Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2021, Vol. 57, No. 1, 169–194
https://doi.org/10.1214/20-AIHP1075
© Association des Publications de l'Institut Henri Poincaré, 2021

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES

# Derivation of viscous Burgers equations from weakly asymmetric exclusion processes

M. Jara[a,*], C. Landim[a,b,†] and K. Tsunoda[c]

[a] *IMPA, Estrada Dona Castorina 110, CEP 22460 Rio de Janeiro, Brasil. E-mail:* [*] *mjara@impa.br;* [†] *landim@impa.br*
[b] *CNRS UMR 6085, Université de Rouen, Avenue de l'Université, BP.12, Technopôle du Madrillet, F76801 Saint-Étienne-du-Rouvray, France*
[c] *Department of Mathematics, Osaka University, Osaka, 560-0043, Japan. E-mail:* *k-tsunoda@math.sci.osaka-u.ac.jp*

Received 19 July 2019; revised 26 May 2020; accepted 29 May 2020

**Abstract.** We consider weakly asymmetric exclusion processes whose initial density profile is a small perturbation of a constant. We show that in the diffusive time-scale, in all dimensions, the density defect evolves as the solution of a viscous Burgers equation.

**Résumé.** Nous examinons le processus d'exclusion simple faiblement asymétrique partant d'une perturbation d'un profil de densité constant. Nous montrons qu'à l'échelle diffusive, en toute dimension, la perturbation évolue selon la solution d'une équation de Burgers visqueuse.

*MSC2020 subject classifications:* Primary 60K35; secondary 82C22

*Keywords:* Viscous Burgers equations; Weakly asymmetric exclusion processes; Incompressible limits

## 1. Introduction

One of the main open problems in nonequilibrium statistical mechanics is the derivation of the hydrodynamic equations of fluids, the so-called Euler and Navier–Stokes equations, from the microscopic Hamiltonian dynamics.

In contrast with the Euler equations, the Navier–Stokes equations are not scale invariant. They are obtained as corrections of the Euler equations by adding a small viscosity, materialized as a second order derivative of the conserved quantities.

Almost thirty years ago, Esposito, Marra and Yau [5,6] initiated the investigation of the time evolution of small perturbations of the density profile around the hydrodynamic limit for stochastic systems, deriving the incompressible limit for asymmetric simple exclusion processes in dimension $d \geq 3$.

To describe their result, fix a scaling parameter $n \in \mathbb{N}$, and denote by $\mathbb{T}_n^d = (\mathbb{Z}/n\mathbb{Z})^d$ the $d$-dimensional discrete torus with $n^d$ points. Elements of $\mathbb{T}_n^d$ are represented by the letters $x$, $y$, $z$. Denote the configuration space by $\Omega_n = \{0,1\}^{\mathbb{T}_n^d}$ and by $\eta = \{\eta_x : x \in \mathbb{T}_n^d\}$ the elements of $\Omega_n$, which describes a configuration on $\mathbb{T}_n^d$ such that $\eta_x = 1$ if there is a particle at $x \in \mathbb{T}_n^d$ and $\eta_x = 0$ otherwise. For a configuration $\eta \in \Omega_n$, let $\sigma^{x,y}\eta$ be the configuration of particles obtained from $\eta$ by exchanging the occupation variables $\eta_x$ and $\eta_y$:

$$\left(\sigma^{x,y}\eta\right)_z = \begin{cases} \eta_y & \text{if } z = x, \\ \eta_x & \text{if } z = y, \\ \eta_z & \text{otherwise.} \end{cases}$$

Consider the asymmetric exclusion process on $\Omega_n$. This is the Markov chain whose generator, denoted by $L_n^A$, applied to a function $f : \Omega_n \to \mathbb{R}$ is given by

$$\left(L_n^A f\right)(\eta) = \sum_{x \in \mathbb{T}_n^d} \sum_{j=1}^{d} r_{x,j}(\eta)\left\{f\left(\sigma^{x,x+e_j}\eta\right) - f(\eta)\right\}, \tag{1.1}$$

where $\{e_j : 1 \le j \le d\}$ represents the canonical basis of $\mathbb{R}^d$, $r_{x,j}(\eta) = p_j\eta_x(1-\eta_{x+e_j}) + q_j\eta_{x+e_j}(1-\eta_x)$ and $0 \le p_j \le 1$, $q_j = 1 - p_j$.

Denote by $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d)$ the points of the $d$-dimensional continuous torus $\mathbb{T}^d = [0,1)^d$ and by $\nabla F$ the gradient of a function $F : \mathbb{T}^d \to \mathbb{R}$, $\nabla F = (\partial_{\theta_1} F, \dots, \partial_{\theta_d} F)$. It is well known [11,20], that in the hyperbolic scaling the density profile evolves according to the inviscid Burger's equation

$$\partial_t u + \mathbf{m} \cdot \nabla \sigma_0(u) = 0,$$

where $\sigma_0(\alpha) = \alpha(1-\alpha)$ is the mobility and $\mathbf{m}$ is the vector whose coordinates are given by $m_j = p_j - q_j$.

In dimension $d \ge 3$, the macroscopic current $\mathbf{m}\sigma_0(u)$ is expected to have a correction of order $1/n$ and be given by $\mathbf{m}\sigma_0(u) - (1/n)\sum_k a_{j,k}(u)\partial_{\theta_k} u$ for some diffusion coefficient $a$. If this is the case, the partial differential equation which describes the evolution of the density becomes

$$\partial_t u + \mathbf{m} \cdot \nabla \sigma_0(u) = \frac{1}{n} \sum_{j,k} \partial_{\theta_j}\big(a_{j,k}(u)\partial_{\theta_k} u\big).$$

If we start from a density which is a $(1/n)$-perturbation of the constant profile equal to $1/2$, $u_0(\boldsymbol{\theta}) = (1/2) + \epsilon_n v_0(\boldsymbol{\theta})$, where $\epsilon_n = 1/n$, if we rescale time by an extra factor $n$ and assume that the density profile remains at all times a $(1/n)$-perturbation of the constant profile equal to $1/2$, $u(t,\boldsymbol{\theta}) = (1/2) + \epsilon_n v(t,\boldsymbol{\theta})$, as $\sigma_0'(1/2) = 0$, a Taylor expansion yields that the perturbation $v$ is expected to solve the viscous Burgers equation

$$\partial_t v = \mathbf{m} \cdot \nabla v^2 + \sum_{j,k} a_{j,k}(1/2)\partial^2_{\theta_j,\theta_k} v. \tag{1.2}$$

This is the content of the main result of Esposito, Marra and Yau [5,6] which we now state. Note that one can consider a perturbation around a general constant profile $\alpha \in (0,1)$ by performing a Galilean transformation [see Remark 2.6].

Recall that a function $f : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$ is said to be a local function or a cylinder function if it depends on the configuration $\eta$ only through a finite number of coordinates.

Denote by $\{\tau_x : x \in \mathbb{Z}^d\}$ the group of translations acting on $\Omega_n$: For a configuration $\eta \in \Omega_n$, $\tau_x\eta$ is the configuration given by $(\tau_x\eta)_z = \eta_{x+z}$, where the sum is taken modulo $n$. We extend the translations to functions $f : \Omega_n \to \mathbb{R}$ by setting $(\tau_x f)(\eta) = f(\tau_x\eta)$, $x \in \mathbb{Z}^d$, $\eta \in \Omega_n$.

Let $\nu_\alpha$, $0 \le \alpha \le 1$, be the product measure on $\{0,1\}^{\mathbb{Z}^d}$ with density $\alpha$. For a continuous function $u : \mathbb{T}^d \to [0,1]$, denote by $\nu^n_{u(\cdot)}$ the Bernoulli product measure on $\Omega_n$ with marginal density $u(x/n)$:

$$\nu^n_{u(\cdot)}\big\{\eta(x) = 1\big\} = u(x/n), \quad x \in \mathbb{T}^d_n. \tag{1.3}$$

Fix a density $v_0 : \mathbb{T}^d \to \mathbb{R}$, and let $\nu^n_t$, $t \ge 0$, be the measure $\nu^n_{(1/2)+\epsilon_n v(t,\cdot)}$, where $v(t,\boldsymbol{\theta})$ is the solution of equation (1.2) with initial condition $v_0$.

Denote by $\eta^n(t)$ the Markov chain on $\Omega_n$ induced by the generator $n^2 L^A_n$, where $L^A_n$ has been introduced in (1.1). Note that time has been rescaled diffusively. For a probability measure $\mu$ on $\Omega_n$, denote by $\mathbb{P}_\mu$ the distribution of the process $\eta^n(t)$ starting from $\mu$. Expectation with respect to $\mathbb{P}_\mu$ is represented by $\mathbb{E}_\mu$.

Fix a smooth density profile $v_0 : \mathbb{T}^d \to \mathbb{R}$, and distribute particles on $\mathbb{T}^d_n$ according to $\nu^n_0 = \nu^n_{(1/2)+\epsilon_n v_0(\cdot)}$. Then, in dimension $d \ge 3$, for every $t > 0$, continuous function $G : \mathbb{T}^d \to \mathbb{R}$, and cylinder function $\Psi : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}_{\nu^n_0}\left[\frac{1}{n^{d-1}}\left|\sum_{x\in\mathbb{T}^d_n} G(x/n)\big\{(\tau_x\Psi)\big(\eta^n(t)\big) - E_{\nu_{\rho_n(t,x)}}[\Psi]\big\}\right|\right] = 0, \tag{1.4}$$

where $\rho_n(t,x) = (1/2) + \epsilon_n v(t,x/n)$ and, recall, $\nu_\alpha$ stands for the Bernoulli product measure with density $\alpha$.

The proof of this result is based on a sharp estimate of the relative entropy. Let $\Sigma_n$ be the set of all probability measures on $\Omega_n$. For a reference measure $\nu \in \Sigma_n$, define the relative entropy $H_n(\cdot|\nu)$ with respect to $\nu$ by

$$H_n(\mu|\nu) = \sup_f \left\{\int_{\Omega_n} f\,d\mu - \log\int_{\Omega_n} e^f\,d\nu\right\},$$

where the supremum is carried over all functions $f : \Omega_n \to \mathbb{R}$. It is well known that

$$H_n(\mu|\nu) = \int_{\Omega_n} \frac{d\mu}{d\nu}\log\frac{d\mu}{d\nu}\,d\nu, \tag{1.5}$$

if $\mu$ is absolutely continuous with respect to $\nu$, while $H_n(\mu|\nu) = \infty$ if this is not the case.

Denote by $\{S^n_t : t \ge 0\}$ the semigroup of the Markov chain $\eta^n_t$ rescaled diffusively. Hence, $\mu^n S^n_t$ represents the state of the process at time $t$ provided the initial state is $\mu^n$. Esposito, Marra and Yau [5,6] proved that in dimension $d \ge 3$,

$$\lim_{n\to\infty} \frac{1}{n^{d-2}} H_n\big(\mu^n S^n_t | \nu^n_t\big) = 0,$$

where $\nu^n_t$ has been introduced just below (1.3). It is not difficult to deduce (1.4) from the previous bound.

The result is restricted to $d \ge 3$, as in dimension 1 and 2 Gaussian fluctuations of order $n^{-d/2}$ appear around the hydrodynamic limit and $n^{-d/2}$ is at least of the order of $1/n$ in dimensions 1 and 2. For more details, see [5, Section 1].

In this article, we pursue the investigation of the time evolution in the hydrodynamic limit of densities in the vicinity of constant profiles by considering weakly asymmetric exclusion processes. These are Markov processes on $\Omega_n$ whose generator $L_n$ acts on cylinder functions as $L_n f = n^2 L^S_n f + n L^T_n f$, where $L^S_n$ represents the generator of the speed-change, symmetric exclusion process given by

$$\big(L^S_n f\big)(\eta) = \sum_{x\in\mathbb{T}^d_n} \sum_{j=1}^{d} c_j(\tau_x\eta)\big\{ f\big(\sigma^{x,x+e_j}\eta\big) - f(\eta)\big\}, \tag{1.6}$$

and $L^T_n$ the generator of the speed-change totally asymmetric exclusion process given by

$$\big(L^T_n f\big)(\eta) = \sum_{x\in\mathbb{T}^d_n} \sum_{j=1}^{d} \mathbf{m}_j c_j(\tau_x\eta)\eta_x(1-\eta_{x+e_j})\big\{ f\big(\sigma^{x,x+e_j}\eta\big) - f(\eta)\big\}. \tag{1.7}$$

In this formula, $c_j : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$, $1 \le j \le d$, are cylinder functions and $\mathbf{m} = (\mathbf{m}_1, \dots, \mathbf{m}_d)$ is a fixed vector in $\mathbb{R}^d$. In this paper, we assume that $c_j$ does not depend on the occupation variables $\eta_0$ and $\eta_{e_j}$ and satisfies the gradient conditions (2.1). Under these conditions, one can see that the generator $L_n$ is invariant with respect to the Bernoulli measures.

Note that the symmetric generator has been speeded-up by $n^2$, while the asymmetric one by $n$. In other words, we consider a weakly asymmetric system in a diffusive time scale $n^2$ with asymmetry strength of order $1/n$.

The hydrodynamic equation of the weakly asymmetric speed-change exclusion process is given by

$$\partial_t u = \nabla\cdot\big[D(u)\nabla u\big] - \nabla\cdot\big[\sigma(u)\mathbf{m}\big],$$

where the matrices $D(\cdot)$ and $\sigma(\cdot)$ represent the diffusivity and the mobility, respectively. By further accelerating the symmetric part of the dynamics by $b_n$, the asymmetric one by $a_n$, and by assuming that the density is an $\epsilon_n$-perturbation of a constant $\alpha$, viz. $u(t,\theta) = \alpha + \epsilon_n v(t,\theta)$, we get from the previous equation that

$$\begin{aligned}\partial_t v = {}& b_n\nabla\cdot\big[D(\alpha)\nabla v\big] + b_n\epsilon_n\nabla\cdot\big[vD'(\alpha)\nabla v\big] \\ &- a_n\nabla\cdot\big[v\sigma'(\alpha)\mathbf{m}\big] - (1/2)a_n\epsilon_n\nabla\cdot\big[v^2\sigma''(\alpha)\mathbf{m}\big].\end{aligned}$$

There are many ways to handle the right-hand side. One of them is to set $b_n = 1$, $a_n = \epsilon_n^{-1}$, and assume that $\sigma'(\alpha) = 0$. In this case, up to smaller order terms, the equation becomes

$$\partial_t v = \nabla\cdot\big[D(\alpha)\nabla v\big] - (1/2)\nabla\cdot\big[v^2\sigma''(\alpha)\mathbf{m}\big]. \tag{1.8}$$

Assume, therefore, that $\sigma'(\alpha) = 0$ for some $\alpha \in (0,1)$. Note that this $\alpha$ always exists since each entry of $\sigma$ is smooth and vanishes at 0 and 1. Consider the weakly asymmetric exclusion process in which the asymmetric part of the generator has been speeded-up by $a_n n$ [instead of $n$] for some sequence $a_n \to \infty$ and $a_n n^{-1} \to 0$. Note that the latter condition ensures that the operator $n^2[L^S_n + (a_n/n)L^T_n]$ becomes a Markovian generator for sufficiently large $n$. Denote by $v = v(t,\theta)$ the solution of (1.8) with a smooth initial condition $v_0 : \mathbb{T}^d \to \mathbb{R}$. Distribute particles on $\mathbb{T}^d_n$ according to $\nu^n_0 = \nu^n_{\alpha+\epsilon_n v_0(\cdot)}$, where $\epsilon_n = 1/a_n$. The first main result of this article states that under some hypotheses on $a_n$, for every $t > 0$, continuous function $G : \mathbb{T}^d \to \mathbb{R}$, and cylinder function $\Psi : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}_{\nu^n_0}\Bigg[\frac{1}{n^d\epsilon_n}\Bigg|\sum_{x\in\mathbb{T}^d_n} G(x/n)\big\{(\tau_x\Psi)\big(\eta^n(t)\big) - E_{\nu_{\rho_n(t,x)}}[\Psi]\big\}\Bigg|\Bigg] = 0, \tag{1.9}$$

where $\rho_n(t,x) = \alpha + \epsilon_n v(t, x/n)$.

