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Markov selection for constrained martingale problems

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Abstract

Constrained Markov processes, such as reflecting diffusions, behave as an unconstrained process in the interior of a domain but upon reaching the boundary are controlled in some way so that they do not leave the closure of the domain. In this paper, the behavior in the interior is specified by a generator of a Markov process, and the constraints are specified by a controlled generator. Together, the generators define a *constrained martingale problem*. The desired constrained processes are constructed by first solving a simpler *controlled martingale problem* and then obtaining the desired process as a time-change of the controlled process. As for ordinary martingale problems, it is rarely obvious that the process constructed in this manner is unique. The primary goal of the paper is to show that from among the processes constructed in this way one can "select", in the sense of Krylov, a strong Markov process. Corollaries to these constructions include the observation that uniqueness among strong Markov solutions implies uniqueness among all solutions. These results provide useful tools for proving uniqueness for constrained processes including reflecting diffusions. The constructions also yield viscosity semisolutions of the resolvent equation and, if uniqueness holds, a viscosity solution, without proving a comparison principle. We illustrate our results by applying them to reflecting diffusions in piecewise smooth domains. We prove existence of a strong Markov solution to the SDE with reflection, under conditions more general than in [13]: In fact our conditions are known to be optimal in the case of simple, convex polyhedrons with constant direction of reflection on each face ([10]). We also indicate how the results can be applied to processes with Wentzell boundary conditions and nonlocal boundary conditions.

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1 Introduction

Let A be an operator determining a Markov process X with state space E as the solution of the martingale problem in which

$$M_f(t) = f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds \tag{1.1}$$

is required to be a martingale with respect to a filtration $\{\mathcal{F}_t\}$ for all $f \in \mathcal{D}(A)$, the domain of A. The study of stochastic processes that behave like the process determined by A when in an open subset $E_0 \subset E$, are constrained to stay in $\overline{E}_0$, and must behave in a prescribed way on ∂E_0, is classically carried out by restricting the domain $\mathcal{D}(A)$ by specifying boundary conditions, typically of the form $Bf(x) = 0$ for $x \in \partial E_0$ for some operator B. Then X is required to remain in $\overline{E}_0$ and (1.1) is required to be a martingale for all functions in $\{f \in \mathcal{D}(A) : Bf(x) = 0, x \in \partial E_0\}$. This approach to constrained Markov processes, however, frequently introduces difficult analytical problems in identifying a set of functions both satisfying the boundary conditions and large enough to characterize the process.

An alternative approach by Stroock and Varadhan [31] introduces a submartingale problem which weakens the restriction on the domain of A to the requirement that $Bf(x) \geq 0$ for $x \in \partial E_0$ and then requires that for all such $f \in \mathcal{D}(A)$, (1.1) is a submartingale. This approach has been used to great effect by a number of authors. See, for example, [37, 20, 21].

Restrictions on the values of Bf on the boundary are dropped altogether in [23, 24] at the cost of introducing a boundary process λ that, in the simplest settings, measures the amount of time the process spends on the boundary in the sense that λ is nondecreasing and increases only when X (or more precisely $X(\cdot-)$) is on the boundary. Then X is required to take values in $\overline{E}_0$ and for each $f \in \mathcal{D}(A) \cap \mathcal{D}(B)$,

$$M_f(t) = f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int Bf(X(s-))d\lambda(s) \tag{1.2}$$

is required to be a martingale. As we will see, the form of the boundary term may be more complicated than this. A process that satisfies these requirements is a solution of the *constrained martingale problem*. Clearly, every solution of the constrained martingale problem is also a solution of the submartingale problem. This approach, or the corresponding one for stochastic equations, has been used, for example, in [10, 5, 7].

Whether the submartingale problem approach or the constrained martingale problem approach is used, the critical issue is uniqueness of the solution, which is still an open question for many examples (see e.g. [17, 18]).

The primary goal of this paper is to prove a Markov selection theorem for solutions of constrained martingale problems. Beyond the intrinsic interest, this selection theorem is frequently a crucial ingredient in proving uniqueness for constrained martingale problems and hence uniqueness for semimartingale reflecting Brownian motion (see, for example, [26, 34, 10]) and reflecting diffusions.

In the unconstrained case, the Markov selection theorem ensures the existence of strong Markov solutions to the martingale problem. The construction of the strong Markov solution also ensures that uniqueness among strong Markov solutions implies uniqueness among all solutions. See [32], Theorems 12.2.3 and 12.2.4, for diffusions and [14], Theorem 4.5.19, for general martingale problems. All these results follow [22]. The observation that uniqueness among strong Markov solutions implies uniqueness among all solutions provides a key tool in uniqueness arguments. Unfortunately, these results do not apply immediately to solutions of submartingale or constrained martingale problems.

We construct solutions of the constrained martingale problem by time-changing solutions of a *controlled martingale problem* (Sections 2 and 3). Solutions of the controlled martingale problem evolve on a slower time scale and may take values in all of E. Their behavior in E_0^c is determined by the operator B. Since solutions of the controlled martingale problem capture the intuition behind the controls that constrain the solution, we will refer to solutions of the constrained martingale problem that arise as time-changes of solutions of the controlled martingale problem as *natural*. We cannot rule out the possibility that there are solutions of the constrained martingale problem which are not natural, but, under very general conditions, uniqueness for natural solutions implies uniqueness for all solutions. See Remark 4.14.

In Section 2.1, we introduce the controlled martingale problem and discuss properties of the collection of solutions. In particular, we prove weak compactness of the collection of solutions. In Section 3, we introduce the time-changed process. Under mild conditions, the time-changed process is a natural solution of the constrained martingale problem. We note however that, even when it is not, the time-changed process still models a process constrained in $\overline{E}_0$, with behavior in the interior determined by A and constraints determined by B.

In Section 4 we prove that there exists a natural strong Markov solution of the constrained martingale problem (Theorem 4.9 and Corollary 4.12) and that uniqueness among natural strong Markov solutions implies uniqueness among all natural solutions (Corollary 4.13).

In Section 5, we discuss connections between solutions of the constrained martingale problem and viscosity semisolutions of the corresponding resolvent equation. In particular, generalizing the results of Section 5 of [6], we see that existence of a comparison principle for the viscosity semisolutions implies uniqueness for natural solutions of the constrained martingale problem. Conversely, uniqueness of natural solutions of the constrained martingale problem gives a viscosity solution of the resolvent equation. Thus one can obtain existence of a viscosity solution from purely probabilistic arguments, without first proving a comparison principle for the resolvent equation.

In Section 6 we apply the results of Section 4 to diffusion processes in piecewise smooth domains of $\mathbb{R}^d$ with varying, oblique directions of reflection on each face. Existence and uniqueness results for these processes have been obtained by many authors ([34, 10] for convex polyhedrons with constant direction of reflection on each face, [33, 28, 4, 13] for nonpolyhedral domains, etc.). For nonpolyhedral domains, [13] is perhaps the most general result, but it still requires a condition that is not satisfied in some very natural examples (see Example 6.1) or is difficult to verify in other ones (see e.g. [18]). In addition, [13] does not cover the case of cusp like singularities, such as in [17] (in dimension 2, cusp like singularities are covered by [7]). In [34] and [10] a key point in proving uniqueness is the fact that there exist strong Markov processes that satisfy the definition of reflecting diffusion and that uniqueness among these strong Markov processes implies uniqueness. By the results of Section 4, we obtain existence of a strong Markov natural solution of the constrained martingale problem under conditions that coincide with those of [10] in the case of simple, convex polyhedrons with constant direction of reflection on each face (see Remark 6.3). In this case, [10] have shown that these conditions are necessary for existence of a semimartingale reflecting Brownian motion. Under the same assumptions, the results of Section 4 ensure also that uniqueness among strong Markov natural solutions implies uniqueness among all natural solutions. Moreover we show that the set of natural solutions of the constrained martingale problem coincides with the set of weak solutions to the corresponding stochastic differential equation with reflection (Theorem 6.12).

Further examples of application of the results of Section 4 are presented in Section 7.

1.1 Notation

For a metric space (E, r), $\mathcal{B}(E)$ will denote the σ-algebra of Borel subsets of E, $B(E)$ will denote the set of bounded, Borel measurable functions on E, and $\|\cdot\|$ will denote the supremum norm on $B(E)$.

$\mathcal{P}(E)$ will denote the set of probability measures on $(E, \mathcal{B}(E))$. For $F \in \mathcal{B}(E)$, with a slight abuse of notation, $\mathcal{P}(F)$ will denote $\{P \in \mathcal{P}(E) : P(F) = 1\}$.

For $x \in E$ and $F \in \mathcal{B}(E)$, $d(x, F)$ will denote the distance from x to F, that is, $d(x, F) = \inf_{y \in F} r(x, y)$.

$\mathbf{1}$ will denote the function identically equal to 1 and, for $F \in \mathcal{B}(E)$, $\mathbf{1}_F$ will denote the indicator function of F.

$|I|$ will denote the cardinality of a finite set I.

For any function or operator, $\mathcal{R}(\cdot)$ will denote the range and $\mathcal{D}(\cdot)$ the domain.

$\mathcal{L}(\cdot)$ will denote the distribution of a stochastic process or a random variable.

If Z is a stochastic process defined on an arbitrary probability space, $\{\mathcal{F}_t^Z\}$ will denote the filtration generated by Z.

If Z is a stochastic process defined on an arbitrary filtered probability space, Z will also denote the canonical process defined on the path space. $\{\mathcal{B}_t\}$ will denote the filtration generated by the canonical process.

2 Controlled martingale problems

We use the control formulation of constrained martingale problems given in [24] rather than the earlier version given in [23] that was based on "patchwork" martingale problems. The control formulation may be less intuitive, but it is more general and notationally simpler, and models described in the earlier manner can be translated to the control formulation.

Let E be a compact metric space, and let E_0 be an open subset of E. The requirement that E be compact is not particularly restrictive since, for example, for most processes in $\mathbb{R}^d$, one can take E to be the one-point compactification of $\mathbb{R}^d$. Let $A \subset C(E) \times C(E)$ with $(1, 0) \in A$.

Let U also be a compact metric space, and let Ξ be a closed subset of $E_0^c \times U$. For each $x \in E_0^c$, let $\xi_x \equiv \{u : (x, u) \in \Xi\}$ be the set of controls that are admissible at x, and define $F_1 \equiv \{x \in E_0^c : \xi_x \neq \emptyset\}$ which is the set of points at which a control exists. Let $B \subset C(E) \times C(\Xi)$ with $(1, 0) \in B$. Using A and B, we define a controlled process Y that outside E_0 evolves on a slower time scale than the desired process X. Like X, inside E_0 the behavior of Y is determined by A, and outside E_0 the behavior of Y is determined by B. In particular, Y may take values in $\overline{E}_0 \cup F_1$.

Let $\mathcal{L}_U$ be the space of measures on $[0, \infty) \times U$ such that $\mu([0, t] \times U) < \infty$ for all $t > 0$. $\mathcal{L}_U$ is topologized so that $\mu_n \in \mathcal{L}_U \to \mu \in \mathcal{L}_U$ if and only if

$$\int_{[0,\infty)\times U} f(s, u)\mu_n(ds \times du) \to \int_{[0,\infty)\times U} f(s, u)\mu(ds \times du)$$

for all continuous f with compact support in $[0, \infty) \times U$. It is possible to define a metric on $\mathcal{L}_U$ that induces the above topology and makes $\mathcal{L}_U$ into a complete, separable metric space. We will say that an $\mathcal{L}_U$-valued random variable Λ_1 is adapted to a filtration $\{\mathcal{F}_t\}$ if

$$\Lambda_1([0, \cdot] \times C) \text{ is } \{\mathcal{F}_t\} - \text{adapted}, \quad \forall C \in \mathcal{B}(U).$$

Definition 2.1. $(Y, \lambda_0, \Lambda_1)$ *is a solution of the* controlled martingale problem *for* (A, E_0, B, Ξ)*, if* Y *is a process in* $D_E[0,\infty)$*,* λ_0 *is nonnegative and nondecreasing and increases only when* $Y \in \overline{E}_0$*,* Λ_1 *is a random measure in* $\mathcal{L}_U$ *such that*

$$\lambda_1(t) \equiv \Lambda_1([0,t]\times U) = \int_{[0,t]\times U} \mathbf{1}_\Xi(Y(s),u)\Lambda_1(ds\times du), \tag{2.1}$$
$$\lambda_0(t) + \lambda_1(t) = t,$$

and there exists a filtration $\{\mathcal{F}_t\}$ *such that* Y*,* λ_0*, and* Λ_1 *are* $\{\mathcal{F}_t\}$*-adapted and*

$$f(Y(t)) - f(Y(0)) - \int_0^t Af(Y(s))d\lambda_0(s) - \int_{[0,t]\times U} Bf(Y(s),u)\Lambda_1(ds\times du) \tag{2.2}$$

is an $\{\mathcal{F}_t\}$*-martingale for all* $f \in \mathcal{D} \equiv \mathcal{D}(A)\cap\mathcal{D}(B)$*. By the continuity of* f*, we can assume, without loss of generality, that* $\{\mathcal{F}_t\}$ *is right continuous.*

Remark 2.2. To get some intuition on λ_0 and Λ_1, consider the case in which A is a bounded Markov process generator and at each point $x \in (E_0)^c$ there is exactly one control $u(x)$, so B is the bounded Markov process generator that, at x, produces a jump $u(x)$. Then Y is the pure jump process with generator $Af(x)\,\mathbf{1}_{E_0}(x)+Bf(x,u(x))\,\mathbf{1}_{(E_0)^c}(x)$. $\lambda_0(t)$ and $\Lambda_1([0,t]\times C)$ are the time that Y spends in E_0 and the time that Y spends in $(E_0)^c$ while the control lies in C, respectively, i.e.

$$\lambda_0(t) \equiv \int_0^t \mathbf{1}_{E_0}(Y(s))\,ds, \quad \Lambda_1([0,t]\times C) \equiv \int_0^t \mathbf{1}_{(E_0)^c}(Y(s))\,\mathbf{1}_C(u(Y(s)))\,ds.$$

For general A and B, frequently $(Y, \lambda_0, \Lambda_1)$ can be obtained as a limit of a sequence $\{(Y^n, \lambda_0^n, \Lambda_1^n)\}$ corresponding to a sequence of bounded Markov process generators $\{(A^n, B^n)\}$ (with jump rates going to infinity, if A, B are not bounded) that approximates (A, B). This construction is carried out rigorously in Theorem 2.2 of [24] and yields a quite general method to obtain solutions of the controlled martingale problem. In the case when there is a corresponding patchwork martingale problem, as defined in [23] (see Definition 6.6), this essentially amounts to constructing a solution of the patchwork martingale problem, which will be a solution of the controlled martingale problem as well: This approach is followed in Section 6. See also Section 7.2 for an example of another construction by approximation.

Remark 2.3. Note that the requirement that $\lambda_0(t)+\lambda_1(t) = t$ implies any solution of the controlled martingale problem for (A, E_0, B, Ξ) must satisfy $Y \in D_{\overline{E}_0\cup F_1}[0,\infty)$. In fact, if $Y(t) \in \left(\overline{E}_0 \cup F_1\right)^c$ for some t, necessarily $Y(s) \in \left(\overline{E}_0 \cup F_1\right)^c$ for all $s \in [t,t')$ for some $t' > t$. Then $\lambda_0(t') - \lambda_0(t) = \lambda_1(t') - \lambda_1(t) = 0$, because λ_1 increases only when $Y \in F_1$, by (2.1), and λ_0 increases only when $Y \in \overline{E}_0$, and this contradicts $t'-t = \big(\lambda_0(t')-\lambda_0(t))\big) + \big(\lambda_1(t')-\lambda_1(t)\big)$.

