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2D ANISOTROPIC KPZ AT STATIONARITY: SCALING, TIGHTNESS AND NONTRIVIALITY

BY GIUSEPPE CANNIZZARO[1], DIRK ERHARD[2] AND PHILIPP SCHÖNBAUER[3]

[1] *Department of Statistics, University of Warwick, giuseppe.cannizzaro@warwick.ac.uk*
[2] *Department of Mathematics, Universidade Federal da Bahia, erharddirk@gmail.com*
[3] *Department of Mathematics, Imperial College London, p.schoenbauer@imperial.ac.uk*

In this work we focus on the two-dimensional anisotropic KPZ (aKPZ) equation, which is formally given by

$$\partial_t h = \frac{\nu}{2}\Delta h + \lambda\big((\partial_1 h)^2 - (\partial_2 h)^2\big) + \nu^{\frac{1}{2}}\xi,$$

where ξ denotes a noise which is white in both space and time, and λ and ν are positive constants. Due to the wild oscillations of the noise and the quadratic nonlinearity, the previous equation is classically ill posed. It is not possible to linearise it via the Cole–Hopf transformation and the pathwise techniques for singular SPDEs (the theory of regularity structures by M. Hairer or the paracontrolled distributions approach of M. Gubinelli, P. Imkeller, N. Perkowski) are not applicable. In the present work we consider a regularised version of aKPZ which preserves its invariant measure. We prove the existence of subsequential limits once the regularisation is removed, provided λ and ν are suitably renormalised. Moreover, we show that, in the regime in which ν is constant and the coupling constant λ converges to 0 as the inverse of the square root logarithm, any limit differs from the solution to the linear equation obtained by simply dropping the nonlinearity in aKPZ.

1. Introduction.

The KPZ equation is a (singular) stochastic partial differential equation (SPDE), whose formal expression is

$$\partial_t h = \nu\Delta h + \langle \nabla h, Q\nabla h\rangle + \sqrt{D}\xi, \tag{1.1}$$

where ξ is a space-time white noise in spatial dimension d, Q is a $d \times d$-matrix and ν and D are positive constants. The importance of this equation stems from the fact that it describes (via Q, ν and D) universal features of randomly evolving surfaces and it is supposed to arise as the limit of a large class of properly rescaled particle systems. The difficulty in establishing its universality is already on the level of the equation since, from an analytic viewpoint, it is ill posed in any dimension. This is due to the fact that the noise ξ is too irregular for the nonlinear term to be canonically defined.

The only dimension in which a rigorous solution theory has been established (for any value of the constants ν, Q and D) and the universality claim corroborated, is $d = 1$. There are by now different approaches that lead to well posedness: the Cole–Hopf transformation that turns (1.1) into the *linear* multiplicative stochastic heat equation [2]; the martingale approach which leads to the notion of energy solution [20, 23]; pathwise techniques, namely, rough paths [25], regularity structures [26] and paracontrolled calculus [19, 22]. In particular, the theory of regularity structures and paracontrolled calculus, additionally, apply to a much larger class of equations, and, since their introduction, the field of (singular) SPDEs has experienced a tremendous growth. That said, their applicability is restricted to those equations that

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are *locally subcritical* which heuristically means that, at small scales, the nonlinearity does not matter much and the solution behaves (regularitywise) as the linear part of the equation. For (1.1) this is the case only for $d = 1$, while in $d = 2$ and $d \geq 3$ (which are said to be the *critical* and *supercritical* regimes, respectively), the pathwise approaches break down.

Only recently, the first mathematically rigorous results in the critical and supercritical regimes have been obtained. In the latter case physicists (see [29]) predict that, for the parameters ν, Q and D in a suitable window, the nonlinearity should not matter much at large scales, so that, taking a smooth noise, rescaling the height function h according to $h^\varepsilon(t,x) \stackrel{\text{def}}{=} \varepsilon^{\frac{d}{2}} h(t/\varepsilon^2, x/\varepsilon)$ and subtracting the average growth, the fluctuations should be the same as those of the solution of the linear stochastic heat equation. Partial results in this direction have been established in the case $Q = \lambda \mathrm{Id}_d$, for Id_d being the $d \times d$ identity matrix and the coupling constant $\lambda > 0$ sufficiently small, first by [33] via renormalisation group techniques and later by [12–14] (see also [18] for the case of the multiplicative stochastic heat equation).[1]

The picture in the critical case, $d = 2$, is more subtle. Indeed, already from the physics perspective this regime is more delicate since finer details of the equation, and, in particular, the sign of $\det Q$, might influence its large scale dynamics. The importance of the matrix Q can be understood from a microscopic viewpoint. Indeed, heuristically speaking, it is expected that the macroscopic *average behaviour* of a microscopic surface is given by the solution of a PDE of the form

$$\partial_t u = v(\nabla u), \tag{1.2}$$

where v is a deterministic scalar valued map depending on the specific (microscopic) features of the model at hand. Now, since (1.1) should represent the (universal) fluctuations of the surface *around* its hydrodynamic limit, a second order expansion of (1.2) leads to the identification of Q with the Hessian of v. Through (nonrigorous) renormalisation group techniques, Wolf showed in [38] that (1.1) gives rise to *two different* universal behaviours depending on the sign of $\det Q$. If $\det Q > 0$, the so-called *isotropic KPZ class*, then the fluctuations should grow in time as t^β for some $\beta > 0$, and the spatial correlation should grow as the distance to the power $\frac{2\beta}{(\beta+1)}$, (see [29]) while for $\det Q \leq 0$, the *anisotropic KPZ class*, the nonlinearity should morally play no role and the behaviour should be the same as the solution to the stochastic heat equation in dimension 2. Note that the latter, in particular, means that the value of β mentioned above should be equal to zero, and the correlations explode logarithmically. We emphasise that it is nowhere stated that the anisotropic KPZ equation *coincides* with the stochastic heat equation, only the correlations should be of the same order. Though this is expected. Indeed, in the works [3, 4] the authors obtained space-time correlations analogous to that of the solution to the stochastic heat equation, for the scaling limit of a certain interacting particle system. That said, such a scaling limit is obtained via a limit transition, namely, a first limit reduces the models to a system of linear SDEs and, thanks to a second limit, the linear stochastic heat equation is derived.

Numerically, the conjecture for the isotropic case was, for instance, confirmed in [36] for two specific models where it turned out that $\beta \approx 0.24$, while that for the anisotropic case is supported by [27].

Mathematically, an even deeper structure has been found for $\det Q > 0$. Indeed, upon choosing $Q = \lambda \mathrm{Id}_2$, and $\lambda \sim \sqrt{\hat{\lambda}/\log N}$, where N is a regularisation parameter, the work of Caravenna, Sun and Zygouras [7] shows that there is a phase transition (for the one point

[1] In the supercritical regime, a phase transition is expected, depending on λ, but the exact value at which the transition happens is still unknown.

distribution) at $\hat{\lambda} = 2\pi$. Later in [10], for $\hat{\lambda} > 0$ sufficiently close to 0, it was shown that a sequence of approximations of (1.1) is tight. The result was then improved in [8], where not only tightness but also uniqueness and characterisation of the limit were obtained in the whole interval $\hat{\lambda} \in (0, 2\pi)$. They proved that the limit is given by the solution of a stochastic heat equation, different from the one obtained by simply dropping the nonlinear term in (1.1) (see also [17]).

In the present paper, we will focus on the anisotropic KPZ class. For numerous (discrete) models the Hessian of v appearing in (1.2) has been computed (see, e.g., [5, 6, 37]) and its determinant proven to be negative. Precise results were obtained concerning the hydrodynamic behaviour and the convergence of the invariant measure to the Gaussian free field (see [5, 32]). What hinders still the progress is that the statements mentioned so far on the fluctuations have been established at *fixed* time and it is not clear how one can show that the time fluctuations are really of the logarithmic order, as expected (some advances have been made in [11, 37] where a $\log t$ upper bound has been obtained for the time increment).

To shed some light on the behaviour as a *process* for a model belonging to the anisotropic KPZ class, we will be working directly at the level of the equation (1.1). We make a specific choice of the matrix Q, that is, $Q = \lambda\,\mathrm{diag}(1, -1)$, and of initial condition, that is, we start from the invariant measure, that with this choice of Q can be shown to exist (see Lemma 3.1 below). The aforementioned paper of Wolf suggests that, in order to see the universal fluctuations, it is necessary to renormalise the coupling constants. Therefore, we were led to study the following family of approximations:

$$(1.3)\qquad \partial_t h^N = \frac{\nu_N}{2}\Delta h^N + \lambda_N \Pi_N\big((\Pi_N \partial_1 h^N)^2 - (\Pi_N \partial_2 h^N)^2\big) + \nu_N^{\frac{1}{2}}\xi, \qquad h_0^N = \tilde{\eta}$$

in which:

- $\tilde{\eta}$ is a Gaussian free field on $\mathbb{T}^2$, that is, a Gaussian field whose covariance function is

$$\mathbb{E}\big[\tilde{\eta}(\varphi)\tilde{\eta}(\psi)\big] = \big\langle(\Delta)^{-1}\varphi, \psi\big\rangle_{L^2(\mathbb{T}^2)}, \qquad \text{for all } \varphi, \psi \in H^{-1}(\mathbb{T}^2),$$

  and it is assumed that the 0 Fourier mode of φ and ψ is 0.
- ξ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{T}^2$ independent of $\tilde{\eta}$ whose 0th Fourier mode is 0, that is, a Gaussian field whose covariance function is

$$\mathbf{E}\big[\xi(\varphi)\xi(\psi)\big] = \Big\langle \varphi - \int_{\mathbb{T}^2}\varphi(x)\,\mathrm{d}x, \psi - \int_{\mathbb{T}^2}\psi(x)\,\mathrm{d}x\Big\rangle_{L^2(\mathbb{R}_+\times\mathbb{T}^2)}$$

  for all $\varphi, \psi \in L^2(\mathbb{R}_+ \times \mathbb{T}^2)$,
- Π_N is the operator acting in Fourier space by cutting the modes higher that N, that is,

$$(\Pi_N w)_k \stackrel{\text{def}}{=} w_k \mathbb{1}_{|k|_\infty \le N}$$

  and w_k is the kth Fourier component of w,
- ν_N and λ_N are positive constants allowed to depend on the regularisation parameter N.

In Theorem 1.1, which is a consequence of Theorem 4.5 and Theorem 4.8 below, we identify a *family* of different scalings for λ_N and ν_N for which the sequence h^N admits subsequential limits in the space of (Hölder-)continuous functions with values in Besov–Hölder spaces of suitable regularity (see (1.11) and below for a precise definition of these spaces).

THEOREM 1.1. *For $N \in \mathbb{N}$, let h^N be the solution of* (1.3) *started from the invariant measure, given by the Gaussian free field $h^N(0) = \tilde{\eta}$. Then, provided that there exists a constant $C > 0$ such that*

$$(1.4)\qquad \lim_{N\to\infty}\sqrt{\log N}\,\lambda_N \nu_N^{-\frac{1}{2}} = C,$$

the sequence $\{h^N\}_N$ *is tight in* $C_T^\gamma \mathcal{C}^\alpha$ *for any* $\gamma < 1/2$ *and* $\alpha < -1$. *Moreover, if* $\nu_N = 1$ *for all* $N \in \mathbb{N}$, *then tightness holds for any* $\alpha < 0$ *and* $\gamma = 0$.

Let us point out some aspects of the previous theorem which mark the difference from the results mentioned above on critical SPDEs. Notice that, for the equation we are considering, there is *no Cole–Hopf transform* which could turn (1.3) into a linear SPDE and, therefore, no explicit representation of the solution is available. Hence, we are forced to work directly with the equation itself and make sense of its nonlinearity. Moreover (at least in the case $\nu_N = 1$ and λ_N satisfies (1.4)), we obtain tightness for the sequence in the space with *optimal* regularity. This can be seen by power counting since ξ has regularity at most -2 and the regularising effect of the Laplacian gains 2. At last, notice that, according to (1.4), we are allowed to take $\lambda_N = \nu_N = (\log N)^{-1}$. The reason why such a scaling is worthy of consideration is the following. Assume for a moment (we will never do in the present work), that we are looking at the smoothened version of the anisotropic KPZ equation on the full space, obtained by convolving the nonlinearity (only in space) with a smooth compactly supported function. To be more precise, let φ be such a function. Then, let $\tilde{h}$ be the solution to the equation

$$\partial_t \tilde{h} = \frac{1}{2}\Delta \tilde{h} + \varphi * \big((\partial_1 \varphi * \tilde{h})^2 - (\partial_2 \varphi * \tilde{h})^2\big) + \xi. \tag{1.5}$$

Consider now the rescaled version of $\tilde{h}$, defined via $\tilde{h}^N(t,x) = \tilde{h}(N^2 t/\log N, Nx)$. It then turns out to be the case that $\tilde{h}^N$ solves

$$\partial_t \tilde{h} = \frac{1}{2\log N}\Delta \tilde{h} + \frac{1}{\log N}\varphi^N * \big((\partial_1 \varphi^N * \tilde{h})^2 - (\partial_2 \varphi^N * \tilde{h})^2\big) + \frac{1}{\sqrt{\log N}}\tilde{\xi}, \tag{1.6}$$

with $\varphi^N(x) = \varphi(Nx)$ and $\tilde{\xi}$ having the same law as ξ. The analog of Theorem 1.1 on $\mathbb{R}^2$ would then imply that the sequence $\tilde{h}^N$ is tight. If one now were able to show that any limit point of the above sequence is not simply a function that is constant in time, then one would have identified the relevant time scale for which a natural smoothing of the original equation shows an interesting behaviour. Although we do not address this problem here, it is currently being investigated by the authors. [2]

The previous statement does not rule out the possibility that the subsequential limits are trivial, that is, simply constant in time or coincide with the solution of an equation in which the summands containing a vanishing factor disappear which would mean that the strength at which they converge to 0 is too strong.

Upon choosing $\nu_N = 1$, we are indeed able to show that any limit point has finite nonzero energy which, in particular, implies that it is not trivial. Here, we say that a stochastic process $\{Y_t\}_{t\in[0,T]}$ has *finite energy* if

$$\sup_{\pi=\{t_i\}_i} \mathbb{E}\Big[\sum_i (Y_{t_{i+1}} - Y_{t_i})^2\Big] < \infty, \tag{1.7}$$

where the supremum is over all the partitions π of $[0,T]$.

THEOREM 1.2. *In the setting of Theorem* 1.1, *assume that* $\nu_N = 1$. *Then, for any test function* φ *any limit point of the sequence*

$$\Big\{\int_0^t \lambda_N \Pi_N\big((\Pi_N \partial_1 h^N)^2 - (\Pi_N \partial_2 h^N)^2\big)(s,\varphi)\,\mathrm{d}s\Big\}_t$$

is a process with finite nonzero energy.

[2] Instead of mollifying the nonlinearity, one may also mollify the noise to work with a well-defined object. We expect the same to hold in that case.

Theorem 1.2 is proved in Proposition 5.5 and Theorem 5.7, where it is actually shown more. In particular, our results suggest that *any* subsequential limit of $\{h^N\}_N$ will contain a *new* noise which is produced by the dynamics itself. Understanding the nature of this new noise (and its relation to the original one) will be crucial in the characterisation of the limit points and is currently being investigated by the authors.

1.1. *Strategy.* Using tools from Malliavin calculus, we show in Lemma 3.1 that the invariant measure of h^N is given by a Gaussian free field $\tilde{\eta}$. Starting from the invariant measure, we use ideas from [20] (established in the study of energy solutions in the one-dimensional case) to show that in the scaling regime (4.16) the sequence of solutions is tight; see Theorem 4.5. The crucial observation (4.9) is that there exists an explicit functional of h^N, called H^N, with the property that the nonlinearity at h^N equals $\mathcal{L}_0^N H^N$, where $\mathcal{L}_0^N$ denotes the generator of the underlying linear equation (3.6). Using martingale techniques, we are able to obtain bounds which are strong enough to control the nonlinearity and to establish tightness of the sequence of solutions (see Lemmas 4.1 and 4.3).

We rule out triviality by establishing a nonvanishing lower bound on the second moment of the integral in time of the nonlinearity; see Corollary 5.4. Inspired by the analysis of the generator for the one-dimensional KPZ equation in [24] and of the diffusion coefficient for the asymmetric simple exclusion process in $d = 1, 2$ of [31], we show that its Laplace transform is nonzero in the limit, as N tends to infinity. The main tool we use for this is the variational formula presented in Lemma 5.2.

REMARK 1.3. We want to stress that, in principle, the techniques we adopt are sufficiently flexible to be used for other equations at criticality for which the invariant measure is explicitly known (e.g., the equations in [20], Sections 6 and 7, but not (1.1) with Q different from that considered above; see Remark 3.2). Moreover, since they were inspired by tools introduced in the particle systems context, we think that our approach might prove useful in establishing existence of subsequential limits for particle systems and improve our understanding of their large scale behaviour (e.g., the time evolution).

1.2. *Structure of the article.* In Section 2 we recall basic facts from Malliavin calculus, which we use in Section 3 to show that the Gaussian free field is indeed invariant for h^N and to analyse the generator of the Markov process $\{h^N(t)\}_t$. In Section 4 we then establish tightness of h^N and prove Theorem 1.1. In Section 5 we show nontriviality of the nonlinearity and prove Theorem 1.2. We conclude the paper with Section 6, in which we explore further consequences of the bounds established in Section 4. In particular, we shed some further light on the behaviour of the nonlinearity by determining the large N limit of the martingales appearing in Section 4.

