Electronic Journal of Statistics
Vol. 14 (2020) 458–502
ISSN: 1935-7524
arXiv: 1803.06446

On polyhedral estimation of signals via indirect observations

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Abstract: We consider the problem of recovering linear image of unknown signal belonging to a given convex compact signal set from noisy observation of another linear image of the signal. We develop a simple generic efficiently computable *non*linear in observations "polyhedral" estimate along with computation-friendly techniques for its design and risk analysis. We demonstrate that under favorable circumstances the resulting estimate is provably near-optimal in the minimax sense, the "favorable circumstances" being less restrictive than the weakest known so far assumptions ensuring near-optimality of estimates which are linear in observations.

MSC 2010 subject classifications: Primary 62J05, 62H12; secondary 62G08.
Keywords and phrases: Functional estimation, statistical linear inverse problems, nonlinear estimation, sketching.

Received April 2019.

Contents

[*]The first author was supported by the PGMO grant 2016-2032H.
[†]Research of the authors was supported by NSF grant CCF-1523768.

1. Introduction

1.1. Motivation

In this paper we consider the estimation problem as follows:

> *Given a noisy observation*
> $$\zeta = Ax + \xi \in \mathbf{R}^m, \tag{1.1}$$
> *of linear image Ax of an unknown signal x, we want to recover the image $w = Bx \in \mathbf{R}^\nu$ of this signal. It is assumed that x is known to belong to a given signal set – a nonempty convex compact set $\mathcal{X} \subset \mathbf{R}^n$, and A and B are given $m\times n$ and $\nu\times n$ matrices; ξ is the observation noise with distribution P_x which may depend on x. In addition, we are given a norm $\|\cdot\|$ on $\mathbf{R}^\nu$ in which the estimation error is measured.*

Estimation problem (1.1) is a (nonparametric) linear inverse problem. Popular approaches to solving statistical inverse problems include, among others, Singular Value Decomposition (SVD), see, e.g., [22, 21, 30], Galerkin projection[32], Wavelet-Vaguelette Decompositions [8, 1, 14, 20], adaptive Galerkin algorithms [6, 19], as well as various iterative regularization techniques [40, 26]. When statistically analysed, those approaches usually assume a special structure of the problem, when matrix A and set $\mathcal{X}$ "fit each other," e.g., there exists a sparse approximation of the set $\mathcal{X}$ in a given basis/pair of bases, in which matrix A is "almost diagonal" (see, e.g. [8, 6] for detail). Under these assumptions, traditional results focus on estimation algorithms which are both numerically straightforward and statistically (asymptotically) optimal with closed form analytical description of estimates and corresponding risks. In the situation considered in this paper, where we do not assume any specific structure of $\mathcal{X}$ apart from convexity, compactness and "computationally tractability,"[1] and A, B are "general" matrices of appropriate dimensions, we cannot expect deriving closed form expressions for estimates and risks. Instead, we adopt an alternative approach initiated in [7] and further developed in [23, 17, 25, 24]. Within this *operational* approach both the estimate and its risk are yielded by efficient computation, usually via convex optimization, rather than by an explicit closed form analytical description; what we know in advance, in good cases, is that the resulting risk, whether large or low, is nearly the best one achievable under the circumstances.

Note that a "standard choice" of statistical techniques which can be applied to solve (1.1) is between Maximum Likelihood and Linear Estimation. Maximal Likelihood Estimation (MLE) [15, 16, 36] is a "universal" statistical tool; when the distribution P of the noise is known, the application to problem (1.1) is straightforward. It is well known that MLE has excellent asymptotical properties in parametric models; however, it is also well known [2, 29, 4] that Maximum Likelihood approach has significant drawbacks. For instance, let us consider the simple situation of direct observation where $A = B = I_n$, observations noise is Gaussian – $\xi \sim \mathcal{N}(0, I_n)$, $\|\cdot\| = \|\cdot\|_2$, and

$$\mathcal{X} = \left\{x = [x_1; ...; x_n] : |x_1| \leq n^{1/4}, \|[x_2; ...; x_n]\|_2 + 2n^{-1/4}|x_1| \leq 2\right\}.$$

In this case, for n large enough, the squared risk of the MLE $\widehat{x}^{ML}$ satisfies [4]

$$\sup_{x\in\mathcal{X}} \mathbf{E}\{\|x - \widehat{x}^{ML}\|_2^2\} \geq \tfrac{3}{4}\sqrt{n},$$

while for the minimax risk it holds

$$\inf_{\widehat{x}} \sup_{x\in\mathcal{X}} \mathbf{E}\{\|x - \widehat{x}\|_2^2\} \leq 5, \tag{1.2}$$

with the upper bound (1.2) attained at the simple linear estimator $\widehat{x} = [\zeta_1; 0; ...; 0]$.

Aside of Maximum Likelihood, very popular, due to their relative simplicity, estimates are the *linear ones* – those of the form $\zeta \mapsto \widehat{w}(\zeta) = G^T\zeta$. Linear estimates have received much attention in the statistical literature (cf. [27, 28, 38, 35, 12, 13, 5, 41, 39] among many others) When designing a linear estimate, the emphasis is on how to specify the matrix G in order to obtain the lowest possible maximal over $\mathcal{X}$ estimation risk, which is then compared to the minimax risk (the infimum, taken w.r.t. all Borel estimates $\widehat{x}(\cdot)$, of the worst case, over signals from $\mathcal{X}$, expected norm of the recovery error). "Near optimality" results for the case of indirect observations (where A and B are arbitrary) are the subject of recent papers [25, 24], where it was shown that in the *spectratopic*

[1]For a brief outline of computational tractability, see Section A.1 of the appendix.

case where $\mathcal{X}$ and the unit ball $\mathcal{B}_*$ of the norm conjugate to $\|\cdot\|$ are *spectratopes*[2] and the noise is zero mean Gaussian, a properly designed, via solving an explicit convex optimization problem, linear estimate is nearly optimal.

What follows is motivated by the desire to build an alternative estimation scheme which works beyond the ellitopic/spectratopic case, where linear estimates can become "heavily nonoptimal."

Motivating example. Consider the simple-looking problem of recovering $Bx = x$ in $\|\cdot\|_2$-norm from *direct* observations ($Ax = x$) corrupted by the standard Gaussian noise $\xi \sim \mathcal{N}(0, \sigma^2 I)$, and let $\mathcal{X}$ be the unit $\|\cdot\|_1$=ball:

$$\mathcal{X} = \Big\{x \in \mathbf{R}^n : \sum_i |x_i| \leq 1\Big\}.$$

In this situation, building the optimal, in terms of the worst-case over $x \in \mathcal{X}$ expected $\|\cdot\|_2^2$-risk, linear estimate $\widehat{x}_H(\zeta) = H^T\zeta$ is extremely simple. Indeed, one easily verifies that the optimal H is a scalar matrix hI, with the optimal h being the minimizer of the univariate quadratic function $(1-h)^2 + \sigma^2 nh^2$. Therefore, in this case, the best achievable with linear estimates expected $\|\cdot\|_2^2$-risk is

$$\max_{x\in\mathcal{X}} \mathbf{E}\left\{\|\widehat{x}_H(\zeta) - x\|_2^2\right\} = \min_h \left[(1-h)^2 + \sigma^2 nh^2\right] = \frac{n\sigma^2}{1+n\sigma^2}.$$

On the other hand, consider *nonlinear* estimate as follows. Given observation ζ, we specify the estimate $\widehat{x}(\zeta)$ as an optimal solution to the optimization problem

$$\mathrm{Opt}(\zeta) = \min_{y\in\mathcal{X}} \|y - \zeta\|_\infty. \tag{1.3}$$

Note that for every $\rho > 0$ the probability for the true signal to satisfy $\|x - \zeta\|_\infty \leq \rho\sigma$ ("event $\mathcal{E}$") is at least $1-\epsilon$ for $\epsilon = 2n\exp\{-\rho^2/2\}$, and if this event occurs then both x and $\widehat{x}$ belong to the box $\{y : \|y-\zeta\|_\infty \leq \rho\sigma\}$, implying that $\|x - \widehat{x}\|_\infty \leq 2\rho\sigma$. Combining the latter bound with the constraint $\|x-\widehat{x}\|_2 \leq \|x-\widehat{x}\|_1 \leq 2$, since $x \in \mathcal{X}$ and $\widehat{x} \in \mathcal{X}$, we obtain

$$\|x-\widehat{x}\|_2 \leq \sqrt{\|x-\widehat{x}\|_\infty \|x-\widehat{x}\|_1} \leq \left\{\begin{array}{ll} 2\sqrt{\rho\sigma}, & \zeta \in \mathcal{E} \\ 2, & \zeta \notin \mathcal{E} \end{array}\right.,$$

whence

$$\mathbf{E}\left\{\|\widehat{x} - x\|_2^2\right\} \leq 4\rho\sigma + 4\epsilon \leq 4\rho\sigma + 8n\exp\{-\rho^2/2\}. \tag{$*$}$$

Assuming $\sigma \leq 2n/\sqrt{\mathrm{e}}$ and specifying ρ as $\sqrt{2\ln(2n/\sigma)}$, we get $\rho \geq 1$ and $2n\exp\{-\rho^2/2\} \leq \sigma$, implying that the right hand side in $(*)$ is at most $8\rho\sigma$. Therefore, the nonlinear estimate $\widehat{x}(\zeta)$ satisfies

$$\max_{x\in\mathcal{X}} \mathbf{E}\left\{\|\widehat{x}(\zeta) - x\|_2^2\right\} \leq 8\sqrt{\ln(2n/\sigma)}\sigma.$$

[2] See [24] or Section 4.5 below for detail. As of now, an instructive example of a spectratope is the intersection of finite family of ellipsoids/elliptic cylinders with common center, or, as a more exotic example, the unit ball of the spectral norm in the space of matrices.

When $n\sigma^2$ is of order of 1, the latter bound is of order of $\sigma\sqrt{\ln(1/\sigma)}$, while the best expected $\|\cdot\|_2^2$-risk attainable with linear estimates under the circumstances is of order of 1. We conclude that when σ is small and n is large (specifically, is of order of $1/\sigma^2$), the best linear estimate is *far inferior* to the nonlinear estimate – the ratio of the corresponding squared risks is as large as $\frac{O(1)}{\sigma\sqrt{\ln(1/\sigma)}}$.

1.2. Polyhedral estimate

The construction of the nonlinear estimate $\widehat{x}$ we have built in the above example[3] admits a natural extension yielding what we call *polyhedral estimate.* The idea underlying polyhedral estimate is quite straightforward. Assuming for present that the observation noise is $\mathcal{N}(0,\sigma^2 I_n)$, observe that there is a spectrum of "easy to estimate" linear forms of signal x underlying observation, namely the forms $g_h^T x = h^T Ax$ with $h \in \mathcal{H} = \{h \in \mathbf{R}^m : \|h\|_2 = 1\}$. Indeed, for a form of this type, the "plug-in" estimate $\widehat{g}_h(\zeta) = h^T\zeta$ is an unbiased estimate of $g_h^T x$ with $\mathcal{N}(0,\sigma^2)$ recovery error. It follows that selecting somehow a *contrast matrix* H – an $m\times M$ matrix with columns from $\mathcal{H}$, the plug-in estimate $H^T\zeta$ recovers well the vector $H^T Ax$ in the uniform norm:

$$\mathrm{Prob}_{\zeta\sim\mathcal{N}(Ax,\sigma^2 I_m)}\left\{\|H^T\zeta - H^T Ax\|_\infty > \sigma\rho\right\} \le 2M\exp\{-\rho^2/2\},\ \rho\ge 0. \quad (1.4)$$

As a result, given a "reliability tolerance" $\epsilon \ll 1$ and setting $\rho = \sqrt{2\ln(2M/\epsilon)}$, the estimate $H^T\zeta$ recovers the vector $H^T Ax$, whatever be $x\in\mathbf{R}^n$, within $\|\cdot\|_\infty$-accuracy $\sigma\rho$ and reliability $1-\epsilon$. When our objective is to recover $w = Bx$, a natural way to combine this estimate with a priori information that $x \in \mathcal{X}$ is to set

$$\widehat{x}^H(\zeta) \in \underset{y}{\mathrm{Argmin}}\left\{\|H^T[Ay-\zeta]\|_\infty : y \in \mathcal{X}\right\},\ \ \widehat{w}^H(\zeta) = B\widehat{x}(\zeta). \quad (1.5)$$

Note that the estimate $\widehat{w}^H(\cdot)$ of w we end up with is defined solely in terms of H and the data $A, B, \mathcal{X}$ of our estimation problem, and that simple estimate (1.3) is nothing but the polyhedral estimate stemming from the unit contrast matrix. The rationale behind polyhedral estimation scheme is the desire to reduce complex estimating problems to those of estimating linear forms. To the best of our knowledge, the idea of polyhedral estimate goes back to [34], see also [33, Chapter 2], where it was shown that when recovering smooth multivariate regression functions known to belong to Sobolev balls from their noisy observations taken along a regular grid Γ, a polyhedral estimate with ad hoc selected contrast matrix is near-optimal in a wide range of smoothness characterizations and norms $\|\cdot\|$. Recently, the ideas underlying the results of [34] have been taken up in the MIND estimator of [18], then applied in the indirect observation setting in [37] in the context of multiple testing.

[3] In fact, this estimate is nearly optimal under the circumstances in a meaningful range of values of n and σ.

The goal of this paper is to investigate characteristics of the polyhedral estimate, with a particular emphasis on efficiently computable upper bounds for the risk of the estimate $\widehat{w}^H(\cdot)$ and design of the *contrast matrix* H resulting in the (nearly) best upper risk bounds. We propose two related approaches. The first of them mimics the mechanism working in the above Motivating example. It allows for a computationally efficient design of estimates and upper bounding their risks whenever the signal set $\mathcal{X}$ is computationally tractable convex and compact, and the recovery error is measured in $\|\cdot\|_r$. In contrast to this "general applicability," (near) minimax optimality of the resulting estimates is be established below only in a very special "standard diagonal case" (a slight generalization of the situation considered in Motivating example), by comparing upper risk bounds for the estimates yielded by the approach in question with well known lower bounds on the minimax risk (in the standard case all these bounds are available in closed analytical form). These near optimality results, due to their severely restricted scope, should be seen as a "proof of concept" — they show that in some cases where linear estimates are "heavily nonoptimal" (e.g., in the Motivating example), polyhedral estimates are near optimal. We believe that identifying essentially more general situations where the estimates supplied by the approach in question are near optimal is a challenging and meaningful open question.

The second, completely different, approach to designing polyhedral estimates and quantifying their risk developed in this paper is based on the notion of a "cone compatible with convex set." It is inspired by the design of nearly optimal linear estimates in the spectratopic case, though its application is *not* restricted to this case. We show that in the spectratopic case application of this approach leads to provably nearly minimax optimal estimation. This result imposes no restrictions on matrices A and B, making seemingly impossible establishing lower and upper risk bounds in closed analytical form, and thus goes beyond the toolbox of traditional nonparametric statistics.

The main body of the paper is organized as follows. We begin in Section 2 with detailed formulation of the estimation problem (see Section 2.1), and present generic polyhedral estimate along with its risk analysis (Section 2.2). This analysis requires from the main ingredient of a polyhedral estimate – the underlying contrast matrix – to be properly adjusted to the structure and the magnitude of observation noise. To allow for this adjustment, we restrict ourselves to three types of *observation schemes* specifying the noise structure, referred to as *sub-Gaussian*, *Discrete*, and *Poisson* cases; description of these cases and of the restrictions they impose on the contrast matrices form the subject of Section 2.3. The subject of the subsequent sections is tuning the polyhedral estimate to the structure of the estimation problem – the design of the contrast matrix aimed at minimizing the risk of the associated polyhedral estimate. Sections 3, 4 are devoted to the first, resp., the second of the aforementioned approaches to the design and analysis of polyhedral estimates. Technical proofs (which are longer than few lines) are relegated to the appendix.

2. Problem of interest and generic polyhedral estimate

2.1. The problem

Suppose that we are given

- a nonempty computationally tractable convex compact *signal set* $\mathcal{X} \subset \mathbf{R}^n$,
- *sensing matrix* $A \in \mathbf{R}^{m\times n}$, *decoding matrix* $B \in \mathbf{R}^{\nu\times n}$, and a norm $\|\cdot\|$ on the space $\mathbf{R}^\nu$,
- a *reliability tolerance* $\epsilon \in (0,1)$,
- a random *observation*
$$\zeta = Ax + \xi_x \tag{2.1}$$
stemming from unknown *signal* x known to belong to $\mathcal{X}$; here ξ_x is a random variable with Borel probability distribution P_x.

Given observation ζ, our goal is to recover $w = Bx$, where x is the signal underlying the observation. A candidate estimate is a Borel function $\widehat{w}(\zeta)$ taking value in $\mathbf{R}^\nu$, and we quantify the performance of such an estimate by its $(\epsilon, \|\cdot\|)$-*risk*

$$\mathrm{Risk}_{\epsilon,\|\cdot\|}[\widehat{w}|\mathcal{X}] = \sup_{x\in\mathcal{X}} \inf\left\{\rho : \mathrm{Prob}_{\xi_x\sim P_x}\{\|Bx - \widehat{w}(Ax+\xi_x)\| > \rho\} \le \epsilon\ \forall x \in \mathcal{X}\right\},$$

that is, the worst, over $x \in \mathcal{X}$, $(1-\epsilon)$-quantile, taken w.r.t. P_x, of the $\|\cdot\|$-magnitude of the recovery error.

Notation. In the sequel, given a convex compact set, say, $\mathcal{Y}$, in $\mathbf{R}^n$, we denote by $\mathcal{Y}_\mathrm{s}$ its symmeterization:
$$\mathcal{Y}_\mathrm{s} = \tfrac{1}{2}(\mathcal{Y} - \mathcal{Y}).$$
Note that whenever $\mathcal{Y}$ is symmetric w.r.t. the origin, we have $\mathcal{Y}_\mathrm{s} = \mathcal{Y}$. We use "MATLAB style" of vector/matrix notation: whenever $H_1, ..., H_k$ are matrices of appropriate dimensions, $[H_1, ..., H_k]$ stands for horizontal and $[H_1; ...; H_k]$ – for their vertical concatenation. $\mathbf{S}^n$ stands for the space of $n\times n$ real symmetric matrices equipped with the Frobenius inner product; $\mathbf{S}^n_+$ is the cone of positive semidefinite matrices from $\mathbf{S}_n$. Relation $A \succeq B$ ($\Leftrightarrow B \preceq A$) means that A and B are real symmetric matrices of common size such that $A-B$ is positive semidefinite, while $A \succ B$ ($\Leftrightarrow B \prec A$) means that A, B are real symmetric matrices of common size such that $A - B$ is positive definite.