Remark 2.4. If $(Y, \lambda_0, \Lambda_1)$ is a solution of the controlled martingale problem for (A, E_0, B, Ξ) with distribution P, the canonical process on $D_E[0,\infty)\times C_{[0,\infty)}[0,\infty)\times\mathcal{L}_U$ under P is also obviously a solution with respect to the filtration $\{\mathcal{B}_t\}$ generated by itself. As mentioned in Section 1.1, we denote the canonical process under P by $(Y, \lambda_0, \Lambda_1)$ as well.

Remark 2.5. One can always assume, without loss of generality, that $\{\mathcal{F}_t\}$ is complete. Then, denoting by $\{\mathcal{F}_t^Y\}$ the smallest complete and right continuous filtration to which Y is adapted, λ_0 and Λ_1 can be replaced by their dual predictable projections on $\{\mathcal{F}_t^Y\}$ so that (2.2) is a $\{\mathcal{F}_t^Y\}$-martingale for each $f \in \mathcal{D}$ (see Lemma 6.1, [25]).

Remark 2.6. Note that the controlled martingale problem can also be formulated by setting

$$Cf(y,u,v) = vAf(y) + (1-v)Bf(y,u)$$

with controls $(u,v)\in U\times[0,1]$. The analog of Ξ is $\Xi_0\subset E\times U\times[0,1]$ such that

$$\begin{aligned}\Xi_0\cap E_0\times U\times[0,1] &= E_0\times U\times\{1\}\\ \Xi_0\cap\partial E_0\times U\times[0,1] &= (\partial E_0\times U\cap\Xi)\times[0,1]|\cup\partial E_0\times U\times\{1\}\\ \Xi_0\cap\overline{E}_0^c\times U\times[0,1] &= (\overline{E}_0^c\times U\cap\Xi)\times\{0\}.\end{aligned}$$

Then (Y,μ), with $Y\in D_E[0,\infty)$ and μ a $\mathcal{P}(U\times[0,1])$-valued process is a solution of the controlled martingale for (C,Ξ_0) if there exists a filtration $\{\mathcal{F}_t\}$ such that (Y,μ) is $\{\mathcal{F}_t\}$-adapted and

$$f(Y(t))-f(Y(0))-\int_0^t\int_{U\times[0,1]}Cf(Y(s),u,v)\mu_s(du\times dv)ds$$

is an $\{\mathcal{F}_t\}$-martingale. Every solution of the controlled martingale problem for (C,Ξ_0) gives a solution for the controlled martingale problem for (A,E_0,B,Ξ) by defining

$$\lambda_0(t)=\int_0^t\int_{U\times[0,1]}v\mu_s(du\times dv)ds$$

and

$$\Lambda_1(D)=\int_0^\infty\int_{U\times[0,1]}(1-v)\mathbf{1}_D(s,u)\mu_s(du\times dv)ds.$$

Conversely, every solution of the controlled martingale problem for (A,E_0,B,Ξ) gives a solution of the controlled martingale problem for (C,Ξ_0).

Definition 2.7. *We define $\Pi\subset\mathcal{P}(D_E[0,\infty)\times C_{[0,\infty)}[0,\infty)\times\mathcal{L}_U)$ to be the collection of the distributions of solutions of the controlled martingale problem for (A,E_0,B,Ξ), and for $\nu\in\mathcal{P}(E)$, $\Pi_\nu\subset\Pi$ to be the collection of distributions such that $Y(0)$ has distribution ν.*

$\mathcal{P}_0$ denotes the collection of $\nu\in\mathcal{P}(\overline{E}_0\cup F_1)$ such that $\Pi_\nu\neq\emptyset$.

Lemma 2.8. *If $\mathcal{D}$ is dense in $C(E)$, then the collection of distributions of solutions (Y,λ_0,Λ_1) of the controlled martingale problem is compact in $\mathcal{P}(D_E[0,\infty)\times C_{[0,\infty)}[0,\infty)\times\mathcal{L}_U)$ in the sense of weak convergence (taking the Skorohod topology on $D_E[0,\infty)$ and the compact uniform topology on $C_{[0,\infty)}[0,\infty)$). Consequently, Π and Π_ν, $\nu\in\mathcal{P}_0$, are compact and convex.*

Proof. Relative compactness for the family of Y follows from Theorems 3.9.4 and 3.9.1 of [14]. The relative compactness of the λ_0 and Λ_1 is immediate, as λ_0 and λ_1 are Lipschitz continuous with Lipschitz constant 1. The fact that every limit point is a solution of the controlled martingale problem follows by standard arguments from the properties of weakly converging measures and from uniform integrability of the martingales in (2.2).

Convexity is immediate. □

2.1 Closure properties of Π

Lemma 2.9. *Let (Y,λ_0,Λ_1) be a solution of the controlled martingale problem for (A,E_0,B,Ξ) with filtration $\{\mathcal{F}_t\}$. Let $H\geq 0$ be a $\mathcal{F}_0$-measurable random variable such that $E[H]=1$. Then $P^H\in\mathcal{P}(D_E[0,\infty)\times C_{[0,\infty)}[0,\infty)\times\mathcal{L}_U)$ defined by*

$$P^H(C)\equiv E[H\,\mathbf{1}_C(Y,\lambda_0,\Lambda_1)],\quad C\in\mathcal{B}(D_E[0,\infty)\times C_{[0,\infty)}[0,\infty)\times\mathcal{L}_U),$$

is in Π.

Proof. If M is a $\{\mathcal{F}_t\}$-martingale under P and $|M(t)|\leq C(1+t)$ for some $C>0$, then M is a $\{\mathcal{F}_t\}$-martingale under P^H. □

Lemma 2.10.

a) If $\nu_1 << \nu_2$ and $\Pi_{\nu_2} \neq \emptyset$, then $\Pi_{\nu_1} \neq \emptyset$.

b) There exists a closed $F_2 \subset \overline{E}_0 \cup F_1$ such that $\mathcal{P}_0 = \mathcal{P}(F_2)$.

Proof. Taking $H = \frac{d\nu_1}{d\nu_2}$, part (a) follows from Lemma 2.9.

Suppose $P \in \Pi_\nu$ and $z \in \text{supp}(\nu)$. Then for each $\epsilon > 0$, $\nu(B_\epsilon(z)) > 0$, and setting $H_\epsilon(x) = \frac{1}{\nu(B_\epsilon(z))}\mathbf{1}_{B_\epsilon(z)}(x)$, by Lemma 2.9, $P^{H_\epsilon} \in \Pi$. By the compactness of Π, P^{H_ϵ} will have at least one limit point P_z as $\epsilon \to 0$, and $P_z \in \Pi_{\delta_z}$.

Let F_2 be the closure of $\cup_{\nu \in \mathcal{P}_0}\text{supp}(\nu)$. Then for each $x \in F_2$, $\Pi_{\delta_x} \neq \emptyset$, and by convexity, for $\nu_{x,p} = \sum_{i=1}^m p_i \delta_{x^i}$, $x^i \in F_2$, $p_i \geq 0$, $\sum_{i=1}^m p_i = 1$, $\Pi_{\nu_{x,p}} \neq \emptyset$. Since every $\nu \in \mathcal{P}(F_2)$ can be approximated by probability measures of this form, $\Pi_\nu \neq \emptyset$ for each $\nu \in \mathcal{P}(F_2)$. □

Lemma 2.11. *Define Y^τ, λ_0^τ, and Λ_1^τ by*

$$\begin{aligned} Y^\tau(t) = Y(\tau + t), \quad \lambda_0^\tau(t) = \lambda_0(\tau + t) - \lambda_0(\tau), \quad t \geq 0, \\ \Lambda_1^\tau([0,t] \times C) = \Lambda_1([\tau, \tau + t] \times C), \quad t \geq 0, C \in \mathcal{B}(U). \end{aligned} \tag{2.3}$$

Note that Y^τ, λ_0^τ, and Λ_1^τ are adapted to the filtration $\{\mathcal{F}_{\tau+t}\}$.

Then the measure $P^{\tau,H} \in \mathcal{P}(D_E[0,\infty) \times C_{[0,\infty)}[0,\infty) \times \mathcal{L}_U)$ defined by

$$P^{\tau,H}(C) = E[H\mathbf{1}_C(Y^\tau, \lambda_0^\tau, \Lambda_1^\tau)], \quad C \in \mathcal{B}(D_E[0,\infty) \times C_{[0,\infty)}[0,\infty) \times \mathcal{L}_U) \tag{2.4}$$

is the distribution of a solution of the controlled martingale problem for (A, E_0, B, Ξ).

Proof. For $0 \leq t < t + r$ and $C \in \mathcal{B}_t$

$$\begin{aligned} &E^{P^{\tau,H}}\Big[\Big\{f(Y(t+r)) - f(Y(t)) - \int_t^{t+r} Af(Y(s))d\lambda_0(s) \\ &\qquad - \int_{(t,t+r]\times U} Bf(Y(s),u)\Lambda_1(ds \times du)\Big\}\mathbf{1}_C(Y, \lambda_0, \Lambda_1)\Big] \\ &= E\Big[\Big\{f(Y^\tau(t+r)) - f(Y^\tau(t)) - \int_t^{t+r} Af(Y^\tau(s))d\lambda_0^\tau(s) \\ &\qquad - \int_{(t,t+r]\times U} Bf(Y^\tau(s),u)\Lambda_1^\tau(ds \times du)\Big\}H\mathbf{1}_C(Y^\tau, \lambda_0^\tau, \Lambda_1^\tau)\Big] \\ &= 0 \end{aligned}$$

by the optional sampling theorem. Therefore, $P^{\tau,H} \in \Pi$. □

Lemma 2.12. *Suppose that $(Y, \lambda_0, \Lambda_1)$ is a solution of the controlled martingale problem with filtration $\{\mathcal{F}_t\}$ and that τ is a finite $\{\mathcal{F}_t\}$-stopping time. Let $P^0 \in \mathcal{P}(D_E[0,\infty) \times C_{[0,\infty)}[0,\infty) \times \mathcal{L}_U \times [0,\infty))$ be the joint distribution of the 4-tuple of random variables $(Y, \lambda_0, \Lambda_1, \tau)$. Let ν be the distribution of $Y(\tau)$, and let $P^1 \in \Pi_\nu$ (not empty by Lemma 2.11). Then there exists $P \in \mathcal{P}(D_E[0,\infty) \times C_{[0,\infty)}[0,\infty) \times \mathcal{L}_U \times [0,\infty))$ and a filtration $\{\mathcal{H}_t\}$ in $D_E[0,\infty) \times C_{[0,\infty)}[0,\infty) \times \mathcal{L}_U \times [0,\infty)$ such that, under P, $(Y, \lambda_0, \Lambda_1)$ is a solution of the controlled martingale problem with filtration $\{\mathcal{H}_t\}$, τ is a $\{\mathcal{H}_t\}$-stopping time, $(Y(\cdot \wedge \tau), \lambda_0(\cdot \wedge \tau), \Lambda_1(\cdot \wedge \tau, \cdot), \tau)$ has the same distribution under P^0 and P and the distribution of $(Y^\tau, \lambda_0^\tau, \Lambda_1^\tau)$ under P is P^1.*

Proof. Let

$$\Omega = D_E[0,\infty) \times C_{[0,\infty)}[0,\infty) \times \mathcal{L}_U \times [0,\infty) \times D_E[0,\infty) \times C_{[0,\infty)}[0,\infty) \times \mathcal{L}_U,$$

and denote the elements by $(Y^0, \lambda_0^0, \Lambda_1^0, \tau^0, Y^1, \lambda_0^1, \Lambda_1^1)$. Apply Lemma 4.5.15 of [14] to P^0 and P^1 to obtain P on Ω such that $Y^0(\tau) = Y^1(0)$ and define

$$\begin{aligned} Y(t) &= \begin{cases} Y^0(t), & t < \tau^0 \\ Y^1(t-\tau^0), & t \geq \tau^0 \end{cases} \\ \lambda_0(t) &= \begin{cases} \lambda_0^0(t), & t < \tau^0 \\ \lambda_0^0(\tau^0) + \lambda_0^1(t-\tau^0), & t \geq \tau^0 \end{cases} \\ \Lambda_1([0,t]\times C) &= \begin{cases} \Lambda_1^0([0,t]\times C), & t < \tau^0 \\ \Lambda_1^0([0,\tau^0]\times C) + \Lambda_1^1([0,t-\tau^0]\times C), & t \geq \tau^0. \end{cases} \end{aligned}$$

The fact that $(Y, \lambda_0, \Lambda_1)$ is a solution of the controlled martingale problem follows as in the proof of Lemma 4.5.16 of [14]. □

3 Constrained martingale problems

As discussed in the Introduction and at the beginning of Section 2, we are interested in processes that in E_0 behave like solutions of the martingale problem for the operator A, are constrained to remain in $\overline{E}_0$, and whose behavior on ∂E_0 is determined by the operator B. In Section 2, we have introduced a controlled process Y with values in all of E, that evolves on a slower time scale and whose behavior in E_0^c is determined by B. Y is the first element of a triple $(Y, \lambda_0, \Lambda_1)$ that is a solution of the controlled martingale problem (Definition 2.1). We now construct the constrained process, X, by time changing Y, where the time change is obtained by inverting λ_0. The following lemma gives conditions that ensure that the process obtained by inverting λ_0 is defined for all time.

Lemma 3.1. *Let $(Y, \lambda_0, \Lambda_1)$ be a solution of the controlled martingale problem for (A, E_0, B, Ξ), and define*

$$\tau(t) = \inf\{s : \lambda_0(s) > t\}, \qquad t \geq 0. \tag{3.1}$$

Suppose there is an $f \in \mathcal{D}$ and $\epsilon > 0$ such that

$$\int_{[0,t]\times U} Bf(Y(s), u)\Lambda_1(ds, du) \geq \epsilon\lambda_1(t). \tag{3.2}$$

Then $\lim_{t\to\infty} \lambda_0(t) = \infty$ almost surely and $E[\tau(t)] < \infty$, for all $t \geq 0$.

Proof. See Lemma 2.9 of [24]. □

Remark 3.2. (3.2) is a natural condition which is also used in the study of PDEs (see, e.g., [9], Lemma 7.6). An example where it is satisfied is a reflecting diffusion in a smooth domain with a nontangential direction of reflection. More precisely, let $E_0 \equiv \{x : \psi(x) > 0\}$ for some function $\psi \in C^2(\mathbb{R}^d)$ such that $\psi(x) = 0$ implies $\nabla\psi(x) \neq 0$, so that, in particular, the unit inward normal at $x \in \partial E_0$ is given by $n(x) \equiv \frac{\nabla\psi(x)}{|\nabla\psi(x)|}$. Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous vector field, of unit length on ∂E_0, such that $\langle g(x), n(x)\rangle > 0$ at every $x \in \partial E_0$. Consider the controlled martingale problem for (A, E_0, B, Ξ), where

$$\begin{gathered} Af(x) \equiv \langle \nabla f(x), b(x)\rangle + \tfrac{1}{2}\mathrm{tr}\big(\sigma(x)\sigma^T(x)D^2f(x)\big), \\ U \equiv \{u \in \mathbb{R}^d : |u| = 1\}, \quad \Xi \equiv \{(x,u) : x \in \partial E_0,\, u = g(x)\}, \\ Bf(x,u) \equiv \langle \nabla f(x), u\rangle, \end{gathered}$$

and $\mathcal{D} \equiv C_c^2(\mathbb{R}^d)$. Then ψ itself satisfies (3.2) (recall that ∂E_0 is compact).