Notations and function spaces. The notation $\mathbb{Z}_0^2$ always refers to $\mathbb{Z}^2 \setminus \{0\}$, and $\mathbb{T}^2$ denotes the two-dimensional torus of side length 2π. We equip the space $L^2(\mathbb{T}^2; \mathbb{C})$ with the Fourier basis $\{e_k\}_{k\in\mathbb{Z}^2}$, defined via $e_k(x) \stackrel{\text{def}}{=} \frac{1}{2\pi} e^{\iota k\cdot x}$, where ι is the imaginary unit. The basis functions e_k can be decomposed in their real and imaginary parts, so that $e_k = a_k + i b_k$ and the system $\{\sqrt{2}a_k\}_{k\in\mathbb{Z}^2_{\text{diag}}} \sqcup \{\sqrt{2}b_k\}_{k\in\mathbb{Z}^2_{\text{diag}}\setminus\{0\}}$ forms a real valued orthonormal basis of $L^2(\mathbb{T}^2)$, where $\mathbb{Z}^2_{\text{diag}} = \{(k_1, k_2) \in \mathbb{Z}^2 : k_1 \geq k_2\}$. The Fourier transform, denoted by $\mathcal{F}$ and at times also by $\hat{\cdot}$, is given by the formula

$$\mathcal{F}(\varphi)(k) \stackrel{\text{def}}{=} \varphi_k = \int_{\mathbb{T}^2} \varphi(x) e_{-k}(x)\, \mathrm{d}x. \tag{1.8}$$

For any real valued distribution $\eta \in \mathcal{D}'(\mathbb{T}^2)$ and $k \in \mathbb{Z}^2$, its Fourier transform is given by the (complex) pairing

$$\eta_k \stackrel{\text{def}}{=} \eta(e_{-k}) = \eta(a_k) - \iota\eta(b_k), \tag{1.9}$$

so that $\overline{\eta(e_k)} = \eta(e_{-k})$. Moreover, we recall that the Laplacian Δ on $\mathbb{T}^2$ has eigenfunctions $\{e_k\}_{k\in\mathbb{Z}^2}$ with eigenvalues $\{-|k|^2 : k \in \mathbb{Z}^2\}$, and, for $\theta > 0$, we define the operator $(-\Delta)^\theta$ by its action on the basis elements

$$(-\Delta)^\theta e_k(x) \stackrel{\text{def}}{=} |k|^{2\theta} e_k(x), \tag{1.10}$$

for $k \neq 0$ and $(-\Delta)^\theta e_0(x) \stackrel{\text{def}}{=} 0$.

We will work mostly in Besov spaces. For a thorough exposition on these spaces and their properties, we refer the interested reader to [1]; see also [19], Appendix A, for a review of the results which we will need below. Besov spaces are defined via a dyadic partition of unity $(\chi, \varrho) \in \mathcal{D}$, that is, χ and ϱ are nonnegative radial functions such that:

- the supports of χ and ϱ are, respectively, contained in a ball and an annulus,
- $\chi(x) + \sum_{j\geq 0} \varrho(2^{-j}x) = 1$ for all $x \in \mathbb{R}^d$,
- $\mathrm{supp}(\chi) \cap \mathrm{supp}\varrho(2^{-j}\cdot) = \varnothing$ for all $j \geq 1$ and $\mathrm{supp}(\varrho(2^{-j}\cdot) \cap \mathrm{supp}\varrho(2^{-i}\cdot)) = \varnothing$ whenever $|i - j| > 1$.

For any distribution $u \in \mathcal{D}'(\mathbb{T}^2)$, the Littlewood–Paley blocks are defined as

$$\Delta_{-1}u = \mathcal{F}^{-1}(\chi\mathcal{F}(u)), \quad \text{and} \quad \Delta_j u = \mathcal{F}^{-1}(\varrho_j\mathcal{F}(u)), \quad j \geq 1,$$

where $\varrho_j \stackrel{\text{def}}{=} \varrho(2^{-j}\cdot)$. Since $K_j \stackrel{\text{def}}{=} \mathcal{F}^{-1}\varrho_j$ is a smooth function, so is $\Delta_j u = K_j * u$. Given $\alpha \in \mathbb{R}$, $p, q \in [1, +\infty)$, the Besov space $B^\alpha_{p,q}$ is given by

$$B^\alpha_{p,q}(\mathbb{T}^2) \stackrel{\text{def}}{=} \Big\{u \in \mathcal{D}'(\mathbb{T}^2) : \|u\|^q_{B^\alpha_{p,q}} \stackrel{\text{def}}{=} \sum_{j\geq -1} 2^{\alpha jq} \|\Delta_j u\|^q_{L^p(\mathbb{T}^2)} < \infty\Big\}. \tag{1.11}$$

In the special case $p = q = \infty$, the norm is

$$\|u\|_{B^\alpha_{\infty,\infty}} \stackrel{\text{def}}{=} \sup_{j\geq -1} 2^{\alpha j} \|\Delta_j u\|_{L^\infty(\mathbb{T}^2)},$$

and, since this is the space with which we will mainly work, we set $\mathcal{C}^\alpha \stackrel{\text{def}}{=} B^\alpha_{\infty,\infty}$ and denote the corresponding norm by $\|u\|_\alpha \stackrel{\text{def}}{=} \|u\|_{B^\alpha_{\infty,\infty}}$. This notation is justified by the fact that, for $\alpha > 0$, $\alpha \notin \mathbb{N}$, the space $B^\alpha_{\infty,\infty}$ coincides with the usual space of α-Hölder continuous functions. We also point out that for $p = q = 2$ and $\alpha \in \mathbb{R}$, $B^\alpha_{2,2} = H^\alpha$, where the latter is the usual Sobolev space of regularity index α, whose norm (on the torus) can be written as

$$\|u\|^2_{\alpha,2} \stackrel{\text{def}}{=} \|u\|^2_{H^\alpha} \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}^2} (1 + |k|^2)^\alpha |u_k|^2.$$

Restricted to the subspace of distributions u with $u_0 = 0$, one may replace $1 + |k|^2$ by $|k|^2$.

We will need to follow classical embedding theorem for Besov spaces (see, e.g., [19], Lemma A.2).

LEMMA 1.4. *For any $\alpha \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, one has*

$$B^\alpha_{p_1,q_1}(\mathbb{T}^2) \hookrightarrow B^{\alpha-2(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,q_2}(\mathbb{T}^2). \tag{1.12}$$

In particular, one has $\|u\|_{\alpha-2/p} \leq \|u\|_{B^\alpha_{p,p}}$.

We will denote the space of γ-Hölder continuous functions on $[0, T]$ with values in a Banach space B by $C_T^\gamma B$.

Throughout the paper we will write $a \lesssim b$, if there exists a constant $C > 0$ such that $a \le Cb$, $a \sim b$ if $a \lesssim b$ and $b \lesssim a$, and for sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$, $a_n \approx b_n$, if $a_n = b_n + o(1)$.

2. A primer on Wiener space analysis and Malliavin calculus. We recall basic tools from Malliavin calculus which we will use below. Most of this is taken from [35], Chapter 1, to which we refer the interested reader (see also [21, 24]).

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space and H a real separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$. A stochastic process $\{\eta(h) : h \in H\}$ is called *isonormal Gaussian process* if η is a family of centred jointly Gaussian random variables whose correlations are given by $\mathbb{E}[\eta(h)\eta(g)] = \langle h, g\rangle$. Given an isonormal Gaussian process η on H and $n \in \mathbb{N}$, we define $\mathcal{H}_n$ as the closed linear subspace of $L^2(\eta) \stackrel{\text{def}}{=} L^2(\Omega)$ generated by the random variables $H_n(\eta(h))$, where H_n is the nth Hermite polynomial, and $h \in H$ is such that $\|h\|_H = 1$. For $m \ne n$, $\mathcal{H}_n$ and $\mathcal{H}_m$ are orthogonal, and $L^2(\eta)$ coincides with the direct orthogonal sum of the $\mathcal{H}_n$'s, that is, $L^2(\eta) = \bigoplus_n \mathcal{H}_n$ (see [35], Theorem 1.1.1). The subspace $\mathcal{H}_n$ is called the *nth homogeneous Wiener chaos*.

When the Hilbert space H is of the form $L^2(T)$, for $(T, \mathcal{B}, \mu)$ a measure space with a σ-finite and atomless measure μ, the decomposition above can be refined. Namely, for every $n \in \mathbb{N}$ there exists a canonical contraction $I : \bigoplus_{n\ge 0} L^2(T^n) \to L^2(\eta)$, called (iterated) Wiener–Itô integral with respect to η, which restricts to an isomorphism $I : \Gamma L^2 \to L^2(\eta)$ on the Fock space $\Gamma L^2 := \bigoplus_{n\ge 0} \Gamma L_n^2$, where ΓL_n^2 denotes the space $L^2_{\text{sym}}(T^n)$ of functions in $L^2(T^n)$ which are symmetric with respect to permutation of variables. Moreover, the restriction of I to ΓL_n^2, denoted by I_n, is an isomorphism onto the nth homogenous Wiener chaos $\mathcal{H}_n$, which satisfies by [35], Proposition 1.1.4,

$$n! H_n\big(\eta(h)\big) = I_n\left(\bigotimes^n h\right), \quad \text{for all } h \in H \text{ such that } \|h\|_H = 1, \tag{2.1}$$

where $\bigotimes^n h$ is the tensor product of n copies of h. We also recall [35], Proposition 1.1.3, that, for $f \in L^2_{\text{sym}}(T^n)$ and $g \in L^2_{\text{sym}}(T^m)$, one has

$$I_n(f) I_m(g) = \sum_{p=0}^{m\wedge n} p! \binom{n}{p}\binom{m}{p} I_{m+n-2p}(f \otimes_p g), \tag{2.2}$$

where

$$(f \otimes_p g)(x_{1:m+n-2p}) \stackrel{\text{def}}{=} \int_{T^p} \mu(dy_1)\dots\mu(dy_p) f(x_{1:n-p}, y_{1:p}) g(x_{n-p+1:m+n-2p}, y_{1:p}).$$

Here, we adopted the shorthand notation $(x_{1:n}) \stackrel{\text{def}}{=} (x_1, \dots, x_n)$.

We call a function $F : \mathcal{D}' \to \mathbb{R}$ a *cylinder function* if there exist $\varphi_1, \dots, \varphi_n \in \mathcal{D}$ and a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with all partial derivatives growing at most polynomially at infinity such that $F(u) = f(u(\varphi_1), \dots, u(\varphi_n))$. Given a cylinder function F as above, we define its "directional derivative" in the direction of $\psi \in L^2(\mathbb{T}^2)$ by $D_\psi F(u) \stackrel{\text{def}}{=} \sum_{i=1}^n \partial_i f(u(\varphi_1), \dots, u(\varphi_n))\langle\varphi_i, \psi\rangle$. If $\{\varphi_i\}_{i\le n}$ forms an orthonormal system in $L^2(\mathbb{R}^n)$, one has the simplified formula $D_{\varphi_i} F(u) = \partial_i f(u(\varphi_1), \dots, u(\varphi_n))$.

Similarly, given Hilbert space H and an isonormal Gaussian process η on H, we call a random variable $X \in L^2(\eta)$ "smooth" (compare [35], (1.28)), if there exist $h_1, \dots, h_n \in H$ and a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with all derivatives growing at most polynomially, such that $X = f(\eta(h_1), \dots, \eta(h_n))$ almost surely. For a smooth random variable F, we define the Malliavin derivative (see [35], Definition 1.2.1) of X by

$$DX \stackrel{\text{def}}{=} \sum_{i=1}^{n} D_{h_i} X(\eta) h_i = \sum_{i=1}^{n} \partial_i f\big(\eta(h_1), \dots, \eta(h_n)\big) h_i. \tag{2.3}$$

In order to manipulate Malliavin derivatives, an important property is the analog of the integration by parts formula, the so-called *Gaussian integration by parts* given in [35], Lemma 1.2.2. Let F and G be smooth random variables on Ω, then

$$\mathbb{E}\big[G\langle DF, h\rangle\big] = \mathbb{E}\big[-F\langle DG, h\rangle + FG\eta(h)\big], \tag{2.4}$$

where $\mathbb{E}$ is the expectation with respect to the law of η.

Throughout the rest of the paper, the isonormal Gaussian process η we will consider is the zero-mean spatial white noise on the two-dimensional torus $\mathbb{T}^2$. To be more precise, η is a centred isonormal Gaussian process on $H \stackrel{\text{def}}{=} L_0^2(\mathbb{T}^2)$, the space of square-integrable functions with *zero* total mass, whose covariance function is given by

$$\mathbb{E}\big[\eta(\varphi)\eta(\psi)\big] = \langle \varphi, \psi\rangle \tag{2.5}$$

for any two functions $\varphi, \psi \in H$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L^2(\mathbb{T}^2)$. We will mainly work with the Fourier representation of η, given by the family of complex valued, centred Gaussian random variables $\{\eta_k\}_{k\in\mathbb{Z}^2}$, where $\eta_0 = 0$, $\overline{\eta_k} = \eta_{-k}$ and $\mathbb{E}[\eta_k \eta_j] = \delta_{k+j=0}$. Since for $k \in \mathbb{Z}^2$ the random variable η_k is complex valued, before proceeding we want to show how the definition of Malliavin derivative can be extended to the complex setting.

Clearly, any complex-valued function f on $\mathbb{C}^n$ can be split into its real and imaginary parts, that is, $f = \Re f + \iota \Im f$, each of which can be analogously treated, so that we can assume f is real valued. Moreover, for any such f, there exists $g_f : \mathbb{R}^{2n} \to \mathbb{R}$ for which

$$f(x_1 + \iota y_1, \dots, x_n + \iota y_n) = g_f(x_1, y_1, \dots, x_n, y_n) \tag{2.6}$$

and, for any $j = 1, \dots, n$, we have

$$\partial_{2j-1} g_f = \partial_j f \quad \text{and} \quad \partial_{2j} g_f = \iota \partial_j f. \tag{2.7}$$

Similarly to what has been done above, we say that a random variable X in $L^2(\eta)$ is smooth if $X = f(\eta_{k_1}, \dots, \eta_{k_n})$, for some $k_1, \dots, k_n \in \mathbb{Z}^2$ and $f : \mathbb{C}^n \to \mathbb{R}$ such that g_f is smooth on $\mathbb{R}^{2n}$ and all its partial derivatives grow at most polynomially at infinity (if f is complex valued, we require the same to hold for both its real and imaginary parts). Thanks to the identification in (2.6), we can define the complex Malliavin derivative of X by using (2.3) on $X = g_f(\eta(a_{k_1}), \dots, \eta(b_{k_n}))$, that is,

$$DX \stackrel{\text{def}}{=} \sum_{j=1}^{n} \partial_{2j-1} g_f\big(\eta(a_{k_1}), \dots, \eta(b_{k_n})\big) a_{k_j} + \sum_{j=1}^{n} \partial_{2j} g_f\big(\eta(a_{k_1}), \dots, \eta(b_{k_n})\big) b_{k_j}. \tag{2.8}$$

For $h \in L^2(\mathbb{T}^2, \mathbb{C})$, we extend definition (2.8) by linearity, so we set

$$D_h X \stackrel{\text{def}}{=} D_{\Re h} X + \iota D_{\Im h} X = \langle DX, \Re h\rangle + \iota \langle DX, \Im h\rangle. \tag{2.9}$$

LEMMA 2.1. [3] *In the above setting, for any $i \in \{1, 2, \dots, n\}$,*

$$D_{k_i} X \stackrel{\text{def}}{=} D_{\bar{e}_{k_i}} X = \partial_i f\big(\eta(e_{k_1}), \dots, \eta(e_{k_n})\big), \tag{2.10}$$

where $\bar{e}_{k_i} = e_{-k_i}$.

---

[3] Personal communication by Nicolas Perkowski.

PROOF. Notice that, by definition (2.8), we have

$$\begin{aligned} DX &\stackrel{\text{def}}{=} \sum_{j=1}^n \partial_{2j-1} g_f\big(\eta(a_{k_1}), \dots, \eta(b_{k_n})\big) a_{k_j} + \sum_{j=1}^n \partial_{2j} g_f\big(\eta(a_{k_1}), \dots, \eta(b_{k_n})\big) b_{k_j} \\ &= \sum_{j=1}^n \partial_j f(\eta_{k_1}, \dots, \eta_{k_n}) a_{k_j} + \iota \sum_{j=1}^n \partial_j f(\eta_{k_1}, \dots, \eta_{k_n}) b_{k_j} \\ &= \sum_{j=1}^n \partial_j f(\eta_{k_1}, \dots, \eta_{k_n}) e_{k_j}, \end{aligned}$$

where the passage from the first to the second line is a consequence of (2.6). Hence, the result follows immediately from the definition (2.9). □

For future use, we remark that, for any (complex- or real-valued) smooth random variable F in $L^2(\eta)$ and $k \in \mathbb{Z}^2$, the integration by parts formula reads

$$\mathbb{E}[G D_k F] = \mathbb{E}\big[G \langle DF, e_k\rangle_{L^2(\mathbb{T}^2;\mathbb{C})}\big] = \mathbb{E}[-F D_k G + F G \eta_k]. \tag{2.11}$$

3. Properties of the approximating equations. In order to simplify our analysis below, we will be working with $u^N \stackrel{\text{def}}{=} (-\Delta)^{\frac{1}{2}} h^N$, which solves

$$\partial_t u^N = \frac{\nu_N}{2} \Delta u^N + \lambda_N \mathcal{N}^N[u^N] + \nu_N^{\frac{1}{2}} (-\Delta)^{\frac{1}{2}} \xi, \tag{3.1}$$

where the nonlinearity $\mathcal{N}^N$ is given by

$$\mathcal{N}^N[u^N] \stackrel{\text{def}}{=} (-\Delta)^{\frac{1}{2}} \Pi_N\big((\Pi_N \partial_1 (-\Delta)^{-\frac{1}{2}} u^N)^2 - (\Pi_N \partial_2 (-\Delta)^{-\frac{1}{2}} u^N)^2\big). \tag{3.2}$$

By definition of the Hölder–Besov spaces (1.11), the fractional Laplacian (1.10) is a continuous and continuously invertible linear bijection $(-\Delta)^{\frac{1}{2}} : \mathcal{C}_0^\alpha \to \mathcal{C}_0^{\alpha-1}$, for any $\alpha \in \mathbb{R}$, where $\mathcal{C}_0^\alpha$ denotes the closed subspace of $\mathcal{C}^\alpha$ spanned by distributions with vanishing 0th Fourier component. Theorem 1.1 therefore reduces to showing tightness of the sequence u^N in $C_T^\gamma \mathcal{C}^{\alpha-1}$ (see Theorem 4.5), for α as in the statement and of the 0th Fourier mode (see Theorem 4.8). As part of this argument, we also show that the anisotropic KPZ equation (1.3) requires no renormalisation other than the coupling constant renormalisation introduced in Theorem 1.1 (see Remark 4.9).