2.2. Generic polyhedral estimate

A generic polyhedral estimate is as follows:

> Given the data $A \in \mathbf{R}^{m\times n}, B \in \mathbf{R}^{\nu\times n}, \mathcal{X} \subset \mathbf{R}^n$ of the estimation problem stated in Section 2.1 and a "reliability tolerance" $\epsilon \in (0,1)$, we somehow specify a positive integer N along with N linear forms $h_\ell^T z$ on the observation space

$\mathbf{R}^m$. These forms define linear forms $g_\ell^T x := h_\ell^T Ax$ on the space of signals $\mathbf{R}^n$. Assume that vectors h_ℓ are selected in such a way that

$$\forall (x \in \mathcal{X}) : \mathrm{Prob}\{|h_\ell^T \xi_x| > 1\} \le \epsilon/N. \tag{2.2}$$

When setting $H = [h_1, ..., h_N]$ (in the sequel, H is referred to as *contrast matrix*), we clearly have for all $x \in \mathcal{X}$:

$$\mathrm{Prob}\left\{\|H^T\zeta - H^T Ax\|_\infty > 1\right\} \le \epsilon. \tag{2.3}$$

With the polyhedral estimation scheme, we act *as if* all information about x contained in our observation ζ were represented by $H^T\zeta$, and we estimate $w = Bx$ by $\widehat{w} = B\widehat{x}$, where $\widehat{x} = \widehat{x}(\zeta)$ is a (whatever) vector from $\mathcal{X}$ compatible with this information, i.e. a solution $\widehat{x}$ to the feasibility problem

$$\text{find } \ u \in \mathcal{X} \text{ such that } \|H^T\zeta - H^T Au\|_\infty \le 1.$$

Note that $\widehat{x}$ not always is well defined: the above feasibility problem may be unsolvable with positive probability (in fact, with probability $\le \epsilon$, since, by construction, the true signal x underlying observation ζ is feasible with probability $1-\epsilon$). To circumvent this difficulty, we define $\widehat{x}$ according to

$$\widehat{x}^H \in \operatorname*{Argmin}_u \left\{\|H^T\zeta - H^T Au\|_\infty : u \in \mathcal{X}\right\} \tag{2.4}$$

so that $\widehat{x}^H$ is always well defined and belongs to $\mathcal{X}$, and estimate w by $\widehat{w}^H = B\widehat{x}^H$.

We have the following immediate observation:

Proposition 2.1. *In the situation in question, given a contrast matrix $H = [h_1, ..., h_N]$ with columns satisfying (2.2), the quantity*

$$\mathfrak{R}[H] := \max_z \left\{\|Bz\| : \|H^T Az\|_\infty \le 2, z \in 2\mathcal{X}_\mathrm{s}\right\} \tag{2.5}$$

is an upper bound on the $(\epsilon, \|\cdot\|)$-risk of the polyhedral estimate $\widehat{w}^H(\cdot)$:

$$\mathrm{Risk}_{\epsilon,\|\cdot\|}[\widehat{w}^H|\mathcal{X}] \le \mathfrak{R}[H]. \tag{2.6}$$

Proof is immediate. Let us fix $x \in \mathcal{X}$, and let $\mathcal{E}$ be the set of all realizations of ξ_x such that $\|H^T\xi_x\|_\infty \le 1$, so that $P_x(\mathcal{E}) \ge 1-\epsilon$ by (2.3). Let us fix a realization $\xi \in \mathcal{E}$ of the observation noise, let $\zeta = Ax + \xi$, and let $\widehat{x} = \widehat{x}^H(Ax+\xi)$. Then $u = x$ is a feasible solution to the optimization problem (2.4) with the value of the objective ≤ 1, implying that the value of this objective at the optimal solution $\widehat{x}$ to the problem is ≤ 1 as well, so that $\|H^T A[x-\widehat{x}]\|_\infty \le 2$. Besides this, $z = x - \widehat{x} \in 2\mathcal{X}_\mathrm{s}$. We see that z is a feasible solution to (2.5), whence

$$\|B[x-\widehat{x}]\| = \|Bx - \widehat{w}^H(\zeta)\| \le \mathfrak{R}[H].$$

It remains to note that the latter relation holds true whenever $\zeta = Ax + \xi$ with $\xi \in \mathcal{E}$, and for any $x \in \mathcal{X}$ the P_x-probability of the latter inclusion is at least $1-\epsilon$. □

What is ahead. In what follows our focus will be on answering the following questions underlying the construction of the polyhedral estimate:

1. Suppose that, given the data of our estimation problem and a tolerance $\delta \in (0,1)$, we can construct a set $\mathcal{H}_\delta$ of vectors $h \in \mathbf{R}^m$ satisfying the relation
$$\forall (x \in \mathcal{X}) : \text{Prob}\left\{|h^T \xi_x| > 1\right\} \leq \delta. \tag{2.7}$$
With our approach, after the number N of columns in a contrast matrix has been selected, we are free to choose these columns from $\mathcal{H}_\delta$, with $\delta = \epsilon/N$, ϵ being a given reliability tolerance of the estimate we are designing. Thus, the problem of building sets $\mathcal{H}_\delta$ satisfying (2.7) arises, the larger $\mathcal{H}_\delta$, the better.
2. The upper bound $\mathfrak{R}[H]$ on the $(\epsilon, \|\cdot\|)$-risk of the polyhedral estimate $\widehat{w}^H$ is, in general, difficult to compute – this is the maximum of a convex function over a computationally tractable convex set. Thus, we need to provide computationally efficient upper bounding of $\mathfrak{R}[\cdot]$.
3. Finally, given the "raw materials" – set $\mathcal{H}_\delta$ and an efficiently computable upper bound on the risk of a candidate polyhedral estimate – how to design the best, in terms of (the upper bound on) its risk, polyhedral estimate?

We are about to consider these questions one by one.

2.3. Specifying sets $\mathcal{H}_\delta$ for basic observation schemes

To specify sets $\mathcal{H}_\delta$ we are to make assumptions on the distributions of observation noise we want to handle. For the sake of conciseness, in the sequel we restrict ourselves with 3 special observation schemes (below called "cases") as follows:

- *Sub-Gaussian case:* For every $x \in \mathcal{X}$, the observation noise ξ_x is sub-Gaussian with parameters $(0, \sigma^2 I_m)$, $\sigma > 0$ (denoted $\xi_x \sim \mathcal{SG}(0, \sigma^2 I_m)$). Let us denote
$$\pi_G(h) = \vartheta_G \|h\|_2 \quad \text{where} \quad \vartheta_G = \sigma\sqrt{2\ln(2/\delta)}.$$
In the sub-Gaussian case we set
$$\mathcal{H}_\delta = \mathcal{H}_\delta^G := \{h : \ \pi_G(h) \leq 1\}. \tag{2.8}$$
- *Discrete case:* $\mathcal{X}$ is a convex compact subset of the probabilistic simplex $\mathbf{\Delta}_n = \{x \in \mathbf{R}^n : x \geq 0, \sum_i x_i = 1\}$, A is column-stochastic matrix, and
$$\zeta = \frac{1}{K}\sum_{k=1}^{K} \omega_k$$
with independent across $k \leq K$ random vectors ω_k, with ω_k taking values e_i with probabilities $[Ax]_i$, $i = 1, ..., m$, e_i being the basic orths in $\mathbf{R}^m$. In this case we put
$$\pi_D(h) = 2\sqrt{\vartheta_D \max_{x\in\mathcal{X}} \sum_i [Ax]_i h_i^2 + \tfrac{16}{9}\vartheta_D^2 \|h\|_\infty^2} \quad \text{with} \quad \vartheta_D = \frac{\ln(2/\delta)}{K},$$

and

$$\mathcal{H}_\delta = \mathcal{H}_\delta^D := \{h : \ \pi_D(h) \leq 1\}. \tag{2.9}$$

- *Poisson case:* $\mathcal{X}$ is a convex compact subset of the nonnegative orthant $\mathbf{R}^n_+$, A is entrywise nonnegative, and the observation ζ stemming from $x \in \mathcal{X}$ is random vector with independent across i entries $\zeta_i \sim \text{Poisson}([Ax]_i)$. In the Poisson case we set

$$\pi_P(h) = 2\sqrt{\vartheta_P \max_{x\in\mathcal{X}} \sum_i [Ax]_i h_i^2 + \tfrac{4}{9}\vartheta_P^2 \|h\|_\infty^2} \ \text{ with } \ \vartheta_P = \ln(2/\delta),$$

and

$$\mathcal{H}_\delta = \mathcal{H}_\delta^P := \{h : \pi_P(h) \leq 1\}. \tag{2.10}$$

We verify in Section A.3.1 that the sets $\mathcal{H}_\delta^G$, $\mathcal{H}_\delta^D$ and $\mathcal{H}_\delta^P$ as given by (2.8)–(2.10) indeed satisfy

$$\forall (h \in \mathcal{H}_\delta, \ x \in \mathcal{X}) \quad \text{Prob}_x\{|h^T\xi_x| \geq 1\} \leq \delta,$$

provided that the observation noises ξ_x, $x \in \mathcal{X}$, stem from the respective observation schemes.

3. Efficient upper-bounding of $\mathfrak{R}[H]$ and contrast design, I.

The scheme for upper-bounding $\mathfrak{R}[H]$ to be presented in this section is inspired by the motivating example from the introduction. Indeed, there is a special case of (2.5) where $\mathfrak{R}[H]$ is easy to compute – the case when $\|\cdot\|$ is the uniform norm $\|\cdot\|_\infty$, whence

$$\mathfrak{R}[H] = \widehat{\mathfrak{R}}[H] := 2\max_{i\leq\nu}\max_x \left\{\text{Row}_i^T[B]x : x \in \mathcal{X}_\text{s}, \|H^TAx\|_\infty \leq 1\right\}$$

is just the maximum of ν efficiently computable convex functions. It turns out that when $\|\cdot\| = \|\cdot\|_\infty$, it is easy not only to compute $\mathfrak{R}[H]$, but to optimize this risk bound in H as well. These observations underly the forthcoming developments in this section: under appropriate assumptions, we bound the risk of a polyhedral estimate stemming from a contrast matrix H via the efficiently computable quantity $\widehat{\mathfrak{R}}[H]$ and then show that the resulting risk bounds can be efficiently optimized w.r.t. H. We shall also see that in some simple situations which allow for analytical analysis, like the one in the motivating example, the resulting estimates turn out to be nearly minimax optimal.

Assumptions. We continue to stay within the setup introduced in Section 2.1 which we now augment with the following assumptions:

A.1. $\|\cdot\| = \|\cdot\|_r$ with $r \in [1, \infty]$.

A.2. We have at our disposal a sequence $\gamma = \{\gamma_i > 0, \ 1 \leq i \leq \nu\}$ and $\rho \in [1, \infty]$ such that the image of $\mathcal{X}_\text{s}$ under the mapping $x \mapsto Bx$ is contained in the "scaled $\|\cdot\|_\rho$-ball"

$$\mathcal{Y} = \{y \in \mathbf{R}^\nu : \|\text{Diag}\{\gamma\}y\|_\rho \leq 1\}. \tag{3.1}$$

3.1. Simple observation

Let B_ℓ^T be ℓ-th row in B, $1 \leq \ell \leq \nu$. Let us make the following observation:

Proposition 3.1. *In the situation described in Section 2.1, assuming that Assumptions **A.1-2** hold, let $\epsilon \in (0,1)$ and a positive real $N \geq \nu$ be given, and let $\pi(\cdot)$ be a norm on $\mathbf{R}^m$ such that*

$$\forall (h: \ \pi(h) \leq 1, x \in \mathcal{X}) : \mathrm{Prob}\{|h^T \xi_x| > 1\} \leq \epsilon/N. \tag{3.2}$$

Let, next, a matrix $H = [H_1, ..., H_\nu]$ with $H_\ell \in \mathbf{R}^{m \times m_\ell}$, $m_\ell \geq 1$, $\sum_\ell m_\ell = N$, and positive reals ς_ℓ, $1 \leq \ell \leq \nu$, satisfy the relations

$$\begin{array}{ll} (a) & \pi(\mathrm{Col}_j[H]) \leq 1, \ 1 \leq j \leq N; \\ (b) & \max_x \left\{ B_\ell^T x : \ x \in \mathcal{X}_\mathrm{s}, \ \|H_\ell^T A x\|_\infty \leq 1 \right\} \leq \varsigma_\ell, \ 1 \leq \ell \leq \nu. \end{array} \tag{3.3}$$

Then the quantity $\mathfrak{R}[H]$ as defined in (2.5) can be upper-bounded as follows:

$$\begin{array}{rcl} \mathfrak{R}[H] \leq \Psi(\varsigma) & := & 2 \max_v \left\{ \|[v_1/\gamma_1; ...; v_\nu/\gamma_\nu]\|_r : \right. \\ & & \left. \qquad \|v\|_\rho \leq 1, \ 0 \leq v_\ell \leq \gamma_\ell \varsigma_\ell, \ 1 \leq \ell \leq \nu \right\}. \end{array} \tag{3.4}$$

This combines with Proposition 2.1 to imply that

$$\mathrm{Risk}_{\epsilon, \|\cdot\|}[\widehat{w}^H | \mathcal{X}] \leq \Psi(\varsigma). \tag{3.5}$$

Function Ψ is nondecreasing on the nonnegative orthant and is easy to compute.

Proof. Let $z = 2\bar{z}$ be a feasible solution to (2.5), so that $\bar{z} \in \mathcal{X}_\mathrm{s}$ and $\|H^T A \bar{z}\|_\infty \leq 1$. Let $y = B\bar{z}$, so that $y \in \mathcal{Y}$ (see (3.1)) due to $\bar{z} \in \mathcal{X}_\mathrm{s}$ and **A.2**. Thus, $\|\mathrm{Diag}\{\gamma\} y\|_\rho \leq 1$. Besides this, by (3.3.$b$) relations $\bar{z} \in \mathcal{X}_\mathrm{s}$ and $\|H^T A \bar{z}\|_\infty \leq 1$ combine with the symmetry of $\mathcal{X}_\mathrm{s}$ to imply that

$$|y_\ell| = |B_\ell^T \bar{z}| \leq \varsigma_\ell, \ \ell \leq \nu.$$

Taking into account that $\|\cdot\| = \|\cdot\|_r$ by **A.1**, we see that

$$\begin{array}{rcl} \mathfrak{R}[H] & = & \max_z \left\{ \|Bz\|_r : z \in 2\mathcal{X}_\mathrm{s}, \|H^T A z\|_\infty \leq 2 \right\} \\ & \leq & 2 \max_y \left\{ \|y\|_r : |y_\ell| \leq \varsigma_\ell, \ell \leq \nu \ \& \ \|\mathrm{Diag}\{\gamma\} y\|_\rho \leq 1 \right\} \\ & = & 2 \max_v \left\{ \|[v_1/\gamma_1; ...; v_\nu/\gamma_\nu]\|_r : \|v\|_\rho \leq 1, 0 \leq v_\ell \leq \gamma_\ell \varsigma_\ell, \ell \leq \nu \right\}, \end{array}$$

as stated in (3.4).

It is evident that Ψ is nondecreasing on the nonnegative orthant. Computation of Ψ can be carried out as follows:

1. When $r = \infty$, we need to compute $\max_{\ell \leq \nu} \max_v \{v_\ell/\gamma_\ell : \|v\|_\rho \leq 1, 0 \leq v_j \leq \gamma_j \varsigma_j, j \leq \nu\}$, so that evaluating Ψ reduces to solving ν simple convex optimization problems;
2. When $\rho = \infty$, we clearly have $\Psi(\varsigma) = \|[\bar{v}_1/\gamma_1; ...; \bar{v}_\nu/\gamma_\nu]\|_r$, $\bar{v}_\ell = \min[1, \gamma_\ell \varsigma_\ell]$;

3. When $1 \leq r, \rho < \infty$, passing from variables v_ℓ to variables $u_\ell = v_\ell^\rho$, we get
$$\Psi^r(\varsigma) = 2^r \max_u \left\{ \sum_\ell \gamma_\ell^{-r} u_\ell^{r/\rho} : \sum_\ell u_\ell \leq 1,\ 0 \leq u_\ell \leq (\gamma_\ell \varsigma_\ell)^\rho \right\}.$$
When $r \leq \rho$, the problem on the right hand side is an easily solvable problem of maximizing a simple concave function over a simple convex compact set. When $\infty > r > \rho$, this problem can be solved by Dynamic Programming. □

3.2. Specifying contrasts

Risk bound (3.5) allows for a straightforward design of contrast matrices. Recalling that Ψ is monotone on the nonnegative orthant, all we need is to select h_ℓ's satisfying (3.3) and resulting in the smallest possible ς_ℓ's, which is what we are about to do now.