Lemma 3.3. *Under the assumptions of Lemma 2.8, if, for each $P \in \Pi$, $P\{\tau(0) < \infty\} = 1$, then for each $P \in \Pi$, $\lim_{t\to\infty} \lambda_0(t) = \infty$ a.s.*

Proof. Let $(Y, \lambda_0, \Lambda_1)$ have distribution in Π. Then by Lemma 2.11 and the compactness of Π (Lemma 2.8), there exists $t_n \to \infty$ such that $(Y^{t_n}, \lambda_0^{t_n}, \Lambda_1^{t_n}) \Rightarrow (Y^\infty, \lambda_0^\infty, \Lambda_1^\infty)$. But

$$P\{\lim_{t\to\infty} \lambda_0(t) < \infty\} \le P\{\lim_{n\to\infty} \lim_{t\to\infty} \lambda_0(t_n + t) - \lambda_0(t_n) = 0\} \le P\{\sup_t \lambda_0^\infty(t) = 0\}.$$

Since by assumption, $P\{\tau^\infty(0) < \infty\} = 1$, $P\{\sup_t \lambda_0^\infty(t) = 0\} = 0$. □

Lemma 3.4. *Suppose every solution of the controlled martingale problem for (A, E_0, B, Ξ) satisfies $\lambda_0(t) > 0$ for all $t > 0$, a.s. (i.e. $P\{\tau(0) = 0\} = 1$ for each $P \in \Pi$). Then, for every solution, λ_0 is a.s. strictly increasing.*

Proof. For each $s > 0$, for every solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for (A, E_0, B, Ξ), with the notation of Lemma 2.11 $(Y^s, \lambda_0^s, \Lambda_1^s)$ is also a solution. □

With Lemmas 2.8, 3.1, 3.4 and 3.3 in mind, throughout the remainder of the paper, we assume the following:

Condition 3.5.

a) *$\mathcal{D}$ is dense in $C(E)$.*

b) *For each $\nu \in \mathcal{P}(\overline{E}_0)$, $\Pi_\nu \neq \emptyset$ (hence $F_2 \supset \overline{E}_0$, where F_2 is defined in Lemma 2.10).*

c) *For each solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for (A, E_0, B, Ξ), $\lim_{t\to\infty} \lambda_0(t) = \infty$ almost surely.*

Theorem 3.6. *Let $(Y, \lambda_0, \Lambda_1)$ be a solution of the controlled martingale problem for (A, E_0, B, Ξ) with right continuous filtration $\{\mathcal{F}_t\}$. Let $\tau(t)$ be given by (3.1), and define $\mathcal{G}_t = \mathcal{F}_{\tau(t)}$. Define*

$$X(t) \equiv Y(\tau(t))$$

and

$$\Lambda([0, t] \times C) \equiv \int_{[0,\tau(t)]\times U} \mathbf{1}_C(Y(s), u)\Lambda_1(ds \times du), \quad C \in \mathcal{B}(\Xi).$$

Suppose there exists a sequence η_n of $\{\mathcal{G}_t\}$-stopping times such that $\eta_n \to \infty$ and, for each n, $E[\tau(\eta_n)] < \infty$.

Then $X \in D_{\overline{E}_0}[0, \infty)$, and, for each $f \in \mathcal{D}$,

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_{[0,t]\times\Xi} Bf(x, u)\Lambda(ds \times dx \times du) \tag{3.3}$$

is a $\{\mathcal{G}_t\}$-local martingale.

Proof. Since $\tau(t)$ must be a point of increase of λ_0, $Y(\tau(t))$ must be in $\overline{E}_0$. Since Y and τ are right continuous, X must be in $D_{\overline{E}_0}[0, \infty)$.

Since

$$\left| \int_{[0,t\wedge\eta_n]\times\Xi} Bf(x, u)\Lambda(ds \times dx \times du) \right| \le \|Bf\|\lambda_1(\tau(t \wedge \eta_n)) \le \|Bf\|\tau(t \wedge \eta_n),$$

(3.3) stopped at η_n is a martingale. □

Remark 3.7. If $\lambda_0(t) > 0$ for all $t > 0$, in particular if $Y(0) \in E_0$, then $X(0) = Y(0)$, but if $\lambda_0(t) = 0$ for some $t > 0$, then $\tau(0) > 0$, and $X(0) = Y(\tau(0))$ may not be $Y(0)$.

Let $\mathcal{Q}_0$ be the collection of $\nu \in \mathcal{P}(\overline{E}_0)$ such that $\nu = \mathcal{L}(X(0)) = \mathcal{L}(Y(\tau(0)))$, for some solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem, i.e. $\mathcal{Q}_0$ is the set of possible initial distributions of the process X constructed in Theorem 3.6. Then, by Lemma 2.11, $\mathcal{Q}_0$ is the collection of $\nu \in \mathcal{P}(\overline{E}_0)$ such that there exists $(Y, \lambda_0, \Lambda_1)$ with initial distribution ν for which $\lambda_0(t) > 0$ for all $t > 0$ a.s. Note that $\mathcal{Q}_0 \supset \mathcal{P}(E_0)$.

Definition 3.8. *A process X in $D_{\overline{E}_0}[0,\infty)$ is a solution of the constrained (local) martingale problem for (A, E_0, B, Ξ) if there exists a random measure Λ in $\mathcal{L}_\Xi$ and a filtration $\{\mathcal{G}_t\}$ such that X and Λ are $\{\mathcal{G}_t\}$-adapted and for each $f \in \mathcal{D}$, (3.3) is a $\{\mathcal{G}_t\}$-(local) martingale. We may assume, without loss of generality, that $\{\mathcal{G}_t\}$ is right continuous.*

A solution obtained as in Theorem 3.6 from a solution of the controlled martingale problem will be called natural. *$\Gamma \subset \mathcal{P}(D_{\overline{E}_0}[0,\infty))$ will denote the set of distributions of natural solutions and, for $\nu \in \mathcal{P}(\overline{E}_0)$, Γ_ν will denote the set of distributions of natural solutions X such that $X(0)$ has distribution ν.*

Corollary 3.9.

a) *For $\nu \in \mathcal{Q}_0$ ($\mathcal{Q}_0$ defined in Remark 3.7), if there exists a solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for (A, E_0, B, Ξ) with initial distribution ν that satisfies the conditions of Lemma 3.1, then there exists a natural solution to the constrained martingale problem for (A, E_0, B, Ξ) with initial distribution ν.*
b) *For $\nu \in \mathcal{P}(\overline{E}_0)$, if there exists a solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for (A, E_0, B, Ξ) with initial distribution ν such that λ_0 is strictly increasing a.s. (see Lemma 3.4 for a sufficient condition), then there exists a natural solution to the constrained local martingale problem for (A, E_0, B, Ξ) with initial distribution ν.*

Proof.

a) Under the conditions of Lemma 3.1, we can take $\eta_n = n$ and (3.3) is actually a martingale.
b) If λ_0 is strictly increasing, then τ is continuous and we can take $\eta_n = \inf\{t : \tau(t) > n\}$. □

We conclude this section with a result giving conditions that imply a solution of the constrained martingale problem is natural.

Proposition 3.10. *Suppose that X is a solution of the constrained martingale problem for (A, E_0, B, Ξ) and Λ is the associated random measure. If $\Lambda([0,\cdot] \times \Xi)$ is continuous and for all $h \in C(\Xi)$ and $t > 0$,*

$$\int_{[0,t]\times\Xi} h(x,u)\Lambda(ds \times dx \times du) = \int_{[0,t]\times\Xi} h(X(s),u)\Lambda(ds \times dx \times du), \tag{3.4}$$

then X is natural.

Proof. Define

$$\lambda_0(t) \equiv \inf\{s : s + \Lambda([0,s] \times \Xi) > t\}, \quad Y(t) \equiv X(\lambda_0(t))$$

and

$$\Lambda_1([0,t] \times C) \equiv \int_{[0,\lambda_0(t)]\times\Xi} \mathbf{1}_C(u)\, \Lambda(ds \times dx \times du), \quad C \in \mathcal{B}(U).$$

Then

$$\begin{aligned} & f(X(\lambda_0(t))) - f(X(0)) - \int_0^{\lambda_0(t)} Af(X(s))ds - \int_{[0,\lambda_0(t)]\times\Xi} Bf(x,u)\Lambda(ds\times dx\times du) \\ &= f(Y(t)) - f(X(0)) - \int_0^t Af(Y(s))d\lambda_0(s) - \int_{[0,\lambda_0(t)]\times\Xi} Bf(X(s),u)\Lambda(ds\times dx\times du) \\ &= f(Y(t)) - f(X(0)) - \int_0^t Af(Y(s))d\lambda_0(s) - \int_{[0,t]\times U} Bf(Y(s),u)\Lambda_1(ds\times du). \quad \square \end{aligned}$$

4 The Markov selection theorem

Our strategy for obtaining a Markov solution for the constrained martingale problem for (A, E_0, B, Ξ) generally follows the approach in Section 4.5 of [14] (which in turn is based on an unpublished paper [16]). With reference to these results, for $h \in C(\overline{E}_0)$, and $\nu \in \mathcal{P}(F_2)$ (F_2 defined in Lemma 2.10), define

$$\gamma(\Pi_\nu, h) \equiv \sup_{P\in\Pi_\nu} E^P[\int_0^\infty e^{-\lambda_0(s)} h(Y(s))d\lambda_0(s)]. \tag{4.1}$$

Recalling that Π_ν is compact (Lemma 2.8), we see that the supremum is achieved.

Lemma 4.1. *For $h \in C(\overline{E}_0)$, there exists $v_h \in B(F_2)$ such that*

$$\gamma(\Pi_\nu, h) = \int_{F_2} v_h(x)\nu(dx), \quad \forall \nu \in \mathcal{P}(F_2),$$

and v_h is upper semicontinuous.

Proof. Suppose first that h is nonnegative. Let $0 < \alpha < 1$ and $\nu, \mu_1, \mu_2 \in \mathcal{P}(F_2)$. Suppose $\nu = \alpha\mu_1 + (1-\alpha)\mu_2$. Then by convexity of Π,

$$\begin{aligned} & \gamma(\Pi_\nu, h) \\ & \geq \sup_{P_1\in\Pi_{\mu_1}, P_2\in\Pi_{\mu_2}} \Big\{ \alpha E^{P_1}[\int_0^\infty e^{-\lambda_0(s)} h(Y(s))d\lambda_0(s)] \\ & \qquad\qquad + (1-\alpha)E^{P_2}[\int_0^\infty e^{-\lambda_0(s)} h(Y(s))d\lambda_0(s)] \Big\} \\ & = \alpha\gamma(\Pi_{\mu_1}, h) + (1-\alpha)\gamma(\Pi_{\mu_2}, h) \end{aligned} \tag{4.2}$$

But μ_1 and μ_2 are absolutely continuos with respect to ν, so setting $H_i = \frac{d\mu_i}{d\nu}$, by Lemma 2.9, for $P \in \Pi_\nu$,

$$\begin{aligned} & E^P[\int_0^\infty e^{-\lambda_0(s)} h(Y(s))d\lambda_0(s)] \\ & = \alpha E^{P^{H_1}}[\int_0^\infty e^{-\lambda_0(s)} h(Y(s))d\lambda_0(s)] + (1-\alpha)E^{P^{H_2}}[\int_0^\infty e^{-\lambda_0(s)} h(Y(s))d\lambda_0(s)] \end{aligned}$$

so the reverse of the previous inequality holds and hence

$$\gamma(\Pi_\nu, h) = \alpha\gamma(\Pi_{\mu_1}, h) + (1-\alpha)\gamma(\Pi_{\mu_2}, h). \tag{4.3}$$

The compactness of Π and the continuity of $(Y, \lambda_0, \Lambda_1) \to \int_0^\infty e^{-\lambda_0(s)} h(Y(s))d\lambda_0(s)$ ensure that the mapping $\nu \to \gamma(\Pi_\nu, h)$ is upper semicontinuous, and the lemma follows by Lemma 4.5.9 of [14].

If h is not nonnegative, take $v_h \equiv v_{h-\inf h} + \inf h$. $\square$

Lemma 4.2. *Let $\Pi_\nu^h \subset \Pi_\nu$ be the subset for which the supremum in (4.1) is achieved, that is, $Q \in \Pi_\nu^h$ if and only if*

$$E^Q[\int_0^\infty e^{-\lambda_0(s)} h(Y(s))d\lambda_0(s)] = \gamma(\Pi_\nu, h) = \int_{F_2} v_h(x)\nu(dx).$$

Defining $\Pi^h = \cup_{\nu\in\mathcal{P}(F_2)}\Pi_\nu^h$, Π^h is convex, and for each ν, Π_ν^h is compact (however, it is not clear whether or not Π^h is compact).

Proof. Let $P_1 \in \Pi_{\mu_1}^h$, $P_2 \in \Pi_{\mu_1}^h$ and $P = \alpha P_1 + (1-\alpha)P_2$, $0 < \alpha < 1$. Setting $\nu = \alpha\mu_1 + (1-\alpha)\mu_2$,

$$E^P[\int_0^\infty e^{-\lambda_0(s)} h(Y(s))d\lambda_0(s)] = \alpha\gamma(\Pi_{\mu_1}, h) + (1-\alpha)\gamma(\Pi_{\mu_2}, h) = \gamma(\Pi_\nu, h),$$

where the last equality follows from (4.3).

Compactness of Π_ν^h follows from the compactness of Π_ν and the continuity of the functional $(Y, \lambda_0) \to \int_0^\infty e^{-\lambda_0(s)} h(Y(s))d\lambda_0(s)$. □

Now consider $\{h_n\} \subset C(\overline{E}_0)$, $h_n \geq 0$, and define $\Pi_\nu^{h_1,h_2}$ to be the subset of distributions $Q \in \Pi_\nu^{h_1}$ such that

$$E^Q[\int_0^\infty e^{-\lambda_0(s)} h_2(Y(s))d\lambda_0(s)] = \gamma(\Pi_\nu^{h_1}, h_2) \equiv \sup_{P\in\Pi_\nu^{h_1}} E^P[\int_0^\infty e^{-\lambda_0(s)} h_2(Y(s))d\lambda_0(s)],$$

and recursively, define $\Pi_\nu^{h_1,\ldots,h_{n+1}}$ to be the subset of distributions $Q \in \Pi_\nu^{h_1,\ldots,h_n}$ such that

$$\begin{aligned} E^Q[\int_0^\infty e^{-\lambda_0(s)} h_{n+1}(Y(s))d\lambda_0(s)] &= \gamma(\Pi_\nu^{h_1,\ldots,h_n}, h_{n+1}) \\ &\equiv \sup_{P\in\Pi_\nu^{h_1,\ldots,h_n}} E^P[\int_0^\infty e^{-\lambda_0(s)} h_{h+1}(Y(s))d\lambda_0(s)] \end{aligned}$$

Inductively, the compactness of $\Pi_\nu^{h_1,\ldots,h_n}$ and the continuity of the functional $(Y, \lambda_0) \to \int_0^\infty e^{-\lambda_0(s)} h_{n+1}(Y(s))d\lambda_0(s)$ ensure that $\Pi_\nu^{h_1,\ldots,h_{n+1}}$ is compact and nonempty. Let $\Pi^{h_1,\ldots,h_n} = \cup_{\nu\in\mathcal{P}(F_2)}\Pi_\nu^{h_1,\ldots,h_n}$.