Passing to Fourier variables, we see that equation (3.1) can be equivalently written as an infinite system of (complex-valued) SDEs

$$\mathrm{d}u_k^N = \Big(-\frac{\nu_N}{2} |k|^2 u_k^N + \lambda_N \mathcal{N}_k^N[u^N]\Big)\,\mathrm{d}t + \nu_N^{\frac{1}{2}} |k|\,\mathrm{d}B_k(t), \quad k \in \mathbb{Z}_0^2, \tag{3.3}$$

where the complex-valued Brownian motions B_k are defined via $B_k(t) \stackrel{\text{def}}{=} \int_0^t \xi_k(s)\,\mathrm{d}s$, ξ_k being the kth Fourier mode of the space-time white noise ξ which, in particular, implies that $\overline{B_k} = B_{-k}$. Hence, their quadratic covariation is given by $\mathrm{d}\langle B_k, B_\ell\rangle_t = \mathbb{1}_{\{k+\ell=0\}}\,\mathrm{d}t$ for $k, l \neq 0$. The kth Fourier component of the nonlinearity is

$$\mathcal{N}_k^N[u^N] \stackrel{\text{def}}{=} \mathcal{N}^N[u^N](e_{-k}) = \sum_{\ell+m=k} \mathcal{K}_{\ell,m}^N u_\ell^N u_m^N, \tag{3.4}$$

$$\mathcal{K}_{\ell,m}^N \stackrel{\text{def}}{=} \frac{1}{2\pi} |\ell+m| \frac{c(\ell,m)}{|\ell||m|} \mathbb{J}_{\ell,m,\ell+m}^N, \tag{3.5}$$

where, for $\ell=(\ell_1,\ell_2)$, $m=(m_1,m_2)\in\mathbb{Z}_0^2$, $c(\ell,m)\stackrel{\text{def}}{=}\ell_2m_2-\ell_1m_1$ and $\mathbb{J}^N_{a,b,\dots}$ is an abbreviation for $\mathbb{1}_{|a|_\infty\le N,|b|_\infty\le N,\dots}$.

This approximation scheme has the advantage that it completely decouples the equations for $\{u_k^N:|k|_\infty\le N\}$ and $\{u_k^N:|k|_\infty>N\}$. The latter is an infinite family of independent Ornstein–Uhlenbeck processes, while the first is a finite-dimensional system of SDEs interacting via a quadratic nonlinearity. Local existence and uniqueness is classical (since the coefficients are locally Lipschitz continuous), and the process $t\mapsto\{u_k^N(t)\}_{k\in\mathbb{Z}_0^2}$ is clearly strong Markov. At this point we refrain from being more specific about the state space for this process. As long as we are working with fixed N, any "reasonable" choice could be used for the sake of the current section (one could take, e.g., H^α, $\alpha<-1$, if one wants to deal with (3.1) directly or $\mathbb{C}^{\mathbb{Z}_0^2}$ with the product topology, if instead one focuses on the system (3.3)). We postpone a detailed discussion of the spaces we actually want to work in to the proof of tightness in Section 4.

Returning to equation (3.1), we can easily determine the generator $\mathcal{L}^N$ for the dynamics of u^N (e.g., by applying Itô's formula to a cylinder function, singling out the drift part (see (4.2) below) and taking the Fourier transform). Let F be a real-valued cylinder function acting on distributions $v\in\mathcal{D}'(\mathbb{T}^2)$ and decompose the generator into $\mathcal{L}^N=\mathcal{L}_0^N+\mathcal{A}^N$, where the action of $\mathcal{L}_0^N$ and $\mathcal{A}^N$ can be written in Fourier as

$$(\mathcal{L}_0^NF)(v)\stackrel{\text{def}}{=}\frac{\nu_N}{2}\sum_{k\in\mathbb{Z}^2}|k|^2(-v_{-k}D_k+D_{-k}D_k)F(v),\tag{3.6}$$

$$(\mathcal{A}^NF)(v)=\lambda_N\sum_{m,l\in\mathbb{Z}_0^2}\mathcal{K}^N_{m,l}v_mv_lD_{-m-l}F(v).\tag{3.7}$$

As a first step of our analysis, we show that the spatial white noise η on $\mathbb{T}^2$ is invariant for the Markov process u^N for all $N\in\mathbb{N}$.

LEMMA 3.1. *For any $N\in\mathbb{N}$, the spatial white noise η, defined in* (2.5), *is invariant for the solution $u^N=\{u_k^N\}_{k\in\mathbb{Z}_0^2}$ of* (3.3). *Moreover, with respect to $L^2(\eta)$ the symmetric and antisymmetric part of $\mathcal{L}^N$ are given by $\mathcal{L}_0^N$ and $\mathcal{A}^N$, respectively.*

PROOF. According to [15], for the first statement it is enough to prove that $\mathbb{E}[\mathcal{L}^NG(\eta)]=0$ for all C^2-cylinder functions, and we will prove the above relation for $\mathcal{L}_0^N$ and $\mathcal{A}^N$ separately, beginning with the first. Let G be a cylinder function. In the Gaussian integration by parts formula (2.11) set $F=D_kG$ and $G\equiv1$, so that we have

$$\mathbb{E}[D_{-k}D_kG(\eta)]=\mathbb{E}[D_kG(\eta)\eta_{-k}],$$

from which $\mathbb{E}\mathcal{L}_0^NG(\eta)=0$ follows. For the operator $\mathcal{A}^N$ we use again Gaussian integration by parts (this time with $F=G$ and $G(\eta)=g(\eta_m,\eta_\ell)=\eta_m\eta_\ell$ in the notation of (2.11)) to obtain

$$\begin{aligned}\mathbb{E}[\eta_m\eta_\ell D_{-m-\ell}G(\eta)]&=\mathbb{E}[\eta_m\eta_\ell\eta_{-m-\ell}G(\eta)-G(\eta)D_{-m-\ell}(\eta_m\eta_\ell)]\\&=\mathbb{E}[\eta_m\eta_\ell\eta_{-m-\ell}G(\eta)],\end{aligned}\tag{3.8}$$

where the last passage is a consequence of the choice $\eta(e_0)=0$. Now, the function G on the right-hand side does not depend on either m or ℓ. We claim that the following stronger statement holds; for any $\{\eta_k\}_{k\in\mathbb{Z}_0^2}$, we have

$$\sum_{m,\ell\in\mathbb{Z}_0^2}\mathcal{K}^N_{m,\ell}\eta_m\eta_\ell\eta_{-m-\ell}=0.\tag{3.9}$$

Assuming (3.9), upon multiplying by $\mathcal{K}^N_{m,l}$ and summing over all ℓ and $m \in \mathbb{Z}^2_0$ on both sides of (3.8), we obtain $\mathbb{E}\mathcal{A}^N G(\eta) = 0$ and thus conclude the proof of invariance of white noise for the dynamics of u^N.

Let us prove (3.9) (an alternative proof is provided in Appendix A). Observe that $f(m,\ell) \stackrel{\text{def}}{=} \eta_m\eta_\ell\eta_{-m-\ell}$ satisfies the symmetry relation $f(m,\ell) = f(-m-\ell,\ell) = f(m,-m-\ell)$, so that it suffices to check that $\mathcal{K}^N_{m,\ell}$ is antisymmetric once we sum over all permutations of m, ℓ and $-m-\ell$. We compute

$$
\begin{aligned}
2\pi\mathcal{K}^N_{m,l} = &- \frac{(m_1+\ell_1)^2 m_1\ell_1}{|m+\ell||m||\ell|} + \frac{(m_2+\ell_2)^2 m_2\ell_2}{|m+\ell||m||\ell|} \\
&+ \frac{(m_1+\ell_1)^2 m_2\ell_2}{|m+\ell||m||\ell|} - \frac{(m_2+\ell_2)^2 m_1\ell_1}{|m+\ell||m||\ell|},
\end{aligned}
\tag{3.10}
$$

where ℓ_1, ℓ_2 and m_1, m_2 are the components of ℓ and m, respectively. Denote these summands by $\mathcal{K}^N_{m,\ell,(i)}$, $i = 1,2,3,4$. Then, for $i = 1,2$, $\mathcal{K}^N_{m,\ell,(i)} + \mathcal{K}^N_{-m-\ell,\ell,(i)} + \mathcal{K}^N_{m,-m-\ell,(i)} = 0$, while

$$
2\pi\big(\mathcal{K}^N_{m,\ell,(3)} + \mathcal{K}^N_{-m-\ell,\ell,(3)} + \mathcal{K}^N_{m,-m-\ell,(3)}\big) = 2m_1\ell_1m_2\ell_2 - m_1^2\ell_2^2 - \ell_1^2m_2^2
$$

which cancels the corresponding term coming from $\mathcal{K}^N_{m,l,(4)}$.

We next show that (3.6) and (3.7) are indeed the symmetric and antisymmetric part of $\mathcal{L}^N$. The first claim follows directly from

$$
\mathbb{E}\big[D_{-k}D_kF(\eta)G(\eta)\big] = \mathbb{E}\big[D_kF(\eta)G(\eta)\eta_{-k}\big] - \mathbb{E}\big[D_kF(\eta)D_{-k}G(\eta)\big],
$$

which is a consequence of Gaussian integration by parts (see (2.11)), so that

$$
\mathbb{E}\big[\mathcal{L}^N_0F(\eta)G(\eta)\big] = -\frac{\nu_N}{2}\sum_{k\in\mathbb{Z}^2}|k|^2\mathbb{E}\big[D_kF(\eta)D_{-k}G(\eta)\big]
$$

and the latter is clearly symmetric. For the antisymmetric part we compute

$$
\mathbb{E}\big[\eta_m\eta_\ell D_{-m-\ell}F(\eta)G(\eta)\big] = \mathbb{E}\big[\eta_m\eta_\ell\eta_{-m-\ell}F(\eta)G(\eta)\big] - \mathbb{E}\big[F(\eta)D_{-m-\ell}\big(G(\eta)\eta_m\eta_\ell\big)\big].
$$

Notice that, summing up over $\ell, m \in \mathbb{Z}^2_0$, the first term on the right-hand side drops out for the same reason as in the proof of stationarity. We can apply the Leibniz rule to the second and, recalling that we chose $\eta(e_0) = 0$, we get

$$
\mathbb{E}\big[\mathcal{A}^NF(\eta)G(\eta)\big] = -\lambda_N\sum_{m,\ell\in\mathbb{Z}^2}\mathcal{K}^N_{m,\ell}\mathbb{E}\big[\eta_m\eta_\ell F(\eta)D_{-m-\ell}G(\eta)\big] = -\mathbb{E}\big[F(\eta)\mathcal{A}^NG(\eta)\big],
$$

so that the proof is concluded. □

REMARK 3.2. Let us point out that the reason why the invariant measure is a spatial white noise is hidden in the algebraic identity (3.9). In particular, if Q where of any form different from $\lambda\,\mathrm{diag}(1,-1)$, for some $\lambda \in \mathbb{R}$, the proof above fails, and an explicit invariant measure is currently unknown.

REMARK 3.3. The previous lemma provides the second advantage of our approximation scheme, namely, the fact that the invariant measure of u^N is independent of N. If we decided, in addition to smoothing the nonlinearity, to cut the high-Fourier modes of the space-time white noise ξ appearing in (3.1) (i.e., replace ξ by $\Pi^N\xi$), so that $u^N_k \equiv 0$ for every k with $|k|_\infty \geq N$, then the same proof shows that the invariant measure would be $\Pi^N\eta$.

Global in-time solutions to (3.1) are a consequence of the following proposition.

PROPOSITION 3.4. *For any deterministic initial condition* $u^N(0) = \{u_k^N(0)\}_{k\in\mathbb{Z}_0^2}$, *the solution* $t \mapsto u^N(t) = \{u_k^N(t)\}_{k\in\mathbb{Z}_0^2}$ *of* (3.3) *exists globally in time.*

PROOF. For $|k|_\infty > N$, this is obvious. For $|k|_\infty \le N$, (3.3) is a system of coupled SDEs driven by complex Brownian motions. Since the drift is locally Lipschitz continuous, local existence and uniqueness is classical, and we only need to guarantee that the solution does not explode for any deterministic initial condition. For this we can proceed as in [20], Section 4. Let $\{u_k^N(0)\}_{|k|_\infty\le N} \subseteq \mathbb{R}$ be the initial condition and $\{u_k^N(t)\}_{|k|_\infty\le N}$ be the local in time solution to the system (3.3). By applying Itô's formula to $A^N(t) \stackrel{\text{def}}{=} \sum_{|k|\le N} |u_k^N(t)|^2$, we obtain

$$\mathrm{d}A^N(t) = \Big(-\sum_{|k|\le N} |k|^2 |u_k^N(t)|^2 + \lambda_N \sum_{|k|\le N} \mathcal{N}_k^N[u^N(t)]u_{-k}^N(t) + C_N\Big)\,\mathrm{d}t + \nu_N^{\frac12} \sum_{|k|\le N} |k| u_k^N(t)\,\mathrm{d}B_{-k}(t),$$

where $C_N \stackrel{\text{def}}{=} \frac{\nu_N}{2}\sum_{|k|_\infty\le N} |k|^2 \le N^4$. Since the first term on the right-hand side is nonpositive and the second vanishes by (3.4), we can conclude that the process $t \mapsto \sum_{|k|_\infty\le N} |u_k^N(t)|^2$ is almost surely bounded on compact time intervals. Hence, we can conclude. □

Similarly to [24], we want to improve our understanding of the generator associated to u^N. More specifically, we would like to know how $\mathcal{L}^N$ acts on elements of $L^2(\eta)$ and ensure that $\mathcal{L}^N$ is reasonably well behaved when applied to elements belonging to a homogeneous Wiener chaos.

For that, recall that the Fourier transform $\mathcal{F}$ maps $\Gamma L_n^2 = L^2_{\mathrm{sym}}(\mathbb{T}^{2n})$ (isometrically) into $\ell^2((\mathbb{Z}^2)^n)$, i.e. $\mathcal{F}(\cdot) = \hat{\cdot} : L^2_{\mathrm{sym}}(\mathbb{T}^{2n}) \to \ell^2((\mathbb{Z}^2)^n)$. Moreover, if $\mathcal{O}$ is an operator acting on (a subspace of) $L^2(\eta)$, we will denote by $\mathfrak{O}$ the operator on ΓL^2 such that, for all $\varphi \in \Gamma L^2$, one has $\mathcal{O} I(\varphi) = I(\mathfrak{O}\varphi)$.

LEMMA 3.5. *For any* $n \in \mathbb{N}$, *the operator* $\mathcal{L}_0^N$ *leaves* $\mathcal{H}^n$ *invariant,* $\mathcal{A}^N$ *maps* $\mathcal{H}^n$ *into* $\mathcal{H}^{n-1} \oplus \mathcal{H}^{n+1}$ *and one has, for any* $K \in \Gamma L_n^2$, *the identity*

$$\mathcal{L}_0^N I_n K = \frac{\nu_N}{2} I_n \Delta K. \tag{3.11}$$

Moreover, one can write $\mathcal{A}^N = \mathcal{A}_+^N + \mathcal{A}_-^N$, *where* $\mathcal{A}_+^N$ *increases and* $\mathcal{A}_-^N$ *decreases the order of the Wiener chaos by one, that is,* $\mathcal{A}_+^N : \mathcal{H}^n \to \mathcal{H}^{n+1}$ *and* $\mathcal{A}_-^N : \mathcal{H}^n \to \mathcal{H}^{n-1}$, *and their action in Fock space representation, denoted by* $\mathfrak{A}_+^N$ *and* $\mathfrak{A}_-^N$, *respectively, satisfy*

$$\mathcal{F}(\mathfrak{A}_+^N K)(k_{1:n+1}) = n\lambda_N \mathcal{K}_{k_1,k_2}^N \hat{K}(k_1 + k_2, k_{3:n+1}), \tag{3.12}$$

$$\mathcal{F}(\mathfrak{A}_-^N K)(k_{1:n-1}) = 2n(n-1)\lambda_N \sum_{\ell+m=k_1} \mathcal{K}_{k_1,-\ell}^N \hat{K}(\ell, m, k_{2:n-1}), \tag{3.13}$$

where we used the shorthand notation $k_{1:n+1} = (k_1, \dots, k_{n+1})$. *Finally, the operator* $-\mathcal{A}_+^N$ *is the adjoint of* $\mathcal{A}_-^N$ *in* $L^2(\eta)$.

PROOF. It suffices to show (3.11), (3.12) and (3.13) for a kernel of the type $K = \bigotimes^n h$ for some $h \in H$, since symmetric functions in $L^2(\mathbb{T}^{2n})$ can always be written as a linear combination of functions of the previous type. As a consequence of (2.1) and the fact that

Hermite polynomials satisfy $H_n' = H_{n-1}$ (see [35], equation (1.2)), the Malliavin derivatives of stochastic integrals of such kernels can be written as

$$D_k I_n\Big(\bigotimes^n h\Big) = n I_{n-1}\Big(\bigotimes^{n-1} h\Big) h_k,$$

$$D_{-k} D_k I_n\Big(\bigotimes^n h\Big) = n(n-1) I_{n-2}\Big(\bigotimes^{n-2} h\Big) h_k h_{-k}.$$

We start by analysing $\mathcal{L}_0^N$. To show (3.11), first note that

$$I_1(\Delta h) = I_1\Big(-\sum_{k\in\mathbb{Z}^2} |k|^2 h_k e_k\Big) = -\sum_{k\in\mathbb{Z}^2} |k|^2 h_k I_1(e_k) = -\sum_{k\in\mathbb{Z}^2} |k|^2 h_k \eta_{-k}, \tag{3.14}$$

where the last equality follows from (2.1), and the fact that H_1 is the identity. Using the above observations and (2.2), we thus see that

$$\begin{aligned} &-\sum_{k\in\mathbb{Z}^2} |k|^2 \eta_{-k} D_k I_n\Big(\bigotimes^n h\Big) \\ &= n I_{n-1}\Big(\bigotimes^{n-1} h\Big) I_1(\Delta h) \\ &= n I_n\Big(\bigotimes^{n-1} h \otimes \Delta h\Big) + n(n-1) I_{n-2}\Big(\bigotimes^{n-2} h\Big)\langle h, \Delta h\rangle. \end{aligned} \tag{3.15}$$

On the other hand, a similar calculation shows

$$\sum_{k\in\mathbb{Z}^2} |k|^2 D_k D_{-k} I_n\Big(\bigotimes^n h\Big) = -n(n-1) I_{n-2}\Big(\bigotimes^{n-2} h\Big)\langle h, \Delta h\rangle.$$

Hence, the second summand in (3.15) drops out, and we obtain the identity (3.11). For the operator $\mathcal{A}^N$, proceeding as above, we see that

$$\begin{aligned} \mathcal{A}^N I_n\Big(\bigotimes^n h\Big) &= \lambda_N n \sum_{\ell,m\in\mathbb{Z}_0^2} \mathcal{K}^N_{\ell,m} \eta_\ell \eta_m I_{n-1}\Big(\bigotimes^{n-1} h\Big) h_{-m-\ell} \\ &= \lambda_N n I_2\Big(\sum_{\ell,m\in\mathbb{Z}_0^2} \mathcal{K}^N_{\ell,m} h_{-m-\ell} e_{-m}\otimes e_{-\ell}\Big) I_{n-1}\Big(\bigotimes^{n-1} h\Big). \end{aligned}$$

By the product rule (2.2), we get

$$\begin{aligned} \mathcal{A}^N I_n\Big(\bigotimes^n h\Big) &= \lambda_N n I_{n+1}\Big(\sum_{\ell,m} \mathcal{K}^N_{l,m} h_{-\ell-m}\Big(e_{-l}\otimes e_{-m}\otimes \bigotimes^{n-1} h\Big)\Big) \\ &\quad + \lambda_N 2n(n-1) I_{n-1}\Big(\sum_{\ell,m} \mathcal{K}^N_{\ell,m} h_\ell h_{-\ell-m}\Big(e_{-m}\otimes \bigotimes^{n-2} h\Big)\Big) \\ &\quad + \lambda_N n(n-1)(n-2)\Big(\sum_{\ell,m} \mathcal{K}^N_{\ell,m} h_\ell h_m h_{-\ell-m}\Big) I_{n-3}\Big(\bigotimes^{n-3} h\Big). \end{aligned}$$

Notice at first that the third summand disappears thanks to (3.9). Moreover, the first and second summand, corresponding, respectively, to $\mathcal{A}^N_+ I_n(\bigotimes^n h)$ and $\mathcal{A}^N_- I_n(\bigotimes^n h)$, live in $\mathcal{H}_{n+1}$

and $\mathcal{H}_{n-1}$, respectively, and, upon taking the Fourier transform of the integrands, we immediately obtain (3.12) and (3.13).