Preliminaries. Given a vector $b \in \mathbf{R}^n$ and a norm $s(\cdot)$ on $\mathbf{R}^m$, consider convex-concave saddle point problem
$$\mathrm{Opt} = \inf_{g \in \mathbf{R}^m} \max_{x \in \mathcal{X}_\mathrm{s}} \left\{ \phi(g,x) := [b - A^T g]^T x + s(g) \right\} \tag{SP}$$
along with the induced primal and dual problems
$$\begin{array}{rcl} \mathrm{Opt}(P) & = & \inf_{g \in \mathbf{R}^m} \left[\overline{\phi}(g) := \max_{x \in \mathcal{X}_\mathrm{s}} \phi(g,x) \right] \\ & = & \inf_{g \in \mathbf{R}^m} \left[s(g) + \max_{x \in \mathcal{X}_\mathrm{s}} [b - A^T g]^T x \right] \end{array} \tag{P}$$
and
$$\begin{array}{rcl} \mathrm{Opt}(D) & = & \max_{x \in \mathcal{X}_\mathrm{s}} \left[\underline{\phi}(g) := \inf_{g \in \mathbf{R}^m} \phi(g,x) \right] \\ & = & \max_{x \in \mathcal{X}_\mathrm{s}} \left[\inf_{g \in \mathbf{R}^m} \left[b^T x - [Ax]^T g + s(g) \right] \right] \\ & = & \max_x \left[b^T x :\ x \in \mathcal{X}_\mathrm{s},\ q(Ax) \leq 1 \right] \end{array} \tag{D}$$
where $q(\cdot)$ is the norm conjugate to $s(\cdot)$ (we have used the evident fact that $\inf_{g \in \mathbf{R}^m} [f^T g + s(g)]$ is either $-\infty$ or 0 depending on whether $q(f) > 1$ or $q(f) \leq 1$). Since $\mathcal{X}_\mathrm{s}$ is compact, we have $\mathrm{Opt}(P) = \mathrm{Opt}(D) = \mathrm{Opt}$ by the Sion-Kakutani theorem. Besides this, (D) is solvable (this is evident) and (P) is solvable as well, since $\overline{\phi}(g)$ is continuous due to the compactness of $\mathcal{X}_\mathrm{s}$, and $\overline{\phi}(g) \geq s(g)$, so that $\overline{\phi}(\cdot)$ has bounded level sets. Let $\bar{g}$ be an optimal solution to (P), let $\bar{x}$ be an optimal solution to (D), and let $\bar{h} = \bar{g}/s(\bar{g})$, so that $s(\bar{h}) = 1$ and $\bar{g} = s(\bar{g})\bar{h}$. Now let us make the observation as follows:

Proposition 3.2. *In the situation in question, we have*
$$\max_x \left\{ |b^T x| : x \in \mathcal{X}_\mathrm{s}, |\bar{h}^T A x| \leq 1 \right\} \leq \mathrm{Opt}. \tag{3.6}$$
In addition, for any matrix $G = [g^1, ..., g^M] \in \mathbf{R}^{m \times M}$ with $s(g^j) \leq 1$, $1 \leq j \leq M$, one has
$$\begin{array}{l} \max_x \left\{ |b^T x| : x \in \mathcal{X}_\mathrm{s}, \|G^T A x\|_\infty \leq 1 \right\} \\ \quad = \max_x \left\{ b^T x : x \in \mathcal{X}_\mathrm{s}, \|G^T A x\|_\infty \leq 1 \right\} \geq \mathrm{Opt}. \end{array} \tag{3.7}$$

Proof. Let x be a feasible solution to the problem in the left hand side of (3.6). Replacing, if necessary, x with $-x$, we can assume that $|b^Tx| = b^Tx$. We now have

$$\begin{array}{rcl} |b^Tx| &=& b^Tx = [\bar{g}^TAx - s(\bar{g})] + \underbrace{[b - A^T\bar{g}]^Tx + s(\bar{g})}_{\leq \overline{\phi}(\bar{g}) = \mathrm{Opt}(P)} \\ &\leq& \mathrm{Opt}(P) + [s(\bar{g})\bar{h}^TAx - s(\bar{g})] \\ &\leq& \mathrm{Opt}(P) + s(\bar{g})\underbrace{|\bar{h}^TAx|}_{\leq 1} - s(\bar{g}) \leq \mathrm{Opt}(P) = \mathrm{Opt}, \end{array}$$

as claimed in (3.6). Now, the equality in (3.7) is due to the symmetry of $\mathcal{X}_{\mathrm{s}}$ w.r.t. the origin. To verify the inequality in (3.7), note that $\bar{x}$ satisfies the relations $\bar{x} \in \mathcal{X}_{\mathrm{s}}$ and $q(A\bar{x}) \leq 1$, implying, due to the fact that the columns of G are of $s(\cdot)$-norm ≤ 1, that $\bar{x}$ is a feasible solution to optimization problems in (3.7). As a result, the second quantity in (3.7) is at least $b^T\bar{x} = \mathrm{Opt}(D) = \mathrm{Opt}$, and (3.7) follows. □

Designing contrasts. Propositions 3.1 and 3.2 allow for a straightforward solution of the associated contrast design problem, at least in the case of Sub-Gaussian, Discrete, and Poisson observation schemes. Indeed, in these cases, when designing a contrast matrix with N columns, we are supposed to select its columns in the respective sets $\mathcal{H}_{\epsilon/N}$, see Section 2.3. Note that these sets are "nearly independent" of N, because the norms π_G, π_D, π_P in the description of the respective sets $\mathcal{H}_\delta^G$, $\mathcal{H}_\delta^D$, $\mathcal{H}_\delta^P$ depend on $1/\delta$ only via logarithmic in $1/\delta$ factors. Thus, we lose nearly nothing when assuming that $N \geq \nu$. So, let us act as follows:

We set $N = \nu$, specify $\overline{\pi}(\cdot)$ as the norm (π_G, or π_D, or π_P) associated with the observation scheme (Sub-Gaussian, or Discrete, or Poisson) in question and $\delta = \epsilon/\nu$, and solve ν convex optimization problems

$$\begin{array}{rcl} \mathrm{Opt}_\ell &=& \min_{g \in \mathbf{R}^m} \left[\overline{\phi}_\ell(g) := \max_{x \in \mathcal{X}_{\mathrm{s}}} \phi_\ell(g,x)\right] \\ \phi_\ell(g,x) &=& [B_\ell - A^Tg]^Tx + \overline{\pi}(g) \end{array} \qquad (P_\ell)$$

Next, we convert optimal solution g_ℓ to (P_ℓ) into a vector $h_\ell \in \mathbf{R}^m$ by representing $g_\ell = \overline{\pi}(g_\ell)h_\ell$ with $\overline{\pi}(h_\ell) = 1$, and set $H_\ell = h_\ell$. As a result, we get an $m \times \nu$ contrast matrix $H = [h_1, ..., h_\nu]$ which, taken along with $N = \nu$, quantities

$$\varsigma_\ell = \mathrm{Opt}_\ell,\ 1 \leq \ell \leq \nu, \qquad (3.8)$$

and $\pi(\cdot) \equiv \overline{\pi}(\cdot)$, in view of the first claim in Proposition 3.2 as applied with $s(\cdot) \equiv \overline{\pi}(\cdot)$, satisfies the premise of Proposition 3.1.
Consequently, by Proposition 3.1 we have

$$\mathrm{Risk}_{\epsilon,\|\cdot\|}[\widehat{w}^H|\mathcal{X}] \leq \Psi([\mathrm{Opt}_1; ...; \mathrm{Opt}_\nu]). \qquad (3.9)$$

Discussion. Within the framework set in Proposition 3.1, optimality of the outlined contrast design for Sub-Gaussian, Discrete and Poisson observation schemes stems from the second claim in Proposition 3.2 which states that when $N \geq \nu$ and the columns of the contrast matrix $H = [H_1, ..., H_\nu]$ belong to the set $\mathcal{H}_{\epsilon/N}$ associated with the observation scheme in question, i.e., the norm $\pi(\cdot)$ in the proposition is the norm π_G, or π_D, or π_P associated with $\delta = \epsilon/N$, the quantities ς_ℓ participating in (3.3.b) cannot be less than Opt_ℓ.

> Indeed, the norm $\pi(\cdot)$ from Proposition 3.1 is $\geq$ the norm $\overline{\pi}(\cdot)$ participating in (P_ℓ) (since the value of ϵ/N corresponding to $\pi(\cdot)$ is at most ϵ/ν), implying, by (3.3.a), that the columns of matrix H obeying the premise of the proposition satisfy the relation $\overline{\pi}(\mathrm{Col}_j[H]) \leq 1$. Invoking the second part of Proposition 3.2 with $s(\cdot) \equiv \overline{\pi}(\cdot)$, $b = B_\ell$, and $G = H_\ell$, and taking (3.3.b) into account, we conclude that $\varsigma_\ell \geq \mathrm{Opt}_\ell$ for all ℓ, as claimed.

Since the bound on the risk of a polyhedral estimate offered by Proposition 3.1 is the better the less are ς_ℓ's, we see that as far as this bound is concerned, the outlined design procedure is the best possible, provided $N \geq \nu$.

An attractive feature of the contrast design we have just presented is that it is completely independent of the entities participating in assumptions **A.1-2** – these entities affect theoretical risk bounds of the resulting polyhedral estimate, but not the estimate itself.

3.3. Illustration: diagonal case

Let us consider the *diagonal case* of our estimation problem, where

- $\mathcal{X} = \{x \in \mathbf{R}^n : \|Dx\|_\rho \leq 1\}$, where D is a diagonal matrix with positive diagonal entries $D_{\ell\ell} =: d_\ell$;
- $m = \nu = n$, and A and B are diagonal matrices with diagonal entries $0 < A_{\ell\ell} =: a_\ell$, $0 < B_{\ell\ell} =: b_\ell$;
- $\|\cdot\| = \|\cdot\|_r$;
- We are in Sub-Gaussian case, that is, $\xi_x \sim \mathcal{SG}(0, \sigma^2 I_n)$ for every $x \in \mathcal{X}$.

Let us implement the approach developed in Sections 3.1 – 3.2.

1. Given reliability tolerance ϵ, we set

$$\delta = \epsilon/n, \;\; \vartheta_G := \sigma\sqrt{2\ln(2/\delta)} = \sigma\sqrt{2\ln(2n/\epsilon)} \tag{3.10}$$

   and
$$\mathcal{H} = \mathcal{H}_\delta^G = \{h \in \mathbf{R}^n : \pi_G(h) := \vartheta_G \|h\|_2 \leq 1\};$$

2. We solve $\nu = n$ convex optimization problems (P_ℓ) associated with $\overline{\pi}(\cdot) \equiv \pi_G(\cdot)$, which is immediate: the resulting contrast matrix is

$$H = \vartheta_G^{-1} I_n,$$

   and
$$\mathrm{Opt}_\ell = \varsigma_\ell := b_\ell \min[\vartheta_G/a_\ell, 1/d_\ell]. \tag{3.11}$$

Risk analysis. The $(\epsilon, \|\cdot\|)$-risk of the resulting polyhedral estimate $\widehat{w}(\cdot)$ can be bounded by Proposition 3.1. Note that setting $\gamma_\ell = d_\ell/b_\ell$, $\;1 \leq \ell \leq n$, we meet assumptions **A.1-2**, and the above choice of H, $N = n$ and ς_ℓ satisfies the premise of Proposition 3.1. By this proposition,

$$\mathrm{Risk}_{\epsilon,\|\cdot\|_r}[\widehat{w}^H|\mathcal{X}] \leq \Psi := 2\max_v \left\{\|[v_1/\gamma_1; ...; v_n/\gamma_n]\|_r : \|v\|_\rho \leq 1, 0 \leq v_\ell \leq \gamma_\ell\varsigma_\ell\right\}. \tag{3.12}$$

Let us work out what happens in the *simple case* where

$$\begin{array}{ll} (a) & 1 \leq \rho \leq r < \infty \\ (b) & a_\ell/d_\ell \text{ and } b_\ell/a_\ell \text{ are nonincreasing in } \ell \end{array} \tag{3.13}$$

Proposition 3.3. *In the just defined simple case, let* $\mathfrak{n} = n$ *when*

$$\sum_{\ell=1}^{n} (\vartheta_G d_\ell/a_\ell)^\rho \leq 1,$$

otherwise let $\mathfrak{n}$ *be the smallest integer such that*

$$\sum_{\ell=1}^{\mathfrak{n}} (\vartheta_G d_\ell/a_\ell)^\rho > 1,$$

with ϑ_G *given by* (3.10)*. Then for the contrast matrix* $H = \vartheta_G^{-1} I_n$ *one has*

$$\mathrm{Risk}_{\epsilon,\|\cdot\|_r}[\widehat{w}^H|\mathcal{X}] \leq \Psi \leq 2\left[\sum\nolimits_{\ell=1}^{\mathfrak{n}} (\vartheta_G b_\ell/a_\ell)^r\right]^{1/r}.$$

For proof, see Section A.3.2

Application. Let us apply the result of Proposition 3.3 to the "standard case" (cf., e.g., [9, 10, 18]) where

$$0 < \sqrt{\ln(2n/\epsilon)}\sigma \leq 1,\ a_\ell = \ell^{-\alpha},\ b_\ell = \ell^{-\beta},\ d_\ell = \ell^\delta \tag{3.14}$$

with

$$\beta \geq \alpha \geq 0,\ \delta \geq 0,\ (\alpha - \beta)r + 1 > 0,\ 1 \leq \rho \leq r, \tag{3.15}$$

so that (3.13) holds true. In this case, for large enough n, namely, for

$$n \geq c\vartheta_G^{-\frac{1}{\alpha+\delta+1/\rho}} \qquad [\vartheta_G = \sigma\sqrt{2\ln(2n/\epsilon)}]$$

(here and in what follows, the factors denoted c and C depend solely on α, β, δ, r, ρ) we get

$$\mathfrak{n} \leq C\vartheta_G^{-\frac{1}{\alpha+\delta+1/\rho}}$$

resulting in[4]

$$\mathrm{Risk}_{\epsilon,\|\cdot\|_r}[\widehat{w}|\mathcal{X}] \leq C\vartheta_G^{\frac{\beta+\delta+1/\rho-1/r}{\alpha+\delta+1/\rho}} \approx \sigma^{\frac{\bar{\beta}+1/\rho-1/r}{\bar{\alpha}+1/\rho}},\ \ \bar{\alpha} = \alpha+\delta,\ \bar{\beta} = \beta+\delta. \tag{3.16}$$

[4]Here and below, "$\simeq$" stands for "equal up to logarithmic in n/ϵ factor."

Note that when setting $x = D^{-1}y$ and treating y, rather than x, as the signal underlying the observation, we obtain an estimation problem which is similar to the original one, in which α, β, δ and $\mathcal{X}$ are replaced, respectively, with $\bar{\alpha}$, $\bar{\beta}$, $\bar{\delta} = 0$, and $\mathcal{Y} = \{y : \|y\|_\rho \le 1\}$, and A, B are replaced with $\overline{A} = \mathrm{Diag}\{\ell^{-\bar{\alpha}}, \ell \le n\}$, and $\overline{B} = \mathrm{Diag}\{\ell^{-\bar{\beta}}, \ell \le n\}$. When n is large enough, namely,

$$n \ge \sigma^{-\frac{1}{\bar{\alpha}+1/\rho}}, \tag{3.17}$$

$\mathcal{Y}$ contains the "coordinate box"

$$\overline{\mathcal{Y}} = \{x : |x_\ell| \le \mathfrak{m}^{-1/\rho}, \mathfrak{m}/2 \le \ell \le \mathfrak{m}, x_\ell = 0 \text{ otherwise}\}$$

of dimension $\ge \mathfrak{m}/2$, where

$$\mathfrak{m} \ge c\sigma^{-\frac{1}{\bar{\alpha}+1/\rho}}.$$

Observe that for all $y \in \overline{\mathcal{Y}}$, $\|\bar{A}y\|_2 \le C\mathfrak{m}^{-\bar{\alpha}}\|y\|_2$, and $\|\bar{B}y\|_r \ge c\mathfrak{m}^{-\bar{\beta}}\|y\|_r$. This observation, when combined with the Fano inequality, implies that for $\epsilon \ll 1$ (cf. [12]) the minimax optimal w.r.t. the family of all Borel estimates $(\epsilon, \|\cdot\|_r)$-risk on the signal set $\overline{\mathcal{X}} = D^{-1}\overline{\mathcal{Y}} \subset \mathcal{X}$ is at least

$$c\sigma^{\frac{\bar{\beta}+1/\rho-1/r}{\bar{\alpha}+1/\rho}}.$$

In other words, in this situation, the upper bound (3.16) on the risk of the polyhedral estimate is within a logarithmic in n/ϵ factor from the minimax risk. In particular, without surprise, in the case of $\beta = 0$ the polyhedral estimates attain, within the same logarithmic in n/ϵ factor, the well known optimal rates [9, 11, 18].

It is instructive to compare the risk bound (3.16) with the best ϵ-risk attainable by linear estimates in the standard case. It is easily seen that when (3.14), (3.15), and (3.17) hold true, the latter risk is $\simeq \sigma^{\frac{\bar{\beta}+1/\rho-1/r}{\bar{\alpha}+1/\rho} - \frac{(r-\rho)((\alpha-\beta)r+1)}{r(1+\bar{\alpha}r)(1+\bar{\alpha}\rho)}}$; when $\rho < r$, this risk is "essentially worse" than the risk (3.16) of the polyhedral estimate.

4. Efficient upper-bounding of $\mathfrak{R}[H]$ and contrast design, II.

4.1. Outline

In this section we develop an approach to the design of polyhedral estimates which is an alternative to that discussed in Section 3. Our present strategy can be outlined as follows. Let us denote by

$$\mathcal{B}_* = \{u \in \mathbf{R}^\nu : \|u\|_* \le 1\}$$

the unit ball of the norm $\|\cdot\|_*$ *conjugate* to the norm $\|\cdot\|$ in the formulation of the estimation problem in Section 2.1. Assume that we have at our disposal a technique for bounding quadratic forms on the set $\mathcal{B}_* \times \mathcal{X}_{\rm s}$, so that there is an efficiently computable convex function $\mathcal{M}(M)$ on $\mathbf{S}^{\nu+n}$ such that

$$\mathcal{M}(M) \geq \max_{[u;z]\in\mathcal{B}_*\times\mathcal{X}_{\rm s}} [u;z]^T M[u;z] \ \forall M \in \mathbf{S}^{\nu+n}. \tag{4.1}$$

Note that the upper bound $\mathfrak{R}[H]$, as defined in (2.5), on the risk of a candidate polyhedral estimate $\widehat{w}^H$ given by (2.5) is nothing but

$$\mathfrak{R}[H] = 2\max_{[u;z]}\left\{[u;z]^T \underbrace{\left[\begin{array}{c|c} & \frac{1}{2}B \\ \hline \frac{1}{2}B^T & \end{array}\right]}_{B_+}[u;z]: \begin{array}{l} u\in\mathcal{B}_*, z\in\mathcal{X}_{\rm s}, \\ z^TA^Th_\ell h_\ell^TAz\leq 1, \ell\leq N\end{array}\right\}. \tag{4.2}$$

When $\lambda \in \mathbf{R}^N_+$, the constraints $z^TA^Th_\ell h_\ell^T Az \leq 1$ in (4.2) can be aggregated to yield the quadratic constraint

$$z^TA^T\Theta_\lambda Az \leq \mu_\lambda,\ \Theta_\lambda = H\mathrm{Diag}\{\lambda\}H^T,\ \mu_\lambda = \sum_\ell \lambda_\ell.$$

Observe that for every $\lambda \geq 0$ we have

$$\mathfrak{R}[H] \leq 2\mathcal{M}\Bigg(\underbrace{\left[\begin{array}{c|c} & \frac{1}{2}B \\ \hline \frac{1}{2}B^T & -A^T\Theta_\lambda A\end{array}\right]}_{B_+[\Theta_\lambda]}\Bigg) + 2\mu_\lambda. \tag{4.3}$$

> Indeed, let $[u;z]$ be a feasible solution to the optimization problem (4.2) specifying $\mathfrak{R}[H]$. Then
>
> $$[u;z]^TB_+[u;z] = [u;z]^TB_+[\Theta_\lambda][u;z] + z^TA^T\Theta_\lambda Az;$$
>
> the first term in the right hand side is $\leq \mathcal{M}(B_+[\Theta_\lambda])$ since $[u;z]\in\mathcal{B}_*\times\mathcal{X}_{\rm s}$, and the second term in the right hand side, as we have already seen, is $\leq \mu_\lambda$, and (4.3) follows.

