal{L}f = \lambda f$, then $PD^{-1/2}f = (1 - \lambda)D^{-1/2}f$. Thus the spectrum of P is $\{1 - \lambda_k(G) : k = 0, 1, \dots, n - 1\}$.

Define the *Rayleigh quotient* $\mathcal{R}_G(f)$ of nonzero $f \in L^2(\pi)$ by

$$\mathcal{R}_G(f) := \frac{\langle D^{1/2}f, \mathcal{L}D^{1/2}f \rangle}{\langle D^{1/2}f, D^{1/2}f \rangle} = \frac{\langle f, Lf \rangle_\pi}{\|f\|_\pi^2} = \frac{\frac{1}{|E|} \sum_{\{x,y\} \in E} |f(x) - f(y)|^2}{\|f\|_\pi^2}.$$

Recall also the variational formula for eigenvalues:

$$(3.1) \quad \lambda_k(G) = \min_{U \subseteq L^2(\pi), 0 \neq f \in U} \max \mathcal{R}_G(f),$$

where the minimum is over all subspaces $U \subseteq L^2(\pi)$ with $\dim(U) = k + 1$. The preceding fact has a useful corollary.

COROLLARY 3.1. *Suppose that $\psi_1, \dots, \psi_r : V \rightarrow \mathbb{R}$ are disjointly supported functions with $\mathcal{R}_G(\psi_i) \leq \theta$ for $i = 1, 2, \dots, r$. Then*

$$\lambda_{r-1}(G) \leq 2\theta.$$

PROOF. Let $U = \text{span}(\psi_1, \dots, \psi_r)$, and note that $\dim(U) = r$ since $\{\psi_i\}$ are mutually orthogonal. Consider $f \in U$ and write $f = \sum_{i=1}^r \alpha_i \psi_i$.

Since the functionals have mutually disjoint supports, for any $x, y \in V$, we have

$$|f(x) - f(y)|^2 \leq 2 \sum_{i=1}^r \alpha_i^2 |\psi_i(x) - \psi_i(y)|^2.$$

Therefore,

$$\mathcal{R}_G(f) \leq \frac{2 \sum_{i=1}^r \alpha_i^2 \|\psi_i(x) - \psi_i(y)\|^2}{\sum_{i=1}^r \alpha_i^2 \|\psi_i\|_\pi^2} \leq 2\theta.$$

Now the claim follows from $\dim(U) = r$ and the variational characterization of eigenvalues. □

Relation to return probabilities. Let $\{\phi_k\}$ be an $L^2(\pi)$ -orthonormal family of eigenfunctions for P such that $P\phi_k = (1 - \lambda_k)\phi_k$ for each $0 \leq k \leq n - 1$. The connection between return probabilities and eigenvalues is straightforward: For any $x \in V$ and $T \geq 1$,

$$(3.2) \quad p_T^G(x, x) = \frac{\langle \mathbb{1}_x, P^T \mathbb{1}_x \rangle_\pi}{\pi(x)} = \sum_{k=0}^{n-1} \pi(x) \phi_k(x)^2 (1 - \lambda_k)^T.$$

3.2. Random partitions. We now introduce a tool that will be used for analyzing the heat kernel in the remainder of this section. Let (X, d) denote a pseudometric space.

Random partitions. For a partition \mathcal{P} of X , we use $\mathcal{P}(x)$ to denote the unique set in \mathcal{P} containing x . We will consider only partitions \mathcal{P} with an at most countable number of elements. Denote

$$\Delta(\mathcal{P}) := \sup\{\text{diam}(S, d) : S \in \mathcal{P}\}.$$

A random partition \mathbf{P} is (τ, α) -padded if it satisfies the following conditions:

1. Almost surely: $\Delta(\mathbf{P}) \leq \tau$.
2. For all $x \in X$ and $\delta > 0$,

$$\mathbb{P}[B(x, \delta\tau/\alpha) \subseteq \mathbf{P}(x)] \geq 1 - \delta.$$

The reader might gain some intuition from considering the case $X = \mathbb{R}^d$ equipped with the Euclidean metric. If one takes \mathbf{P} to be a randomly translated partition of \mathbb{R}^d into axis-aligned cubes of side-length L , then \mathbf{P} is $(L\sqrt{k}, k\sqrt{k})$ -padded.

Uniformly decomposable graph families. We say that a family \mathcal{F} of locally finite, connected graphs is α -decomposable if there is an $\alpha > 0$ such that for every $G \in \mathcal{F}$, every conformal weight $\omega : V(G) \rightarrow \mathbb{R}_+$, and every $\tau > 0$, the metric space $(V(G), \text{dist}_\omega)$ admits a (τ, α) -padded random partition. We say that \mathcal{F} is *uniformly decomposable* if it is α -decomposable for some $\alpha > 0$.

The next result is proved in [48]. The special case for graphs G that themselves exclude K_h as a minor was established much earlier in [42]. For such graphs, the bound $\alpha \leq O(h^2)$ was established in [31], and this was improved to $\alpha \leq O(h)$ in [1]. Let $\mathcal{F}(K_h)$ denote the family of connected, locally finite graphs that exclude K_h as a minor, and denote $\text{rig}(\mathcal{F}(K_h)) := \bigcup_{G_0 \in \mathcal{F}(K_h)} \text{rig}(G_0)$.

THEOREM 3.2 ([48]). *For every $h \geq 1$, the family $\text{rig}(\mathcal{F}(K_h))$ is α -decomposable for some $\alpha \leq O(h^2)$.*

We say that a unimodular random graph (G, ρ) is *uniformly decomposable* if there is an $\alpha > 0$ such that G is almost surely α -decomposable.

We remark that often in the literature (e.g., in [48]), one only exhibits random partitions that satisfy property (2) of a padded partition with $\delta = 1/2$. The following (unpublished) lemma of the author and A. Naor shows that this is sufficient to conclude that it holds for all $\delta \in [0, 1]$, with a small loss in parameters.

LEMMA 3.3. *Suppose that a metric space (X, d) admits a random partition \mathbf{P} with $\Delta(\mathbf{P}) \leq \tau$ almost surely, and for every $x \in X$,*

$$(3.3) \quad \mathbb{P}[B(x, \tau/\alpha) \subseteq \mathbf{P}(x)] \geq \frac{1}{2}.$$

Then there is a random partition \mathbf{P}' with $\Delta(\mathbf{P}') \leq \tau$ almost surely, and such that for every $\delta > 0$ and $x \in X$,

$$(3.4) \quad \mathbb{P}[B(x, \delta\tau/\alpha) \subseteq \mathbf{P}'(x)] \geq 1 - 4\delta.$$

PROOF. For a subset $S \subseteq X$ and a number $\lambda > 0$, denote

$$S_{-\lambda} := \{x \in S : B(x, \lambda) \subseteq S\}.$$

Let $\{\mathbf{P}_k\}$ be an infinite sequence of i.i.d. random partitions with the law of \mathbf{P} . Let $\{\varepsilon_k\}$ be an independent infinite sequence of i.i.d. random variables where $\varepsilon_k \in [0, 1]$ is chosen uniformly

at random. We will define a sequence $\{A_k\}$ where each A_k is a collection of disjoint subsets of X and define $\mathbf{P}' = \bigcup_{k \geq 1} A_k$.

Denote $A_0 = \emptyset, X_0 = \emptyset$ and for $k \geq 1$,

$$A_k = \{S_{-\varepsilon_k \tau / \alpha} \setminus X_{k-1} : S \in \mathbf{P}_k\},$$

$$X_k = X_{k-1} \cup \bigcup_{S \in A_k} S.$$

First, observe that for every $x \in X$, it holds that almost surely $x \in S \in A_k$ for some $k \in \mathbb{N}$. This is because for every $k \geq 1$,

$$(3.5) \quad \mathbb{P}[x \in (P_k(x))_{-\varepsilon_k \tau / \alpha}] \geq \mathbb{P}[B(x, \tau / \alpha) \subseteq P_k(x)] \geq \frac{1}{2},$$

where the latter inequality is from (3.3). Since we deal only with separable metric spaces, to verify that \mathbf{P}' is almost surely a partition, it suffices to consider a dense countable subset of X . Similarly, we can conclude that \mathbf{P}' almost surely satisfies $\Delta(\mathbf{P}') \leq \tau$.

So now we move on to verifying (3.4). Fix $x \in X$ and $\delta \in [0, 1/4]$. Let $R = \delta \tau / \alpha$. Then

$$\mathbb{P}[B(x, R) \not\subseteq \mathbf{P}'(x)] \leq \sum_{k \geq 1} \mathbb{P}[B(x, R) \cap X_{k-1} = \emptyset] \cdot \mathbb{P}[B(x, R) \cap X_k \neq \emptyset \wedge B(x, R) \not\subseteq (P_k(x))_{-\varepsilon_k \tau / \alpha}].$$

Now observe that (3.5) implies $\mathbb{P}[B(x, R) \cap X_{k-1} = \emptyset] \leq 2^{1-k}$.

For $y \in X$, let $\eta_k(y) = \sup\{\eta \geq 0 : B(y, \eta) \subseteq P_k(y)\}$. Note that, conditioned on P_k , η_k is a 1-Lipschitz function. Therefore,

$$\mathbb{P}[B(x, R) \cap X_k \neq \emptyset \wedge B(x, R) \not\subseteq (P_k(x))_{-\varepsilon_k \tau / \alpha}] \leq \mathbb{P}\left(\varepsilon_k \in \left[\frac{\eta_k(x)}{\tau / \alpha} - \delta, \frac{\eta_k(x)}{\tau / \alpha} + \delta\right]\right) \leq 2\delta.$$

We conclude that

$$\mathbb{P}[B(x, R) \not\subseteq \mathbf{P}'(x)] \leq 2\delta \sum_{k \geq 1} 2^{1-k} = 4\delta,$$

completing the proof. \square

3.3. *Eigenvalues and the degree distribution.* Let us define $\Delta_G : \mathbb{Z}_+ \rightarrow \mathbb{N}$ by

$$\Delta_G(k) = \max \left\{ \sum_{x \in S} \deg_G(x) : S \subseteq V, |S| \leq k \right\}.$$

Momentarily, we will prove the following theorem. Say that a graph is (κ, α) -controlled if it is α -decomposable and (κ, R) -quadratic for all $R \geq 1$.

THEOREM 3.4. *Suppose that a family \mathcal{F} of finite graphs has uniform gQCG and is uniformly decomposable. Then there is a constant $c > 0$ such that for every $G \in \mathcal{F}$ and $k = 0, 1, \dots, |V(G)| - 1$,*

$$\lambda_k(G) \leq c \frac{\Delta_G(k)}{|V(G)|}.$$

Quantitatively, if a finite graph G is (κ, α) -controlled, then

$$\lambda_k(G) \lesssim \alpha^2 \kappa \frac{\Delta_G(k)}{|V(G)|}.$$

We present an illustrative corollary of Theorem 2.10 and Theorem 3.2 in conjunction with Theorem 3.4.

COROLLARY 3.5. *Suppose G is an n -vertex graph that excludes K_h as a minor. Then there is a constant $c_h \leq O(h^6 \log h)$ such that for every $k = 0, 1, \dots, n - 1$,*

$$\lambda_k(G) \leq c_h \frac{\Delta_G(k)}{n}.$$

Note that a weaker statement was established in [41] with $\Delta_G(k)$ replaced by $k \cdot d_{\max}(G)$. The proof of Theorem 3.4 is immediate from Corollary 3.1 and the following result.

THEOREM 3.6. *Suppose G is an n -vertex graph and $\omega : V(G) \rightarrow \mathbb{R}_+$ is a normalized conformal weight such that:*

1. For all $x \in V(G)$,

$$|B_\omega(x, R_*)| \leq \kappa R_*^2,$$

where $R_* = \sqrt{\frac{n}{16k \cdot \kappa}}$.

2. $(V(G), \text{dist}_\omega)$ admits an $(\alpha, R_*/2)$ -padded random partition.

Then for every $k \leq n$, there are disjointly supported functions $\psi_1, \dots, \psi_k : V \rightarrow \mathbb{R}$ such that

$$\mathcal{R}_G(\psi_i) \lesssim \alpha^2 \kappa \frac{\Delta_G(k)}{n}.$$

Section 3.5 is devoted to the construction of the bump functions ψ_1, \dots, ψ_k . As discussed in the Introduction, the spectral bounds from Theorem 3.4 are not strong enough to yield almost sure bounds on the heat kernel of a distributional limit. Consulting (3.2), one sees that to control the return probabilities for most vertices $x \in V(G)$ requires us to say something about the distribution of the low-frequency eigenfunctions of G .

3.4. Return probabilities and spectral delocalization. Let us now indicate how a sufficient strengthening of Theorem 3.4 will allow us to control return probabilities for most of the vertices. For our finite graph $G = (V, E)$, it will help to define for every $\varepsilon > 0$:

$$\pi_G^*(\varepsilon) := \max\{\pi(S) : |S| \leq \varepsilon|V|\}.$$

THEOREM 3.7. *Let $G = (V, E)$ be an n -vertex graph. Suppose that for some $k \leq n$, there are capacitors $(A_1, \Omega_1), \dots, (A_k, \Omega_k)$ so that $\{\Omega_i\}$ are pairwise disjoint and $|\Omega_i| \leq M$ for all $i = 1, \dots, k$. Then for all $\varepsilon > 0$ and $T \geq 1$:*

$$(3.6) \quad \pi\left(\left\{x \in V : \frac{p_{2T}^G(x, x)}{\pi(x)} \geq \frac{\varepsilon|V|}{4M}\right\}\right) \geq -2\pi_G^*(\varepsilon) + \sum_{i=1}^k \pi(A_i) - 2T \sum_{i=1}^k \text{cap}_{\Omega_i}^G(A_i).$$

In particular, for any $\beta > 0$,

$$(3.7) \quad \pi\left(\left\{x \in V : p_{2T}^G(x, x) \geq \frac{\varepsilon\beta}{4M}\right\}\right) \geq -2\pi_G^*(\varepsilon) - \beta + \sum_{i=1}^k \pi(A_i) - 2T \sum_{i=1}^k \text{cap}_{\Omega_i}^G(A_i).$$

For illustration, consider a bounded-degree graph planar graph. In Section 3.5, we will show that under this assumption, for every $\varepsilon > 0$ and $M \ll n$, we can find such a family $\{(A_i, \Omega_i)\}$ satisfying

$$\sum_{i=1}^k \pi(A_i) \geq 1 - \varepsilon,$$

and for each $i = 1, \dots, k$,

$$(3.8) \quad \text{cap}_{\Omega_i}^G(A_i) \leq \frac{c(\varepsilon)}{M} \pi(A_i),$$

where $c(\varepsilon)$ is some function of ε . Since our graph has bounded degrees, we have $\pi_G^*(\varepsilon) \leq O(\varepsilon)$, so choosing $T \leq \frac{\varepsilon^2 M}{c(\varepsilon)}$ yields

$$\pi \left(\left\{ x \in V : p_{2T}^G(x, x) \geq \frac{c'(\varepsilon)}{T} \right\} \right) \geq 1 - O(\varepsilon)$$

for some other function $c'(\varepsilon)$.

We will require the following two preliminary results.

LEMMA 3.8. *For any capacitor (A, Ω) and $T \geq 1$,*

$$\langle \mathbb{1}_\Omega, P^T \mathbb{1}_\Omega \rangle_\pi \geq \pi(A) - T \cdot \text{cap}_\Omega^G(A).$$

REMARK 3.9. We remark that the same argument gives an identical lower bound on $\langle \mathbb{1}_\Omega, (I_\Omega P I_\Omega)^T \mathbb{1}_\Omega \rangle_\pi$ where I_Ω is the multiplication operator on $L^2(\pi)$ defined by $I_\Omega f(x) = \mathbb{1}_\Omega(x) f(x)$. Then $I_\Omega P I_\Omega$ is the operator of the walk killed off Ω .

Using reversibility, a lower bound on $\langle \mathbb{1}_S, P^T \mathbb{1}_S \rangle_\pi$ will give us control on return probabilities.

LEMMA 3.10. *Suppose that, for some $S \subseteq V$, we have*

$$\langle \mathbb{1}_S, P^T \mathbb{1}_S \rangle_\pi \geq (1 - \delta)\pi(S).$$

Then for any $\gamma > 0$,

$$\pi \left(\left\{ x \in S : p_{2T}(x, x) \geq \frac{\pi(x)}{4\gamma|S|} \right\} \right) \geq (1 - 2\delta)\pi(S) - 2\pi(\{x \in S : \pi(x) > \gamma\}).$$

PROOF OF THEOREM 3.7. Apply Lemma 3.8 and Lemma 3.10 to each (A_i, Ω_i) with $\delta_i = T \frac{\text{cap}_{\Omega_i}^G(A_i)}{\pi(A_i)}$ and sum over $i = 1, \dots, k$, yielding

$$\sum_{i=1}^k \pi \left(\left\{ x \in A_i : p_{2T}^G(x, x) \geq \frac{\pi(x)}{4\gamma|\Omega_i|} \right\} \right) \geq \sum_{i=1}^k (1 - 2\delta_i)\pi(A_i) - 2\pi(\{x \in V : \pi(x) > \gamma\}),$$

where we have used that the sets $\{A_i\}$ are pairwise disjoint.

The former sum is precisely

$$\sum_{i=1}^k \pi(A_i) - 2T \text{cap}_{\Omega_i}^G(A_i),$$

and $|\Omega_i| \leq M$ for all $i = 1, \dots, k$ by assumption. Conclude the proof of (3.6) by setting $\gamma = 1/(\varepsilon|V|)$ so that the second term is at least $-2\pi_G^*(\varepsilon)$. To obtain (3.7), remove all $x \in V$ with $\pi(x) < \beta/|V|$. \square

Let us now prove the lemmas.

PROOF OF LEMMA 3.8. We need the following basic fact.