At last, $\mathcal{A}_-^N$ and $-\mathcal{A}_+^N$ are adjoint to each other since $\mathcal{A}^N = \mathcal{A}_-^N + \mathcal{A}_+^N$ is an antisymmetric operator on $L^2(\mu)$ and, as noted above, one has $\mathcal{A}_-^N : \mathcal{H}^n \to \mathcal{H}^{n-1}$ and $\mathcal{A}_+^N : \mathcal{H}^n \to \mathcal{H}^{n+1}$. Indeed, if $F(\eta) = \sum_n I_n(f_n)$ and $G(\eta) = \sum_n I_n(g_n)$, then

$$\begin{aligned}\mathbb{E}[\mathcal{A}_+^N F(\eta)G(\eta)] &= \sum_n \mathbb{E}[\mathcal{A}_+^N I_n(f_n) I_{n+1}(g_{n+1})]\\ &= \sum_n \mathbb{E}[\mathcal{A}^N I_n(f_n) I_{n+1}(g_{n+1})]\\ &= -\sum_n \mathbb{E}[I_n(f_n)\mathcal{A}^N I_{n+1}(g_{n+1})]\\ &= -\sum_n \mathbb{E}[I_n(f_n)\mathcal{A}_-^N I_{n+1}(g_{n+1})]\\ &= -\mathbb{E}[F(\eta)\mathcal{A}_-^N G(\eta)],\end{aligned}$$

and the proof is concluded. □

4. Upper bounds and tightness of the approximating sequence. In this section we want to show how to obtain suitable bounds (depending on the coupling constants λ_N and ν_N) on the time integral of (nonlinear) functionals of the solution u^N of (3.1). The point we want to make is that the technique exploited in [20] is sufficiently flexible to be able to handle even cases in which the limiting equation is *critical*.

To get a feeling of the procedure followed in the aforementioned paper, consider a generic functional F in the domain of the generator $\mathcal{L}^N$ of the Markov process $\{u^N(t)\}_{t\in\mathbb{R}_+}$ solving (3.3), whose symmetric and antisymmetric part, with respect to the invariant measure η, are $\mathcal{L}_0^N$ and $\mathcal{A}^N$, respectively (see Lemma 3.1). The main idea is that the relation between the forward and the backward processes ($u^N(t)$ and $u^N(T-t)$) can be used in the representation of $F(u^N)$ given by Dynkin's (or Itô's) formula (see (4.2) and (4.5)) in order to get rid of both the boundary terms and the terms containing $\mathcal{A}^N F(u^N)$. In this way the time average of $\mathcal{L}_0^N F(u^N)$ can be expressed as the sum of two martingales (see (4.6)) which in turn can be controlled via their quadratic variation. The latter is explicit and depends only on u^N evaluated at a *single point in time*. The knowledge of the invariant measure for the process is then the key to obtain a bound on (moments of) the quadratic variation of these martingales (see Lemma 4.1). At last, once estimates for quantities of the form $\int_0^T \mathcal{L}_0^N F(u^N(s))\,\mathrm{d}s$ are available, analogous estimates for $\int_0^T V(u^N(s))\,\mathrm{d}s$, for more general functionals V, can be consequently achieved if one is able to determine a solution F to the *Poisson equation* given by

$$\mathcal{L}_0^N F = V. \tag{4.1}$$

In what follows we will first describe in more detail the strategy outlined above and then show how we can take advantage of these techniques in the context of the anisotropic KPZ equation.

Let $F = F(t,\cdot)$ be a real-valued cylinder function depending smoothly on time. Thanks to Itô's formula (and the Fourier representation of u^N given in (3.3)), we can write

$$F(t,u^N(t)) = F(0,u^N(0)) + \int_0^t (\partial_s + \mathcal{L}^N)F(s,u^N(s))\,\mathrm{d}s + \nu_N^{\frac{1}{2}} M_t^N(F), \tag{4.2}$$

where $M^N_\cdot(F)$ is the martingale (depending on F) defined by

$$\mathrm{d}M_t^N(F) = \sum_{k\in\mathbb{Z}^2} (D_k F)\big(t, u^N(t)\big)|k|\,\mathrm{d}B_k(t) \tag{4.3}$$

whose quadratic variation is

$$\mathrm{d}\langle M^N(F)\rangle_t = \mathcal{E}^N(F)\big(t, u^N(t)\big)\,\mathrm{d}t \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}^2} |k|^2\big|D_k F\big(t, u^N(t)\big)\big|^2\,\mathrm{d}t, \tag{4.4}$$

where the equality above is due to the fact that F is real valued.[4] For fixed $T > 0$, it follows from Lemma 3.1 that the backward process $\bar{u}^N(t) \stackrel{\text{def}}{=} u^N(T-t)$ is itself a Markov process whose generator is given by the adjoint of $\mathcal{L}^N$, $(\mathcal{L}^N)^* = \mathcal{L}_0^N - \mathcal{A}^N$. In particular, applying again Itô's formula but this time on $G(t, \bar{u}^N(t))$ for some cylinder function depending smoothly on time, we get

$$\begin{aligned} G\big(t, \bar{u}^N(t)\big) = G\big(0, \bar{u}^N(0)\big) + \int_0^t \big(\partial_s + \mathcal{L}_0^N - \mathcal{A}^N\big)G\big(s, \bar{u}^N(s)\big)\,\mathrm{d}s \\ + \nu_N^{\frac{1}{2}}\bar{M}_t^N(G), \end{aligned} \tag{4.5}$$

where $\bar{M}^N(G)$ is a martingale with respect to the *backward* filtration, generated by the process $\bar{u}^N$, and its quadratic variation is given by $\mathrm{d}\langle \bar{M}^N(G)\rangle_t = \mathcal{E}^N(G)(t, u^N(T-t))\,\mathrm{d}t$. Summing up (4.2) and (4.5) with $G(t,x) = F(T-t,x)$, one obtains the following analog of [20], equation (10):

$$2\int_0^t \mathcal{L}_0^N F\big(s, u^N(s)\big)\,\mathrm{d}s = \nu_N^{\frac{1}{2}}\big(-M_t^N(F) + \bar{M}_{T-t}^N(G) - \bar{M}_T^N(G)\big). \tag{4.6}$$

The right-hand side of (4.6) can be bounded by the Burkholder–Davis–Gundy inequality; this yields [20], Lemma 2, (which in turn was inspired by [9], Lemma 4.4) that we here recall.

LEMMA 4.1 (Itô-trick). *For any $p \geq 2$, $T > 0$ and cylinder function $F = F(t,\cdot)$ smoothly depending on time, the following estimate holds*:

$$\mathbf{E}\bigg[\sup_{t\leq T}\bigg|\int_0^t \mathcal{L}_0^N F\big(s, u^N(s)\big)\,\mathrm{d}s\bigg|^p\bigg]^{\frac{1}{p}} \lesssim \nu_N^{\frac{1}{2}} T^{\frac{1}{2}} \sup_{s\in[0,T]} \mathbb{E}\big[\big|\mathcal{E}^N(F)(s,\eta)\big|^{\frac{p}{2}}\big]^{\frac{1}{p}}. \tag{4.7}$$

Moreover, in the specific case in which $F(t,x) = \sum_{i\in I} e^{a_i(T-t)}\tilde{F}_i(x)$ where I is an index set, $a_i \in \mathbb{R}$ and F_i is a cylinder function for every $i \in I$, we have

$$\begin{aligned} &\mathbf{E}\bigg[\bigg|\int_0^T \sum_{i\in I} e^{a_i(T-s)}\mathcal{L}_0^N \tilde{F}_i\big(u^N(s)\big)\,\mathrm{d}s\bigg|^p\bigg]^{\frac{1}{p}} \\ &\quad\lesssim \nu_N^{\frac{1}{2}}\bigg(\sum_{i\in I}\bigg(\frac{e^{2a_iT}-1}{2a_i}\bigg)\mathbb{E}\big[\big|\mathcal{E}^N(\tilde{F}_i)(\eta)\big|^{\frac{p}{2}}\big]^{\frac{2}{p}}\bigg)^{\frac{1}{2}}. \end{aligned} \tag{4.8}$$

In both cases the proportionality constant hidden in $\lesssim$ is independent of both N and F. Here and below, we use the symbol $\mathbf{E}$ to denote expectations with respect to the law of $\{u^N(t)\}_{t\in\mathbb{R}_+}$ and $\mathbb{E}$ for expectations with respect to the law of η.

[4] In case F were complex valued, then, instead of $|D_k F(t, u^N(t))|^2$, we would have $D_k F(t, u^N(t)) D_{-k} F(t, u^N(t))$.

REMARK 4.2. The crucial aspect of the previous lemma is that we are able to bound the expectation of functionals of u^N with respect to the *space-time* law of u^N in terms of the expectation with respect to the *sole* invariant measure, so that explicit computations become indeed possible.

PROOF. The proof of (4.7) is that of equation (11) in [20], Lemma 2. The second bound can be obtained following the proof of [20], Lemma 2, equation (12), and we provide the details for completeness. Notice that, given $t \in [0, T]$ and F as in the statement, the left-hand side of (4.8) is bounded from above by

$$\begin{aligned}
&\mathbf{E}\Big[\sup_{t\le T}\Big|\int_0^t \sum_{i\in I} e^{a_i(T-s)}\mathcal{L}_0^N \tilde{F}_i\big(u^N(s)\big)\,\mathrm{d}s\Big|^p\Big]^{\frac{1}{p}} \\
&\quad = \mathbf{E}\Big[\sup_{t\le T}\Big|\int_0^t \mathcal{L}_0^N F\big(s, u^N(s)\big)\,\mathrm{d}s\Big|^p\Big]^{\frac{1}{p}} \\
&\quad \lesssim \nu_N^{\frac{1}{2}}\mathbf{E}\Big[\Big|\int_0^T \mathcal{E}^N\big(F(s,\cdot)\big)\big(u^N(s)\big)\,\mathrm{d}s\Big|^{\frac{p}{2}}\Big]^{\frac{1}{p}} \\
&\quad = \nu_N^{\frac{1}{2}}\mathbf{E}\Big[\Big|\int_0^T \sum_{i\in I} e^{2a_i(T-s)}\mathcal{E}^N(\tilde{F}_i)\big(u^N(s)\big)\,\mathrm{d}s\Big|^{\frac{p}{2}}\Big]^{\frac{1}{p}},
\end{aligned}$$

where for the first equality we used the fact that $\mathcal{L}_0^N$ acts only on the spatial variable, the subsequent bound follows by Burkholder–Davis–Gundy inequality and the last comes from the fact that $\mathrm{d}\langle M_\cdot(F)\rangle_t = \sum_{i\in I} e^{2a_i(T-t)}\mathcal{E}^N(\tilde{F}_i)(u^N(t))\,\mathrm{d}t$. The right-hand side of the latter is trivially bounded by

$$\begin{aligned}
&\Big(\int_0^T \sum_{i\in I} e^{2a_i(T-s)}\mathbf{E}\big[\big|\mathcal{E}^N(\tilde{F}_i)\big(u^N(s)\big)\big|^{\frac{p}{2}}\big]^{\frac{2}{p}}\,\mathrm{d}s\Big)^{\frac{1}{2}} \\
&\quad = \nu_N^{\frac{1}{2}}\Big(\sum_{i\in I}\int_0^T e^{2a_i(T-s)}\,\mathrm{d}s\,\mathbb{E}\big[\big|\mathcal{E}^N(\tilde{F}_i)(\eta)\big|^{\frac{p}{2}}\big]^{\frac{2}{p}}\Big)^{\frac{1}{2}}.
\end{aligned}$$

The equality comes from the fact that u^N appears only evaluated at a single point in time, and its law is that of η. By evaluating the integral we obtain (4.8). □

For any test function ϕ we are interested in uniform bounds of the linear and the nonlinear part of (3.1) tested against ϕ, that is, on $\frac{\nu_N}{2}\eta(\Delta\varphi)$ and $\lambda_N\mathcal{N}^N[\eta](\varphi)$, respectively. As mentioned above, they will be obtained by combining the Itô-trick, Lemma 4.1, with an explicit solution of the Poisson equation.

Thanks to the Fourier representation of the nonlinearity given in (3.4), we can write $\mathcal{N}^N[\eta](\varphi)$ as a second order Wiener–Itô integral of the form

$$\mathcal{N}^N[\eta](\varphi) = \sum_{\ell,m\in\mathbb{Z}_0^2}\mathcal{K}_{\ell,m}^N\eta_l\eta_m\varphi_{-\ell-m} = I_2\Big(\sum_{\ell,m\in\mathbb{Z}_0^2}\mathcal{K}_{\ell,m}^N\varphi_{-\ell-m}e_{-\ell}\otimes e_{-m}\Big).$$

Using (3.11), it is easy to see that the solution of the Poisson equation $\mathcal{L}_0^N H^N[\eta](\varphi) = \lambda_N\mathcal{N}^N[\eta](\varphi)$ is the cylinder function (clearly, depending on N) given by

$$H^N[\eta](\varphi) \stackrel{\text{def}}{=} 2\lambda_N\nu_N^{-1}\sum_{\ell,m\in\mathbb{Z}_0^2}\frac{\mathcal{K}_{\ell,m}^N}{|\ell|^2+|m|^2}\eta_l\eta_m\varphi_{-\ell-m}. \tag{4.9}$$

On the other hand, the solution $K^N[\eta](\varphi)$ of $\mathcal{L}_0^N K^N[\eta](\varphi) = \frac{\nu_N}{2}\eta(\Delta\varphi)$ is, again by (3.11), simply $K^N[\eta](\varphi) = \eta(\varphi)$. We are now ready to state and prove the following lemma.

LEMMA 4.3 (Energy estimates). *Let $T > 0$ be fixed, $\varphi \in H^1$ and u^N be the solution to* (3.1). *Let $\mathcal{N}^N$ be defined according to* (3.2). *Then, for any $p \geq 2$, the following estimates hold*:

$$\mathbf{E}\Big[\sup_{t\leq T}\Big|\int_0^t \lambda_N \mathcal{N}^N[u^N(s)](\varphi)\,\mathrm{d}s\Big|^p\Big]^{\frac{1}{p}} \lesssim_p T^{\frac{1}{2}}\lambda_N \nu_N^{-\frac{1}{2}}(\log N)^{\frac{1}{2}}\|\varphi\|_{1,2}, \tag{4.10}$$

$$\mathbf{E}\Big[\sup_{t\leq T}\Big|\int_0^t \frac{\nu_N}{2} u^N(s,\Delta\varphi)\,\mathrm{d}s\Big|^p\Big]^{\frac{1}{p}} \lesssim_p T^{\frac{1}{2}}\nu_N^{\frac{1}{2}}\|\varphi\|_{1,2}, \tag{4.11}$$

where in both cases the implicit constant does not depend on φ, T and N.