We now need to show the existence of a function $v_{h_{n+1}}^{h_1,\ldots,h_n}$ such that

$$\gamma(\Pi_\nu^{h_1,\ldots h_n}, h_{n+1}) = \int_{F_2} v_{h_{n+1}}^{h_1,\ldots,h_n}(x)\nu(dx), \quad \forall\nu \in \mathcal{P}(F_2).$$

If $\nu = \alpha\mu_1 + (1-\alpha)\mu_2$, then, by the same argument used for (4.3),

$$\gamma(\Pi_\nu^{h_1,\ldots h_n}, h_{n+1}) = \alpha\gamma(\Pi_{\mu_1}^{h_1,\ldots h_n}, h_{n+1}) + (1-\alpha)\gamma(\Pi_{\mu_2}^{h_1,\ldots h_n}, h_{n+1});$$

however, we do not know the upper semicontinuity of $\gamma(\Pi_\nu^{h_1,\ldots h_n}, h_{n+1})$ as a function of ν, because it is not clear whether or not $\Pi^{h_1,\ldots h_n}$ is compact. Consequently, we cannot apply Lemma 4.5.9 of [14] as we did in Lemma 4.1.

Lemma 4.3. *For each $n = 1, 2, \ldots$, $\nu \in \mathcal{P}(F_2)$, and $g \in C(\overline{E}_0)$, there exists $v_g^{h_1,\ldots,h_n} \equiv v_g^{n+1} \in B(F_2)$ such that*

$$\gamma(\Pi_\nu^{h_1,\ldots h_n}, g) = \int_{E_2} v_g^{n+1}(x)\nu(dx). \tag{4.4}$$

Proof. Suppose first that $g \geq 0$. Following the argument on page 214 of [14], we proceed by induction. For $n = 1$, (4.4) is given by Lemma 4.1. Assuming (4.4) holds for n, we claim

$$v_g^{n+1}(x) \equiv \lim_{\epsilon \to 0+} \epsilon^{-1}(v^n_{h_n+\epsilon g}(x) - v^n_{h_n}(x))$$

satisfies (4.4). Note that for all $\nu \in \mathcal{P}(F_2)$,

$$\int_{F_2} v^n_{h_n+\epsilon g}(x)\nu(dx) \geq \int_{F_2} v^n_{h_n}(x)\nu(dx) + \epsilon\gamma(\Pi_\nu^{h_1,\ldots,h_n}, g), \tag{4.5}$$

and hence, for all $x \in F_2$,

$$\liminf_{\epsilon \to 0} \epsilon^{-1}(v^n_{h_n+\epsilon g}(x) - v^n_{h_n}(x)) \geq \gamma(\Pi_{\delta_x}^{h_1,\ldots,h_n}, g).$$

For each $\epsilon > 0$, let $P_\nu^\epsilon \in \Pi_\nu^{h_1,\ldots,h_{n-1}}$ satisfy

$$\begin{aligned}\int_{F_2} v^n_{h_n+\epsilon g}(x)\nu(dx) &= E^{P_\nu^\epsilon}[\int_0^\infty e^{-\lambda_0(s)}(h_n+\epsilon g)(Y(s))d\lambda_0(s)]] \\ &\leq \int_{F_2} v^n_{h_n}(x)\nu dx) + \epsilon E^{P_\nu^\epsilon}[\int_0^\infty e^{-\lambda_0(s)} g(Y(s))d\lambda_0(s)]]. \end{aligned}\tag{4.6}$$

By (4.5) and (4.6), all limit points of P_ν^ϵ as $\epsilon \to 0$ are in $\Pi_\nu^{h_1,\ldots,h_n}$, so

$$\begin{aligned}\limsup_{\epsilon\to 0}\int_{F_2} \epsilon^{-1}(v^n_{h_n+\epsilon g}(x) - v^n_{h_n}(x))\nu(dx) &\leq \limsup_{\epsilon\to 0} E^{P_\nu^\epsilon}[\int_0^\infty e^{-\lambda_0(s)} g(Y(s))d\lambda_0(s)]] \\ &\leq \gamma(\Pi_\nu^{h_1,\ldots,h_n}, g).\end{aligned}$$

Therefore,

$$v_g^{n+1}(x) \equiv \lim_{\epsilon \to 0} \epsilon^{-1}(v^n_{h_n+\epsilon g}(x) - v^n_{h_n}(x))$$

exists, and since, again by (4.5) and (4.6),

$$0 \leq \epsilon^{-1}(v^n_{h_n+\epsilon g}(x) - v^n_{h_n}(x)) \leq \sup_z g(z),$$

(4.4) holds by the dominated convergence theorem.

If g is not nonnegative, take $v_g^{n+1} \equiv v_{g-\inf g} + \inf g$. □

4.1 Closure properties of $\Pi^{h_1,\ldots,h_n}$

Lemma 4.4. *Suppose $(Y, \lambda_0, \Lambda_1)$ is a solution of the controlled martingale problem with filtration $\{\mathcal{F}_t\}$ and distribution $P \in \Pi^{h_1,\ldots,h_n}$. Let $H \geq 0$ be $\mathcal{F}_0$-measurable with $E[H] = 1$. Then P^H defined as in Lemma 2.9 is in $\Pi^{h_1,\ldots,h_n}$.*

Proof. Let $c > 0$, $H^c = \frac{H\wedge c}{E[H\wedge c]}$, and $G^c = \frac{c-H\wedge c}{E[c-H\wedge c]}$. Then

$$\begin{aligned} E^P[v^n_{h_n}(Y(0))] &= \frac{E[H\wedge c]}{c} E^{P^{H^c}}[\int_0^\infty e^{-\lambda_0(s)} h_n(Y(s))d\lambda_0(s)] \\ &\qquad + \frac{E[c-H\wedge c]}{c} E^{P^{G^c}}[\int_0^\infty e^{-\lambda_0(s)} h_n(Y(s))d\lambda_0(s)] \\ &\leq \frac{E[H\wedge c]}{c} E^{P^{H^c}}[v^n_{h_n}(Y(0))] + \frac{E[c-H\wedge c]}{c} E^{P^{G^c}}[v^n_{h_n}(Y(0))] \\ &= E^P[v^n_{h_n}(Y(0))], \end{aligned}$$

and since the inequality is termwise, we must have

$$E^{P^{H^c}}[\int_0^\infty e^{-\lambda_0(s)} h_n(Y(s))d\lambda_0(s)] = E^{P^{H^c}}[v^n_{h_n}(Y(0))].$$

Letting $c \to \infty$, the monotone convergence theorem implies

$$E^{P^H}[\int_0^\infty e^{-\lambda_0(s)} h_n(Y(s))d\lambda_0(s)] = E^{P^H}[v_{h_n}^n(Y(0))], \tag{4.7}$$

and $P^H \in \Pi^{h_1,\ldots,h_n}$. □

Remark 4.5. Note that (4.7) implies

$$E^P[\int_0^\infty e^{-\lambda_0(s)} h_n(Y(s))d\lambda_0(s)|\mathcal{B}_0] = v_{h_n}^n(Y(0)).$$

In particular

$$v_{h_n}^n(x) = E^P[\int_0^\infty e^{-\lambda_0(s)} h_n(Y(s))d\lambda_0(s)], \quad P \in \Pi_{\delta_x}^{h_1,\ldots,h_n}, \quad x \in F_2.$$

Lemma 4.6. *Suppose $(Y, \lambda_0, \Lambda_1)$ is a solution of the controlled martingale problem with filtration $\{\mathcal{F}_t\}$ with distribution $P \in \Pi^{h_1,\ldots,h_n}$. Let τ be a finite $\{\mathcal{F}_t\}$-stopping time and let $H \geq 0$ be $\mathcal{F}_\tau$-measurable with $E[H] = 1$. Then, for $(Y^\tau, \lambda_0^\tau, \Lambda_1^\tau)$ defined by (2.3), $P^{\tau,H}$ defined by (2.4) is in $\Pi^{h_1,\ldots,h_n}$ and $\Pi^{h_1,\ldots,h_n}$ is closed under the pasting operation in Lemma 2.12.*

Proof. Again we proceed by induction. By Lemma 2.11,

$$\begin{aligned}
&\gamma(\Pi_\nu, h_1) \\
&= E[\int_0^\infty e^{-\lambda_0(s)} h_1(Y(s))d\lambda_0(s)] \\
&= E[\int_0^\tau e^{-\lambda_0(s)} h_1(Y(s))d\lambda_0(s)] + E[e^{-\lambda_0(\tau)} \int_0^\infty e^{-\lambda_0^\tau(s)} h_1(Y^\tau(s))d\lambda_0^\tau(s)] \qquad (4.8) \\
&= E[\int_0^\tau e^{-\lambda_0(s)} h_1(Y(s))d\lambda_0(s)] + E[e^{-\lambda_0(\tau)}]E^{P^{\tau,H_0}}[\int_0^\infty e^{-\lambda_0^\tau(s)} h_1(Y^\tau(s))d\lambda_0^\tau(s)] \\
&\leq E[\int_0^\tau e^{-\lambda_0(s)} h_1(Y(s))d\lambda_0(s)] + E[e^{-\lambda_0(\tau)}]\gamma(\Pi_\mu, h_1),
\end{aligned}$$

where

$$H_0 \equiv \frac{e^{-\lambda_0(\tau)}}{E[e^{-\lambda_0(\tau)}]}, \quad \mu(C) \equiv E[H_0 \mathbf{1}_C(Y(\tau))].$$

Let ζ be the distribution of $Y(\tau)$ and let $P^1 \in \Pi_\zeta^{h_1}$. Taking $(Y^0, \lambda_0^0, \Lambda_1^0, \tau^0)$ with the same distribution as $(Y, \lambda_0, \Lambda_1, \tau)$ and $(Y^1, \lambda_0^1, \Lambda_1^1)$ with distribution P^1, let $(\widehat{Y}, \widehat{\lambda}_0, \widehat{\Lambda}_1, \widehat{\tau})$ be given by Lemma 2.12. Then, for $\widehat{H}_0 \equiv \frac{e^{-\widehat{\lambda}_0(\widehat{\tau})}}{E[e^{-\widehat{\lambda}_0(\widehat{\tau})}]}$,

$$\begin{aligned}
&E[\int_0^\infty e^{-\widehat{\lambda}_0(s)} h_1(\widehat{Y}(s))ds] \\
&= E[\int_0^\tau e^{-\lambda_0(s)} h_1(Y(s))d\lambda_0(s)] + E[e^{-\widehat{\lambda}_0(\widehat{\tau})}]E[\widehat{H}_0 \int_0^\infty e^{-\widehat{\lambda}_0^{\widehat{\tau}}(s)} h_1(\widehat{Y}^{\widehat{\tau}}(s))d\widehat{\lambda}_0^{\widehat{\tau}}(s)] \\
&= E[\int_0^\tau e^{-\lambda_0(s)} h_1(Y(s))d\lambda_0(s)] + E[e^{-\widehat{\lambda}_0(\widehat{\tau})}]E^{P^{\widehat{H}_0}}[\int_0^\infty e^{-\widehat{\lambda}_0^{\widehat{\tau}}(s)} h_1(\widehat{Y}^{\widehat{\tau}}(s))d\widehat{\lambda}_0^{\widehat{\tau}}(s)] \\
&= E[\int_0^\tau e^{-\lambda_0(s)} h_1(Y(s))d\lambda_0(s)] + E[e^{-\lambda_0(\tau)}]\gamma(\Pi_\mu, h_1) \\
&\geq \gamma(\Pi_\nu, h_1),
\end{aligned}$$

where the third equality holds by Lemma 4.4 and the inequality is given by (4.8). Consequently, equality must hold here and in (4.8), giving both that P^{τ,H_0} is in Π^{h_1}

and that Π^{h_1} is closed under the pasting operation. Now for an arbitrary H as in the statement of the theorem, note that the probability measure P^H can be written as

$$P^H(C) = E^{P^{H_0}}[HH_0^{-1}\mathbf{1}_C], \quad C \in \mathcal{B}(D_E[0,\infty) \times C_{[0,\infty)}[0,\infty) \times \mathcal{L}_U),$$

where $E^{P^{H_0}}[HH_0^{-1}] = 1$. Since P^{τ,H_0} is the distribution of $(Y^\tau, \lambda_0^\tau, \Lambda_1^\tau)$ under P^{H_0}, Lemma 4.4 yields that the distribution of $(Y^\tau, \lambda_0^\tau, \Lambda_1^\tau)$ under P^H is in Π^{h_1}, i.e. $P^{\tau,H}$ is in Π^{h_1}.

Now suppose that the result holds for $1 \le k \le n-1$. In particular, if the distribution of $(Y, \lambda_0, \Lambda_1)$ is in $\Pi^{h_1,\dots,h_{n-1}}$, then the distribution of $(Y^\tau, \lambda_0^\tau, \Lambda_1^\tau)$ under P^{H_0} is in $\Pi^{h_1,\dots,h_{n-1}}$. With this observation, the proof of the result for n follows. □

4.2 The martingale property and the Markov selection theorem

Lemma 4.7. *Let $(Y, \lambda_0, \Lambda_1)$ be a solution of the controlled martingale problem for (A, E_0, B, Ξ) with filtration $\{\mathcal{F}_t\}$ and distribution in $\Pi^{h_1,\dots,h_n}$. For $v_{h_n}^n$ given by Lemma 4.3,*

$$e^{-\lambda_0(t)} v_{h_n}^n(Y(t)) + \int_0^t e^{-\lambda_0(s)} h_n(Y(s)) d\lambda_0(s)$$

is a $\{\mathcal{F}_t\}$-martingale, and

$$v_{h_n}(Y(0)) = E[\int_0^\infty e^{-\lambda_0(t)} h_n(Y(s)) d\lambda_0(s) | \mathcal{F}_0]. \tag{4.9}$$

Proof. For $t \ge 0$ and H bounded and $\mathcal{F}_t$-measurable, by Lemma 4.6 and Remark 4.5

$$\begin{aligned} E[\int_t^\infty e^{-\lambda_0(s)} h_n(Y(s) d\lambda_0(s) H] &= E[e^{-\lambda_0(t)} \int_0^\infty e^{-\lambda_0^t} h_n(Y^t(s) d\lambda_0^t(s) H] \\ &= E[e^{-\lambda_0(t)} v_{h_n}^n(Y(t)) H], \end{aligned}$$

and hence

$$E[\int_t^\infty e^{-\lambda_0(s)} h_n(Y(s) d\lambda_0(s) | \mathcal{F}_t] = e^{-\lambda_0(t)} v_{h_n}^n(Y(t))$$

and

$$E[\int_0^\infty e^{-\lambda_0(s)} h_n(Y(s) d\lambda_0(s) | \mathcal{F}_t] = e^{-\lambda_0 t} v_{h_n}^n(Y(t)) + \int_0^t e^{-\lambda_0(s)} h_n(Y(s)) d\lambda_0(s).$$

The left side is clearly a martingale, and (4.9) follows by taking $t = 0$. □

Recall that we are assuming Condition 3.5. In particular, we are assuming that for all solutions of the controlled martingale problem, $\lambda_0(t) \to \infty$.