LEMMA 3.11 ([20, 56]). *Suppose Q is a self-adjoint operator on $L^2(\pi)$ with $\langle \mathbb{1}_u, Q\mathbb{1}_v \rangle_\pi \geq 0$ for all $u, v \in V$, and $\psi \in L^2(\pi)$ satisfies $\psi \geq 0$ and $\|\psi\|_\pi = 1$. Then for every integer $T \geq 1$,*

$$\langle \psi, Q^T \psi \rangle_\pi \geq (\langle \psi, Q\psi \rangle_\pi)^T.$$

Now let $\varphi : V \rightarrow [0, 1]$ be any function satisfying $\text{supp } \varphi \subseteq \Omega$. Define $\psi := \varphi / \|\varphi\|_\pi$. Using Lemma 3.11 and $\varphi \leq \mathbb{1}_\Omega$, we have

$$\frac{\langle \mathbb{1}_\Omega, P^T \mathbb{1}_\Omega \rangle_\pi}{\|\varphi\|_\pi^2} \geq \langle \psi, P^T \psi \rangle_\pi \geq \langle \psi, P\psi \rangle_\pi^T \geq (1 - \langle \psi, (I - P)\psi \rangle_\pi)^T = (1 - \mathcal{R}_G(\psi))^T.$$

Since $\mathcal{R}_G(\varphi) = \mathcal{R}_G(\psi)$,

$$\langle \mathbb{1}_\Omega, P^T \mathbb{1}_\Omega \rangle_\pi \geq \|\varphi\|_\pi^2 (1 - \mathcal{R}_G(\varphi))^T \geq \|\varphi\|_\pi^2 (1 - T\mathcal{R}_G(\varphi)) = \|\varphi\|_\pi^2 - T\mathcal{E}_G(\varphi).$$

Using the definition of the capacity, take now a φ that additionally satisfies $\varphi|_A \equiv 1$ and $\mathcal{E}_G(\varphi) = \text{cap}_\Omega^G(A)$, yielding

$$\langle \mathbb{1}_\Omega, P^T \mathbb{1}_\Omega \rangle_\pi \geq \pi(A) - T \cdot \text{cap}_\Omega^G(A). \quad \square$$

PROOF OF LEMMA 3.10. Let $L_\gamma(S) := \{y \in S : \pi(y) \leq \gamma\}$. Using reversibility, write

$$\begin{aligned} p_{2T}(x, x) &\geq \sum_{y \in S} p_T(x, y)p_T(y, x) = \sum_{y \in S} p_T(x, y)^2 \frac{\pi(x)}{\pi(y)} \\ &\geq \frac{\pi(x)}{\gamma} \sum_{y \in S: \pi(y) \leq \gamma} p_T(x, y)^2 \\ &\geq \frac{\pi(x)}{\gamma|S|} \left(\sum_{y \in S: \pi(y) \leq \gamma} p_T(x, y) \right)^2 \\ &= \frac{\pi(x)}{\gamma|S|} p_T(x, L_\gamma(S))^2. \end{aligned}$$

This gives

$$(3.9) \quad \pi \left(\left\{ x \in S : p_{2T}(x, x) \geq \frac{1}{4} \frac{\pi(x)}{\gamma|S|} \right\} \right) \geq \pi \left(\left\{ x \in S : p_T(x, L_\gamma(S)) \geq \frac{1}{2} \right\} \right).$$

On the other hand, note that

$$\sum_{x \in S} \pi(x) p_T(x, S) = \sum_{x, y \in S} \langle \mathbb{1}_x, P^T \mathbb{1}_y \rangle_\pi = \langle \mathbb{1}_S, P^T \mathbb{1}_S \rangle_\pi \geq (1 - \delta)\pi(S).$$

Therefore,

$$\sum_{x \in S} \pi(x) p_T(x, L_\gamma(S)) \geq (1 - \delta)\pi(S) - \pi(S \setminus L_\gamma(S)),$$

and Markov's inequality yields

$$\pi \left(\left\{ x \in S : p_T(x, L_\gamma(S)) \geq \frac{1}{2} \right\} \right) \geq (1 - 2\delta)\pi(S) - 2\pi(S \setminus L_\gamma(S)).$$

Combining this with (3.9) yields the claimed inequality. \square

3.5. *Constructing bump functions.* We will now show that, given a conformal metric $\omega : V \rightarrow \mathbb{R}_+$ with sufficiently nice properties, we can construct many disjoint bump functions with small Rayleigh quotient. Our main geometric tool will be random partitions of metric spaces (cf. Section 3.2).

It will be easier to first prove Theorem 3.6, and then to perform the more complicated construction needed for Theorem 3.7. Let us define the function $\bar{d}_G : [0, 1] \rightarrow \mathbb{N}$ by

$$\bar{d}_G(\varepsilon) := \frac{\Delta_G(\varepsilon n)}{\varepsilon n},$$

which is the average degree among the εn vertices of largest degree in G . It is useful to observe that following simple fact: For every $C > 1$,

$$(3.10) \quad \#\{x \in V : \deg_G(x) \geq C\bar{d}_G(\varepsilon)\} \leq \frac{\varepsilon n}{C}.$$

Indeed, if $N := \#\{x \in V : \deg_G(x) \geq C\bar{d}_G(\varepsilon)\}$, then $\min(N, \varepsilon n)C\bar{d}_G(\varepsilon) \leq \Delta_G(\varepsilon n)$, implying $\min(N, \varepsilon n) \leq \frac{\varepsilon n}{C}$. For $C > 1$, this gives (3.10).

3.5.1. *Many disjoint bumps.* Suppose we have a conformal metric $\omega : V \rightarrow \mathbb{R}_+$ that satisfies the following assumptions: For some numbers $R > 0$ and $\alpha, K \geq 1$,

(A0) $K \leq n/2$.

(A1) For all $x \in V$, it holds that $|B_\omega(x, R)| \leq K$.

(A2) The space (V, dist_ω) admits an $(R/2, \alpha)$ -padded random partition.

Define the quantity

$$(3.11) \quad \eta := R/(12\alpha).$$

When dealing with unbounded degrees, we have to be careful about handling vertices of large conformal weight. To this end, for $\eta > 0$, define the set

$$V_L := \{x \in V : \omega(x) \geq \eta\}.$$

For a subset $S \subseteq V$, define

$$\mathcal{A}_\omega^\eta(S) := 16\bar{d}_G(1/K)\mathcal{A}_\omega(S) + \eta^2 \cdot |E_G(S, V_L)|.$$

Observe that \mathcal{A}_ω^η is a measure on V , and

$$\mathcal{A}_\omega^\eta(V) \leq 16\bar{d}_G(1/K)\|\omega\|_{\ell^2(V)}^2 + \eta^2 \cdot |E_G(V, V_L)|.$$

Since $|V_L| \leq \frac{\|\omega\|_{\ell^2(V)}^2}{\eta^2}$, it holds that

$$(3.12) \quad \mathcal{A}_\omega^\eta(V) \leq \|\omega\|_{\ell^2(V)}^2 \left(16\bar{d}_G(1/K) + \bar{d}_G \left(\frac{1}{n} \|\omega\|_{\ell^2(V)}^2 / \eta^2 \right) \right).$$

LEMMA 3.12. *Under Assumptions (A0)–(A2), there exist disjoint subsets $T_1, T_2, \dots, T_r \subseteq V$ such that $r \geq n/8K$ and, moreover:*

1. For all $i = 1, \dots, r$, it holds that $\frac{K}{2} \leq |T_i| \leq K$, and

$$\mathcal{A}_\omega^\eta(B_\omega(T_i, R/6\alpha)) \leq \frac{3}{r} \mathcal{A}_\omega^\eta(V).$$

2. For all $i \neq j$,

$$\text{dist}_\omega(T_i, T_j) \geq \frac{R}{2\alpha}.$$

PROOF. Let $\mathcal{P} = \{S_1, S_2, \dots, S_m\}$ be a partition of V such that $\text{diam}_\omega(S_i) \leq R/2$ for each i . By property (A1), it holds that

$$(3.13) \quad |S_i| \leq K.$$

For each $i = 1, \dots, m$, define

$$\hat{S}_i = \{x \in S_i : B_\omega(x, R/4\alpha) \subseteq S_i\}.$$

Observe that for $i \neq j$, we have $\text{dist}_\omega(\hat{S}_i, \hat{S}_j) \geq R/2\alpha$ by construction.

Let $N_{\mathcal{P}} = |\hat{S}_1| + \dots + |\hat{S}_m|$. Suppose now that \mathbf{P} is an $(R/2, \alpha)$ -padded random partition. From the definition and linearity of expectation, we have

$$\mathbb{E}[N_{\mathbf{P}}] \geq \frac{1}{2}|V|.$$

So let us fix a partition \mathcal{P} satisfying $N_{\mathcal{P}} \geq \frac{1}{2}|V|$ for the remainder of the proof.

Using (3.13), it is possible to take unions of the sets $\{\hat{S}_i : i \in I\}$ to form disjoint sets T_1, T_2, \dots, T_r with $\frac{K}{2} \leq |T_i| \leq K$ and such that for $i \neq j$, $\text{dist}_\omega(T_i, T_j) \geq R/(2\alpha)$. In this process, we discard at most $K/2$ points, thus

$$|T_1| + \dots + |T_r| \geq \frac{1}{2}|V| - \frac{K}{2} \geq \frac{1}{4}|V|,$$

where the final inequality uses assumption (A0). In particular, we have $r \geq n/4K$.

Let us now sort the sets so that

$$\mathcal{A}_\omega^\eta(B_\omega(T_1, R/6\alpha)) \leq \mathcal{A}_\omega^\eta(B_\omega(T_2, R/6\alpha)) \leq \dots \leq \mathcal{A}_\omega^\eta(B_\omega(T_r, R/6\alpha)).$$

Then for $i \in \{1, 2, \dots, \lceil r/2 \rceil\}$, since the sets $\{B_\omega(T_j, R/6\alpha) : j \in [r]\}$ are pairwise disjoint by construction, it must be that $\mathcal{A}_\omega^\eta(B_\omega(T_i, R/6\alpha)) \leq \frac{3}{r}\mathcal{A}_\omega^\eta(V)$. Thus the statement of the lemma is satisfied by the sets $\{T_i : i < r/2 + 1\}$. \square

Next, we observe that we can remove sets that have a vertex of large degree.

LEMMA 3.13. *Under assumptions (A0)–(A2), there exist disjoint subsets $T_1, T_2, \dots, T_r \subseteq V$ with $r \geq n/16K$, satisfying properties (1) and (2) of Lemma 3.12, and furthermore,*

$$(3.14) \quad \max\{\text{deg}_G(x) : x \in B_\omega(T_i, R/6\alpha)\} \leq 16\bar{d}_G(1/K), \quad i = 1, 2, \dots, r.$$

PROOF. Recalling (3.10), there are at most $n/16K$ vertices with degree larger than $16\bar{d}_G(1/K)$. Thus one can apply Lemma 3.12 and then remove at most $n/16K$ of the sets that contain a vertex of large degree. \square

We are now ready to construct the bump functions.

THEOREM 3.14. *If $\omega : V \rightarrow \mathbb{R}_+$ is a normalized conformal metric on G satisfying assumptions (A0)–(A2), then then there exist disjointly supported functions $\psi_1, \psi_2, \dots, \psi_r : V \rightarrow \mathbb{R}_+$ with $r \geq n/16K$, and such that for all $i = 1, \dots, r$,*

$$(3.15) \quad \mathcal{R}_G(\psi_i) \lesssim \frac{\alpha^2(\bar{d}_G(1/K) + \bar{d}_G(\alpha^2/R^2))}{R^2}.$$

PROOF. Let $T_1, T_2, \dots, T_r \subseteq V$ be the subsets guaranteed by Lemma 3.13. For each $i \in [r]$, define

$$\psi_i(x) = \max\{0, \eta - \text{dist}_\omega(x, T_i)\}.$$

By construction, $T_i \subseteq \text{supp}(\psi_i) \subseteq B_\omega(T_i, \eta)$, hence by Lemma 3.12(2), the functions $\{\psi_i : i \in [r]\}$ are disjointly supported. (Recall that $\eta = R/(12\alpha)$.)

Consider $\{x, y\} \in E$. If $|\psi_i(x) - \psi_i(y)| > 0$, then at least one endpoint must lie in $\text{supp}(\psi_i) \subseteq B_\omega(T_i, \eta)$. Suppose $x \in B_\omega(T_i, \eta)$. If $y \notin V_L$, then $\omega(y) < \eta$, implying that $y \in B_\omega(T_i, 2\eta)$. Therefore, we can bound

$$\sum_{\{x,y\} \in E} |\psi_i(x) - \psi_i(y)|^2 \leq \eta^2 |E_G(B_\omega(T_i, \eta), V_L)| + \sum_{\substack{\{x,y\} \in E: \\ \{x,y\} \subseteq B_\omega(T_i, 2\eta)}} |\psi_i(x) - \psi_i(y)|^2,$$

where we have used $|\psi_i(x) - \psi_i(y)| \leq \eta$ for all $x, y \in V$. Now use the fact that each ψ_i is 1-Lipschitz to write

$$\begin{aligned} \sum_{\{x,y\} \in E} |\psi_i(x) - \psi_i(y)|^2 &\leq \eta^2 |E_G(B_\omega(T_i, \eta), V_L)| + \sum_{\substack{\{x,y\} \in E: \\ \{x,y\} \subseteq B_\omega(T_i, 2\eta)}} \text{dist}_\omega(x, y)^2 \\ &\stackrel{(3.14)}{\leq} \eta^2 |E_G(B_\omega(T_i, \eta), V_L)| + 16 \bar{d}_G(1/K) \mathcal{A}_\omega(B_\omega(T_i, R/6\alpha)) \\ &\leq \mathcal{A}_\omega^\eta(B_\omega(T_i, R/6\alpha)), \end{aligned}$$

where the second inequality uses $\text{dist}_\omega(x, y)^2 \leq (\omega(x)^2 + \omega(y)^2)/2$ for $\{x, y\} \in E$, and we recall that $2\eta = R/6\alpha$.

Combining this with Lemma 3.12(1) yields

$$\mathcal{R}_G(\psi_i) = \frac{2 \sum_{\{x,y\} \in E} |\psi(x) - \psi(y)|^2}{\sum_{x \in V} \text{deg}_G(x) \psi(x)^2} \leq \frac{6 \mathcal{A}_\omega^\eta(V)}{r \eta^2 |T_i|} \leq \frac{864 \alpha^2 \mathcal{A}_\omega^\eta(V)}{R^2 |V|}.$$

To arrive at the statement of the theorem, use (3.12) and the assumption that $|V|^{-1} \|\omega\|_{\ell^2(V)}^2 = 1$. \square

Let us now use this to prove Theorem 3.6.

PROOF OF THEOREM 3.6. Consider $R = R_* = \sqrt{n/(16\kappa \cdot k)}$. By assumption, there is a normalized conformal metric $\omega : V \rightarrow \mathbb{R}_+$ satisfying $\max_{x \in V} |B_\omega(x, R)| \leq \kappa R^2$.

We may assume that $\kappa \geq 1$. Let $K = \kappa R^2$. Again by assumption, (V, dist_ω) admits an $(R/2, \alpha)$ -padded random partition. Now apply Theorem 3.14 to find $r \geq |V|/(16\kappa R_*^2)$ disjointly supported test functions $\{\psi_i\}$, each with

$$\begin{aligned} \mathcal{R}_G(\psi_i) &\lesssim \frac{\alpha^2 (\bar{d}_G(1/(\kappa R_*^2)) + \bar{d}_G(\alpha^2/R_*^2))}{R_*^2} \leq 2 \frac{\alpha^2 \bar{d}_G(k/n)}{R_*^2} \\ &\lesssim \alpha^2 \kappa \bar{d}_G(k/n) \frac{k}{n} = \alpha^2 \kappa \frac{\Delta_G(k)}{n}. \end{aligned}$$

We may assume that $k \leq n/(16\kappa)$. (Otherwise, we can just take n functions—one supported on each vertex of the graph—since the bound we are required to prove on the Rayleigh quotient is trivial.) Note that in this case, $r \geq |V|/(16\kappa R_*^2) \geq k$, completing the proof. \square

3.5.2. *Exhausting the stationary measure by capacitors.* Our arguments will follow along similar lines to those of the preceding section although things will be somewhat more delicate. $G = (V, E)$ is an n -vertex, connected graph. Suppose we have a conformal metric $\omega : V \rightarrow \mathbb{R}_+$ that satisfies (A1) and (A2) for some numbers $R > 0, \alpha, K \geq 1$.

Consider a number $\delta > 0$ and define

$$\eta := \frac{\delta R}{18\alpha},$$

$$V_L := \{x \in V : \omega(v) \geq \eta\}.$$

LEMMA 3.15. *For any $\delta > 0$, it holds that under assumptions (A1) and (A2), there are pairwise disjoint sets S_1, \dots, S_r satisfying $\text{diam}_\omega(S_i) \leq R/2$ for each $i = 1, \dots, r$, and such that*

$$(3.16) \quad \sum_{i=1}^r \pi(\hat{S}_i) \geq 1 - \delta - \pi_G^*(\delta),$$

where for a subset $S \subseteq V$, we denote

$$\hat{S} := \{x \in S : B_\omega(x, \delta R/6\alpha) \subseteq S\}.$$

Moreover, it holds that

$$(3.17) \quad \max\{\text{deg}_G(x) : x \in S_1 \cup \dots \cup S_r\} \leq \bar{d}_G(\delta/K),$$

and

$$(3.18) \quad \pi(\hat{S}_i) \geq \frac{1}{2}\pi(S_i) \quad \forall i = 1, \dots, r.$$

PROOF. Let \mathbf{P} denote an $(R/2, \alpha)$ -padded random partition of (V, dist_ω) . Using linearity of expectation and the definition of a padded random partition yields

$$\mathbb{E}\left[\sum_{S \in \mathbf{P}} \pi(\hat{S})\right] \geq 1 - \delta/3.$$

Let us fix a partition \mathcal{P} in the support of \mathbf{P} satisfying $\sum_{S \in \mathcal{P}} \pi(\hat{S}) \geq 1 - \delta/3$.

Let $\mathcal{P}' = \{S \in \mathcal{P} : \pi(\hat{S}) \geq \frac{1}{2}\pi(S)\}$ and note that

$$\sum_{S \in \mathcal{P}'} \pi(\hat{S}) \geq 1 - (\delta/3) - 2(\delta/3) = 1 - \delta.$$

Finally, denote

$$\{S_1, \dots, S_r\} = \{S \in \mathcal{P}' : \max\{\text{deg}_G(x) : x \in S\} \leq \bar{d}_G(\delta/K)\}.$$

Recalling (3.10), there are at most $\frac{\delta}{K}|V|$ vertices in G with degree larger than $\bar{d}_G(\delta/K)$. Therefore,

$$\sum_{i=1}^r \pi(\hat{S}_i) \geq \sum_{S \in \mathcal{P}'} \pi(\hat{S}) - \sum_{S \in \mathcal{P} \setminus \{S_1, \dots, S_r\}} \pi(S) \geq 1 - \delta - \pi_G^*\left(\frac{\delta}{K} \max_{S \in \mathcal{P}} |S_i|\right) \geq 1 - \delta - \pi_G^*(\delta),$$

where the final inequality uses the fact that $|S| \leq K$ for $S \in \mathcal{P}$ which follows from $\text{diam}_\omega(S) \leq R/2$ and (A1). \square

We are now ready to construct the capacitors.

