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The non-linear sewing lemma I: weak formulation

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Abstract

We introduce a new framework to deal with rough differential equations based on flows and their approximations. Our main result is to prove that measurable flows exist under weak conditions, even if solutions to the corresponding rough differential equations are not unique. We show that under additional conditions of the approximation, there exists a unique Lipschitz flow. Then, a perturbation formula is given. Finally, we link our approach to the additive, multiplicative sewing lemmas and the rough Euler scheme.

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1 Introduction

1.1 Motivations

The theory of rough paths allows one to define the solution to a differential equation of type

$$y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s, \tag{1.1}$$

for a path x which is irregular, say α-Hölder continuous. Such an equation is then called a *Rough Differential Equation* (RDE) [20, 21, 25]. The key point of this theory is to show that such a solution can be defined provided that x is extended to a path x, called a *rough path*, living in a larger space that depends on the integer part of $1/\alpha$. When $\alpha > 1/2$, no such extension is needed. This case is referred as the *Young case*, as the integrals

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are constructed in the sense given by L.C. Young [24, 28]. Provided that one considers a rough path, integrals and differential equations are natural extensions of ordinary ones.

The first proof of existence of a solution to (1.1) from T. Lyons relied on a fixed point [25–27]. It was quickly shown that RDE shares the same properties as ordinary differential equations, including the flow property. In [16], A.M. Davie gives an alternative proof based on an Euler type approximation as well as counter-example to uniqueness. More recently, I. Bailleul gave a direct construction through the flow property [3, 4, 6].

A *flow* in a metric space V is a family $\{\psi_{t,s}\}_{0\leq s\leq t\leq T}$ of maps from V to V such that $\psi_{t,s}\circ\psi_{s,r}=\psi_{t,r}$ for any $0\leq r\leq s\leq t\leq T$. When $y_t(s,a)$ is a family of solutions to differential equations with $y_s(s,a)=a$, the element $\psi_{t,s}(a)$ can be seen as a map which carries a to $y_t(s,a)$. Flows are related to dynamical systems. They differ from solutions. One of their interest lies in their characterization as lipeomorphims (Lipschitz functions with a Lipschitz inverse), diffeomorphisms...

In this article, we develop a generic framework to construct flows from approximations. We do not focus on a particular form of the solutions, so that our construction is a *non-linear sewing lemma*, modelled after the additive and multiplicative sewing lemmas developed in [19, 20, 22, 26].

In this first part, we study flows under weak conditions and prove existence of a measurable flow even when the solutions of RDE are not necessarily unique. This is based on a selection theorem [11] due to J.E. Cardona and L. Kapitanski. Such a result is new in the literature where existence of flows was only proved under stronger regularity conditions (the many approaches are summarized in [15]). Besides, our approach also contains the additive and multiplicative sewing lemmas [13, 19]. The rough equivalent of the Duhamel formula for solving linear RDE [14] with a perturbative terms follows directly from our construction.

In a second part [9], we provide conditions for uniqueness and continuity. Besides, we show that our construction encompass many of the previous approaches or results: A.M. Davie [16], I. Bailleul [2, 3, 5] and P. Friz & N. Victoir [21].

Our starting point in the world of classical analysis is the *product formula* which relates how the iterated product of an approximation of a flow converge to the flow [1, 12]. It is important both from the theoretical and numerical point of view.

On a Banach space V, let us consider a family $(\epsilon,a)\in\mathbb{R}_+\times\mathrm{V}\mapsto\Phi(\epsilon,a)$, called an *algorithm* of class $\mathcal{C}^1$ in (ϵ,a) such that $\Phi(0,a)=a$. The parameter ϵ is related to the quality of the approximation.

The algorithm Φ is *consistent* with a vector field f when

$$f(a)=\frac{\partial\Phi}{\partial\epsilon}(0,a),\ \forall a\in\mathrm{V}. \tag{1.2}$$

For a consider algorithm, when $\phi_t(a)$ is the solution to $\phi_t(a)=a+\int_0^t f(\phi_s(a))\,\mathrm{d}s$,

$$\underbrace{\Phi(t/n,\Phi(t/n,\cdots\Phi(t/n,a)))}_{n\text{ times}}\text{ converges to }\phi_t(a)\text{ as }n\to\infty. \tag{1.3}$$

Eq. (1.3) is called the product formula.

The Euler scheme for solving ODE is the prototypical example of such behavior. Set $\Phi(\epsilon,a):=a+f(a)\epsilon$ so that (1.2) holds. In this case, (1.3) expresses the convergence of the Euler scheme.

Using the product formula, one recovers easily the proof of Lie's theorem on matrices: If A is a matrix, then $\exp(tA)$ is the solution to $\dot{Y}_t=AY_t$ with $Y_0=\mathrm{Id}$ and then

$$\exp(tA)=\lim_{n\to\infty}\left(\mathrm{Id}+\frac{t}{n}A\right)^n.$$

For two matrices A and B, $\exp(t(A+B))$ is given by

$$\exp(t(A+B)) = \lim_{n\to\infty}\left(\exp\left(\frac{tA}{n}\right)\exp\left(\frac{tB}{n}\right)\right)^n.$$

To prove the later statement, we consider $\Phi(\epsilon, a) = \exp(\epsilon A)\exp(\epsilon B)a$ and we verify that $\partial_\epsilon\Phi(0,a) = (A+B)a$ for any matrix a.

For unbounded operators, it is also related to Chernoff and Trotter's results on the approximation of semi-groups [18,29]. The product formula also justifies the construction of some splitting schemes [8].

In this article, we consider as an algorithm a family $\{\phi_{t,s}\}_{0\le s\le t\le T}$ of functions from V to V which is close to the identity map in short time and such that $\phi_{t,s}\circ\phi_{s,r}$ is close to $\phi_{t,r}$ for any time $s\le r\le t$. For a path x of finite p-variation, $1\le p<2$ with values in $\mathbb{R}^d$ and a smooth enough function $f:\mathbb{R}^m\to \mathrm{L}(\mathbb{R}^d,\mathbb{R}^m)$, such an example is given by $\phi_{t,s}(a) = a + f(a)x_{s,t}$.

We then study the behavior of the composition ϕ^π of the ϕ_{t_{i+1},t_i} along a partition $\pi = \{t_i\}_{i=0,\dots,n}$. Clearly, as the mesh of the partition π goes to 0, the limit, when it exists is a candidate to be a flow. In the example given above, it will be the flow associated to the family of Young differential equations $y_t(a) = a + \int_0^t f(y_s(a))\,\mathrm{d}x_s$, which means according to A.M. Davie [16] that

$$|y_t(a) - \phi_{t,s}(y_s(a))| \le L(a)\varpi(\omega_{s,t}). \tag{1.4}$$

We show that measurable flows may exist for Young or Rough Differential Equations even when several paths satisfying (1.4) exist.

In [9, 10], we exhibit a condition on almost flow that ensure existence of Lipschitz flows. Such an almost flow is called a *stable almost flow*. Besides, we study further the connection between almost flows and solutions in the sense of (1.4). In particular, when an almost flow is stable, solutions exist and are unique. Stronger convergence rate of numerical approximations, as well as continuity results are then given.

In order to present our main results, we introduce some necessary notations as well as some central notions such as galaxies.

1.2 Notations, definitions and concepts

The following notations and hypotheses will be constantly used throughout all this article.

1.2.1 Hölder and Lipschitz continuous functions

Let V and W be two metric spaces. The space of continuous functions from V to W is denoted by $\mathcal{C}(\mathrm{V},\mathrm{W})$.

Let d (resp. d') be a distance on V (resp. W). For $\gamma\in(0,1]$, we say that a function $f:\mathrm{V}\to\mathrm{W}$ is *γ-Hölder* if

$$\|f\|_\gamma := \sup_{\substack{a,b\in\mathrm{V},\\ a\neq b}} \frac{d'(f(a),f(b))}{d(a,b)^\gamma} < +\infty.$$

If $\gamma = 1$ we say that f is *Lipschitz*. We then set $\|f\|_{\mathrm{Lip}} := \|f\|_1$.