Theorem 4.8. *For $\nu \in \mathcal{P}(F_2)$, let $\Pi_\nu^\infty \equiv \cap_n \Pi_\nu^{h_1,\dots,h_n}$ (note that $\Pi_\nu^\infty \ne \emptyset$) and $\Pi^\infty \equiv \cup_{\nu \in \mathcal{P}(F_2)} \Pi_\nu^\infty$. Let $(Y, \lambda_0, \Lambda_1)$ be a solution of the controlled martingale problem with filtration $\{\mathcal{F}_t\}$ and distribution in Π^∞. Define, as in Theorem 3.6, $\tau(t) \equiv \inf\{s : \lambda_0(s) > t\}$ and $X(t) \equiv Y(\tau(t))$. Then for all h_n,*

$$v_{h_n}^n(X(t)) - \int_0^t (v_{h_n}^n(X(s)) - h_n(X(s))) ds,$$

is a $\{\mathcal{F}_{\tau(t)}\}$-martingale.

Proof. For each h_n, by Lemma 4.7

$$e^{-\lambda_0(t)} v_{h_n}^n(Y(t)) + \int_0^t e^{-\lambda_0(s)} h_n(Y(s)) d\lambda_0(s)$$

is a $\{\mathcal{F}_t\}$-martingale, so the time changed process

$$e^{-t}v^n_{h_n}(X(t)) + \int_0^t e^{-s}h_n(X(s))ds$$

is a $\{\mathcal{F}_{\tau(t)}\}$-martingale. Hence by Lemma 4.3.2 in [14],

$$v^n_{h_n}(X(t)) - \int_0^t (v^n_{h_n}(X(s)) - h_n(X(s)))ds,$$

is a $\{\mathcal{F}_{\tau(t)}\}$-martingale. □

Let $\mathcal{Q}_0^\infty$ be the collection of $\nu \in \mathcal{P}(\overline{E}_0)$ such that $\nu = \mathcal{L}(X(0)) = \mathcal{L}(Y(\tau(0)))$, for some $(Y, \lambda_0, \Lambda_1)$ with distribution in Π^∞ and τ and X as in Theorem 4.8. Then, by Lemma 4.6, $\mathcal{Q}_0^\infty$ is the collection of $\nu \in \mathcal{P}(\overline{E}_0)$ such that there exists $(Y, \lambda_0, \Lambda_1)$ with distribution in Π^∞_ν for which $\lambda_0(t) > 0$ for all $t > 0$ a.s. Note that $\mathcal{Q}_0^\infty \supset \mathcal{P}(E_0)$. In particular $\delta_x \in \mathcal{Q}_0^\infty$ for every $x \in E_0$.

Theorem 4.9. *Let $\{h_n\} \subset C(\overline{E}_0)$ be such that its linear span is dense in $B(\overline{E}_0)$ under bounded pointwise convergence. For $\nu \in \mathcal{Q}_0^\infty$, let Γ^∞_ν be the collection of distributions of processes $X \equiv Y \circ \tau$ defined as in Theorem 4.8 with $\nu = \mathcal{L}(X(0))$ and $(Y, \lambda_0, \Lambda_1)$ with distribution in Π^∞. Then, there exists one and only one distribution in Γ^∞_ν and it is the distribution of a strong Markov process.*

Proof. By Remark 4.5 and Theorem 4.8, for each n, $(v^n_{h_n}, h_n)$ is a pair(v_h, h) such that

$$v_h(Y(0)) = E^P[\int_0^\infty e^{-\lambda_0(s)}h(Y(s))d\lambda_0(s)|\mathcal{B}_0], \quad \forall P \in \Pi^\infty, \tag{4.10}$$

and

$$v_h(X(t)) - \int_0^t [v_h(X(s)) - h(X(s)]ds \tag{4.11}$$

is a $\{\mathcal{B}_{\tau(t)}\}$-martingale for each $X = Y \circ \tau$, $(Y, \lambda_0, \Lambda_1)$ with distribution in Π^∞. Let

$\mathbb{A} = \{(\,v_h|_{\overline{E}_0}\,, v_h|_{\overline{E}_0} - h)$: such that $(v_h, h) \in B(F_2) \times B(\overline{E}_0)$ satisfies (4.10) and (4.11$\}$.

$\mathbb{A}$ is linear and closed under bounded pointwise convergence.

For (v_h, h) such that $(\,v_h|_{\overline{E}_0}\,, v_h|_{\overline{E}_0} - h) \in \mathbb{A}$, by Lemma 4.3.2 of [14], for each $\eta > 0$ and $X = Y \circ \tau$ as in (4.11),

$$e^{-\eta t}v_h(X(t)) + \int_0^t e^{-\eta s}(\eta v_h(X(s)) - v_h(X(s)) + h(X(s)))ds$$

is a $\{\mathcal{B}_{\tau(t)}\}$-martingale, and hence

$$v_h(X(0)) = E[\int_0^\infty e^{-\eta s}(\eta v_h(X(s)) - v_h(X(s)) + h(X(s)))ds|\mathcal{B}_{\tau(0)}]. \tag{4.12}$$

(4.12) with $\eta = 1$ and (4.10) imply

$$v_h(Y(0)) = E[v_h(X(0))|\mathcal{B}_0].$$

Consequently, for each $x \in \overline{E}_0$, for $(Y, \lambda_0, \Lambda_1)$ with distribution in $\Pi^\infty_{\delta_x}$, $X = Y \circ \tau$,

$$v_h(x) = E[\int_0^\infty e^{-\eta s}(\eta v_h(X(s)) - v_h(X(s)) + h(X(s)))ds],$$

and, as in Proposition 4.3.5 of [14], this implies that $\mathbb{A}$ is dissipative.

Since $\mathcal{R}(I-\mathbb{A})\supset\{h_n\}$ and the linear span of $\{h_n\}$ is bounded pointwise dense in $B(\overline{E}_0)$, we have $\overline{\mathcal{R}(I-\mathbb{A})}^{bp}=B(\overline{E}_0)$. The properties of resolvents of dissipative operators (for example, Lemma 1.2.3 of [14]) ensure that $\overline{\mathcal{R}(\eta I-\mathbb{A})}^{bp}=B(\overline{E}_0)$ for all $\eta>0$. Therefore, by Corollary 4.4.4 of [14], for each $\nu\in\mathcal{Q}_0^\infty$ uniqueness holds for the martingale problem for $\mathbb{A}$ with initial distribution ν, and, by construction, the distribution of the solution is the unique distribution in Γ_ν^∞.

Now let (Y,λ_0,Λ_1) be the canonical process with distribution $P\in\Pi^\infty$ such that $\mathcal{L}(Y(\tau(0)))=\nu$, so that the distribution of $X\equiv Y\circ\tau$, defined as in Theorem 4.8, is the unique distribution in Γ_ν^∞. In order to show that X is a strong Markov process we need to show that, for each $\{\mathcal{B}_{\tau(t)}\}$ finite stopping time σ, $\tau(\sigma)$ is a $\{\mathcal{B}_t\}$-stopping time and, setting $X^\sigma(\cdot)=X(\sigma+\cdot)$, for every $F\in\mathcal{B}_{\tau(\sigma)}$,

$$E^P[\mathbf{1}_F\mathbf{1}_B(X^\sigma)]=E^P[\mathbf{1}_F E[\mathbf{1}_B(X^\sigma)|X(\sigma)]],\qquad\forall B\in\mathcal{B}(\mathcal{D}_{\overline{E}_0}[0,\infty)). \tag{4.13}$$

The fact that $\tau(\sigma)$ is a $\{\mathcal{B}_t\}$-stopping time follows by the right continuity of $\{\mathcal{B}_t\}$ and the observation that

$$\{\tau(\sigma)<s\}=\cup_{t\in\mathbb{Q}\cap[0,\infty)}\{\sigma\le t\}\cap\{\tau(t)<s\},\qquad s>0.$$

Fix $F\in\mathcal{B}_{\tau(\sigma)}$ with $P(F)>0$, and define two probability measures P_1 and P_2 on $D_E[0,\infty)\times C_{[0,\infty)}[0,\infty)\times\mathcal{L}_U$ by

$$P_1(C)\equiv\frac{1}{P(F)}E^P[\mathbf{1}_F\mathbf{1}_C],\quad P_2(C)\equiv\frac{1}{P(F)}E^P[\mathbf{1}_F E[\mathbf{1}_C|Y(\tau(\sigma))]].$$

Note that

$$\mathcal{L}^{P_1}(X^\sigma(0))=\mathcal{L}^{P_1}(Y(\tau(\sigma))=\mathcal{L}^{P_2}(Y(\tau(\sigma))=\mathcal{L}^{P_2}(X^\sigma(0))\equiv\mu.$$

Since

$$X^\sigma(t)=Y^{\tau(\sigma)}(\tau^\sigma(t)),$$

where τ^σ is given by

$$\tau^\sigma(t)\equiv\inf\{s:\lambda_0^{\tau(\sigma)}(s)>t\},$$

and $(Y^{\tau(\sigma)},\lambda_0^{\tau(\sigma)},\Lambda_1^{\tau(\sigma)})$ is defined as in (2.3), Lemma 4.6 yields that $\mathcal{L}^{P_1}(X^\sigma)\in\Gamma_\mu^\infty$. On the other hand $\mathcal{L}^{P_2}(Y^{\tau(\sigma)},\lambda_0^{\tau(\sigma)},\Lambda_1^{\tau(\sigma)})\in\Pi$ by the optional sampling theorem. Moreover, for each n,

$$\begin{aligned}
&E^{P_2}[\int_0^\infty e^{-\lambda_0^{\tau(\sigma)}(s)}h_n(Y^{\tau(\sigma)}(s))d\lambda_0^{\tau(\sigma)}(s)]\\
&=\frac{1}{P(F)}E^P[\mathbf{1}_F E[\int_0^\infty e^{-\lambda_0^{\tau(\sigma)}(s)}h_n(Y^{\tau(\sigma)}(s))d\lambda_0^{\tau(\sigma)}(s)|Y(\tau(\sigma))]]\\
&=\frac{1}{P(F)}E^P[E[\mathbf{1}_F|Y(\tau(\sigma))]E[\int_0^\infty e^{-\lambda_0^{\tau(\sigma)}(s)}h_n(Y^{\tau(\sigma)}(s))d\lambda_0^{\tau(\sigma)}(s)|Y(\tau(\sigma))]]\\
&=\frac{1}{P(F)}E^P[E[\mathbf{1}_F|Y(\tau(\sigma))]\int_0^\infty e^{-\lambda_0^{\tau(\sigma)}(s)}h_n(Y^{\tau(\sigma)}(s))d\lambda_0^{\tau(\sigma)}(s)]\\
&=\gamma(\Pi_\mu^{h_1,\dots,h_{n-1}},h_n),
\end{aligned}$$

where the last equality follows from Lemma 4.6. Therefore the distribution of $(Y^{\tau(\sigma)}, \lambda_0^{\tau(\sigma)}, \Lambda_1^{\tau(\sigma)})$ under P_2 belongs to Π^∞, so that $\mathcal{L}^{P_2}(X^\sigma) \in \Gamma_\mu^\infty$. Then, by uniqueness of the distribution in Γ_μ^∞, it must hold $\mathcal{L}^{P_1}(X^\sigma) = \mathcal{L}^{P_2}(X^\sigma)$, which gives (4.13). □

Remark 4.10. The process constructed in Theorem 4.9 may not be a solution of the constrained (local) martingale problem because (3.3) is not necessarily a (local) martingale for all $f \in \mathcal{D}$. However it is, by construction, a solution of the martingale problem for $\mathbb{A}$. Note that $\mathcal{D}(\mathbb{A}) \supset \{f|_{\overline{E}_0} : f \in \mathcal{D}$ and $Bf(x,u) = 0, \forall (x,u) \in \Xi \cap \partial E_0\}$.

Lemma 4.11. *Let $\nu \in \mathcal{P}(\overline{E}_0)$. Suppose every solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for (A, E_0, B, Ξ) with initial distribution ν satisfies $\lambda_0(t) > 0$ for all $t > 0$ a.s. Then, for every choice of the $\{h_n\}$ in Theorem 4.9, $\nu \in \mathcal{Q}_0^\infty$.*

Proof. If $(Y, \lambda_0, \Lambda_1)$ has distribution in Π_ν^∞, then $\tau(0) = 0$. □

Corollary 4.12.

a) *Let $\nu \in \mathcal{P}(\overline{E}_0)$. If every solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for (A, E_0, B, Ξ) with initial distribution ν satisfies the conditions of Lemma 4.11 and Lemma 3.1, then there exists a strong Markov, natural solution to the constrained martingale problem for (A, E_0, B, Ξ) with initial distribution ν.*

b) *Let $\nu \in \mathcal{P}(\overline{E}_0)$. If λ_0 is a.s. strictly increasing for every solution of the controlled martingale problem for (A, E_0, B, Ξ) with initial distribution ν (see Lemma 3.4 for a sufficient condition), then there exists a strong Markov, natural solution to the constrained local martingale problem for (A, E_0, B, Ξ) with initial distribution ν.*

Proof. By Lemma 4.11, $\nu \in \mathcal{Q}_0^\infty$, and the assertion follows immediately from Theorem 4.9 by the same arguments as in Corollary 3.9. □

Corollary 4.13. *Assume Condition 3.5. Let $\nu \in \mathcal{P}(\overline{E}_0)$. If every solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for (A, E_0, B, Ξ) with initial distribution $\nu \in \mathcal{P}(\overline{E}_0)$ satisfies the conditions of Lemma 4.11 and there is a unique (in distribution) strong Markov process $X = Y \circ \tau$ with initial distribution ν that can be obtained from a solution of the controlled martingale problem as in Theorem 3.6, then there is a unique (in distribution) process that can be obtained in this way.*

In particular, under either condition a) or b) of Corollary 4.12, if there is a unique strong Markov, natural solution of the constrained (local) martingale problem with initial distribution ν, then there exists a unique natural solution.

Proof. If Γ_ν contains more than one distribution, then, by selecting appropriate sequences $\{h_n\}$, more than one strong Markov solution can be constructed. □

Remark 4.14. We can't rule out the possibility that there exist solutions of the constrained martingale problem that are not natural, but, under Condition 1.2 of [25], Theorem 2.2 of that paper yields that for any solution of the constrained martingale problem there exists a natural solution that has the same one dimensional distributions. By Theorem 3.2 of [24], uniqueness of one dimensional distributions for solutions with any given initial distribution implies uniqueness of finite dimensional distributions, so under Condition 1.2 of [25], uniqueness among natural solutions will imply uniqueness among all solutions.