As above, the proof of this result is based on an estimate of the relative entropy of the state of the process with respect to a product measure. We start the presentation of this bound with a remark which elucidates what is needed. In Lemma 2.1 below, we show that in order to single out an $\epsilon_n$-perturbation of the density around a constant profile we need the entropy of the state of the process with respect to the inhomogeneous product measure associated to the density profile $\alpha + \epsilon_n v(t, x/n)$ to be of an order much smaller than $n^d \epsilon_n^2$.

To state the entropy bound, denote by $d$ the dimension, and let $(g_d(n) : n \ge 1)$ be the sequences given by

$$g_d(n) = \begin{cases} n & \text{if } d = 1, \\ \log n & \text{if } d = 2, \\ 1 & \text{for } d \ge 3. \end{cases} \tag{1.10}$$

Following Jara and Menezes in [10], we prove in Theorem 2.2 that under certain assumptions on the initial profile $v_0$, the sequence $a_n$ and the initial distribution of particles, for all $t > 0$ there exists a finite constant $C = C(t)$, such that

$$H_n\big(\mu^n S_t^n | \nu_t^n\big) \le C n^{d-2} g_d(n),$$

where $\nu_t^n$ stands for the inhomogeneous product measure associated to the density profile $\alpha + \epsilon_n v(t, x/n)$. This entropy estimate and a simple argument, presented in the proof of Corollary 2.3, yield (1.9). Lemma 2.1 and (1.10) yield some restrictions on $\epsilon_n$ discussed in Remark 2.4 below.

We here mention related results, which establish the incompressible limits for interacting particle systems: Esposito, Marra and Yau [5,6], Quastel and Yau [19], Beltrán and Landim [1]. We also mention recent results, which study the entropy estimate as in Theorem 2.2. The entropy estimate as in Theorem 2.2 has been established in Jara and Menezes [9,10] to study the nonequilibrium fluctuations for interacting particle systems. By establishing a similar entropy estimate, Funaki and Tsunoda [7] derived the motion by mean curvature from Glauber-Kawasaki processes, and Jara and Landim [8] the stochastic heat equation from a stirring dynamics perturbed by a voter model, respectively.

We conclude this introduction mentioning two other ways to detect the evolution of small perturbations around the hydrodynamic limit. Dobrushin [2], Dobrushin, Pellegrinotti, Suhov and Triolo [4], Dobrushin, Pellegrinotti, Suhov [3] and Landim, Olla, Yau [15,16] investigated the first order correction to the hydrodynamic equation. Landim, Valle and Sued [17] examined the evolution of the density profile in the orthogonal direction to the drift when the initial condition is constant along the drift direction. Versions of these results might be problems for future investigation.

## 2. Notation and results

### 2.1. *Model*

Recall that we denote by $\{e_j : j = 1, \dots, d\}$ the canonical basis of $\mathbb{R}^d$. Fix cylinder functions $c_j : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}_+$, $1 \le j \le d$. Assume that $c_j$ does not depend on $\eta_0$, $\eta_{e_j}$ and that the gradient conditions are in force: For each $j$, there exist cylinder functions $g_{j,p}$ and finitely-supported signed measures $m_{j,p}$, $1 \le p \le n_j$, such that

$$c_j(\eta)[\eta_0 - \eta_{e_j}] = \sum_{p=1}^{n_j} \sum_{y \in \mathbb{Z}^d} m_{j,p}(y)(\tau_y g_{j,p})(\eta), \qquad \sum_{y \in \mathbb{Z}^d} m_{j,p}(y) = 0. \tag{2.1}$$

Denote by $\ell_0$ the size of the support of the measures $m_{j,p}$. This is the smallest integer such that

$$m_{j,p}(y) = 0 \quad \text{if } y \notin \Lambda_{\ell_0} := \{-\ell_0, \dots, \ell_0\}^d.$$

Let $L_n^S$ be the generator of the speed-change exclusion process in $\Omega_n$ introduced in (1.6), and let $L_n^T$ be the generator of the speed-change totally asymmetric exclusion process in $\Omega_n$, introduced in (1.7).

Recall that we denote by $\nu_\alpha = \nu_\alpha^n$, $0 \le \alpha \le 1$, the Bernoulli product measure on $\Omega_n$ or on $\{0,1\}^{\mathbb{Z}^d}$ with density $\alpha$. Since we assume that $c_j$ does not depend on $\eta_0, \eta_{e_j}$, for any $\alpha$, $L_n^S$ is reversible with respect to $\nu_\alpha$. Moreover, this assumption together with the gradient conditions (2.1) ensures that $L_n^T$ is invariant with respect to $\nu_\alpha$. For a cylinder function $g : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$, let $\widetilde{g} : [0,1] \to \mathbb{R}$ be the polynomial function given by

$$\widetilde{g}(\alpha) = E_{\nu_\alpha}[g], \quad \alpha \in [0,1]. \tag{2.2}$$

Denote by $D(\rho) = (D_{j,k}(\rho))_{1\le j,k\le d}$, the *diffusivity* of the exclusion process, the matrix whose entries are given by

$$D_{j,k}(\rho) = \sum_{p=1}^{n_j} D_p(j,k)\widetilde{g}'_{j,p}(\rho) \quad \text{where } D_p(j,k) = -\sum_y y_k m_{j,p}(y). \tag{2.3}$$

In this formula, $\widetilde{g}'_{j,p}$ represents the derivative of the function $\widetilde{g}_{j,p}$. This later one is obtained through equation (2.2) from the cylinder functions $g_{j,p}$ introduced in (2.1). We prove in Proposition 5.7 that $D(\rho)$ is a diagonal matrix:

$$\sum_{p=1}^{n_j} D_p(j,k)\widetilde{g}'_{j,p}(\rho) = 0 \quad \text{for } k \ne j. \tag{2.4}$$

Denote by $\sigma(\rho) = (\sigma_{i,j}(\rho))_{1\le i,j\le d}$ the *mobility*, the diagonal matrix whose entries are given by

$$\sigma_{j,j}(\rho) = \rho(1-\rho)\widetilde{c}_j(\rho). \tag{2.5}$$

We prove in Proposition 5.7 the Einstein relation, which in the present context reads that for every $\rho \in (0,1)$, $1 \le j \le d$,

$$\widetilde{c}_j(\rho) = \sum_{p=1}^{n_j} D_p(j,j)\widetilde{g}'_{j,p}(\rho) \quad \text{so that } \frac{1}{\chi(\rho)}\sigma(\rho) = D(\rho), \tag{2.6}$$

where $\chi(\rho) = \rho(1-\rho)$ is the *static compressibility*.

Recall that we denote by $\mathbb{T}^d = [0,1)^d$ the $d$-dimensional torus and by the symbol $\theta = (\theta_1, \dots, \theta_d)$ elements of $\mathbb{T}^d$. For a smooth function $u : \mathbb{T}^d \to \mathbb{R}$, let $\partial_{\theta_j} u$ be the partial derivative of $u$ in the $j$th direction and let $\nabla u = (\partial_{\theta_1} u, \dots, \partial_{\theta_d} u)$ be the gradient of $u$. Similarly, for a smooth vector field $b = (b_1, \dots, b_d) : \mathbb{T}^d \to \mathbb{R}^d$, denote by $\nabla \cdot b$ its divergence: $\nabla \cdot b = \sum_j \partial_{\theta_j} b_j$.

Fix a sequence $(a_n : n \ge 1)$ such that $a_n \uparrow \infty$, and let $\epsilon_n = 1/a_n$. Denote by $\{\eta^n(t) : t \ge 0\}$ the Markov process on $\Omega_n$ generated by the operator

$$L_n = n^2\left[L_n^S + \frac{a_n}{n} L_n^T\right].$$

As mentioned in the Introduction, throughout the paper we assume $a_n n^{-1} \to 0$, and this condition ensures that the operator $n^2[L_n^S + (a_n/n)L_n^T]$ becomes a Markovian generator for sufficiently large $n$. If $a_n$ is constant in $n$, then the process is a weakly asymmetric speed-change exclusion process. Therefore, formally, the hydrodynamic equation is given by

$$\partial_t u = \nabla \cdot \big[D(u)\nabla u\big] - a_n \nabla \cdot \big[\sigma(u)\mathbf{m}\big]. \tag{2.7}$$

Assume that there exists $\alpha_0 \in (0,1)$ such that

$$\sigma'(\alpha_0) = 0: \quad \sigma'_{j,j}(\alpha_0) = 0 \quad \text{for } 1 \le j \le d. \tag{2.8}$$

Assume, furthermore, that the initial condition $u_0^n$ is given by $u_0^n = \alpha_0 + \epsilon_n v_0$, where $v_0 : \mathbb{T}^d \to \mathbb{R}$ is a smooth profile, and, recall, $\epsilon_n = 1/a_n$. Write the solution $u$ as $\alpha_0 + \epsilon_n v$. Since $\sigma'(\alpha_0) = 0$, a straightforward computation yields that, up to lower order terms, $v : \mathbb{T}^d \times [0,\infty) \to \mathbb{R}$ is the solution of the Cauchy problem

$$\begin{cases} \partial_t v = \nabla \cdot [D(\alpha_0)\nabla v] - (1/2)\nabla \cdot [v^2 \sigma''(\alpha_0)\mathbf{m}], \\ v(0,\cdot) = v_0(\cdot). \end{cases} \tag{2.9}$$

From these observations, one might expect that the empirical measure of the weakly asymmetric exclusion process suitably rescaled converges to the solution of the viscous Burgers equation (2.9). As mentioned in the Introduction, one can consider a perturbation around a general constant profile $\alpha \in (0,1)$ by performing a Galilean transformation [see Remark 2.6].

## 2.2. *Main results*

Let $u : \mathbb{T}^d \to [0,1]$ be a continuous function. Denote by $\|u\|_\infty$ the supremum norm: $\|u\|_\infty = \sup_{\theta\in\mathbb{T}^d} |u(\theta)|$. Let $u_j : \mathbb{T}^d \to \mathbb{R}$, $j = 1, 2$, be two continuous functions and let $u_j^n(\theta) = u(\theta) + \kappa_n u_j(\theta)$, where $\lim_n \kappa_n = 0$. Assume that there exists $\delta > 0$ such that $\delta \le u_j(\theta) \le 1-\delta$ for all $\theta \in \mathbb{T}^d$, $j = 1, 2$. The proof of the next lemma relies on a simple Taylor expansion.

**Lemma 2.1.** *There exists a finite constant $C_0$, depending only on $\delta$ and $\|u_1\|_\infty$, $\|u_2\|_\infty$, such that*

$$H_n\big(\nu^n_{u_2^n(\cdot)} | \nu^n_{u_1^n(\cdot)}\big) = \frac{\kappa_n^2}{2} \sum_{x\in\mathbb{T}_n^d} \frac{[u_2(x/n) - u_1(x/n)]^2}{\chi(u(x/n))} + R_n,$$

*where $|R_n| \le C_0 \kappa_n^3 n^d$.*

This result states that $H_n(\nu^n_{u_2^n(\cdot)} | \nu^n_{u_1^n(\cdot)})$ is of order $\kappa_n^2 n^d$. In particular, the density profile at the scale $\kappa_n$ of a probability measure $\mu_n$ is not characterized if its relative entropy with respect to $\nu^n_{u_1^n(\cdot)}$ is of order $\kappa_n^2 n^d$.

Denote by $C^m(\mathbb{T}^d)$, $m \ge 1$, the set of $m$-times continuously differentiable functions on $\mathbb{T}^d$, and by $C^{m+\beta}(\mathbb{T}^d)$, $0 < \beta < 1$, the set of functions in $C^m(\mathbb{T}^d)$ whose $m$th derivatives are Hölder-continuous with exponent $\beta$. Fix a function $v_0$ in $C^3(\mathbb{T}^d)$. By [14, Theorem V.6.1], for each $T > 0$, there exists a unique solution, represented by $v(t,x)$, of (2.9). Denote by $(S_t^n : t \ge 0)$ the semigroup associated to the generator $L_n$, and recall from (1.10) the definition of the sequence $g_d(n)$.

**Theorem 2.2.** *Assume that $\epsilon_n \downarrow 0$ and that $n^2\epsilon_n^4 \le C_0 g_d(n)$ for some finite constant $C_0$. Recall hypothesis (2.8). Suppose that $v_0$ belongs to $C^{3+\beta}(\mathbb{T}^d)$ for some $0 < \beta < 1$. Let $v_t$ be the solution of (2.9), $u_t^n = \alpha_0 + \epsilon_n v_t$ and $\nu_t^n = \nu^n_{u_t^n(\cdot)}$. Consider a sequence of probability measures $\{\mu^n : n \ge 1\}$ on $\Omega_n$ such that*

$$H_n\big(\mu^n | \nu_0^n\big) \le C_1 n^{d-2} g_d(n)$$

*for some finite constant $C_1$. Then, for every $T > 0$, there exists a finite constant $C_2 = C_2(T, v_0, C_0, C_1)$, such that for every $0 \le t \le T$,*

$$H_n\big(\mu^n S_t^n | \nu_t^n\big) \le C_2 n^{d-2} g_d(n).$$

The proof of this result is based on a two-blocks estimate due to Jara and Menezes [10] and stated below in Lemma 4.2.

For two sequences $(b_n : n \ge 1)$, $(c_n : n \ge 1)$ of non-negative real numbers, we write $b_n \ll c_n$ to mean that $\lim_n b_n/c_n = 0$. In view of Lemma 2.1 and Theorem 2.2, to characterize the density profile at the scale $\epsilon_n$, we need at least $n^{d-2} g_d(n) \ll n^d \epsilon_n^2$. This is exactly the extra assumption of the next corollary.

**Corollary 2.3.** *Besides the assumptions of Theorem 2.2, assume that $g_d(n) \ll n^2\epsilon_n^2$. Then, for every $t \ge 0$, every function $H$ in $C^2(\mathbb{T}^d)$ and every cylinder function $\Psi : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$,*

$$\lim_{n\to\infty} E_{\mu^n S_t^n}\left[\left| \frac{1}{n^d \epsilon_n} \sum_{x\in\mathbb{T}_n^d} H(x/n)(\tau_x\Psi)(\eta) - \int_{\mathbb{T}^d} H(x) E_{\nu_t^n}[\Psi]\, dx \right|\right] = 0.$$

**Remark 2.4.** The conditions $g_d(n) \ll n^2\epsilon_n^2$ and $n^2\epsilon_n^4 \le C_0 g_d(n)$ in Theorem 2.2 and Corollary 2.3 read as follows, respectively. There exists a finite constant $C_0$ such that

(a) in dimension 1, $n^{-1/2} \ll \epsilon_n$ and $\epsilon_n \le C_0 n^{-1/4}$;
(b) in dimension 2, $(\log n)^{1/2} n^{-1} \ll \epsilon_n$ and $\epsilon_n \le C_0 (\log n)^{1/4} n^{-1/2}$;
(c) in dimension $d \ge 3$, $n^{-1} \ll \epsilon_n$ and $\epsilon_n \le C_0 n^{-1/2}$.