For any integer $r\ge 0$ and $\gamma\in(0,1]$, we denote by $\mathcal{C}^{r+\gamma}(\mathrm{V},\mathrm{W})$ the subspace of functions from V to W whose derivatives $\mathrm{d}^k f$ of order $k\le r$ are continuous and such that $\mathrm{d}^r f$ is γ-Hölder.

We denote by $\mathcal{C}^{r+\gamma}_{\mathrm{b}}(\mathrm{V},\mathrm{W})$ the subset of $\mathcal{C}^{r+\gamma}(\mathrm{V},\mathrm{W})$ of bounded functions with bounded derivatives up to order r.

1.2.2 Controls and remainders

From now, V is a topological, complete metric space with a distance d. A distinguished point of V is denoted by 0.

We fix $\gamma \in (0,1]$. Let $N_\gamma : \mathrm{V} \to [1,+\infty)$ be a γ-Hölder continuous function with constant $\|N\|_\gamma$. The index γ in N_γ refers to its regularity. If N_γ is Lipschitz continuous ($\gamma = 1$), then we simply write N.

Let us fix a *time horizon* T. We write $\mathbb{T} := [0,T]$ as well as

$$\mathbb{T}_2^+ := \{(s,t) \in \mathbb{T}^2 \mid s \leq t\} \text{ and } \mathbb{T}_3^+ := \{(r,s,t) \in \mathbb{T}^3 \mid r \leq s \leq t\}. \tag{1.5}$$

A *control* $\omega : \mathbb{T}_2^+ \to \mathbb{R}_+$ is a *super-additive* family, *i.e.*,

$$\omega_{r,s} + \omega_{s,t} \leq \omega_{r,t},\ \forall (r,s,t) \in \mathbb{T}_3^+$$

with $\omega_{s,s} = 0$ for all $s \in \mathbb{T}$, and for any $\delta > 0$, there exists $\epsilon > 0$ such that $\omega_{s,t} < \delta$ whenever $0 \leq s \leq t \leq s + \epsilon$. A typical example of a control is $\omega_{s,t} = C|t-s|$ for a constant $C \geq 0$.

For $p \geq 1$, we say that a path $x \in \mathcal{C}(\mathbb{T}, \mathrm{V})$ is a *path of finite p-variation controlled by ω* if

$$\|x\|_p := \sup_{\substack{(s,t)\in\mathbb{T}_2^+,\\ s\neq t}} \frac{d(x_s, x_t)}{\omega_{s,t}^{1/p}} < +\infty.$$

A *remainder* is a function $\varpi : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous, increasing and such that for some $0 < \varkappa < 1$,

$$2\varpi\left(\frac{\delta}{2}\right) \leq \varkappa\varpi(\delta),\ \forall \delta > 0. \tag{1.6}$$

A typical example for a remainder is $\varpi(\delta) = \delta^\theta$ for any $\theta > 1$.

We consider that $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function with $\lim_{T\to 0} \delta_T = 0$.

Finally, let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, increasing function such that for all $(s,t) \in \mathbb{T}_2^+$,

$$\eta(\omega_{s,t})\varpi(\omega_{s,t})^\gamma \leq \delta_T \varpi(\omega_{s,t}). \tag{1.7}$$

Partitions of $\mathbb{T}$ are customary denoted by $\pi = \{t_i\}_{i=0,\dots,n}$. The *mesh* $|\pi|$ of a partition π is $|\pi| := \max_{i=0,\dots,n}(t_{i+1} - t_i)$. The discrete simplices π_2^+ and π_3^+ are defined similarly to $\mathbb{T}_2^+$ and $\mathbb{T}_3^+$ in (1.5).

1.2.3 Galaxies

Notation 1.1. *We denote by $\mathcal{F}(\mathrm{V})$ the set of functions $\{\phi_{t,s}\}_{(s,t)\in\mathbb{T}_2^+}$ from V to V which are continuous in (s,t), i.e. for any $a \in \mathrm{V}$, the map $(s,t) \in \mathbb{T}_2^+ \mapsto \phi_{t,s}(a)$ is continuous.*

Notation 1.2 (Iterated products)**.** *For any $\phi \in \mathcal{F}(\mathrm{V})$, any partition π of $\mathbb{T}$ and any $(s,t) \in \mathbb{T}_2^+$, we write*

$$\phi_{t,s}^\pi := \phi_{t,t_j} \circ \phi_{t_j,t_{j-1}} \circ \cdots \circ \phi_{t_{i+1},t_i} \circ \phi_{t_i,s}, \tag{1.8}$$

where $[t_i, t_j]$ is the biggest interval of such kind contained in $[s,t] \subset \mathbb{T}$ (possibly, $t_i = t_j$). If no such interval exists, then $\phi_{t,s}^\pi = \phi_{t,s}$.

Clearly, for any partition, $\phi^\pi \in \mathcal{F}(\mathrm{V})$. A trivial but important remark is that from the very construction,

$$\phi_{t,r}^\pi = \phi_{t,s}^\pi \circ \phi_{s,r}^\pi \text{ for any } s \in \pi,\ r \leq s \leq t. \tag{1.9}$$

In particular, $\{\phi^{\pi}_{t,s}\}_{(s,t)\in\pi^2_+}$ enjoys a (semi-)flow property when the times are restricted to the elements of π.

The article is mainly devoted to study the possible limits of ϕ^{π} as the mesh of π decreases to 0.

Notation 1.3. *From a distance d on V, we define the distance Δ_{N_γ} on the space of functions from V to V by*

$$\Delta_{N_\gamma}(f,g) := \sup_{a\in\mathrm{V}} \frac{d(f(a),g(a))}{N_\gamma(a)},$$

where N_γ is defined in Section 1.2.

This distance is extended on $\mathcal{F}(\mathrm{V})$ by

$$\Delta_{N_\gamma,\varpi}(\phi,\psi) := \sup_{\substack{(s,t)\in\mathbb{T}_2^+\\ s\neq t}} \frac{\Delta_{N_\gamma}(\phi_{t,s},\psi_{t,s})}{\varpi(\omega_{s,t})},$$

where ω, ϖ are defined in Section 1.2. The distance $\Delta_{N_\gamma,\varpi}$ may take infinite values.

If d is a distance for which (V,d) is complete, then $(\mathcal{C}(\mathrm{V},\mathrm{V}),\Delta_N)$ and $(\mathcal{F}(\mathrm{V}),\Delta_{N,\varpi})$ are complete.

Definition 1.4 (Galaxy)**.** *We define the equivalence relation $\sim$ on $\mathcal{F}(\mathrm{V})$ by $\phi\sim\psi$ if and only if there exists a constant C such that*

$$d(\phi_{t,s}(a),\psi_{t,s}(a)) \leq CN_\gamma(a)\varpi(\omega_{s,t}),\ \forall a\in\mathrm{V},\ \forall(s,t)\in\mathbb{T}_2^+.$$

In other words, $\phi\sim\psi$ if and only if $\Delta_{N_\gamma,\varpi}(\phi,\psi)<+\infty$. Each quotient class of $\mathcal{F}(\mathrm{V})/\sim$ is called a galaxy*, which contains elements of $\mathcal{F}(\mathrm{V})$ which are at finite distance from each others.*

1.3 Summary of the main results

The galaxies partition the space $\mathcal{F}(\mathrm{V})$. Each galaxy may contain two classes of elements on which we focus on this article:

1. The *flows*, that is the families of maps $\psi:\mathrm{V}\to\mathrm{V}$ which satisfy

$$\psi_{t,r} = \psi_{t,s}\circ\psi_{s,r},\ \forall(r,s,t)\in\mathbb{T}_3^+, \tag{1.10}$$

   or equivalently, $\psi^{\pi}=\psi$ (See (1.8)) for any partition π.