5 Viscosity solutions

The approach taken above in the construction of a strong Markov solution to the constrained martingale problem simplifies the proof of existence of viscosity semisolutions to the problem

$$\begin{aligned} &v(x) - Av(x) = h(x), && \text{for } x \in E_0, \\ &Bv(x,u) = 0, && \text{for } x \in \partial E_0 \text{ and some } u \in \xi_x \end{aligned} \tag{5.1}$$

given in [6], Section 5. In fact Theorem 5.1 below shows that the function v_h defined by (4.1) and Lemma 4.1 is a viscosity subsolution of (5.1), and hence the function $-v_{-h}$ is a viscosity supersolution. As a consequence, under mild assumptions, uniqueness of the strong Markov solution of the constrained martingale problem starting at each $x \in \overline{E}_0$ implies existence of a viscosity solution (Corollary 5.3). This construction is a "probabilistic" alternative to Perron's method, and it does not require proving the comparison principle for (5.1).

For unconstrained martingale problems, the analogous result follows immediately from Section 3 of [6]. For a class of jump-diffusion processes, for which uniqueness in law holds, [8] proves existence of a viscosity solution to the backward Kolmogorov equation directly, and then uniqueness of the viscosity solution by the comparison principle. The fact that the comparison principle for (5.1) implies uniqueness of the solution to the constrained (or unconstrained) martingale problem is the object of [6].

Theorem 5.1. *Let $(Y, \lambda_0, \Lambda_1)$ be a solution to the controlled martingale problem for (A, E_0, B, Ξ). For $h \in C(\overline{E}_0)$, let $v \equiv v_h$ be the function defined by* (4.1) *and Lemma 4.1.*

Then $v\big|_{\overline{E}_0}$ is a viscosity subsolution of (5.1)*, that is, it is upper semicontinuous, and if $f \in \mathcal{D}$ and $x \in \overline{E}_0$ satisfy*

$$\sup_{z \in \overline{E}_0} (v - f)(z) = (v - f)(x), \tag{5.2}$$

then

$$\begin{aligned} &v(x) - Af(x) \le h(x), && \text{if } x \in E_0 \cup (\partial E_0 - F_1), \\ &(v(x) - Af(x) - h(x)) \wedge (-\max_{u \in \xi_x} Bf(x,u)) \le 0, && \text{if } x \in \partial E_0 \cap F_1, \end{aligned}$$

(ξ_x and F_1 being defined at the beginning of Section 2).

Proof. v is upper semicontinuous by Lemma 4.1.

Suppose x is a point such that $v(x) - f(x) = \sup_z (v(z) - f(z))$. As we can always add a constant to f, we can assume $v(x) - f(x) = 0$. By compactness, we have

$$v(x) = E^P \left[\int_0^\infty e^{-\lambda_0(s)} h(Y(s)) d\lambda_0(s) \right]$$

for some $P \in \Pi_{\delta_x}$. For $\epsilon > 0$, define

$$\tau_\epsilon = \epsilon \wedge \inf\{t > 0 : r(Y(t), x) \ge \epsilon \text{ or } r(Y(t-), x) \ge \epsilon\},$$

where r is the metric in E, and let $H_\epsilon = e^{-\lambda_0(\tau_\epsilon)}$. Since $(Y, \lambda_0, \Lambda_1)$ is a solution to the controlled martingale problem for (A, E_0, B, Ξ), we have

$$
\begin{aligned}
&0\\
&= v(x) - f(x)\\
&= E^P\left[\int_0^\infty e^{-\lambda_0(s)}(h - f + Af)(Y(s))\,d\lambda_0(s) + \int_{[0,\infty)\times U} e^{-\lambda_0(s)} Bf(Y(s),u)\Lambda_1(ds\times du)\right]\\
&= E^P\left[\int_0^{\tau_\epsilon} e^{-\lambda_0(s)}(h - f + Af)(Y(s))\,d\lambda_0(s) + \int_{[0,\tau_\epsilon]\times U} e^{-\lambda_0(s)} Bf(Y(s),u)\Lambda_1(ds\times du)\right]\\
&\qquad + E^P\left[e^{-\lambda_0(\tau_\epsilon)}\int_0^\infty e^{-\lambda_0^{\tau_\epsilon}(s)}(h - f + Af)(Y^{\tau_\epsilon}(s))\,d\lambda_0^{\tau_\epsilon}(s)\right]\\
&\qquad + E^P\left[e^{-\lambda_0(\tau_\epsilon)}\int_{[0,\infty)\times U} e^{-\lambda_0^{\tau_\epsilon}(s)} Bf(Y^{\tau_\epsilon}(s),u)\Lambda_1^{\tau_\epsilon}(ds\times du)\right]\\
&= E^P\left[\int_0^{\tau_\epsilon} e^{-\lambda_0(s)}(h - f + Af)(Y(s))\,d\lambda_0(s)\right]\\
&\qquad + E^P\left[\int_{[0,\tau_\epsilon]\times U} e^{-\lambda_0(s)} Bf(Y(s),u)\Lambda_1(ds\times du)\right]\\
&\qquad + E^P[H_\epsilon]E^{P^{\tau_\epsilon,H_\epsilon}}\left[\int_0^\infty e^{-\lambda_0(s)}(h - f + Af)(Y(s))\,d\lambda_0(s)\right]\\
&\qquad + E^P[H_\epsilon]E^{P^{\tau_\epsilon,H_\epsilon}}\left[\int_{[0,\infty)\times U} e^{-\lambda_0(s)} Bf(Y(s),u)\Lambda_1(ds\times du)\right],
\end{aligned}
$$

with $(Y^{\tau_\epsilon}, \lambda_0^{\tau_\epsilon}(s), \Lambda_1^{\tau_\epsilon})$ and $P^{\tau_\epsilon,H_\epsilon}$ as in Lemma 2.11. Setting $\mu_\epsilon(\cdot) \equiv P^{\tau_\epsilon,H_\epsilon}(Y(0)\in\cdot) = P(Y(\tau_\epsilon)\in\cdot)$, and denoting $\mu_\epsilon f \equiv \int_{F_2} f(z)\,\mu_\epsilon(dz)$, by Lemma 2.11 and Lemma 4.1 we have

$$
\begin{aligned}
&E^P[H_\epsilon]E^{P^{\tau_\epsilon,H_\epsilon}}\left[\int_0^\infty e^{-\lambda_0(s)}\left(h(Y(s)) - f(Y(s)) + Af(Y(s))\right)d\lambda_0(s)\right]\\
&\qquad\qquad + E^P[H_\epsilon]E^{P^{\tau_\epsilon,H_\epsilon}}\left[\int_{[0,\infty)\times U} e^{-\lambda_0(s)} Bf(Y(s),u)\Lambda_1(ds\times du)\right]\\
&\le E^P[H_\epsilon](\gamma(\Pi_{\mu_\epsilon}, h) - \mu_\epsilon f) = E^P[H_\epsilon](\mu_\epsilon v - \mu_\epsilon f)\\
&\le 0,
\end{aligned}
$$

where the last inequality uses the fact that $v - f \le 0$. Therefore

$$
\begin{aligned}
&0\\
&\le \lim_{\epsilon\to 0}\frac{E^P\left[\int_0^{\tau_\epsilon} e^{-\lambda_0(s)}(h - f + Af)(Y(s))d\lambda_0(s) + \int_{[0,\tau_\epsilon]\times U} e^{-\lambda_0(s)} Bf(Y(s),u)\Lambda_1(ds\times du)\right]}{E^P[\tau_\epsilon]}\\
&= h(x) - f(x) + Af(x_0) = h(x) - v(x) + Af(x)
\end{aligned}
$$

if $x \in E_0 \cup (\partial E_0 - F_1)$, and

$$
0 \le (h(x) - v(x) + Af(x)) \vee \max_{u\in\xi_x} Bf(x,u)),
$$

if $x \in \partial E_0 \cap F_1$. □

Remark 5.2. Note that, for each $x \in \overline{E}_0$,

$$
v(x) \equiv v_h(x) = E\left[\int_0^\infty e^{-s} h(X^h(s))ds\right]
$$

for some strong Markov process $X^h = Y \circ \tau$ obtained from a solution of the controlled martingale problem as in Theorem 3.6 with $Y(0) = x$.

Corollary 5.3.

a) *If, for each* $x \in \overline{E}_0$*, there is a unique solution* $(Y, \lambda_0, \Lambda_1)$ *of the controlled martingale problem for* (A, E_0, B, Ξ) *with* $Y(0) = x$*, then there exists a viscosity solution to* (5.1).

b) *If the assumptions of Corollary 4.12 a) or b) are satisfied for each* δ_x*,* $x \in \overline{E}_0$*, and there is a unique strong Markov, natural solution to the (local) constrained martingale problem with* $X(0) = x$*, then there exists a viscosity solution to* (5.1).

Proof. For each $x \in \overline{E}_0$, let $v \equiv v_h$ be the function defined by (4.1) and Lemma 4.1. Then, by uniqueness of the solution to the controlled martingale problem for (A, E_0, B, Ξ),

$$v(x) \equiv v_h(x) = -v_{-h}(x),$$

and, as noted at the beginning of this subsection, $-v_{-h}$ is a supersolution of (5.1).

The second assertion follows from Remark 5.2 by the same argument. □

6 Diffusions with oblique reflection in piecewise smooth domains: existence and Markov property

Let E_0 be a bounded, simply connected, open subset of $\mathbb{R}^d$ such that $E_0 \equiv \cap_{i=1}^m E_0^i$, where E_0^i, $i = 1, ..., m$, are simply connected open sets in $\mathbb{R}^d$ with C^1 boundaries. Specifically, we will assume that for each i there is a function $\psi_i \in C^1(\mathbb{R}^d)$ such that $E_0^i = \{x : \psi_i(x) > 0\}$ and that $\psi_i(x) = 0$ implies $\nabla\psi_i(x) \neq 0$. In particular, $\partial E_0^i = \{x : \psi_i(x) = 0\}$, and the inward normal at $x \in \partial E_0^i$ is $n^i(x) = \frac{\nabla\psi_i(x)}{|\nabla\psi_i(x)|}$. We will assume that

$$\overline{E_0} = \cap_{i=1}^m \overline{E_0^i}. \tag{6.1}$$

Suppose that on ∂E_0^i a variable direction of reflection g^i is assigned. We assume that g^i is continuous on ∂E_0^i and $\langle \nabla\psi^i(x), g^i(x)\rangle > 0$, $x \in \partial E_0^i$. It is convenient, and no loss of generality to assume that $g^i : \mathbb{R}^d \to \mathbb{R}^d$ and is continuous on all of $\mathbb{R}^d$ with $\langle \nabla\psi^i(x), g^i(x)\rangle \geq 0$ (allowing 0 away from ∂E_0^i). Noting that $x \in \partial E_0$ may be in more than one ∂E_0^i, for $x \in \partial E_0$, we define the cone of possible directions of reflection

$$G(x) \equiv \left\{ \sum_{i: x \in \partial E_0^i} \eta_i g^i(x),\ \eta_i \geq 0 \right\} \tag{6.2}$$

and also define

$$N(x) \equiv \left\{ \sum_{i: x \in \partial E_0^i} \eta_i n^i(x), \eta_i \geq 0 \right\}. \tag{6.3}$$

Starting from the late '70s, there has been a considerable amount of work devoted to proving existence and uniqueness of reflecting diffusions in $\overline{E}_0$ with direction of reflection g^i on ∂E_0^i. Perhaps the most general result in this sense is [13]. However the assumptions in [13] are not satisfied in many natural situations, as in the following example.

Example 6.1. Let $E_0 \equiv E_0^1 \cap E_0^2$, where E_0^1 is the unit ball centered at $(1, 0)$ and E_0^2 is the upper half plane. Let n^i, $i = 1, 2$, denote the unit, inward normal to $\overline{E_0^i}$, and

$$g^i(x) \equiv \begin{bmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{bmatrix} n^i(x), \qquad \vartheta \text{ a constant angle, } \frac{\pi}{4} \leq \vartheta < \frac{\pi}{2}.$$

Then, at $x^0 = 0$, it can be proved by contradiction that there is no convex compact set that satisfies (3.7) of [13].

In addition [13] does not cover the case of cusp like singularities (covered by [7] in dimension 2).

[10] considers convex polyhedrons (take $\psi_i(x) = \langle n^i, x\rangle - b^i$, n^i and b^i constant) with constant direction of reflection g^i on each face. In this context, [10] proves existence and uniqueness (in distribution) of semimartingale reflecting Brownian motion under a condition which, in the case of simple polyhedrons, reduces to the assumption that, for every $x \in \partial E_0$, there exists $e(x) \in N(x)$, $|e(x)| = 1$, such that

$$\langle g, e(x)\rangle > 0, \quad \forall g \in G(x) - \{0\}. \tag{6.4}$$

Moreover, for simple polyhedrons, [10], Propositions 1.1 and 1.2, shows that (6.4) is necessary for existence of semimartingale reflecting Brownian motion. (Non-semimartingale reflecting Brownian motion, which is studied, for example, in [19], [21] and [27], is not considered here.) Note that (6.4) is satisfied in Example 6.1.

In [10], a key point in proving uniqueness is the fact that there exist strong Markov processes that satisfy the definition of semimartingale reflecting Brownian motion and that uniqueness among these strong Markov processes implies uniqueness among all processes that satisfy the definition (analogously in [26] and [34]). Our goal here is to prove that this key point holds for general diffusion processes on domains E_0 as defined above under Condition 6.2 below, thus providing the first step in extending proofs of uniqueness to this more general setting

In [13], [10] and in most of the literature, reflecting diffusions are defined as (weak) solutions of stochastic differential equations with reflection. Here we start by studying the corresponding controlled martingale problem and constrained martingale problem, and then show that the set of natural solutions to the constrained martingale problem coincides with the set of solutions of the stochastic differential equation with reflection.

We consider the controlled martingale problem for (A, E_0, B, Ξ), with

$$\begin{aligned} Af(x) &\equiv \langle \nabla f(x), b(x)\rangle + \tfrac{1}{2}\mathrm{tr}\big(\sigma(x)\sigma^T(x)D^2 f(x)\big), \\ Bf(x,u) &\equiv \langle \nabla f(x), u\rangle, \\ U &\equiv \{u \in \mathbb{R}^d : \, |u| = 1\}, \\ \Xi &\equiv \{(x,u) \in \partial E_0 \times U : \, u \in G(x)\}, \end{aligned} \tag{6.5}$$

$\mathcal{D} \equiv C_c^2(\mathbb{R}^d)$, and we assume that σ and b are bounded and continuous on $\mathbb{R}^d$.

Note that F_1, defined at the beginning of Section 2, in this case is ∂E_0, so a solution of the controlled martingale problem must take values in $\overline{E}_0$ (Remark 2.3).

For $x \in (E_0)^c = \cup_{i=1}^m (E_0^i)^c$, let

$$I(x) \equiv \{i : x \in (E_0^i)^c\}. \tag{6.6}$$

Since $(E_0^j)^c$ is closed, if $j \in I(z^k)$ for some sequence $z^k \to x$, then $j \in I(x)$. Consequently, for each $x \in (E_0)^c$ there exists $\delta(x)$ such that

$$I(z) \subset I(x), \quad \text{for } z \in (E_0)^c \text{ with } |z - x| < \delta(x). \tag{6.7}$$

Note that, for $x \in \partial E_0$,

$$I(x) \equiv \{i : x \in \partial E_0^i\}. \tag{6.8}$$

Define also, for $x \in \partial E_0$,

$$\mathcal{I}(x) \equiv \{I \subset I(x) : \, \exists z \in (\overline{E_0})^c, \, |z - x| < \delta(x), \, s.t. \, I = I(z)\}. \tag{6.9}$$

We assume that E_0^i and g^i, $i = 1, ..., m$, satisfy the following condition.