**Remark 2.5.** In all dimensions, in the scaling $\epsilon_n = n^{-d/2}$ one observes the fluctuations of the density field. In dimension 1, the condition $n^{-1/2} \ll \epsilon_n$ is therefore optimal, while in dimension 2, there is an extra factor $(\log n)^{1/2}$. In dimension $d \ge 3$, Esposito, Marra and Yau [5,6] examined the incompressible limit of the asymmetric simple exclusion process. They proved that a perturbation of size $1/n$ of the density profile around a constant evolves in the diffusive time-scale as the solution of (2.9).

In particular, we believe that to reach perturbations of size $1/n$ in dimension $d \geq 3$ we have to improve Theorem 2.2 by adding "non-gradient corrections", that is, to add a local perturbation of the state of the process, as it has been done in [12,18,21] to derive the hydrodynamic behavior of non-gradient interacting particle systems [cf. Chapter 7 of [11]].

The diffusive behavior of the asymmetric exclusion process has been further investigated in [16,17].

**Remark 2.6.** Hypothesis (2.8) can be circumvented by performing a Galilean transformation. Indeed, writing the solution of (2.7) as $\alpha_0 + \epsilon_n v(t, x - \epsilon_n^{-1}\sigma'(\alpha)\boldsymbol{m}t)$, we get, from a straightforward computation, that $v$ is the solution of the Cauchy problem (2.9). This computation does not require hypothesis (2.8), as the higher order terms in $\epsilon_n$ cancel [one of them being $\nabla \cdot [v\sigma'(\alpha)\boldsymbol{m}]$].

**Remark 2.7.** The assumption that $n^2\epsilon_n^4 \leq C_0 g_d(n)$ for some finite constant $C_0$ is needed to estimate the linear terms of the time-derivative of the relative entropy. This issue is further discussed in Remarks 3.6 and 3.7 below.

The paper is organized as follows. In Section 3, we compute the time derivative of the entropy $H_n(\mu^n S_t^n | \nu_t^n)$. In Section 4, we estimate the time derivative of the entropy and we prove Theorem 2.2 and Corollary 2.3. In Section 5, we present the results on the viscous Burger's equation (2.9) needed in the proofs of the main results, and, in Section 6, we compute the adjoint of the generator $L_n$ in $L^2(\nu_{u(\cdot)}^n)$.

## 3. Entropy production

We estimate in this section the time derivative of the relative entropy. Fix $n \geq 1$, and recall that we denote by $(S_t^n : t \geq 0)$ the semigroup associated to the generator $L_n$. Fix a stationary state $\nu_\alpha$, $0 < \alpha < 1$, and a probability measure $\mu$ on $\Omega_n$. Denote by $f_t$ the Radon–Nikodym derivative of $\mu S_t^n$ with respect to $\nu_\alpha$. An elementary computation yields that

$$\frac{d}{dt} f_t = L_n^* f_t,$$

where $L_n^*$ stands for the adjoint of $L_n$ in $L^2(\nu_\alpha)$.

For a function $f : \Omega_n \to \mathbb{R}$ and a probability measure $\nu$ on $\Omega_n$, denote by $I(f; \nu)$ the Dirichlet form given by

$$I(f;\nu) = \sum_{x \in \mathbb{T}_n^d} \sum_{j=1}^{d} \int \left\{ \sqrt{f\left(\sigma^{x,x+e_j}\eta\right)} - \sqrt{f(\eta)} \right\}^2 \nu(d\eta). \tag{3.1}$$

The proof of the next result, which is similar to the one of Lemma 6.1.4 in [11], is left to the reader. Recall from (1.3) the definition of the product measure $\nu_{u(\cdot)}^n$ associated to a function $u : \mathbb{T}_n^d \to (0,1)$. For a function $w : \mathbb{R}_+ \times \mathbb{T}_n^d \to (0,1)$, let $\nu_{w(t)}^n = \nu_{w(t,\cdot)}^n$.

**Lemma 3.1.** *Fix $n \geq 1$ and $0 < \alpha < 1$. Let $w : \mathbb{R}_+ \times \mathbb{T}_n^d \to (0,1)$ be a differentiable function in time, and let $\mu$ be a probability measure on $\Omega_n$. Then,*

$$\frac{d}{dt} H_n\left(\mu S_t^n | \nu_{w(t)}^n\right) \leq -n^2 I\left(g_t; \nu_{w(t)}^n\right) + \int \left\{ L_{w(t)}^* \mathbf{1} - \partial_t \log \psi_t \right\} d\mu S_t^n,$$

*where $g_t$ represents the Radon–Nikodym derivative of $\mu S_t^n$ with respect to $\nu_{w(t)}^n$, $g_t = d\mu S_t^n / d\nu_{w(t)}^n$, $L_{w(t)}^*$ the adjoint operator of $L_n$ in $L^2(\nu_{w(t)}^n)$ and $\psi_t$ the density given by $\psi_t = d\nu_{w(t)}^n / d\nu_\alpha^n$.*

In view of the previous lemma, we need to compute the integrand in the right hand side of the statement of the lemma. To state the explicit formula of $L_\varrho^* \mathbf{1} - \partial_t \log \psi_t$ for a function $\varrho : \mathbb{T}_n^d \to (0,1)$, we need to introduce several notations. This computation will be postponed to Section 6.

Consider a cylinder function $f : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$ and a function $\varrho : \mathbb{T}_n^d \to (0,1)$. Fix a positive integer $n$ large enough for $\{-n/2, \dots, n/2\}^d$ to contain the support of $f$. We also introduce the notion of Fourier coefficients for local functions, c.f. [13, Section 5.4]. For each $x \in \mathbb{T}_n^d$ and subset $B$ of $\mathbb{T}_n^d$, let

$$\mathfrak{f}(x, \varnothing) := E_{\nu_\varrho}[\tau_x f], \qquad \mathfrak{f}(x, B) := E_{\nu_\varrho}\left[(\tau_x f)\xi_\varrho(B + x)\right], \tag{3.2}$$

where, for a subset $D$ of $\mathbb{T}_n^d$,

$$D + x = \{y + x : y \in D\}, \xi_\varrho(D) = \prod_{y\in D} \big[\eta(y) - \varrho(y)\big].$$

When $D = \{x\}$ for some $x \in \mathbb{T}_n^d$, we shall denote $\xi_\varrho(D)$ by $\xi_\varrho(x)$ for simplicity.

Note that $\mathfrak{f}(x, B) = 0$ if the set $B$ is not contained in the support of $f$. More precisely, assume that $f$ depends on $\eta$ only through $\{\eta(x) : x \in \Lambda_\ell\}$, where $\Lambda_\ell = \{-\ell, \dots, \ell\}^d$. Then,

$$\mathfrak{f}(x, B) = 0 \quad \text{if } B \text{ is not a subset of } \Lambda_\ell. \tag{3.3}$$

With these notations, we may write

$$(\tau_x f)(\eta) = \mathfrak{f}(x, \varnothing) + \sum_A \mathfrak{f}(x, A)\omega_\varrho(A + x), \tag{3.4}$$

where the sum is performed over all non-empty subsets $A$ of $\mathbb{T}_n^d$ and

$$\omega_\varrho(D) = \prod_{y\in D} \frac{\eta(y) - \varrho(y)}{\varrho(y)[1 - \varrho(y)]}.$$

Note that $\{\omega_\varrho(D) : D \subset \mathbb{T}_n^d\}$ forms an orthogonal basis of $L^2(\nu_\varrho^n)$. Denote by $\mathcal{E}_k = \mathcal{E}_{n,k}$ all subsets of $\mathbb{T}_n^d$ with $k$ elements: $\mathcal{E}_k = \{A \subset \mathbb{T}_n^d : |A| = k\}$. A cylinder function $\tau_x f$, $x \in \mathbb{T}_n^d$, is said to be of degree $k$ if $\mathfrak{f}(x, A) = 0$ for all $A \notin \mathcal{E}_k$.

In Section 6, we compute $L_\varrho^* \mathbf{1}$ for a function $\varrho : \mathbb{T}_n^d \to [0, 1]$ and the results are stated in terms of coefficients $A_j(x)$, $B_{j,p}^{(i)}(x)$, $\dots$, which are defined there. Since we apply the results to the function $w(t)$, to stress the dependence we denote them with $t$ in the following paragraphs. Moreover, as we shall consider the case $w(t) = (1/2) + \epsilon_n v^n(t)$ in the following subsection, we shall denote them with $n$. For instance, $A_j(x)$ will be denoted by $A_j(t, x)$, $A_j^n(t, x)$ when $\varrho = w(t)$, $w(t) = (1/2) + \epsilon_n v^n(t)$, respectively. Moreover, the contributions coming from the symmetric part or the asymmetric part will be denoted with the superscript $i = 1, 2$, respectively.

The explicit expression of $L_{w(t)}^* \mathbf{1}$ requires some notation. Some of the notation below is borrowed from Section 6. Denote by $D_j$ the difference operator defined by

$$(D_j F)(x) = F(x + e_j) - F(x), \quad x \in \mathbb{T}_n^d. \tag{3.5}$$

For $1 \le j \le d$, $1 \le p \le n_j$, $x \in \mathbb{T}_n^d$, $t \ge 0$ $A \subset \mathbb{T}_n^d$, let $\mathfrak{c}_j(t, x, A)$, $\mathfrak{g}_{j,p}(t, x, A)$ be the Fourier coefficients, introduced in (3.2), of the cylinder functions $\tau_x c_j$, $\tau_x g_{j,p}$, respectively, with respect to the measure $\nu_{w(t)}^n$:

$$\mathfrak{c}_j(t, x, A) = E_{\nu_{w(t)}^n}\big[(\tau_x c_j)\xi_{w(t)}(A + x)\big],$$
$$\mathfrak{g}_{j,p}(t, x, A) = E_{\nu_{w(t)}^n}\big[(\tau_x g_{j,p})\xi_{w(t)}(A + x)\big],$$

where $\xi_{w(t)}(B) = \prod_{x\in B}[\eta_x - w(t, x)]$, $B \subset \mathbb{T}_n^d$.

For $i = 1, 2$, let $A_j(t, x)$, $B_{j,p}^{(i)}(t, x)$, $E_j^{(i)}(t, x)$, $F_j^{(i)}(t, x)$, $G_j^{(i)}(t, x)$, $H_j^{(i)}(t, x, A)$, $I_j(t, x)$ be the functions obtained from (6.1), (6.8), (6.9) by replacing $\varrho(x)$ by $w(t, x)$. For example,

$$A_j(t, x) = \frac{\chi(w(t, x)) + \chi(w(t, x + e_j))}{\chi(w(t, x))\chi(w(t, x + e_j))},$$
$$E_j^{(2)}(t, x) = -\frac{\mathbf{m}_j}{2} A_j(t, x)(D_j w_t)(x)w(t, x)\big[1 - w(t, x + e_j)\big].$$

In the case of $H_j^{(i)}(t, x, A)$ one has also to replace the Fourier coefficients $\mathfrak{c}_j(x, A)$, $\mathfrak{g}_{j,p}(x, A)$, computed with respect to $\nu_\varrho$, by $\mathfrak{c}_j(t, x, A)$, $\mathfrak{g}_{j,p}(t, x, A)$, respectively.

Let

$$H_j(t, x, A) = n^2 H_j^{(1)}(t, x, A) + a_n n H_j^{(2)}(t, x, A).$$

As mentioned before, note that the first or the second term in the right-hand side comes from the contribution of the symmetric or the asymmetric part of the generator, respectively.

The functions $H_j^{(1)}$, $H_j^{(2)}$, introduced in (6.2), (6.10), are defined in terms of the Fourier coefficients $\mathfrak{c}_j(t,x,A)$ and $\mathfrak{g}_{j,p}(t,x,A)$. Since the functions $c_j$, $g_{j,p}$ are cylinder functions, there exists $\ell \ge 1$ such that $\mathfrak{c}_j(t,x,A) = 0$ and $\mathfrak{g}_{j,p}(t,x,A) = 0$ for all sets $A$ which are not contained in $\{-\ell,\dots,\ell\}^d$ [cf. remark (3.3)]. Hence, there exists $\ell \ge 1$ such that $H_j(t,x,A) = 0$ if $A \not\subset \{-\ell,\dots,\ell\}^d$.

It also follows from the definitions of $H_j^{(1)}$, $H_j^{(2)}$, given in (6.2), (6.10), that the functions of $x$ which appear in the previous formula either contain the product of derivatives [this is the case of $E_j$, $F_j$ and $G_j$] or a second discrete derivative, which is the case of $B_{j,p}$ [see also the paragraph before Lemma 6.1].

Denote by $j_{0,e_j}$ the instantaneous current over the bond $(0,e_j)$. This is the rate at which a particle jumps from 0 to $e_j$ minus the rate at which it jumps from $e_j$ to 0. It is given by

$$j_{0,e_j} = c_j(\eta)[\eta_0 - \eta_{e_j}]. \quad \text{Let } j_{x,x+e_j} = \tau_x j_{0,e_j}, \quad x \in \mathbb{T}_n^d. \tag{3.6}$$

The gradient conditions (2.1) assert that this current can be written as a mean-zero average of translations of cylinder functions.

Next result is a consequence of Lemmata 6.1 and 6.6.

**Lemma 3.2.** *We have that*

$$L_{w(t)}^* \mathbf{1} = \sum_{j=1}^d \sum_{x\in\mathbb{T}_n^d} K_j(t,x)\omega_x + \sum_{j=1}^d \sum_{A:|A|\ge 2} \sum_{x\in\mathbb{T}_n^d} H_j(t,x,A)\omega(A+x),$$

*where*

$$K_j(t,x) = n^2\big\{E_{\nu^n_{w(t)}}[j_{x-e_j,x}] - E_{\nu^n_{w(t)}}[j_{x,x+e_j}]\big\} - a_n n (D_j I_j)(t, x-e_j),$$

*the sum over $A$ is performed over finite subsets $A$ with at least two elements*, *and*

$$\omega_x = \frac{\eta_x - w(t,x)}{\chi(w(t,x))}, \qquad \omega(B) = \prod_{x\in B}\omega_x, \quad B \subset \mathbb{Z}^d.$$

Note that $\omega_x$ depends on time, but this dependency is frequently omitted from the notation to avoid long formulas. Also, to stress the point at which it is evaluated, we write sometimes $\omega(x)$ for $\omega_x$.

**Lemma 3.3.** *Under the assumptions of Lemma 3.1*, *for every $t \ge 0$*,

$$\partial_t \log\psi_t = \sum_{x\in\mathbb{T}_n^d} (\partial_t w)(t,x)\omega_x.$$

It follows from Lemmata 3.1, 3.2, 3.3 that $L_{w(t)}^*\mathbf{1} - \partial_t \log\psi_t$ presents only terms of degree 2 or higher if $w(t,x)$ solves the semi-discrete equation

$$(\partial_t w)(t,x) = \sum_{j=1}^d K_j(t,x).$$

### 3.1. *Perturbations of constant profiles*

We turn to the setting of Theorem 2.2, and assume, without loss of generality, that in hypothesis (2.8), $\alpha_0 = 1/2$. Recall that $\epsilon_n = 1/a_n$ and assume, throughout this subsection, that the function $w(t)$ of Lemma 3.1 is given by $w(t) = (1/2) + \epsilon_n v^n(t)$ for some function $v^n : \mathbb{R}_+ \times \mathbb{T}_n^d \to \mathbb{R}$. At this point we do not suppose yet that $v^n(t)$ is the solution of (2.9).

Lemma 3.2 provides a formula for $L_{w(t)}^*\mathbf{1}$. Many terms cancel or simplify due to the special form of $w(t)$. In the next lemma we present the result of these reductions. As mentioned before, the coefficients $A_j^n(t,x),\dots$, which will be defined below, formally coincide with $A_j(t,x),\dots$, respectively.