THEOREM 3.16. *If $\omega : V \rightarrow \mathbb{R}_+$ is a conformal metric on G satisfying assumptions (A1) and (A2), then there exist pairwise disjoint capacitors $(A_1, \Omega_1), \dots, (A_k, \Omega_k)$ with*

$$(3.19) \quad \sum_{i=1}^k \pi(A_i) \geq 1 - \delta - \pi_G^*(\delta),$$

and such that for all $i = 1, \dots, k$,

$$(3.20) \quad \text{diam}_\omega(\Omega_i) \leq R/2,$$

$$(3.21) \quad |\Omega_i| \leq K.$$

And furthermore,

$$(3.22) \quad \sum_{i=1}^k \text{cap}_{\Omega_i}^G(A_i) \leq \frac{18^2 \alpha^2}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2 \frac{\bar{d}_G(\frac{18^2 \alpha^2}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2) + \bar{d}_G(\delta/K)}{\bar{d}_G(1)}.$$

PROOF OF THEOREM 3.16. Let $S_1, S_2, \dots, S_k \subseteq V$ be the subsets guaranteed by Lemma 3.15. For each $i = 1, \dots, k$, define

$$\psi_i(x) := \frac{1}{\eta} \max\{0, \eta - \text{dist}_\omega(x, \hat{S}_i)\}.$$

We denote $A_i := \hat{S}_i$ and $\Omega_i := S_i$. By construction, we have $0 \leq \psi_i \leq 1$, as well as $\text{supp } \psi_i \subseteq \Omega_i$ and $\psi_i|_{A_i} \equiv 1$. One observes that (3.19) follows from (3.16). Similarly, (3.20) follows from Lemma 3.15, and (3.21) then follows from assumption (A1).

Use the fact that $\eta\psi_i$ is 1-Lipschitz to calculate

$$\begin{aligned} \eta^2 \sum_{\{x,y\} \in E} |\psi_i(x) - \psi_i(y)|^2 &\leq \eta^2 |E_G(B_\omega(\hat{S}_i, \eta), V_L)| + \sum_{\substack{\{x,y\} \in E: \\ \{x,y\} \subseteq B_\omega(\hat{S}_i, \delta R/6\alpha)}} \text{dist}_\omega(x, y)^2 \\ &\leq \eta^2 |E_G(S_i, V_L)| + \bar{d}_G(\delta/K) \mathcal{A}_\omega(B_\omega(\hat{S}_i, \delta R/6\alpha)) \\ &\leq \eta^2 |E_G(S_i, V_L)| + \bar{d}_G(\delta/K) \mathcal{A}_\omega(S_i). \end{aligned}$$

Therefore,

$$(3.23) \quad \sum_{i=1}^k \text{cap}_{\Omega_i}^G(A_i) \leq \sum_{i=1}^k \mathcal{E}_G(\psi_i) \leq \frac{1}{|E|} (|E_G(V, V_L)| + \bar{d}_G(\delta/K) \eta^{-2} \|\omega\|_{\ell^2(V)}^2).$$

Note that $|V_L| \leq \eta^{-2} \|\omega\|_{\ell^2(V)}^2$, yielding

$$|E_G(V, V_L)| \leq \Delta_G(\eta^{-2} \|\omega\|_{\ell^2(V)}^2) = \eta^{-2} \|\omega\|_{\ell^2(V)} \bar{d}_G(\eta^{-2} \|\omega\|_{L^2(V)}^2).$$

Finally, note that

$$\frac{\|\omega\|_{\ell^2(V)}^2}{|E|} = \frac{\|\omega\|_{L^2(V)}^2}{\bar{d}_G(1)}.$$

Plugging these into (3.23) and using the definition of η yields

$$\sum_{i=1}^k \text{cap}_{\Omega_i}^G(A_i) \leq \frac{18^2 \alpha^2}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2 \frac{\bar{d}(\frac{18\alpha^2}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2) + \bar{d}(\delta/K)}{\bar{d}_G(1)}.$$

□

Combining Theorem 3.16 with Theorem 3.7 yields the following corollary.

COROLLARY 3.17. *There is a universal constant $C \geq 1$ such that if $\omega : V \rightarrow \mathbb{R}_+$ is a conformal metric satisfying assumptions (A1) and (A2), then for every $\delta, \beta > 0$ and $T \geq 1$,*

$$\pi \left(\left\{ x \in V : p_{2T}^G(x, x) < \frac{\delta\beta}{4K} \right\} \right) \leq \beta + \delta + 3\pi_G^*(\delta) + \frac{C\alpha T}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2 \frac{\bar{d}(\frac{\alpha^2}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2) + \bar{d}(\delta/K)}{\bar{d}_G(1)}.$$

If additionally, $\|\omega\|_{L^2(V)} \geq 1/2$, then the bound simplifies to

$$\pi \left(\left\{ x \in V : p_{2T}^G(x, x) < \frac{\delta\beta}{4K} \right\} \right) \leq \beta + \delta + 3\pi_G^*(\delta) + \frac{C\alpha T}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2 \bar{d}(\delta/(K + R^2)).$$

4. Conformal growth rates and random walks. We will now apply the tools of the previous section to establish our main claims on spectral dimension and the diagonal heat kernel. Toward this end, it will be convenient to start with a unimodular random graph (G, ρ) and derive from it a sequence $\{G_n\}$ of finite unimodular random graphs such that $\{G_n\} \Rightarrow (G, \rho)$.

4.1. *Invariant amenability and soficity.* A unimodular random graph is called *sofic* if it is the distributional limit of finite graphs. It is an open question whether every unimodular random graph is sofic (see, e.g., [3], Section 10). But as one might expect, for the proper definition of “amenable,” it turns out that all amenable graphs are sofic.

The invariant Cheeger constant. A percolation on (G, ρ) is a $\{0, 1\}$ -marking $\xi : E(G) \cup V(G) \rightarrow \{0, 1\}$ of the edges and vertices such that (G, ρ, ξ) is unimodular as a marked graph. One thinks of ξ as specifying a (random) subgraph of G corresponding to all the edges and vertices with $\xi = 1$. One calls ξ a *bond percolation* if $\xi(v) = 1$ almost surely for all $v \in V(G)$. The *cluster of vertex v* is the connected component $K_\xi(v)$ of v in the ξ -percolated graph. Finally, one says that ξ is *finitary* if almost surely all its clusters are finite.

For a graph G and a finite subset $W \subseteq V(G)$, we write $\partial_G^E W$ for the *edge boundary of W* : The subset of edges $\partial_G^E W \subseteq E(G)$ that have exactly one endpoint in W . The *invariant Cheeger constant* of a unimodular random graph (G, ρ) is the quantity

$$\Phi^{\text{inv}}(G, \rho) := \inf \left\{ \mathbb{E} \left[\frac{|\partial_G^E K_\xi(\rho)|}{|K_\xi(\rho)|} \right] : \xi \text{ is a finitary percolation on } G \right\}.$$

One says that (G, ρ) is *invariantly amenable* if $\Phi^{\text{inv}}(G, \rho) = 0$. Conversely, (G, ρ) is *invariantly nonamenable* if it is not invariantly amenable.

Hyperfiniteness. A unimodular random graph (G, ρ) is *hyperfinites* if there is a sequence $\langle \xi_i \rangle_{i \geq 1}$ of percolations such that each ξ_i is finitary, $\xi_i \subseteq \xi_{i+1}$ almost surely, and almost surely $\bigcup_{i \geq 1} \xi_i = G$. In this case, $\langle \xi_i \rangle_{i \geq 1}$ is called a *finitary exhaustion of (G, ρ)* . One can consult [9] for a proof of the following (stated without proof in [3]).

THEOREM 4.1 ([3], Theorem 8.5). *If (G, ρ) is a unimodular random graph with $\mathbb{E}[\text{deg}_G(\rho)] < \infty$, then (G, ρ) is invariantly amenable if and only if it is hyperfinite.*

The main point for us is that if $\langle \xi_i \rangle_{i \geq 1}$ is a finitary exhaustion of (G, ρ) , then one has an approximation by finite unimodular random graphs:

$$(4.1) \quad \{G[K_{\xi_i}(\rho)] : i \geq 1\} \Rightarrow (G, \rho).$$

To state the next corollary, let us recall that if (G, ρ) is a unimodular random graph with law μ , then

$$\bar{d}_\mu(\varepsilon) := \sup\{\mathbb{E}[\deg_G(\rho) \mid \mathcal{E}] : \mathbb{P}(\mathcal{E}) \geq \varepsilon\},$$

where the supremum is over all measurable sets \mathcal{E} with $\mathbb{P}(\mathcal{E}) \geq \varepsilon$.

COROLLARY 4.2. *If (G, ρ) is a hyperfinite unimodular random graph, then there is a sequence $\{(G_n, \rho_n)\}$ of finite unimodular random graphs such that $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$ and, moreover:*

1. *If (G, ρ) is α -decomposable, then for each $n \geq 1$, the unimodular random graph (G_n, ρ_n) is α -decomposable.*
2. *For any $R \geq 1$ and any normalized metric ω on (G, ρ) , there is a sequence $\{\omega_n\}$ of normalized metrics on $\{G_n\}$ such that for each $n \geq 1$,*

(a) *Almost surely, $\|\omega_n\|_{L^2(V(G_n))}^2 \geq 1/2$.*

(b) *It holds that*

$$\| \#B_{\omega_n}(\rho_n, R/\sqrt{2}) \|_{L^\infty} \leq \| \#B_\omega(\rho, R) \|_{L^\infty}.$$

PROOF. Property (1) follows from the definition of α -decomposability. Suppose G is α -decomposable; then so is $G[S]$ for every finite, connected subset $S \subseteq V(G)$, by simply extending any weight $\omega : S \rightarrow \mathbb{R}_+$ to a weight $\hat{\omega} : V(G) \rightarrow \mathbb{R}_+$ defined by $\hat{\omega}(x) = \omega(x)$ if $x \in S$ and $\hat{\omega}(x) = \text{diam}_\omega(S)$ otherwise. In this case, $\text{dist}_{\hat{\omega}}|_{S \times S} = \text{dist}_\omega$.

Denote $\hat{\omega} = \sqrt{(\omega^2 + 1)}/2$. By assumption, $\hat{\omega}$ is normalized. The mass-transport principle implies that (see, e.g., [9], Lemma 3.1) if ξ is finitary, then ρ is uniformly distributed on its component $K_\xi(\rho)$ and, therefore, the unimodular random conformal graph $(G[K_\xi(\rho)], \hat{\omega}|_{K_\xi(\rho)}, \rho)$ is normalized as well. Moreover,

$$\| \#B_{\omega_n}(\rho_n, R/\sqrt{2}) \|_{L^\infty} \leq \| \#B_{\hat{\omega}}(\rho, R/\sqrt{2}) \|_{L^\infty} \leq \| \#B_\omega(\rho, R) \|_{L^\infty},$$

verifying property (2). \square

Subexponential conformal growth and invariant amenability. In conjunction with Corollary 4.2, the next result will allow us to approximate a unimodular random graph with bounded conformal growth exponent by a sequence of finite unimodular random graphs.

LEMMA 4.3. *If (G, ρ) is a unimodular random graph with $\mathbb{E}[\deg_G(\rho)^2] < \infty$ and $\underline{\text{dim}}_{\text{cg}}(G, \rho) < \infty$, then (G, ρ) is invariantly amenable.*

PROOF. Suppose that (G, ρ) is invariantly nonamenable, ω is a normalized conformal metric on (G, ρ) . We will show that $\|B_\omega(\rho, R)\|_{L^\infty}$ grows at least exponentially in R , implying that $\underline{\text{dim}}_{\text{cg}}(G, \rho) = \infty$.

Let $h > 0$ be such that $\Phi^{\text{inv}}(G, \rho) \geq h$. For some $K > 0$ to be specified soon, define a bond percolation $\xi : V(G) \rightarrow \{0, 1\}$ by

$$\xi(x) = \mathbb{1}_{\{\omega(x) \leq K\}}.$$

For a vertex $x \in V(G)$, define $\deg_\xi(x) := \sum_{y:\{x,y\} \in E(G)} \xi(y)$.

From [3], Theorem 8.13(i), one concludes that if

$$(4.2) \quad \mathbb{E}[\deg_\xi(\rho) \mid \xi(\rho) = 1] > \mathbb{E}[\deg_G(\rho)] - h,$$

then with positive probability, the subgraph $\{x \in V(G) : \xi(x) = 1\}$ is nonamenable. Since a nonamenable subgraph has exponential growth and $\xi(x) = 1 \implies \omega(x) \leq K$, we conclude that $\|B_\omega(\rho, R)\|_{L^\infty}$ grows (at least) exponentially as $R \rightarrow \infty$. We are thus left to verify (4.2) for some $K > 0$.

Using Chebyshev’s inequality and the fact that ω is normalized:

$$(4.3) \quad \mathbb{P}[\xi(\rho) = 0] = \mathbb{P}[\omega(\rho) > K] \leq \frac{1}{K^2}.$$

Now applying the mass-transport principle yields

$$(4.4) \quad \begin{aligned} \mathbb{E}[\deg_G(\rho) - \deg_\xi(\rho)] &= \mathbb{E}\left[\sum_{x:\{x,\rho\} \in E(G)} (1 - \xi(x))\right] \\ &= \mathbb{E}[\deg_G(\rho)(1 - \xi(\rho))] \\ &\leq \sqrt{\mathbb{E}[\deg_G(\rho)^2]} \sqrt{\mathbb{P}[\xi(\rho) = 0]} \\ &\leq \frac{C}{K}, \end{aligned}$$

where $C := (\mathbb{E}[\deg_G(\rho)^2])^{1/2}$. This gives

$$\begin{aligned} \mathbb{E}[\deg_\xi(\rho) \mid \xi(\rho) = 1] &\geq \mathbb{E}[\deg_\xi(\rho)\xi(\rho)] \\ &\geq \mathbb{E}[\deg_\xi(\rho)] - \mathbb{E}[\deg_G(\rho)(1 - \xi(\rho))] \\ &\stackrel{(4.4)}{\geq} \mathbb{E}[\deg_G(\rho)] - \frac{2C}{K}. \end{aligned}$$

Choosing K large enough verifies (4.2), completing the proof. \square

The preceding argument was suggested to us by Tom Hutchcroft, replacing a considerably more complicated proof (a variant of which appears in Lemma 5.5 below).

4.2. *Conformal growth exponent bounds the spectral dimension.* Let us now prove Theorem 1.2. In fact, we will establish the following quantitative strengthening. If (G, ρ) is a unimodular random graph with law μ , let us define

$$\psi_\mu(R) := \inf_\omega \|B_\omega(\rho, R)\|_{L^\infty},$$

where the infimum is over all normalized conformal metrics ω on (G, ρ) .

THEOREM 4.4. *Suppose (G, ρ) is a unimodular random graph and $\deg_G(\rho)$ has negligible tails. Assume furthermore that for some number $d > 0$ and an increasing sequence $\{R_n\}$ of radii, it holds that*

$$(4.5) \quad \psi_\mu(R_n) \leq R_n^{d+o(1)} \quad \text{as } n \rightarrow \infty.$$

Then for any sequence $\{\varepsilon_n\}$ with $\varepsilon_n \geq R_n^{-d-o(1)}$, there is a sequence of times $\{T_n\}$ such that $T_n \geq \varepsilon_n^9 R_n^{2-o(1)}$, and

$$(4.6) \quad \mathbb{P}\left[p_{2T_n}^G(\rho, \rho) \leq \frac{\varepsilon_n^6}{\psi_\mu(R_n)}\right] \leq \varepsilon_n^{1-o(1)} \quad \text{as } n \rightarrow \infty.$$

If $\deg_G(\rho)$ has exponential tails, then the corrections are polylogarithmic: There is a sequence $\{T_n\}$ satisfying $T_n \geq (\log R_n)^{-O(1)} \varepsilon_n^9 R_n^2$, and

$$\mathbb{P}\left[p_{2T_n}^G(\rho, \rho) \leq \frac{\varepsilon_n^6}{\psi_\mu(R_n)}\right] \leq \varepsilon_n (\log R_n)^{O(1)} \quad \text{as } n \rightarrow \infty.$$

PROOF OF THEOREM 1.2. Choose $\varepsilon_n := (\log R_n)^{-2}$. Then (4.5) asserts the existence of a function $h(n) \leq T_n^{o(1)}$ such that

$$(4.7) \quad \mathbb{P}\left[p_{2T_n}^G(\rho, \rho) \geq \frac{h(n)}{T_n^{d/2}}\right] \geq 1 - (\log R_n)^{-2+o(1)}.$$

If $\underline{\dim}_{\text{cg}}(G) \leq d$, then there is an unbounded sequence $\{R_n\}$ of radii satisfying (4.5). Thus there exists an unbounded sequence $\{T_n\}$ satisfying (4.7). It follows that almost surely (over the choice of (G, ρ)), there is an infinite subsequence $\{T_{n_j}\}$ such that $p_{2T_{n_j}}^G(\rho, \rho) \geq h(n)T_{n_j}^{-d/2}$, implying that $\underline{\dim}_{\text{sp}}(G) \leq d$ as well.

If $\overline{\dim}_{\text{cg}}(G) \leq d$, then one can take $\{R_n\}$ satisfying (4.5) so that $\mathbb{N} \setminus \{R_n\}$ is finite. Note that the even return times are monotone (see, e.g., (4.28)): For all integers $t \geq s \geq 1$,

$$(4.8) \quad p_{2^s}^G(\rho, \rho) \geq p_{2^t}^G(\rho, \rho),$$

hence we can assume that $T_n = R_n^2 g(n)$ for some $g(n) \leq R_n^{o(1)}$ as $n \rightarrow \infty$, and (4.7) holds. We may similarly assume that $g(n + 1) \leq 2g(n)$, otherwise replace $g(n)$ by $\hat{g}(n) := \min(g(n), 2\hat{g}(n - 1))$.