Condition 6.2.

a) *For* $i = 1, ..., m$*,* $g^i : \mathbb{R}^d \to \mathbb{R}^d$ *are continuous vector fields of unit length on* ∂E_0^i*, that satisfy*

$$\langle g^i(x), n^i(x)\rangle > 0, \quad \forall x \in \partial E_0^i.$$

b) *For each* $x \in \partial E_0$*, there exists* $e(x) \in N(x)$*,* $|e(x)| = 1$*, that satisfies*

$$\langle g, e(x)\rangle > 0, \quad \forall g \in G(x) - \{0\}.$$

c) *For each* $x \in \partial E_0$*,* $I \in \mathcal{I}(x)$*, and* $n = \sum_{i\in I} \eta_i n^i(x)$*,* $\eta_i \geq 0$*,* $\sum_{i\in I} \eta_i > 0$*, there exists* $j \in I$ *such that*

$$\langle n, g^j(x)\rangle > 0.$$

Remark 6.3. In the case of simple, convex polyhedrons with constant direction of reflection on each face, Condition 6.2 b) becomes **(S.b)** of [10] and Condition 6.2 c) is immediately implied by **(S.a)** of [10]. In fact, since **(S.a)** and **(S.b)** are equivalent for simple polyhedrons ([10], Proposition 1.1), in this case Condition 6.2 is equivalent to the assumptions of [10].

Example 6.4. For domains with curved boundaries and singularities, e.g. cusp-like singularities, Condition 6.2 may be satisfied, whereas **(S.a)** and **(S.b)** of [10] are not. As an example, consider the domain

$$E_0 \equiv \{x \in \mathbb{R}^2 :\, 0 < x_1,\, -x_1^4 < x_2 < x_1^2,\, x_1^2 + x_2^2 < 1\}.$$

Then $E_0 = \cap_{i=1}^4 E_0^i$ with

$$\psi_1(x) \equiv x_2 + x_1^4,\;\; \psi_2(x) \equiv x_1^2 - x_2,\;\; \psi_3(x) \equiv 1 - x_1^2 - x_2^2,\;\; \psi_4(x) \equiv x_1.$$

Let g^1 and g^2 be continuous vector fields defined on ∂E_0^1 and ∂E_0^2, respectively, such that $g^1(0) = [-\frac{1}{2}, \frac{\sqrt{3}}{2}]^T$, $g^2(0) = [\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}]^T$, and take $g^4(0) \equiv [1, 0]^T$. Then $\mathcal{I}(0) = \big\{\{1\}, \{2\}, \{4\}, \{1, 4\}, \{2, 4\}\big\}$ and it is easy to check that Condition 6.2 is satisfied at 0.

Remark 6.5. In general, there are multiple possible choices of E_0^i and g^i, $i = 1, ..., m$, that determine the same domain $E_0 = \cap_{i=1}^m E_0^i$ and the same direction of reflection at each point of the smooth part of the boundary of E_0. In some cases, some of these choices satisfy Condition 6.2 and others do not. For instance, in Example 6.4 one can take $E_0 = \cap_{i=1}^3 \widetilde{E}_0^i$ with $\widetilde{\psi}_3 = \psi_3$ and

$$\widetilde{\psi}_1(x) \equiv \begin{cases} x_2 + x_1^4, & x_1 \geq 0, \\ x_2 - x_1^4, & x_1 < 0, \end{cases} \quad , \quad \widetilde{\psi}_2(x) \equiv \begin{cases} x_1^2 - x_2, & x_1 \geq 0, \\ -x_1^2 - x_2, & x_1 < 0. \end{cases}$$

Then $\mathcal{I}(0) = \big\{\{1\}, \{2\}, \{1, 2\}\big\}$ and, with the same $g^1(0)$ and $g^2(0)$ as above, Condition 6.2 is not satisfied at 0.

As anticipated in Remark 2.2, we will obtain a solution to the controlled martingale problem (6.5) by constructing a solution to the corresponding *patchwork martingale problem* ([23]), which will also be a solution to the controlled martingale problem.

Definition 6.6 ([23], Lemma 1.1)**.** *Given a complete, separable metric space* E*, an open subset* E_0 *of* E*, a partition of* $E - E_0$ *into Borel sets* $\{E_1, ..., E_m\}$ *and dissipative operators* $A, B_1, ..., B_m \subset C(E) \times C(E)$*, each containing* $(1, 0)$ *and with a common domain* $\mathcal{D}$ *dense in* $C(E)$*, a solution to the patchwork martingale problem for* $(A, E_0, B_1, E_1, ...B_m, E_m)$ *is a process* $(Y, \lambda_0, l_1, ..., l_m)$ *such that* Y *has paths in* $\mathcal{D}_E[0, \infty)$*,* $\lambda_0, l_1, ..., l_m$ *are nondecreasing,*

l_1 increases only when $Y \in \overline{E}_i$, $\lambda_0(t) + \sum_{i=1}^m l_i(t) = t$, and there exists a filtration $\{\mathcal{F}_t\}$ such that $(Y, \lambda_0, l_1, ..., l_m)$ is $\{\mathcal{F}_t\}$-adapted and

$$f(Y(t)) - f(Y(0)) - \int_0^t Af(Y(s))d\lambda_0(s) - \sum_{i=1}^m \int_0^t B_i f(Y(s))dl_i(s)$$

is a $\{\mathcal{F}_t\}$-martingale for all $f \in \mathcal{D}$.

Theorem 6.7. *For each $\nu \in \mathcal{P}(\overline{E}_0)$, there exists a solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for (A, E_0, B, Ξ) defined by* (6.5)*, with initial distribution ν.*

Proof. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\chi(r) = 0$ for $r \le 0$, $\chi(r) = 1$ for $r \ge 1$, $\chi'(r) > 0$ for $0 < r < 1$, and define

$$\phi(x) \equiv \sum_{i=1}^m \chi(-\psi_i(x)). \tag{6.10}$$

For $x \in \partial E_0$ and $I \in \mathcal{I}(x)$, let

$$N_1^I(x) = \{n : \, n = \sum_{i\in I} \eta_i n^i(x), \, \eta_i \ge 0, \, \sum_{i\in I} \eta_i = 1\},$$

and define

$$\beta \equiv \inf_{x\in\partial E_0} \min_{I\in\mathcal{I}(x)} \inf_{n\in N_1^I(x)} \max_{j\in I} \langle n, g^j(x)\rangle.$$

By Condition 6.2c) and compactness,

$$\beta > 0. \tag{6.11}$$

Let $0 < \epsilon_0 \le 1$ be sufficiently small so that for all $i = 1, ..., m$,

$$\inf_{x\in(\overline{E_0})^c: \, d(x,\overline{E_0})\le\epsilon_0} \psi_i(x) > -1, \quad \inf_{x\in(\overline{E_0})^c: \, d(x,\overline{E_0})\le\epsilon_0} |\nabla\psi_i(x)| > 0,$$

and for $|x - z| \le \epsilon_0$, $d(x, \partial E_0^i \cap \partial E_0) \le \epsilon_0$, $d(z, \partial E_0^i \cap \partial E_0) \le \epsilon_0$,

$$|g^i(x) - g^i(z)| \le \frac{\beta}{4} \quad \text{and} \quad |n^i(x) - n^i(z)| \le \frac{\beta}{4}.$$

Then, in particular, by (6.1), for $x \in (\overline{E_0})^c$, $d(x, \overline{E_0}) \le \epsilon_0$,

$$\phi(x) > 0, \qquad \sum_{i\in I(x)} \chi'(-\psi_i(x))|\nabla\psi_i(x)| > 0. \tag{6.12}$$

For $z \in \partial E_0$, let $\delta(z)$ be as in (6.7). By compactness, there exists $\delta_0 > 0$ such that, for every $x \in (\overline{E_0})^c$ with $d(x, \overline{E_0}) \le \delta_0$, there exists $z \in \partial E_0$ such that $|x - z| < \delta(z)$, hence $I(x) \in \mathcal{I}(z)$.

For each $j = 1, ..., m$, $x \in (\overline{E_0})^c$ with $d(x, \overline{E}_0) \le \delta_0 \wedge \epsilon_0$, and $z \in \partial E_0$ with $|x - z| < \delta(z)$,

$$\begin{aligned}
&- \langle \nabla\phi(x), g^j(x)\rangle \\
&= \sum_{i\in I(x)} \chi'(-\psi_i(x))|\nabla\psi_i(x)| \, \frac{\langle \nabla\psi_i(x), g^j(x)\rangle}{|\nabla\psi_i(x)|} \\
&= \sum_{i\in I(x)} \chi'(-\psi_i(x))|\nabla\psi_i(x)| \, \langle n^i(x), g^j(x)\rangle \\
&\ge \sum_{i\in I(x)} \chi'(-\psi_i(x))|\nabla\psi_i(x)| \, \big(\langle n^i(z), g^j(z)\rangle - |\langle n^i(x), g^j(x)\rangle - \langle n^i(z), g^j(z)\rangle|\big) \\
&\ge \sum_{i\in I(x)} \chi'(-\psi_i(x))|\nabla\psi_i(x)| \left(\langle n, g^j(z)\rangle - \frac{\beta}{2}\right),
\end{aligned}$$

where
$$n \equiv \frac{1}{\sum_{i\in I(x)}\chi'(-\psi_i(x))|\nabla\psi_i(x)|}\sum_{i\in I(x)}\chi'(-\psi_i(x))|\nabla\psi_i(x)|n^i(z)$$
belongs to $N_1^{I(x)}(z)$ since $I(x)\in\mathcal{I}(z)$. (6.11) implies that for some $j\in I(x)$,
$$-\langle\nabla\phi(x),g^j(x)\rangle \geq \sum_{i\in I(x)}\chi'(-\psi_i(x))|\nabla\psi_i(x)|\,\frac{\beta}{2}>0. \tag{6.13}$$

Define
$$E\equiv\{x:\ d(x,\overline{E}_0)\leq\delta_0\wedge\epsilon_0\}$$
and
$$\widetilde{F}_i=\{x\in E:\psi_i(x)\leq 0,\ \langle\nabla\phi(x),\ g^i(x)\rangle\leq 0\}. \tag{6.14}$$
By (6.13), each $x\in E-E_0$ is in at least one of the $\widetilde{F}_i$, so defining
$$\widetilde{E}_1=\{x\in E:\psi_1(x)\leq 0,\ \langle\nabla\phi(x),\ g^1(x)\rangle\leq 0\}$$
and
$$\widetilde{E}_i=\{x\in E:\psi_i(x)\leq 0,\ \langle\nabla\phi(x),\ g^i(x)\rangle\leq 0\}-\cup_{j<i}\widetilde{E}_j,\quad i=2,\dots,m,$$
$E_0,\widetilde{E}_1,\dots,\widetilde{E}_m$ are disjoint and
$$E=E_0\cup\bigcup_{i=1}^m\widetilde{E}_i.$$
Setting $\widetilde{\mathcal{D}}=C^2(E)$, $\rho(x)=[1-\chi(\frac{d(x,\overline{E}_0)}{\delta_0\wedge\epsilon_0})]$, $\widetilde{A}f(x)=\rho(x)Af(x)$, and $\widetilde{B}_if=\rho(x)\langle\nabla f(x),g^i(x)\rangle$, $\widetilde{A}$ and the $\widetilde{B}_i$ are dissipative, and Lemma 1.1 of [23] yields that, for each $\nu\in\mathcal{P}(\overline{E}_0)$, there exists a solution, $(Y,\lambda_0,l_1,...,l_m)$, of the patchwork martingale problem for $(\widetilde{A},E_0,\widetilde{B}_1,\widetilde{E}_1,...\widetilde{B}_m,\widetilde{E}_m)$ with initial distribution ν. Then, for $f\in\widetilde{\mathcal{D}}$
$$\begin{aligned}M_f(t) &= f(Y(t))-\int_0^t Af(Y(s))d\lambda_0(s)-\sum_{i=1}^m\int_0^t\widetilde{B}_if(Y(s))dl_i(s)\\ &= f(Y(t))-\int_0^t Af(Y(s))d\lambda_0(s)-\sum_{i=1}^m\int_0^t\rho(Y(s))\langle\nabla f(Y(s),g^i(Y(s)\rangle dl_i(s)\end{aligned}$$
is a $\{\mathcal{F}_t\}$-martingale. (We can write A rather than $\widetilde{A}$ since $Af=\widetilde{A}f$ on $\overline{E}_0$.)

Since ϕ is constant on $\overline{E}_0$, if ϕ were C^2, then
$$M_\phi(t)=\phi(Y(t))-\sum_{i=1}^m\int_0^t\rho(Y(s))\langle\nabla\phi(Y(s),g^i(Y(s)\rangle dl_i(s) \tag{6.15}$$
would be a martingale. Since we can approximate ϕ by C^2 functions $\{\phi^n\}$ in such a way that ϕ^n is constant on $\overline{E}_0$ and $\nabla\phi^n\to\nabla\phi$ uniformly on E, M_ϕ is a martingale even if ϕ is not C^2. M_ϕ is a nonnegative martingale because $\langle\nabla\phi,g^i\rangle\leq 0$ on $\overline{\widetilde{E}}_i$. If $Y(0)\in\overline{E}_0$, then $M_\phi(0)=0$ so, as in the proof of Lemma 1.4 of [23], $M_\phi(t)=0$ for all $t\geq 0$. As all terms in M_ϕ are nonnegative, $\phi(Y(t))$ must be zero for all $t\geq 0$, and hence, by (6.12), $Y(t)\in\overline{E}_0$ for all $t\geq 0$. Therefore $(Y,\lambda_0,l_1,...,l_m)$ is a solution of the patchwork martingale problem for $(A,E_0,B_1,E_1,...B_m,E_m)$, where
$$\begin{gathered}E_1\equiv\{x\in\partial E_0:\ \psi_1(x)=0\}\\ E_i\equiv\{x\in\partial E_0:\ \psi_1(x)>0,\ ..,\ \psi_{i-1}(x)>0,\ \psi_i(x)=0\},\quad i=2,...,m,\\ B_if(x)\equiv\langle\nabla f(x),g^i(x)\rangle\end{gathered}\tag{6.16}$$
.

If we define

$$\Lambda_1([0,t] \times C) \equiv \sum_{i=1}^{m} \int_0^t \mathbf{1}_C(g^i(Y(s))dl_i(s) \tag{6.17}$$

then $(Y, \lambda_0, \Lambda_1)$ is a solution of the controlled martingale problem for (A, E_0, B, Ξ). $\square$

Let $(Y, \lambda_0, \Lambda_1)$ be a solution of the controlled martingale problem for (A, E_0, B, Ξ). It is easy to verify that Y is continuous and

$$Y(t) - Y(0) - \int_0^t b(Y(s))d\lambda_0(s) - \int_{[0,t]\times U} u\,\Lambda_1(ds \times du) \equiv M(t) \tag{6.18}$$

is a continuous martingale with $[M](t) = \int_0^t (\sigma\sigma^T)(Y(s))d\lambda_0(s)$.

The following lemma is the analog of Lemma 3.1 of [10] and its proof is based on similar arguments.