Now assume that we have at our disposal a computationally tractable cone

$$\mathbf{H} \subset \mathbf{S}^N_+ \times \mathbf{R}_+$$

satisfying the following assumption

> **Assumption C.** *Whenever $(\Theta,\mu)\in\mathbf{H}$, we can efficiently find an $n\times N$ matrix $H=[h_1,...,h_N]$ and a nonnegative vector $\lambda\in\mathbf{R}^N_+$ such that*
>
> $$\begin{array}{ll} (a) & \text{the columns } h_\ell \text{ of } H \text{ satisfy (2.2)} \\ (b) & \Theta = H\mathrm{Diag}\{\lambda\}H^T \\ (c) & \sum_i \lambda_i \leq \mu \end{array} \tag{4.4}$$

The following simple observation is crucial to what follows:

Proposition 4.1. *Consider the estimation problem posed in Section 2.1, and let efficiently computable convex function $\mathcal{M}$ and computationally tractable closed* *convex cone* $\mathbf{H}$ *satisfy* (4.1) *and Assumption* $\mathbf{C}$*, respectively. Consider the convex optimization problem*

$$\begin{array}{c}\mathrm{Opt} = \min_{\tau,\Theta,\mu}\left\{2\tau + 2\mu : (\Theta,\mu) \in \mathbf{H},\, \mathcal{M}(B_+[\Theta]) \le \tau\right\} \\ \left[B_+[\Theta] = \left[\begin{array}{c|c} & \frac{1}{2}B \\ \hline \frac{1}{2}B^T & -A^T\Theta A\end{array}\right]\right]\end{array} \tag{4.5}$$

Given a feasible solution (τ,Θ,μ) *to this problem, by* $\mathbf{C}$ *we can efficiently convert it to* (H,λ) *such that* $H = [h_1, ..., h_N]$ *with* h_ℓ *satisfying* (2.2) *and* $\lambda \ge 0$ *with* $\sum_\ell \lambda_\ell \le \mu$*. We have*

$$\mathfrak{R}[H] \le 2\tau + 2\mu,$$

whence the $(\epsilon, \|\cdot\|)$*-risk of the polyhedral estimate* $\widehat{w}^H$ *satisfies the bound*

$$\mathrm{Risk}_{\epsilon,\|\cdot\|}[\widehat{w}^H|\mathcal{X}] \le 2\tau + 2\mu. \tag{4.6}$$

As a result, we can construct efficiently a polyhedral estimate with $(\epsilon, \|\cdot\|)$*-risk arbitrarily close to* Opt *(and with risk exactly* Opt *if problem* (4.5) *is solvable).*

Proof is readily given by the reasoning preceding the proposition. Indeed, with $\tau, \Theta, \mu, H, \lambda$ as in the premise of the proposition, the columns h_ℓ of H satisfy (2.2) by $\mathbf{C}$, implying, by Proposition 2.1, that $\mathrm{Risk}_{\epsilon,\|\cdot\|}[\widehat{w}^H|\mathcal{X}] \le \mathfrak{R}[H]$. Besides this, $\mathbf{C}$ says that for these H, λ it holds $\Theta = \Theta_\lambda$ and $\mu_\lambda \le \mu$. Therefore, (4.3) combines with the constraints of (4.5) to imply that $\mathfrak{R}[H] \le 2\tau + 2\mu$, and (4.6) follows by Proposition 2.1. □

The approach to the design of polyhedral estimate we develop in this section amounts to reducing the construction of the estimate (i.e., construction of the contrast matrix H) to finding (nearly) optimal solutions to (4.5). Implementing the proposed approach requires devising techniques for constructing cones $\mathbf{H}$ satisfying $\mathbf{C}$ along with efficiently computable functions $\mathcal{M}(\cdot)$ satisfying (4.1). These tasks are the subjects of the sections to follow.

4.2. Specifying cones $\mathbf{H}$

We specify cones $\mathbf{H}$ in the case when the number N of columns in the candidate contrast matrices is m and under the following assumption on the observation scheme in question:

> **Assumption D**. *There is a computationally tractable convex compact subset* $Z \subset \mathbf{R}^m_+$ *intersecting* $\mathrm{int}\mathbf{R}^m_+$ *such that the norm* $\pi(\cdot)$
>
> $$\pi(h) = \sqrt{\max_{z\in Z}\sum_i z_i h_i^2}$$
>
> *induced by* Z *satisfies the relation*
>
> $$\pi(h) \le 1 \;\Rightarrow\; \mathrm{Prob}\{|h^T\xi_x| > 1\} \le \epsilon/m \;\; \forall x \in \mathcal{X}.$$

Note that Assumption $\mathbf{D}$ is satisfied for Sub-Gaussian, Discrete, and Poisson observation schemes: according to the results of Section 2.3,

- in the Sub-Gaussian case, it suffices to take

$$Z = \{2\sigma^2 \ln(2m/\epsilon)[1; ...; 1]\};$$

- in the Discrete case, it suffices to take

$$Z = \frac{4\ln(2m/\epsilon)}{K} A\mathcal{X} + \frac{64\ln^2(2m/\epsilon)}{9K^2}\mathbf{\Delta}_m$$

  where

$$A\mathcal{X} = \{Ax : x \in \mathcal{X}\}, \quad \mathbf{\Delta}_m = \{y \in \mathbf{R}^m : y \geq 0, \sum_i y_i = 1\};$$

- finally, in the Poisson case, it suffices to take

$$Z = 4\ln(2m/\epsilon)A\mathcal{X} + \tfrac{16}{9}\ln^2(2m/\epsilon)\mathbf{\Delta}_m,$$

  with $A\mathcal{X}$ and $\mathbf{\Delta}_m$ as above.

Note that in all these cases Z only "marginally" – logarithmically – depends on ϵ and m.

Under Assumption **D**, the cone **H** can be built as follows:

- When $\mathcal{Z}$ is a singleton: $\mathcal{Z} = \{\bar{z}\}$, so that $\pi(\cdot)$ is a scaled Euclidean norm, we set

$$\mathbf{H} = \{(\Theta, \mu) \in \mathbf{S}^m_+ \times \mathbf{R}_+ : \mu \geq \sum_i \bar{z}_i \Theta_{ii}\}.$$

  Given $(\Theta, \mu) \in \mathbf{H}$, the $m \times m$ matrix H and $\lambda \in \mathbf{R}^m_+$ are defined as follows. We set $S = \mathrm{Diag}\{\sqrt{\bar{z}_1}, ..., \sqrt{\bar{z}_m}\}$ and compute the eigenvalue decomposition of the matrix $S\Theta S$:

$$S\Theta S = U\mathrm{Diag}\{\lambda\}U^T,$$

  where U is orthogonal, and set $H = S^{-1}U$, thus ensuring that $\Theta = H\mathrm{Diag}\{\lambda\}H^T$. Since $\mu \geq \sum_i \bar{z}_i\Theta_{ii}$, we have $\sum_i \lambda_i = \mathrm{Tr}(S\Theta S) \leq \mu$. Finally, a column h of H is of the form $S^{-1}f$ with a $\|\cdot\|_2$-unit vector f, implying that

$$\pi(h) = \sqrt{\sum_i \bar{z}_i [S^{-1}f]_i^2} = \sqrt{\sum_i f_i^2} = 1,$$

  so that h satisfies (2.2) by Assumption **D**.

- When Z is not a singleton, we set

$$\begin{array}{rcl} \phi(r) &=& \max_{z\in Z} z^T r, \\ \varkappa &=& 6\ln(2\sqrt{3}m^2), \\ \mathbf{H} &=& \{(\Theta, \mu) \in \mathbf{S}^m_+ \times \mathbf{R}_+ : \mu \geq \varkappa\phi(\mathrm{diag}(\Theta))\}, \end{array} \tag{4.7}$$

where $\mathrm{diag}(Q)$ is the diagonal of a (square) matrix Q. Note that $\phi(r) > 0$ whenever $r \geq 0$, $r \neq 0$, since Z contains a positive vector.
The justification of this construction and the efficient (randomized) algorithm for converting a pair $(\Theta, \mu) \in \mathbf{H}$ into (H, λ) satisfying, along with (Θ, μ), Assumption **C** are given by the following

Lemma 4.1. *Let the norm $\pi(\cdot)$ satisfies Assumption D.*
(i) *Whenever H is an $m \times m$ matrix with columns h_ℓ satisfying $\pi(h_\ell) \leq 1$ and $\lambda \in \mathbf{R}^m_+$, we have*

$$(\Theta_\lambda = H\mathrm{Diag}\{\lambda\}H^T,\ \mu = \varkappa \sum_i \lambda_i) \in \mathbf{H}.$$

(ii) *Given $(\Theta, \mu) \in \mathbf{H}$ with $\Theta \neq 0$, we find decomposition $\Theta = QQ^T$ with $m \times m$ matrix Q and fix an orthogonal $m \times m$ matrix V with magnitudes of entries not exceeding $\sqrt{2/m}$ (e.g., the orthogonal scaling of the matrix of the cosine transform).*
When $\mu > 0$, let us set $\lambda = \frac{\mu}{m}[1; ...; 1] \in \mathbf{R}^m$ and consider the random matrix

$$H_\chi = \sqrt{\frac{m}{\mu}} Q\mathrm{Diag}\{\chi\}V,$$

where χ is the m-dimensional Rademacher random vector (i.e., the entries in χ are independent of each other random variables taking values ± 1 with probabilities $1/2$). We have

$$H_\chi \mathrm{Diag}\{\lambda\}H_\chi^T \equiv \Theta,\ \lambda \geq 0, \sum_i \lambda_i = \mu. \tag{4.8}$$

Moreover, the probability of the event

$$\pi(\mathrm{Col}_\ell[H_\chi]) \leq 1\ \ \forall \ell \leq m \tag{4.9}$$

*is at least 1/2. Thus, generating independent samples of χ and terminating with $H = H_\chi$ when the latter matrix satisfies (4.9), we with probability 1 terminate with (H, λ) satisfying Assumption **C**. Moreover, the probability for the procedure to terminate in course of the first $M = 1, 2, ...$ steps is at least $1 - 2^{-M}$.*
When $\mu = 0$, we have $\Theta = 0$ (since $\mu = 0$ implies $\phi(\mathrm{diag}(\Theta)) = 0$, which with $\Theta \succeq 0$ is possible only when $\Theta = 0$); thus, when $\mu = 0$, we set $H = 0_{m\times m}$ and $\lambda = 0_{m\times 1}$.

For proof, see Section A.3.3.
Note that the lemma states, essentially, that the cone $\mathbf{H}$ is a tight, up to a logarithmic in m factor, inner approximation of the set

$$\left\{ (\Theta, \mu) : \exists (\lambda \in \mathbf{R}^m_+, H \in \mathbf{R}^{m\times m}) : \begin{array}{l} \Theta = H\mathrm{Diag}\{\lambda\}H^T, \\ \pi(\mathrm{Col}_\ell[H]) \leq 1,\ \ell \leq m, \\ \mu \geq \sum_\ell \lambda_\ell \end{array} \right\}$$

4.3. Specifying functions $\mathcal{M}$

In this section we focus on computationally efficient upper-bounding of maxima of quadratic forms over symmetric w.r.t. the origin convex compact sets, our goal being to specify efficiently computable convex function $\mathcal{M}(\cdot)$ satisfying (4.1).

Cones compatible with convex sets. Given a nonempty convex compact set $\mathcal{Y} \subset \mathbf{R}^N$, we say that a cone $\mathbf{Y}$ is *compatible* with $\mathcal{Y}$, if

- $\mathbf{Y}$ is a closed convex computationally tractable cone contained in $\mathbf{S}^N_+ \times \mathbf{R}_+$
- one has
$$\forall (V,\tau) \in \mathbf{Y}: \ \max_{y\in\mathcal{Y}} y^T V y \leq \tau \tag{4.10}$$
- $\mathbf{Y}$ contains a pair (V,τ) with $V \succ 0$.
- relations $(V,\tau) \in \mathbf{Y}$ and $\tau' \geq \tau$ imply that $(V,\tau') \in \mathbf{Y}$.[5]

We call a cone $\mathbf{Y}$ *sharp*, if $\mathbf{Y}$ is a closed convex cone contained in $\mathbf{S}^N_+ \times \mathbf{R}_+$ and such that the only pair $(V,\tau) \in \mathbf{Y}$ with $\tau = 0$ is the pair $(0,0)$, or, equivalently, a sequence $\{(V_i,\tau_i) \in \mathbf{Y}, i \geq 1\}$ is bounded if and only if the sequence $\{\tau_i, i \geq 1\}$ is bounded.

Note that whenever the linear span of $\mathcal{Y}$ is the entire $\mathbf{R}^N$, every compatible with $\mathcal{Y}$ cone is sharp.

Observe that *if $\mathcal{Y} \subset \mathbf{R}^N$ is a nonempty convex compact set and $\mathbf{Y}$ is a cone compatible with a shift $\mathcal{Y} - a$ of $\mathcal{Y}$, then $\mathbf{Y}$ is compatible with $\mathcal{Y}_\mathrm{s}$* (a symmetrisation of $\mathcal{Y}$).

Indeed, when shifting a set $\mathcal{Y}$, its symmeterization $\frac{1}{2}[\mathcal{Y}-\mathcal{Y}]$ remains intact, so that we can assume that $\mathbf{Y}$ is compatible with $\mathcal{Y}$. Now let $(V,\tau) \in \mathbf{Y}$ and $y, y' \in \mathcal{Y}$. We have
$$[y-y']^T V[y-y'] + \underbrace{[y+y']^T V[y+y']}_{\geq 0} = 2[y^T V y + [y']^T V y'] \leq 4\tau,$$
whence for $z = \frac{1}{2}[y-y']$ it holds $z^T V z \leq \tau$. Since every $z \in \mathcal{Y}_\mathrm{s}$ is of the form $\frac{1}{2}[y-y']$ with $y, y' \in \mathcal{Y}$, the claim follows.
Note that the claim can be "nearly inverted:" *if $0 \in \mathcal{Y}$ and $\mathbf{Y}$ is compatible with $\mathcal{Y}_\mathrm{s}$, then the "widening" of $\mathbf{Y}$ – the cone*
$$\mathbf{Y}^+ = \{(V,\tau): \ (V,\tau/4) \in \mathbf{Y}\}$$
is compatible with $\mathcal{Y}$ (evident, since when $0 \in \mathcal{Y}$, every vector from $\mathcal{Y}$ is proportional, with coefficient 2, to a vector from $\mathcal{Y}_\mathrm{s}$).

Constructing functions $\mathcal{M}$. The role of compatibility in our context becomes clear from the following observation:

[5] The latter requirement is "for free" – passing from a computationally tractable closed convex cone $\mathbf{Y} \subset \mathbf{S}^N_+ \times \mathbf{R}_+$ satisfying (4.10) to the cone $\mathbf{Y}^+ = \{(V,\tau) : \exists \bar\tau \leq \tau : (V,\bar\tau) \in \mathbf{Y}\}$, we get a larger than $\mathbf{Y}$ cone compatible with $\mathcal{Y}$. It will be clear from the sequel that in our context, the larger is a cone compatible with $\mathcal{Y}$, the better, so that this extension makes no harm.

Proposition 4.2. *In the situation described in Section 2.1, assume that we have at our disposal cones* $\mathbf{X}$ *and* $\mathbf{U}$ *compatible, respectively, with* $\mathcal{X}_\mathrm{s}$ *and with the unit ball*

$$\mathcal{B}_* = \{v \in \mathbf{R}^\nu : \|u\|_* \le 1\}$$

of the norm $\|\cdot\|_*$ *conjugate to* $\|\cdot\|$. *Given* $M \in \mathbf{S}^{\nu+n}$, *let us set*

$$\mathcal{M}(M) = \inf_{X,t,U,s} \{t + s : \ (X,t) \in \mathbf{X}, (U,s) \in \mathbf{U}, \, \mathrm{Diag}\{U, X\} \succeq M\} \tag{4.11}$$

Then $\mathcal{M}$ *is a real-valued efficiently computable convex function on* $\mathbf{S}^{\nu+n}$ *such that* (4.1) *takes place: for every* $M \in \mathbf{S}^{n+\nu}$ *it holds*

$$\mathcal{M}(M) \ge \max_{[u;z]\in\mathcal{B}_*\times\mathcal{X}_\mathrm{s}} [u;z]^T M[u;z].$$

In addition, when $\mathbf{X}$ *and* $\mathbf{U}$ *are sharp, the infimum in* (4.11) *is attained.*

Proof is immediate. Given that the objective of the optimization problem specifying $\mathcal{M}(M)$ is nonnegative on the feasible set, the fact that $\mathcal{M}$ is real-valued is equivalent to problem's feasibility, and the latter is readily given by the fact that $\mathbf{X}$ is a cone containing a pair (X,t) with $X \succ 0$ and similarly for $\mathbf{U}$. Convexity of $\mathcal{M}$ is evident. To verify (4.1), let (X,t,U,s) form a feasible solution to the optimization problem in (4.11). When $[u;z] \in \mathcal{B}_* \times \mathcal{X}_\mathrm{s}$ we have

$$[u;z]^T M[u;z] \le u^T U u + z^T X z \le s + t,$$

where the first inequality is due to the $\succeq$-constraint in (4.11), and the second is due to the fact that $\mathbf{U}$ is compatible with $\mathcal{B}_*$, and $\mathbf{X}$ is compatible with $\mathcal{X}_\mathrm{s}$. Since the resulting inequality holds true for all feasible solutions to the optimization problem in (4.11), (4.1) follows. Finally, when $\mathbf{X}$ and $\mathbf{U}$ are sharp, (4.11) is a feasible conic problem with bounded level sets of the objective and as such is solvable. □

4.4. Putting things together

The following statement summarizes our second approach to the design of polyhedral estimate.