In particular, since $\mathbb{N} \setminus \{R_n\}$ is unbounded, it holds that for some $h_1(n) \leq n^{o(1)}$,

$$\mathbb{P}\left[p_{2 \cdot 4^k}^G(\rho, \rho) \geq \frac{h_1(4^k)}{4^{kd/2}}\right] \geq 1 - k^{-2+o(1)} \quad \text{as } k \rightarrow \infty.$$

Now the Borel–Cantelli lemma asserts that almost surely $p_{2 \cdot 4^k}^G(\rho, \rho) \geq h_1(4^k)/4^{kd/2}$ for all but finitely many k . Recalling again (4.8), this gives $\overline{\dim}_{\text{sp}}(G) \leq d$, as desired. \square

We record first some preliminary results. The following lemma is well known; see, for example, [51], Lemma 3.11. Let (X, dist) be a metric space and consider $R > r > 0$. Let $\mathcal{C}(X; R, r)$ denote the largest cardinality of a set $S \subseteq X$ such that $x \neq y \in S \implies r \leq \text{dist}(x, y) \leq R$.

LEMMA 4.5. For any metric space (X, dist) and $\tau > 0$, it holds that (X, dist) admits a (τ, α) -padded random partition for some $\alpha \leq O(\log \mathcal{C}(X; 2\tau, \tau/4))$.

LEMMA 4.6. Suppose that (G, ρ) is a finite unimodular random graph with stationary measure π_G , and such that $E[\deg_G(\rho)] < \infty$. Then for any $\delta > 0$,

$$(4.9) \quad \pi_G^*(\delta) = \delta \frac{\bar{d}_G(\delta)}{\bar{d}_G(1)} \leq \delta \bar{d}_G(\delta).$$

(Recall the definition of π_G^* from Section 3.4.) Moreover, for any set of vertices $U_G \subseteq V(G)$, it holds that

$$\mathbb{P}[\rho \in U_G \mid G] \leq \pi_G(U_G) \bar{d}_G(1).$$

PROOF. The first fact follows directly from the definitions and $\bar{d}_G(1) \geq 1$ since G is almost surely connected. The second fact is a reformulation of $|U_G| \leq \sum_{x \in U_G} \deg_G(x)$. \square

LEMMA 4.7. Suppose that (G, ρ) is a unimodular random graph with law μ . Then there exists a sequence of finite, normalized unimodular random conformal graphs $\{(G_k, \omega_k, \rho_k)\}$ so that:

1. $\{(G_k, \rho_k)\} \Rightarrow (G, \rho)$.
2. $\|\omega_k\|_{L^2(V(G_k))}^2 \geq 1/2$ almost surely.

3. For every $R > 0$,

$$\limsup_{k \rightarrow \infty} \|\#B_{\omega_k}(\rho_k, R)\|_{L^\infty} \leq \psi_\mu(\sqrt{2}R).$$

4. Almost surely over the choice of (G_k, ρ_k) : For every $R > 0$, $(V(G_k), \text{dist}_{\omega_k})$ admits an $(R/2, \alpha_{R,k})$ -padded random partition with

$$\alpha_{R,k} \lesssim \log \|\#B_{\omega_k}(\rho_k, R)\|_{L^\infty}$$

PROOF. Combining Lemma 4.3 with Corollary 4.2, we may take a sequence of finite normalized unimodular random conformal graphs $\{(G_k, \omega_k, \rho_k)\}$ so that $\{(G_k, \rho_k)\} \Rightarrow (G, \rho)$ and (2) and (3) are satisfied.

Note that Lemma 4.5 implies that $(V(G_k), \text{dist}_{\omega_k})$ almost surely admits, for every $n \geq 1$, an $(R_n/2, \alpha_n)$ -padded partition with

$$\alpha_n \lesssim \log \|\#B_{\omega_k}(\rho_k, R_n)\|_{L^\infty}. \quad \square$$

We will also require the following basic fact.

LEMMA 4.8. Suppose G is a random finite graph and $\rho \in V(G)$ is chosen uniformly at random. Let μ be the law of the unimodular random graph (G, ρ) . Then for every $\varepsilon > 0$, it holds that

$$\mathbb{E}[\bar{d}_G(\varepsilon)] \leq \bar{d}_\mu(\varepsilon/2).$$

PROOF. Let S_G^ε be the $\lfloor \varepsilon |V(G)| \rfloor$ vertices of largest degree in G , and define the event $\mathcal{E} := \{\rho \in S_G^\varepsilon\}$. Then $\mathbb{P}(\mathcal{E} \mid G) = \frac{|S_G^\varepsilon|}{|V(G)|} \geq \varepsilon/2$; hence, $\mathbb{P}(\mathcal{E}) \geq \varepsilon/2$. By definition, it follows that

$$\bar{d}_\mu(\varepsilon/2) \geq \mathbb{E}\left[\frac{1}{|S_G^\varepsilon|} \sum_{x \in S_G^\varepsilon} \text{deg}_G(x)\right] = \mathbb{E}[\bar{d}_G(\varepsilon)]. \quad \square$$

PROOF OF THEOREM 4.4. Let μ be the law of (G, ρ) . Apply Lemma 4.7 to obtain a sequence $\{(G_k, \omega_k, \rho_k)\}$ satisfying conclusions (1)–(4). Define $N_{R,k} := \|\#B_{\omega_k}(\rho_k, R)\|_{L^\infty}$ for $R, k \geq 1$, and fix some $\delta > 0$. Denote the event

$$\begin{aligned} \mathcal{Q}_{R,k} := & \{\bar{d}_{G_k}(1) > \delta^{-1/3} \bar{d}_\mu(1/2)\} \cup \{\bar{d}_{G_k}(\delta) > \delta^{-1/3} \bar{d}_\mu(\delta/2)\} \\ & \cup \{\bar{d}_{G_k}(\delta/(N_{R,k} + R^2)) > \delta^{-1/3} \bar{d}_\mu(\delta/2(N_{R,k} + R^2))\}. \end{aligned}$$

Condition on (G_k, ω_k, ρ_k) , and let π_k denote the stationary measure on G_k . Apply Corollary 3.17 with $\alpha = \alpha_{R,k}$, $K = N_{R,k}$ to obtain for any $T \geq 1$,

$$\begin{aligned} & \pi_k\left(\left\{x \in V(G_k) : p_{2T}^{G_k}(x, x) < \frac{\delta^2}{4N_{R,k}}\right\}\right) \\ & \lesssim \delta + \pi_{G_k}^*(\delta) + \frac{\alpha_{R,k}^2}{\delta^2} \frac{T}{R^2} \|\omega_k\|_{L^2(V(G_k))}^2 \bar{d}_{G_k}(\delta/(N_{R,k} + R^2)). \end{aligned}$$

Observe that $\pi_{G_k}^*(\delta) \leq \delta \bar{d}_{G_k}(\delta)$ and use Lemma 4.6 to change from the stationary to uniform measure, yielding

$$\begin{aligned} & \frac{\#\{x \in V(G_k) : p_{2T}^{G_k}(x, x) < \frac{\delta^2}{4N_{R,k}}\}}{|V(G_k)|} \\ & \lesssim \bar{d}_{G_k}(1) \left(\delta(1 + \bar{d}_{G_k}(\delta)) + \frac{\alpha_{R,k}^2}{\delta^2} \frac{T}{R^2} \|\omega_k\|_{L^2(V(G_k))}^2 \bar{d}_{G_k}(\delta/(N_{R,k} + R^2)) \right). \end{aligned}$$

Taking expectation over (G_k, ω_k, ρ_k) and using that ω_k is normalized yields

$$\begin{aligned} & \mathbb{P}\left(p_{2T}^{G_k}(\rho_k, \rho_k) < \frac{\delta^2}{4N_{R,k}}\right) \\ & \lesssim \bar{d}_\mu(1/2)\left(\delta^{2/3}(1 + \delta^{-1/3}\bar{d}_\mu(\delta/2)) + \frac{\alpha_{R,k}^2}{\delta^{8/3}}\frac{T}{R^2}\bar{d}_\mu(\delta/2(N_{R,k} + R^2))\right) + \mathbb{P}[\mathcal{Q}_{R,k}] \\ & \lesssim \delta^{2/3}(1 + \delta^{-1/3}\bar{d}_\mu(\delta/2)) + \frac{\alpha_{R,k}^2}{\delta^{8/3}}\frac{T}{R^2}\bar{d}_\mu(\delta/2(N_{R,k} + R^2)) + \mathbb{P}[\mathcal{Q}_{R,k}], \end{aligned}$$

where the last inequality uses $\bar{d}_\mu(1/2) \leq O(1)$, since $\text{deg}_G(\rho)$ has negligible tails.

Let us now take $k \rightarrow \infty$ and use the fact that $\{(G_k, \rho_k)\} \Rightarrow (G, \rho)$. Since $\bar{d}_{\mu_k}(\varepsilon) \rightarrow \bar{d}_\mu(\varepsilon)$ for every $\varepsilon > 0$, Lemma 4.8 and Markov’s inequality show that $\limsup_{k \rightarrow \infty} \mathbb{P}[\mathcal{Q}_{R,k}] \leq 3\delta^{1/3}$. Using additionally Lemma 4.7(3)–(4) gives, for all $R, T \geq 1$,

$$\begin{aligned} (4.10) \quad & \mathbb{P}\left[p_{2T}^G(\rho, \rho) < \frac{\delta^2}{4\psi_\mu(\sqrt{2}R)}\right] \\ & \lesssim \delta^{2/3}(1 + \delta^{-1/3}\bar{d}_\mu(\delta)) + \frac{\alpha_R^2}{\delta^{8/3}}\frac{T}{R^2}\bar{d}_\mu(\delta/(\psi_\mu(\sqrt{2}R) + R^2)) + \delta^{1/3}, \end{aligned}$$

where $\alpha_R := \log \psi_\mu(\sqrt{2}R)$.

Let $\{R_n\}$ be an increasing sequence of radii satisfying (4.5), and $\{\varepsilon_n\}$ a sequence satisfying $\varepsilon_n \geq R_n^{-d+o(1)}$. The assumption that $\text{deg}_G(\rho)$ has negligible tails (recall (1.3)) yields

$$\bar{d}_\mu(\beta) \leq \beta^{-o(1)} \quad \text{as } \beta \rightarrow 0,$$

hence applying (4.10) with $R = R_n/\sqrt{2}$ and $\delta = 2\varepsilon_n^3$ gives for all $T \geq 1$, as $n \rightarrow \infty$:

$$\mathbb{P}\left[p_{2T}^G(\rho, \rho) < \frac{\varepsilon_n^6}{\psi_\mu(R_n)}\right] \leq \varepsilon_n g(n) + \varepsilon_n^{-8}\frac{T}{R_n^2}h(n),$$

where $g(n) \leq \varepsilon_n^{-o(1)}$ and $h(n) \leq R_n^{o(1)}$ as $n \rightarrow \infty$. If $\text{deg}_G(\rho)$ additionally has exponential tails, one has $g(n) \leq O(\log(1/\varepsilon_n)) \leq O(\log R_n)$ and $h(n) \leq (\log R_n)^{O(1)}$.

If we now define, for $n \geq 1$,

$$(4.11) \quad T_n := \left\lceil \varepsilon_n^9 \frac{R_n^2}{h(n)} \right\rceil,$$

we arrive at

$$\mathbb{P}\left[p_{2T_n}^G(\rho, \rho) < \frac{\varepsilon_n^6}{\psi_\mu(R_n)}\right] \leq \varepsilon_n(1 + g(n)) \quad \text{as } n \rightarrow \infty,$$

yielding (4.6). \square

4.3. *On-diagonal heat kernel bounds.* Our goal now is to prove Theorem 1.7 and Theorem 1.8. We start with the former and restate it here for ease of reference.

THEOREM 4.9 (Restatement of Theorem 1.7). *Suppose that (G, ρ) satisfies the the conditions:*

1. (G, ρ) has gauged quadratic conformal growth and is uniformly decomposable,
2. $\mathbb{E}[\text{deg}_G(\rho)^2] < \infty$.

Then there is a constant $C = C(\mu)$ such that for every $\delta > 0$ and all $T \geq C/\delta^2$,

$$(4.12) \quad \mathbb{P}\left[p_{2T}^G(\rho, \rho) < \frac{\delta}{T\bar{d}_\mu(1/T^3)}\right] \leq C\delta^{1/17}.$$

PROOF. Let $C_\mu = \mathbb{E}[\text{deg}_G(\rho)^2]$. Then for $\varepsilon > 0$,

$$(4.13) \quad C_\mu \geq \varepsilon \bar{d}_\mu(\varepsilon)^2.$$

From Corollary 4.2, we can take a sequence $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$ such that (G_n, ρ_n) is a finite unimodular random graph that is almost surely:

1. α -decomposable,
2. (κ, R) -quadratic for every $R \geq 1$,

where $\alpha, \kappa > 0$ are some constants depending on μ .

Consider $\delta > 0$, $T \geq 1$ and $n \geq 1$. Let $R := \sqrt{\gamma T \bar{d}_\mu(1/T^3)}$ for some number $\gamma > 0$ to be chosen soon. Recall from Corollary 4.2 that we may assume that $\|\omega_n\|_{L^2(V(G_n))}^2 \geq 1/2$ almost surely.

Set $K = \kappa R^2$ and apply Corollary 3.17 with $\beta = \sqrt{\delta}$ to obtain, for some constant $C_1 = C_1(\alpha, \kappa)$, almost surely over the choice of (G_n, ω_n, ρ_n) :

$$\pi_{G_n}\left(\left\{x \in V : p_{2T}^{G_n}(x, x) < \frac{\delta^{3/2}}{4K}\right\}\right) \lesssim \sqrt{\delta} + \pi_{G_n}^*(\delta) + \frac{C_1}{\delta^2 \gamma} \|\omega_n\|_{L^2(V_n)}^2 \frac{\bar{d}_{G_n}\left(\frac{\delta}{C_1 \gamma T \bar{d}_\mu(1/T^3)}\right)}{\bar{d}_\mu(1/T^3)}.$$

Observe that

$$(4.14) \quad \bar{d}_{G_n}\left(\frac{\delta}{C_1 \gamma T \bar{d}_\mu(1/T^3)}\right) \stackrel{(4.13)}{\leq} C_1 \bar{d}_{G_n}\left(\frac{\delta}{C_1 C_\mu \gamma T^{2.5}}\right).$$

From Lemma 4.6, we have

$$(4.15) \quad \pi_{G_n}^*(\delta) \leq \delta \bar{d}_\mu(\delta) \stackrel{(4.13)}{\leq} 2\delta \sqrt{\frac{C_\mu}{\delta}} = 2\sqrt{C_\mu} \delta.$$

Using (4.14) and (4.15), along with Lemma 4.6 to change from the stationary measure to the uniform measure, gives

$$\frac{\#\{x \in V : p_{2T}^{G_n}(x, x) < \frac{\delta^{3/2}}{4K}\}}{|V(G_n)|} \lesssim \bar{d}_{G_n}(1) \left((1 + 2\sqrt{C_\mu})\sqrt{\delta} + \frac{C_1}{\delta^2 \gamma} \|\omega_n\|_{L^2(V_n)}^2 \frac{\bar{d}_{G_n}\left(\frac{\delta}{C_1 C_\mu \gamma T^{2.5}}\right)}{\bar{d}_\mu(1/T^3)} \right).$$

Define the event

$$\mathcal{Q}_n := \{\bar{d}_{G_n}(\eta) > \delta^{-1/4} \bar{d}_\mu(\eta/2)\} \cup \{\bar{d}_{G_n}(1) > \delta^{-1/4} \bar{d}_\mu(1/2)\},$$

where $\eta := \frac{\delta}{C_1 C_\mu \gamma T^{2.5}}$. Taking expectation over (G_n, ω_n, ρ_n) gives

$$\mathbb{P}\left(p_{2T}^{G_n}(\rho_n, \rho_n) < \frac{\delta^{3/2}}{4K}\right) \leq O(\bar{d}_\mu(1/2)\delta^{-1/4}) \left((1 + 2\sqrt{C_\mu})\sqrt{\delta} + \frac{C_1}{\delta^{9/4} \gamma} \frac{\bar{d}_\mu(\eta/2)}{\bar{d}_\mu(1/T^3)} \right) + \mathbb{P}[\mathcal{Q}_n].$$

Now set $\gamma := \delta^{-11/4}$ and take $C_2 = C_2(\mu)$ sufficiently large so that for $T \geq C_2/\delta^8$, we have $\eta \geq 2T^{-3}$, and

$$\mathbb{P}\left(p_{2T}^{G_n}(\rho_n, \rho_n) < \frac{\delta^{3/2}}{4K}\right) \leq C_2 \delta^{1/4} + \mathbb{P}[\mathcal{Q}_n].$$

Recalling that $K = \kappa R^2 = \kappa \gamma T \bar{d}_\mu(1/T^3)$, this gives

$$\mathbb{P}\left(p_{2T}^{G_n}(\rho_n, \rho_n) < \frac{\delta^{17/4}}{T \bar{d}_\mu(1/T^3)}\right) \leq C_2 \delta^{1/4} + \mathbb{P}[Q_n].$$

Let μ_n denote the law of (G_n, ρ_n) . Since $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$, it holds that $\bar{d}_{\mu_n}(\varepsilon) \rightarrow \bar{d}_\mu(\varepsilon)$ for every $\varepsilon > 0$; hence, Markov's inequality in conjunction with Lemma 4.8 gives $\lim_{n \rightarrow \infty} \mathbb{P}[Q_n] \leq 2\delta^{1/4}$, yielding

$$\mathbb{P}\left(p_{2T}^G(\rho, \rho) < \frac{\delta^{17/4}}{T \bar{d}_\mu(1/T^3)}\right) \leq (2 + C_2)\delta^{1/4}. \quad \square$$

Now we move on to the proof of Theorem 1.8.

PROOF OF THEOREM 1.8. Let $d_t = \bar{d}_\mu(1/t)$ and observe that since $\{d_t\}$ is monotone increasing,

$$\sum_{t \geq 1} \frac{1}{td_t} \geq \sum_{t \geq 1} \frac{1}{td_{t^3}} \geq \frac{1}{8} \sum_{k \geq 0} \frac{1}{d_{2^{3k}}} \geq \frac{1}{48} \sum_{k \geq 0} \frac{1}{d_{2^k}} \geq \frac{1}{96} \sum_{t \geq 1} \frac{1}{td_t}.$$

Define $c_t := \frac{1}{td_\mu(1/t^3)}$. From the preceding inequalities, it suffices to consider $\hat{g}(T) = \sum_{t=1}^T c_t$ in place of $g(T)$.