PROOF. Let φ be a test function in H^1, and recall the solutions $H^N(\varphi)$ and $K^N(\varphi)$ of the Poisson equations defined in (and directly below of) (4.9), so that with the aid of Lemma 4.1 the proof of (4.10) and (4.11) boils down to bounding the moments of $\mathcal{E}^N(H^N(\varphi))(\eta)$ and $\mathcal{E}^N(K^N(\varphi))(\eta)$ with respect to the white noise measure. Since this measure is Gaussian and both $H^N(\varphi)$ and $K^N(\varphi)$ live in a homogeneous Wiener chaos (of order 2 and 1, respectively), by Gaussian hypercontractivity [28], Theorem 3.50, it suffices to bound the first moment of $\mathcal{E}^N(H^N(\varphi))(\eta)$ and $\mathcal{E}^N(K^N(\varphi))(\eta)$. Let us begin with the former. Notice that

$$D_\ell H^N[\eta](\varphi) = 2\lambda_N\nu_N^{-1}\sum_{k,j\in\mathbb{Z}_0^2}\tilde{\mathcal{K}}^N_{j,k-j}D_\ell(\eta_j\eta_{k-j})\varphi_{-k},$$

where we set $\tilde{\mathcal{K}}^N_{\ell,m} \stackrel{\text{def}}{=} (|\ell|^2+|m|^2)^{-1}\mathcal{K}^N_{\ell,m}$. Now, $D_\ell(\eta_j\eta_{k-j}) = \eta_{k-\ell}$ in two cases, namely, $j=\ell$ and $k-j=\ell$, while if $j=k-j=\ell$, then $D_\ell(\eta_j\eta_{k-j}) = 2\eta_\ell = 2\eta_{k-\ell}$. Putting these together, we get

$$D_\ell H^N[\eta](\varphi) = 4\lambda_N\nu_N^{-1}\sum_{k\in\mathbb{Z}_0^2}\tilde{\mathcal{K}}^N_{\ell,k-\ell}\eta_{k-\ell}\varphi_{-k} = 4\lambda_N\nu_N^{-1}\sum_{m\in\mathbb{Z}_0^2}\tilde{\mathcal{K}}^N_{\ell,m}\eta_m\varphi_{-\ell-m};$$

therefore,

$$\mathcal{E}^N\big(H^N(\varphi)\big)(\eta) = 16\lambda_N^2\nu_N^{-2}\sum_{\ell\in\mathbb{Z}_0^2}|\ell|^2\Big|\sum_{m\in\mathbb{Z}_0^2}\tilde{\mathcal{K}}^N_{\ell,m}\eta_m\varphi_{-\ell-m}\Big|^2,$$

where we set $\tilde{\mathcal{K}}^N_{\ell,m} \stackrel{\text{def}}{=} (|\ell|^2+|m|^2)^{-1}\mathcal{K}^N_{\ell,m}$. Upon taking expectation, we get

$$\begin{aligned}\mathbb{E}\big|\mathcal{E}^N\big(H^N(\varphi)\big)(\eta)\big| &= 16\lambda_N^2\nu_N^{-2}\sum_{\ell,m\in\mathbb{Z}_0^2}|\ell|^2|\tilde{\mathcal{K}}^N_{l,m}|^2|\varphi_{-\ell-m}|^2\\ &= 16\lambda_N^2\nu_N^{-2}\sum_{k\in\mathbb{Z}_0^2}|k|^2\Big(\sum_{\ell\in\mathbb{Z}_0^2}\frac{|\ell|^2}{|k|^2}|\tilde{\mathcal{K}}^N_{\ell,k-\ell}|^2\Big)|\varphi_k|^2,\end{aligned} \tag{4.12}$$

where the first equality is a consequence of the fact that $\{\eta_m\}_m$ is a family of standard complex valued Gaussian random variables such that $\mathbb{E}[\eta_m\eta_\ell] = \mathbb{1}_{\ell+m=0}$ for $l,m\neq 0$. In order to bound the quantity in the parenthesis, recall (3.5), and set

$$f_k(z) \stackrel{\text{def}}{=} \frac{1}{4\pi^2}\frac{c(z,\tilde{k}-z)^2}{(|z|^2+|\tilde{k}-z|^2)^2|\tilde{k}-z|^2}, \qquad \tilde{k} \stackrel{\text{def}}{=} \frac{k}{|k|}, \tag{4.13}$$

for $z \in \mathbb{R}^2$ and $k \in \mathbb{Z}_0^2$, and notice that $|f_k(z)| \lesssim g(z) \stackrel{\text{def}}{=} \frac{1}{|z|^2}\mathbb{1}_{|z|>1} + \mathbb{1}_{|z|\le 1}$ uniformly in $k \in \mathbb{Z}_0^2$ and $z \in \mathbb{R}^2$. Now, plugging in the definition of $\mathcal{K}^N_{\ell,k-\ell}$, we have

$$\sum_{\ell\in\mathbb{Z}_0^2} \frac{|\ell|^2}{|k|^2}|\tilde{\mathcal{K}}^N_{\ell,k-\ell}|^2 = \sum_{\ell\in\mathbb{Z}_0^2} \frac{1}{|k|^2} f_k\left(\frac{\ell}{|k|}\right) \lesssim \int_{|z|\le N} g(z)\,\mathrm{d}z \lesssim \log N.$$

Therefore, we conclude that

$$\mathbb{E}\big[\big|\mathcal{E}^N\big(H^N(\varphi)\big)(\eta)\big|^p\big]^{\frac{1}{p}} \lesssim \lambda_N^2 \nu_N^{-2} \log N \|\varphi\|_{1,2}^2 \tag{4.14}$$

from which (4.10) follows.

The proof of (4.11) is straightforward, since in this case the quadratic variation of $M_\cdot(K^N(\varphi))$ is deterministic, and we have $\mathcal{E}^N(K^N(\varphi))(\eta) = \|\varphi\|_{1,2}^2$. □

REMARK 4.4. At first sight, estimate (4.11) might come as a surprise. Indeed, if we take $\nu_N \equiv 1$ it shows that *no matter how λ_N behaves as $N \uparrow \infty$*, the bound would provide tightness for the sequence of approximations $\{u^N\}_N$ in a suitable space of *space-time distributions* (replace $\Delta\varphi$ with any ψ smooth in (4.11)). To understand this behaviour, consider, as an example, the family of SDEs

$$\mathrm{d}X_t^N = -X_t^N + C_N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X_t^N \,\mathrm{d}t + \mathrm{d}B_t,$$

where B is a two-dimensional Brownian motion. Thanks to the Itô trick, it is easy to see that the time average of X^N stays uniformly bounded, *independently of the value of C_N*, thus giving tightness of X^N in a space of distributions. That said, the time integral of X^N represents a poor description of its actual behaviour since, in case C_N goes to ∞, X^N is oscillating increasingly fast, and the time integral simply converges to its average.

Lemma 4.3 suggests that, in order to control the nonlinearity in (3.1) *uniformly* in N, we need to tune λ_N and ν_N in such a way that the logarithmic factor on the right-hand side of (4.10) disappears. Let us define the integral in time of $\lambda_N \mathcal{N}^N[u^N]$ as

$$\mathcal{B}_t^N[u^N](\varphi) \stackrel{\text{def}}{=} \int_0^t \lambda_N \mathcal{N}^N[u^N(s)](\varphi)\,\mathrm{d}s \tag{4.15}$$

for any test function $\varphi \in H^1$. In the following theorem we show that, under this scaling, the couple $\{(u^N, \mathcal{B}^N[u^N])\}_N$ admits subsequential limits in a (product) space of continuous functions in time with values in a space of distributions of suitable regularity.

THEOREM 4.5. *Let $T > 0$; for $N \in \mathbb{N}$, let u^N be the stationary solution of* (3.1) *and $\mathcal{B}^N$ be the functional defined in* (4.15). *Let $C > 0$, and assume that λ_N and ν_N satisfy*

$$\lim_{N\to\infty} \sqrt{\frac{\log N}{4\pi^2 C}}\lambda_N \nu_N^{-\frac{1}{2}} = 1. \tag{4.16}$$

Then, the sequence $\{(u^N, \mathcal{B}^N[u^N])\}_N$ is tight in $C_T^\gamma \mathcal{C}^\alpha \times C_T^\gamma \mathcal{C}^\alpha$ for any $\gamma < 1/2$ and $\alpha < -2$.

Moreover, if $\nu_N = 1$ for all $N \in \mathbb{N}$, then the sequence $\{(u^N, \mathcal{B}^N[u^N])\}_N$ is tight in $C_T\mathcal{C}^\alpha \times C_T\mathcal{C}^\beta$ for any $\alpha < -1$ and $\beta < -2$.

REMARK 4.6. It is not surprising that, in case ν_N goes to 0, we can prove tightness only in the same space where the space-time white noise lives. Indeed, although in this scenario the noise disappears in the limit, we also lose the smoothing effect of the Laplacian, so that we cannot expect any regularisation coming from it. Since we are starting from a space white noise η whose regularity is $-1-\varepsilon$ for any $\varepsilon > 0$, then, heuristically, power-counting suggests that the regularity of the nonlinearity (and, consequently, of the limit of u^N) should be $-2-2\varepsilon$.

PROOF OF THEOREM 4.5. Choose sequences of coupling constants λ_N and ν_N such that (4.16) holds, let u^N be the stationary solution of (3.1) and let $\mathcal{B}^N$ be given by (4.15). A natural way to establish tightness for a sequence of random processes is Kolmogorov's criterion which, in the present context, requires a uniform control over the moments of the $\mathcal{C}^\alpha \times \mathcal{C}^\beta$-norm of the time increments of $(u^N, \mathcal{B}^N[u^N])$. Thanks to the Markov property and the fact that u^N is stationary for any $N \in \mathbb{N}$, we have

$$\begin{aligned}\mathbf{E}[\|u^N(t)-u^N(r)\|_\alpha^p] &= \mathbf{E}[\mathbf{E}[\|u^N(t)-u^N(r)\|_\alpha^p|\mathcal{G}_r]]\\ &= \mathbf{E}[\mathbf{E}^{u^N(r)}[\|u^N(t-r)-u^N(0)\|_\alpha^p]]\\ &= \mathbf{E}[\|u^N(t-r)-u^N(0)\|_\alpha^p],\end{aligned}$$

where $\{\mathcal{G}_r\}_r$ is the filtration generated by u^N and the previous holds for all $0 \le r < t \le T$. An analogous computation can be carried out for $\mathcal{B}^N[u^N]$, so that, for both, we can simply focus on the case $r = 0$.

Now, in order to obtain uniform bounds on $(u^N(t)-u^N(0), \mathcal{B}_t^N[u^N])$ (clearly, $\mathcal{B}_0^N[u^N] = 0$) in a Besov space we need to understand the behaviour of their Littlewood–Paley blocks. For $\mathcal{B}^N[u^N]$, we can immediately exploit Lemma 4.3 and, in particular, (4.10). Indeed, it suffices to choose φ to be jth Littlewood–Paley kernel ($j \ge -1$), so that

$$\mathbf{E}[\|\Delta_j \mathcal{B}_t^N[u^N]\|_{L^p(\mathbb{T}^2)}^p] = \int_{\mathbb{T}^2} \mathbb{E}[|\mathcal{B}_t^N[u^N](K_j(x-\cdot))|^p]\,\mathrm{d}x \lesssim t^{\frac{p}{2}} \lambda_N^p \nu_N^{-\frac{p}{2}} (\log N)^{\frac{p}{2}} 2^{2jp},$$

where K_j was defined above (1.11) and we used that $\|K_j\|_{1,2}^2 \sim \sum_{|k|\sim 2^j} |k|^2 \sim 2^{4j}$. Hence, by Besov embedding (1.12) we have

$$\begin{aligned}\mathbf{E}[\|\mathcal{B}_t^N[u^N]\|_\alpha^p] &\lesssim \mathbf{E}[\|\mathcal{B}_t^N[u^N]\|_{B_{p,p}^{\alpha+d/p}}^p]\\ &= \sum_{j\ge -1} 2^{(\alpha+d/p)jp}\mathbf{E}[\|\Delta_j \mathcal{B}_t^N[u^N]\|_{L^p(\mathbb{T}^2)}^p]\\ &\lesssim t^{\frac{p}{2}} \lambda_N^p \nu_N^{-\frac{p}{2}} (\log N)^{\frac{p}{2}} \sum_{j\ge -1} 2^{(\alpha+d/p+2)jp},\end{aligned}$$

and the latter sum converges if and only if $\alpha < -2 - d/p$. Since the previous bound holds for any $p \ge 2$, by choosing the renormalisation constants λ_N and ν_N, according to (4.16), Kolmogorov implies that $\{\mathcal{B}^N[u^N]\}_N$ is tight in $C_T^\gamma \mathcal{C}^\alpha$ for any $\alpha < -2$ and $\gamma < 1/2$.

We now focus on u^N. By writing (3.1) in its mild formulation and convolving both sides of the resulting expression with the jth Littlewood–Paley kernel ($j \ge -1$), we obtain

$$\begin{aligned}\Delta_j(u^N(t)-\eta) = \Delta_j(P^N\eta(t)-\eta) + \lambda_N \Delta_j P^N \mathcal{N}^N[u^N](t)\\ + \nu_N^{\frac{1}{2}}(-\Delta)^{\frac{1}{2}}\Delta_j P^N \xi(t),\end{aligned} \tag{4.17}$$

where P^N is the fundamental solution of $(\partial_t - \frac{\nu_N}{2}\Delta)P^N = 0$ and, for any space-time distribution f, $P^N f$ denotes the space-time convolution between P^N and f. At first we want to determine bounds on the pth moment of the L^p norm of the three summands on the right-hand side. For the first, using Gaussian hypercontractivity of η, we have

$$\begin{aligned}\mathbb{E}\big[\|\Delta_j(P^N\eta(t)-\eta)\|^p_{L^p(\mathbb{T}^2)}\big] &= \int_{\mathbb{T}^2}\mathbb{E}\big[|\Delta_j(P^N\eta(t)-\eta)(x)|^p\big]\,\mathrm{d}x\\ &\lesssim \int_{\mathbb{T}^2}\mathbb{E}\big[|\Delta_j(P^N\eta(t)-\eta)(x)|^2\big]^{\frac{p}{2}}\,\mathrm{d}x\\ &\lesssim \Big(\sum_{k\in\mathbb{Z}^2_0}\varrho_j(k)\big(e^{-\frac{\nu_N}{2}|k|^2t}-1\big)^2\Big)^{\frac{p}{2}}\\ &\lesssim t^{\frac{\kappa}{2}p}\nu_N^{\frac{\kappa}{2}p}2^{jp(1+\kappa)},\end{aligned}$$

where the last bound is a consequence of the fact that ϱ_j is supported on those $k\in\mathbb{Z}^2$ such that $|k|\sim 2^j$ and the geometric interpolation inequality, that is, $1-e^{-\nu_N|k|^2t}\le\min\{1,\nu_N|k|^2t\}\lesssim(\nu_N|k|^2t)^\kappa$, valid for any $\kappa\in[0,1]$ (applied above for $\tilde\kappa\stackrel{\text{def}}{=}\kappa/2$, $\kappa\in[0,2]$).

To treat the second summand in (4.17), we want to rewrite it in such a way that Lemma 4.1 is applicable. This is indeed possible since

$$\lambda_N\Delta_jP^N\mathcal{N}^N[u^N](t,x)=\int_0^t\mathcal{L}_0^N\tilde H^N_{(t,x)}(s,u^N(s))\,\mathrm{d}s,$$

where $\tilde H^N_{(t,x)}$ is the cylinder function depending smoothly on time defined by

$$\tilde H^N_{(t,x)}(s,\eta)\stackrel{\text{def}}{=}\sum_{k\in\mathbb{Z}^2_0}e^{-\frac{\nu_N}{2}|k|^2(t-s)}H^N_k(\eta)\varrho_j(k)e_k(x)$$

and H^N_k is the kth Fourier component of the solution of the Poisson equation, that is,

$$H^N_k=2\lambda_N\nu_N^{-1}\sum_{\ell+m=k}\frac{\mathcal{K}^N_{\ell,m}}{|\ell|^2+|m|^2}\eta_\ell\eta_m.$$

Hence, the L^p norm of the Littlewood–Paley block is controlled by

$$\begin{aligned}&\mathbf{E}\big[\|\lambda_N\Delta_jP^N\mathcal{N}^N[u^N](t,\cdot)\|^p_{L^p(\mathbb{T}^2)}\big]\\ &=\int_{\mathbb{T}^2}\mathbf{E}\Big[\Big|\int_0^t\mathcal{L}_0^N\tilde H^N_{(t,x)}(s,u^N(s))\,\mathrm{d}s\Big|^p\Big]\,\mathrm{d}x\\ &\lesssim \nu_N^{\frac{p}{2}}\Big(\sum_{k\in\mathbb{Z}^2_0}\varrho_j(k)\Big(\frac{1-e^{-\nu_N|k|^2t}}{\nu_N|k|^2}\Big)\mathbb{E}\big[|\mathcal{E}^N(H^N_k)(\eta)|^{\frac{p}{2}}\big]^{\frac{2}{p}}\Big)^{\frac{p}{2}}\\ &\lesssim \nu_N^{\frac{p}{2}}\Big(\sum_{|k|\sim 2^j}\Big(\frac{1-e^{-\nu_N|k|^2t}}{\nu_N|k|^2}\Big)\lambda_N^2\nu_N^{-2}\log N|k|^2\Big)^{\frac{p}{2}}\\ &\lesssim t^{\frac{\kappa}{2}p}\big(\lambda_N\nu_N^{-1+\frac{\kappa}{2}}(\log N)^{\frac12}\big)^p2^{jp(1+\kappa)},\end{aligned}$$

where we went from the first to the second line via (4.8). We subsequently bounded the $p/2$-moment of the energy through (4.14), and, in the last line, we used the same interpolation inequality as above.

For the last term in (4.17), we can apply once more Gaussian hypercontractivity, but this time for the space-time white noise ξ, to get

$$\begin{aligned}\mathbf{E}\big[\|\nu_N^{\frac{1}{2}}\Delta_j P^N\xi(t,\cdot)\|_{L^p(\mathbb{T}^2)}^p\big] &= \int_{\mathbb{T}^2}\mathbf{E}\big[|\nu_N^{\frac{1}{2}}\Delta_j P^N\xi(t,x)|^p\big]\,\mathrm{d}x\\ &\lesssim \int_{\mathbb{T}^2}\mathbf{E}\big[|\nu_N^{\frac{1}{2}}\Delta_j P^N\xi(t,x)|^2\big]^{\frac{p}{2}}\,\mathrm{d}x\\ &\lesssim \Big(\sum_{|k|\sim 2^j}\big(1-e^{-\nu_N|k|^2 t}\big)\Big)^{\frac{p}{2}} \lesssim t^{\frac{\kappa}{2}p}\nu_N^{\frac{\kappa}{2}p}2^{jp(1+\kappa)},\end{aligned}$$

where the last passage is again a consequence of the interpolation inequality.

Putting these three bounds together and applying the Besov embedding (1.12), we see that, for any $t > 0$ and $p \geq 2$, we have

$$\begin{aligned}\mathbf{E}\big[\|u^N(t)-u^N(0)\|_\alpha^p\big] &\lesssim \mathbf{E}\big[\|u^N(t)-u^N(0)\|_{B^{\alpha+d/2}_{p,p}}^p\big]\\ &= \sum_{j\geq -1}2^{(\alpha+d/p)jp}\mathbf{E}\big[\|\Delta_j\big(u^N(t)-u^N(0)\big)\|_{L^p}^p\big]\\ &= t^{\frac{\kappa}{2}p}\nu_N^{\frac{\kappa}{2}p}\big(2+\big(\lambda_N\nu_N^{-1}(\log N)^{\frac{1}{2}}\big)^p\big)\sum_{j\geq -1}2^{jp(\alpha+d/p+1+\kappa)}.\end{aligned}$$

Now, notice that the last sum converges as soon as $\alpha < -1-\kappa-d/p$. Hence, if ν_N is a constant independent of N and $\lambda_N \sim (\log N)^{-\frac{1}{2}}$, we can conclude, by Kolmogorov's criterion, that the sequence $\{u^N\}_N$ is tight in the space $C_T\mathcal{C}^\alpha$, with α arbitrarily close to (but strictly smaller than) -1. Otherwise, to take advantage of condition (4.16), we are forced to choose $\kappa = 1$, and the sequence $\{u^N\}_N$ is tight in $C_T^\gamma\mathcal{C}^\alpha$ for all $\gamma < 1/2$ and $\alpha < -2$. □

REMARK 4.7. The previous theorem guarantees that if λ_N and ν_N satisfy (4.16), then the couple $(u^N, \mathcal{B}^N)$ converges (at least along a subsequence) to some limit $(u, \mathcal{B})$. In case that $\nu_N \to 0$, the energy estimate (4.11) of Lemma 4.3 implies that, for any test function $\varphi \in H^1$, one has

$$u_t(\varphi) - u_0(\varphi) = \mathcal{B}_t[u](\varphi).$$

Hence, a characterisation of the limit u is connected to a deeper understanding of the process $\mathcal{B}$. We are currently neither able to show that $\mathcal{B}$ is 0 nor are we able to define its law, so we leave its study to future investigations.