2. The *almost flows* which we see as time-inhomogeneous algorithms. Besides some conditions on the continuity and the growth given in Definition 2.1 below, an almost flow ϕ is close to a flow with the difference that

$$d(\phi_{t,s}\circ\phi_{s,r}(a),\phi_{t,r}(a)) \leq N_\gamma(a)\varpi(\omega_{r,t}),\ \forall(r,s,t)\in\mathbb{T}_3^+,\ a\in\mathrm{V},$$

   for a suitable function $N_\gamma:\mathrm{V}\to[1,+\infty)$.

Along with an almost flow ϕ comes the notion of *manifold of D-solutions*, that is a family $y:=\{y_t(a)\}_{t\geq 0,\ a\in\mathrm{V}}$ of paths such that

$$d(y_t(a),\phi_{t,s}(y_s(a))) \leq C\varpi(\omega_{s,t}),\ \forall(s,t)\in\mathbb{T}_2^+. \tag{1.11}$$

Each path $y(a)$ that satisfies (1.11) is called a *D-solution*. This definition expands naturally the one introduced by A.M. Davie in [16].

Clearly, a manifold of D-solutions associated to an almost flow ϕ is also associated to any almost flow in the same galaxy as ϕ. Besides, if a flow ψ exists in the same galaxy as ϕ, then $z_t(a) = \psi_{t,0}(a)$ defines a manifold of D-solutions z. Flows are more constrained objects than solutions as (1.10) implies some compatibility conditions, while it is possible to create new D-solutions by splicing two different ones. As it will be shown in [9], uniqueness of manifold of D-solutions is strongly related to existence of a flow.

Given an almost flow ϕ, it is natural to study the limit of the net $\{\phi^\pi\}_\pi$ as the mesh of π decreases to 0. Any limit will be a good candidate to be a flow.

Our first main result (Theorem 2.5) asserts that for any almost flow ϕ in a galaxy G, any iterated map ϕ^π belongs to G whatever the partition π although the map ϕ^π is not necessarily an almost flow. More precisely, one controls $\Delta_{N_\gamma,\varpi}(\phi^\pi, \phi)$ uniformly in the partition π.

An immediate corollary is that any possible limit of $\{\phi^\pi\}_\pi$ as the mesh of the partition decreases to 0 also belongs to G.

Our second main result (Theorem 3.10) is that when the underlying Banach space V is finite-dimensional, there exists at least one measurable flow in a galaxy containing an almost flow, even when several manifolds of D-solutions may exist. Our proof uses a recent result of J.E. Cardona and L. Kapitanski [11] on selection theorems.

Our third result is to give conditions ensuring the existence of at most one flow in a galaxy. A sufficient condition is given for the galaxy G contains a Lipschitz flow ψ. In this case, $\{\phi^\pi\}_\pi$ converges to ψ whatever the almost flow ϕ in G. The rate of convergence is also quantified.

Finally, we apply our results to the additive, multiplicative sewing lemmas [19] as well as to the algorithms proposed by A.M. Davie in [16] to show existence of measurable flows even without uniqueness. In the sequel [9], we study in details the properties of Lipschitz flows and give some conditions on almost Lipschitz flows to generate them. In addition, we also apply our results to other approximations of RDE, namely the one proposed by P. Friz & N. Victoir [21] and the one proposed by I. Bailleul [3] by using perturbation arguments.

1.4 Outline

We show in Section 2 that a uniform control of the iterated product of approximation of flows with respect to the subdivision. In Section 3, we prove our main result: the existence of a measurable flow under weak conditions of regularity. Then, in Section 4 we show the existence and uniqueness of a Lipschitz flow under stronger assumptions. Moreover, we give a rate of the convergence of the iterated product to the flow. In Section 5, we show that additive perturbations preserve the convergence of iterated products of approximations of flows. Finally, we recover in Section 6 the additive [26], multiplicative [14, 19] and Davie's sewing lemmas [16].

2 A uniform control over almost flows

In this section, we define almost flows which serve as approximations. The properties of an almost flow ϕ are weaker than the minimal condition to obtain the convergence of the iterated product ϕ^π as the mesh of the partitions π decreases to 0. However, we prove in Theorem 2.5 that we can control ϕ^π uniformly over the partitions π. This justifies our definition.

2.1 Definition of almost flows

Definition 2.1 (Almost flow)**.** *An element $\phi \in \mathcal{F}(\mathrm{V})$ is an* almost flow *if for any $(r, s, t) \in \mathbb{T}_3^+$, $a \in \mathrm{V}$,*

$$\phi_{t,t}(a) = a, \tag{2.1}$$

$$d(\phi_{t,s}(a), a) \leq \delta_T N_\gamma(a), \tag{2.2}$$
$$d(\phi_{t,s}(a), \phi_{t,s}(b)) \leq (1 + \delta_T) d(a, b) + \eta(\omega_{s,t}) d(a, b)^\gamma, \tag{2.3}$$
$$d(\phi_{t,s} \circ \phi_{s,r}(a), \phi_{t,r}(a)) \leq N_\gamma(a) \varpi(\omega_{r,t}). \tag{2.4}$$

Remark 2.2. A family $\phi \in \mathcal{F}(\mathrm{V})$ satisfying condition (2.3) with $\gamma = 1$ is said to be *quasi-contractive*. This notion plays an important role in the fixed point theory [7].

Definition 2.3 (Iterated almost flow)**.** *For a partition π and an almost flow ϕ, we call ϕ^π an* iterated almost flow*, where ϕ^π is the iterated product defined in* (1.8)*.*

Definition 2.4 (A flow)**.** *A* flow *ψ is a family of functions $\{\psi_{t,s}\}_{(s,t)\in\mathbb{T}_2^+}$ from V to V (not necessarily continuous in (s,t)) such that $\psi_{t,t}(a) = a$ and $\psi_{t,s} \circ \psi_{s,r}(a) = \psi_{t,r}(a)$ for any $a \in \mathrm{V}$ and $(r,s,t) \in \mathbb{T}_3^+$.*

In this paper, we consider almost flows which are continuous although flows may *a priori* be discontinuous (See Theorem 3.10).

2.2 A uniform control on iterated almost flows

Before proving our main result in Section 3, we prove an important uniform control over ϕ^π.

Theorem 2.5. *Let ϕ be an almost flow. Then there exists a time horizon T small enough and constants $L_T \geq 1$ as well as $K_T \geq 1$ that decrease to 1 as T decreases to 0 such that*

$$d(\phi^\pi_{t,s}(a), \phi_{t,s}(a)) \leq L_T N_\gamma(a) \varpi(\omega_{s,t}), \tag{2.5}$$
$$N_\gamma(\phi^\pi_{t,s}(a)) \leq K_T N_\gamma(a), \tag{2.6}$$

for any $(s,t) \in \mathbb{T}^2_+$, $a \in \mathrm{V}$ and any partition π of $\mathbb{T}$. The choice of T, L_T and K_T depend only on δ, ϖ, ω, γ and $\|N_\gamma\|_\gamma$.

Remark 2.6. The distance d may be replaced by a pseudo-distance in the statement of Theorem 2.5.

The proof of Theorem 2.5 is inspired by the one of the Claim in the proof of Lemma 2.4 in [16]. With respect to the one of A.M. Davie, we consider obtaining a uniform control over a family of elements indexed by $(s,t) \in \mathbb{T}_2^+$ which are also parametrized by points in V.

Definition 2.7 (Successive points / distance between two points)**.** *Let π be a partition of $[0,T]$. Two points s and t of π are said to be* at distance k *if there are exactly $k-1$ points between them in π. We write $d_\pi(t,s) = k$. Points at distance 1 are called* successive points *in π.*

Proof. For $a \in \mathrm{V}$, $(r,t) \in \mathbb{T}_2^+$ and a partition π, we set

$$U_{r,t}(a) := d(\phi^\pi_{t,r}(a), \phi_{t,r}(a)).$$

We now restrict ourselves to the case $(r,t) \in \pi_2^+$. To control $U_{r,t}(a)$ in a way that does not depend on π, we use an induction in the distance between r and t.