Let $C = C(\mu)$ be the constant from (4.12). Fix $\delta > 0$. For $N \geq 1$, let $T_N = \min\{T : \hat{g}(T) \geq N\}$. Choose $N(\delta)$ large enough so that for $N \geq N(\delta)$, we have $T_N \geq C/\delta^2$ and

$$N \leq \hat{g}(T_N) \leq (1 + \delta)N.$$

Define the random variable

$$Z_N = \sum_{t=1}^{T_N} \min\{p_{2t}^G(\rho, \rho), \delta c_t\}.$$

Then by definition, $Z_N \leq \delta \hat{g}(T_N)$, and (4.12) implies that $\mathbb{E}[Z_N] \geq \delta(1 - C\delta^{1/8})\hat{g}(T_N)$; hence,

$$\mathbb{P}\left[Z_N \geq \frac{\delta}{2}\hat{g}(T_N)\right] \geq 1 - 2C\delta^{1/8}.$$

Define the sequence $\{Y_N\}$ by

$$Y_N := \frac{1}{N} \sum_{n \leq N} \mathbb{1}_{\{Z_n \geq \frac{\delta}{2}\hat{g}(T_n)\}}.$$

Since $0 \leq Y_N \leq 1$ almost surely, Fatou's lemma yields

$$(4.16) \quad \mathbb{P}\left[\limsup_{N \rightarrow \infty} Y_N > 0\right] \geq \mathbb{E}\left[\limsup_{N \rightarrow \infty} Y_N\right] \geq \limsup_{N \rightarrow \infty} \mathbb{E}[Y_N] \geq 1 - 2C\delta^{1/8}.$$

By construction, this implies

$$\mathbb{P}\left[\limsup_{T \rightarrow \infty} \frac{\sum_{t=1}^T p_{2t}^G(\rho, \rho)}{g_\mu(T)} > 0\right] \geq 1 - 2C\delta^{1/8}.$$

Now send $\delta \rightarrow 0$, concluding the proof. \square

4.3.1. *Fatter degree tails and transience.* We generalize the example from [35], Section 1.3.

LEMMA 4.10. *For every monotonically increasing sequence $\{d_t : t = 1, 2, \dots\}$ of positive integers such that $\sum_{t \geq 1} \frac{1}{td_t} < \infty$, there is a unimodular random planar graph (G, ρ) with law μ such that for all t sufficiently large,*

$$(4.17) \quad \bar{d}_\mu(1/t) \leq d_t,$$

$\mathbb{E}[\deg_G(\rho)^2] < \infty$, and G is almost surely transient.

PROOF. We may replace d_t by $\min(d_t, t^{1/4})$ so that $d_t \leq t^{1/4}$. We may also assume that $d_{2t} \leq 2d_t$ for all $t \geq 1$ without affecting convergence of the sequence $\sum_{t \geq 1} \frac{1}{td_t}$.

Consider an increasing function $f : \mathbb{N} \rightarrow \mathbb{N}$. Let \mathcal{T}_n be a complete binary tree of height n and replace each edge at height $k = 1, 2, \dots, n$ from the leaves by $f(k)$ parallel edges (at the end of the proof, we indicate how to convert the construction into a simple graph).

Let (\mathcal{T}, ρ) be the distributional limit of $\{\mathcal{T}_n\}$, and let μ be the law of (\mathcal{T}, ρ) . If $f(k) \leq 2^{o(k)}$ as $k \rightarrow \infty$, then the distributional limit exists and, moreover, the unique path to infinity is the one moving away from the leaves. Thus almost surely,

$$(4.18) \quad R_{\text{eff}}^{\mathcal{T}}(\rho \leftrightarrow \infty) \leq \sum_{k=1}^{\infty} \frac{1}{f(k)}.$$

Let us now define $f(k) := 2d_{2k} - d_{2k+1}$ so that

$$(4.19) \quad d_{2^k} = 2^k \sum_{j=k+1}^{\infty} (2^{-j+1}d_{2^j} - 2^{-j}d_{2^{j+1}}) = \sum_{j=1}^{\infty} f(k+j)2^{-j},$$

where convergence of the telescopic sum follows from our assumption that $d_{2^j} \leq 2^{j/4}$. Now observe if $t > 2^{k-1}$, then

$$\bar{d}_\mu(1/t) \leq \sum_{j=1}^{\infty} f(k+j)2^{-j} = d_{2^k},$$

hence (4.17) is satisfied.

We also have

$$\sum_{k=1}^{\infty} \frac{1}{f(k)} \stackrel{(4.19)}{\lesssim} \sum_{k=1}^{\infty} \frac{1}{d_{2^k}} \leq 2 \sum_{t=1}^{\infty} \frac{1}{td_t} < \infty.$$

From (4.18), this implies that almost surely \mathcal{T} is transient. Finally, note that $\mathbb{E}[\deg_G(\rho)^2] \leq 2 \sum_{k \geq 1} 2^{-k} (d_{2^k})^2 < \infty$ since we assumed that $d_{2^k} \leq 2^{k/4}$.

We may replace every parallel edge by a path of length two while affecting the degree distribution only by a factor of 2 (and one can rescale f accordingly to maintain property (4.17)). \square

4.4. *Spectrally heterogeneous graphs.* There are unimodular random graphs (G, ρ) with $\deg_G(\rho) \leq O(1)$ and $\dim_{\text{sp}}(G) \leq O(1)$ almost surely, but $\dim_{\text{cg}}(G, \rho) = \infty$. Indeed, there exist invariantly nonamenable graphs (G, ρ) for which $\dim_{\text{sp}}(G) \leq O(1)$ almost surely. This is asserted in [9].

An invariantly nonamenable graph with bounded spectral dimension. We recall the construction alluded to there. Fix a value $\alpha > 0$. Let \mathcal{T} denote the infinite 3-regular tree and fix

a vertex $v_0 \in V(\mathcal{T})$. To each $v \in V(\mathcal{T})$, we attach a random path of length L_v , where the random variables $\{L_v : v \in V(\mathcal{T})\}$ are independent and satisfy, for $\ell \geq 1$:

$$\mathbb{P}(L_v = \ell) = \begin{cases} c_1(\ell + 1)\ell^{-2-\alpha}, & v = v_0, \\ c_2\ell^{-2-\alpha}, & v \in V(\mathcal{T}) \setminus \{v_0\}, \end{cases}$$

where $c_1, c_2 > 0$ are the unique values that give rise to probability measures. Let \mathcal{G} denote the resulting graph, and let P_0 denote the path attached to v_0 (so that P_0 contains $L_{v_0} + 1$ vertices).

It is not difficult to verify that (\mathcal{G}, ρ) is unimodular when $\rho \in V(P_0)$ is chosen uniformly at random. Indeed, (\mathcal{G}, ρ) is the distributional limit of finite graphs with uniformly random roots. To see this, consider a sequence $\{G_n\}$ of 3-regular graphs with girth tending to infinity (see, e.g., [40]). Let \mathcal{G}_n denote the random graph in which we attach to every vertex of G_n a path of length L_v , where $\{L_v : v \in V(G_n)\}$ is a family of independent random variables with law $\mathbb{P}(L_v = \ell) = c_2\ell^{-2-\alpha}$ for $\ell \geq 1$. Then \mathcal{G}_n is almost surely finite. Let $u_n \in V(G_n)$ be chosen uniformly at random.

CLAIM 4.11. $\{(\mathcal{G}_n, u_n)\} \Rightarrow (\mathcal{G}, \rho)$.

PROOF. Let $\{P_v : v \in V(G_n)\}$ be the collection of attached paths. Calculate

$$\begin{aligned} \mathbb{P}(L_v = \ell \mid u_n \in P_v) &= \frac{\mathbb{P}(L_v = \ell)}{\mathbb{P}(u_n \in P_v)} \mathbb{P}(u_n \in P_v \mid L_v = \ell) \\ &= c_2\ell^{-2-\alpha} |V(G_n)| \mathbb{E}\left[\frac{\ell + 1}{|V(\mathcal{G}_n)|} \mid L_v = \ell\right]. \end{aligned}$$

Note that the law of $|V(\mathcal{G}_n)|$ conditioned on $\{L_v = \ell\}$ is $(\ell + 1) + \sum_{u \in V(G_n) \setminus \{v\}} L_u$. Thus by the law of large numbers, it holds that

$$\lim_{n \rightarrow \infty} \mathbb{P}(L_v = \ell \mid u_n \in P_v) = c_2(\ell + 1)\ell^{-2-\alpha} \lim_{n \rightarrow \infty} \frac{|V(G_n)|}{\mathbb{E}[|V(\mathcal{G}_n)| \mid L_v = \ell]} = c_1(\ell + 1)\ell^{-2-\alpha}. \quad \square$$

An interesting feature of (\mathcal{G}, ρ) is that the mean return probability is dominated by a small set of vertices (of measure $\approx T^{-\alpha/2}$):

$$\mathbb{E}[p_{2T}^{\mathcal{G}}(\rho, \rho)] \approx \mathbb{P}[L_{v_0} \geq \sqrt{T}] \cdot \frac{1}{\sqrt{T}} \approx T^{-(1+\alpha)/2}.$$

It turns out that one can obtain polynomial conformal volume growth if they are willing to ignore the “spectrally insignificant” vertices. Moreover, if (G, ρ) is spectrally homogeneous in a strong sense ((4.22) below), one can reverse the bound of Theorem 1.2 and obtain $\dim_{\text{cg}}(G, \rho) \leq \text{a.s.} \dim_{\text{sp}}(G)$.

Consider a monotone nondecreasing function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $h(n) \leq n^{o(1)}$ as $n \rightarrow \infty$ and a number $d > 0$. For $T \geq 1$, define the set of vertices with d -dimensional lower bounds on the diagonal heat kernel:

$$H_G(T) := \left\{ x \in V(G) : p_{2T}^G(x, x) \geq \deg_G(x) \frac{T^{-d/2}}{h(T)} \right\}.$$

Define also, for $R \geq 0$,

$$\widehat{H}_G(R) := H_G(R^2 h(R)^4 (\log R)^4).$$

THEOREM 4.12. *Let (G, ρ) be a unimodular random graph and suppose that for all $T \geq 1$,*

$$(4.20) \quad \mathbb{E}[p_{2T}^G(\rho, \rho)] \leq h(T)T^{-d/2}.$$

Then there is a normalized conformal metric $\omega : V(G) \rightarrow \mathbb{R}_+$ such that

$$(4.21) \quad \|\mathbb{1}_{\widehat{H}_G(R)}(\rho) \cdot \#(B_\omega(\rho, R) \cap \widehat{H}_G(R))\|_{L^\infty} \leq R^{d+o(1)} \quad \text{as } R \rightarrow \infty.$$

Moreover, if it holds that

$$(4.22) \quad \mathbb{P}[\rho \notin H_G(T)] \leq \frac{h(T)}{T} \quad \text{for all } T \geq 1,$$

then $\overline{\dim}_{\text{cg}}(G, \rho) \leq d$.

In order to prove Theorem 4.12, we will need to recall some background on the spectral measures of infinite graphs.

4.4.1. *Spectral measures on infinite graphs.* Fix a connected, locally-finite graph G . We use $\ell^2(G)$ for the Hilbert space of real-valued functions $f : V(G) \rightarrow \mathbb{R}$ equipped with the inner product

$$\langle f, g \rangle_{\ell^2(G)} = \sum_{x \in V(G)} \deg_G(x) f(x)g(x).$$

For a graph G , define the averaging operator $P_G : \ell^2(G) \rightarrow \ell^2(G)$ by

$$P_G \psi(u) := \frac{1}{\deg_G(u)} \sum_{v: \{u,v\} \in E(G)} \psi(v).$$

Observe that P_G is self-adjoint:

$$\langle \varphi, P_G \psi \rangle_{\ell^2(G)} = \sum_{x \in V(G)} \deg_G(x) \varphi(x) \frac{1}{\deg_G(x)} \sum_{y: \{x,y\} \in E(G)} \psi(y) = 2 \sum_{\{x,y\} \in E(G)} \varphi(x) \psi(y).$$

Since P_G is an averaging operator, it is also bounded and, therefore, the spectral theorem yields a resolution of the identity I_{P_G} so that $P_G = \int_{\mathbb{R}} \lambda dI_{P_G}(\lambda)$.

Given a vertex $v \in V(G)$, one can define the associated *spectral measure* μ_G^v at v by

$$\mu_G^v((-\infty, \lambda)) = \deg_G(v)^{-1} \langle \mathbb{1}_v, I_{P_G}((-\infty, \lambda)) \mathbb{1}_v \rangle_{\ell^2(G)}.$$

This is the unique probability measure μ_G^v on \mathbb{R} such that

$$(4.23) \quad \deg_G(v) \int_{[-1,1]} \lambda^T d\mu_G^v(\lambda) = \langle \mathbb{1}_v, P_G^T \mathbb{1}_v \rangle_{\ell^2(G)}$$

for all integers $T \geq 1$.

Let us record a few additional equalities. Fix $\rho \in V(G)$. Then by self-adjointness, for any $T \geq 1$, we have:

$$(4.24) \quad \begin{aligned} \frac{\|P_G^T \mathbb{1}_\rho\|_{\ell^2(G)}^2}{\deg_G(\rho)} - \sum_{x \sim \rho} \frac{\langle P_G^T \mathbb{1}_x, P_G^T \mathbb{1}_\rho \rangle_{\ell^2(G)}}{\deg_G(\rho) \deg_G(x)} &= \frac{\langle \mathbb{1}_\rho, (I - P_G) P_G^{2T} \mathbb{1}_\rho \rangle_{\ell^2(G)}}{\deg_G(\rho)} \\ &= \int (1 - \lambda) \lambda^{2T} d\mu_G^\rho(\lambda), \end{aligned}$$

where we have used $P_G \mathbb{1}_\rho = \sum_{x \sim \rho} \frac{\mathbb{1}_x}{\deg_G(x)}$. Moreover,

$$(4.25) \quad \sum_{x \sim \rho} \left\| \frac{P_G^T \mathbb{1}_\rho}{\deg_G(\rho)} - \frac{P_G^T \mathbb{1}_x}{\deg_G(x)} \right\|_{\ell^2(G)}^2 = \frac{\|P_G^T \mathbb{1}_\rho\|_{\ell^2(G)}^2}{\deg_G(\rho)} + \sum_{x \sim \rho} \frac{\|P_G^T \mathbb{1}_x\|_{\ell^2(G)}^2}{\deg_G(x)^2} - 2 \sum_{x \sim \rho} \frac{\langle P_G^T \mathbb{1}_x, P_G^T \mathbb{1}_\rho \rangle_{\ell^2(G)}}{\deg_G(\rho) \deg_G(x)}.$$

For any $x \in V(G)$ and integer $T \geq 0$, we have

$$(4.26) \quad \|P_G^T \mathbb{1}_x\|_{\ell^2(G)}^2 = \langle \mathbb{1}_x, P_G^{2T} \mathbb{1}_x \rangle_{\ell^2(G)} = \deg_G(x) \cdot p_{2T}^G(x, x).$$

Moreover, observe that $\{\mathbb{1}_x / \sqrt{\deg_G(x)} : x \in V(G)\}$ forms an orthonormal basis for $\ell^2(G)$; hence,

$$(4.27) \quad \sum_{x \in V(G)} \frac{\langle P_G^T \mathbb{1}_x, P_G^T \mathbb{1}_\rho \rangle_{\ell^2(G)}^2}{\deg_G(x)} = \sum_{x \in V(G)} \frac{\langle \mathbb{1}_x, P_G^{2T} \mathbb{1}_\rho \rangle_{\ell^2(G)}^2}{\deg_G(x)} = \|P_G^{2T} \mathbb{1}_\rho\|_{\ell^2(G)}^2.$$

Note also that since P_G is a Markov operator, it is a contraction on $\ell^2(G)$, hence for all integers $T \geq 1$,

$$(4.28) \quad p_{2T}^G(x, x) = \deg_G(x) \|P_G^T \mathbb{1}_x\|_{\ell^2(G)}^2 \geq \deg_G(x) \|P_G^{2T} \mathbb{1}_x\|_{\ell^2(G)}^2 = p_{4T}^G(x, x).$$

The heat kernel embedding and growth rates. Suppose that G is a connected, locally finite graph, and let $d > 0$ and $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be as in Section 4.4. Let us define $\Phi_T^G : V(G) \rightarrow \ell^2(G)$ by

$$\Phi_T^G(x) := \frac{P_G^T \mathbb{1}_x}{\deg_G(x)}.$$

The heat-kernel embedding is closely related to the spectral embedding which can be described as follows. Then for $\delta > 0$, the spectral embedding $\Psi_\delta^G : V(G) \rightarrow \ell^2(G)$ is given by

$$\Psi_\delta^G(x) := \frac{I_{P_G}([1 - \delta, 1]) \mathbb{1}_x}{\sqrt{\deg_G(x)}}.$$

Clearly, these two embeddings are closely related for $T \asymp \frac{1}{\delta}$. The geometry of the spectral embedding has been used in work on higher-order Cheeger inequalities [50] and in connection with return probabilities [52].

For $x \in V(G)$, also define the set of points that are closer to $\Phi_T^G(x)$ than the origin in the heat kernel embedding:

$$\mathcal{C}_T^G(x) := \{y \in V(G) : \|\Phi_T^G(x) - \Phi_T^G(y)\|_{\ell^2(G)} \leq \|\Phi_T^G(y)\|_{\ell^2(G)}\}.$$

The next lemma gives a relationship between return probabilities and the size of $\mathcal{C}_T^G(x)$. This is inspired by the ‘‘mass spreading’’ property of the spectral embedding employed in [50]. For a subset $S \subseteq V(G)$, we will use the notation $\deg_G(S) = \sum_{x \in S} \deg_G(x)$.