We define the integral in time of the nonlinearity of the solution h^N of (1.3), as

$$\tilde{\mathcal{B}}_t^N[h^N](\varphi) \stackrel{\text{def}}{=} \int_0^t \lambda_N\tilde{\mathcal{N}}^N[h^N(s)](\varphi)\,\mathrm{d}s. \tag{4.18}$$

Above, φ is a generic test function, and

$$\tilde{\mathcal{N}}^N[h^N] \stackrel{\text{def}}{=} \Pi_N\big((\Pi_N\partial_1 h^N)^2 - (\Pi_N\partial_2 h^N)^2\big). \tag{4.19}$$

In the following theorem we prove joint tightness for the sequence $\{(h^N, \tilde{\mathcal{B}}^N[h^N])\}_N$.

THEOREM 4.8. *Let $T > 0$, and, for $N \in \mathbb{N}$, let h^N be the solution of (1.3) started at 0 from $\tilde{\eta}$, where for all $k \in \mathbb{Z}_0^2$, $\tilde{\eta}_k \stackrel{\text{def}}{=} |k|^{-1}\eta_k$, η a space white noise and $\tilde{\eta}_0 = 0$, and $\tilde{\mathcal{B}}^N[h^N]$ be defined according to (4.18). Let $C > 0$, and assume λ_N and ν_N satisfy (4.16).*

Then, the sequence $\{(h^N, \tilde{\mathcal{B}}^N[h^N])\}_N$ is tight in $C_T^\gamma\mathcal{C}^{\alpha+1} \times C_T^\gamma\mathcal{C}^{\alpha+1}$ for any $\gamma < 1/2$ and $\alpha < -2$. Moreover, if (4.16) is satisfied with ν_N (a constant that is independent of N), then the sequence $\{(h^N, \tilde{\mathcal{B}}^N[h^N])\}_N$ is tight in $C_T\mathcal{C}^{\alpha+1} \times C_T\mathcal{C}^{\beta+1}$ for any $\alpha < -1$ and $\beta < -2$.

PROOF. For $\alpha \in \mathbb{R}$, define $\mathcal{C}_0^\alpha$ as the set of functions in $\mathcal{C}^\alpha$ whose 0th Fourier mode is 0. Then, $\Delta^{1/2}$ is a homeomorphism between $\mathcal{C}_0^\alpha$ and $\mathcal{C}_0^{\alpha-1}$. Since by definition $(u^N, \mathcal{B}^N[u^N]) = (\Delta^{1/2} h^N, \Delta^{1/2}\tilde{\mathcal{B}}^N[h^N])$ and by Theorem 4.5 $(u^N, \mathcal{B}^N[u^N])$ is tight in $C_T^\gamma\mathcal{C}^\alpha \times C_T^\gamma\mathcal{C}^\alpha$ for any $\gamma < 1/2$ and $\alpha < -2$ (resp., $C_T\mathcal{C}^\alpha \times C_T\mathcal{C}^\beta$ for any $\alpha < -1$ and $\beta < -2$, if ν_N constant), then the sequence $(h^N - h^N(e_0), \tilde{\mathcal{B}}^N[h^N] - \tilde{\mathcal{B}}^N[h^N](e_0))$ is tight in $C_T^\gamma\mathcal{C}^{\alpha+1} \times C_T^\gamma\mathcal{C}^{\alpha+1}$ (resp., $C_T\mathcal{C}^{\alpha+1} \times C_T\mathcal{C}^{\beta+1}$). Therefore, it suffices to focus on $(h^N(e_0), \tilde{\mathcal{B}}^N[h^N](e_0))$. Notice also that, since we chose $\tilde{\eta}_0 = 0$, $h_t^N(e_0) = \tilde{\mathcal{B}}_t^N[h^N](e_0)$.

In order to show tightness for the 0th Fourier mode of h^N, we want to again apply Lemma 4.1. To do so, we need to solve the Poisson equation $\mathcal{L}_0^N \tilde{H}_0^N = \lambda_N \tilde{\mathcal{N}}_0^N$. Notice that

$$\begin{aligned}\tilde{\mathcal{N}}_0^N[\eta] &= \sum_{\ell\in\mathbb{Z}_0^2} c(\ell,-\ell)|\eta_\ell|^2 \\ &= \sum_{\ell\in\mathbb{Z}_0^2} c(\ell,-\ell)\big(|\eta_\ell|^2-1\big) + \sum_{\ell\in\mathbb{Z}_0^2} c(\ell,-\ell) \\ &= I_2\bigg(\sum_{\ell\in\mathbb{Z}_0^2} \mathcal{K}_{\ell,-\ell}^N e_{-\ell}\otimes e_\ell\bigg),\end{aligned}$$

where, both here and below, the sum in ℓ is restricted to $|\ell| \le N$ and the last passage is a consequence of the fact that, for every $N \in \mathbb{N}$, we have

$$\sum_{\ell=(\ell_1,\ell_2)} c(\ell,-\ell) = -\sum_{\ell=(\ell_1,\ell_2)} \ell_2^2 + \sum_{\ell=(\ell_1,\ell_2)} (\ell_1)^2 = 0. \tag{4.20}$$

We can now proceed as in (4.9), so that we get

$$\tilde{H}_0^N[\eta] = \frac{2}{2\pi}\lambda_N \nu_N^{-1} \sum_{\substack{\ell,m\in\mathbb{Z}_0^2\\ \ell+m=0}} \frac{c(\ell,m)}{(|\ell|^2+|m|^2)|\ell||m|}\eta_\ell\eta_m \tag{4.21}$$

and

$$\mathbb{E}[\mathcal{E}^N(\tilde{H}_0^N)(\eta)] = \sum_{k\in\mathbb{Z}_0^2} |k|^2 \mathbb{E}\big[\big|D_k\tilde{H}_0^N[\eta]\big|^2\big] = \frac{16}{4\pi^2}\lambda_N^2\nu_N^{-2}\sum_{|k|\le N}\frac{c(k,-k)^2}{|k|^6}$$

which, by (4.7), implies

$$\mathbf{E}\big[\big|h_t^N(e_0)\big|^p\big]^{\frac{1}{p}} = \mathbf{E}\big[\big|\tilde{\mathcal{B}}^N[h^N](e_0)\big|^p\big]^{\frac{1}{p}} \lesssim t^{\frac{1}{2}}\lambda_N^2\nu_N^{-1}\log N$$

and tightness follows. □

REMARK 4.9. As opposed to the isotropic KPZ equation treated in [8], [17] and [10], in the present context there is no average growth that needs to be subtracted in order to guarantee the convergence of the approximation. This is due to the fact that the nonlinearity in (1.3) is antisymmetric with respect to the change of variables $\mathbb{R}^2 \ni (x_1, x_2) \mapsto (x_2, x_1)$, as can be seen in (4.20).

5. Lower bounds and nontriviality. Throughout this section we will be assuming that, for every $N \in \mathbb{N}$, $\nu_N = 1$, so that the only renormalisation constant that we allow to vanish is λ_N. Notice that in this case the symmetric part of the generator $\mathcal{L}^N$, $\mathcal{L}_0^N$, does not depend on N, so we will simply denote it by $\mathcal{L}_0$.

We aim at obtaining lower bounds on functionals of the solution u^N to (3.1) and to show that any subsequential limit u is not trivial. By "trivial," here we mean that u is the solution of the original equation without the nonlinearity, a scenario that could materialise in case λ_N converges to 0 too fast.

To do so, we apply a technique, coming from particle systems (see [31]), which consists in determining (and bounding) a variational formula for the Laplace transform of the integral in time of a suitable functional of our process. We begin with the following lemma.

LEMMA 5.1. *Let $\{u^N(t)\}_{t\geq 0}$ be the stationary solution to* (3.1) *and $F \in L^2(\eta)$. Then, for every $\lambda > 0$, the following equality holds*:

$$\int_0^\infty \mathrm{d}t e^{-\lambda t}\mathbf{E}\Big[\Big(\int_0^t F\big(u^N(s)\big)\,\mathrm{d}s\Big)^2\Big] = \frac{2}{\lambda^2}\mathbb{E}\big[F(\eta)\big(\lambda - \mathcal{L}^N\big)^{-1}F(\eta)\big]. \tag{5.1}$$

PROOF. Notice that we can rewrite the expectation at the left-hand side of (5.1) as

$$\begin{aligned}\mathbf{E}\Big[\Big(\int_0^t F\big(u^N(s)\big)\,\mathrm{d}s\Big)^2\Big] &= 2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\mathbf{E}\big[F\big(u^N(r)\big)F\big(u^N(s)\big)\big]\\ &= 2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\mathbf{E}\big[F\big(u^N(r)\big)\mathbf{E}\big[F\big(u^N(s)\big)|\mathcal{G}_r\big]\big],\end{aligned}$$

where $\mathcal{G}$ denotes the natural filtration of the process $\{u^N(t)\}_{t\geq 0}$. Now, u^N is a Markov process; it generates a semigroup, which we denote by $\{e^{t\mathcal{L}^N}\}_{t\geq 0}$, and at any fixed time is distributed according to the law of η. Therefore, the right-hand side of the previous is equal to

$$\begin{aligned}2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\mathbb{E}\big[F(\eta)\mathbf{E}^\eta\big[F\big(u^N(s-r)\big)\big]\big] &= 2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\mathbb{E}\big[F(\eta)e^{(s-r)\mathcal{L}^N}F(\eta)\big]\\ &= 2\int_0^t \mathrm{d}r(t-r)\mathbb{E}\big[F(\eta)e^{r\mathcal{L}^N}F(\eta)\big].\end{aligned}$$

Here, we use the symbol $\mathbf{E}^\eta$ to denote the expectation with respect to the law of the process $\{u^N(t)\}_{t\geq 0}$ conditioned to start at $t=0$ from η. Notice that the expectation in the last term above does not depend on t, hence the Laplace transform of the last integral above equals

$$\begin{aligned}&2\int_0^\infty \mathrm{d}t\int_0^t \mathrm{d}r(t-r)e^{-\lambda(t-r)}\mathbb{E}\big[F(\eta)e^{-r(\lambda-\mathcal{L}^N)}F(\eta)\big]\\ &\quad= \frac{2}{\lambda^2}\mathbb{E}\Big[F(\eta)\int_0^\infty dre^{-r(\lambda-\mathcal{L}^N)}F(\eta)\Big],\end{aligned}$$

where the equality is obtained by simply changing the order of integration. The conclusion now follows by applying the equality $\int_0^\infty dre^{-r(\lambda-\mathcal{L}^N)} = (\lambda-\mathcal{L}^N)^{-1}$. □

The advantage of the previous statement is twofold. At first, notice that, while in principle the expectation at the left-hand side of (5.1) depends on the distribution of the solution at different (at least *two*) points in time the right-hand side only depends on the law of the invariant measure which is explicitly known. Moreover, even though it is hard, in general, to invert the full generator (which is what seems to be required in order to exploit Lemma 5.1), the expression on the right-hand side of (5.1) allows for a variational formulation which turns out to be easier to manipulate.

This variational formula is given in [30], Theorem 4.1, and, below, we state it in the way in which we will use it in the remainder of the section.

LEMMA 5.2 (Variational formula). *Let $\mathcal{L}^N$ be the generator of the Markov process $\{u^N(t)\}_{t\geq 0}$, and let $\mathcal{L}_0$ and $\mathcal{A}^N$, defined in (3.6) and (3.7), be its symmetric and antisymmetric parts with respect to the white noise measure η. Let $F \in L^2(\eta)$, and denote by $\langle \cdot, \cdot \rangle_\eta$ the scalar product in $L^2(\eta)$. Then, for every $\lambda > 0$, one has*

$$\langle F, (\lambda - \mathcal{L}^N)^{-1} F \rangle_\eta = \sup_G \{ 2\langle F, G\rangle_\eta - \langle (\lambda - \mathcal{L}_0)G, G\rangle_\eta - \langle \mathcal{A}^N G, (\lambda - \mathcal{L}_0)^{-1} \mathcal{A}^N G \rangle_\eta \}, \tag{5.2}$$

where G ranges over a fixed core of $\mathcal{L}^N$.

PROOF. The lemma is a direct consequence of [30], Theorem 4.1. Indeed, it suffices to apply the first equality in [30], Theorem 4.1, twice so as to simplify the term $\|Ag\|^2_{-1,\lambda}$ (in the notation of the reference). □

Thanks to the variational formula above, in order to obtain the lower bounds we are looking for, it suffices to find *one* G for which the quantity in brackets in (5.2) is bounded from below by a positive constant uniformly in N. The functional F, to which we will apply Lemmas 5.1 and 5.2, is the nonlinearity $\lambda_N \mathcal{N}^N$ which, for fixed N, is a cylinder function belonging to a *fixed* (the second) Wiener chaos.

Using the explicit expressions for $\mathcal{L}_0^N$ and $\mathcal{A}^N$ and the decomposition of $\mathcal{A}^N$ from Lemma 3.5, we are indeed able to determine such a function G and, consequently, prove the following proposition.

PROPOSITION 5.3. *Let $\varphi \in H^1$ and $\mathcal{N}^N$ be defined according to (3.2). Let $C > 0$, and assume*

$$\lim_{N\to\infty} \sqrt{\frac{\log N}{4\pi^2 C}} \lambda_N = 1. \tag{5.3}$$

Then, there exists a constant $\delta > 0$ independent of N and φ, such that

$$\mathbb{E}[\lambda_N \mathcal{N}^N[\eta](\varphi)(\lambda - \mathcal{L}^N)^{-1} \lambda_N \mathcal{N}^N[\eta](\varphi)] \geq C\pi\delta \|\varphi\|^2_{1,2} \tag{5.4}$$

for all $\lambda > 0$.

PROOF. First, we obtain a lower bound of the right-hand side of (5.2) by restricting the supremum to (smooth) random variables living in $\mathcal{H}_2$, the second homogeneous Wiener chaos of η. With this choice, since, by Lemma 3.5, $\mathcal{A}^N = \mathcal{A}^N_+ + \mathcal{A}^N_-$ and $\mathcal{A}^N_+$ maps $\mathcal{H}_2$ to $\mathcal{H}_3$ while $\mathcal{A}^N_-$ maps $\mathcal{H}_2$ to $\mathcal{H}_1$, the quantity inside the brackets can be rewritten as

$$2\langle \lambda_N \mathcal{N}^N(\varphi), G\rangle - \langle (\lambda - \mathcal{L}_0)G, G\rangle_\eta - \|(\lambda - \mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}^N_+ G\|^2_\eta - \|(\lambda - \mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}^N_- G\|^2_\eta, \tag{5.5}$$

where $\|\cdot\|_\eta \overset{\text{def}}{=} \|\cdot\|_{L^2(\eta)}$. Denote by (I), (II), (III), (IV) each of the summands in (5.5), so that it equals 2(I) − (II) − (III) − (IV).

Notice that (I) is linear in G while the others are quadratic. In order to take advantage of this fact, it suffices to determine a function G^N, allowed to depend on N, such that, under the scaling (5.3), (II)–(IV) are bounded uniformly, while (I) is bounded uniformly from below by a positive constant.

Hence, let $\delta > 0$ be a constant to be fixed later, and take G to be the solution of the Poisson equation (4.9) with $\nu_N = 1$, multiplied by δ, that is, $G(\eta) \stackrel{\text{def}}{=} \delta H^N[\eta](\varphi)$. Then, we have

$$\text{(I)} = \langle \lambda_N \mathcal{N}^N(\varphi), \delta H^N(\varphi)\rangle_\eta = 4\delta \sum_{k\in\mathbb{Z}_0^2} |k|^2 \Big(\tilde{\lambda}_N^2 \sum_{\ell+m=k} \frac{1}{|k|^2} \frac{(\mathcal{K}^N_{\ell,m})^2}{|\ell|^2+|m|^2}\Big)|\varphi_{-k}|^2,$$

where $\tilde{\lambda}_N \stackrel{\text{def}}{=} \lambda_N/2\pi$. Now, the quantity in brackets can be analysed with the same tools used in the proof of (6.2), so we address the reader to the section below for the details and, here, limit ourselves to outline the procedure and highlight the main steps. By a Riemann sum approximation we have

$$\begin{aligned}
&\tilde{\lambda}_N^2 \sum_{\ell+m=k} \frac{1}{|k|^2} \frac{(\mathcal{K}^N_{\ell,m})^2}{|\ell|^2+|m|^2} \\
&\quad = \tilde{\lambda}_N^2 \sum_{|\ell|,|k-\ell|\le N} \frac{c(\ell,k-\ell)}{|\ell|^2|k-\ell|^2(|\ell|^2+|k-\ell|^2)} \\
&\quad = \tilde{\lambda}_N^2 \sum_{|\ell|,|k-\ell|\le N} \frac{1}{|k|^2} \tilde{f}_k\Big(\frac{\ell}{|k|}\Big) \approx \tilde{\lambda}_N^2 \int_{\frac{5}{2}\le|x|\le\frac{N}{|k|}} \frac{c(x,\tilde{k}-x)^2}{|x|^2|\tilde{k}-x|^2(|x|^2+|\tilde{k}-x|^2)}\,\mathrm{d}x,
\end{aligned}$$

where $\tilde{f}_k(x)$ coincides with the integrand in the last term of the previous equality and $\tilde{k} = k/|k|$. Since $\tilde{k}$ has norm one, let $\theta_k \in [0,2\pi)$ be such that $\tilde{k} = (\cos\theta_k, \sin\theta_k)$. Then, passing to polar coordinates and neglecting all terms in the integral which are uniformly bounded in N (since they are then killed by the vanishing constant $\tilde{\lambda}_N$), the previous is approximated by

$$\begin{aligned}
&\frac{\tilde{\lambda}_N^2}{4} \int_0^{2\pi} \int_{\frac{5}{2}}^{\frac{N}{|k|}} \frac{r\cos^2(2\theta)}{(r-\cos(\theta-\theta_k))^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&\qquad \approx \frac{1}{4}\int_0^{2\pi} \cos^2(2\theta)\Big(\tilde{\lambda}_N^2 \log\Big(\frac{N/|k|-\cos(\theta-\theta_k)}{5/2-\cos(\theta-\theta_k)}\Big)\Big)\,\mathrm{d}\theta \stackrel{N\to\infty}{\longrightarrow} C\frac{\pi}{4},
\end{aligned}$$

which implies that, as N goes to ∞,

$$\text{(I)} \sim \delta C_{\text{(I)}} \|\varphi\|^2_{1,2}, \tag{5.6}$$

where $C_{\text{(I)}} \stackrel{\text{def}}{=} C\pi$. For (II), notice that, by definition of $H^N(\varphi)$, as $N \to \infty$ we have

$$\text{(II)} = \delta^2 \langle (\lambda - \mathcal{L}_0) H^N(\varphi), H^N(\varphi)\rangle_\eta = \lambda\delta^2 \|H^N(\varphi)\|^2_\eta - \delta\text{(I)} \approx -\delta^2 C_{\text{(I)}} \|\varphi\|^2_{1,2} \tag{5.7}$$

locally uniform in λ. The last passage will be justified in detail in the proof of Corollary 6.3 where we will see that the $L^2(\eta)$-norm of $H^N(\varphi)$ converges to 0 as $N \to \infty$ (see (6.4)).