Proposition 4.3. *In the situation of Section 2.1, assume that we have at our disposal cones* $\mathbf{X}$ *and* $\mathbf{U}$ *compatible, respectively, with* $\mathcal{X}_\mathrm{s}$ *and with the unit ball* $\mathcal{B}_*$ *of the norm conjugate to* $\|\cdot\|$. *Given reliability tolerance* $\epsilon \in (0,1)$, *assume that we have at our disposal a positive integer* N *and a computationally tractable cone* $\mathbf{H}$ *satisfying, along with* ϵ, *Assumption* ***C***. *Consider (clearly feasible) convex optimization problem*

$$\mathrm{Opt} = \min_{\Theta,\mu,X,t,U,s} \left\{ f(t,s,\mu) := 2(t+s+\mu) : \begin{array}{c} (\Theta,\mu) \in \mathbf{H}, (X,t) \in \mathbf{X}, (U,s) \in \mathbf{U} \\ \left[\begin{array}{c|c} U & \frac{1}{2}B \\ \hline \frac{1}{2}B^T & A^T\Theta A + X \end{array}\right] \succeq 0 \end{array} \right\} \tag{4.12}$$

Given a feasible solution Θ, μ, X, t, U, s *to* (4.12)*, due to* $\boldsymbol{C}$*, we can convert, in a computationally efficient manner,* (Θ, μ) *into* (H, λ) *such that the columns of the* $m \times N$ *contrast matrix* H *satisfy* (2.2)*,* $\Theta = H\mathrm{Diag}\{\lambda\}H^T$*, and* $\mu \geq \sum_\ell \lambda_\ell$*. In this case, the* $(\epsilon, \|\cdot\|)$*-risk of the polyhedral estimate* $\widehat{w}^H$ *satisfies the bound*

$$\mathrm{Risk}_{\epsilon,\|\cdot\|}[\widehat{w}^H|\mathcal{X}] \leq f(t, s, \mu). \tag{4.13}$$

In particular, we can build, in a computationally efficient manner, polyhedral estimates with risks arbitrarily close to Opt *(and with risk* Opt*, provided that* (4.12) *is solvable).*

Proof. Let Θ, μ, X, t, U, s form a feasible solution to (4.12). By the semidefinite constraint in (4.12) we have

$$0 \preceq \left[\begin{array}{c|c} U & -\frac{1}{2}B \\ \hline -\frac{1}{2}B^T & A^T\Theta A + X \end{array}\right] = \mathrm{Diag}\{U, X\} - \underbrace{\left[\begin{array}{c|c} & \frac{1}{2}B \\ \hline \frac{1}{2}B^T & -A^T\Theta A \end{array}\right]}_{=:M}.$$

Hence, for the function $\mathcal{M}$ defined in (4.11) one has

$$\mathcal{M}(M) \leq t + s.$$

Since, by Proposition 4.2, $\mathcal{M}$ satisfies (4.1), invoking Proposition 4.1 we arrive at

$$\mathfrak{R}[H] \leq 2(\mu + \mathcal{M}(M)) \leq f(t, s, \mu).$$

By Proposition 2.1, this implies the target relation (4.13). □

4.5. Compatibility

The approach to design of polyhedral estimates utilizing the recipe described in Proposition 4.3 relies upon our ability to equip convex "sets of interest" (in our context, these are the symmeterization $\mathcal{X}_\mathrm{s}$ of the signal set and the unit ball $\mathcal{B}_*$ of the norm conjugate to the norm $\|\cdot\|$) with compatible cones.[6] Below, we discuss two principal sources of such cones, namely (a) spectratopes/ellitopes, and (b) absolute norms.

More examples of compatible cones can be constructed using a "compatibility calculus." Namely, let us assume that we are given a collection of convex sets (operands) and apply to them some basic operation, such as taking a finite intersection, or arithmetic sum, direct or inverse linear image, or convex hull of a finite union of the sets. It turns out that cones compatible with the results of such operations can be easily (in a fully algorithmic fashion) obtained from the cones compatible with the operands; see Appendix A.2 for principal calculus rules.

[6] Recall that we already know how to specify the second element of the construction, the cone $\mathbf{H}$.

In view of Proposition 4.3, the larger are the cones $\mathbf{X}$ and $\mathbf{U}$ compatible with $\mathcal{X}_{\rm s}$ and $\mathcal{B}_*$, the better – the wider is the optimization domain in (4.12) and, consequently, the less is (the best) attainable risk bound. Given convex compact set $\mathcal{Y} \in \mathbf{R}^N$, the "ideal" – the largest candidate to the role of the cone compatible with $\mathcal{Y}$ would be

$$\mathbf{Y}^* = \{(V,\tau) \in \mathbf{S}^N_+ \times \mathbf{R}_+ :\ \tau \geq \max_{y\in\mathcal{Y}} y^T V y\}.$$

However, this cone is typically intractable; therefore we look for "as large as possible" *tractable* inner approximations of $\mathbf{Y}^*$.

4.5.1. Cones compatible with ellitopes/spectratopes

The ellitopes and spectratopes as introduced in [25, 24] are sets $\mathcal{X} \subset \mathbf{R}^n$ representable as linear images

$$\mathcal{X} = M\mathcal{Y}$$

of *basic ellitopes/spectratopes*, that is, *bounded* sets $\mathcal{Y}$ representable as

$$\begin{array}{llll}
(a) & \mathcal{Y} = \{y \in \mathbf{R}^N : \exists r \in \mathcal{R} : y^T R_\ell y \leq r_\ell, \ell \leq L\} & \text{[basic ellitope]} \\
 & \qquad R_\ell \succeq 0,\ \ell \leq L & \\
(b) & \mathcal{Y} = \{y \in \mathbf{R}^N : \exists r \in \mathcal{R} : R_\ell^2[y] \preceq r_\ell I_{d_\ell}, \ell \leq L\} & \text{[basic spectratope]} \\
 & \qquad R_\ell[y] = \sum_{i=1}^N y_i R^{\ell i},\ R^{\ell i} \in \mathbf{S}^{d_\ell}, &
\end{array}, \tag{4.14}$$

where $\mathcal{R} \subset \mathbf{R}^L_+$ is a convex compact set intersecting with $\mathrm{int}\mathbf{R}^L_+$ and such that $0 \leq r' \leq r \in \mathcal{R}$ implies that $r' \in \mathcal{R}$.

An ellitope/spectratope is a convex compact set symmetric w.r.t. the origin; as shown in [25, 24], the families of ellitopes/spectratopes admit a kind of fully algorithmic "calculus" which demonstrates that these families are closed w.r.t. nearly all basic operations preserving convexity, compactness, and symmetry w.r.t. the origin, including taking finite intersections, direct products, linear images, inverse linear images under linear embeddings, and arithmetic summation. The "raw materials" this calculus can be applied to include: for ellitopes – $\|\cdot\|_p$-balls with $p \in [2,\infty]$, and for spectratopes – the unit ball of the spectral norm in the space of matrices. In addition, every ellitope is a spectratope as well.

The importance of ellitopes/spectratopes in our context stems from the fact that it is easy to point out cones compatible with these sets. Specifically, it is shown in [24][7] that if $\mathcal{X} = M\mathcal{Y}$, with $\mathcal{Y}$ given by (4.14.b), is a spectratope, then the set

$$\begin{aligned}\mathbf{X} = \Big\{&(V,\tau) \in \mathbf{S}^n_+ \times \mathbf{R}_+ : \exists \Lambda = \{\Lambda_\ell \in \mathbf{S}^{d_\ell}_+, \ell \leq L\} : \\ & M^T V M \preceq \sum_\ell \mathcal{R}^*_\ell[\Lambda_\ell],\ \phi_{\mathcal{R}}(\lambda[\Lambda]) \leq \tau\Big\}\end{aligned} \tag{4.15}$$

[7]To make the paper self-contained, we reproduce the derivations in Section A.3.4

where

$$[\mathcal{R}_\ell^*[\Lambda_\ell]]_{ij} = \mathrm{Tr}(R^{\ell i}\Lambda_\ell R^{\ell j}),\ \lambda[\Lambda] = [\mathrm{Tr}(\Lambda_1);...;\mathrm{Tr}(\Lambda_L)],\ \text{and}\ \phi_{\mathcal{R}}(\lambda) = \max_{r\in\mathcal{R}} r^T\lambda,$$

Let us look at the proposed construction in the case where $p(\cdot) = \|\cdot\|_s$, $s \in [1,\infty]$, and let $r(\cdot) = \|\cdot\|_{\bar{s}}$, $\bar{s} = \max[s/2, 1]$. Setting $s_* = \frac{s}{s-1}$, $\bar{s}_* = \frac{\bar{s}}{\bar{s}-1}$, we clearly have

$$[p^+]_*(W) = \|W\|_{s_*} := \begin{cases} \left(\sum_{i,j}|W_{ij}|^{s_*}\right)^{1/s_*}, & s_* < \infty \\ \max_{i,j}|W_{ij}|, & s_* = \infty, \end{cases},\ r_*(w) = \|w\|_{\bar{s}_*},$$

resulting in

$$\begin{array}{rcl} \mathbf{P}^s & := & \mathbf{P}_{\|\cdot\|_s,\|\cdot\|_{\bar{s}}} \\ & = & \left\{(V,\tau) : V \in \mathbf{S}^N_+, \exists (W \in \mathbf{S}^N, w \in \mathbf{R}^N_+) : \begin{array}{l} V \preceq W + \mathrm{Diag}\{w\}, \\ \|W\|_{s_*} + \|w\|_{\bar{s}_*} \le \tau \end{array}\right\}. \end{array} \tag{4.16}$$

By Proposition 4.4, $\mathbf{P}^s$ is compatible with the unit ball of $\|\cdot\|_s$-norm on $\mathbf{R}^N$ (and therefore with every closed convex subset of this ball).

When $s = 1$, that is, $s_* = \bar{s}_* = \infty$, (4.16) results in

$$\begin{array}{rcl} \mathbf{P}^1 & = & \left\{(V,\tau) : V \succeq 0, \exists (W \in \mathbf{S}^N, w \in \mathbf{R}^N_+) : \begin{array}{l} V \preceq W + \mathrm{Diag}\{w\}, \\ \|W\|_\infty + \|w\|_\infty \le \tau \end{array}\right\} \\ & = & \{(V,\tau) : V \succeq 0, \|V\|_\infty \le \tau\}, \end{array} \tag{4.17}$$

and it is easily seen that the situation is a good as it could be, namely,

$$\mathbf{P}^1 = \left\{(V,\tau) : V \succeq 0, \max_{\|x\|_1 \le 1} x^TVx \le \tau\right\}.$$

It can be shown (see Section A.3.5) that when $s \in [2,\infty]$, so that $\bar{s}_* = \frac{s}{s-2}$, (4.16) results in

$$\mathbf{P}^s = \left\{(V,\tau) : V \succeq 0, \exists (w \in \mathbf{R}^N_+) :\ V \preceq \mathrm{Diag}\{w\}\ \&\ \|w\|_{\frac{s}{s-2}} \le \tau\right\}. \tag{4.18}$$

Note that

$$\mathbf{P}^2 = \{(V,\tau) :\ V \succeq 0,\ \|V\|_{\mathrm{sp}} \le \tau\}$$

where $\|\cdot\|_{\mathrm{sp}}$ is the spectral norm, and this is *exactly* the largest cone compatible with the unit Euclidean ball.

When $s \ge 2$, the unit ball $\mathcal{Y}$ of the norm $\|\cdot\|_s$ is an ellitope:

$$\{y \in \mathbf{R}^N : \|y\|_s \le 1\} = \{y \in \mathbf{R}^N : \exists (t \ge 0, \|t\|_{\bar{s}} \le 1) : y^TR_\ell y := y_\ell^2 \le t_\ell,\ \ell \le L = N\},$$

so that a compatible with $\mathcal{Y}$ cone is given by (4.19) with $M = I_N$. It comes as no surprise that, as it is immediately seen, the latter cone is nothing but the cone given by (4.18). is a closed convex cone which is compatible with $\mathcal{Y}$. Similarly, when $\mathcal{X} = M\mathcal{Y}$ with $\mathcal{Y}$ given by (4.14.a), is an ellitope, the set

$$\mathbf{X} = \left\{ (V,\tau) \in \mathbf{S}^n_+ \times \mathbf{R}_+ : \exists \lambda \in \mathbf{R}^L_+ : M^T V M \preceq \sum_\ell \lambda_\ell R_\ell,\ \phi_{\mathcal{R}}(\lambda) \le \tau \right\} \tag{4.19}$$

is a closed convex cone compatible with $\mathcal{Y}$. In both cases, $\mathbf{X}$ is sharp, provided that the image space of M is the entire $\mathbf{R}^n$.

Note that in both these cases $\mathbf{X}$ is a reasonably tight inner approximation of

$$\mathbf{X}^* = \{(V,\tau) \in \mathbf{S}^n_+ \times \mathbf{R}_+ :\ \tau \ge \max_{x\in\mathcal{X}} x^T V x\},$$

whenever $(V,\tau) \in \mathbf{X}^*$, we have $(V,\theta\tau) \in \mathbf{X}$, with a moderate θ (namely, $\theta = O(1)\ln(2\sum_\ell d_\ell)$ in the spectratopic, and $\theta = O(1)\ln(2L)$ in the ellitopic case, see [24, Proposition 2.1] and [25, Proposition 3.3], respectively).

4.5.2. Compatibility via absolute norms

Preliminaries. Recall that a norm $p(\cdot)$ on $\mathbf{R}^N$ is called *absolute*, if $p(x)$ is a function of the vector $\mathrm{abs}[x] := [|x_1|;...;|x_N|]$ of the magnitudes of entries in x. It is well known that an absolute norm p is monotone on $\mathbf{R}^N_+$, so that $\mathrm{abs}[x] \le \mathrm{abs}[x']$ implies that $p(x) \le p(x')$, and that the norm

$$p_*(x) = \max_{y:\, p(y)\le 1} x^T y$$

conjugate to $p(\cdot)$ is absolute along with p.

Let us say that an absolute norm $r(\cdot)$ *fits* an absolute norm $p(\cdot)$ on $\mathbf{R}^N$, if for every vector x with $p(x) \le 1$ the entrywise square $[x]^2 = [x_1^2;...;x_N^2]$ of x satisfies $r([x]^2) \le 1$. For example, the largest norm $r(\cdot)$ which fits the absolute norm $p(\cdot) = \|\cdot\|_s$, $s \in [1,\infty]$, is

$$r(\cdot) = \begin{cases} \|\cdot\|_1, & 1 \le s \le 2 \\ \|\cdot\|_{s/2}, & s \ge 2 \end{cases}$$

An immediate observation is that an absolute norm $p(\cdot)$ on $\mathbf{R}^N$ can be "lifted" to a norm on $\mathbf{S}^N$, specifically, the norm

$$p^+(Y) = p([p(\mathrm{Col}_1[Y]);...;p(\mathrm{Col}_N[Y])]) : \mathbf{S}^N \to \mathbf{R}_+, \tag{4.20}$$

where $\mathrm{Col}_j[Y]$ is jth column in Y. It is immediately seen that when p is an absolute norm, the right hand side in (4.20) indeed is a norm on $\mathbf{S}^N$ satisfying the identity

$$p^+(xx^T) = p^2(x),\ x \in \mathbf{R}^N. \tag{4.21}$$

Absolute norms and compatibility. Our interest in absolute norms is motivated by the following immediate

Proposition 4.4. *Let $p(\cdot)$ be an absolute norm on $\mathbf{R}^N$, and $r(\cdot)$ be another absolute norm which fits $p(\cdot)$, both norms being computationally tractable. These norms give rise to the computationally tractable and sharp closed convex cone*

$$\mathbf{P} = \mathbf{P}_{p,r} = \left\{ (V,\tau) \in \mathbf{S}^N_+ \times \mathbf{R}_+ : \exists (W \in \mathbf{S}^N, w \in \mathbf{R}^N_+) : \begin{array}{l} V \preceq W + \mathrm{Diag}\{w\} \\ [p^+]_*(W) + r_*(w) \leq \tau \end{array} \right\}, \tag{4.22}$$

where $[p^+]_*(\cdot)$ is the norm on $\mathbf{S}^N$ conjugate to the norm $p^+(\cdot)$, and $r_*(\cdot)$ is the norm on $\mathbf{R}^N$ conjugate to the norm $r(\cdot)$, and this cone is compatible with the unit ball of the norm $p(\cdot)$ (and thus – with any convex compact subset of this ball).

Verification is immediate. The fact that $\mathbf{P}$ is a computationally tractable and closed convex cone is evident. Now let $(V,\tau) \in \mathbf{P}$, so that $V \succeq 0$ and $V \preceq W + \mathrm{Diag}\{w\}$ with $[p^+]_*(W) + r_*(w) \leq \tau$. For x with $p(x) \leq 1$ we have

$$\begin{array}{rcl} x^T V x & \leq & x^T[W + \mathrm{Diag}\{w\}]x = \mathrm{Tr}(W[xx^T]) + w^T[x]^2 \\ & \leq & p^+(xx^T)[p^+]_*(W) + r([x]^2)r_*(w) = p^2(x)[p^+]_*(W) + r_*(w) \\ & \leq & [p^+]_*(W) + r_*(w) \leq \tau, \end{array}$$

whence $x^T V x \leq \tau$ for all x with $p(x) \leq 1$. □

4.6. Near-optimality of polyhedral estimate in the spectratopic sub-Gaussian case

As an instructive application of the design of polyhedral estimate developed in this section, consider the special case of the estimation problem stated in Section 2.1, where

1. The signal set $\mathcal{X}$ and the unit ball $\mathcal{B}_*$ of the norm conjugate to $\|\cdot\|$ are spectratopes:

$$\begin{array}{rcl} \mathcal{X} & = & \{x \in \mathbf{R}^n : \exists t \in \mathcal{T} : R_k^2[x] \preceq t_k I_{d_k},\ 1 \leq k \leq K\}, \\ \mathcal{B}_* & = & \{z \in \mathbf{R}^\nu : \exists y \in \mathcal{Y} : z = My\}, \\ & & \mathcal{Y} := \{y \in \mathbf{R}^q : \exists r \in \mathcal{R} : S_\ell^2[y] \preceq r_\ell I_{f_\ell},\ 1 \leq \ell \leq L\}, \end{array}$$

2. For every $x \in \mathcal{X}$, observation noise is sub-Gaussian, i.e. $\xi_x \sim \mathcal{SG}(0, \sigma^2 I_m)$.

We are about to show that in the present situation, *the polyhedral estimate yielded by the approach described in Sections 4.1–4.4, i.e., yielded by the efficiently computable (high accuracy near-) optimal solution to the optimization problem (4.12) is near-optimal in the minimax sense.*

Given reliability tolerance $\epsilon \in (0,1)$, the recipe for constructing a $m \times m$ contrast matrix H as prescribed by Proposition 4.3 is as follows:

- Set

$$Z = \{\vartheta^2[1;...;1]\},\ \vartheta = \sigma\kappa,\ \kappa = \sqrt{2\ln(2m/\epsilon)},$$

and utilize the construction from Section 4.2, thus arriving at the cone

$$\mathbf{H} = \{(\Theta, \mu) \in \mathbf{S}^m_+ \times \mathbf{R}_+ : \sigma^2\kappa^2\mathrm{Tr}(\Theta) \leq \mu\}$$

satisfying the requirements of Assumption **C**.

- Specify the cones $\mathbf{X}$ and $\mathbf{U}$ compatible with $\mathcal{X}_\mathrm{s} = \mathcal{X}$, and $\mathcal{B}_*$, respectively, according to (4.15).