Our induction hypothesis is that there exist constants $L_T \geq 0$ and $K_T \geq 1$ independent from the partition π such that for any $(r,t) \in \pi_2^+$ at distance at most $m \geq 1$,

$$U_{r,t}(a) \leq L_T N_\gamma(a) \varpi(\omega_{r,t}), \tag{2.7}$$
$$N_\gamma(\phi^\pi_{t,s}(a)) \leq K_T N_\gamma(a), \tag{2.8}$$

with K_T decreases to 1 at T goes to 0.

The induction hypothesis is true for $m = 0$, since $U_{r,r}(a) = 0$ and $N_\gamma(\phi^\pi_{r,r}(a)) = N_\gamma(a)$.

If r and t are successive points, $\phi^\pi_{t,r} = \phi_{t,r}$ so that $U_{r,t}(a) = 0$. Thus, (2.7) is true for $m = 1$. With (2.2) and since $N_\gamma(a)^\gamma \leq N_\gamma(a)$ as $N_\gamma(a) \geq 1$ by hypothesis,

$$\begin{aligned} N_\gamma(\phi_{t,s}(a)) \leq N_\gamma(\phi_{t,s}(a)) - N_\gamma(a) + N_\gamma(a) &\leq \|N_\gamma\|_\gamma d(\phi_{t,s}(a), a)^\gamma + N_\gamma(a) \\ &\leq \|N_\gamma\|_\gamma \delta_T^\gamma N_\gamma(a)^\gamma + N_\gamma(a) \leq (\|N_\gamma\|_\gamma \delta_T^\gamma + 1) N_\gamma(a). \end{aligned}$$

For $m = 2$, it is also true with $L_T = 1$ as $U_{r,t}(a) = d(\phi_{t,s} \circ \phi_{s,r}(a), \phi_{t,r}(a))$ where s is the point in the middle of r and t. This proves (2.7).

In addition, using (2.7),

$$\begin{aligned} N_\gamma(\phi^\pi_{t,s}(a)) &\leq \|N\|_\gamma d(\phi^\pi_{t,s}(a), \phi_{t,s}(a))^\gamma + N_\gamma(\phi_{t,s}(a)) - N_\gamma(a) + N_\gamma(a) \\ &\leq K_T N_\gamma(a) \text{ with } K_T := \|N_\gamma\|_\gamma \varpi(\omega_{0,T})^\gamma + \|N_\gamma\|_\gamma \delta_T^\gamma + 1. \end{aligned} \tag{2.9}$$

Clearly, K_T decreases to 1 at T decreases to 0. This proves (2.8) whenever (2.7), in particular for $m = 1$.

Assume that (2.7)-(2.8) when the distance between r and t is smaller than m for some $m \geq 2$.

For $s \in \pi$, $r \leq s \leq t$, with (1.9),

$$\begin{aligned} U_{r,t}(a) \leq d(\phi^\pi_{t,s} \circ \phi^\pi_{s,r}(a), \phi_{t,s} \circ \phi^\pi_{s,r}(a)) \\ + d(\phi_{t,s} \circ \phi^\pi_{s,r}(a), \phi_{t,s} \circ \phi_{s,r}(a)) + d(\phi_{t,s} \circ \phi_{s,r}(a), \phi_{t,r}(a)). \end{aligned}$$

With (2.3) and (2.4),

$$U_{r,t}(a) \leq U_{s,t}(\phi^\pi_{s,r}(a)) + (1 + \delta_T) U_{r,s}(a) + \eta(\omega_{s,t}) U_{r,s}(a)^\gamma + N_\gamma(a) \varpi(\omega_{r,t}). \tag{2.10}$$

Using (2.10) on $U_{s,t}(\phi^\pi_{s,r}(a))$ by replacing (r, s, t) by (s, s', t), where s and s' are two successive points of π leads to

$$U_{s,t}(\phi^\pi_{s,r}(a)) \leq U_{s',t}(\phi^\pi_{s',r}(a)) + N_\gamma(\phi^\pi_{s,r}(a)) \varpi(\omega_{s,t})$$

since $\phi_{s',s} \circ \phi^\pi_{s,r}(a) = \phi^\pi_{s',r}(a)$ and $U_{s,s'}(a) = 0$.

With (2.7)-(2.8) and our hypothesis on η, again since $N_\gamma \geq 1$ and s, s' are at distance less than m provided that s is at distance at most 2 from t,

$$U_{r,t}(a) \leq N_\gamma(a) \left(K_T L_T \varpi(\omega_{s',t}) + (L_T + \delta_T(1 + L_T^\gamma)) \varpi(\omega_{r,s}) + (1 + K_T) \varpi(\omega_{r,t}) \right). \tag{2.11}$$

This inequality also holds true for $r = s$ or $s' = t$.

We proceed as in [16] to split π in "essentially" two parts. We set

$$s' := \min \left\{ \tau \in \pi \;\middle|\; \omega_{r,\tau} \geq \frac{\omega_{r,t}}{2} \right\} \text{ and } s := \max\{\tau \in \pi \mid \tau < s'\}.$$

Hence, s and s' are successive points with $r \leq s < s' \leq t$ and $\omega_{r,s} \leq \omega_{r,t}/2$. Besides, since ω is super-additive, $\omega_{r,s'} + \omega_{s',t} \leq \omega_{r,t}$. Therefore,

$$\omega_{r,s} \leq \frac{\omega_{r,t}}{2} \text{ and } \omega_{s',t} \leq \frac{\omega_{r,t}}{2}. \tag{2.12}$$

With such a choice, since $L_T \geq 1$, (2.11) becomes with (1.6):

$$U_{r,t}(a) \leq \left(L_T \frac{K_T + 1 + 2\delta_T}{2} \varkappa + 1 + K_T \right) N_\gamma(a) \varpi(\omega_{r,t}).$$

If T is small enough so that

$$\alpha := \frac{1 + K_T + 2\delta_T}{2}\varkappa < 1 \text{ and } L_T\alpha + 1 + K_T \leq L_T,$$

then one may choose L_T so that $L_T\alpha+1+K_T \leq L_T$, that is $L_T \geq \max\{1, (1+K_T)/(1-\alpha)\}$. This choice ensures that $U_{r,t}(a) \leq L_T N_\gamma(a)\varpi(\omega_{r,t})$ when r and t are at distance m. Condition (2.8) follows from (2.9) and (2.7).

The choice of T, K_T and L_T does not depend on π. In particular, $d(\phi^\pi_{t,s}(a), \phi_{t,s}(a)) \leq L_T N_\gamma(a)\varpi(\omega_{s,t})$ becomes true for any $(s,t) \in \mathbb{T}_2^+$ (it is sufficient to add the points s and t to π). □

Corollary 2.8. *Let ϕ be an almost flow and π be a partition of $\mathbb{T}$. Then $\phi^\pi \sim \phi$ (we have not proved that ϕ^π is itself an almost flow).*

Notation 2.9. *For an almost flow ϕ, let us denote by $S_\phi(s,a)$ the set of all the possible limits of the net $\{\phi^\pi_{\cdot,s}(a)\}_\pi$ in $(\mathcal{C}([s,T], \mathrm{V}), \|\cdot\|_\infty)$ for nested partitions.*

When V is a finite dimensional space with the norm $|\cdot|$, $S_\phi(s,a) \neq \emptyset$. We start with a lemma which will be useful to prove some equi-continuity. We denote $\bar{B}(0,R)$ the closed ball centered in 0 of radius $R > 0$.