Lemma 6.8. *For every solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for (A, E_0, B, Ξ) defined by (6.5), $\lambda_0(t) > 0$ for all $t > 0$, a.s.*

Proof. By (6.18), for $\tau(0) = \inf\{t \geq 0 :\, \lambda_0(t) > 0\}$,

$$Y(t \wedge \tau(0)) = Y(0) + \int_{[0,t\wedge\tau(0)]\times U} u\,\Lambda_1(ds \times du). \tag{6.19}$$

For every path such that $\tau(0) > 0$, $\lambda_1(t \wedge \tau(0)) = t \wedge \tau(0)$, and we must have $Y(t) \in \partial E_0$ for all $t \in [0, \tau(0))$. Setting, for $I \subset \{1, ..., m\}$,

$$\partial_I E_0 \equiv \{x \in \partial E_0 :\, I(x) = I\}$$

there must exist a k such that

$$Y(t) \in \bigcup_{I:\,|I|\geq k} \partial_I E_0 \tag{6.20}$$

for all $t \in [0, \tau(0))$. Let k_0 be the maximal such k, that is, $k = k_0$ satisfies (6.20) and there exists $t \in [0, \tau(0))$ such that $|I(Y(t))| = k_0$.

By (6.7) and the continuity of Y, for $s > t$ close enough to t, $I(Y(r)) \subset I(Y(t))$ for all $r \in [t, s]$. Since by definition of k_0, $|I(Y(r))| \geq k_0$, we must have $I(Y(r)) = I(Y(t))$ for all $r \in [t, s]$.

Since ∂E_0^i is C^1,

$$|\langle Y(s) - Y(t), n^i(Y(t))\rangle| = o(|Y(s) - Y(t)|), \quad \forall i \in I(Y(t)).$$

In addition, by (6.19) and the fact that $\lambda_1(t) = \Lambda_1([0,t] \times U)$ is Lipschitz,

$$|Y(s) - Y(t)| \leq O(s - t),$$

so that

$$\Big|\int_{(s,t]\times U} \langle u, n\rangle\,\Lambda_1(ds \times du)\Big| = |\langle Y(s) - Y(t), n\rangle| = o(|s - t|), \quad \forall n \in N(Y(t)).$$

On the other hand, setting $G^{I(y)}(x) \equiv \big\{\sum_{i\in I(y)} \eta_i g^i(x),\, \eta_i \geq 0\big\}$, (6.7) implies $G(Y(r)) \subset G^{I(Y(t))}(Y(r))$ for all $r \in [t, s]$. Since the Hausdorff distance $d(G^{I(y)}(x) \cap U, G(y) \cap U) \to 0$ as $x \to y$, if s is close enough to t, by 6.2 b) we have, for some $e \in N(Y(t))$, $|e| = 1$,

$$\langle Y(s) - Y(t), e\rangle = \int_{(t,s]\times U} \langle u, e\rangle\,\mathbf{1}_\Xi(Y(r), u)\,\Lambda_1(dr \times du) \geq \frac{1}{2} \inf_{u\in G(Y(t))\cap U} \langle u, e\rangle (s - t).$$

Consequently, by contradiction, $\tau(0)$ must be zero almost surely. $\square$

Lemma 6.9. *The controlled martingale problem for* (A, E_0, B, Ξ) *defined by (6.5) satisfies Condition 3.5.*

Proof. Condition 3.5 a) is clearly satisfied. Condition 3.5 b) is satisfied by Theorem 6.7, while Condition 3.5 c) is satisfied by Lemma 6.8 and Lemma 3.3. ☐

Theorem 6.10. *For each* $\nu \in \mathcal{P}(\overline{E}_0)$ *there exists a natural solution of the constrained martingale problem for* (A, E_0, B, Ξ) *defined by* (6.5).

Proof. By Lemma 6.8 and Lemma 3.4, Corollary 3.9 b) applies. ☐

As mentioned at the beginning of this section, a reflecting diffusion in $\overline{E}_0$ with direction of reflection g^i on $\{x \in \partial E_0 : \psi_i(x) = 0, \psi_j(x) > 0, \text{ for } j \neq i\}$, $i = 1, ..., m$, is often defined as a weak solution of a stochastic differential equation with reflection of the form

$$\begin{gathered} X(t) = X(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \int_0^t \gamma(s)\, d\lambda(s), \quad t \geq 0, \\ \gamma(t) \in G(X(t)), \quad |\gamma(t)| = 1, \quad d\lambda - a.e., \quad t \geq 0, \\ X(t) \in \overline{E}_0, \quad \lambda(t) = \int_0^t \mathbf{1}_{\partial E_0}(X(s))d\lambda(s), \quad t \geq 0. \end{gathered} \tag{6.21}$$

Definition 6.11. X*, defined on some probability space, is a weak solution of* (6.21) *if there are* λ *a.s. continuous and nondecreasing,* γ *a.s. measurable and a standard Brownian motion* W*, all defined on the same probability space as* X*, such that* (X, γ, λ) *is compatible with* W *(i.e.* $W(t + \cdot) - W(t)$ *is independent of* $\mathcal{F}_t^{W,X,\gamma,\lambda}$*, where* $\{\mathcal{F}_t^{W,X,\gamma,\lambda}\}$ *is the filtration generated by* (W, X, γ, λ)*) and* (6.21) *is satisfied.*

Theorem 6.12. *Every weak solution of* (6.21) *is a natural solution of the constrained martingale problem for* (A, E_0, B, Ξ) *defined by* (6.5).

Conversely, for every natural solution, X*, of the constrained martingale problem for* (A, E_0, B, Ξ) *there exists a weak solution of* (6.21) *with the same distribution as* X*.*

Proof. Let X be a weak solution of (6.21). Setting

$$\Lambda([0, t] \times C) \equiv \int_0^t \mathbf{1}_C(X(s), \gamma(s))d\lambda(s), \quad C \in \mathcal{B}(\Xi),$$

we see that X is a solution of the constrained martingale problem for (A, E_0, B, Ξ). Since $\Lambda([0, \cdot] \times \Xi)$ is continuous and (3.4) is satisfied, by Proposition 3.10, X is a natural solution.

Conversely, let $X = Y \circ \tau$, where $(Y, \lambda_0, \Lambda_1)$ is a solution of the controlled martingale problem (A, E_0, B, Ξ) with filtration $\{\mathcal{F}_t\}$ and τ is given by (3.1). Without loss of generality we can suppose $\{\mathcal{F}_t\}$ complete. Then (see [24], page 141) there is a $\{\mathcal{F}_t\}$-predictable, $\mathcal{P}(U)$-valued process L such that, in particular,

$$\int_{[0,t]\times U} u\, \Lambda_1(ds \times du) = \int_0^t \int_U u\, L(s, du)\, d\lambda_1(s) = \int_0^t \int_{U\cap G(Y(s))} u\, L(s, du)\, d\lambda_1(s).$$

Note that $\big| \int_{U\cap G(Y(s))} u\, L(s, du)\big| > 0\ d\lambda_1$-a.e. by Condition 6.2 b) and (2.4) of [24]. Then, setting

$$\widetilde{\gamma}(s) \equiv \frac{\int_{U\cap G(Y(s))} u\, L(s, du)}{\big| \int_{U\cap G(Y(s))} u\, L(s, du)\big|}, \quad \widetilde{\lambda}_1(t) \equiv \int_0^t \big| \int_{U\cap G(Y(s))} u\, L(s, du)\big| d\lambda_1(s),$$

we see that (6.18) can be written as

$$Y(t) = Y(0) + \int_0^t b(Y(s))d\lambda_0(s) + \int_0^t \widetilde{\gamma}(s)d\widetilde{\lambda}_1(s) + M(t).$$

By Lemma 6.8 and Lemma 3.4, λ_0 is strictly increasing, therefore $\tau = (\lambda_0)^{-1}$ and X satisfies

$$X(t) = X(0) + \int_0^t b(X(s))ds + \int_0^t \gamma(s)d\lambda(s) + N(t),$$

where $\gamma \equiv \widetilde{\gamma} \circ \tau$, $\lambda \equiv \widetilde{\lambda}_1 \circ \tau$ and $N \equiv M \circ \tau$ is a continuous martingale with $[N](t) = \int_0^t (\sigma\sigma^T)(X(s))ds$. Then the assertion follows by classical arguments. □

Theorem 6.13. *For each $\nu \in \mathcal{P}(\overline{E}_0)$, there exists a strong Markov solution of* (6.21). *If uniqueness in distribution holds among strong Markov solutions of* (6.21)*, then it holds among all solutions.*

Proof. The assertion follows from Theorem 6.10, Theorem 6.12, Corollary 4.12 and Corollary 4.13. □

We conclude this section with the proof of the equivalence between the controlled martingale problem (6.5) and the corresponding patchwork martingale problem (6.16) (see Definition 6.6). This equivalence is a valuable tool. For instance, in the last step of the proof of Theorem 6.7) we have already used one direction of the equivalence, which is immediate to see, namely the fact that every solution of the patchwork martingale problem yields a solution of the controlled martingale problem. On the contrary, the other direction of the equivalence is nontrivial and is proved in the following theorem.

Theorem 6.14. *For every solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for (A, E_0, B, Ξ) defined by* (6.5) *there exist $l_1, ..., l_m$ such that $(Y, \lambda_0, l_1, ..., l_m)$ is a solution of the patchwork martingale for $(A, E_0, B_1, E_1, ...B_m, E_m)$ defined by* (6.16).

Proof. First, we show that there is a Borel mapping $\Theta : \Xi \to \{\eta \in [0,\infty)^m : \sum_{i=1}^m \eta_i = 1\}$ such that

$$u = \sum_{i=1}^m \Theta_i(x,u)g^i(x), \quad \Theta_i(x,u) = 0 \text{ for } i \notin I(x).$$

Let $\mathbb{G}(x) \equiv [g_1(x), \ldots, g_m(x))]$, and let $\mathbb{G}^+(x)$ be the Moore-Penrose pseudo-inverse (see [3], Chapter 1). $\mathbb{G}^+(x)$ is a Borel function of $\mathbb{G}(x)$, hence of x. Then, for each $u \in \mathbb{R}^d$ such that $\mathbb{G}(x)\eta = u$ has at least one solution, all solutions have the form

$$\eta(w) = \mathbb{G}^+(x)u + (I - \mathbb{G}^+(x)\mathbb{G}(x))w, \quad w \in \mathbb{R}^m.$$

For $(x, u) \in \Xi$, let $w^0(x, u) \equiv$ argmin$|w|$, where the minimum is taken over all w such that $\eta_i(w) \geq 0$, $i = 1, ..., m$, $\eta_i(w) = 0$, $i \notin I(x)$, $\sum_{i=1}^m \eta_i(w) = 1$. w^0 is a Borel function ([12]). Then the mapping

$$\Theta(x,u) \equiv \mathbb{G}^+(x)u + (I - \mathbb{G}^+(x)\mathbb{G}(x))w^0(x,u)$$

has the desired properties.

The assertion follows by defining

$$l_i(t) = \int_{[0,t]\times U} \Theta_i(Y(s), u)\Lambda_1(ds \times du), \quad i = 1, \ldots, m. \tag{6.22}$$

□

7 Examples of application to other boundary conditions

7.1 Non-local boundary conditions

Let $A \subset C(E) \times C(E)$ with $\mathcal{D}(A)$ dense in $C(E)$, and assume that there exist solutions of the martingale problem for A with sample paths in $D_E[0,\infty)$ for all initial distributions $\nu \in \mathcal{P}(E)$.

Let $U \equiv \{1\}$ and B be defined by

$$Bf(x,1) \equiv Bf(x) \equiv \int (f(y) - f(x))\eta(x,dy),$$

where η is a transition function on E and, for all $x \in E$,

$$\eta(x, E_0) = \eta(x, E) = 1.$$

Then the controlled martingale problem requires

$$f(Y(t)) - f(Y(0)) - \int_0^t Af(Y(s))d\lambda_0(s) - \int_0^t Bf(Y(s))d\lambda_1(s)$$

to be a martingale. Note that the assumption that $\eta(x, E_0) = 1$ implies that for every solution of the controlled martingale problem $P\{\tau(0) < \infty\} = 1$. In fact, if $Y(0) \in E_0^c$, $P\{\tau(0) > t\} \leq e^{-t}$, since B is a generator of a pure jump process with unit exponential holding times. Consequently, by Lemma 3.3, $\lambda_0(t) \to \infty$.

Processes of this type have been considered in a variety of settings, for example [11, 30]. Semigroups corresponding to processes with nonlocal boundary conditions of this type have been considered in [2]. Related models are considered in [29].

7.2 Wentzell boundary conditions

Let $A \subset C(E) \times C(E)$ and $B \subset C(E) \times C(E)$ be generators such that for every $\mu \in \mathcal{P}(E)$ there exist solutions of the martingale problem for (A, μ) and (B, μ), every solution of the martingale problem for A has continuous sample paths and every solution for B has cadlag sample paths. In addition, assume that if Z is a solution of the martingale problem for B with $Z(0) \in \overline{E}_0$, then $Z(t) \in E_0$ for all $t > 0$. Set $U \equiv \{1\}$ and $Bf(\cdot, 1) \equiv Bf$.

Let $\mu \in \mathcal{P}(\overline{E}_0)$, let $Y^\epsilon(0)$ have distribution μ and let Y^ϵ evolve as a solution of the martingale problem for A until the first time τ_1^ϵ that Y^ϵ hits ∂E_0. After time τ_1^ϵ, let Y^ϵ evolve as a solution of the martingale problem for B until $\sigma_1^\epsilon \equiv \inf\{t > \tau_1^\epsilon : \inf_{x \in \partial E_0} |Y^\epsilon(t) - x| \geq \epsilon\}$. Recursively define τ_k^ϵ and σ_k^ϵ and assume $\sigma_0^\epsilon = 0$. By pasting, Y^ϵ is constructed so that for $f \in \mathcal{D}$,

$$f(Y^\epsilon(t)) - f(Y(0)) - \int_0^t \left(\sum_{k=0}^\infty \mathbf{1}_{[\sigma_k^\epsilon, \tau_{k+1}^\epsilon)}(s) Af(Y^\epsilon(s)) + \sum_{k=1}^\infty \mathbf{1}_{[\tau_k^\epsilon, \sigma_k^\epsilon)}(s) Bf(Y^\epsilon(s)) \right) ds$$

is a martingale. Define

$$\lambda_0^\epsilon(t) = \int_0^t \sum_{k=0}^\infty \mathbf{1}_{[\sigma_k^\epsilon, \tau_{k+1}^\epsilon)}(s) ds.$$

Assume that $\mathcal{D} = \mathcal{D}(A) = \mathcal{D}(B)$ is dense in $C(E)$. Then, by Theorem 3.9.4 of [14], $\{(Y^\epsilon, \lambda_0^\epsilon, \lambda_1^\epsilon), \epsilon > 0\}$ is relatively compact, and every limit point $(Y, \lambda_0, \lambda_1)$ will give a solution of the controlled martingale problem, that is, for every $f \in \mathcal{D}$

$$f(Y(t)) - f(Y(0)) - \int_0^t Af(Y(s))d\lambda_0(s) - \int_0^t Bf(Y(s))d\lambda_1(s)$$

is a $\{\mathcal{F}_t^{(Y,\lambda_0,\lambda_1)}\}$-martingale.

Our assumptions imply that λ_0 is strictly increasing, so $\lambda_0(t) \to \infty$ by Lemma 3.3.

Diffusions with Wentzell boundary conditions have been studied in [35, 36, 1]. Note that [35, 36] study the models using stochastic differential equations while [1] uses submartingale problems. [15] formulates what we call the constrained martingale problem.

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