The resulting problem (4.12), after immediate straightforward simplifications, reads

$$\begin{array}{rcl}\mathrm{Opt} & = & \min\limits_{\Theta,U,\Lambda,\Upsilon} \left\{ 2\left[\phi_{\mathcal{R}}(\lambda[\Upsilon]) + \phi_{\mathcal{T}}(\lambda[\Lambda]) + \sigma^2\kappa^2\mathrm{Tr}(\Theta)\right] : \right. \\ & & \Theta \succeq 0,\, U \succeq 0, \Lambda = \{\Lambda_k \succeq 0, k \leq K\},\, \Upsilon = \{\Upsilon_\ell \succeq 0, \ell \leq L\}, \\ & & \left. \left[\begin{array}{c|c} U & \frac{1}{2}B \\ \hline \frac{1}{2}B^T & A^T\Theta A + \sum_k \mathcal{R}_k^*[\Lambda_k]\end{array}\right] \succeq 0,\; M^TUM \preceq \sum_\ell \mathcal{S}_\ell^*[\Upsilon_\ell] \right\}\end{array} \tag{4.23}$$

where

$$\begin{array}{rcl}[\mathcal{R}_k^*[\Lambda_k]]_{ij} & = & \mathrm{Tr}(R^{ki}\Lambda_k R^{kj}) \quad \left[R_k[x] = \sum_i x_i R^{ki}\right], \\ {}[\mathcal{S}_\ell^*[\Upsilon_\ell]]_{ij} & = & \mathrm{Tr}(S^{\ell i}\Upsilon_\ell S^{\ell j}) \quad \left[S_\ell[u] = \sum_i u_i S^{\ell i}\right],\end{array}$$

and

$$\lambda[\Lambda] = [\mathrm{Tr}(\Lambda_1); ...; \mathrm{Tr}(\Lambda_K)],\ \lambda[\Upsilon] = [\mathrm{Tr}(\Upsilon_1); ...; \mathrm{Tr}(\Upsilon_L)],\ \phi_W(f) = \max_{w\in W} w^T f.$$

Let now

$$\mathrm{RiskOpt}_\epsilon = \inf_{\widehat{w}(\cdot)} \sup_{x\in\mathcal{X}} \inf\left\{\rho : \mathrm{Prob}_{\xi\sim\mathcal{N}(0,\sigma^2 I)}\{\|Bx - \widehat{w}(Ax+\xi)\| > \rho\} \leq \epsilon\ \forall x \in \mathcal{X}\right\},$$

be the minimax optimal $(\epsilon, \|\cdot\|)$-risk of estimating $w = Bx$ in Gaussian observation scheme where $\xi_x \sim \mathcal{N}(0, \sigma^2 I_m)$ independently of $x \in X$.

Proposition 4.5. *When $\epsilon \leq 1/8$, the polyhedral estimate $\widehat{w}^H$ yielded by a feasible near optimal, in terms of the objective, solution to problem (4.23) is minimax optimal within the logarithmic factor, namely*

$$\begin{array}{rcl}\mathrm{Risk}_{\epsilon,\|\cdot\|}[\widehat{w}^H|\mathcal{X}] & \leq & O(1)\sqrt{\ln\left(\sum_k d_k\right)\ln\left(\sum_\ell f_\ell\right)\ln(2m/\epsilon)}\,\mathrm{RiskOpt}_{\frac{1}{8}} \\ & \leq & O(1)\sqrt{\ln\left(\sum_k d_k\right)\ln\left(\sum_\ell f_\ell\right)\ln(2m/\epsilon)}\,\mathrm{RiskOpt}_\epsilon,\end{array}$$

where $O(1)$ is an absolute constant.

For proof, see Section A.3.6.

Discussion. It is worth mentioning that the approach described in Section 3 is complementary to the approach discussed in this section. In fact, it is easily seen that the bound Opt for the risk of the polyhedral estimate stemming from (4.12) is suboptimal in the simple situation described in the motivating example from Introduction. Indeed, let $\mathcal{X}$ be the unit $\|\cdot\|_1$-ball, let $\|\cdot\| = \|\cdot\|_2$, and let us consider the problem of estimating $x \in \mathcal{X}$ from the direct observation $\zeta = x + \xi$ with Gaussian observation noise $\xi \sim \mathcal{N}(0, \sigma^2 I)$. We equip the ball $\mathcal{B}_* = \{u \in \mathbf{R}^n : \|u_2\|_2 \le 1\}$ with the cone

$$\mathbf{U} = \mathbf{P}^2 = \{(U, \tau) : U \succeq 0, \|U\|_{\rm sp} \le \tau\}$$

and $\mathcal{X}$ – with the cone

$$\mathbf{X} = \mathbf{P}^1 = \{(X, t) : X \succeq 0, \|X\|_\infty \le t\},$$

(note that both cones are the largest w.r.t. inclusion cones compatible with the respective sets). The corresponding problem (4.12) reads

$$\begin{array}{rcl}\mathrm{Opt} &=& \min\limits_{\Theta,X,U}\left\{2\Big(\kappa^2\sigma^2\mathrm{Tr}(\Theta)+\max\limits_i X_{ii}+\|U\|_{\rm sp}\Big):\ \begin{array}{c}\Theta\succeq 0,\ X\succeq 0,\ U\succeq 0,\\ \left[\begin{array}{c|c}U & \frac{1}{2}I_n\\ \hline \frac{1}{2}I_n & \Theta+X\end{array}\right]\succeq 0\end{array}\right\}\\ &=& \min\limits_{\Theta,X,U}\left\{2\Big(\kappa^2\sigma^2\mathrm{Tr}(\Theta)+\max\limits_i X_{ii}+\tau\Big):\ \begin{array}{c}\Theta\succeq 0,\ X\succeq 0,\ U\succeq 0,\\ \left[\begin{array}{c|c}\tau I_n & \frac{1}{2}I_n\\ \hline \frac{1}{2}I_n & \Theta+X\end{array}\right]\succeq 0\end{array}\right\}\end{array} \tag{4.24}$$

Observe that every $n \times n$ matrix of the form $Q = EP$, where E is diagonal with diagonal entries ± 1, and P is a permutation matrix, induces a symmetry $(\Theta, X, \tau) \mapsto (Q\Theta Q^T, QXQ^T, \tau)$ of the second optimization problem in (4.24), that is, a transformation which maps the feasible set onto itself and keeps the objective intact. Since the problem is convex and solvable, we conclude that it has an optimal solution which remains intact under the symmetries in question, i.e., solution with scalar matrices $\Theta = \theta I_n$ and $X = uI_n$. As a result,

$$\mathrm{Opt} = \min_{\theta\ge 0, u\ge 0, \tau}\left\{2(\kappa^2\sigma^2 n\theta + u + \tau) : \tau(\theta + u) \ge \tfrac{1}{4}\right\} = 2\min\left[\kappa\sigma\sqrt{n}, 1\right]. \tag{4.25}$$

A similar derivation shows that the value Opt remains intact if we replace the set $\mathcal{X} = \{x : \|x\|_1 \le 1\}$ with $\mathcal{X} = \{x : \|x\|_s \le 1\}$, $s \in [1, 2]$, and the cone $\mathbf{X} = \mathbf{P}^1$ with $\mathbf{X} = \mathbf{P}^s$, see (4.16). Since the Θ-component of an optimal solution to (4.24) can be selected to be scalar, the contrast matrix H we end up with can be selected to be the unit matrix. An unpleasant observation is that when $s < 2$, the quantity Opt given by (4.25) "heavily overestimates" the actual risk of the polyhedral estimate with $H = I_n$. Indeed, the analysis of this estimate in Section 3.3 results in the risk bound (up to a logarithmic in n factor) $\min[\sigma^{1-s/2}, \sigma\sqrt{n}]$, what can be much less than $\mathrm{Opt} = 2\min[\kappa\sigma\sqrt{n}, 1]$, e.g., in the case of large n, and $\sigma\sqrt{n} = O(1)$.

The good news is that whenever the approaches developed in this section and in Section 3 are applicable, they can be utilized simultaneously. The underlying observation is that

(!) *In the problem setting described in Section 2, a collection of K candidate polyhedral estimates can be assembled into a single polyhedral estimate with the (upper bound on the) risk, as given by Proposition 2.1, being nearly the minimum of the risks of estimates we aggregate.*

Indeed, given an observation scheme (that is, collection of probability distributions P_x of noises ξ_x, $x \in \mathcal{X}$), assume we have at our disposal norms

$$\pi_\delta(\cdot) : \mathbf{R}^m \to \mathbf{R}$$

parameterized by $\delta \in (0, 1)$ such that $\pi_\delta(h)$, for every h, is the larger the less is δ, and (cf. Section 4.2)

$$\forall (x \in \mathcal{X}, \delta \in (0,1), h \in \mathbf{R}^m) : \pi_\delta(h) \le 1 \Rightarrow \mathrm{Prob}_{\xi \sim P_x}\{\xi : |h^T \xi| > 1\} \le \delta.$$

To ensure that the columns h_ℓ of an $m \times N$ contrast matrix H satisfy the relations (2.2) we impose on h_ℓ the restrictions $\pi_{\epsilon/N}(h_\ell) \le 1$.

Now suppose that given the risk tolerance $\epsilon \in (0, 1)$, we have generated somehow K candidate contrast matrices $H_k \in \mathbf{R}^{m \times N_k}$ such that

$$\pi_{\epsilon/N_k}(\mathrm{Col}_j[H_k]) \le 1,\ j \le N_k,$$

so that the $(\epsilon, \|\cdot\|)$-risk of the polyhedral estimate yielded by the contrast matrix H_k does not exceed

$$\mathfrak{R}_k = \max_x \left\{ \|Bx\| : x \in 2\mathcal{X}_{\mathrm{s}}, \|H_k^T A x\|_\infty \le 2 \right\}.$$

Let us combine the contrast matrices $H_1, ..., H_K$ into a single contrast matrix H with $N = N_1 + ... + N_K$ columns by normalizing the columns of the matrix $[H_1, ..., H_K]$ to have $\pi_{\epsilon/N}$-norms equal to 1, so that

$$H = [\bar{H}_1, ..., \bar{H}_K],\ \mathrm{Col}_j[\bar{H}_k] = \theta_{jk} \mathrm{Col}_j[H_k]\ \forall (k \le K, j \le N_k)$$

with

$$\theta_{jk} = \frac{1}{\pi_{\epsilon/N}(\mathrm{Col}_j[H_k])} \ge \vartheta_k := \min_{h \ne 0} \frac{\pi_{\epsilon/N_k}(h)}{\pi_{\epsilon/N}(h)},$$

where the concluding $\ge$ is due to $\pi_{\epsilon/N_k}(\mathrm{Col}_j[H_k]) \le 1$. We claim that in terms of $(\epsilon, \|\cdot\|)$-risk, the polyhedral estimate yielded by H is "almost as good" as the best of the polyhedral estimates yielded by the contrast matrices $H_1, ..., H_K$, specifically,

$$\mathfrak{R}[H] := \max_x \left\{ \|Bx\| : x \in 2\mathcal{X}_{\mathrm{s}}, \|H^T A x\|_\infty \le 2 \right\} \le \max_k \vartheta_k^{-1} \mathfrak{R}_k.$$ [8]

The justification is readily given by the following observation: when $\vartheta \in (0, 1)$, we have

$$\mathfrak{R}_{k,\vartheta} := \max_x \left\{ \|Bx\| : x \in 2\mathcal{X}_{\mathrm{s}}, \|H_k^T A x\|_\infty \le 2/\vartheta \right\} \le \mathfrak{R}_k/\vartheta.$$

[8] This is the precise "quantitative expression" of the observation (!).

Indeed, when x is a feasible solution to the maximization problem specifying $\mathfrak{R}_{k,\vartheta}$, ϑx is a feasible solution to the problem specifying $\mathfrak{R}_k$, implying that $\vartheta\|Bx\| \leq \mathfrak{R}_k$. It remains to note that we clearly have $\mathfrak{R}[H] \leq \min_k \mathfrak{R}_{k,\vartheta_k}$.

The bottom line is that the just described aggregation of contrast matrices $H_1, ..., H_K$ into a single contrast matrix H results in polyhedral estimate which in terms of upper bound $\mathfrak{R}[\cdot]$ on its $(\epsilon, \|\cdot\|)$-risk is, up to the factor $\bar{\vartheta} = \max_k \vartheta_k^{-1}$, not worse than the best of the K estimates yielded by the original contrast matrices. Consequently, whenever $\pi_\delta(\cdot)$ grows slowly as δ decreases, the "price" $\bar{\vartheta}$ of assembling the original estimates is quite moderate. For example, in the case of basic observation schemes (Sub-Gaussian, Discrete, and Poisson) described in Section 2.3, $\bar{\vartheta}$ is logarithmic in $\max_k N_k^{-1}(N_1 + ... + N_K)$, and $\bar{\vartheta} = 1 + o(1)$ as $\epsilon \to +0$ for $N_1, ..., N_K$ fixed.

4.7. Numerical illustration

To illustrate the performance of the polyhedral estimates we compare it numerically with a "presumably good" linear estimate. Our setup is deliberately simple: the signal set $\mathcal{X}$ is just the unit box $\{x \in \mathbf{R}^n : \|x\|_\infty \leq 1\}$, $B \in \mathbf{R}^{n\times n}$ is "numerical double integration:" for a $\delta > 0$,

$$B_{ij} = \begin{cases} \delta^2(i-j+1), & j \leq i \\ 0, & j > i \end{cases},$$

so that x, modulo boundary effects, is the second order finite difference derivative of $w = Bx$:

$$x_i = \frac{w_i - 2w_{i-1} + w_{i-2}}{\delta^2},\ 2 < i \leq n,$$

and Ax is comprised of m randomly selected entries of Bx. The observation is

$$\zeta = Ax + \xi,\ \xi \sim \mathcal{N}(0, \sigma^2 I_m).$$

and the recovery norm is $\|\cdot\|_2$. In other words, we want to recover a restriction of twice differentiable function of one variable on the n-point regular grid on the segment $\Delta = [0, n\delta]$ from noisy observations of this restriction taken along m randomly selected points of the grid. A priori information on the function is that the magnitude of its second order derivative does not exceed 1.

Note that in the considered situation both linear estimate $\widehat{w}_H$ yielded by Proposition 3.3 in [24] and polyhedral estimate $\widehat{w}^H$ yielded by Proposition 4.2, are near-optimal in the minimax sense in terms of their $\|\cdot\|_2$-, resp., $(\epsilon, \|\cdot\|_2)$-risk.

In the experiments reported in Figure 1, we used $n = 64$, $m = 32$, and $\delta = 4/n$ (i.e., $\Delta = [0, 4]$); the reliability parameter for the polyhedral estimate was set to $\epsilon = 0.1$. For different noise levels $\sigma = \{0.1, 0.01, 0.001, 0.0001\}$ we generate 20 random signals x from $\mathcal{X}$ and record the $\|\cdot\|_2$-recovery errors of the linear and the polyhedral estimates. In addition to testing the nearly optimal polyhedral estimate *PolyII* yielded by Proposition 4.3 as applied in the framework of Section 4.5.1, we also record the performance of the polyhedral estimate

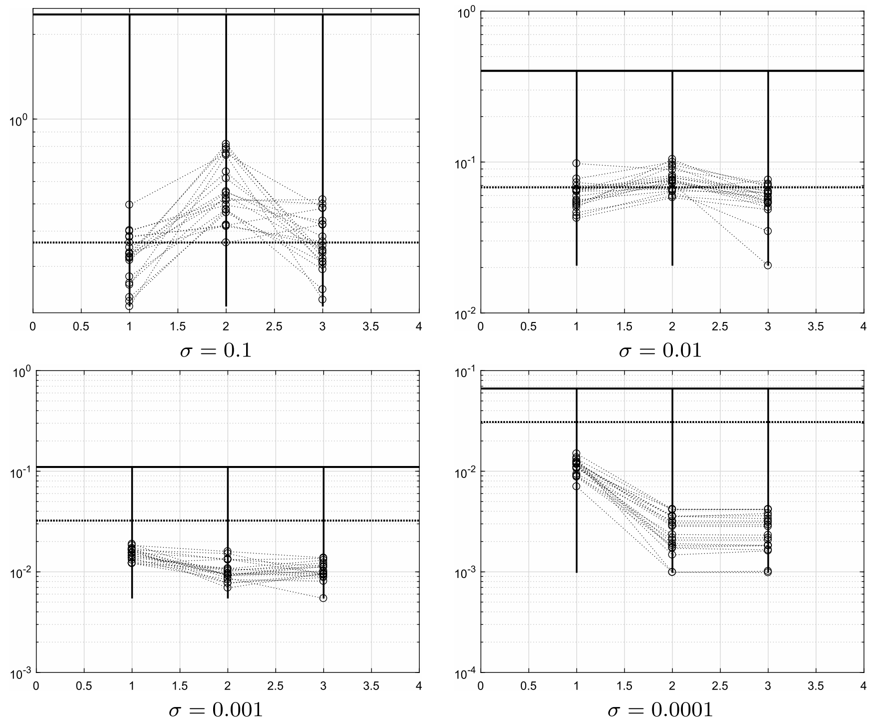

FIG 1. *Recovery errors for near-optimal linear estimate (left), for polyhedral estimates yielded by Proposition 4.3 (*PolyII*, middle), and by the construction from Section 3 (*PolyI*, right), 20 simulations per each value of σ.*
Horizontal lines: solid — upper bound on 0.9*-risk of* PolyII*; dotted — upper bound on expected $\|\cdot\|_2$ recovery error for linear estimate.*

PolyI yielded by the construction from Section 3. The observed $\|\cdot\|_2$-recovery errors of the three estimates are plotted in Figure 1.

In these simulations, the three estimates exhibit similar empirical performance. However, when the noise level becomes small, polyhedral estimates seem to outperform the linear one. In addition, the estimate *PolyI* seems to "work" better than or, at the very worst, similarly to *PolyII* in spite of the fact that in the situation in question the estimate *PolyII,* in contrast to *PolyI,* is provably near-optimal.

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Appendix A: Appendix

A.1. Executive summary on computationally tractable convex sets

Detailed description of what "computationally tractable convex set/function" is goes beyond the scope of this paper; an interested reader is referred to [3]. Here we restrict ourselves with "executive summary" as follows: "for all practical purposes" (including all applications considered in this paper), a computationally tractable convex set is a set $X \subset \mathbf{R}^N$ given by *semidefinite representation*

$$X = \{x \in \mathbf{R}^N : \exists u \in \mathbf{R}^M : \mathcal{A}(x,u) \succeq 0\},$$

where $\mathcal{A}(\cdot)$ is a symmetric matrix affinely depending on $[x;u]$; sets of this type automatically are convex. In simple words: if you can feed the problem of minimizing a linear function over X to `CVX` [9], your X is computationally tractable, and nearly vice versa.

A.2. Calculus of compatibility

The principal rules of the calculus of compatibility are as follows (verification of the rules is straightforward and is therefore skipped):

1. [passing to a subset] When $\mathcal{Y}' \subset \mathcal{Y}$ are convex compact subsets of $\mathbf{R}^N$ and a cone $\mathbf{Y}$ is compatible with $\mathcal{Y}$, the cone is compatible with $\mathcal{Y}'$ as well.
2. [finite intersection] Let cones $\mathbf{Y}^j$ be compatible with convex compact sets $\mathcal{Y}_j \subset \mathbf{R}^N$, $j = 1, ..., J$. Then the cone

   $$\mathbf{Y} = \mathrm{cl}\,\{(V,\tau) \in \mathbf{S}^N_+ \times \mathbf{R}_+ : \exists ((V_j,\tau_j) \in \mathbf{Y}^j, j \le J) : V \preceq \sum_j V_j, \sum_j \tau_j \le \tau\}$$

   is compatible with $\mathcal{Y} = \bigcap_j \mathcal{Y}_j$. The closure operation can be skipped when all cones $\mathbf{Y}^j$ are sharp, in which case $\mathbf{Y}$ is sharp as well.