LEMMA 4.13. *For any $x \in V(G)$, it holds that*

$$\deg_G(\mathcal{C}_T^G(x)) \leq \frac{4 \deg_G(x)}{p_{2T}^G(x, x)}.$$

PROOF. Note that $\langle u, v \rangle = \frac{1}{2}(\|u\|^2 + \|v\|^2 - \|u - v\|^2) \geq \frac{1}{2}\|u\|^2$ whenever $\|u - v\| \leq \|v\|$. Employ this in conjunction with (4.27) to write and

$$\|\Phi_{2T}^G(\rho)\|_{\ell^2(G)}^2 = \sum_{x \in V(G)} \deg_G(x) \langle \Phi_T^G(x), \Phi_T^G(\rho) \rangle_{\ell^2(G)}^2 \geq \deg(\mathcal{C}_T^G(\rho)) \frac{\|\Phi_T^G(\rho)\|_{\ell^2(G)}^4}{4}.$$

To finish, use the fact that P_G is a contraction on $\ell^2(G)$: $\|\Phi_{2T}^G(\rho)\|_{\ell^2(G)} \leq \|\Phi_T^G(\rho)\|_{\ell^2(G)}$, hence

$$\deg_G(\mathcal{C}_T^G(\rho)) \leq \frac{4}{\|\Phi_T^G(\rho)\|_{\ell^2(G)}^2} \stackrel{(4.26)}{=} \frac{4 \deg_G(\rho)}{p_{2T}^G(\rho, \rho)}. \quad \square$$

4.4.2. *Spectrally significant vertices in unimodular random graphs.* If (G, ρ) is a random rooted graph such that P_G is almost surely self-adjoint, one defines the *spectral measure* of (G, ρ) by

$$\mu := \mathbb{E}[\mu_G^\rho].$$

Let (G, ρ) be a unimodular random graph with spectral measure μ . Taking expectations in (4.25) gives

$$\begin{aligned} & \mathbb{E} \left[\sum_{x \sim \rho} \|\Phi_T^G(x) - \Phi_T^G(\rho)\|_{\ell^2(G)}^2 \right] \\ &= \mathbb{E} \left[\sum_{x \sim \rho} \left\| \frac{P_G^T \mathbb{1}_\rho}{\deg_G(\rho)} - \frac{P_G^T \mathbb{1}_x}{\deg_G(x)} \right\|_{\ell^2(G)}^2 \right] \\ &= \mathbb{E} \left[\frac{\|P_G^T \mathbb{1}_\rho\|_{\ell^2(G)}^2}{\deg_G(\rho)} + \sum_{x \sim \rho} \frac{\|P_G^T \mathbb{1}_x\|_{\ell^2(G)}^2}{\deg_G(x)^2} - 2 \sum_{x \sim \rho} \frac{\langle P_G^T \mathbb{1}_x, P_G^T \mathbb{1}_\rho \rangle_{\ell^2(G)}}{\deg_G(\rho) \deg_G(x)} \right]. \end{aligned}$$

The mass-transport principle applied to the functional $F(G, \rho, x) := \frac{\mathbb{1}_{E(G)}(\{x, \rho\})}{\deg_G(x)^2} \|P_G^T \mathbb{1}_x\|_{\ell^2(G)}^2$ gives

$$\mathbb{E} \left[\sum_{x \sim \rho} \frac{\|P_G^T \mathbb{1}_x\|_{\ell^2(G)}^2}{\deg_G(x)^2} \right] = \mathbb{E} \left[\frac{\|P_G^T \mathbb{1}_\rho\|_{\ell^2(G)}^2}{\deg_G(\rho)} \right],$$

so that applying (4.24) shows that for all $T \geq 1$,

$$(4.29) \quad \mathbb{E} \left[\sum_{x \sim \rho} \|\Phi_T^G(x) - \Phi_T^G(\rho)\|_{\ell^2(G)}^2 \right] = 2 \int (1 - \lambda) \lambda^{2T} d\mu(\lambda).$$

Consider some $d > 0$ and split the latter integral into two pieces, depending on whether $\lambda \leq 1 - \frac{(d+1) \log T}{T}$:

$$(4.30) \quad \begin{aligned} \int (1 - \lambda) \lambda^{2T} d\mu(\lambda) &\leq T^{-d-1} + \frac{(d+1) \log T}{T} \int \lambda^{2T} d\mu(\lambda) \\ &\stackrel{(4.23)}{=} T^{-d-1} + \frac{(d+1) \log T}{T} \mathbb{E}[p_{2T}^G(\rho, \rho)]. \end{aligned}$$

The remainder of this section is devoted to the proof of Theorem 4.12.

PROOF OF THEOREM 4.12. For $k \geq 1$, define the weights $\omega_k : V(G) \rightarrow \mathbb{R}_+$ by

$$\omega_k(x) := \sqrt{\sum_{y: \{x, y\} \in E(G)} \|\Phi_{2^k}^G(x) - \Phi_{2^k}^G(y)\|_{\ell^2(G)}^2}.$$

Under assumption (4.20), we can employ (4.29) and (4.30) to write

$$\mathbb{E} \omega_k(\rho)^2 \leq 2^{-k(d+1)} + \frac{(d+1)k}{2^k} 2^{-kd/2} h(2^k).$$

Define now

$$\omega := \sqrt{\sum_{k \geq 1} \frac{2^{k(1+d/2)}}{k^3 h(2^k)} \omega_k^2}$$

so that

$$\mathbb{E} \omega(\rho)^2 \lesssim \sum_{k \geq 1} \frac{1}{k^2} \lesssim 1.$$

By construction, we have, for every $k \geq 1$ and $x, y \in V(G)$,

$$(4.31) \quad \text{dist}_\omega(x, y)^2 \geq \frac{2^{k(1+d/2)}}{k^3 h(2^k)} \|\Phi_{2^k}^G(x) - \Phi_{2^k}^G(y)\|_{\ell^2(G)}^2.$$

LEMMA 4.14. *For all $k \geq 1$, if $x \in H_G(2^k)$, then*

$$\left| B_\omega\left(x, \frac{2^{k/2}}{h(2^k)k^{3/2}}\right) \cap H_G(2^k) \right| \leq |\mathcal{C}_{2^k}^G(x)| \leq h(2^k) 2^{kd/2+2}.$$

PROOF. From (4.31),

$$y \in B_\omega\left(x, \frac{2^{k/2}}{h(2^k)k^{3/2}}\right) \implies \|\Phi_{2^k}^G(x) - \Phi_{2^k}^G(y)\|_{\ell^2(G)}^2 \leq \frac{2^{-kd/2}}{h(2^k)}.$$

By definition, $y \in H_G(2^k) \implies p_{2^{k+1}}^G(y, y) \geq \text{deg}_G(y) \frac{2^{-kd/2}}{h(2^k)}$. Therefore,

$$\begin{aligned} y \in B_\omega\left(x, \frac{2^{k/2}}{h(2^k)k^{3/2}}\right) \cap H_G(2^k) \\ \implies \|\Phi_{2^k}^G(x) - \Phi_{2^k}^G(y)\|_{\ell^2(G)}^2 \leq \frac{p_{2^{k+1}}^G(y, y)}{\text{deg}_G(y)} \stackrel{(4.26)}{=} \|\Phi_{2^k}^G(y)\|_{\ell^2(G)}^2, \end{aligned}$$

which yields $y \in \mathcal{C}_{2^k}^G(x)$. Now Lemma 4.13 gives

$$|\mathcal{C}_{2^k}^G(x)| \leq \frac{4 \text{deg}_G(x)}{p_{2^{k+1}}^G(x, x)} \leq 4h(2^k) 2^{kd/2},$$

where the last inequality uses $x \in H_G(2^k)$. \square

COROLLARY 4.15. *For all R sufficiently large, if $x \in \widehat{H}_G(R)$, then*

$$|B_\omega(x, R) \cap \widehat{H}_G(R)| \leq R^{d+o(1)}.$$

This confirms (4.21). To verify that $\overline{\dim}_{\text{cg}}(G, \rho) \leq d$ under (4.22), we define

$$\widehat{\omega}_k = \frac{2^k}{\sqrt{h(4^k h(2^k) 4k^4)}} \mathbb{1}_{V(G) \setminus \widehat{H}_G(2^k)}.$$

Observe that from (4.22),

$$\mathbb{E} \widehat{\omega}_k(\rho)^2 = \frac{4^k}{h(4^k h(2^k) 4k^4)} \mathbb{P}[\rho \notin \widehat{H}_G(2^k)] \lesssim 1.$$

Define

$$\hat{\omega} := \sqrt{\sum_{k \geq 1} \frac{\hat{\omega}_k^2}{k^2}}$$

so that $\mathbb{E} \hat{\omega}(\rho)^2 \lesssim 1$. Finally, note that for any $k \geq 1$:

$$x \notin \hat{H}_G(2^k) \implies B_{\hat{\omega}}(x, 2^k / \sqrt{h(4^k h(2^k) k^4)}) = \{x\}.$$

Thus taking the final weight $\omega_0 = \sqrt{\omega^2 + \hat{\omega}^2}$ verifies that $\overline{\dim}_{\text{cg}}(G, \rho) \leq d$. \square

5. Markov type and speed of the random walk. We will now address the speed of the random walk on unimodular random graphs. Our approach is to first establish that the random walk is at most diffusive under any normalized conformal metric, and our main tool will be the theory of Markov type. We then use separators to construct conformal metrics with Hölder-type comparisons to the graph metric, allowing us to establish subdiffusive speed in certain settings.

5.1. *Diffusive bounds in the conformal metric.* In this section, it will be helpful to think about *reversible* random graphs. Suppose (G, ρ) is a random rooted graph and let $\{X_t\}$ denote the random walk on G (conditioned on (G, ρ)) with $X_0 = \rho$. Then (G, ρ) is *reversible* if we have the identity of laws:

$$(G, X_0, X_1) \stackrel{\text{law}}{=} (G, X_1, X_0).$$

The following lemma is from [15], Proposition 2.5.

LEMMA 5.1. *There is a correspondence between unimodular random graphs with $\mathbb{E}[\text{deg}_G(\rho)] < \infty$ and reversible random graphs: (G, ρ) is unimodular if and only if $(\tilde{G}, \tilde{\rho})$ is reversible, where $(\tilde{G}, \tilde{\rho})$ has the law of (G, ρ) biased by $\text{deg}_G(\rho)$.*

We first prove a general result for the case when a reversible conformal random graph (G, ω, ρ) is such that the Markov type 2 constant of the metric space $(V(G), \text{dist}_\omega)$ is essentially bounded. For instance, by [28], this is true when G is almost surely planar. Then we move on to graphs of annealed polynomial growth. One should recall the definition of the Markov type 2 constant M_2 from Definition 1.14.

THEOREM 5.2. *Suppose that (G, ρ) is an invariantly amenable (cf. Section 4.1) reversible random graph. Then for any conformal metric ω on (G, ρ) , the following holds: For all $T \geq 1$,*

$$\mathbb{E}[\text{dist}_\omega(X_0, X_T)^2 \mid X_0 = \rho] \leq T \|M_2(V(G), \text{dist}_\omega)\|_{L^\infty}^2 \cdot \mathbb{E}[\omega(\rho)^2].$$

PROOF. Let $\{\xi_j : j \geq 1\}$ denote a finitary exhaustion (G, ρ) . Then Lemma 5.1 gives the variant of (4.1) for reversible random graphs:

$$\{(G[K_{\xi_j}(\rho)], \rho) : j \geq 1\} \Rightarrow (G, \rho).$$

In particular, we have

$$(5.1) \quad \{(G[K_{\xi_j}(\rho)], \rho_j)\} \Rightarrow (G, \rho),$$

where ρ_j is distributed according to the stationary measure on $G[K_{\xi_j}(\rho)]$.

Let $\{X_t^j\}$ denote the random walk conditioned on $G[K_{\xi_j}(\rho)]$, where X_0^j has the law of the stationary measure on $G[K_{\xi_j}(\rho)]$. If we let $M := \|M_2(V(G), \text{dist}_\omega)\|_{L^\infty}$, then the definition of Markov type yields, for any $j \geq 1$:

$$\mathbb{E}[\text{dist}_\omega(X_0^j, X_T^j)^2 \mid G[K_{\xi_j}(\rho)]] \leq TM^2 \mathbb{E}[\text{dist}_\omega(X_0^j, X_1^j)^2 \mid G[K_{\xi_j}(\rho)]].$$

Recall that if $\{x, y\} \in E(G)$, then $\text{dist}_\omega(x, y) \leq \frac{1}{2}(\omega(x)^2 + \omega(y)^2)$, hence by stationarity, the latter quantity is bounded by

$$\mathbb{E}[\text{dist}_\omega(X_0^j, X_1^j)^2 \mid G[K_{\xi_j}(\rho)]] \leq \mathbb{E}[\omega(X_0^j)^2 \mid G[K_{\xi_j}(\rho)]].$$

Taking expectation over $G[K_{\xi_j}(\rho)]$ yields

$$\mathbb{E}[\text{dist}_\omega(X_0^j, X_T^j)^2] \leq TM^2 \mathbb{E}[\omega(X_0^j)^2].$$

If we replace the weight ω by $\tilde{\omega} := \min(\omega, \tau)$ for some $\tau > 0$, this argument yields

$$\mathbb{E}[\text{dist}_{\tilde{\omega}}(X_0^j, X_T^j)^2] \leq TM^2 \mathbb{E}[\tilde{\omega}(X_0^j)^2].$$

Since $\tilde{\omega}$ is essentially bounded, taking a limit as $j \rightarrow \infty$ yields

$$\mathbb{E}[\text{dist}_{\tilde{\omega}}(X_0, X_T)^2 \mid X_0 = \rho] \leq TM^2 \mathbb{E}[\tilde{\omega}(\rho)^2] \leq TM^2 \mathbb{E}[\omega(\rho)^2].$$

Now taking the truncation parameter $\tau \rightarrow \infty$ yields the desired result. \square

In order to prove a similar result for reversible random graphs of polynomial growth, we need a result about the Markov type 2 constants of finite metric spaces. It is an immediate consequence of the following facts: (1) Hilbert spaces have Markov type 2 with constant 1 [11], (2) Markov type is a bi-Lipschitz invariant, (3) every n -point metric space embeds into a Hilbert space with $O(\log n)$ bi-Lipschitz distortion [25].

THEOREM 5.3. *If (X, d) is an n -point metric space with $n \geq 2$, then $M_2(X, d) \leq O(\log n)$.*

The next lemma will allow us to choose a (unimodular) finitary exhaustion whose sets have controlled diameters.

LEMMA 5.4. *Suppose that (G, ρ) is a unimodular random graph. Then there is a sequence of bond percolations $\{\xi_j : j \geq 1\}$ with the following properties for every $j \geq 1$:*

1. $(G, \rho, \{\xi_j : j \geq 1\})$ is unimodular as a marked network.
2. Almost surely, $\text{diam}_G(K_{\xi_j}(\rho)) \leq 2j$.
3. For every $r > 0$, it holds that

$$\mathbb{P}[B_G(\rho, r) \not\subseteq K_{\xi_j}(\rho)] \leq \frac{1}{|B_G(\rho, j)|^2} + \frac{12r}{j} \log|B_G(\rho, 3j)|.$$

PROOF. Fix a parameter $R \geq 1$ and let $\{R_x : x \in V(G)\}$ denote a sequence of independent exponential random variables where R_x has mean

$$(5.2) \quad \frac{R}{3 \log|B_G(x, 2R)|}.$$

Let $\{\beta_x \in [0, 1] : x \in V(G)\}$ denote an independent family of i.i.d. uniform random variables.

Let $\hat{R}_x := \min(R_x, R)$ and define for every $x \in V(G)$:

$$\Theta_x := \{y \in B_G(x, R) : \hat{R}_y \geq \text{dist}_G(x, y)\}.$$

Let $\theta(x) \in \Theta_x$ be the vertex $y \in \Theta_x$ with β_y minimal. By construction, $\{\theta^{-1}(y) : y \in V(G)\}$ is almost surely a partition of $V(G)$ and $\theta^{-1}(y) \subseteq B_G(y, R)$ for every $y \in V(G)$.

Define a bond percolation $\xi_R : E(G) \rightarrow \{0, 1\}$ on G by $\xi_R(\{u, v\}) = \mathbb{1}_{\{\theta(u)=\theta(v)\}}$. By construction, every cluster $K_{\xi_R}(x)$ is of the form $\theta^{-1}(y)$ for some $y \in V(G)$, and thus $\text{diam}_G(K_{\xi_R}(x)) \leq 2R$ for every $x \in V(G)$.

The marked network (G, ρ, ξ_R) is unimodular because the law of ξ_R depends only on the isomorphism class of G . Let ν_G be the law of ξ_R given G . Then for any functional $F(G, x, y, \xi)$, one can define

$$\hat{F}(G, x, y) := \mathbb{E}_{\nu_G}[F(G, x, y, \xi_R)],$$

and then apply the mass-transport principle to \hat{F} . The next lemma verifies (3) and completes the proof.

LEMMA 5.5. *For every $x \in V(G)$, the following holds:*

$$\mathbb{P}[B_G(x, r) \not\subseteq K_{\xi_R}(x)] \leq \frac{1}{|B_G(x, R)|^2} + \frac{12r}{R} \log |B_G(x, 3R)|.$$

PROOF. Denote the event $\mathcal{E} = \{\exists y \in B_G(x, R) : R_y > R\}$. Note that $y \in B_G(x, R) \implies |B_G(y, 2R)| \geq |B_G(x, R)|$, and thus reviewing the mean of each R_y in (5.2), a union bound yields

$$\mathbb{P}[\mathcal{E}] \leq e^{-3 \log |B_G(x, R)|} |B_G(x, R)| \leq \frac{1}{|B_G(x, R)|^2}.$$

Define the set

$$U_x := \{y \in B_G(x, R) : R_y \geq \text{dist}_G(x, y) - r\}.$$

For $y \in B_G(x, R)$, let \mathcal{E}'_y denote the event $\{R_y > \text{dist}_G(x, y) + r\}$, and note that by the memoryless property of the exponential distribution,

$$\begin{aligned} \mathbb{P}[\neg \mathcal{E}'_y \mid U_x, y \in U_x] &= \mathbb{P}[\neg \mathcal{E}'_y \mid R_y \geq \text{dist}_G(x, y) - r] \\ (5.3) \quad &= \frac{6 \log |B_G(y, 2R)|}{R} \int_0^{2r} \exp\left(\frac{-6t \log |B_G(y, 2R)|}{R}\right) dt \\ &\leq \frac{12r}{R} \log |B_G(y_*, 2R)| \leq \frac{12r}{R} \log |B_G(x, 3R)|. \end{aligned}$$

Now let $y_* \in V(G)$ denote the vertex with β_{y_*} minimal in the set U_x . Note that U_x is always nonempty since $x \in U_x$. Moreover,

$$\mathcal{E}'_{y_*} \wedge \neg \mathcal{E} \implies B_G(x, r) \subseteq B_G(y_*, R_{y_*}) \implies B_G(x, r) \subseteq K_{\xi_R}(x),$$

where the second implication follows from $\beta_{y_*} \leq \min\{\beta_z : z \in \bigcup_{v \in B_G(x, r)} \Theta_v\}$. Since y_* is independent of $\{R_y : y \in U_x\}$ conditioned on U_x , (5.3) gives

$$\mathbb{P}[\neg \mathcal{E}'_{y_*}] \leq \frac{12r}{R} \log |B_G(x, 3R)|,$$

completing the proof. \square

We will soon define a finite-state Markov chain on the cluster $K_\xi(\rho)$; the following definition will be helpful.