Lemma 2.10. *Let $R > 0$. We assume that V is a finite dimensional vector space and ϕ is an almost flow. Then, for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $(s,t), (s',t') \in \mathbb{T}_2^+$ with $|t-t'|, |s-s'| < \delta$ and $a \in \bar{B}(0,R)$,*

$$|\phi_{t,s}(a) - \phi_{t',s'}(a)| \leq \epsilon.$$

Proof. In all the proof $(s,t), (s',t') \in \mathbb{T}_2^+$. For any $a \in \bar{B}(0,R)$, $(s,t) \in \mathbb{T}_2^+ \mapsto \phi_{t,s}(a)$ is continuous, so uniformly continuous on the compact $\mathbb{T}_2^+$. Let $\epsilon > 0$, there is δ_a such that for all $|t-t'|, |s-s'| < \delta_a$,

$$|\phi_{t,s}(a) - \phi_{t',s'}(a)| \leq \frac{\epsilon}{3}. \tag{2.13}$$

For a $\rho > 0$ with (2.3), for all $b \in B(a,\rho)$,

$$|\phi_{t,s}(a) - \phi_{t,s}(b)| \leq (1+\delta_T)\rho + \eta(\omega_{0,T})\rho^\gamma,$$

where $B(a,\rho)$ denotes the open ball centred in a of radius ρ. We choose $\rho(\epsilon) > 0$ such that $(1+\delta_T)\rho(\epsilon) + \eta(\omega_{0,T})\rho(\epsilon)^\gamma \leq \epsilon/3$ to obtain for all $b \in B(a,\rho(\epsilon))$,

$$|\phi_{t,s}(a) - \phi_{t,s}(b)| \leq \frac{\epsilon}{3}. \tag{2.14}$$

We note that $\bigcup_{a\in\bar{B}(0,R)} B(a,\rho(\epsilon))$ is a covering of $\bar{B}(0,R)$ which is a compact of V. There exist an integer N and a finite family of balls $B(a_i,\rho(\epsilon))$ for $i \in \{1,\dots,N\}$ such that $\bar{B}(0,R) \subset \bigcup_{i\in\{1,\dots,N\}} B(a_i,\rho(\epsilon))$.

It follows that for all $b \in \bar{B}(0,R)$ there exists $i \in \{1,\dots,N\}$ such that $b \in B(a_i,\rho(\epsilon))$. From (2.13)-(2.14) there exists $\delta_{a_i} > 0$ such that for all $|t-t'|, |s-s'| < \delta_{a_i}$,

$$|\phi_{t,s}(b) - \phi_{t',s'}(b)| \leq \epsilon.$$

Taking $\delta := \min_{i\in\{1,\dots,N\}} \delta_{a_i}$, we obtain that for all $|t-t'|, |s-s'| < \delta$ and $b \in \bar{B}(0,R)$,

$$|\phi_{t,s}(b) - \phi_{t',s'}(b)| \leq \epsilon,$$

which achieves the proof. □

Proposition 2.11. *Assume that* V *is finite-dimensional and that* ϕ *is an almost flow. Then* $S_\phi(r,a)$ *is not empty for any* $(r,a)\in\mathbb{T}\times\mathrm{V}$.

Proof. Let us show for an almost flow ϕ, $\{\phi^\pi_{\cdot,r}(a)\}_\pi$ is equi-continuous and bounded. The result is then a direct consequence of the Ascoli-Arzelà theorem.

Let $(\pi_n)_{n\in\mathbb{N}}$ be an increasing sequence of partitions of $\mathbb{T}$ such that $|\pi_n|\to 0$ when $n\to+\infty$ and $\bigcup_{n\in\mathbb{N}}\pi_n$ dense in $\mathbb{T}$. Let $R>0$ and $R':=N_\gamma(R)(L_T\varpi(\omega_{0,T})+\delta_T)+R$. From Theorem 2.5, for any $a\in\bar{B}(0,R)$ and $(s,t)\in\mathbb{T}_2^+$,

$$|\phi^{\pi_n}_{t,s}(a)|\leq|\phi^{\pi_n}_{t,s}(a)-\phi_{t,s}(a)|+|\phi_{t,s}(a)-a|+|a|\leq N_\gamma(a)(L_T\varpi(\omega_{0,T})+\delta_T)+|a|\leq R'.$$

Let $(r,s,t)\in\mathbb{T}_3^+$ and $s,t\in\bigcup_{n\in\mathbb{N}}\pi_n$, let $\epsilon>0$ and $\delta>0$ given by Lemma 2.10 for $\epsilon/2$. For $|t-s|<\delta$, let m an integer such that $s,t\in\pi_m$.

We differentiate two cases. If $m\leq n$, then $\pi_m\subset\pi_n$, which implies that $\phi^{\pi_n}_{t,r}=\phi^{\pi_n}_{t,s}\circ\phi^{\pi_n}_{s,r}$. From Theorem 2.5 and Lemma 2.10, for all $a\in\bar{B}(0,R)$, for all $|t-s|<\delta$

$$|\phi^{\pi_n}_{t,r}(a)-\phi^{\pi_n}_{s,r}(a)|\leq|\phi^{\pi_n}_{t,s}\circ\phi^{\pi_n}_{s,r}(a)-\phi^{\pi_n}_{s,r}(a)|\leq\frac{\epsilon}{2}+L_TN_\gamma(\phi^{\pi_n}_{s,r}(a))\varpi(\omega_{s,t}).$$

Since, $\omega_{s,s}=0$ and ω is continuous close to diagonal, there exists $\delta'>0$ such that for all $|t-s|<\delta'$, $L_TN_\gamma(\phi^{\pi_n}_{s,r}(a))\varpi(\omega_{s,t})<\epsilon/2$. Thus for all $|t-s|<\min(\delta,\delta')$, it holds that $|\phi^{\pi_n}_{t,r}(a)-\phi^{\pi_n}_{s,r}(a)|\leq\epsilon$.

In the case $m>n$, let s_-,s_+ be successive points in π_n such that $[s,t]\subset[s_-,s_+]$. Then, $\phi^{\pi_n}_{t,r}(a)=\phi_{t,s_-}\circ\phi^{\pi_n}_{s_-,r}(a)$ and $\phi^{\pi_n}_{s,r}(a)=\phi_{s,s_-}\circ\phi^{\pi_n}_{s_-,r}(a)$. According to Lemma 2.10, for $|t-s|<\delta$ with $s,t\in\pi_m$, and all $a\in\bar{B}(0,R)$,

$$|\phi^{\pi_n}_{t,r}(a)-\phi^{\pi_n}_{s,r}(a)|=|\phi_{t,s_-}\circ\phi^{\pi_n}_{s_-,r}(a)-\phi_{s,s_-}\circ\phi^{\pi_n}_{s_-,r}(a)|\leq\epsilon. \tag{2.15}$$

By continuity of $t\mapsto\phi^{\pi_n}_{t,r}(a)$, and density of $\bigcup_{m\in\mathbb{N}}\pi_m$ in $\mathbb{T}$, we obtain (2.15) for all $(s,t)\in\mathbb{T}_2^+$ with $r\leq s$ and $|t-s|<\delta$. This proves that $\{t\mapsto\phi^{\pi_n}_{t,r}(a)\}_n$ is uniformly equi-continuous for all $a\in\mathrm{V}$. We conclude the proof with the Ascoli-Arzelà theorem. □

3 The non-linear sewing lemma

We now show that in the finite dimensional case, we can build a flow from ϕ^π using a selection principle [11]. In this section, we consider that almost flows ϕ are defined for $\mathbb{T}:=[0,+\infty)$.

Definition 3.1 (Solution in the sense of Davie or D-solution)**.** *For an almost flow* ϕ*, a time* $r\in\mathbb{T}$ *and a point* $a\in\mathrm{V}$*, a* solution in the sense of Davie *(or a D-solution) is a path* $y\in\mathcal{C}(\mathbb{S},\mathrm{V})$ *with* $\mathbb{S}=[r,r+T]\subset\mathbb{T}$ *such that*

$$d(y_t,\phi_{t,s}(y_s))\leq KN_\gamma(a)\varpi(\omega_{s,t}),\ \forall(s,t)\in\mathbb{S}^2_+, \tag{3.1}$$

where $K\geq 0$ *is a constant.*

Remark 3.2. Our definition of D-solutions extends the one of Davie in [16] to a metric space V and a general almost flow ϕ.

Remark 3.3. When ϕ is only an almost flow, it is not guaranteed that a D-solution exists or is unique. When $S_\phi(r,a)\neq\emptyset$ (see Notation 2.9), we prove below in Lemma 3.14 that a D-solution exists. However, even if for all $(r,a)\in\mathbb{T}\times\mathrm{V}$, $S_\phi(r,a)\neq\emptyset$ this does not imply the existence of families of solutions $\{\psi_{\cdot,r}(a)\}_{r\in\mathbb{T},a\in\mathrm{V}}$ which satisfies the flow property.