[9] M. Grant and S. Boyd. *The* `CVX` *Users' Guide. Release 2.1*, 2014. http://web.cvxr.com/cvx/doc/CVX.pdf

3. [convex hulls of finite union] Let cones $\mathbf{Y}^j$ be compatible with convex compact sets $\mathcal{Y}_j \subset \mathbf{R}^N$, $j = 1, ..., J$, and let there exist (V, τ) such that $V \succ 0$ and
$$(V, \tau) \in \mathbf{Y} := \bigcap_j \mathbf{Y}^j.$$
Then $\mathbf{Y}$ is compatible with $\mathcal{Y} = \mathrm{Conv}\{\bigcup_j \mathcal{Y}_j\}$ and, in addition, is sharp provided that at least one of $\mathbf{Y}^j$ is sharp.
4. [direct product] Let cones $\mathbf{Y}^j$ be compatible with convex compact sets $\mathcal{Y}_j \subset \mathbf{R}^{N_j}$, $j = 1, ..., J$. Then the cone
$$\begin{array}{rcl}\mathbf{Y} & = & \{(V, \tau) \in \mathbf{S}_+^{N_1 + ... + N_J} \times \mathbf{R}_+ : \exists (V_j, \tau_j) \in \mathbf{Y}^j : \\ & & V \preceq \mathrm{Diag}\{V_1, ..., V_J\} \ \&\ \tau \geq \sum_j \tau_j\}\end{array}$$
is compatible with $\mathcal{Y} = \mathcal{Y}_1 \times ... \times \mathcal{Y}_J$. This cone is sharp, provided that all $\mathbf{Y}^j$ are so.
5. [linear image] Let cone $\mathbf{Y}$ be compatible with convex compact set $\mathcal{Y} \subset \mathbf{R}^N$, let A be a $K \times N$ matrix, and let $\mathcal{Z} = A\mathcal{Y}$. The cone
$$\mathbf{Z} = \mathrm{cl}\,\{(V, \tau) \in \mathbf{S}_+^K \times \mathbf{R}_+ : \exists U \succeq A^T V A : (U, \tau) \in \mathbf{Y}\}$$
is compatible with $\mathcal{Z}$. The closure operation can be skipped whenever $\mathbf{Y}$ is either sharp, or *complete*, completeness meaning that $(V, \tau) \in \mathbf{Y}$ and $0 \preceq V' \preceq V$ imply that $(V', \tau) \in \mathbf{Y}$. The cone $\mathbf{Z}$ is sharp, provided $\mathbf{Y}$ is so and the rank of A is K.
6. [inverse linear image] Let cone $\mathbf{Y}$ be compatible with convex compact set $\mathcal{Y} \subset \mathbf{R}^N$, let A be a $N \times K$ matrix with trivial kernel, and let $\mathcal{Z} = A^{-1}\mathcal{Y} := \{z \in \mathbf{R}^K : Az \in \mathcal{Y}\}$. The cone
$$\mathbf{Z} = \mathrm{cl}\,\{(V, \tau) \in \mathbf{S}_+^K \times \mathbf{R}_+ : \exists U : A^T U A \succeq V \ \&\ (U, \tau) \in \mathbf{Y}\}$$
is compatible with $\mathcal{Z}$. The closure operations can be skipped whenever $\mathbf{Y}$ is sharp, in which case $\mathbf{Z}$ is sharp as well.
7. [arithmetic summation] Let cones $\mathbf{Y}^j$ be compatible with convex compact sets $\mathcal{Y}_j \subset \mathbf{R}^N$, $j = 1, ..., J$. Then the arithmetic sum $\mathcal{Y} = \mathcal{Y}_1 + ... + \mathcal{Y}_J$ of the sets $\mathcal{Y}_j$ can be equipped with compatible cone readily given by the cones $\mathbf{Y}^j$; this cone is sharp, provided all $\mathbf{Y}^j$ are so.
Indeed, the arithmetic sum of $\mathcal{Y}_j$ is the linear image of the direct product of $\mathcal{Y}_j$'s under the mapping $[y^1; ...; y^J] \mapsto y^1 + ... + y^J$, and it remains to combine rules 4 and 5; note the cone yielded by rule 4 is complete, so that when applying rule 5, the closure operation can be skipped.

A.3. Proofs

A.3.1. Proofs for Section 2.3

1^o Sub-Gaussian case. Note that when $h\in\mathbf{R}^n$ is deterministic and $\xi\sim\mathcal{SG}(0,\sigma^2 I_m)$ we have

$$\mathrm{Prob}\{|h^T\xi|>1\}\leq 2\exp\left\{-\frac{1}{2\sigma^2\|h\|_2^2}\right\}.$$

Indeed, when $h\neq 0$ and $\gamma>0$,

$$\mathrm{Prob}\{h^T\xi>1\}\leq\exp\{-\gamma\}\mathbf{E}\left\{\exp\{\gamma h^T\xi\}\right\}\leq\exp\{\tfrac12\sigma^2\gamma^2\|h\|_2^2-\gamma\}.$$

When minimizing the resulting bound in $\gamma>0$, we get $\mathrm{Prob}\{h^T\xi>1\}\leq\exp\left\{-\frac{1}{2\|h\|_2^2\sigma^2}\right\}$; and the same reasoning as applied to $-h$ in the role of h results in $\mathrm{Prob}\{h^T\xi<-1\}\leq\exp\left\{-\frac{1}{2\|h\|_2^2\sigma^2}\right\}$. Consequently

$$\pi_G(h):=\underbrace{\sigma\sqrt{2\ln(2/\delta)}}_{\vartheta_G}\|h\|_2\leq 1\;\;\Rightarrow\;\;\mathrm{Prob}\{|h^T\xi|>1\}\leq\delta,$$

implying that for $\mathcal{H}_\delta^G$ as in (2.8) we indeed have

$$h\in\mathcal{H}_\delta^G\;\;\Rightarrow\;\;\mathrm{Prob}\{|h^T\xi|>1\}\leq\delta.$$

2^o Discrete case. Given $x\in\mathcal{X}$, setting $\mu=Ax$ and $\eta_k=\omega_k-\mu$, we get

$$\zeta=Ax+\underbrace{\frac{1}{K}\sum_{k=1}^K\eta_k}_{\xi_x}.$$

Given $h\in\mathbf{R}^m$,

$$h^T\xi_x=\frac1K\sum_k\underbrace{h^T\eta_k}_{\chi_k}.$$

Random variables $\chi_1,...,\chi_K$ are independent zero mean and clearly satisfy

$$\mathbf{E}\left\{\chi_k^2\right\}\leq\sum_i[Ax]_ih_i^2,\;\;|\chi_k|\leq 2\|h\|_\infty.$$

Applying the Bernstein inequality we get

$$\mathrm{Prob}\{|h^T\xi_x|>1\}=\mathrm{Prob}\left\{|\textstyle\sum_k\chi_k|>K\right\}\leq 2\exp\left\{-\frac{K}{2\sum_i[Ax]_ih_i^2+\frac43\|h\|_\infty}\right\}. \tag{A.1}$$

Let now

$$\pi_D(h) = 2\sqrt{\vartheta_D \max_{x\in\mathcal{X}} \sum_i [Ax]_i h_i^2 + \tfrac{16}{9}\vartheta_D^2 \|h\|_\infty^2} \text{ with } \vartheta_D = \frac{\ln(2/\delta)}{K}.$$

After a straightforward computation, we conclude from (A.1) that

$$\pi_D(h) \le 1 \;\;\Rightarrow\;\; \mathrm{Prob}\{|h^T\xi_x| > 1\} \le \delta,\ \forall x \in \mathcal{X}.$$

Thus, in the Discrete case we can set

$$\mathcal{H}_\delta = \mathcal{H}_\delta^D := \{h:\ \pi_D(h) \le 1\}.$$

Poisson case. In the Poisson case, for $x \in \mathcal{X}$, setting $\mu = Ax$, we have

$$\zeta = Ax + \xi_x,\ \xi_x = \zeta - \mu.$$

In this case, for all $h \in \mathbf{R}^m$ one has

$$\forall t \ge 0: \mathrm{Prob}\left\{|h^T\xi_x| \ge t\right\} \le 2\exp\left\{-\frac{t^2}{2[\sum_i h_i^2\mu_i + \|h\|_\infty t/3]}\right\} \tag{A.2}$$

Indeed, let $h \in \mathbf{R}^m$, and let ζ be random vector with independent across i entries $\zeta_i \sim \mathrm{Poisson}(\mu_i)$. We have

$$\begin{aligned}\mathbf{E}\left\{\exp\{\gamma h^T\zeta\}\right\} &= \prod_i \mathbf{E}\left\{\gamma h_i \zeta_i\right\} = \prod_i \exp\{[\exp\{\gamma h_i\} - 1]\mu_i\} \\ &= \exp\left\{\sum_i [\exp\{\gamma h_i\} - 1]\mu_i\right\}.\end{aligned}$$

Hence, due to the Markov inequality for $\gamma \ge 0$ it holds

$$\begin{aligned}&\mathrm{Prob}\{h^T\zeta > h^T\mu + t\} = \mathrm{Prob}\{\gamma h^T\zeta > \gamma h^T\mu + \gamma t\} \\ &\le\ \mathbf{E}\left\{\exp\{\gamma h^T\zeta\}\right\}\exp\{-\gamma h^T\mu - \gamma t\} \\ &\le\ \exp\left\{\sum_i [\exp\{\gamma h_i\} - \gamma h_i - 1]\mu_i - \gamma t\right\}.\end{aligned} \tag{A.3}$$

Now the standard computation (see, e.g., [31]) shows that

$$\gamma \le 3/\|h\|_\infty \;\;\Rightarrow\;\; \left|\mathrm{e}^{\gamma h_i} - \gamma h_i - 1\right| \le \frac{\gamma^2 h_i^2}{2(1 - \gamma\|h\|_\infty/3)}$$

which combines with (A.3) to imply

$$\ln\left(\mathrm{Prob}\{h^T\zeta > h^T\mu + t\}\right) \le \frac{\gamma^2\sum_i h_i^2\mu_i}{2(1 - \gamma\|h\|_\infty/3)} - \gamma t. \tag{A.4}$$

For $\gamma_* = \frac{t}{\sum_i h_i^2\mu_i + \|h\|_\infty t/3} \in [0, 3/\|h\|_\infty]$ we get:

$$\mathrm{Prob}\left\{h^T\zeta > h^T\mu + t\right\} \le \exp\left\{-\frac{t^2}{2[\sum_i h_i^2\mu_i + \|h\|_\infty t/3]}\right\}.$$

This inequality combines with the same inequality applied to $-h$ in the role of h to imply (A.2). □

From (A.2), we conclude via a straightforward computation that setting

$$\pi_P(h) = \sqrt{4\vartheta_P \max_{x\in\mathcal{X}} \sum_i [Ax]_i h_i^2 + \tfrac{16}{9}\vartheta_P^2 \|h\|_\infty^2} \text{ with } \vartheta_P = \ln(2/\delta),$$

we ensure that

$$\pi_P(h) \le 1 \Rightarrow \mathrm{Prob}\{|h^T\xi_x| > 1\} \le \delta,\ \forall x \in \mathcal{X}.$$

Thus, in the Poisson case we can set

$$\mathcal{H}_\delta = \mathcal{H}_\delta^P := \{h : \pi_P(h) \le 1\}.$$

A.3.2. Proof of Proposition 3.3

Consider optimization problem specifying Ψ in (3.12). Setting $\theta = r/\rho \ge 1$, let us pass in this problem from variables v_ℓ to variables $z_\ell = v_\ell^\rho$, so that

$$\Psi^r = 2^r \max_z \left\{ \sum_\ell z_\ell^\theta (b_\ell/d_\ell)^r : \sum_\ell z_\ell \le 1,\ 0 \le z_\ell \le (d_\ell \varsigma_\ell / b_\ell)^\rho \right\} \le 2^r \Gamma,$$

where

$$\Gamma = \max_z \left\{ \sum_\ell z_\ell^\theta (b_\ell/d_\ell)^r : \sum_\ell z_\ell \le 1,\ 0 \le z_\ell \le \chi_\ell := (\vartheta_G d_\ell / a_\ell)^\rho \right\}$$

(we have used (3.11)). Note that Γ is the optimal value in the problem of maximizing a convex (since $\theta \ge 1$) function $\sum_\ell z_\ell^\theta (b_\ell/d_\ell)^r$ over a bounded polyhedral set, so that the maximum is achieved at an extreme point $\bar{z}$ of the feasible set. By the standard characterization of extreme points, the (clearly nonempty) set I of positive entries in $\bar{z}$ is as follows: denoting by I' the set of indexes $\ell \in I$ such that $\bar{z}_\ell$ is on its upper bound $\bar{z}_\ell = \chi_\ell$, its cardinality $|I'|$ is at least $|I| - 1$. Since $\sum_{\ell\in I'} \bar{z}_\ell = \sum_{\ell \in I'} \chi_\ell \le 1$ and χ_ℓ are nondecreasing in ℓ by (3.13.b), we conclude that

$$\sum_{\ell=1}^{|I'|} \chi_\ell \le 1,$$

implying that $|I'| < \mathfrak{n}$ when $\mathfrak{n} < n$, so that in this case $|I| \le \mathfrak{n}$; and of course $|I| \le \mathfrak{n}$ when $\mathfrak{n} = n$. Next, we have

$$\Gamma = \sum_{\ell \in I} \bar{z}_\ell^\theta (b_\ell/d_\ell)^r \le \sum_{\ell\in I} \chi_\ell^\theta (b_\ell/d_\ell)^r = \sum_{\ell \in I} (\vartheta_G b_\ell / a_\ell)^r,$$

and since b_ℓ/a_ℓ is nonincreasing in ℓ and $|I| \le \mathfrak{n}$, the latter quantity is at most $\sum_{\ell=1}^{\mathfrak{n}} (\vartheta_G b_\ell/a_\ell)^r$. □

A.3.3. Proof of Lemma 4.1

(i): When $\pi(\mathrm{Col}_\ell[H])) \leq 1$ for all ℓ and $\lambda \geq 0$, denoting by $[h]^2$ the vector comprised of squares of the entries in h, we have

$$\begin{aligned}\phi(\mathrm{diag}(H\mathrm{Diag}\{\lambda\}H^T)) &= \phi(\sum_\ell \lambda_\ell[\mathrm{Col}_\ell[H]]^2) \leq \sum_\ell \lambda_\ell \phi([\mathrm{Col}_\ell[H]]^2) \\ &= \sum_\ell \lambda_\ell \pi^2(\mathrm{Col}_\ell[H]) \leq \sum_\ell \lambda_\ell,\end{aligned}$$

implying that $(H^T\mathrm{Diag}\{\lambda\}H^T,\ \varkappa\sum_\ell \lambda_\ell)$ belongs to $\mathbf{H}$.

(ii): Let Θ, μ, Q, V be as stated in (ii); there is nothing to prove when $\mu = 0$, thus assume that $\mu > 0$. Let $d = \mathrm{diag}(\Theta)$, so that

$$d_i = \sum_j Q_{ij}^2, \text{ and } \varkappa\phi(d) \leq \mu \tag{A.5}$$

(the second relation is due to $(\Theta, \mu) \in \mathbf{H}$). (4.8) is evident. We have

$$[H_\chi]_{ij} = \sqrt{m/\mu}[G_\chi]_{ij},\ \ G_\chi = Q\mathrm{Diag}\{\chi\}V = \left[\sum_{k=1}^m Q_{ik}\chi_k V_{kj}\right]_{i,j}.$$

We claim that for every i it holds

$$\forall \gamma > 0: \ \mathrm{Prob}\left\{[G_\chi]_{ij}^2 > 3\gamma d_i/m\right\} \leq \sqrt{3}\exp\{-\gamma/2\}. \tag{A.6}$$

Indeed, let us fix i. There is nothing to prove when $d_i = 0$, since in this case $Q_{ij} = 0$ for all j and therefore $[G_\chi]_{ij} \equiv 0$. When $d_i > 0$, by homogeneity in Q, it suffices to verify (A.6) when $d_i/m = 1/3$. Assuming that this is the case, let $\eta \sim \mathcal{N}(0,1)$ be independent of χ. We have

$$\begin{aligned}&\mathbf{E}_\eta\left\{\mathbf{E}_\chi\{\exp\{\eta[G_\chi]_{ij}\}\}\right\} = \mathbf{E}_\eta\left\{\prod_k \cosh(\eta Q_{ik}V_{kj})\right\} \\ &\leq \ \mathbf{E}_\eta\left\{\prod_k \exp\{\tfrac{1}{2}\eta^2 Q_{ik}^2 V_{kj}^2\}\right\} = \mathbf{E}_\eta\big\{\exp\{\tfrac{1}{2}\eta^2 \underbrace{\textstyle\sum_k Q_{ik}^2 V_{kj}^2}_{\leq 2d_i/m}\}\big\} \\ &\leq \ \mathbf{E}_\eta\left\{\eta^2 d_i/m\right\} = \mathbf{E}_\eta\left\{\exp\{\eta^2/3\}\right\} = \sqrt{3}.\end{aligned}$$

On the other hand,

$$\mathbf{E}_\chi\left\{\mathbf{E}_\eta\{\exp\{\eta[G_\chi]_{ij}\}\}\right\} = \mathbf{E}_\chi\left\{\exp\{\tfrac{1}{2}[G_\chi]_{ij}^2\}\right\},$$

implying that

$$\mathbf{E}_\chi\left\{\exp\{\tfrac{1}{2}[G_\chi]_{ij}^2\}\right\} \leq \sqrt{3}.$$

Therefore in the case of $d_i/m = 1/3$ for all $s > 0$ it holds

$$\mathrm{Prob}\{\chi : [G_\chi]_{ij}^2 > s\} \le \sqrt{3}\exp\{-s/2\},$$

and (A.6) follows. Recalling the relation between H and G, we get from (A.6) that for all $\gamma > 0$

$$\mathrm{Prob}\{\chi : [H_\chi]_{ij}^2 > 3\gamma d_i/\mu\} \le \sqrt{3}\exp\{-\gamma/2\}.$$

Therefore, with $\varkappa$ given by (4.7), the probability of the event

$$\forall i,j : [H_\chi]_{ij}^2 \le \varkappa \frac{d_i}{\mu}$$

is at least $1/2$. Let this event take place; in this case we have $[\mathrm{Col}_\ell[H_\chi]]^2 \le \varkappa d/\mu$, whence, by definition of the norm $\pi(\cdot)$, $\pi^2(\mathrm{Col}_\ell[H_\chi]) \le \varkappa\phi(d)/\mu \le 1$ (see the inequality in (A.5)). Thus, the probability of the event (4.9) is at least $1/2$. $\square$