Denote by $\nabla_j^n$ the discrete partial derivative in the $j$th direction. For a function $\varrho : \mathbb{T}_n^d \to \mathbb{R}$, $\nabla_j^n \varrho$ is given by

$$\big(\nabla_j^n \varrho\big)(x) = n\big[\varrho(x+e_j) - \varrho(x)\big], \quad x \in \mathbb{T}_n^d. \tag{3.7}$$

For $1 \le j \le d$, $1 \le p \le n_j$, $x \in \mathbb{T}_n^d$, let

$$\begin{aligned} A_j^n(t,x) &= \frac{\chi(w_t(x)) + \chi(w_t(x+e_j))}{\chi(w_t(x))\chi(w_t(x+e_j))}, \\ C_j^n(t,x) &= \mathbf{m}_j w_t(x)\big[1 - w_t(x+e_j)\big]. \end{aligned} \tag{3.8}$$

Let $U_j^{n,(1)}(t,x) = (\epsilon_n/n)(\nabla_j^n v^n)(t,x)$, $U_j^{n,(2)}(t,x) = -C_j^n(t,x)$ and

$$V_j^{n,(i)}(t,x) = \big[U_j^{n,(1)}(t,x)\big]^{3-i}\big[U_j^{n,(2)}(t,x)\big]^{i-1}, \quad i = 1,2.$$

For $i = 1, 2$, $B_{j,p}^{n,(i)}$, $E_{j,p}^{n,(i)}$, $F_{j,p}^{n,(i)}$, $G_{j,p}^{n,(i)}$, are defined as

$$B_{j,p}^{n,(i)}(t,x) = \frac{1}{2}\sum_{y\in\mathbb{T}_n^d} m_{j,p}(y) A_j^n(t,x-y) U_j^{n,(i)}(t,x-y),$$

$$E_j^{n,(i)}(t,x) = \frac{1}{2} A_j^n(t,x) V_j^{n,(i)}(t,x),$$

$$F_j^{n,(i)}(t,x) = -\frac{\epsilon_n}{2}\frac{V_j^{n,(i)}(t,x)}{\chi(w_t(x+e_j))}\big\{v^n(t,x) + v^n(t,x+e_j)\big\},$$

$$G_j^{n,(i)}(t,x) = -\frac{\epsilon_n}{2}\frac{V_j^{n,(i)}(t,x)}{\chi(w_t(x))}\big\{v^n(t,x) + v^n(t,x+e_j)\big\},$$

respectively. For $A \subset \mathbb{T}_n^d$ and $i = 1, 2$, let $J_j^{n,(i)}(t,x,A)$ be given

$$\begin{aligned} J_j^{n,(i)}(t,x,A) = &-\Upsilon_{\{0,e_j\}}(A) V_j^{n,(i)}(t,x)\mathfrak{c}_j\big(t,x,A\setminus\{0,e_j\}\big) \\ &+ \Upsilon_{\{0\}}(A) F_j^{n,(i)}(t,x)\mathfrak{c}_j\big(t,x,A\setminus\{0\}\big) \\ &+ \Upsilon_{\{e_j\}}(A) G_j^{n,(i)}(t,x)\mathfrak{c}_j\big(t,x,A\setminus\{e_j\}\big), \end{aligned}$$

where, for two subsets $A$, $B$ of $\mathbb{Z}^d$,

$$\Upsilon_B(A) = 1 \quad \text{if } B \subset A, \quad \text{and} \quad \Upsilon_B(A) = 0 \quad \text{otherwise}. \tag{3.9}$$

Here, the Fourier coefficients $\mathfrak{c}_j(t,x,A)$, $\mathfrak{g}_{j,p}(t,x,A)$ are computed with respect to the product measure $\nu_{w(t)}^n$. Finally, let

$$\begin{aligned} H_j^{n,(i)}(t,x,A) &= E_j^{n,(i)}(t,x)\mathfrak{c}_j(t,x,A) + \sum_{p=1}^{n_j} B_{j,p}^{n,(i)}(t,x)\mathfrak{g}_{j,p}(t,x,A) + J_j^{n,(i)}(t,x,A), \\ H_j^n(t,x,A) &= n^2 H_j^{n,(1)}(t,x,A) + a_n n H_j^{n,(2)}(t,x,A). \end{aligned}$$

In the case where $w(t) = (1/2) + \epsilon_n v^n(t)$, Lemma 3.3 and Lemma 3.2 become

$$\partial_t \log \psi_t = \epsilon_n \sum_{x\in\mathbb{T}_n^d} \big(\partial_t v^n\big)(t,x)\omega_x. \tag{3.10}$$

**Lemma 3.4.** *Suppose that $w(t) = (1/2) + \epsilon_n v^n(t)$. Then,*

$$L_{w(t)}^* \mathbf{1} = \sum_{j=1}^d \sum_{x\in\mathbb{T}_n^d} K_j^n(t,x)\omega_x + \sum_{j=1}^d \sum_{A:|A|\ge 2} \sum_{x\in\mathbb{T}_n^d} H_j^n(t,x,A)\omega(A+x),$$

*where*

$$K_j^n(t,x) = n^2\big\{E_{\nu^n_{w(t)}}[j_{x-e_j,x}] - E_{\nu^n_{w(t)}}[j_{x,x+e_j}]\big\} - a_n\big(\nabla_j^n I_j^n\big)(t,x-e_j),$$
$$I_j^n(t,x) = E_{\nu^n_{w(t)}}[\tau_x c_j]C_j^n(t,x), \quad \text{and}$$

*$\omega_x$ and $\omega(B)$ are defined in Lemma 3.2.*

The next result is a consequence of Lemmata 3.1, 3.3, 3.4.

**Corollary 3.5.** *Suppose that $w(t) = (1/2) + \epsilon_n v^n(t)$. All terms of degree 1 of $L^*_{w(t)}\mathbf{1} - \partial_t \log\psi_t$ vanish as long as $v^n(t,x)$ is the solution of the semi-discrete equation*

$$\big(\partial_t v^n\big)(t,x) = a_n \sum_{j=1}^d K_j^n(t,x), \quad t\ge 0, x\in\mathbb{T}_n^d. \tag{3.11}$$

**Remark 3.6.** Note that the computation of $L^*_{w(t)}\mathbf{1}$ for an arbitrary profile $w(t):\mathbb{T}_n^d\to(0,1)$ reveals the semi-discrete partial differential equation which describes the macroscopic evolution of the density.

At this point, there are two possible choices. In Lemma 3.4, we may consider as reference state the product measure $\nu^n_{w(t)}$ whose density profile $w(t)$ is given by $(1/2)+\epsilon_n v^n(t)$, where $v^n(t)$ is the solution of the semi-discrete equation (3.11), or the one given by $(1/2)+\epsilon_n v(t)$, where $v(t)$ is the solution of the semi-linear equation (2.9).

With the first choice, the terms of degree one in the expression $L^*_{w(t)}\mathbf{1} - \partial_t\log\psi_t$ vanish. To estimate the terms of order 2 or higher, uniform bounds of the discrete derivatives of the solutions of the semi-discrete equation (3.11) are needed.

With the second choice, the terms of degree one appear multiplied by a small constant, but do not vanish and need to be estimated. In contrast, the terms of degree 2 or higher can be estimated with bounds on the derivatives of the solutions of the semi-linear equation (2.9) provided by [14].

We followed here the approach adopted by the previous authors and sticked to the second choice.

**Remark 3.7.** The assumption that $n^2\epsilon_n^4 \le C_0 g_d(n)$ for some finite constant $C_0$ is needed to estimate the linear terms of the time-derivative of the relative entropy [the linear terms of $L^*_{w(t)}\mathbf{1} - \partial_t\log\psi_t$, computed in Lemmata 3.2 and 3.3]. Actually, equation (2.9) is a continuous version of the semi-discrete equation obtained by considering the linear terms (in $\eta$) of the identity

$$L^*_{w(t)}\mathbf{1} - \partial_t\log\psi_t = 0. \tag{3.12}$$

One may try to weaken or remove the hypothesis $n^2\epsilon_n^4 \le C_0 g_d(n)$ by replacing equation (2.9) by the one obtained restricting (3.12) to the linear terms. In this case, however, estimating the quadratic terms of (3.12) might be more demanding. One may also try to weaken this hypothesis by adding to equation (2.9) terms of order $\epsilon_n^k$, $k\ge 2$.

**Remark 3.8.** In the case where $c_j(\eta)=1$, $\mathbf{m}_j = 1$ for all $j$, the semi-discrete equation (3.11) becomes

$$(\partial_t v)(t,x) = \left(1+\frac{a_n}{n}\right)(\Delta_n v)(t,x) + n\sum_{j=1}^d \big\{v(t,x)v(t,x+e_j) - v(t,x-e_j)v(t,x)\big\},$$

where $\Delta_n\varrho$ stands for the discrete Laplacian:

$$(\Delta_n\varrho)(x) = n^2\sum_{j=1}^d\big\{\varrho(x+e_j)+\varrho(x-e_j)-2\varrho(x)\big\}.$$

## 4. Proof of Theorems 2.2 and Corollary 2.3

Assume, without loss of generality, that in hypothesis (2.8), $\alpha_0 = 1/2$. Assume, furthermore, that $v:\mathbb{R}_+\times\mathbb{T}^d\to\mathbb{R}$ is the solution of the semi-linear equation (2.9) and that $w^n(t,x) = (1/2)+\epsilon_n v(t,x/n)$. We refer constantly to Section 5 for properties of the solutions of the viscous Burgers equation (2.9).

By Lemma 5.1, for all $T > 0$, there exists $\delta > 0$ such that

$$\delta \le w^n(t,x) \le 1-\delta \tag{4.1}$$

for all $0 \le t \le T$, $x \in \mathbb{T}_n^d$ and $n$ sufficiently large.

Let $L^n : \mathbb{R}_+ \times \mathbb{T}_n^d \to \mathbb{R}$ be given by

$$L^n(t,x) = \sum_{j=1}^d K_j^n(t,x) - \epsilon_n (\partial_t v)(t,x/n). \tag{4.2}$$

**Lemma 4.1.** *Fix a density profile $v_0$ in $C^{3+\beta}(\mathbb{T}^d)$ for some $0<\beta<1$. For every $T>0$, there exists a finite constant $C_0$, depending only on $v_0$ and $T$, such that for all $0 \le t \le T$, $\gamma > 0$,*

$$\int \sum_{x\in\mathbb{T}_n^d} L^n(t,x)\omega_x \, d\mu S_t^n \le \frac{1}{\gamma} H_n\big(\mu S_t^n | \nu_{w^n(t)}^n\big) + C_0 \gamma n^{d-2}\big(1+n^2\epsilon_n^4\big) e^{C_0\gamma\kappa_n},$$

*where $\kappa_n = \epsilon_n^2 + (1/n)$.*

**Proof.** By the entropy inequality, the left-hand side is bounded by

$$\frac{1}{\gamma} H_n\big(\mu S_t^n | \nu_{w^n(t)}^n\big) + \frac{1}{\gamma} \log \int \exp\Big\{\gamma \sum_{x\in\mathbb{T}_n^d} L^n(t,x)\omega_x\Big\} d\nu_{w^n(t)}^n$$

for all $\gamma > 0$. As $\nu_{w^n(t)}^n$ is a product measure, we may move the sum outside the logarithm. Since $e^x \le 1+x+(1/2)x^2 e^{|x|}$, $\log(1+a) \le a$, $a>0$, and since $\omega_x$ has mean zero with respect to $\nu_{w^n(t)}^n$, the second term of the previous formula is bounded above by

$$\frac{\gamma}{2} \sum_{x\in\mathbb{T}_n^d} \frac{L^n(t,x)^2}{\chi(w^n(t,x))} \exp\big\{\gamma \big|L^n(t,x)\big| / \chi\big(w^n(t,x)\big)\big\},$$

because $E_{\nu_{w^n(t)}^n}[\omega_x^2] = 1/\chi(w^n(t,x))$. By Lemma 5.2 and by (4.1), the previous expression is bounded by

$$C_0 \gamma n^{d-2}\big(1+n^2\epsilon_n^4\big) e^{C_0\gamma[\epsilon_n^2+(1/n)]}$$

for some finite constant $C_0$ which depends only on $v_0$ and $T$. This completes the proof of the lemma. □

We turn to the quadratic or higher order term $H_j^n(t,x,A)$. The estimation is based on the following bound due to Jara and Menezes [10, Lemma 3.1].

**Proposition 4.2.** *Fix a finite subset $A$ of $\mathbb{Z}^d$ with at least two elements. For every $\delta>0$, $a>0$ and $C_1<\infty$, there exists a finite constant $C_0$, depending only on $\delta$, $A$, $C_1$ and $a$ such that the following holds. For all $n \ge 1$, probability measures $\mu$ on $\Omega_n$, functions $u$, $J : \mathbb{T}_n^d \to \mathbb{R}$ such that $\delta \le u(x) \le 1-\delta$ for all $x \in \mathbb{T}_n^d$, and*

$$\max_{x\in\mathbb{T}_n^d} \max_{1\le j\le d} \big|\big(\nabla_j^n u\big)(x)\big| \le C_1, \qquad \max_{x\in\mathbb{T}_n^d} \big|J(x)\big| \le C_1,$$

*we have that*

$$\int \sum_{x\in\mathbb{T}_n^d} J(x)\omega(A+x)\, d\mu \le a n^2 I\big(g;\nu_{u(\cdot)}^n\big) + C_0\big\{H_n\big(\mu|\nu_{u(\cdot)}^n\big) + n^{d-2} g_d(n)\big\},$$

*where*

$$\omega_x = \frac{\eta_x - u(x)}{\chi(u(x))}, \qquad \omega(B) = \prod_{x\in B} \omega_x, \quad B \subset \mathbb{Z}^d,$$

*and $g = d\mu/d\nu_{u(\cdot)}^n$.*

We show in the next paragraphs that the hypotheses of this proposition are fulfilled for $u(x) = w^n(t,x)$, $J(x) = H_j^n(t,x,A)$. We first prove the bounds for $u$ and then the ones for $J$.

By definition, $|(\nabla_j^n w^n)(t,x)| \le \epsilon_n \sup_{\theta\in\mathbb{T}^d} |(\partial_{\theta_j} v)(t,\theta)|$. Hence, by Lemma 5.1, for every $T>0$, there exists a finite constant $C_1 = C_1(T, v_0)$ such that for all $n \ge 1$,

$$\sup_{0\le t\le T} \max_{x\in\mathbb{T}_n^d} \max_{1\le j\le d} \left|\left(\nabla_j^n w^n\right)(t,x)\right| \le C_1\epsilon_n. \tag{4.3}$$

On the other hand, we have seen in (4.1) that for all $T>0$ there exists $\delta>0$ such that $\delta \le w^n(t,x) \le 1-\delta$ for all $x \in \mathbb{T}_n^d$, $0\le t\le T$ and $n$ sufficiently large.

The next lemma provides an estimate for the term $J(x) = H_j^n(t,x,A)$.

**Lemma 4.3.** *For each $T>0$, there exists a finite constant $C_0 = C_0(T,v_0)$ such that for all $n\ge 1$,*

$$\sup_{0\le t\le T} \max_{x\in\mathbb{T}_n^d} \sup_{A\subset\mathbb{Z}^d} \max_{1\le j\le d} \left|H_j^n(t,x,A)\right| \le C_0,$$

*where the supremum is carried over all finite subsets $A$ of $\mathbb{Z}^d$.*

**Proof.** The proof is long, elementary and tedious. It follows from Lemma 5.1 and from the definitions (3.8) of the terms $A_j^n$, $C_j^n$ that for each $T>0$, there exists a finite constant $C_0 = C_0(T,v_0)$ such that for all $n\ge 1$,

$$\sup_{0\le t\le T} \max_{x\in\mathbb{T}_n^d} \max_{1\le j\le d} \left|A_j^n(t,x)\right| \le C_0, \qquad \sup_{0\le t\le T} \max_{x\in\mathbb{T}_n^d} \max_{1\le j\le d} \left|C_j^n(t,x)\right| \le C_0.$$

Furthermore, as $v(t,x)$ remains bounded in bounded time-intervals, for each $T>0$, there exists a finite constant $C_0 = C_0(T,v_0)$ such that for all $n\ge 1$,

$$\sup_{0\le t\le T} \max_{x\in\mathbb{T}_n^d} \max_{|y|\le \ell_0} \max_{1\le j\le d} n\left|A_j^n(t,x-y) - A_j^n(t,x)\right| \le C_0\epsilon_n,$$

where $\ell_0$, introduced just after (2.1), represents the size of the support of the measures $m_{j,p}$.