We can now focus on (III). By Lemma 3.5 the Fourier transform of the kernel (in Fock space representation) of $(\lambda - \mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}_+^N H^N(\varphi)$ is given by

$$\mathcal{F}((\lambda - \mathfrak{L}_0)^{-\frac{1}{2}} \mathfrak{A}_+^N \mathfrak{H}_\varphi^N)(\ell,m,n) = 4\lambda_N^2 \frac{\mathcal{K}^N_{\ell,m}\mathcal{K}^N_{\ell+m,n}\varphi_{-\ell-m-n}}{(\lambda+\frac{1}{2}(|\ell|^2+|m|^2+|n|^2))^{\frac{1}{2}}(|\ell+m|^2+|n|^2)},$$

where we denoted by $\mathfrak{H}_\varphi^N$ the kernel of $H^N(\varphi)$. Hence, we get

$$\begin{aligned}
&\|(\lambda - \mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}_+^N H^N(\varphi)\|^2_\eta \\
&\qquad \lesssim \sum_{k\in\mathbb{Z}_0^2} \Big(\lambda_N^4 \sum_{\ell+m+n=k} \frac{(\mathcal{K}^N_{\ell,m}\mathcal{K}^N_{\ell+m,n})^2}{(\lambda+\frac{1}{2}(|\ell|^2+|m|^2+|n|^2))(|\ell+m|^2+|n|^2)^2}\Big)|\varphi_{-k}|^2.
\end{aligned}$$

By bounding brutally $(\mathcal{K}^N_{\ell,m}\mathcal{K}^N_{\ell+m,n})^2 \lesssim |k|^2|n|^{-2}$, the quantity in brackets above can be treated as

$$\lambda_N^4 \sum_{\ell+m+n=k} \frac{(\mathcal{K}^N_{\ell,m}\mathcal{K}^N_{\ell+m,n})^2}{(\lambda+\frac{1}{2}(|\ell|^2+|m|^2+|n|^2))(|\ell+m|^2+|n|^2)^2}$$
$$\lesssim |k|^2\lambda_N^2 \sum_{|n|\le N} \frac{1}{|n|^2}\lambda_N^2 \sum_{|\ell|\le N} \frac{1}{\lambda+\frac{1}{2}(|\ell|^2+|m|^2+|n|^2)} \le |k|^2\Big(\lambda_N^2 \sum_{|\ell|\le N}\frac{1}{|\ell|^2}\Big)^2,$$

and the choice of λ_N guarantees that the previous is uniformly bounded by $|k|^2$. Therefore, there exists a constant $C_{(\mathrm{III})}>0$ independent of N, λ, φ, such that

$$(\mathrm{III}) \le \delta^2 C_{(\mathrm{III})}\|\varphi\|_{1,2}^2. \tag{5.8}$$

It remains to study (IV). Again, by Lemma 3.5 the Fourier transform of the kernel (in Fock space representation) of $\mathcal{A}_-^N H^N(\varphi)$ is given by

$$\mathcal{F}(\mathfrak{A}_-^N\mathfrak{H}_\varphi^N)(k) = 4\lambda_N \sum_{\ell+m=k} \mathcal{K}^N_{k,-\ell}\mathcal{F}(\mathfrak{H}_\varphi^N)(\ell,m) = 8\tilde\lambda_N^2 \sum_{\ell+m=k} \frac{c(k,-\ell)c(\ell,m)}{|\ell|^2(|\ell|^2+|m|^2)}\varphi_{-k},$$

where, again, $\tilde\lambda_N \stackrel{\mathrm{def}}{=} \lambda_N/2\pi$. Let us observe the inner sum more carefully. Define $K(\ell,m) \stackrel{\mathrm{def}}{=} c(\ell,m)(|\ell|^2+|m|^2)^{-1}$ which is clearly symmetric in ℓ and m. Then, by changing variables in the sum ($\ell \to k-\ell$) and using the fact that c is a symmetric bilinear form in its arguments (the antisymmetry is only by swapping the coordinates of both variables) so that in particular $c(\ell,-m) = -c(\ell,m)$, we have

$$\begin{aligned}
&\sum_{\ell+m=k} \frac{c(k,-\ell)}{|\ell|^2} K(\ell,k-\ell)\\
&\quad= \frac{1}{2}\sum_{|\ell|,|k-\ell|\le N}\Big(\frac{c(k,-\ell)}{|\ell|^2} - \frac{c(k,k-\ell)}{|k-\ell|^2}\Big)K(\ell,k-\ell)\\
&\quad= -\frac{c(k,k)}{2}\sum_{|\ell|,|k-\ell|\le N}\frac{K(\ell,k-\ell)}{|k-\ell|^2}\\
&\qquad+ \frac{1}{2}\sum_{|\ell|,|k-\ell|\le N} c(k,\ell)\Big(\frac{1}{|k-\ell|^2}-\frac{1}{|\ell|^2}\Big)K(\ell,k-\ell).
\end{aligned}$$

Now, since for any ℓ, m, $|K(\ell,m)|\lesssim 1$ and $|c(\ell,m)|\le|\ell||m|$, it is immediate to see that the first summand is bounded by $|k|^2\log N$. For the second, by Taylor's formula (holding at least for $|\ell|$ large enough, say $|\ell|>2|k|$) we have

$$\Big|\sum_{|\ell|,|k-\ell|\le N} c(k,\ell)\Big(\frac{1}{|k-\ell|^2}-\frac{1}{|\ell|^2}\Big)K(\ell,k-\ell)\Big| \lesssim |k|^2\sum_{|\ell|\le N}\frac{1}{|\ell|^2} \lesssim |k|^2\log N.$$

Exploiting (5.3) to get rid of the log divergence, it follows that there exists a constant $C_{(\mathrm{IV})}>0$ for which

$$\|(\lambda-\mathcal{L}_0)^{-\frac{1}{2}}\mathcal{A}_-^N H^N(\varphi)\|_\eta^2 = \sum_{k\in\mathbb{Z}_0^2}\frac{|\mathcal{F}(\mathfrak{A}_-^N\mathfrak{H}_\varphi^N)(k)|^2}{\lambda+|k|^2} \le C_{(\mathrm{IV})}\sum_{k\in\mathbb{Z}_0^2}\frac{|k|^4|\varphi_{-k}|^2}{\lambda+|k|^2} \le C_{(\mathrm{IV})}\|\varphi\|_{1,2}^2$$

and, consequently,

$$(\mathrm{IV}) \le \delta^2 C_{(\mathrm{IV})}\|\varphi\|_{1,2}^2. \tag{5.9}$$

Collecting (5.6), (5.7), (5.8), (5.9), we see that (5.5) is bounded below by $\delta C_{(\mathrm{I})}(2-\delta\tilde{C})\|\varphi\|_{1,2}^2$, where $\tilde{C} \stackrel{\mathrm{def}}{=} 2C_{(\mathrm{I})}^{-1}\max\{C_{(\mathrm{I})}, C_{(\mathrm{III})}, C_{(\mathrm{IV})}\} - 1$ and therefore, for any $\delta \in (0, 2\tilde{C}^{-1})$, (5.4) holds. □

As an immediate consequence of the previous proposition, we can show that the Laplace transform of the integral in time of the (rescaled) nonlinearity of (3.1) is (uniformly) bounded from above and below.

COROLLARY 5.4. *For any $N \in \mathbb{N}$, let u^N be the solution of* (3.1) *and $\mathcal{N}^N$ be the functional defined by* (3.2). *Assume $\nu_N = 1$ for all N, and the sequence of positive constants λ_N satisfies the scaling relation* (5.3). *Then, there exists a constant $\delta > 0$ such that, for any $\lambda > 0$, $\varphi \in H^1$ and $N \in \mathbb{N}$ we have*

$$\frac{\delta}{\lambda^2}\|\varphi\|_{1,2}^2 \le \int_0^\infty e^{-\lambda t}\mathbf{E}\Big[\Big(\int_0^t \lambda_N \mathcal{N}^N[u^N(s)](\varphi)\,\mathrm{d}s\Big)^2\Big]\,\mathrm{d}t \le \frac{\delta^{-1}}{\lambda^2}\|\varphi\|_{1,2}^2. \tag{5.10}$$

PROOF. The proof of the lower bound in (5.10) is a direct consequence of Lemma 5.1 and Proposition 5.3. The upper bound instead follows by (4.10) in Lemma 4.3, upon taking $p = 2$ and evaluating the Laplace transform of $f(t) = t$. □

In the following proposition we collect the results obtained so far and provide a description of the limit points of the sequence u^N.

PROPOSITION 5.5. *For $N \in \mathbb{N}$, let u^N be the stationary solution of* (3.1). *Assume $\nu_N = 1$, and the sequence of constants λ_N satisfies* (5.3). *Then, any subsequential limit $(u, \mathcal{B}^N[u])$ of $\{(u^N, \mathcal{B}^N[u^N])\}_N$ satisfies*

$$u_t(\varphi) - u_0(\varphi) = \frac{1}{2}\int_0^t u_s(\Delta\varphi)\,\mathrm{d}s + \mathcal{B}_t[u](\varphi) + B_t(\varphi) \tag{5.11}$$

for any $\varphi \in H^1$, and $\mathcal{B}_t[u](\varphi)$ is a stationary stochastic process such that

$$\mathbf{E}\big[\mathcal{B}_t[u](\varphi)^2\big] \sim t, \quad \text{as } t \to 0. \tag{5.12}$$

In particular, it has nonzero finite energy, as defined in (1.7).

PROOF. The validity of (5.11) is a consequence of Theorem 4.5, hence the only thing to prove is that the process $\{\mathcal{B}_t[u](\varphi)\}_{t\ge 0}$ satisfies (5.12). Once the latter is established, we can immediately conclude that the process has nonzero energy, and the fact that it is finite follows by (4.10).

Now, for (5.12) we need the estimate (5.10), which clearly holds also for $\mathcal{B}_\cdot(\varphi)$, that is,

$$\frac{\delta}{\lambda^2}\|\varphi\|_{1,2}^2 \le \int_0^\infty e^{-\lambda t}\mathbf{E}\big[\mathcal{B}_t(\varphi)^2\big]\,\mathrm{d}t \le \frac{\delta^{-1}}{\lambda^2}\|\varphi\|_{1,2}^2$$

and (4.10). Lemma B.1, whose proof is provided in Appendix B, allows us to conclude. □

REMARK 5.6. The previous proposition marks the difference between the *one*− and the *two*-dimensional cases. Indeed, for $d = 1$, [20] shows that the solution of the KPZ equation (or stochastic Burgers) is a Dirichlet process, that is, the sum of a martingale and a zero-quadratic variation process. In particular, the integral in time of the nonlinearity converges to a *zero*-quadratic variation process. In the two dimensional anisotropic case instead, the relation (5.12) suggests that the integral in time of the nonlinearity should morally contain a martingale part (hence, in particular, if it admitted quadratic variation, it would be nonzero) whose understanding would represent the main step in the characterisation of the limit points.

We conclude this section by stating (and proving) an analogous result at the level of the anisotropic KPZ equation. In this context we show that, assuming the noise to have zero average, the time increment of the average of the solution does not vanish, thus distinguishing it from the solution of the stochastic heat equation X defined in (6.3).

THEOREM 5.7. *For $N \in \mathbb{N}$, let h^N be the solution of the smoothened anisotropic KPZ equation* (1.3) *and $\tilde{X}$ be the solution of the stochastic heat equation obtained by setting $\lambda_N = 0$ in* (1.3), *both started at* 0 *from $\tilde{\eta}$, defined as in the statement of Theorem* 4.8. *Assume that the constants ν_N and λ_N are such that $\nu_N = 1$ and λ_N satisfies* (5.3).

Then, any limit point $\{h(t, \cdot)\}_{t\geq 0}$ of the sequence $\{h^N\}_N$ is a stochastic processes different in law from $\{\tilde{X}(t, \cdot)\}_{t\geq 0}$.

PROOF. Let h be a limit point of the sequence $\{h^N\}_{N\in\mathbb{N}}$ and $\tilde{X}$ be the solution of the stochastic heat equation. In order to prove the statement, it suffices to exhibit *any* observable which is different for h and $\tilde{X}$. An observable easy to treat, considered also in [10], is the 0th Fourier mode h_0 and $\tilde{X}_0$ of h and X, that is, their spatial average. Notice that, by construction $\tilde{X}_0 = 0$, while, by (1.3), one has $h_0^N(t) = \int_0^t \lambda_N \tilde{\mathcal{N}}_0^N[h^N(s)]\,\mathrm{d}s$. In Theorem 4.8 we have shown that the right-hand side of the latter has finite moments of any order; therefore, by [10], Lemma 9.7, (applied taking $B_n = 0$ for all n) it is enough to determine the existence of a $\delta > 0$ (a priori depending on t) for which

$$\mathbf{E}\left[\left(\int_0^t \lambda_N \tilde{\mathcal{N}}_0^N[h^N(s)]\,\mathrm{d}s\right)^2\right] > \delta. \tag{5.13}$$

For this we will exploit the same strategy as in the proof of Proposition 5.3 and Corollary 5.4. To be more precise, we consider the Laplace transform of $\mathbf{E}[(\int_0^t \lambda_N \tilde{\mathcal{N}}_0^N[h^N(s)]\,\mathrm{d}s)^2]$, to which we apply Lemmas 5.1 and 5.2. In the variational problem we take G to be $\theta \tilde{H}_0^N$, where θ is a positive constant and $\tilde{H}_0^N$ is the solution of the Poisson equation $\mathcal{L}_0 \tilde{H}_0^N(\eta) = \lambda_N \mathcal{N}^N[\eta]$ obtained in (4.21). We now need to control the four terms in the brackets of the right-hand side of (5.2). We treat the second summand as in (5.7), and, since $\mathcal{A}_-^N \tilde{H}_0^N = 0$, we are left to consider the first and the third, which give

$$\langle \lambda_N \tilde{\mathcal{N}}_0^N, \tilde{H}_0^N \rangle = 2\theta \tilde{\lambda}_N^2 \sum_{|\ell|\leq N} \frac{c(\ell,\ell)^2}{|\ell|^6} \xrightarrow{N\to\infty} 2\theta C\pi$$

for $\tilde{\lambda}_N = \lambda_N / 2\pi$, while

$$\begin{aligned}\|(\lambda - \mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}_+^N \tilde{H}_0^N\|_{L^2(\eta)}^2 &= \theta^2 \tilde{\lambda}_N^4 \sum_{\ell+m=n} \frac{c(n,n)^2}{|n|^6} \frac{c(\ell,m)^2}{(\lambda + \frac{1}{2}(|\ell|^2 + |m|^2 + |n|^2)|m|^2|\ell|^2} \\ &\lesssim \theta^2 \lambda_N^4 \left(\sum_{|n|\leq N} \frac{1}{|n|^2}\right)^2 \lesssim \theta^2.\end{aligned}$$

Hence, following the same steps as in the proof of Proposition 5.3, we conclude that there exists $t > 0$ and $\delta > 0$ (a priori depending on t) for which (5.13) holds and the proof is concluded. □

6. Further consequences of the Itô trick. In this section we want to make some further observation on the martingales appearing on the right-hand side of (4.6). This is done in order to shed some light on the behaviour we might expect for these limit points and could represent a starting point for their characterisation.

We want to analyse the martingale associated to the solution of the Poisson equation (4.1) for V given by $\lambda_N \mathcal{N}^N[\eta](\varphi)$, where φ is some test function, say $\varphi \in H^1$. We define

$$\mathcal{M}_t^N(\varphi) \stackrel{\text{def}}{=} \nu_N^{\frac{1}{2}} M_t^N\big(H^N[u^N](\varphi)\big), \tag{6.1}$$

where the definition of the martingale M^N on the right-hand side and $H^N(\varphi)$ can be found in (4.3) and (4.9), respectively. In the following proposition we show that, upon choosing the renormalising constants in such a way that the right-hand side of (4.10) is uniformly bounded in the limit as $N \to \infty$, $\mathcal{M}_t^N(\varphi)$ converges to a Brownian motion.