DEFINITION 5.6 (Restricted random walk). Consider a graph $G = (V, E)$ and a finite subset $S \subseteq V$. Let

$$N_G(x) = \{y \in V : \{x, y\} \in E\}$$

denote the neighborhood of a vertex $x \in V$.

Extend deg_G to a measure on subsets $S \subseteq V(G)$ in the obvious way: $\text{deg}_G(S) := \sum_{x \in S} \text{deg}_G(x)$. Define a measure π_S on S by

$$(5.4) \quad \pi_S(x) := \frac{\text{deg}_G(x)}{\text{deg}_G(S)} \mathbb{1}_S(x).$$

We define the random walk restricted to S as the following process $\{Z_t\}$: For $t \geq 0$, put

$$\mathbb{P}(Z_{t+1} = y \mid Z_t = x) = \begin{cases} \frac{|N_G(x) \setminus S|}{\text{deg}_G(x)}, & y = x, \\ \frac{1}{\text{deg}_G(x)}, & y \in N_G(x) \cap S, \\ 0, & \text{otherwise.} \end{cases}$$

It is straightforward to check that $\{Z_t\}$ is a reversible Markov chain on S with stationary measure π_S . If Z_0 has law π_S , we say that $\{Z_t\}$ is the stationary random walk restricted to S .

We now prove the following theorem; it immediately yields Theorem 1.15 (since the assumption of the latter theorem implies that $\text{deg}_G(\rho)$ is essentially bounded).

THEOREM 5.7. Suppose that (G, ρ) is a random rooted graph and for some constants $C, q \geq 1$, it holds that

$$(5.5) \quad \mathbb{E}|B_G(\rho, r)| \leq Cr^q \quad \forall r \geq 1.$$

Then:

1. If (G, ρ) is reversible, then for any conformal metric ω on (G, ρ) : For any $q' \geq 1$ and $T \geq 2$,

$$(5.6) \quad \mathbb{E}[T^{2q'} \wedge \text{dist}_\omega(X_0, X_T)^2 \mid X_0 = \rho] \leq C'T(\log T)^2 \cdot \mathbb{E}[\omega(\rho)^2] + 1,$$

where $C' = C'(C, q, q')$ is number depending only on C, q, q' .

2. If (G, ρ) is unimodular and

$$(5.7) \quad \mathbb{P}[\text{deg}_G(\rho) > \lambda] \leq e^{\lambda/C} \quad \forall \lambda \geq 1,$$

then for any conformal metric ω on (G, ρ) : For any $q' \geq 1$ and $T \geq 2$,

$$(5.8) \quad \mathbb{E}[T^{2q'} \wedge \text{dist}_\omega(X_0, X_T)^2 \mid X_0 = \rho] \leq C'T(\log T)^4 \cdot \mathbb{E}[\omega(\rho)^2] + 1,$$

where $C' = C'(C, q, q')$.

If, additionally, G is almost surely planar, then the bound improves to

$$(5.9) \quad \mathbb{E}[T^{2q'} \wedge \text{dist}_\omega(X_0, X_T)^2 \mid X_0 = \rho] \leq C'T(\log T)^2 \cdot \mathbb{E}[\omega(\rho)^2] + 1.$$

PROOF. Let $\langle \xi_j : j \geq 1 \rangle$ denote the sequence of bond percolations guaranteed by Lemma 5.4. Note that Lemma 5.4(1) assures that each ξ_j is almost surely finitary. Assume first that (G, ρ) is reversible. Then the (degree-biased) mass-transport principle shows that if we choose $\hat{\rho}$ according to the measure $\pi_{K_{\xi_j}(\rho)}$ (recall (5.4)), then (G, ρ) and $(G, \hat{\rho})$ have the same law.

Let $\{Z_t^j\}$ denote the stationary random walk restricted to $K_{\xi_j}(\rho)$ (conditioned on $(G, \rho), \xi_j$). Therefore, Theorem 5.3 yields, for any $T, j \geq 1$:

$$(5.10) \quad \mathbb{E}[\text{dist}_\omega(Z_0^j, Z_T^j)^2 \mid (G, \rho), \xi_j] \lesssim T(\log|K_{\xi_j}(\rho)|)^2 \mathbb{E}[\text{dist}_\omega(Z_0^j, Z_1^j)^2 \mid (G, \rho), \xi_j].$$

Define the events

$$\begin{aligned} \mathcal{E}_T &:= \{ |K_{\xi_j}(\rho)| \leq 2T^{2q'} \mathbb{E}|B_G(\rho, j)| \}, \\ \mathcal{B}_T &:= \{ B_G(\hat{\rho}, T) \subseteq K_{\xi_j}(\rho) \}, \end{aligned}$$

and let $\{X_t\}$ denote the random walk on G . Observe now that for $j \geq 2$:

$$\begin{aligned} &\mathbb{E}[\text{dist}_\omega(X_0, X_T)^2 \mathbb{1}_{\mathcal{B}_T} \mathbb{1}_{\mathcal{E}_T} \mid X_0 = \hat{\rho}, (G, \rho), \xi_j] \\ &= \mathbb{E}[\text{dist}_\omega(Z_0^j, Z_T^j)^2 \mathbb{1}_{\mathcal{B}_T} \mathbb{1}_{\mathcal{E}_T} \mid Z_0^j = \hat{\rho}, (G, \rho), \xi_j] \\ &\leq \mathbb{E}[\text{dist}_\omega(Z_0^j, Z_T^j)^2 \mathbb{1}_{\mathcal{E}_T} \mid Z_0^j = \hat{\rho}, (G, \rho), \xi_j] \\ &\lesssim T(q' \log T + \log(\mathbb{E}|B_G(\rho, j)|))^2 \mathbb{E}[\omega(\hat{\rho})^2 \mid (G, \rho), \xi_j]. \end{aligned}$$

Taking expectations and using (5.5) yields, for $T \geq 2$:

$$\mathbb{E}[\text{dist}_\omega(X_0, X_T)^2 \mathbb{1}_{\mathcal{B}_T} \mathbb{1}_{\mathcal{E}_T} \mid X_0 = \rho] \lesssim CqT(q' \log T + \log j)^2 \mathbb{E}[\omega(\rho)^2].$$

Using assumptions (5.5) and Lemma 5.4(2), we can choose $j \leq T^{O(q')}$ so that

$$\mathbb{P}[\mathcal{B}_T \text{ and } \mathcal{E}_T] \geq 1 - \frac{1}{T^{2q'}},$$

verifying (5.6).

Let us now assume that (G, ρ) is unimodular and verify (5.8). In this case, if we choose $\hat{\rho} \in K_{\xi_j}(\rho)$ uniformly at random, then the mass-transport principle shows that (G, ρ) and $(G, \hat{\rho})$ have the same law.

Return momentarily to (5.10) and observe that we can replace the stationary measure with the uniform measure on both the left and right, losing two factors of $d_{\max}(K_{\xi_j}(\rho))$:

$$\begin{aligned} &\mathbb{E}[\text{dist}_\omega(Z_0^j, Z_T^j)^2 \mid Z_0^j = \hat{\rho}, (G, \rho), \xi_j] \\ &\lesssim (d_{\max}(K_{\xi_j}(\rho)))^2 (\log|K_{\xi_j}(\rho)|)^2 \cdot T \mathbb{E}[\omega(\hat{\rho})^2 \mid (G, \rho), \xi_j]. \end{aligned}$$

Define instead

$$\mathcal{E}_T := \{ |K_{\xi_j}(\rho)| \leq 2T^{2q'} \mathbb{E}|B_G(\rho, j)| \wedge d_{\max}(K_{\xi_j}(\rho)) \leq c' \log T \}.$$

Using (5.7), along with assumptions (5.5) and Lemma 5.4(2), we can choose $c' \leq O(q')$ and $j \leq T^{O(q')}$ so that

$$\mathbb{P}[\mathcal{B}_T \text{ and } \mathcal{E}_T] \geq 1 - \frac{1}{T^{2q'}},$$

and the proof is completed as before.

Finally, to verify (5.9), note that when G is planar, for any conformal metric $\omega : V(G) \rightarrow \mathbb{R}_+$, it holds that $M_2(V(G), \text{dist}_\omega) \leq O(1)$ by the results of [28], which establishes that planar graphs metrics have Markov type 2 with a uniform constant. Appealing to this fact instead of Theorem 5.3 yields the desired improvement. \square

5.2. *Weights from separators.* We now turn to the proofs of Lemma 1.16 and Theorem 1.17. For a graph G , $x \in V(G)$, and $r' > r > 0$, define

$$q_G(x; r, r') := \max \left\{ \sum_{y \in S} \frac{1}{|B_G(y, r)|} : S \subseteq B_G(x, r'), |S| = \kappa_G(\rho; r, r') \right\}.$$

Note that under the assumptions of Theorem 1.13, we have almost surely:

$$q_G(\rho; r, h(r)r) \leq r^{k-1-d+o(1)} \quad \forall r \geq 1.$$

For a given $s \geq 1$, define

$$r(s) := \sup \{ r : h(r)r \leq s \}.$$

Since $h(r) \leq r^{o(1)}$ by assumption, we have $s \leq r(s)^{1+o(1)}$. Applying the following lemma with $r = r(s/2)$ and $r' = s/2$ yields Lemma 1.16.

LEMMA 5.8. *Suppose (G, ρ) is a unimodular random graph. Then for every $r' > r > 0$, there is a subset $W_{r,r'} \subseteq V(G)$ such that $(G, \rho, W_{r,r'})$ is unimodular as a marked network, and the following hold:*

1. $\mathbb{P}[\rho \in W_{r,r'}] \leq \mathbb{E}[q_G(\rho; r, r')]$.
2. Every connected component of $G[V(G) \setminus W_{r,r'}]$ has diameter at most $2r'$ in dist_G .

PROOF. Fix $r' > r \geq 1$. For each $x \in V(G)$, let $U_x \subseteq B_G(x, r') \setminus B_G(x, r)$ denote a separator achieving $\kappa_G(x; r, r')$. Let $\{\beta_x \in [0, 1] : x \in V(G)\}$ be a collection of i.i.d. uniform random variables. Define, for every $x \in V(G)$, the set

$$\hat{U}_x := U_x \setminus \bigcup_{y: \beta_y < \beta_x} B_G(y, r),$$

and

$$W_{r,r'} := \bigcup_{x \in V(G)} \hat{U}_x.$$

To see that almost surely every connected component of $G[V(G) \setminus W_{r,r'}]$ has diameter at most $2r'$ in dist_G , consider $x, y \in V(G)$ with $\text{dist}_G(x, y) > 2r'$. Let γ be a simple path from x to y in G . We will show that almost surely $\gamma \cap W_{r,r'} \neq \emptyset$. Let $z \in V(G)$ be the vertex in the finite set $\{z : \gamma \cap B_G(z, r) \neq \emptyset\}$ with β_z minimal. Since $\text{dist}_G(x, y) > 2r'$, it cannot be that both $x, y \in B_G(z, r')$; hence, it must be that $\gamma \cap U_z \neq \emptyset$. By minimality of β_z , we have $\gamma \cap \hat{U}_z \neq \emptyset$ as well.

Now note that for any $x \in V(G)$,

$$\mathbb{E}[|\hat{U}_x| \mid (G, \rho)] \leq \sum_{y \in U_x} \frac{1}{|B_G(y, r)|}.$$

If we define a flow $F(G, x, y) := \mathbb{1}_{\hat{U}_x}(y)$, then combining the preceding inequality with the mass-transport principle yields

$$\mathbb{P}[\rho \in W_{r,r'}] \leq \mathbb{E} \left[\sum_{y \in U_\rho} \frac{1}{|B_G(y, r)|} \right]. \quad \square$$

Finally, we move on to the proof of Theorem 1.17.

PROOF OF THEOREM 1.17. We begin by showing that assumption (A) implies assumption (B).

LEMMA 5.9 ([17]). *Let H be a planar graph. Consider $x \in V(H)$ and $\tau \geq 1$. If $B_H(x, 4\tau)$ can be covered by λ balls of radius τ , then there is a subset $W \subseteq B_H(x, 6\tau) \setminus B_H(x, \tau)$ whose removal separates $B_H(x, \tau)$ and $V(H) \setminus B_H(x, 6\tau)$, and furthermore, $|W| \leq (\lambda + 1)(2\tau + 1)$.*

Define

$$v_G(x, r) := \min\{|B_G(y, r)| : y \in B_G(x, 6r)\}.$$

By a straightforward packing/covering argument, Lemma 5.9 yields the following if G is almost surely planar: For any $r \geq 2$, it holds that

$$\kappa_G(\rho; r, 6r) \lesssim r \frac{|B_G(\rho, 6r)|}{v_G(x, r)},$$

and thus

$$q_G(\rho; r, 6r) \lesssim r \frac{|B_G(\rho, 6r)|}{v_G(x, r)^2}.$$

Combining this with Lemma 5.8 and the next lemma shows that assumption (A) implies assumption (B).

LEMMA 5.10. *Suppose (G, ρ) is a unimodular random graph and the following two conditions hold for some $C \geq 1, q > 1, \beta, \gamma > 0$ and $r \geq 2$:*

$$(5.11) \quad (\mathbb{E}|B_G(\rho, 6r)|^q)^{1/q} \leq Cr^d,$$

$$(5.12) \quad \mathbb{P}\left[|B_G(\rho, r)| < \frac{\varepsilon}{C(\log r)^\gamma} r^d\right] \leq \exp(-1/\varepsilon^\beta) \quad \forall \varepsilon > 0.$$

Then

$$\mathbb{E}\left[\frac{|B_G(\rho, 6r)|}{v_G(\rho, r)^2}\right] \leq C'r^{-d}(\log r)^{2/\beta+2\gamma}$$

for some $C' = C'(C, \beta, \gamma, d, q)$.

PROOF. Fix $\varepsilon > 0$. Let us define a flow

$$F(G, x, y) := \mathbb{1}_{\{|B_G(x, r)| < \varepsilon/(C(\log r)^\gamma)r^d\}} \mathbb{1}_{\{\text{dist}_G(x, y) \leq 6r\}}.$$

Then we can apply Hölder's inequality with q' such that $1/q + 1/q' = 1$ to conclude that

$$\begin{aligned} \mathbb{E}\left[\sum_{x \in V(G)} F(G, \rho, x)\right] &= \mathbb{E}[|B_G(\rho, 6r)| \mathbb{1}_{\{|B_G(\rho, r)| < \varepsilon/(C(\log r)^\gamma)r^d\}}] \\ &\leq (\mathbb{E}|B_G(\rho, 6r)|^q)^{1/q} \mathbb{P}[|B_G(\rho, r)| < \varepsilon/(C(\log r)^\gamma)r^d]^{1/q'} \\ &\leq Cr^d \exp(-1/(q'\varepsilon^\beta)). \end{aligned}$$

By the mass-transport principle, this bounds $\mathbb{E}[\sum_{x \in V(G)} F(G, x, \rho)] = \mathbb{P}[v_r(\rho) < \varepsilon/(C(\log r)^\gamma)r^d]$; hence, for every $\varepsilon > 0$:

$$\mathbb{P}\left[v_r(\rho) < \frac{\varepsilon}{C(\log r)^\gamma (q'd \log r)^{1/\beta}} r^d\right] \leq Cr^{d(1-\varepsilon^{-\beta})}.$$

Thus we can bound:

$$(5.13) \quad \left(\mathbb{E}\left[\frac{1}{v_r(\rho)^{2q'}}\right]\right)^{1/q'} \leq C_1(\log r)^{2/\beta+2\gamma} r^{-2d},$$

where $C_1 = C_1(C, q, \beta, \gamma, d)$.

Now apply Hölder’s inequality to (5.11) and (5.13) yields

$$\mathbb{E} \left[\frac{|B_G(\rho, 6r)|}{v_r(\rho)^2} \right] \leq Cr^d \cdot C_1(\log r)^{2/\beta+2\gamma} r^{-2d},$$

which gives the desired bound. \square

Therefore, it suffices to prove the desired conclusion under assumption (B). Let us now apply (B) to obtain, for every $r \geq 2$, a marked unimodular network (G, ρ, W_r) satisfying properties (B)(i) and (B)(ii).

Define normalized conformal metrics $\{\omega_j : j \in \mathbb{N}\}$ on (G, ρ) by

$$\omega_j := \frac{\mathbb{1}_{W_{2^j}}}{\sqrt{\mathbb{P}[\rho \in W_{2^j}]}}$$

and note that from (B)(i), we have

$$\omega_j \geq j^{-\alpha/2} 2^{j(d-1)/2} \mathbb{1}_{W_{2^j}} \quad \forall j \geq 1.$$

Combining this with (B)(ii) gives for every $x, y \in V(G)$:

$$(5.14) \quad \text{dist}_G(x, y) \geq 6 \cdot 2^j \implies \text{dist}_{\omega_j}(x, y) \geq j^{-\alpha/2} 2^{j(d-1)/2}.$$

Consider now the unnormalized metric

$$\omega^{(T)} := \sqrt{\sum_{j=1}^{\lceil \log_2 T \rceil} \omega_j^2},$$

and observe the bound: For $T \geq 2$,

$$(5.15) \quad \mathbb{E}[\omega^{(T)}(\rho)^2] \lesssim \log T.$$

Finally, we apply Theorem 5.7(5.8) to give: For every $T \geq 2$,

$$\begin{aligned} \mathbb{E}[\text{dist}_G(X_0, X_T)^{d-1}] &= \mathbb{E}[\min\{T^{d-1}, \text{dist}_G(X_0, X_T)^{d-1}\}] \\ &\stackrel{(5.14)}{\lesssim} (\log T)^{\alpha(d-1)} \mathbb{E}[\min\{T^{(d-1)/2}, \text{dist}_{\omega^{(T)}}(X_0, X_T)\}^2] \\ &\lesssim T(\log T)^{\alpha(d-1)+5}, \end{aligned}$$

where our application of Theorem 5.7(5.8) in the final inequality uses (5.15). \square

APPENDIX: GROWTH BOUNDS FOR UIPT/UIPQ

We now sketch a justification for why Theorem 1.17 applies to UIPT/UIPQ with $d = 4$.