Similar bounds hold for the functions $U_j^{n,(i)}$. For each $T>0$, there exists a finite constant $C_0 = C_0(T,v_0)$ such that for all $n\ge 1$, $i=1,2$,

$$\sup_{0\le t\le T} \max_{x\in\mathbb{T}_n^d} \max_{1\le j\le d} \max_{1\le p\le n_j} n^{2-i}\left|U_j^{n,(i)}(t,x)\right| \le C_0\epsilon_n^{2-i},$$

$$\sup_{0\le t\le T} \max_{x\in\mathbb{T}_n^d} \max_{|y|\le\ell_0} \max_{1\le j\le d} \max_{1\le p\le n_j} n^{3-i}\left|U_j^{n,(i)}(t,x-y) - U_j^{n,(i)}(t,x)\right| \le C_0\epsilon_n^{3-i}.$$

It follows from the estimates on $A_j^n(t,x)$ and $U_j^{n,(i)}(t,x)$ that for each $T>0$, there exists a finite constant $C_0 = C_0(T,v_0)$ such that for all $n\ge 1$, $i=1,2$,

$$\sup_{0\le t\le T} \max_{x\in\mathbb{T}_n^d} \max_{1\le j\le d} \max_{1\le p\le n_j} n^{3-i}\left|B_{j,p}^{n,(i)}(t,x)\right| \le C_0\epsilon_n.$$

Similarly, for each $T>0$, there exists a finite constant $C_0 = C_0(T,v_0)$ such that for each $i=1,2$ and all $n\ge 1$,

$$\sup_{0\le t\le T} \max_{x\in\mathbb{T}_n^d} \max_{1\le j\le d} n^{3-i}\left|E_j^{n,(i)}(t,x)\right| \le C_0\epsilon_n^{3-i},$$

$$\sup_{0\le t\le T} \max_{x\in\mathbb{T}_n^d} \max_{1\le j\le d} n^{3-i}\left|F_j^{n,(i)}(t,x)\right| \le C_0\epsilon_n^{4-i},$$

$$\sup_{0\le t\le T} \max_{x\in\mathbb{T}_n^d} \max_{1\le j\le d} n^{3-i}\left|G_j^{n,(i)}(t,x)\right| \le C_0\epsilon_n^{4-i}.$$

Let $f$ be a cylinder function. Denote by $\mathfrak{f}(t,x,A)$ the Fourier coefficients of $f$ with respect to the measure $\nu_{w^n(t)}^n$, $w^n(t) = (1/2) + \epsilon_n v(t)$. It is clear, from the definition (3.2), that for all $n\ge 1$, $t\ge 0$, $x\in\mathbb{T}_n^d$, $A\subset\mathbb{Z}^d$,

$$\left|\mathfrak{f}(t,x,A)\right| \le \|f\|_\infty := \sup_\eta \left|f(\eta)\right|. \tag{4.4}$$

It follows from the previous estimate on the Fourier coefficients of cylinder functions and from the bounds on $F_j^{n,(q)}$, $G_j^{n,(q)}$ that for each $T>0$, there exists a finite constant $C_0=C_0(T,v_0)$ such that for each $i=1,2$, and all $n\geq 1$,

$$\sup_{0\leq t\leq T}\max_{x\in\mathbb{T}_n^d}\sup_{A\subset\mathbb{Z}^d}\max_{1\leq j\leq d} n^{3-i}\big|J_j^{n,(i)}(t,x,A)\big|\leq C_0\epsilon_n^{3-i},$$

where the supremum is carried over all finite subsets $A$ of $\mathbb{Z}^d$.

To complete the proof of the lemma, it remains to put together all previous estimates. □

**Proof of Theorem 2.2.** Let $\{\mu^n: n\geq 1\}$ be a sequence of probability measures on $\Omega_n$ satisfying the assumptions of the theorem. Let $\mu_t^n=\mu^n S_t^n$ and $H_n(t)=H_n(\mu_t^n|\nu_t^n)$.

Lemma 3.1, equation (3.10) and Lemma 3.4 provide a formula for the derivative of $H_n(t)$. Fix $T>0$. By (4.1), there exists $\delta>0$ such that $\delta\leq w^n(t,x)\leq 1-\delta$ for all $x\in\mathbb{T}_n^d$, $0\leq t\leq T$. By (4.3),

$$\kappa_T:=\sup_{0\leq t\leq T}\max_{x\in\mathbb{T}_n^d}\max_{1\leq j\leq d}\big|\big(\nabla_j^n w^n\big)(t,x)\big|<\infty,$$

and by Lemma 4.3,

$$H_T:=\sup_{0\leq t\leq T}\sup_{n\geq 0}\max_{A\subset\mathbb{Z}^d}\max_{x\in\mathbb{T}_n^d}\max_{1\leq j\leq d}\big|H_j^n(t,x,A)\big|<\infty.$$

Therefore, the hypotheses of Proposition 4.2 are in force for $u(x)=w(t,x)$, $J(x)=H_j^n(t,x,A)$.

By hypothesis, $n^2\epsilon_n^4\leq g_d(n)$. Hence, the second term on the right-hand side of the statement of Lemma 4.1 is bounded by $C_0\gamma n^{d-2}g_d(n)\exp\{C_0\gamma\}$. In particular, by Lemma 4.1 with $\gamma=1$ and by Proposition 4.2 with $a=1/2$ applied to $\mu=\mu_t^n$, $u(x)=w^n(t,x)$, $J(x)=H_j^n(t,x,A)$, there exists a finite constant $C_0$ such that

$$H_n'(t)\leq C_0H_n(t)+C_0n^{d-2}g_d(n)-\frac{1}{2}n^2I\big(g_t^n;\nu_t^n\big),$$

where $g_t^n=d\mu_t^n/d\nu_t^n$. At this point the assertion of the theorem follows from Gronwall's lemma. □

**Proof of Corollary 2.3.** For simplicity, we prove the corollary in the case $\Psi(\eta)=\eta_0$. Since $v_t$ is Lipschitz-continuous and $H$ is of class $C^2(\mathbb{T}^d)$,

$$a_n\int_{\mathbb{T}^d}H(\theta)\bigg\{\frac{1}{2}+\epsilon_n v(t,\theta)\bigg\}\,d\theta=\frac{a_n}{n^d}\sum_{x\in\mathbb{T}_n^d}H(x/n)\bigg\{\frac{1}{2}+\epsilon_n v(t,x/n)\bigg\}+O\bigg(\frac{a_n}{n}\bigg).$$

For each $x\in\mathbb{T}_n^d$, let $J_x^n(t)=H(x/n)(\eta_x^n(t)-1/2-\epsilon_n v(t,x/n))$. Since $a_n/n\to 0$, to conclude the proof it is enough to show that

$$\lim_{n\to\infty}E_{\mu^n S_t^n}\Bigg[\Bigg|\frac{a_n}{n^d}\sum_{x\in\mathbb{T}_n^d}J_x^n(t)\Bigg|\Bigg]=0.$$

By the entropy inequality and Theorem 2.2, the expectation appearing in the left-hand side can be bounded above by

$$\frac{C_0}{K}+\frac{1}{Kn^{d-2}g_d(n)}\log E_{\nu_t^n}\Bigg[\exp\Bigg\{\Bigg|\frac{Ka_ng_d(n)}{n^2}\sum_{x\in\mathbb{T}_n^d}J_x^n(t)\Bigg|\Bigg\}\Bigg]$$

for all $K>0$ and some finite constant $C_0>0$. Using $\exp\{|x|\}\leq\exp\{x\}+\exp\{-x\}$, it is enough to estimate the previous expression without the absolute value. Indeed, the other term can be handled by the following argument similarly.

As $\nu_t^n$ is a product measure, the second term of the previous displayed expression without the absolute value is equal to

$$\frac{1}{Kn^{d-2}g_d(n)}\sum_{x\in\mathbb{T}_n^d}\log E_{\nu_t^n}\bigg[\exp\bigg\{\frac{Ka_ng_d(n)}{n^2}J_x^n(t)\bigg\}\bigg].$$

Since $\exp x \le 1+x+2^{-1}x^2\exp|x|$ and $\log(1+y)\le y$, as $J_x^n(t)$ has mean zero with respect to $\nu_t^n$, the previous displayed expression is bounded above by

$$\frac{Ka_n^2 g_d(n)}{2n^{d+2}}\sum_{x\in\mathbb{T}_n^d} E_{\nu_t^n}\big[J_x^n(t)^2\big]\exp\bigg\{\frac{Ka_n g_d(n)}{n^2}\|H\|_\infty\bigg\},$$

because $\upsilon_t$ is bounded. Since $a_n^2 g_d(n)/n^2\to 0$, to conclude the proof of the corollary, it remains to let $n\to\infty$ and then $K\to\infty$. □

## 5. The Burgers viscous equation

We present in this section the properties of the solutions of the Burgers viscous equation (2.9) needed in the proof of Theorem 2.2. Without loss of generality, we assume that in hypothesis (2.8), $\alpha_0=1/2$.

Recall the definition of the space $C^{m+\beta}(\mathbb{T}^d)$ introduced just above Theorem 2.2. Fix a function $\upsilon_0$ in $C^{3+\beta}(\mathbb{T}^d)$ for some $0<\beta<1$. According to [14, Theorem V.6.1] there exists a unique solution to (2.9). Moreover, the partial derivatives of the solution are uniformly bounded on bounded time intervals. This later result is summarized in the next lemma.

**Lemma 5.1.** *Assume that $\upsilon_0$ belongs to $C^{3+\beta}(\mathbb{T}^d)$ for some $0<\beta<1$. For every $T>0$, there is a finite constant $C_0=C_0(T)$, depending only on $\upsilon_0$ and $T$, such that*

$$\sup_{0\le t\le T}\sup_{\theta\in\mathbb{T}^d}\big|\upsilon(t,\theta)\big|\le C_0,\qquad \max_{1\le j\le d}\sup_{0\le t\le T}\sup_{\theta\in\mathbb{T}^d}\big|(\partial_{\theta_j}\upsilon)(t,\theta)\big|\le C_0,$$
$$\max_{1\le i,j\le d}\sup_{0\le t\le T}\sup_{\theta\in\mathbb{T}^d}\big|\big(\partial^2_{\theta_i,\theta_j}\upsilon\big)(t,\theta)\big|\le C_0,$$
$$\max_{1\le i,j,k\le d}\sup_{0\le t\le T}\sup_{\theta\in\mathbb{T}^d}\big|\big(\partial^3_{\theta_i,\theta_j,\theta_k}\upsilon\big)(t,\theta)\big|\le C_0.$$

Recall the definition of the function $L_n:\mathbb{R}_+\times\mathbb{T}_n^d\to\mathbb{R}$ introduced in (4.2).

**Lemma 5.2.** *Let $\upsilon:\mathbb{R}_+\times\mathbb{T}^d\to\mathbb{R}$ be the solution of (2.9) and set $w(t,x)=(1/2)+\epsilon_n\upsilon(t,x/n)$, $x\in\mathbb{T}_n^d$. Then, for every $T>0$, there is a finite constant $C(T)$, depending only on $T$ and $\upsilon_0$, such that*

$$\sup_{0\le t\le T}\max_{x\in\mathbb{T}_n^d}\big|L^n(t,x)\big|\le C(T)\bigg(\epsilon_n^2+\frac{1}{n}\bigg)$$

*for all $n\ge 1$.*

The proof of this lemma is divided in several steps.

**Lemma 5.3.** *Fix $x\in\mathbb{T}_n^d$, $1\le j\le d$ and $0\le t\le T$. We claim that*

$$\begin{aligned}&n^2\big\{E_{\nu^n_{w(t)}}[j_{x-e_j,x}]-E_{\nu^n_{w(t)}}[j_{x,x+e_j}]\big\}\\&\quad=\epsilon_n D_{j,j}(1/2)\big(\partial^2_{x_j}\upsilon\big)(t,x/n)+\bigg(\epsilon_n^2+\frac{\epsilon_n}{n}\bigg)R_n,\end{aligned}$$

*where $R_n$ is a remainder whose absolute value is bounded by a finite constant $C(T)$ which depends only on $T$ and on $\upsilon$ through the $L^\infty$ norm of its first three derivatives.*

**Proof.** By definition of the current and by assumption (2.1), the difference inside braces is equal to

$$\sum_{p=1}^{n_j}\sum_{y\in\mathbb{Z}^d} m_{j,p}(y)E_{\nu^n_{w(t)}}[\tau_{x+y-e_j}g_{j,p}-\tau_{x+y}g_{j,p}]. \tag{5.1}$$

We may rewrite the previous expectation as $E_{\nu^n_{w_{t,x}(\cdot)}}[\tau_{y-e_j}g_{j,p}-\tau_y g_{j,p}]$, where $w_{t,x}(z)=w(t,x+z)$, $z\in\mathbb{T}_n^d$. By Corollary 5.6, this expectation can be written as the sum of two expressions and a remainder. We consider them separately.

The contribution to (5.1) of the first expression in Corollary 5.6 is equal to

$$\frac{-\epsilon_n}{n}\sum_{p=1}^{n_j}\sum_{y\in\mathbb{Z}^d} m_{j,p}(y)\sum_z \big[\nabla_j^n v\big]\big(t,[x+z-e_j]/n\big)E_{\nu^n_{w(t,x)}}[\tau_y g_{j,p}\omega_z],$$

where $\nu^n_{w(t,x)}$ is the homogeneous product Bernoulli measure with density $w(t,x)$. Fix $p$ and $y$. Performing a change of variables we may rewrite the sum over $z$ as

$$\sum_z \big[\nabla_j^n v\big]\big(t,[x+z+y-e_j]/n\big)E_{\nu^n_{w(t,x)}}[g_{j,p}\omega_z]. \tag{5.2}$$

Performing a Taylor expansion around $(t,x/n)$,

$$\begin{aligned}\big[\nabla_x^n v\big]\big(x'\big) &:= n\big[v\big(t,\big[x+x'\big]/n\big)-v(t,x/n)\big]\\ &= \sum_{k=1}^d x'_k(\partial_{x_k}v)(t,x/n)\\ &\quad + \frac{1}{n}\sum_{k,k'} x'_k x'_{k'}(\partial_{x_k}v)(t,x/n)(\partial_{x_{k'}}v)(t,x/n),\end{aligned}$$

plus $R_n/n^2$, where $R_n$ is a remainder whose absolute value is bounded by $C_0$, for some constant $C_0$ depending only on $T$ and on the $L^\infty$ norm of the first three derivatives of $v$. The expression of the remainder $R_n$ may change below from line to line. Note that $[\nabla_j^n v](t,[x+z+y-e_j]/n)=[\nabla_x^n v](z+y)-[\nabla_x^n v](z+y-e_j)$. Therefore an easy computation yields that the sum in (5.2) becomes

$$\begin{aligned}&(\partial_{x_j}v)(t,x/n)\sum_z E_{\nu^n_{w(t,x)}}[g_{j,p}\omega_z]\\ &\quad -\frac{1}{2n}\sum_z\Big\{\big(\partial^2_{x_j}v\big)(t,x/n)-2\sum_{k=1}^d(y_k+z_k)\big(\partial^2_{x_j,x_k}v\big)(t,x/n)\Big\}E_{\nu^n_{w(t,x)}}[g_{j,p}\omega_z],\end{aligned}$$

plus $R_n/n^2$.