Notation 3.4. *We denote* $\Omega(r,a)$ *the set of continuous paths such that* $y\in\mathcal{C}(\mathbb{S},\mathrm{V})$ *verifies* $y_r=a$*. We denote by* $G^K_\phi(r,a)$ *the set of paths in* $\Omega(r,a)$ *verifying* (3.1) *for the constant* K*.*

Definition 3.5 (Splicing of paths)**.** *For $r \leq s$, let us consider $y(r,a) \in \Omega(r,a)$ and $z(s,b) \in \Omega(s,b)$ with $b = y_s(r,a)$. Their* splicing *is*

$$(y \bowtie_s z)_t(r,a) = \begin{cases} y_t(r,a) & \text{if } t \leq s, \\ z_t(s, y_s(r,a)) & \text{if } t \geq s. \end{cases}$$

We restate here, the definition of a family of abstract integral local funnels (Definition 2 in [11]), which leads to the existence of a measurable flow.

Definition 3.6. *A family $F(r,a)$ with $r \in [0,+\infty)$, $a \in \mathrm{V}$, will be called a* family of abstract local integral funnels *with* terminal time $T(r,a) \in (0,+\infty)$ *if*

H0 *The map $(r,a) \in [0,+\infty) \times \mathrm{V} \mapsto T(r,a)$ is lower semi-continuous in the sense that if $(r_n,a_n) \to (r,a)$, then $T(r,a) \leq \liminf_n T(r_n,a_n)$.*

H1 *Every set $F(r,a)$ is a non-empty compact in the space $\mathcal{C}([r,r+T(r,a)],\mathrm{V})$ and every path $y(r,a) \in F(r,a)$ is a continuous map from $[r,r+T(r,a)]$ to V with $y_r(r,a) = a$.*

H2 *For all $r \geq 0$, the map $a \in \mathrm{V} \mapsto F(r,a)$ is measurable in the sense that for any closed subset $\mathcal{K} \subset \mathcal{C}([0,1],\mathrm{V})$, $\{a \in \mathrm{V} \mid \tilde{F}(r,a) \bigcap \mathcal{K} \neq \emptyset\}$ is Borel, where $\tilde{F}(r,a)$ is the set of re-parametrizations of paths of $F(r,a)$ on $[0,1]$.*

H3 *If $y(r,a) \in F(r,a)$ and $\tau < T(r,a)$, then $T(r+\tau, y_{r+\tau}(r,a)) \geq T(r,a) - \tau$ and $t \in [r+\tau, r+\tau+T(r+\tau, y_{r+\tau}(r,a))] \mapsto y_t(r,a)$ belongs to $F(r+\tau, y_{r+\tau}(r,a))$.*

H4 *If $y(r,a) \in F(r,a)$ and $\tau < T(r,a)$, and $z \in F(r+\tau, y_{t+\tau}(r,a))$ then the spliced path (Definition 3.5) $x := y \bowtie_{r+\tau} z$ belongs to $F(r,a)$.*

Definition 3.7 (Lipschitz almost flow)**.** *A* Lipschitz almost flow *is an almost flow for which* (2.3) *is satisfied with $\eta = 0$, and $N_\gamma = N_1 = N$ is Lipschitz continuous.*

A flow is constructed by assigning to each point of the space a particular D-solution, in a sense which is compatible.

Hypothesis 3.8. *Let V be a finite dimensional vector space. Let $\phi := \{\phi_{t,s}\}_{0 \leq s \leq t < +\infty}$ be a Lipschitz almost flow (Definition 3.7) with N bounded. We fix a time horizon $T > 0$ such that $\varkappa(1+\delta_T) < 1$.*

Remark 3.9. When N bounded, we can choose $K_T = 1$ and $L_T = 2/(1-\varkappa(1+\delta_T))$ where K_T and L_T are the constants of Theorem 2.5.

The main theorem of this paper is the following one.

Theorem 3.10 (Non-linear sewing lemma, weak formulation)**.** *Under Hypothesis 3.8, there exists $\psi \in \mathcal{F}(\mathrm{V})$ in the same galaxy as ϕ satisfying the flow property and such that $\psi_{t,s}$ is Borel measurable for any $(s,t) \in \mathbb{T}_+^2$.*

Remark 3.11. Proving such a result with a general Banach space V is false as even existence of solutions to ordinary differential equations may fail [17, 23].

Remark 3.12. To prove Theorem 3.10, we show that $(G^{L_T}(r,a))_{r\in[0,+\infty),a\in\mathrm{V}}$ is a family of abstract local integral funnels in the sense of Definition 3.6. Then, we use Theorem 2 of [11].

Lemmas 3.13-3.18 prove that $G^{L_T}(r,a)$ is a family of local abstract funnels in the sense of the Definition 2 in [11]. Then we apply Theorem 2 in [11] to obtain the above theorem.

Lemma 3.13. *Under Hypothesis 3.8, the terminal time $T(r,a) := T$ is independent of the starting time r and the starting point a. In particular,* H0 *holds for $F = G_\phi^{L_T}$.*

Proof. It is sufficient to notice that the constants $\varkappa$ and δ_T do not depend on $a \in \mathrm{V}$ neither on r. $\square$

We recall that $S_\phi(r,a)$ is defined in Notation 2.9. Our first result is that when $S_\phi(r,a) \neq \emptyset$, then there exists at least one D-solution in $G_\phi^{L_T}(r,a)$.

Lemma 3.14. *Assume that $K \geq K_T L_T$ in Definition 3.1, where K_T and L_T are constants in Theorem 2.5. For any $(r,a) \in \mathbb{T} \times \mathrm{V}$, $S_\phi(r,a) \subset G_\phi^K(r,a)$ for an almost flow ϕ (note that $S_\phi(r,a)$ may be empty).*

Proof. If $y \in S_\phi(r,a)$ when $S_\phi(r,a) \neq \emptyset$, then there exists a sequence $\{\pi_k\}_{k\in\mathbb{N}}$ of partitions such that $y_t = \lim \phi_{t,r}^{\pi_k}(a)$ uniformly in $t \in [r,T]$. We note that $y_r = a$. For $k \in \mathbb{N}$ and $s_k \in \pi_k$, with (1.9) and Theorem 2.5,

$$\begin{aligned} d(\phi_{t,r}^{\pi_k}(a), \phi_{t,s_k} \circ \phi_{s_k,r}^{\pi_k}(a)) &= d(\phi_{t,s_k}^{\pi_k} \circ \phi_{s_k,r}^{\pi_k}(a), \phi_{t,s_k} \circ \phi_{s_k,r}^{\pi_k}(a)) \\ &\leq L_T N_\gamma(\phi_{s_k,r}^{\pi_k}(a)) \varpi(\omega_{s_k,t}) \\ &\leq L_T K_T N_\gamma(a) \varpi(\omega_{s_k,t}). \end{aligned} \tag{3.2}$$

Moreover, fixing $s \in [r,T]$ and using (2.2),

$$\begin{aligned} &d(\phi_{t,s_k} \circ \phi_{s_k,r}^{\pi_k}(a), \phi_{t,s}(y_s(a))) \\ &\qquad \leq d(\phi_{t,s_k} \circ \phi_{s_k,r}^{\pi_k}(a), \phi_{t,s_k} \circ y_s(a)) + d(\phi_{t,s_k} \circ y_s(a), \phi_{t,s} \circ y_s(a)) \\ &\qquad \leq (1+\delta_T) d(\phi_{s_k,r}^{\pi_k}(a), y_s(a)) + \eta(\omega_{0,T}) d(\phi_{s_k,r}^{\pi_k}(a), y_s(a))^\gamma \\ &\qquad\qquad + d(\phi_{t,s_k} \circ y_s(a), \phi_{t,s} \circ y_s(a)). \end{aligned} \tag{3.3}$$