PROPOSITION 6.1. *Let $\varphi \in H^1$ and, for any $N \in \mathbb{N}$, $\{\mathcal{M}_t^N(\varphi)\}_{t\geq 0}$ be the martingale defined in* (6.1). *Let $C > 0$ be a real constant for which* (4.16) *holds. Then, the sequence of martingales $\{\mathcal{M}^N(\varphi)\}_N$ converges in distribution to a Brownian motion whose quadratic variation is given by $tQ(\varphi)$, where $Q(\varphi)$ is defined as*

$$Q(\varphi) = 4C\pi \|\varphi\|_{1,2}^2. \tag{6.2}$$

PROOF. According to [16], Theorem 7.1.4, since for every N the martingale $\mathcal{M}^N(\varphi)$ is continuous, the proof of the statement follows once we show that its quadratic variation converges in probability to a deterministic function of time. Now, the quadratic variation of $\mathcal{M}^N(\varphi)$ is explicit and can be deduced by (4.4). The choice of the renormalisation constants in (4.16) and (4.14) imply that $\langle \mathcal{M}_\cdot^N(\varphi)\rangle$ has bounded moments of all orders, so we are left to prove that its variance vanishes in the limit $N \to \infty$ and show (6.2). Notice that, by (4.12), we have

$$\mathbf{E}\big[\langle \mathcal{M}_\cdot^N(\varphi)\rangle_t\big] = t\nu_N \mathbb{E}\big[\mathcal{E}^N\big(H^N(\varphi)\big)(\eta)\big]$$

and

$$\begin{aligned}\mathbf{E}\big[\langle \mathcal{M}_\cdot^N(\varphi)\rangle_t^2\big]^{\frac{1}{2}} &= \nu_N \mathbf{E}\bigg[\bigg(\int_0^t \mathcal{E}^N\big(H^N[u^N(s)](\varphi)\big)\,\mathrm{d}s\bigg)^2\bigg]^{\frac{1}{2}} \\ &\lesssim t\nu_N \mathbb{E}\big[\mathcal{E}^N\big(H^N(\varphi)\big)(\eta)^2\big]^{\frac{1}{2}},\end{aligned}$$

and we can compute the last expectation explicitly. Wick's theorem for the product of Gaussian random variables [28], Theorem 1.36, gives

$$\nu_N^2 \mathbb{E}\big[\mathcal{E}^N\big(H^N(\varphi)\big)(\eta)^2\big] = \nu_N^2\big(\mathbb{E}\big[\mathcal{E}^N\big(H^N(\varphi)\big)(\eta)\big]\big)^2 + R_N(\varphi),$$

where the remainder $R_N(\varphi)$ is given by

$$R_N(\varphi) \stackrel{\text{def}}{=} 4^4\lambda_N^4\nu_N^{-2} \sum_{\substack{\ell_1,\ell_2\in\mathbb{Z}_0^2 \\ k_1,\dots,k_4\in\mathbb{Z}_0^2}} \bigg(\prod_{\substack{i=1,2 \\ j=1,\dots,4}} |\ell_i| \tilde{\mathcal{K}}^N_{\ell_i,k_j-\ell_i}\bigg)\varphi_{k_1}\varphi_{-k_2}\varphi_{k_3}\varphi_{-k_4}(\mathbb{1}_A + \mathbb{1}_B)$$

and, to shorten the notation, we set $\tilde{\mathcal{K}}^N_{\ell,m} \stackrel{\text{def}}{=} (|\ell|^2 + |m|^2)^{-1}\mathcal{K}^N_{\ell,m}$, $\mathcal{K}^N$ as in (3.5); the two sets appearing at the right-hand side are $A = \{\ell_1, \ell_2, k_1, \dots, k_4 \in \mathbb{Z}_0^2 : k_1 - \ell_1 + k_3 - \ell_2 = 0 = \ell_1 - k_2 + k_4 - \ell_2\}$ and $B = \{\ell_1, \ell_2, k_1, \dots, k_4 \in \mathbb{Z}_0^2 : k_1 - \ell_1 + \ell_2 - k_4 = 0 = \ell_1 - k_2 + k_3 - \ell_2\}$. The two terms can the treated similarly, so we will focus on the second. Brutally bounding $|\tilde{\mathcal{K}}^N_{\ell_i,k_j-\ell_i}| \lesssim |k_j|(|\ell_i||k_j - \ell_i|)^{-1}$ and using that, when restricted to B, we can express ℓ_2 and

k_4 in terms of ℓ_1 and k_1, k_2, k_3, respectively, we can estimate the above sum by

$$\lambda_N^4 \nu_N^{-2} \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}_0^2 \\ k_4=k_3+k_2-k_1}} \prod_{i=1}^4 |k_i| \Big(\sum_{\ell\in\mathbb{Z}_0^2} \frac{1}{|k_1-k_2+\ell|^2|\ell|^2} \Big) \varphi_{k_1}\varphi_{-k_2}\varphi_{k_3}\varphi_{-k_4}$$
$$\lesssim \lambda_N^4 \nu_N^{-2} \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}_0^2 \\ k_4=k_3+k_2-k_1}} \prod_{i=1}^4 |k_i| \varphi_{k_1}\varphi_{-k_2}\varphi_{k_3}\varphi_{-k_4}$$
$$\lesssim \lambda_N^4 \nu_N^{-2} \|\varphi\|_{1,2}^4,$$

where the passage from the first to the second line is due to the fact that the inner sum converges while the second is a consequence of Young's convolution inequality.

Therefore, collecting the observations made so far, we have

$$\mathrm{Var}\big[\langle \mathcal{M}_\cdot^N(\varphi)\rangle_t\big] = \mathbf{E}\big[\langle \mathcal{M}_\cdot^N(\varphi)\rangle_t^2\big] - \mathbf{E}\big[\langle \mathcal{M}_\cdot^N(\varphi)\rangle_t\big] \lesssim t^2 \lambda_N^4 \nu_N^{-2} \|\varphi\|_{1,2}^4.$$

Now, by (4.16), the right-hand side converges to 0 as N tends to ∞, for every $t > 0$. In particular, by dominated convergence this means that $\langle \mathcal{M}_\cdot^N(\varphi)\rangle_t$ converges to the limit of its expectation (i.e., to a deterministic function of time). Therefore, it remains to identify $\lim_N \mathbf{E}[\langle \mathcal{M}_\cdot^N(\varphi)\rangle_t]$ for which we need to refine the estimates in the proof of Lemma 4.3. In (4.12) we showed the following identity:

$$\nu_N \mathbb{E}\big[\mathcal{E}^N\big(H^N(\varphi)\big)(\eta)\big] = 4^2 \sum_{k\in\mathbb{Z}_0^2} |k|^2 \Big(\lambda_N^2 \nu_N^{-1} \sum_{\ell\in\mathbb{Z}_0^2} \frac{|\ell|^2}{|k|^2} \big|\tilde{\mathcal{K}}_{\ell,k-\ell}^N\big|^2 \Big) |\varphi_k|^2$$

and the part to control is the one in brackets. By Riemann-sum approximation we can rewrite the latter (for $k \in \mathbb{Z}_0^2$ fixed such that $|k| \le N$) as

$$\lambda_N^2 \nu_N^{-1} \sum_{\ell\in\mathbb{Z}_0^2} \frac{|\ell|^2}{|k|^2} \big|\tilde{\mathcal{K}}_{\ell,k-\ell}^N\big|^2 = \tilde{\lambda}_N^2 \nu_N^{-1} \sum_{|\ell|,|k-\ell|\le N} \frac{c(\ell,k-\ell)^2}{|\ell|^2(|\ell|^2+|k-\ell|^2)}$$
$$= \tilde{\lambda}_N^2 \nu_N^{-1} \sum_{|\ell|,|k-\ell|\le N} \frac{1}{|k|^2} f_k\Big(\frac{\ell}{|k|}\Big)$$
$$\approx \tilde{\lambda}_N^2 \nu_N^{-1} \int_{\frac{3}{2}\le|x|\le\frac{N}{|k|}} \frac{c(x,\tilde{k}-x)^2}{|x|^2(|x|^2+|\tilde{k}-x|^2)^2}\,\mathrm{d}x,$$

where f_k was defined in (4.13) and $\tilde{k} = k/|k|$, while $\tilde{\lambda}_N = \lambda_N/2\pi$. Since $\tilde{k}$ has Euclidean norm 1, let $\theta_k \in [0, 2\pi)$ be such that $\tilde{k} = (\cos\theta_k, \sin\theta_k)$, so that, by passing to polar coordinates (and exploiting basic trigonometric identities), the integral becomes

$$\tilde{\lambda}_N^2 \nu_N^{-1} \int_0^{2\pi} \int_{\frac{3}{2}}^{\frac{N}{|k|}} \frac{(r\cos(2\theta)-\cos(\theta+\theta_k))^2}{(2r^2-2r\cos(\theta-\theta_k)+1)^2} r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$\approx \tilde{\lambda}_N^2 \nu_N^{-1} \int_0^{2\pi} \int_{\frac{3}{2}}^{\frac{N}{|k|}} \frac{r\cos^2(2\theta)}{4(r-\cos(\theta-\theta_k))^2}\,\mathrm{d}r\,\mathrm{d}\theta,$$

and the last approximation holds since the integrals, in which at the numerator r is raised to a power smaller than 2, are uniformly bounded in N and therefore they converge to 0 because of the prefactor $\lambda_N^2 \nu_N^{-1}$. Now, adding and subtracting $\cos(\theta-\theta_k)\cos^2(2\theta)$ at the numerator and arguing, as above, we can further approximate the quantity above by

$$\begin{aligned}&\tilde{\lambda}_N^2 \nu_N^{-1} \int_0^{2\pi} \int_{\frac{3}{2}}^{\frac{N}{|k|}} \frac{\cos^2(2\theta)}{4(r-\cos(\theta-\theta_k))} \, \mathrm{d}r \, \mathrm{d}\theta \\ &\quad = \frac{1}{4} \int_0^{2\pi} \cos^2(2\theta) \left(\tilde{\lambda}_N^2 \nu_N^{-1} \log\left(\frac{N/|k| - \cos(\theta-\theta_k)}{3/2 - \cos(\theta-\theta_k)} \right) \right) \mathrm{d}\theta \stackrel{N\to\infty}{\longrightarrow} C \frac{\pi}{4},\end{aligned}$$

where in the last passage we used dominated convergence theorem and the proportionality constant $C > 0$ in (4.16). Hence, we conclude that

$$\mathbf{E}\left[\langle \mathcal{M}^N_\cdot(\varphi) \rangle_t\right] \stackrel{N\to\infty}{\longrightarrow} 4\pi C t \|\varphi\|_{1,2}^2$$

which completes the proof. □

REMARK 6.2. Notice that the previous proposition allows to understand the behaviour of the two martingales on the right-hand side of (4.2). In order to obtain a characterisation of the limit points of the nonlinearity, we would need to understand the joint correlation between them. The problem is not easy and out of reach of the techniques of the present paper.

As an easy corollary of the previous proposition, we show how to construct the time average of nonlinear unbounded functionals of the solution to the stochastic heat equation, purely by martingale techniques. The stochastic heat equation we have in mind is the stochastic PDE, whose expression is given by

$$\partial_t X = \frac{1}{2} \Delta X + (-\Delta)^{\frac{1}{2}} \xi, \qquad X(0,\cdot) = \eta, \tag{6.3}$$

where ξ and η are, respectively, a space-time and a space white noise on $\mathbb{T}^2$. Existence and uniqueness of a probabilistically strong solution is well known and a martingale characterisation can be found, for example, in [34], Appendix D. We are now ready to state and prove the following.

COROLLARY 6.3. *Let X be the unique stochastic process solving the stochastic heat equation (6.3) started at 0 from the stationary measure η, a space white noise on $\mathbb{T}^2$. Assume there exists $C > 0$ such that λ_N satisfies (5.3). Then, for any $\varphi \in H^1$, $\{\mathcal{B}_t^N[X^N](\varphi)\}_{t\geq 0}$ converges in distribution to a Brownian motion independent from $B_t(\varphi) \stackrel{\text{def}}{=} \int_0^t (-\Delta)^{\frac{1}{2}} \xi(\mathrm{d}s, \varphi)$ whose quadratic variation is given in (6.2).*

PROOF. Let $\varphi \in H^1$ and $H^N[\eta](\varphi)$ be the solution of the Poisson equation determined in (4.9). Notice that the generator of the process X^N is $\mathcal{L}_0$ which coincides with $\mathcal{L}_0^N$ once we choose $\nu_N = 1$. $H^N(\varphi)$ is a cylinder function and, therefore (for N fixed), belongs to the domain of $\mathcal{L}_0$. By Dynkin's formula we have

$$H^N[X(t)](\varphi) = H^N[X(0)](\varphi) + \int_0^t \mathcal{L}_0 H^N[X(s)](\varphi) \, \mathrm{d}s + M_t^N(\varphi),$$

where the martingale $M_t^N(\varphi)$ equals the right-hand side of (6.1) upon replacing u^N by X^N (to see this, apply Itô's formula to $H^N[X(t)](\varphi)$). From the previous we deduce

$$\int_0^t \lambda_N \mathcal{N}^N[X(s)](\varphi) \, \mathrm{d}s = H^N[X(0)](\varphi) - H^N[X(t)](\varphi) - M_t^N(\varphi),$$

so that it suffices to study the terms appearing on the right-hand side. Since the proof of Proposition 6.1 does not depend on the law of u^N as a process, but only on its invariant measure, we conclude that $\{M_t^N(\varphi)\}_{t\geq 0}$ converges to a Brownian motion with the covariance prescribed by the statement. Concerning the boundary conditions, notice that

$$\mathbf{E}[H^N[X(t)](\varphi)^2] = \mathbf{E}[H^N[X(0)](\varphi)^2] = \mathbb{E}[H^N[\eta](\varphi)^2]$$

since X is started at the invariant measure. But now, by Wick's theorem we have

$$\mathbb{E}[H^N[\eta](\varphi)^2] \lesssim \lambda_N^2 \sum_k |k|^2 \sum_{\ell+m=k} \frac{1}{(|\ell|^2+|m|^2)^2} |\varphi_k|^2 \lesssim \lambda_N^2 \|\varphi\|_{1,2}^2 \tag{6.4}$$

which converges to 0. Collecting the observations made so far, we see that $\{\mathcal{B}_t^N[X^N](\varphi)\}_{t\geq 0}$ converges to a Brownian motion with the covariance prescribed by the statement.

Independence is a consequence of the fact that, for any N, $\mathcal{B}_t^N[X^N](\varphi)$ belongs to the second homogeneous Wiener chaos associated to ξ and η together. Hence, $\mathcal{B}_t^N[X^N](\varphi)$ and ξ are uncorrelated, and, since the former is bounded in L^p, their covariance also converges to 0. Since the limit of $\mathcal{B}_t^N[X^N](\varphi)$ is Gaussian and it is uncorrelated from ξ, the two are independent. □

REMARK 6.4. The interest in the previous corollary is twofold. First, it provides an example of a situation in which a deterministic ill-posed operation (in this case the AKPZ nonlinearity $\mathcal{N}^N$) when suitably rescaled and evaluated at a Gaussian measure, produces a *new* noise *independent* from the one with which we started. Similar phenomena are observed in situations in which the nonlinearity becomes *critical* (in terms of regularity) for the equation and have been observed also in the context of the Isotropic KPZ equation; see [8, 17].

On a different note, Corollary 6.3, provides a purely probabilistic construction of the first (relevant) stochastic process one would need to analyse in the context of regularity structures [26], also informally referred to as "cherry," namely, $\partial_x(X^2)$, and (one of) the reason why the theory is not expected to work if applied to the *two*-dimensional (A)KPZ. Indeed, without entering the details, the approach is based on the ability of performing a partial expansion of the solution around the solution of the linearised equation, in which the terms appearing can be obtained via a Picard iteration and are increasingly more regular. The expansion is *partial* since from some point on there is no more gain in regularity and a deterministic argument needs to be invoked in order to conclude the fixed point argument.

Now, the problem here is that the Picard iteration does not provide any gain in regularity, and, consequently, there is also no point where one could stop. Hence, one would end up with an infinite series of stochastic processes, each of which could, in principle (as it happens for the cherry), converge to a new white noise, potentially independent of the others, and there is no hope for such a series to be summable.

APPENDIX A: AN ALTERNATIVE PROOF OF THE INVARIANCE OF THE SPATIAL WHITE NOISE

In this section we give an alternative proof of Lemma 3.1 which boils down to give an alternative proof of (3.9). To that end, it suffices to notice that the right-hand side of (3.9) is simply the scalar product of the nonlinearity evaluated at η, that is, $\mathcal{N}^N(\eta)$ and η itself. Let ϱ^N be a function, such that $\varrho_k^N = 1$ for all $k \neq 0$ such that $|k| \leq N$ and 0 elsewhere. Setting

$\mu^N \stackrel{\text{def}}{=} (-\Delta)^{-\frac{1}{2}}\varrho^N * \eta$, we have

$$\begin{aligned}\sum_{m,l\in\mathbb{Z}_0^2} \mathcal{K}^N_{m,l}\eta_m\eta_l\eta_{-m-l} &= \langle \mathcal{N}^N(\eta), \varrho^N * \eta\rangle \\ &= \langle(-\Delta)\big((\partial_1\mu^N)^2 - (\partial_2\mu^N)^2\big), \mu^N\rangle \\ &= \sum_{i=1}^2 \langle \partial_i(\partial_i\mu^N)^2, \partial_i\mu^N\rangle + \sum_{\substack{i,j\in\{1,2\}\\ i\neq j}} (-1)^i \langle \partial_i(\partial_j\mu^N)^2, \partial_i\mu^N\rangle \\ &= \frac{1}{3}\sum_{i=1}^2 \langle \partial_i(\partial_i\mu^N)^3, 1\rangle + 2\sum_{\substack{i,j\in\{1,2\}\\ i\neq j}} (-1)^i \langle \partial_i\mu^N\partial_j\mu^N\partial_{i,j}\mu^N), 1\rangle,\end{aligned}$$

from which we see that the first sum is 0, since each summand is, while the second sum vanishes because the two summands are the same but they have opposite sign.

APPENDIX B: LAPLACE TRANSFORM AND SHORT-TIME BEHAVIOUR

In this appendix we provide a proof of the following lemma.

LEMMA B.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nonnegative function. Assume there exist two strictly positive constants, $c < C$, such that*

$$f(t) \leq Ct \qquad \text{for all } t \geq 0, \tag{B.1}$$

$$\int_0^\infty e^{-\lambda t} f(t)\,\mathrm{d}t \geq \frac{c}{\lambda^2} \qquad \text{for all } \lambda > 0, \tag{B.2}$$

then, $f(t) \sim t$ as t converges to 0.

PROOF. Notice that it suffices to prove that $\limsup_{t\to 0} t^{-1} f(t) > 0$. We argue by contradiction. Assume $\lim_{t\to 0} t^{-1} f(t) = 0$, so that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all $t \leq \delta(\varepsilon)$, $f(t) < \varepsilon t$. Let $\varepsilon > 0$ (to be fixed later) and $a, \lambda > 0$ be such that $a/\lambda \leq \delta(\varepsilon)$. Then, an easy computation shows that

$$\int_0^{a/\lambda} e^{-\lambda t} f(t)\,\mathrm{d}t \leq \varepsilon \int_0^{a/\lambda} e^{-\lambda t} t\,\mathrm{d}t = \frac{\varepsilon(1 - ae^{-a} - e^{-a})}{\lambda^2} \leq \frac{\varepsilon}{\lambda^2},$$

while, by (B.1), we have

$$\int_{a/\lambda}^\infty e^{-\lambda t} f(t)\,\mathrm{d}t \leq C \int_{a/\lambda}^\infty e^{-\lambda t} t\,\mathrm{d}t = C\frac{(a+1)e^{-a}}{\lambda^2}$$

which implies

$$\int_0^\infty e^{-\lambda t} f(t)\,\mathrm{d}t \leq \frac{\varepsilon + C(a+1)e^{-a}}{\lambda^2}.$$

But now, if we choose $\varepsilon = c/4$, where c is the constant in (B.2), and a and λ sufficiently large so that $a/\lambda < \delta(c/4)$ and $C(a+1)e^{-a} < c/4$, then, by (B.2), we obtain the desired contradiction. □

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