Since for each $j$ and $p$, $\sum_y m_{j,p}(y)=0$, in view of (5.4), the contribution to (5.1) of the first expression in Corollary 5.6 is equal to

$$\begin{aligned}&\frac{\epsilon_n}{n^2}\sum_{p=1}^{n_j}\sum_k D_p(j,k)\big(\partial^2_{x_j,x_k}v\big)(t,x/n)\widetilde{g}'_{j,p}\big(w(t,x)\big)+\frac{\epsilon_n}{n^3}R_n\\ &\quad=\frac{\epsilon_n}{n^2}D_{j,j}\big(w(t,x)\big)\big(\partial^2_{x_j}v\big)(t,x/n)+\frac{\epsilon_n}{n^3}R_n,\end{aligned}$$

where $D_p(j,k)$, $D_{j,j}(\rho)$ have been introduced in (2.3). We used in the previous step the identities (2.4). As $w(t,x)=(1/2)+\epsilon_n v(t,x/n)$, by a Taylor expansion, the previous expression is equal to

$$\frac{\epsilon_n}{n^2}D_{j,j}(1/2)\big(\partial^2_{x_j}v\big)(t,x/n)+\left(\frac{\epsilon_n^2}{n^2}+\frac{\epsilon_n}{n^3}\right)R_n.$$

We turn to the contribution to (5.1) of the second expression in Corollary 5.6. It is equal to

$$\frac{\epsilon_n^2}{2n}\sum_{p=1}^{n_j}\sum_{y\in\mathbb{Z}^d} m_{j,p}(y)\sum_{z\neq z'} c_{z,z'}E_{\nu^n_{w(t,x)}}\big[(\tau_y g_{j,p})\omega_z\omega_{z'}\big],$$

where

$$\begin{aligned}c_{z,z'} &= \frac{1}{n}\big(\nabla_j^n v\big)\big([z-e_j]/n\big)\big(\nabla_j^n v\big)\big([z'-e_j]/n\big)\\&\quad - \big(\nabla_j^n v\big)\big([z-e_j]/n\big)\big[v\big(z'/n\big)-v(0)\big]\\&\quad - \big(\nabla_j^n v\big)\big([z'-e_j]/n\big)\big[v(z/n)-v(0)\big].\end{aligned}\tag{5.3}$$

By a change of variables, we may write this expression as

$$\frac{\epsilon_n^2}{2n}\sum_{p=1}^{n_j}\sum_{y\in\mathbb{Z}^d} m_{j,p}(y)\sum_{z\neq z'} c_{z+y,z'+y}E_{\nu^n_{w(t,x)}}[g_{j,p}\omega_z\omega_{z'}].$$

The fact that $\sum_y m_{j,p}(y)=0$ yields that this sum is equal to

$$\frac{\epsilon_n^2}{2n}\sum_{p=1}^{n_j}\sum_{y\in\mathbb{Z}^d} m_{j,p}(y)\sum_{z\neq z'}[c_{z+y,z'+y}-c_{z,z'}]E_{\nu^n_{w(t,x)}}[g_{j,p}\omega_z\omega_{z'}].$$

Note that $m_{j,p}(y)$ and the last expectation vanish except for a finite number of $y,z,z'$. For such $y,z,z'$, a Taylor expansion shows that $c_{z+y,z'+y}-c_{z,z'}$ is of order $n^{-2}$, uniformly in $y,z,z'$. Therefore this sum is bounded in absolute value by $C(T)\epsilon_n^2/n^3$. Since the third expression in Corollary 5.6 is bounded by $C(T)\epsilon_n^3/n^3$, the proof is complete. □

**Lemma 5.4.** *Fix $x\in\mathbb{T}_n^d$, $1\le j\le d$ and $0\le t\le T$. We claim that*

$$\begin{aligned}&a_n\big(\nabla_j^n I_j^n\big)(t,x-e_j)\\&\quad=\epsilon_n\mathbf{m}_j\sigma''_{j,j}(1/2)v(t,x/n)(\partial_{x_j}v)(t,x/n)+\left(\epsilon_n^2+\frac{1}{n}\right)R_n,\end{aligned}$$

*where $R_n$ is a remainder whose absolute value is bounded by $C(T)$, where $C(T)$ is a finite constant which depends only on $T$ and on $v$ through the $L^\infty$ norm of its first three derivatives.*

**Proof.** Let $d_j$ be the cylinder function defined by $d_j(\eta)=c_j(\eta)\eta_0[1-\eta_{e_j}]$. With this notation and since $c_j$ does not depend on $\eta_0,\eta_{e_j}$, we may rewrite the left-hand side of the statement of the lemma as

$$\frac{n}{\epsilon_n}\mathbf{m}_j\big\{E_{\nu^n_{w(t)}}[\tau_x d_j]-E_{\nu^n_{w(t)}}[\tau_{x-e_j}d_j]\big\}.$$

Recall the definition of the measure $\nu^n_{w_{t,x}(\cdot)}$, introduced just after (5.1), and that $\nu^n_{w(t,x)}$ represents the homogeneous product Bernoulli measure with density $w(t,x)$. By Corollary 5.6 and since the absolute value of $c_{z,z'}$ is bounded by $C(T)/n$, the previous expression is equal to

$$\mathbf{m}_j\sum_z\big[\nabla_j^n v\big]\big(t,[x+z-e_j]/n\big)E_{\nu^n_{w(t,x)}}[d_j\omega_z]+\frac{\epsilon_n}{n}R_n.$$

In this formula and below, $R_n$ is a remainder whose absolute value is bounded by $C_0$, for some constant $C_0$ depending only on $T$ and on the $L^\infty$ norm of the first three derivatives of $v$. The exact expression of the remainder $R_n$ may change from line to line.

A Taylor expansion around $x/n$ yields that the previous sum is equal to

$$\mathbf{m}_j(\partial_{x_j}v)(t,x/n)\sum_z E_{\nu^n_{w(t,x)}}[d_j\omega_z]+\frac{1}{n}R_n.$$

By definition of $d_j$ and by (2.5), $\widetilde{d}_j(\rho)=\widetilde{c}_j(\rho)\rho[1-\rho]=\sigma_{j,j}(\rho)$. Hence, by (5.4), the sum over $z$ is equal to $\sigma'_{j,j}(w(t,x/n))$. By (2.8) and a Taylor expansion, this later expression is equal to $\epsilon_n\sigma''_{j,j}(1/2)v(t,x/n)+\epsilon_n^2R_n$. This completes the proof of the lemma. □

**Proof of Lemma 5.2.** The proof is a straightforward consequence of Lemmata 5.3 and 5.4 and from the fact that $v$ is the solution of the equation (2.9). In both lemmata, the constant depends on the $L^\infty$ norm of the first three derivatives of $v$. Lemma 5.1 states that these derivatives are bounded by a constant which depends on $v_0$. $\square$

We conclude this section with some results used above. Let $v : \mathbb{T}^d \to \mathbb{R}$ be a function in $C^1(\mathbb{T}^d)$, and let $w : \mathbb{T}_n^d \to \mathbb{R}$ be given by $w(x) = (1/2) + \epsilon_n v(x/n)$. Recall from (1.3) that we denote by $\nu^n_{w(\cdot)}$ the product measure on $\Omega_n$ in which the density of $\eta_x$ is $w(x/n)$, while $\nu^n_{w(0)}$ represents the homogeneous product measure with constant density equal to $w(0)$.

**Lemma 5.5.** *Let $g : \Omega_n \to \mathbb{R}$ be a local function. Then, there exists a constant $C_0$, depending only on the cylinder function $g$ and on $\|\nabla v\|_\infty$, such that*

$$E_{\nu^n_{w(\cdot)}}[g] = E_{\nu^n_{w(0)}}[g] + \epsilon_n \sum_z \big[v(z/n) - v(0)\big] E_{\nu^n_{w(0)}}[g\omega_z]$$
$$+ \frac{1}{2}\epsilon_n^2 \sum_{z \neq z'} \big[v(z/n) - v(0)\big]\big[v\big(z'/n\big) - v(0)\big] E_{\nu^n_{w(0)}}[g\omega_z\omega_{z'}] + R_n,$$

*where $|R_n| \le C_0(\epsilon_n/n)^3$, $\omega_z = [\eta_z - w(0)]/w(0)[1 - w(0)]$. On the right hand side, the sum is carried out over all $z$ (and $z' \neq z$) in the support of $g$.*

**Proof.** Fix a local function $g : \Omega_n \to \mathbb{R}$, and denote by $\Lambda(g)$ its support. Clearly, as $\nu^n_{w(\cdot)}$, $\nu^n_{w(0)}$ are product measures,

$$E_{\nu^n_{w(\cdot)}}[g] = E_{\nu^n_{w(0)}}\big[g e^H\big],$$

where

$$H(\eta) = \sum_{z \in \Lambda(g)} \eta_z \log\left(1 + \frac{\epsilon_n[v(z/n) - v(0)]}{w(0)}\right)$$
$$+ \sum_{z \in \Lambda(g)} [1 - \eta_z] \log\left(1 - \frac{\epsilon_n[v(z/n) - v(0)]}{1 - w(0)}\right).$$

The result follows from a Taylor expansion up to the third order. $\square$

Recall from (3.7) the definition of the discrete partial derivative in the $j$th direction represented by $\nabla_j^n$, and from (5.3) the definition of $c_{z,z'}$.

**Corollary 5.6.** *Let $g : \Omega_n \to \mathbb{R}$ be a local function. Then, there exists a constant $C_0$, depending only on the cylinder function $g$ and on $\|\nabla v\|_\infty$, such that*

$$E_{\nu^n_{w(\cdot)}}[\tau_{-e_j} g - g] = \frac{-\epsilon_n}{n} \sum_z \big[\nabla_j^n v\big]\big([z - e_j]/n\big) E_{\nu^n_{w(0)}}[g\omega_z]$$
$$+ \frac{\epsilon_n^2}{2n} \sum_{z \neq z'} c_{z,z'} E_{\nu^n_{w(0)}}[g\omega_z\omega_{z'}] + R_n,$$

*where $|R_n| \le C_0(\epsilon_n/n)^3$.*

**Proof.** Fix a local function $g : \Omega_n \to \mathbb{R}$. According to the previous lemma, the expectation appearing on the left-hand side of the statement is equal to

$$\epsilon_n \sum_z \big[v(z/n) - v(0)\big] E_{\nu^n_{w(0)}}\big[[\tau_{-e_j} g - g]\omega_z\big]$$
$$+ \frac{1}{2}\epsilon_n^2 \sum_{z \neq z'} \big[v(z/n) - v(0)\big]\big[v\big(z'/n\big) - v(0)\big] E_{\nu^n_{w(0)}}\big[[\tau_{-e_j} g - g]\omega_z\omega_{z'}\big] + R_n,$$

where $|R_n| \le C_0(\epsilon_n/n)^3$, for some constant $C_0$ which depends only on $g$ and $\|\nabla v\|_\infty$. Here, the sum over $z$ is carried out over all $z$ (and $z' \neq z$) in the support of $\tau_{-e_j} g - g$. As the measure $\nu^n_{w(0)}$ is homogeneous, a change of variables permits to complete the proof of the lemma. □

Let $g : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$ be a local function. Recall from (2.2) the definition of the smooth function $\widetilde{g} : [0,1] \to \mathbb{R}$. A similar computation to the one presented in the proof of Lemma 5.5 yields that

$$\widetilde{g}'(\theta) = \sum_z E_{\nu_\theta}[g\omega_z], \quad \theta \in [0,1]. \tag{5.4}$$

Along the same lines, we may also prove the Einstein relation.

**Proposition 5.7.** *For every $\alpha \in (0,1)$, $1 \le j \le d$,*

$$\widetilde{c}_j(\alpha) = \sum_{p=1}^{n_j} D_p(j,j)\widetilde{g}'_{j,p}(\alpha) \quad \text{and} \quad \sum_{p=1}^{n_j} D_p(j,k)\widetilde{g}'_{j,p}(\alpha) = 0 \quad \text{for } k \neq j.$$

**Proof.** Fix $1 \le j \le d$, $\alpha \in (0,1)$ and let $u : \mathbb{T}^d \to (0,1)$ be a differentiable function such that $u(0) = \alpha$, $(\partial_{x_j} u)(0) \neq 0$. Take the expectation with respect to $\nu^n_{u(\cdot)}$ on both sides of (2.1).

First, note that $E_{\nu_\alpha}[c_j(\eta)[\eta_0 - \eta_{e_j}]] = 0$ since $c_j$ does not depend on $\eta_0$ and $\eta_{e_j}$. For the left-hand side, by the proof of Lemma 5.5 and since $u(0) = \alpha$,

$$E_{\nu^n_{u(\cdot)}}\big[c_j(\eta)[\eta_0 - \eta_{e_j}]\big] = \sum_z \big[u(z/n) - \alpha\big] E_{\nu_\alpha}\big[c_j(\eta)[\eta_0 - \eta_{e_j}]\omega_z\big] + O\big(1/n^2\big),$$

where $\omega_z = [\eta_z - \alpha]/\alpha(1-\alpha)$. Since $c_j$ does not depend on $\eta_0$ and $\eta_{e_j}$, for $z \neq 0, e_j$,

$$E_{\nu_\alpha}\big[c_j(\eta)[\eta_0 - \eta_{e_j}]\omega_z\big] = 0.$$

As $u(0) = \alpha$, the sum in the penultimate line is equal to

$$\big[u(e_j/n) - \alpha\big] E_{\nu_\alpha}\big[c_j(\eta)[\eta_0 - \eta_{e_j}]\omega_{e_j}\big] = -\big[u(e_j/n) - \alpha\big] E_{\nu_\alpha}\big[c_j(\eta)\big].$$

We turn to the expectation of the right-hand side of (2.1). By the proof of Lemma 5.5 and since $\sum_y m_{j,p}(y) = 0$, the first term in the expansion vanishes so that

$$\sum_{p=1}^{n_j} \sum_{y \in \mathbb{Z}^d} m_{j,p}(y) E_{\nu^n_{u(\cdot)}}[\tau_y g_{j,p}] = \sum_{p=1}^{n_j} \sum_{y \in \mathbb{Z}^d} m_{j,p}(y) \sum_z \big[u(z/n) - \alpha\big] E_{\nu_\alpha}\big[(\tau_y g_{j,p})\omega_z\big] + O\big(1/n^2\big).$$

A change of variables $\eta \mapsto \tau_y \eta$ and a Taylor expansion permit to rewrite the sum as

$$\frac{1}{n} \sum_{p=1}^{n_j} \sum_{y \in \mathbb{Z}^d} m_{j,p}(y) \sum_z (z+y) \cdot (\nabla u)(0) E_{\nu_\alpha}[g_{j,p}\omega_z] + O\big(1/n^2\big).$$

Since $\sum_y m_{j,p}(y) = 0$ and, by definition, $\sum_y y_k m_{j,p}(y) = -D_p(j,k)$, the last expression is equal to

$$-\frac{1}{n} \sum_{p=1}^{n_j} \big[D_p(j,\cdot) \cdot (\nabla u)(0)\big] \widetilde{g}'_{j,p}(\alpha) + O\big(1/n^2\big).$$

Putting together the previous estimates, we conclude that for every $v \in \mathbb{R}^d$,

$$v_j \widetilde{c}_j(\alpha) = \sum_{p=1}^{n_j} \sum_k D_p(j,k) v_k \widetilde{g}'_{j,p}(\alpha).$$

This completes the proof of the proposition. □

## 6. The adjoint generator

Fix a function $\varrho : \mathbb{T}_n^d \to (0,1)$. Throughout this section, $\nu_\varrho$ is a product measure on $\Omega_n$ with marginals given by $E_{\nu_\varrho}[\eta(x)] = \varrho(x)$, $x \in \mathbb{T}_n^d$. Recall that we denote by $\chi(\alpha)$ the static compressibility, $\chi(\alpha) = \alpha[1-\alpha]$.