Choosing $\{s_k\}_{k\in\mathbb{N}}$ so that s_k decreases to s and passing to the limit, we obtain with (3.3) that $\phi_{t,s_k} \circ \phi_{s_k,r}^{\pi_k}(a)$ converges uniformly to $\phi_{t,s} \circ y_s(a)$. Thus, when $k \to +\infty$, (3.2) shows that y is a D-solution. $\square$

Lemma 3.15. *Under Hypothesis 1, $G_\phi^{L_T}(r,a)$ is a non-empty compact subset of the set of paths $y \in \mathcal{C}(\mathbb{S},\mathrm{V})$ such that $y_r = a$ for any $r \in \mathbb{T}$ and $a \in \mathrm{V}$. It shows that H1 holds for $F := G_\phi^{L_T}$.*

Proof. It follows directly from Proposition 2.11 and Lemma 3.14 (with $K_T = 1$ and $K = L_T$) that $G_\phi^K(r,a)$ is not empty. Now, if $\{y^k\}_k$ is a sequence in $G_\phi^K(r,a)$ then $\{y^k\}_k$ is equi-continuous with the same argument as in the proof of Proposition 2.11. The subsequence of $\{y^k\}_k$ converges in $G_\phi^K(r,a)$ because $a \in \mathrm{V} \mapsto \phi_{t,s}(a)$ is continuous for any $(s,t) \in \mathbb{T}_2^+$. $\square$

Let us denote $\tilde{G}_\phi^{L_T}(r,a)$, the set of paths $y \in G_\phi^{L_T}(r,a)$ reparametrised on $[0,1]$ as $t \in [0,1] \mapsto \tilde{y}_t := y_{r+t(T-r)}$.

Lemma 3.16. *Let us assume Hypothesis 3.8. Let $r \geq 0$, for any closed subset $\mathcal{K} \subset \mathcal{C}([0,1],\mathrm{V})$, the set $S'(r) := \{a \in \mathrm{V} \mid \tilde{G}_\phi^{L_T}(r,a) \bigcap \mathcal{K} \neq \emptyset\}$ is closed in V, in particular it is a Borel set in V. It shows that H2 holds for $F = G_\phi^{L_T}$.*

Proof. Let $\{a_k\}_{k\in\mathbb{N}}$ be a convergent sequence of $S'(r)$. For each $k \in \mathbb{N}$, we choose a path $\tilde{y}^k \in \tilde{G}_\phi^{L_T}(r,a_k) \bigcap \mathcal{K}$ (which is not empty by definition). Then, for every $s,t \in [0,1]$, $s \leq t$,

$$d(\tilde{y}_t^k, \tilde{y}_s^k) \leq d(\tilde{y}_t^k, \phi_{\tilde{t},\tilde{s}}(\tilde{y}_s^k)) + d(\phi_{\tilde{t},\tilde{s}}(\tilde{y}_s^k), \tilde{y}_s^k) \leq [L_T \varpi(\omega_{\tilde{s},\tilde{t}}) + \delta_{\tilde{t}-\tilde{s}}] \|N\|_\infty, \tag{3.4}$$

where $\tilde{t} := r + t(T-r)$ and $\tilde{s} := r + s(T-r)$. Since $\tilde{t} - \tilde{s}$ goes to zero when $t - s \to 0$, it follows that $\{t \in [0,1] \mapsto \tilde{y}_t^k\}_{k\in\mathbb{N}}$ is equi-continuous.

The sequence $\{a_k\}_{k\in\mathbb{N}}$ converges, so it is bounded by a constant $A \geq 0$. Applying (3.4) between $s = 0$ and t, we get $|\tilde{y}_t^k| \leq (L_T\varpi(\omega_{0,1}) + \delta_T)\|N\|_\infty + A$, which proves that $t \in [0,1] \mapsto y^k$ is uniformly bounded.

By Ascoli-Arzelà theorem, there is a convergent subsequence $\{\tilde{y}^{k_i}\}_{i\in\mathbb{N}}$ in $(\mathcal{C}([0,1],\mathrm{V}), \|\cdot\|_\infty)$ to a path y. This path belongs to $\mathcal{K}$ since $\mathcal{K}$ is closed. Because $\phi_{t,s}$ is continuous, $\tilde{y} \in \tilde{G}_\phi^{L_T}(r,a)$. Hence $S'(r)$ is closed and then Borel. □

The proof of the next lemma is an immediate consequence of the definition of D-solutions.

Lemma 3.17. *If $t \in [r, r+T] \mapsto y_t(r,a)$ belongs to $G_\phi^{L_T}(r,a)$, then for any $r' \geq 0$, its restriction $t \in [r+r', r+r'+T] \mapsto y_t(r,a)$ belongs to $G_\phi^{L_T}(r+r', y_{r+r'}(r,a))$. It shows that* H3 *holds for $F = G_\phi^{L_T}$.*

Lemma 3.18. *We assume that Hypothesis 3.8 hold. For $r' \geq 0$, if $y \in G_\phi^{L_T}(r,a)$ and $z \in G_\phi^{L_T}(r+r', y_{r+r'}(r,a))$, then $y \bowtie_{r+r'} z \in G_\phi^{L_T}(r,a)$. It shows that* H4 *holds for $F := G_\phi^{L_T}$*

Proof. Let us write $x := y \bowtie_s z$ where $s := r + r'$ and $U_{\tau,t} := d(x_t, \phi_{t,\tau}(x_\tau))$ for $\tau \leq t$. On the one hand, for any $r \leq \tau \leq s \leq t$ with (3.1), (2.3) and (2.4),

$$\begin{aligned} U_{\tau,t} &\leq d(x_t, \phi_{t,s}(x_s)) + d(\phi_{t,s}(x_s), \phi_{t,s} \circ \phi_{s,\tau}(x_\tau)) + d(\phi_{t,s} \circ \phi_{s,\tau}(x_\tau), \phi_{t,\tau}(x_\tau)) \\ &\leq \|N\|_\infty (2 + \delta_T) L_T \varpi(\omega_{\tau,t}). \end{aligned} \tag{3.5}$$

On the other hand, for $s \leq \tau \leq t$ or $\tau \leq t \leq s$ with (3.1)

$$U_{\tau,t} \leq \|N\|_\infty L_T \varpi(\omega_{\tau,t}). \tag{3.6}$$

Thus, combining (3.5) and (3.6), for any $r \leq \tau \leq t \leq T$,

$$U_{\tau,t} \leq \|N\|_\infty (2 + \delta_T) L_T \varpi(\omega_{\tau,t}).$$

Besides, for any $r \leq \tau \leq u \leq t \leq T$ with (2.3) and (2.4),

$$U_{\tau,t} \leq U_{u,t} + (1 + \delta_T) U_{\tau,u} + \|N\|_\infty \varpi(\omega_{\tau,t}). \tag{3.7}$$

Let $\lambda \in (0,1)$ such that ϖ^λ satisfies (1.6) with $\varkappa_\lambda := 2^{1-\lambda}\varkappa^\lambda < 1$. Let $T(\lambda) > 0$ be a real number such that $\varkappa_\lambda(1 + \delta_{T(\lambda)}) < 1$. For any, two successive points τ, t of a subdivision π,

$$U_{\tau,t} \leq D_{L_T}(\pi, \lambda) \varpi^\lambda(\omega_{\tau,t}), \tag{3.8}$$

where $D_{L_T}(\pi,\lambda) := \|N\|_\infty (2 + \delta_T) L_T \sup_{\tau,t \text{ successive points of } \pi} \varpi^{1-\lambda}(\omega_{\tau,t})$.