Upper tail (5.11). It is known that q th moment bounds of the form (5.11) hold for the size of the “ r -hull” (which contains $B_G(\rho, r)$) for $d = 4$ and every $q < 3/2$. See [46] for UIPQ and [55] for UIPT.

Lower tail (5.12). The stretched exponential lower tail is more delicate. We sketch here the verification for UIPQ. One can use Schaeffer’s correspondence to establish that the law of $|B_G(\rho, r)|$ is the law of the number of nodes N_r of label at most r in a random labeled tree (see [27], Section 5, and [47], Section 2). The spine of the tree is governed by a Markov process $\{X_n \in \mathbb{N} : n \geq 0\}$, and conditioned on $\{X_n\}$, one has

$$N_r = \sum_{n: X_n \leq r} Y_{X_n}^{(n)},$$

where for each $i \geq 1$, $\{Y_i^{(n)} : n \geq 0\}$ are independent copies of a nonnegative random variable Y_i .

The law of Y_i is sampled as follows: Let T_i be a critical Galton–Watson tree where the number of offspring is geometric with parameter $1/2$. Inductively label the vertices of T_i by a map $\ell : V(T_i) \rightarrow \mathbb{Z}$ as follows: The root is labeled i . If a node has label ℓ , the labels of its offspring are independent and uniform in the set $\{\ell - 1, \ell, \ell + 1\}$. Let \hat{T}_i have the law of T_i conditioned on the event $\{\ell(x) > 0 \forall x \in V(T_i)\}$. Then Y_i has the law of $|\{x \in V(\hat{T}_i) : \ell(x) \leq r\}|$. See, for example, [47], Section 2.3.

The tree T_i satisfies, for all $N \geq 1$,

$$\mathbb{P}(|V(T_i)| = N) \gtrsim N^{-3/2}.$$

Let $W(h)$ denote the number of nodes in T_i of height h . Then there is a constant $c_1 > 0$ such that

$$\mathbb{P}\left[\sum_{h=1}^N W(h) \geq c_1 N^2 \mid |V(T_i)| \geq N^2\right] \geq c_1.$$

See, for instance, [29].

From this, it is elementary to establish that if $i \in [r/4, r/2]$, then Y_i satisfies

$$(A.1) \quad \mathbb{P}[Y_i \geq \varepsilon r^4] \gtrsim c_2 \frac{r^{-2}}{\sqrt{\varepsilon}} \quad \forall \varepsilon > 0, r \geq \varepsilon^{-4}.$$

Therefore, if $Z = \#\{n \geq 1 : X_n \in [r/4, r/2]\}$, then we have

$$(A.2) \quad \mathbb{P}[N_r < \varepsilon r^4 \mid Z] \leq \left(1 - c_2 \frac{r^{-2}}{\sqrt{\varepsilon}}\right)^Z \leq \exp\left(-c_2 \frac{Zr^{-2}}{\sqrt{\varepsilon}}\right).$$

The Markov process $\{X_n\}$ is a birth and death chain whose transition probabilities converge to a limiting distribution as $X_n \rightarrow \infty$. When scaled appropriately, $\{X_n\}$ converges to a Bessel process [54], Proposition 5, and satisfies, for some $c > 0$ and r sufficiently large:

$$\mathbb{P}[Z < cr^2] \lesssim e^{-cr^2}.$$

Combining this with (A.2) yields

$$\mathbb{P}[N_r < \varepsilon r^4] \leq \mathbb{P}[Z_n < cr^2] + \exp\left(-c_2 \frac{c}{\sqrt{\varepsilon}}\right) \lesssim e^{-cr^2} + \exp\left(-c_2 \frac{c}{\sqrt{\varepsilon}}\right).$$

This verifies (5.12) for $\beta = 1/2$ and $\gamma = 0$.

Acknowledgments. I would like to thank Itai Benjamini for his support and for sharing his many insights, along with a stream of inspiring open questions. My thanks to Asaf Nachmias for reading many drafts of this manuscript and sharing his wisdom on circle packings, and to Omer Angel and Nicolas Curien for invaluable discussions on the geometry of random planar maps. In particular, Nicolas offered crucial guidance for the references and calculations in the [Appendix](#).

I am grateful to Jian Ding, Russ Lyons, Steffen Rohde and Lior Silberman for comments at various stages, and to Tom Hutchcroft who pointed out an error in an initial draft of this manuscript and explained the construction found at the beginning of Section 4.4, as well as a simpler proof of Lemma 4.3. Lastly, I am grateful to the anonymous referees for their detailed, insightful feedback.

REFERENCES

- [1] ABRAHAM, I., GAVOILLE, C., GUPTA, A., NEIMAN, O. and TALWAR, K. (2014). Cops, robbers, and threatening skeletons: Padded decomposition for minor-free graphs. In *Symposium on Theory of Computing, STOC 2014, New York, NY, USA, May 31–June 03, 2014* 79–88.
- [2] AIZENMAN, M. and WARZEL, S. (2006). The canopy graph and level statistics for random operators on trees. *Math. Phys. Anal. Geom.* **9** 291–333. MR2329431 <https://doi.org/10.1007/s11040-007-9018-3>
- [3] ALDOUS, D. and LYONS, R. (2007). Processes on unimodular random networks. *Electron. J. Probab.* **12** 1454–1508. MR2354165 <https://doi.org/10.1214/EJP.v12-463>
- [4] AMBJØRN, J., ANAGNOSTOPOULOS, K. N., JENSEN, L., ICHIHARA, T. and WATABIKI, Y. (1998). Quantum geometry and diffusion. *J. High Energy Phys.* **11** Paper 22, 16 pp. MR1667363 <https://doi.org/10.1088/1126-6708/1998/11/022>
- [5] AMBJØRN, J., DURHUUS, B. and JONSSON, T. (1997). *Quantum Geometry: A Statistical Field Theory Approach. Cambridge Monographs on Mathematical Physics.* Cambridge Univ. Press, Cambridge. MR1465433 <https://doi.org/10.1017/CBO9780511524417>
- [6] AMBJØRN, J., NIELSEN, J. L., ROLF, J., BOULATOV, D. and WATABIKI, Y. (1998). The spectral dimension of 2D quantum gravity. *J. High Energy Phys.* **2** Paper 10, 8 pp. MR1613966 <https://doi.org/10.1088/1126-6708/1998/02/010>
- [7] ANGEL, O. (2003). Growth and percolation on the uniform infinite planar triangulation. *Geom. Funct. Anal.* **13** 935–974. MR2024412 <https://doi.org/10.1007/s00039-003-0436-5>
- [8] ANGEL, O., BARLOW, M. T., GUREL-GUREVICH, O. and NACHMIAS, A. (2016). Boundaries of planar graphs, via circle packings. *Ann. Probab.* **44** 1956–1984. MR3502598 <https://doi.org/10.1214/15-AOP1014>
- [9] ANGEL, O., HUTCHCROFT, T., NACHMIAS, A. and RAY, G. (2018). Hyperbolic and parabolic unimodular random maps. *Geom. Funct. Anal.* **28** 879–942. MR3820434 <https://doi.org/10.1007/s00039-018-0446-y>
- [10] ANGEL, O. and SCHRAMM, O. (2003). Uniform infinite planar triangulations. *Comm. Math. Phys.* **241** 191–213. MR2013797 https://doi.org/10.1007/978-1-4419-9675-6_16
- [11] BALL, K. (1992). Markov chains, Riesz transforms and Lipschitz maps. *Geom. Funct. Anal.* **2** 137–172. MR1159828 <https://doi.org/10.1007/BF01896971>
- [12] BARLOW, M. T. (2004). Which values of the volume growth and escape time exponent are possible for a graph? *Rev. Mat. Iberoam.* **20** 1–31. MR2076770 <https://doi.org/10.4171/RMI/378>
- [13] BARLOW, M. T., COULHON, T. and KUMAGAI, T. (2005). Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs. *Comm. Pure Appl. Math.* **58** 1642–1677. MR2177164 <https://doi.org/10.1002/cpa.20091>
- [14] BENJAMINI, I. and CURIEN, N. (2011). On limits of graphs sphere packed in Euclidean space and applications. *European J. Combin.* **32** 975–984. MR2825530 <https://doi.org/10.1016/j.ejc.2011.03.016>
- [15] BENJAMINI, I. and CURIEN, N. (2012). Ergodic theory on stationary random graphs. *Electron. J. Probab.* **17** no. 93, 20 pp. MR2994841 <https://doi.org/10.1214/EJP.v17-2401>
- [16] BENJAMINI, I. and CURIEN, N. (2013). Simple random walk on the uniform infinite planar quadrangulation: Subdiffusivity via pioneer points. *Geom. Funct. Anal.* **23** 501–531. MR3053754 <https://doi.org/10.1007/s00039-013-0212-0>
- [17] BENJAMINI, I. and PAPASOGLU, P. (2011). Growth and isoperimetric profile of planar graphs. *Proc. Amer. Math. Soc.* **139** 4105–4111. MR2823055 <https://doi.org/10.1090/S0002-9939-2011-10810-4>
- [18] BENJAMINI, I. and SCHRAMM, O. (2001). Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.* **6** no. 23, 13 pp. MR1873300 <https://doi.org/10.1214/EJP.v6-96>
- [19] BISWAL, P., LEE, J. R. and RAO, S. (2010). Eigenvalue bounds, spectral partitioning, and metrical deformations via flows. *J. ACM* **57** Art. 13, 23 pp. MR2665882 <https://doi.org/10.1145/1706591.1706593>
- [20] BLAKLEY, G. R. and ROY, P. (1965). A Hölder type inequality for symmetric matrices with nonnegative entries. *Proc. Amer. Math. Soc.* **16** 1244–1245. MR0184950 <https://doi.org/10.2307/2035908>
- [21] BONK, M. (2006). Quasiconformal geometry of fractals. In *International Congress of Mathematicians. Vol. II* 1349–1373. Eur. Math. Soc., Zürich. MR2275649
- [22] BONK, M. and KLEINER, B. (2002). Quasisymmetric parametrizations of two-dimensional metric spheres. *Invent. Math.* **150** 127–183. MR1930885 <https://doi.org/10.1007/s00222-002-0233-z>
- [23] BONK, M. and KLEINER, B. (2005). Conformal dimension and Gromov hyperbolic groups with 2-sphere boundary. *Geom. Topol.* **9** 219–246. MR2116315 <https://doi.org/10.2140/gt.2005.9.219>
- [24] BOURDON, M. and PAJOT, H. (2003). Cohomologie l_p et espaces de Besov. *J. Reine Angew. Math.* **558** 85–108. MR1979183 <https://doi.org/10.1515/crll.2003.043>
- [25] BOURGAIN, J. (1985). On Lipschitz embedding of finite metric spaces in Hilbert space. *Israel J. Math.* **52** 46–52. MR0815600 <https://doi.org/10.1007/BF02776078>

- [26] CANNON, J. W. (1994). The combinatorial Riemann mapping theorem. *Acta Math.* **173** 155–234. MR1301392 <https://doi.org/10.1007/BF02398434>
- [27] CHASSAING, P. and DURHUUS, B. (2006). Local limit of labeled trees and expected volume growth in a random quadrangulation. *Ann. Probab.* **34** 879–917. MR2243873 <https://doi.org/10.1214/009117905000000774>
- [28] DING, J., LEE, J. R. and PERES, Y. (2013). Markov type and threshold embeddings. *Geom. Funct. Anal.* **23** 1207–1229. MR3077911 <https://doi.org/10.1007/s00039-013-0234-7>
- [29] DRMOTA, M. (2009). *Random Trees: An Interplay Between Combinatorics and Probability*. SpringerWien-NewYork, Vienna. MR2484382 <https://doi.org/10.1007/978-3-211-75357-6>
- [30] EBRAHIMNEJAD, F. and LEE, J. R. (2021). On planar graphs of uniform polynomial growth. *Probab. Theory Related Fields* **180** 955–984. MR4288335 <https://doi.org/10.1007/s00440-021-01045-5>
- [31] FAKCHAROENPHOL, J. and TALWAR, K. (2003). An improved decomposition theorem for graphs excluding a fixed minor. In *Approximation, Randomization, and Combinatorial Optimization. Lecture Notes in Computer Science* **2764** 36–46. Springer, Berlin. MR2080779 https://doi.org/10.1007/978-3-540-45198-3_4
- [32] FITZNER, R. and VAN DER HOFSTAD, R. (2017). Mean-field behavior for nearest-neighbor percolation in $d > 10$. *Electron. J. Probab.* **22** Paper No. 43, 65 pp. MR3646069 <https://doi.org/10.1214/17-EJP56>
- [33] GANGULY, S. and LEE, J. R. (2020). Chemical subdiffusivity of critical 2D percolation. Preprint.
- [34] GRIGORYAN, A., NETRUSOV, Y. and YAU, S.-T. (2004). Eigenvalues of elliptic operators and geometric applications. In *Surveys in Differential Geometry. Vol. IX. Surv. Differ. Geom.* **9** 147–217. Int. Press, Somerville, MA. MR2195408 <https://doi.org/10.4310/SDG.2004.v9.n1.a5>
- [35] GUREL-GUREVICH, O. and NACHMIAS, A. (2013). Recurrence of planar graph limits. *Ann. of Math. (2)* **177** 761–781. MR3010812 <https://doi.org/10.4007/annals.2013.177.2.10>
- [36] GWYNNE, E. and HUTCHCROFT, T. (2020). Anomalous diffusion of random walk on random planar maps. *Probab. Theory Related Fields* **178** 567–611. MR4146545 <https://doi.org/10.1007/s00440-020-00986-7>
- [37] GWYNNE, E. and MILLER, J. (2017). Random walk on random planar maps: Spectral dimension, resistance, and displacement. Preprint. Available at arXiv:1711.00836.
- [38] HEINONEN, J. (2001). *Lectures on Analysis on Metric Spaces. Universitext*. Springer, New York. MR1800917 <https://doi.org/10.1007/978-1-4613-0131-8>
- [39] HERSCH, J. (1970). Quatre propriétés isopérimétriques de membranes sphériques homogènes. *C. R. Acad. Sci. Paris, Sér. A–B* **270** A1645–A1648. MR0292357
- [40] IMRICH, W. (1984). Explicit construction of regular graphs without small cycles. *Combinatorica* **4** 53–59. MR0739413 <https://doi.org/10.1007/BF02579157>
- [41] KELNER, J. A., LEE, J. R., PRICE, G. N. and TENG, S.-H. (2011). Metric uniformization and spectral bounds for graphs. *Geom. Funct. Anal.* **21** 1117–1143. MR2846385 <https://doi.org/10.1007/s00039-011-0132-9>
- [42] KLEIN, P., PLOTKIN, S. A. and RAO, S. (1993). Excluded minors, network decomposition, and multicommodity flow. In *25th Annual ACM Symposium on Theory of Computing* 682–690.
- [43] KLEINER, B. (2006). The asymptotic geometry of negatively curved spaces: Uniformization, geometrization and rigidity. In *International Congress of Mathematicians. Vol. II* 743–768. Eur. Math. Soc., Zürich. MR2275621
- [44] KOREVAAR, N. (1993). Upper bounds for eigenvalues of conformal metrics. *J. Differential Geom.* **37** 73–93. MR1198600
- [45] KOZMA, G. and NACHMIAS, A. (2009). The Alexander–Orbach conjecture holds in high dimensions. *Invent. Math.* **178** 635–654. MR2551766 <https://doi.org/10.1007/s00222-009-0208-4>
- [46] KRIKUN, M. (2008). Local structure of random quadrangulations. Preprint. Available at arXiv:math/0512304.
- [47] LE GALL, J.-F. and MÉNARD, L. (2010). Scaling limits for the uniform infinite quadrangulation. *Illinois J. Math.* **54** 1163–1203. MR2928350
- [48] LEE, J. R. (2016). Separators in region intersection graphs. Preprint. Available at arXiv:1608.01612.
- [49] LEE, J. R. (2018). Discrete uniformizing metrics on distributional limits of sphere packings. *Geom. Funct. Anal.* **28** 1091–1130. MR3820440 <https://doi.org/10.1007/s00039-018-0442-2>
- [50] LEE, J. R., GHARAN, S. O. and TREVISAN, L. (2014). Multiway spectral partitioning and higher-order Cheeger inequalities. *J. ACM* **61** Art. 37, 30 pp. MR3293072 <https://doi.org/10.1145/2665063>
- [51] LEE, J. R. and NAOR, A. (2005). Extending Lipschitz functions via random metric partitions. *Invent. Math.* **160** 59–95. MR2129708 <https://doi.org/10.1007/s00222-004-0400-5>
- [52] LYONS, R. and OVEIS GHARAN, S. (2018). Sharp bounds on random walk eigenvalues via spectral embedding. *Int. Math. Res. Not. IMRN* **24** 7555–7605. MR3892273 <https://doi.org/10.1093/imrn/rnx082>

- [53] LYONS, R. and PERES, Y. (2016). *Probability on Trees and Networks. Cambridge Series in Statistical and Probabilistic Mathematics* **42**. Cambridge Univ. Press, New York. Available at <http://pages.iu.edu/~rdlyons/>. MR3616205 <https://doi.org/10.1017/9781316672815>
- [54] MÉNARD, L. (2010). The two uniform infinite quadrangulations of the plane have the same law. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 190–208. MR2641776 <https://doi.org/10.1214/09-AIHP313>
- [55] MÉNARD, L. (2016). Volumes in the uniform infinite planar triangulation: From skeletons to generating functions. Preprint. Available at [arXiv:1604.00908](https://arxiv.org/abs/1604.00908).
- [56] MULHOLLAND, H. P. and SMITH, C. A. B. (1959). An inequality arising in genetical theory. *Amer. Math. Monthly* **66** 673–683. MR0110721 <https://doi.org/10.2307/2309342>
- [57] NACHMIAS, A. (2017). Personal communication.
- [58] PANSU, P. (1989). Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)* **129** 1–60. MR0979599 <https://doi.org/10.2307/1971484>
- [59] ROBERTSON, N. and SEYMOUR, P. D. (2004). Graph minors. XX. Wagner’s conjecture. *J. Combin. Theory Ser. B* **92** 325–357. MR2099147 <https://doi.org/10.1016/j.jctb.2004.08.001>
- [60] ROHDE, S. (2011). Oded Schramm: From circle packing to SLE. *Ann. Probab.* **39** 1621–1667. MR2884870 <https://doi.org/10.1214/10-AOP590>
- [61] SCHOEN, R. and YAU, S.-T. (1994). *Lectures on Differential Geometry. Conference Proceedings and Lecture Notes in Geometry and Topology, I*. International Press, Cambridge, MA. MR1333601