For each $q \geq 0$, recall the definition of the set $\mathcal{E}_q$: $\mathcal{E}_q = \{A \subset \mathbb{T}_n^d : |A| = k\}$. Denote by $\mathbf{P}_\varrho^{(q)}(\tau_x f)$ the projection of the cylinder function $\tau_x f$ over the linear set of functions of degree $q$:

$$\big[\mathbf{P}_\varrho^{(q)}(\tau_x f)\big](\eta) = \sum_{A \in \mathcal{E}_q} \mathfrak{f}(x, A)\omega_\varrho(A + x).$$

In particular, $\mathbf{P}_\varrho^{(0)}(\tau_x f) = E_{\nu_\varrho}[\tau_x f]$. Let $\mathbf{P}_\varrho^{(+q)} = \sum_{p \geq q} \mathbf{P}_\varrho^{(p)}$ so that

$$\big[\mathbf{P}_\varrho^{(+q)}(\tau_x f)\big](\eta) = \sum_{p \geq q} \sum_{A \in \mathcal{E}_p} \mathfrak{f}(x, A)\omega_\varrho(A + x).$$

We represent $\mathbf{P}_\varrho^{(+1)}$ by $\mathbf{P}_\varrho$:

$$\big[\mathbf{P}_\varrho(\tau_x f)\big](\eta) = (\tau_x f)(\eta) - E_{\nu_\varrho}[\tau_x f].$$

The statement of Lemma 6.1 requires some notation. Recall from (3.5) that $D_j$ stands for the difference operator, and from (3.6) that we denote by $j_{x,x+e_j}$ the instantaneous current over the bond $(x, x+e_j)$.

For $1 \leq j \leq d$, $1 \leq p \leq n_j$, $x \in \mathbb{T}_n^d$, let

$$\begin{aligned}
&A_j(x) = \frac{\chi(\varrho(x)) + \chi(\varrho(x+e_j))}{\chi(\varrho(x))\chi(\varrho(x+e_j))}, \\
&B_{j,p}^{(1)}(x) = \frac{1}{2} \sum_{y \in \mathbb{T}_n^d} m_{j,p}(y) A_j(x-y)(D_j\varrho)(x-y), \\
&E_j^{(1)}(x) = \frac{1}{2} A_j(x)\big[(D_j\varrho)(x)\big]^2, \qquad F_j^{(1)}(x) = \frac{[D_j(\chi \circ \varrho)](x)(D_j\varrho)(x)}{2\chi(\varrho(x+e_j))}, \\
&G_j^{(1)}(x) = \frac{[D_j(\chi \circ \varrho)](x)(D_j\varrho)(x)}{2\chi(\varrho(x))}.
\end{aligned} \tag{6.1}$$

Finally, for $A \subset \mathbb{T}_n^d$, let

$$H_j^{(1)}(\varrho, x, A) = E_j^{(1)}(x)\mathfrak{c}_j(x, A) + \sum_{p=1}^{n_j} B_{j,p}^{(1)}(x)\mathfrak{g}_{j,p}(x, A) + J_j^{(1)}(x, A), \tag{6.2}$$

where

$$\begin{aligned}
J_j^{(1)}(x, A) = &-\Upsilon_{\{0,e_j\}}(A)\big[(D_j\varrho)(x)\big]^2 \mathfrak{c}_j\big(x, A \setminus \{0, e_j\}\big) \\
&+ \Upsilon_{\{0\}}(A) F_j^{(1)}(x)\mathfrak{c}_j\big(x, A \setminus \{0\}\big) \\
&+ \Upsilon_{\{e_j\}}(A) G_j^{(1)}(x)\mathfrak{c}_j\big(x, A \setminus \{e_j\}\big).
\end{aligned}$$

In this formula, $\mathfrak{c}_j(x, A)$, $\mathfrak{g}_{j,p}(x, A)$ represent the Fourier coefficients, introduced in (3.2), of the cylinder functions $c_j$, $g_{j,p}$, respectively; and $\Upsilon_B$ stand for the function introduced in (3.9).

It follows from (3.3) that there exists $\ell \geq 1$ such that $H_j^{(1)}(\varrho, x, A) = 0$ if $A \not\subset \Lambda_\ell$. Note that the functions of $x$ which appear in the previous formula either contain the product of derivatives [this is the case of $E_j^{(1)}$, $F_j^{(1)}$ and $G_j^{(1)}$] or a mean-zero sum of discrete derivatives, which is the case of $B_{j,p}^{(1)}$. This structure makes $n^2 H_j^{(1)}(\varrho, x, A)$ bounded in $n$ if the reference density is good enough since these derivatives absorb the speeded-up factor $n^2$.

**Lemma 6.1.** *Denote by $L^{S,*}_{n,\nu_\varrho}$ the adjoint of $L^S_n$ in $L^2(\nu_\varrho)$. Then,*

$$L^{S,*}_{n,\nu_\varrho}\mathbf{1} = \sum_{j=1}^d \sum_{x\in\mathbb{T}^d_n} \{E_{\nu_\varrho}[j_{x-e_j,x}] - E_{\nu_\varrho}[j_{x,x+e_j}]\}\omega_\varrho(x) + \sum_{j=1}^d \sum_{A:|A|\geq 2} \sum_{x\in\mathbb{T}^d_n} H^{(1)}_j(\varrho, x, A)\omega_\varrho(A+x),$$

*where the (finite) sum over $A$ is performed over finite subsets $A$ with at least two elements.*

Note that the first term on the right-hand side contains only terms of degree 1, while the second one only terms of degree 2 or higher.

The proof of this lemma is divided in four Lemmata and one identity, presented in (6.3). We first compute the adjoint $L^{S,*}_{n,\nu_\varrho}$ of $L^S_n$.

**Lemma 6.2.** *For $x\in\mathbb{T}^d_n$ and $1\leq j\leq d$, let*

$$J_{x,x+e_j}(\eta) = \frac{\nu_\varrho(\sigma^{x,x+e_j}\eta)}{\nu_\varrho(\eta)}.$$

*Then, for any $f\in L^2(\nu_\varrho)$,*

$$\big(L^{S,*}_{n,\nu_\varrho} f\big)(\eta) = \sum_{x\in\mathbb{T}^d_n} \sum_{j=1}^d c_j(\tau_x\eta) J_{x,x+e_j}(\eta)\big\{f\big(\sigma^{x,x+e_j}\eta\big) - f(\eta)\big\} + \sum_{x\in\mathbb{T}^d_n} \sum_{j=1}^d c_j(\tau_x\eta)\big\{J_{x,x+e_j}(\eta) - 1\big\} f(\eta).$$

The proof of this lemma is elementary and left to the reader.

**Lemma 6.3.** *We have that*

$$\begin{aligned}\big(L^{S,*}_{n,\nu_\varrho}\mathbf{1}\big)(\eta) &= \sum_{j=1}^d \sum_{x\in\mathbb{T}^d_n} \{E_{\nu_\varrho}[j_{x-e_j,x}] - E_{\nu_\varrho}[j_{x,x+e_j}]\}\omega_\varrho(x) \\ &\quad + \sum_{x\in\mathbb{T}^d_n} \sum_{j=1}^d (\mathbf{P}_\varrho \tau_x c_j)(\eta)(D_j\varrho)(x)\big[\omega_\varrho(x) - \omega_\varrho(x+e_j)\big] \\ &\quad - \sum_{x\in\mathbb{T}^d_n} \sum_{j=1}^d c_j(\tau_x\eta)\big[(D_j\varrho)(x)\big]^2 \omega_\varrho(x)\omega_\varrho(x+e_j).\end{aligned}$$

**Proof.** By Lemma 6.2,

$$L^{S,*}_{n,\nu_\varrho}\mathbf{1} = \sum_{x\in\mathbb{T}^d_n} \sum_{j=1}^d c_j(\tau_x\eta)\big\{J_{x,x+e_j}(\eta) - 1\big\}.$$

The definition of $J_{x,x+e_j}$ and a straightforward computation yield that this expression is equal to

$$\sum_{x\in\mathbb{T}^d_n} \sum_{j=1}^d c_j(\tau_x\eta)(D_j\varrho)(x)\left\{\frac{\eta_x(1-\eta_{x+e_j})}{\varrho(x)[1-\varrho(x+e_j)]} - \frac{\eta_{x+e_j}(1-\eta_x)}{\varrho(x+e_j)[1-\varrho(x)]}\right\}.$$

Recall that $\omega_\varrho(x) = [\eta(x) - \varrho(x)]/\chi(\varrho(x))$. The expression inside braces can be written as

$$\omega_\varrho(x) - \omega_\varrho(x+e_j) - (D_j\varrho)(x)\omega_\varrho(x)\omega_\varrho(x+e_j).$$

Therefore,

$$\begin{aligned}\big(L^{S,*}_{n,\nu_\varrho}\mathbf{1}\big)(\eta) = &\sum_{x\in\mathbb{T}^d_n}\sum_{j=1}^{d} E_{\nu_\varrho}\big[c_j(\tau_x\eta)\big](D_j\varrho)(x)\big[\omega_\varrho(x) - \omega_\varrho(x+e_j)\big] \\ &+ \sum_{x\in\mathbb{T}^d_n}\sum_{j=1}^{d}(\mathbf{P}_\varrho\tau_x c_j)(\eta)(D_j\varrho)(x)\big[\omega_\varrho(x) - \omega_\varrho(x+e_j)\big] \\ &- \sum_{x\in\mathbb{T}^d_n}\sum_{j=1}^{d} c_j(\tau_x\eta)\big[(D_j\varrho)(x)\big]^2\omega_\varrho(x)\omega_\varrho(x+e_j).\end{aligned}$$

Note that the second and third lines contain only terms of degree 2 or more, while the first line have only terms of degree 1.

Since $c_j$ does not depend on $\eta(0)$ and $\eta(e_j)$, by definition of the instantaneous current $j_{x,x+e_j}$,

$$E_{\nu_\varrho}\big[c_j(\tau_x\eta)\big](D_j\varrho)(x) = -E_{\nu_\varrho}\big[c_j(\tau_x\eta)\big[\eta(x) - \eta(x+e_j)\big]\big] = -E_{\nu_\varrho}[j_{x,x+e_j}].$$

To complete the proof, it remains to insert this expression in the first line of the formula for $(L^{S,*}_{n,\nu_\varrho}\mathbf{1})(\eta)$ and to sum by parts. □

In view of (3.4), the third term of Lemma 6.3 can be written as

$$-\sum_{j=1}^{d}\sum_{x\in\mathbb{T}^d_n}\sum_{A}\big[(D_j\varrho)(x)\big]^2\mathfrak{c}_j(x,A)\omega_\varrho(A+x)\omega_\varrho(x)\omega_\varrho(x+e_j),$$

where $\mathfrak{c}_j(x,A)$ stands for the Fourier coefficients of $\tau_x c_j$, given by (3.2). As $c_j$ does not depend on $\eta(0)$ and $\eta(e_j)$, $\mathfrak{c}_j(x,A) = 0$ if $A$ contains 0 or $e_j$. We may therefore restrict the sum to sets which do not contain these points and rewrite the previous expression as

$$\begin{aligned}&-\sum_{j=1}^{d}\sum_{x\in\mathbb{T}^d_n}\sum_{A:A\cap\{0,e_j\}=\varnothing}\big[(D_j\varrho)(x)\big]^2\mathfrak{c}_j(x,A)\omega_\varrho\big(\big[A\cup\{0,e_j\}\big]+x\big) \\ &\quad= -\sum_{j=1}^{d}\sum_{x\in\mathbb{T}^d_n}\sum_{A:A\supset\{0,e_j\}}\big[(D_j\varrho)(x)\big]^2\mathfrak{c}_j\big(x,A\setminus\{0,e_j\}\big)\omega_\varrho(A+x).\end{aligned}\tag{6.3}$$

We turn to the second term of Lemma 6.3.

**Lemma 6.4.** *For each* $1\le j\le d$,

$$\begin{aligned}&\sum_{x\in\mathbb{T}^d_n}\big[\mathbf{P}_\varrho(\tau_x c_j)\big](\eta)(D_j\varrho)(x)\big[\omega_\varrho(x) - \omega_\varrho(x+e_j)\big] \\ &\quad= \frac{1}{2}\sum_{x\in\mathbb{T}^d_n}\big[\mathbf{P}^{(+2)}_\varrho(\tau_x j_{0,e_j})\big](\eta)A_j(x)(D_j\varrho)(x) \\ &\qquad+ \frac{1}{2}\sum_{x\in\mathbb{T}^d_n}\big[\mathbf{P}^{(+2)}_\varrho(\tau_x c_j)\big](\eta)A_j(x)\big[(D_j\varrho)(x)\big]^2 \\ &\qquad+ \frac{1}{2}\sum_{x\in\mathbb{T}^d_n}\big[\mathbf{P}_\varrho(\tau_x c_j)\big](\eta)\bigg\{\frac{\omega(x)}{\chi(\varrho(x+e_j))} + \frac{\omega(x+e_j)}{\chi(\varrho(x))}\bigg\}\big[D_j(\chi\circ\varrho)\big](x)(D_j\varrho)(x).\end{aligned}\tag{6.4}$$

**Proof.** Recall the definition of $\xi_\varrho(x) : \xi_\varrho(x) = \eta(x) - \varrho(x)$, $x \in \mathbb{T}_n^d$. Fix $j$ and write $\omega_\varrho(x) - \omega_\varrho(x+e_j)$ as

$$\begin{aligned} &\frac{1}{2\chi(\varrho(x))\chi(\varrho(x+e_j))}\big[\xi_\varrho(x) - \xi_\varrho(x+e_j)\big]\big[\chi\big(\varrho(x+e_j)\big) + \chi\big(\varrho(x)\big)\big] \\ &\quad + \frac{1}{2\chi(\varrho(x))\chi(\varrho(x+e_j))}\big[\xi_\varrho(x) + \xi_\varrho(x+e_j)\big]\big[\chi\big(\varrho(x+e_j)\big) - \chi\big(\varrho(x)\big)\big]. \end{aligned} \tag{6.5}$$

On the other hand, taking the operator $\mathbf{P}_\varrho \circ \tau_x$ for (3.6), one can obtain

$$\begin{aligned} \big[\mathbf{P}_\varrho(\tau_x j_{0,e_j})\big](\eta) = {} & E_{\nu_\varrho}[\tau_x c_j]\big[\xi_\varrho(x) - \xi_\varrho(x+e_j)\big] \\ & + \big[\mathbf{P}_\varrho(\tau_x c_j)\big](\eta)\big[\xi_\varrho(x) - \xi_\varrho(x+e_j)\big] \\ & - \big[\mathbf{P}_\varrho(\tau_x c_j)\big](\eta)(D_j\varrho)(x). \end{aligned} \tag{6.6}$$

From (6.5) and (6.6), the left-hand side of (6.4) becomes

$$\begin{aligned} &\frac{1}{2}\sum_{x\in\mathbb{T}_n^d}\big[\mathbf{P}_\varrho(\tau_x j_{0,e_j})\big](\eta)A_j(x)(D_j\varrho)(x) \\ &\quad - \frac{1}{2}\sum_{x\in\mathbb{T}_n^d} E_{\nu_\varrho}[\tau_x c_j]\big[\xi_\varrho(x) - \xi_\varrho(x+e_j)\big]A_j(x)(D_j\varrho)(x) \\ &\quad + \frac{1}{2}\sum_{x\in\mathbb{T}_n^d}\big[\mathbf{P}_\varrho(\tau_x c_j)\big](\eta)A_j(x)\big[(D_j\varrho)(x)\big]^2 + L_3, \end{aligned}$$

where $L_3$ is the last term appearing on the right-hand side of (6.4) and $A_j(x)$ has been introduced in (6.1).