A.3.4. Verification of (4.15), (4.19)

Let us verify that (4.15) is a closed convex cone compatible with the spectratope $\mathcal{X} = M\mathcal{Y}$, with $\mathcal{Y}$ given by (4.14.b); verification of the "ellitopic" version of this claim can be obtained from what follows by straightforward simplifications. The only non-evident fact in the claim is that whenever $(V,\tau) \in \mathbf{X}$, we have

$$x^T V x \le \tau \;\forall x \in \mathcal{X}. \tag{A.7}$$

To justify (A.7), let $(V,\tau) \in \mathbf{X}$, so that

$$M^T V M \preceq \sum_\ell \mathcal{R}_\ell^*[\Lambda_\ell] \text{ and } \phi_{\mathcal{R}}(\lambda[\Lambda]) \le \tau \tag{A.8}$$

for properly selected $\Lambda_\ell \in \mathbf{S}^{d_\ell}$, and let $x \in \mathcal{X}$, so that $x = My$ for some y satisfying the relations

$$\sum_{i,j} y_i y_j R^{\ell i} R^{\ell j} = R_\ell^2[y] \preceq r_\ell I_{d_\ell},\ \ell \le L \tag{A.9}$$

for some properly selected $r \in \mathcal{R}$. Taking into account that $R^{\ell i}$ are symmetric and $\Lambda_\ell \succeq 0$, (A.9) implies that

$$y^T \mathcal{R}_\ell^*[\Lambda_\ell] y = \sum_{i,j} y_i y_j \mathrm{Tr}(R^{\ell i}\Lambda_\ell R^{\ell j}) = \mathrm{Tr}(R_\ell[y]\Lambda_\ell R_\ell[y]) = \mathrm{Tr}(\Lambda_\ell R_\ell^2[y]) \le r_\ell \mathrm{Tr}(\lambda_\ell),$$

which combines with the first relation in (A.8) to imply that

$$x^T V x = y^T[M^T V M]y \le \sum_\ell y^T \mathcal{R}_\ell^*[\Lambda_\ell] y \le \sum_\ell r_\ell \mathrm{Tr}(\lambda_\ell) \le \phi_{\mathcal{R}}(\lambda[\Lambda]),$$

where the concluding inequality is due to $r \in \mathcal{R}$. Invoking the second relation in (A.8) and recalling that $x \in \mathcal{X}$ is arbitrary, we arrive at (A.7). $\square$

A.3.5. Verification of (4.18)

Given $s \in [2,\infty]$ and setting $\bar{s} = s/2$, $s_* = \frac{s}{s-1}$, $\bar{s}_* = \frac{\bar{s}}{\bar{s}-1}$, we want to prove that

$$\begin{array}{l} \left\{(V,\tau) \in \mathbf{S}^N_+ \times \mathbf{R}_+ : \exists (W \in \mathbf{S}^N, w \in \mathbf{R}^N_+) : \right. \\ \qquad\qquad\qquad \left. V \preceq W + \mathrm{Diag}\{w\} \ \&\ \|W\|_{s_*} + \|w\|_{\bar{s}_*} \leq \tau \right\} \\ \quad = \ \left\{(V,\tau) \in \mathbf{S}^N_+ \times \mathbf{R}_+ : \exists w \in \mathbf{R}^N_+ : V \preceq \mathrm{Diag}\{w\}, \|w\|_{\bar{s}_*} \leq \tau \right\}. \end{array}$$

To this end it suffices to check that whenever $W \in \mathbf{S}^N$ there exists $w \in \mathbf{R}^N$ satisfying

$$W \preceq \mathrm{Diag}\{w\}, \ \ \|w\|_{\bar{s}_*} \leq \|W\|_{s_*}.$$

The latter is equivalent to saying that for all $W \in \mathbf{S}^N$ such that $\|W\|_{s_*} \leq 1$, the conic optimization problem

$$\mathrm{Opt} = \min_{t,w} \{t : t \geq \|w\|_{\bar{s}_*}, \mathrm{Diag}\{w\} \succeq W\} \tag{A.10}$$

is solvable (which is evident) with optimal value ≤ 1. To see that the latter indeed is the case, note that the problem clearly is strictly feasible, whence its optimal value is the same as the optimal value in the conic problem

$$\begin{array}{c} \mathrm{Opt} = \max_P \left\{\mathrm{Tr}(PW) : P \succeq 0, \ \|\mathrm{diag}(P)\|_{\bar{s}_*/(\bar{s}_*-1)} \leq 1 \right\} \\ [\mathrm{diag}(P) = [P_{11}; P_{22}; ...; P_{NN}]] \end{array}$$

dual to (A.10). Since

$$\mathrm{Tr}(PW) \leq \|P\|_{s_*/(s_*-1)} \|W\|_{s_*} \leq \|P\|_{s_*/(s_*-1)},$$

when recalling what s_* and $\bar{s}_*$ are, our task boils down to verifying that whenever a matrix $P \succeq 0$ satisfies $\|\mathrm{diag}(P)\|_{s/2} \leq 1$, one has also $\|P\|_s \leq 1$. This is immediate: since P is positive semidefinite, we have $|P_{ij}| \leq P_{ii}^{1/2} P_{jj}^{1/2}$, whence, assuming $s < \infty$,

$$\|P\|_s^s = \sum_{i,j} |P_{ij}|^s \leq \sum_{i,j} P_{ii}^{s/2} P_{jj}^{s/2} = \left(\sum_i P_{ii}^{s/2}\right)^2 \leq 1.$$

When $s = \infty$, the same argument leads to

$$\|P\|_\infty = \max_{i,j} |P_{ij}| = \max_i |P_{ii}| = \|\mathrm{diag}(P)\|_\infty. \qquad \square$$

A.3.6. Proof of Proposition 4.5

1^o. Let us consider the optimization problem, as defined in [24, relation (26)] (where one should set $\mathcal{Q} = \sigma^2 I_m$) which under the circumstances is responsible for building a nearly optimal *linear* estimate of $w = Bx$, namely,

$$\begin{array}{rcl}\mathrm{Opt}_* & = & \min\limits_{\Theta,H,\Lambda,\Upsilon',\Upsilon''}\left\{\phi_{\mathcal{T}}(\lambda[\Lambda]) + \phi_{\mathcal{R}}(\lambda[\Upsilon']) + \phi_{\mathcal{R}}(\lambda[\Upsilon'']) + \sigma^2\mathrm{Tr}(\Theta):\right.\\ && \Lambda = \{\Lambda_k \succeq 0, k \le K\}, \Upsilon' = \{\Upsilon'_\ell \succeq 0, \ell \le L\},\\ && \Upsilon'' = \{\Upsilon''_\ell \succeq 0, \ell \le L\}, \left[\begin{array}{c|c}\sum_\ell \mathcal{S}^*_\ell[\Upsilon''_\ell] & \frac{1}{2}M^TH^T\\ \hline \frac{1}{2}HM & \Theta\end{array}\right] \succeq 0\\ && \left.\left[\begin{array}{c|c}\sum_\ell \mathcal{S}^*_\ell[\Upsilon'_\ell] & \frac{1}{2}M^T[B - H^TA]\\ \hline \frac{1}{2}[B - H^TA]^TM & \sum_k \mathcal{R}^*_k[\Lambda_k]\end{array}\right] \succeq 0.\right\}\end{array} \tag{A.11}$$

Let us show that the optimal value Opt of (4.23) satisfies

$$\mathrm{Opt} \le 2\kappa\mathrm{Opt}_* = 2\sqrt{2\ln(2m/\epsilon)}\mathrm{Opt}_*. \tag{A.12}$$

To this end, observe that the matrices

$$Q := \left[\begin{array}{c|c} U & \frac{1}{2}B\\ \hline \frac{1}{2}B^T & A^T\Theta A + \sum_k \mathcal{R}^*_k[\Lambda_k]\end{array}\right]$$

and

$$\left[\begin{array}{c|c} M^TUM & \frac{1}{2}M^TB\\ \hline \frac{1}{2}B^TM & A^T\Theta A + \sum_k \mathcal{R}^*_k[\Lambda_k]\end{array}\right] = \left[\begin{array}{c|c} M^T & \\ \hline & I_n\end{array}\right] Q \left[\begin{array}{c|c} M & \\ \hline & I_n\end{array}\right]$$

simultaneously are/are not positive semidefinite due to the fact that the image space of M contains the full-dimensional set $\mathcal{B}_*$ and thus is the entire $\mathbf{R}^\nu$, so that the image space of $\left[\begin{array}{c|c} M & \\ \hline & I_n\end{array}\right]$ is the entire $\mathbf{R}^\nu \times \mathbf{R}^n$. Therefore

$$\begin{array}{rcl}\mathrm{Opt} & = & \min\limits_{\Theta,U,\Lambda,\Upsilon}\left\{2\left[\phi_{\mathcal{R}}(\lambda[\Upsilon]) + \phi_{\mathcal{T}}(\lambda[\Lambda]) + \sigma^2\kappa^2\mathrm{Tr}(\Theta)\right]:\right.\\ && \Theta \succeq 0, U \succeq 0, \Lambda = \{\Lambda_k \succeq 0, k \le K\}, \Upsilon = \{\Upsilon_\ell \succeq 0, \ell \le L\},\\ && \left.\left[\begin{array}{c|c} M^TUM & \frac{1}{2}M^TB\\ \hline \frac{1}{2}B^TM & A^T\Theta A + \sum_k \mathcal{R}^*_k[\Lambda_k]\end{array}\right] \succeq 0,\ M^TUM \preceq \sum_\ell \mathcal{S}^*_\ell[\Upsilon_\ell]\right\}\end{array}$$

Further, note that if a collection $\Theta, U, \{\Lambda_k\}, \{\Upsilon_\ell\}$ is a feasible solution to the latter problem and $\theta > 0$, the scaled collection $\theta\Theta, \theta^{-1}U, \{\theta\Lambda_k\}, \{\theta^{-1}\Upsilon_\ell\}$ is also a feasible solution. When optimizing with respect to the scaling, we get

$$\begin{array}{rcl}\mathrm{Opt} & = & \inf\limits_{\Theta,U,\Lambda,\Upsilon}\left\{4\sqrt{\phi_{\mathcal{R}}(\lambda[\Upsilon])\left[\phi_{\mathcal{T}}(\lambda[\Lambda] + \sigma^2\kappa^2\mathrm{Tr}(\Theta)\right]}:\right.\\ && \Theta \succeq 0, U \succeq 0, \Lambda = \{\Lambda_k \succeq 0, k \le K\}, \Upsilon = \{\Upsilon_\ell \succeq 0, \ell \le L\},\\ && \left.\left[\begin{array}{c|c} M^TUM & \frac{1}{2}M^TB\\ \hline \frac{1}{2}B^TM & A^T\Theta A + \sum_k \mathcal{R}^*_k[\Lambda_k]\end{array}\right] \succeq 0,\ M^TUM \preceq \sum_\ell \mathcal{S}^*_\ell[\Upsilon_\ell]\right\}\\ & \le & 2\kappa\mathrm{Opt}_+,\end{array} \tag{A.13}$$

where

$$\mathrm{Opt}_+ = \inf\limits_{\Theta,U,\Lambda,\Upsilon}\left\{2\sqrt{\phi_{\mathcal{R}}(\lambda[\Upsilon])\left[\phi_{\mathcal{T}}(\lambda[\Lambda]) + \sigma^2\mathrm{Tr}(\Theta)\right]}:\right.$$

$$\left.\begin{array}{l}\Theta\succeq 0, U\succeq 0, \Lambda=\{\Lambda_k\succeq 0, k\leq K\},\\ \Upsilon=\{\Upsilon_\ell\succeq 0,\ell\leq L\},\ M^TUM\preceq\sum_\ell\mathcal{S}^*_\ell[\Upsilon_\ell],\\ \left[\begin{array}{c|c}M^TUM & \frac{1}{2}M^TB\\ \hline \frac{1}{2}B^TM & A^T\Theta A+\sum_k\mathcal{R}^*_k[\Lambda_k]\end{array}\right]\succeq 0,\end{array}\right\}\qquad (A.14)$$

[note that $\kappa>1$]

On the other hand, when strengthening the constraint $\Lambda_k\succeq 0$ of (A.11) to $\Lambda_k\succ 0$, we still have

$$\begin{array}{rl}\mathrm{Opt}_* & = \inf\limits_{\Theta,H,\Lambda,\Upsilon',\Upsilon''}\Bigg\{\phi_{\mathcal{T}}(\lambda[\Lambda])+\phi_{\mathcal{R}}(\lambda[\Upsilon'])+\phi_{\mathcal{R}}(\lambda[\Upsilon''])+\sigma^2\mathrm{Tr}(\Theta):\\ & \left.\begin{array}{l}\Lambda=\{\Lambda_k\succ 0,k\leq K\},\Upsilon'=\{\Upsilon'_\ell\succeq 0,\ell\leq L\},\\ \Upsilon''=\{\Upsilon''_\ell\succeq 0,\ell\leq L\},\ \left[\begin{array}{c|c}\sum_\ell\mathcal{S}^*_\ell[\Upsilon''_\ell] & \frac{1}{2}M^TH^T\\ \hline \frac{1}{2}HM & \Theta\end{array}\right]\succeq 0,\\ \left[\begin{array}{c|c}\sum_\ell\mathcal{S}^*_\ell[\Upsilon'_\ell] & \frac{1}{2}M^T[B-H^TA]\\ \hline \frac{1}{2}[B-H^TA]^TM & \sum_k\mathcal{R}^*_k[\Lambda_k]\end{array}\right]\succeq 0.\end{array}\right\}\end{array}\qquad (A.15)$$

Now let $\Theta,H,\Lambda,\Upsilon',\Upsilon''$ be a feasible solution to the latter problem. By the second semidefinite constraint in (A.15) we have

$$\begin{array}{l}\left[\begin{array}{c|c}\sum_\ell\mathcal{S}^*_\ell[\Upsilon''_\ell] & \frac{1}{2}M^TH^TA\\ \hline \frac{1}{2}A^THM & A^T\Theta A\end{array}\right]\\ \quad=\ \left[\begin{array}{c|c}I & \\ \hline & A\end{array}\right]^T\left[\begin{array}{c|c}\sum_\ell\mathcal{S}^*_\ell[\Upsilon''_\ell] & \frac{1}{2}M^TH^T\\ \hline \frac{1}{2}HM & \Theta\end{array}\right]\left[\begin{array}{c|c}I & \\ \hline & A\end{array}\right]\succeq 0\end{array}$$

which combines with the first semidefinite constraint in (A.15) to imply that

$$\left[\begin{array}{c|c}\sum_\ell\mathcal{S}^*_\ell[\Upsilon'_\ell+\Upsilon''_\ell] & \frac{1}{2}M^TB\\ \hline \frac{1}{2}B^TM & A^T\Theta A+\sum_k\mathcal{R}^*_k[\Lambda_k]\end{array}\right]\succeq 0.$$

Next, by the Schur Complement Lemma (which is applicable due to

$$A^T\Theta A+\sum_k\mathcal{R}^*_k[\Lambda_k]\succeq\sum_k\mathcal{R}^*_k[\Lambda_k]\succ 0,$$

where the concluding $\succ$ is due to [24, Lemma 5.1] combined with $\Lambda_k\succ 0$), this relation implies that for

$$\Upsilon_\ell=\Upsilon'_\ell+\Upsilon''_\ell,$$

we have

$$\sum_\ell\mathcal{S}^*_\ell[\Upsilon_\ell]\succeq M^T\underbrace{\left[\tfrac{1}{4}B[A^T\Theta A+\sum_k\mathcal{R}^*_k[\Lambda_k]]^{-1}B^T\right]}_{U}M.$$

Using the Schur Complement Lemma again, for the just defined $U\succeq 0$ we obtain

$$\left[\begin{array}{c|c}M^TUM & \frac{1}{2}M^TB\\ \hline \frac{1}{2}B^TM & A^T\Theta A+\sum_k\mathcal{R}^*_k[\Lambda_k]\end{array}\right]\succeq 0,$$

and in addition, by the definition of U,

$$M^T U M \preceq \sum_\ell \mathcal{S}_\ell^*[\Upsilon_\ell].$$

We conclude that

$$(\Theta, U, \Lambda, \Upsilon := \{\Upsilon_\ell = \Upsilon'_\ell + \Upsilon''_\ell, \ell \leq L\})$$

is a feasible solution to optimization problem (A.14) specifying Opt_+. The value of the objective of the latter problem at this feasible solution is

$$\begin{array}{l} 2\sqrt{\phi_{\mathcal{R}}(\lambda[\Upsilon'] + \lambda[\Upsilon''])\,[\phi_{\mathcal{T}}(\lambda[\Lambda]) + \sigma^2 \mathrm{Tr}(\Theta)]} \\ \quad \leq \quad \phi_{\mathcal{R}}(\lambda[\Upsilon'] + \lambda[\Upsilon'']) + \phi_{\mathcal{T}}(\lambda[\Lambda]) + \sigma^2 \mathrm{Tr}(\Theta) \\ \quad \leq \quad \phi_{\mathcal{R}}(\lambda[\Upsilon']) + \phi_{\mathcal{R}}(\lambda[\Upsilon'']) + \phi_{\mathcal{T}}(\lambda[\Lambda]) + \sigma^2 \mathrm{Tr}(\Theta), \end{array}$$

the concluding quantity in the chain being the value of the objective of problem (A.15) at the feasible solution $\Theta, H, \Lambda, \Upsilon', \Upsilon''$ to this problem. Since the resulting inequality holds true for every feasible solution to (A.15), we conclude that $\mathrm{Opt}_+ \leq \mathrm{Opt}_*$, and we arrive at (A.12) due to (A.13).

2^o. Now, from Proposition 3.3 in the latest arXiv version of [24, Section 5.7], we conclude that Opt_* is within a logarithmic factor of the minimax optimal $(\frac{1}{8}, \|\cdot\|)$-risk corresponding to the case of Gaussian noise $\xi_x \sim \mathcal{N}(0, \sigma^2 I_m)$ for all x:

$$\mathrm{Opt}_* \leq \theta_* \mathrm{RiskOpt}_{1/8},$$

where

$$\theta_* = 8\sqrt{(2\ln F + 10\ln 2)\,(2\ln D + 10\ln 2)}, \quad F = \sum_\ell f_\ell,\ D = \sum_k d_k.$$

Since the minimax optimal $(\epsilon, \|\cdot\|)$-risk clearly only grows when ϵ decreases, we conclude that for $\epsilon \leq 1/8$ a feasible near optimal solution to (4.23) is minimax optimal within the factor $2\theta^*\kappa$. □