Since $c_j$ does not depend on $\eta(0)$, $\eta(e_j)$, the expectation with respect to $\nu_\varrho$ of the left-had side of (6.4) vanishes. It is also clear that the covariance of this sum with respect to $\xi_\varrho(z)$ vanishes for all $z \in \mathbb{T}_n^d$. We may therefore introduce the operator $\mathbf{P}_\varrho^{(+2)}$ in front of the sum. By doing so, the second sum of the previous formula vanishes because it contains only terms of degree 1. This completes the proof of the lemma. □

We further express the sums on the right-hand side of (6.4) in terms of the Fourier coefficients of the cylinder functions. Recall the notation introduced in (6.1) and below.

**Lemma 6.5.** *For each* $1 \le j \le d$,

$$\begin{aligned} &\sum_{x\in\mathbb{T}_n^d}\big[\mathbf{P}_\varrho(\tau_x c_j)\big](\eta)(D_j\varrho)(x)\big[\omega_\varrho(x) - \omega_\varrho(x+e_j)\big] \\ &\quad = \sum_{A:|A|\ge 2}\sum_{x\in\mathbb{T}_n^d}\sum_{p=1}^{n_j} B_{j,p}^{(1)}(x)\mathfrak{g}_{j,p}(x,A)\omega_\varrho(A+x) \\ &\qquad + \sum_{A:|A|\ge 2}\sum_{x\in\mathbb{T}_n^d} E_j^{(1)}(x)\mathfrak{c}_j(x,A)\omega_\varrho(A+x) \\ &\qquad + \sum_{\substack{A:|A|\ge 2\\ A\ni 0}}\sum_{x\in\mathbb{T}_n^d} F_j^{(1)}(x)\mathfrak{c}_j\big(x, A\setminus\{0\}\big)\omega_\varrho(A+x) \\ &\qquad + \sum_{\substack{A:|A|\ge 2\\ A\ni e_j}}\sum_{x\in\mathbb{T}_n^d} G_j^{(1)}(x)\mathfrak{c}_j\big(x, A\setminus\{e_j\}\big)\omega_\varrho(A+x). \end{aligned} \tag{6.7}$$

**Proof.** Fix $1 \le j \le d$. We consider separately each term on the right-hand side of (6.4). Let $B_j(x) = A_j(x)(D_j\varrho)(x)$. By the gradient conditions (2.1), the first term can be written as

$$\frac{1}{2}\sum_{x\in\mathbb{T}_n^d}\sum_{p=1}^{n_j}\sum_{y\in\mathbb{T}_n^d} m_{j,p}(y)\big[\mathbf{P}_\varrho^{(+2)}(\tau_{x+y}g_{j,p})\big](\eta)B_j(x).$$

Perform the change of variables $x' = x + y$ and express the cylinder function $g_{j,p}$ in terms of its Fourier coefficients to rewrite this expression as

$$\frac{1}{2}\sum_{x\in\mathbb{T}_n^d}\sum_{p=1}^{n_j}\bigg(\sum_{y\in\mathbb{T}_n^d} m_{j,p}(y)B_j(x-y)\bigg)\sum_{A:|A|\ge 2}\mathfrak{g}_{j,p}(x,A)\omega_\varrho(A+x).$$

This expression corresponds to the first one on the right-hand side of (6.7). The other three can be obtained easily. □

Recall the definition of the asymmetric part of the generator introduced in (1.7). For $1 \le j \le d$, let $C_j, I_j : \mathbb{T}_n^d \to \mathbb{R}$ be given by

$$C_j(x) = \mathbf{m}_j\varrho(x)\big[1-\varrho(x+e_j)\big], \qquad I_j(x) = E_{\nu_\varrho}[\tau_x c_j]C_j(x). \tag{6.8}$$

For $1 \le j \le d$, $1 \le p \le n_j$, $x \in \mathbb{T}_n^d$, let

$$\begin{aligned}
B_{j,p}^{(2)}(x) &= -\frac{1}{2}\sum_{y\in\mathbb{T}_n^d} m_{j,p}(y)A_j(x-y)(C_j\varrho)(x-y),\\
E_j^{(2)}(x) &= -\frac{1}{2}A_j(x)(D_j\varrho)(x)C_j(x),\\
F_j^{(2)}(x) &= -\frac{[D_j(\chi\circ\varrho)](x)(C_j\varrho)(x)}{2\chi(\varrho(x+e_j))},\\
G_j^{(2)}(x) &= -\frac{[D_j(\chi\circ\varrho)](x)(C_j\varrho)(x)}{2\chi(\varrho(x))}.
\end{aligned} \tag{6.9}$$

For $A \subset \mathbb{T}_n^d$, let

$$H_j^{(2)}(\varrho,x,A) = E_j^{(2)}(x)\mathfrak{c}_j(x,A) + \sum_{p=1}^{n_j} B_{j,p}^{(2)}(x)\mathfrak{g}_{j,p}(x,A) + J_j^{(2)}(x,A), \tag{6.10}$$

where

$$\begin{aligned}
J_j^{(2)}(x,A) = {} & \Upsilon_{\{0,e_j\}}(A)(D_j\varrho)(x)C_j(x)\mathfrak{c}_j\big(x,A\setminus\{0,e_j\}\big)\\
& + \Upsilon_{\{0\}}(A)F_j^{(2)}(x)\mathfrak{c}_j\big(x,A\setminus\{0\}\big)\\
& + \Upsilon_{\{e_j\}}(A)G_j^{(2)}(x)\mathfrak{c}_j\big(x,A\setminus\{e_j\}\big).
\end{aligned}$$

**Lemma 6.6.** *Let $L_{n,\nu_\varrho}^{T,*}$ be the adjoint of $L_n^T$ in $L^2(\nu_\varrho)$. Then,*

$$\begin{aligned}
L_{n,\nu_\varrho}^{T,*}\mathbf{1} = {} & -\sum_{x\in\mathbb{T}_n^d}\sum_{j=1}^{d}(D_jI_j)(x-e_j)\omega_\varrho(x)\\
& + \sum_{j=1}^{d}\sum_{A:|A|\ge 2}\sum_{x\in\mathbb{T}_n^d} H_j^{(2)}(\varrho,x,A)\omega_\varrho(A+x),
\end{aligned}$$

*where the (finite) sum over A is performed over finite subsets A with at least two elements.*

The proof of this lemma relies on the next two lemmata.

**Lemma 6.7.** *Recall the definition of $J_{x,x+e_j}$ given in Lemma 6.2. Then, for any $f \in L^2(\nu_\varrho)$,*

$$\big(L^{T,*}_{n,\nu_\varrho} f\big)(\eta) = \sum_{x\in\mathbb{T}^d_n}\sum_{j=1}^{d} \mathbf{m}_j(\tau_x c_j)(\eta) J_{x,x+e_j}(\eta)(1-\eta_x)\eta_{x+e_j}\big\{f\big(\sigma^{x,x+e_j}\eta\big) - f(\eta)\big\}$$
$$+ \sum_{x\in\mathbb{T}^d_n}\sum_{j=1}^{d} \mathbf{m}_j(\tau_x c_j)(\eta)\big\{(1-\eta_x)\eta_{x+e_j} J_{x,x+e_j}(\eta) - \eta_x(1-\eta_{x+e_j})\big\} f(\eta).$$

**Lemma 6.8.** *We have that*

$$L^{T,*}_{n,\nu_\varrho}\mathbf{1} = -\sum_{x\in\mathbb{T}^d_n}\sum_{j=1}^{d}(D_j I_j)(x-e_j)\omega_\varrho(x)$$
$$-\sum_{x\in\mathbb{T}^d_n}\sum_{j=1}^{d}\big[\mathbf{P}_\varrho(\tau_x c_j)\big](\eta) C_j(x)\big[\omega_\varrho(x) - \omega_\varrho(x+e_j)\big]$$
$$+\sum_{x\in\mathbb{T}^d_n}\sum_{j=1}^{d}(\tau_x c_j)(\eta) C_j(x)(D_j\varrho)(x)\omega_\varrho(x)\omega_\varrho(x+e_j).$$

*where $I_j(x) = E_{\nu_\varrho}[\tau_x c_j] C_j(x)$.*

**Proof.** Recall the definition of $C_j$. It follows from the previous lemma and a straightforward computation that

$$L^{T,*}_{n,\nu_\varrho}\mathbf{1} = \sum_{x\in\mathbb{T}^d_n}\sum_{j=1}^{d}(\tau_x c_j)(\eta) C_j(x)\big[\omega_\varrho(x+e_j) - \omega_\varrho(x)\big]$$
$$+\sum_{x\in\mathbb{T}^d_n}\sum_{j=1}^{d}(\tau_x c_j)(\eta) C_j(x)(D_j\varrho)(x)\omega_\varrho(x)\omega_\varrho(x+e_j).$$

It remains to add and subtract $E_{\nu_\varrho}[\tau_x c_j]$ in the first term and to sum by parts. □

**Proof of Lemma 6.6.** The expression of $L^{T,*}_{n,\nu_\varrho}\mathbf{1}$ is similar to the one of $L^{S,*}_{n,\nu_\varrho}\mathbf{1}$. In the second and third terms one has to replace $D_j\varrho$ by $-C_j$. We may thus follow the arguments presented for the symmetric part to complete the proof of Lemma 6.6. □

## Acknowledgements

Part of this work was done during K. Tsunoda's visit to IMPA. He would like to thank IMPA for numerous support and warm hospitality during his visit. M. Jara acknowledges CNPq for its support through the Grant 305075/2017-9, FAPERJ for its support through the Grant E-29/203.012/2018 and ERC for its support through the European Unions Horizon 2020 research and innovative programme (Grant Agreement No. 715734). C. Landim has been partially supported by FAPERJ CNE E-26/201.207/2014, by CNPq Bolsa de Produtividade em Pesquisa PQ 303538/2014-7, and by ANR-15-CE40-0020-01 LSD of the French National Research Agency. K. Tsunoda has been partially supported by JSPS KAKENHI, Grant-in-Aid for Early-Career Scientists 18K13426. The authors are grateful to the anonymous referees for the careful reading and their comments.

## Conflict of interest

The authors declare that they have no conflict of interest.

## References


[1] J. Beltrán and C. Landim. A lattice gas model for the incompressible Navier–Stokes equation. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** (2008) 886–914. MR2453775 https://doi.org/10.1214/07-AIHP125

[2] R. L. Dobrushin. Caricature of hydrodynamics. In *IXth International Congress on Mathematical Physics (Swansea, 1988)* 117–132. Adam Hilger, Bristol, 1989. MR1033758

[3] R. L. Dobrushin, A. Pellegrinotti and Yu. M. Suhov. One-dimensional harmonic lattice caricature of hydrodynamics: A higher correction. *J. Stat. Phys.* **61** (1990) 387–402. MR1084285 https://doi.org/10.1007/BF01013971

[4] R. L. Dobrushin, A. Pellegrinotti, Yu. M. Suhov and L. Triolo. One-dimensional harmonic lattice caricature of hydrodynamics: Second approximation. *J. Stat. Phys.* **52** (1988) 423–439. MR0968594 https://doi.org/10.1007/BF01016423

[5] R. Esposito, R. Marra and H.-T. Yau. Diffusive limit of asymmetric simple exclusion. *Rev. Math. Phys.* **6** (1994) 1233–1267. MR1301374 https://doi.org/10.1142/S0129055X94000444

[6] R. Esposito, R. Marra and H.-T. Yau. Navier–Stokes equations for stochastic lattice gases. *Comm. Math. Phys.* **182** (1996) 395–456. MR1388943 https://doi.org/10.1103/PhysRevE.53.4486

[7] T. Funaki and K. Tsunoda. Motion by mean curvature from Glauber–Kawasaki dynamics. *J. Stat. Phys.* **177** (2019) 183–208. MR4010776 https://doi.org/10.1007/s10955-019-02364-7

[8] M. Jara and C. Landim. The stochastic heat equation as the limit of a stirring dynamics perturbed by a voter model, 2018. Available at arXiv:2008.03076.

[9] M. Jara and O. Menezes. Non-equilibrium fluctuations for a reaction–diffusion model via relative entropy. *Markov Processes Relat. Fields* **26** (2020) 95–124.

[10] M. Jara and O. Menezes. Nonequilibrium fluctuations of interacting particle systems, 2018. Available at arXiv:1810.09526.

[11] C. Kipnis and C. Landim. *Scaling Limits of Interacting Particle Systems*. *Grundlheren der mathematischen Wissenschaften* **320**. Springer, Berlin, 1999. MR1707314 https://doi.org/10.1007/978-3-662-03752-2

[12] C. Kipnis, C. Landim and S. Olla. Macroscopic properties of a stationary non-equilibrium distribution for a non-gradient interacting particle system. *Ann. Inst. Henri Poincaré B, Calc. Probab. Stat.* **31** (1995) 191–221. MR1340037

[13] T. Komorowski, C. Landim and S. Olla. *Fluctuations in Markov Processes. Time Symmetry and Martingale Approximation*. *Grundlheren der mathematischen Wissenschaften* **345**. Springer, Heidelberg, 2012. MR2952852 https://doi.org/10.1007/978-3-642-29880-6

[14] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uraltseva. *Linear and Quasilinear Equations of Parabolic Type*. *Translations of Mathematical Monographs* **23**. American Mathematical Society, Providence, RI, 1968. MR0241822

[15] C. Landim, S. Olla and H. T. Yau. Some properties of the diffusion coefficient for asymmetric simple exclusion processes. *Ann. Probab.* **24** (1996) 1779–1807. MR1415229 https://doi.org/10.1214/aop/1041903206

[16] C. Landim, S. Olla and H. T. Yau. First-order correction for the hydrodynamic limit of asymmetric simple exclusion processes in dimension $d \geq 3$. *Comm. Pure Appl. Math.* **50** (1997) 149–203. MR1426710 https://doi.org/10.1002/(SICI)1097-0312(199702)50:2<149::AID-CPA2>3.3.CO;2-Q

[17] C. Landim, R. M. Sued and G. Valle. Hydrodynamic limit of asymmetric exclusion processes under diffusive scaling in $d \geq 3$. *Comm. Math. Phys.* **249** (2004) 215–247. MR2080952 https://doi.org/10.1007/s00220-004-1076-9

[18] J. Quastel. Diffusion of color in the simple exclusion process. *Comm. Pure Appl. Math.* **45** (1992) 623–679. MR1162368 https://doi.org/10.1002/cpa.3160450602

[19] J. Quastel and H.-T. Yau. Lattice gases, large deviations, and the incompressible Navier–Stokes equations. *Ann. of Math.* **148** (1998) 51–108. MR1652971 https://doi.org/10.2307/120992

[20] F. Rezakhanlou. Hydrodynamic limit for attractive particle systems on $\mathbf{Z}^d$. *Comm. Math. Phys.* **140** (1991) 417–448. MR1130693

[21] S. R. S. Varadhan. Nonlinear diffusion limit for a system with nearest neighbor interactions II. In *Asymptotic Problems in Probability Theory: Stochastic Models and Diffusions on Fractals (Sanda/Kyoto, 1990)* 75–128. K. D. Elworthy and N. Ikeda (Eds). *Pitman Res. Notes Math. Ser.* **283**. Longman Sci. Tech., Harlow, 1993. MR1354152