Let us show by induction over the distance m between points τ and t in $\pi \cap [0, T(\lambda)]$ that

$$U_{\tau,t} \leq A_{L_T}(\pi, \lambda) \varpi^\lambda(\omega_{\tau,t}), \tag{3.9}$$

where

$$A_{L_T}(\pi,\lambda) := \frac{D_{L_T}(\pi,\lambda)(1 + \delta_{T(\lambda)}) + 2\|N\|_\infty \varpi^\lambda(\omega_{0,T(\lambda)})}{1 - \varkappa_\lambda(1 + \delta_{T(\lambda)})}.$$

When $m = 0$, $U_{\tau,\tau} = 0$ so that (3.9) holds. For $m = 1$, τ and t are successive points then (3.9) holds with (3.8). Now, we assume that (3.9) holds for any two points at distance m. Let τ and t be two points at distance $m+1$ in $\pi \cap [0, T(\lambda)]$. Since ω is super-additive, one may choose two successive points s and s' in π with $\tau < s < s' < t$ such that $\omega_{\tau,s} \leq \omega_{\tau,t}/2$

and $\omega_{s',t} \leq \omega_{\tau,t}/2$, as in the proof of Theorem 2.5. Then, by applying (3.7) between (τ, s, s') and (s, s', t) we obtain,

$$\begin{aligned} U_{\tau,t} &\leq U_{s,t} + (1+\delta_{T(\lambda)})U_{\tau,s} + \|N\|_\infty \varpi^{1-\lambda}(\omega_{0,T(\lambda)})\varpi^\lambda(\omega_{\tau,t}) \\ &\leq U_{s',t} + (1+\delta_{T(\lambda)})U_{s,s'} + (1+\delta_{T(\lambda)})U_{\tau,s} + 2\|N\|_\infty \varpi^{1-\lambda}(\omega_{0,T(\lambda)})\varpi^\lambda(\omega_{\tau,t}) \\ &\leq [A_{L_T}(\pi,\lambda)\varkappa_\lambda(1+\delta_{T(\lambda)}) + (1+\delta_{T(\lambda)})D_{L_T}(\pi,\lambda) + 2\|N\|_\infty \varpi^{1-\lambda}(\omega_{0,T(\lambda)})]\varpi^\lambda(\omega_{\tau,t}) \\ &\leq A_{L_T}(\pi,\lambda)\varpi^\lambda(\omega_{\tau,t}), \end{aligned}$$

with our choice of $A_{L_T}(\pi,\lambda)$. This concludes the induction, so (3.9) holds for any $\tau, t \in \pi \cap [0, T(\lambda)]^2$.

Clearly, $D_{L_T}(\pi,\lambda) \to 0$ when the mesh of π goes to zero. Then, $A_{L_T}(\pi,\lambda) \to A(\lambda) := 2\|N\|_\infty \varpi^\lambda(\omega_{0,T(\lambda)})/(1-\varkappa_\lambda(1+\delta_{T(\lambda)}))$ when the mesh of π goes to zero.

By continuity of $(\tau,t) \mapsto U_{\tau,t}$, considering finer and finer partitions leads to $U_{\tau,t} \leq A(\lambda)\varpi^\lambda(\omega_{\tau,t})$ for any $r \leq \tau \leq t \leq T(\lambda)$.

Finally, choosing $T(\lambda)$ so that $T(\lambda)$ increases to T defined in Hypothesis 3.8 when λ goes to 1, we conclude that for any $r \leq \tau \leq t \leq T$,

$$U_{\tau,t} \leq \|N\|_\infty L_T \varpi(\omega_{\tau,t}),$$

where L_T is defined in Hypothesis 3.8. This proves that $z \in G_\phi^{L_T}(r,a)$. □

Proof of Theorem 3.10. Lemma 3.13-3.18 prove that conditions H0-H4 of Definition 3.6 hold for $F = G_\phi^{L_T}$. This means that $G^{L_T}(r,a)$ is a family of abstract local integral funnels. We apply Theorem 1 in [11]. For any $(r,a) \in \mathbb{T} \times \mathrm{V}$, there exists a measurable map $a \mapsto (t \mapsto \psi_{t,r}(a))$ with respect to the Borel subsets of $\mathcal{C}^0(\mathbb{T},\mathrm{V})$ with the property that $\psi_{r,r}(a) = a$ and $\psi_{t,s} \circ \psi_{s,r}(a) = \psi_{t,r}(a)$, $t \geq r$. □

4 Lipschitz flows

A Lipschitz almost flow which has the flow property is said to be a Lipschitz flow. We recast the definition.

Definition 4.1 (Lipschitz flow)**.** *A flow $\psi \in \mathcal{F}(\mathrm{V})$ is said to be a* Lipschitz flow *if for any $(s,t) \in \mathbb{T}_2^+$, $\psi_{t,s}$ is Lipschitz in space with $\|\psi_{t,s}\|_{\mathrm{Lip}} \leq 1 + \delta_T$.*

In this section, we consider galaxies that contain a Lipschitz flow.

We prove that such a Lipschitz flow ψ is the only possible flow in the galaxy (Theorem 4.5), and that the iterated almost flow ϕ^π of any almost flow ϕ converges to ψ (Theorem 4.2). We also characterize the rate of convergence (Theorem 4.3).

Let us choose $\lambda \in (0,1)$ such that ϖ^λ satisfies the same properties as ϖ up to changing $\varkappa$ to $\varkappa_\lambda := 2^{1-\lambda}\varkappa^\lambda$, provided $\varkappa_\lambda < 1$. This is possible as soon as $\lambda > 1/(1 - \log_2(\varkappa))$ with (1.6).

Clearly, if for $\psi, \chi \in \mathcal{F}(\mathrm{V})$, $\Delta_{N,\varpi}(\psi,\chi) < +\infty$, then

$$\Delta_{N,\varpi^\lambda}(\psi,\chi) \leq \Delta_{N,\varpi}(\psi,\chi)\varpi^{1-\lambda}(\omega_{0,T}) < +\infty, \tag{4.1}$$

where ϖ is defined by (1.6). Hence, the galaxies remain the same when ϖ is changed to ϖ^λ. We define

$$\Theta(\pi) := \sup_{d_\pi(s,s')=1} \varpi^{1-\lambda}(\omega_{s,s'}). \tag{4.2}$$

Theorem 4.2. *Let ϕ be an almost flow such that $\|\phi^\pi\|_{\mathrm{Lip}} \leq 1+\delta_T$ whatever the partition π, we say that ϕ satisfies the uniform Lipschitz (UL) condition. Then there exists a Lipschitz flow $\zeta \in \mathcal{F}(\mathrm{V})$ with $\|\zeta_{s,t}\|_{\mathrm{Lip}} \leq 1 + \delta_T$ such that $\{\phi^\pi\}$ converges to ζ as $|\pi| \to 0$.*

Theorem 4.3. *Let ϕ be an almost flow and ψ be a Lipschitz flow with $\psi \sim \phi$. Then there exists a constant K that depends only on λ, $\Delta_{N,\varpi}(\phi,\psi)$, $\varkappa$ and T (assumed to be small enough) so that*

$$\Delta_{N,\varpi^\lambda}(\psi,\phi^\pi) \leq K\Theta(\pi).$$

In particular, $\{\phi^\pi\}_\pi$ converges to ψ as $|\pi| \to 0$.

Remark 4.4. In [9, 10], we develop the notion of *stable almost flow* around a necessary condition for an almost flow to be associated to a Lipschitz flow. Under such a condition, a stronger rate of convergence may be achieved (see [10]) by taking

$$\Theta(\pi) := \sup_{d_\pi(s,s')=1} \varpi(\omega_{s,s'})/\omega_{s,s'}.$$

Theorem 4.5 (Uniqueness of Lipschitz flows)**.** *If ψ is a Lipschitz flow and χ is a flow (not necessarily Lipschitz a priori) in the same galaxy as ψ, that is $\chi \sim \psi$, then $\chi = \psi$.*

Hypothesis 4.6. *Let us fix a partition π. We consider ψ and χ in $\mathcal{F}(\mathrm{V})$ such that $\psi \sim \chi$ and for any $(r,s,t) \in \pi^3_+$,*

$$\|\psi_{t,s}\|_{\mathrm{Lip}} \leq 1 + \delta_T, \tag{4.3}$$

$$N(\chi_{t,s}(a)) \leq (1+\delta_T)N(a),\ \forall a \in \mathrm{V}, \tag{4.4}$$

$$\Delta_N(\psi_{t,s}\circ\psi_{s,r},\psi_{t,r}) \leq \beta_\psi\varpi(\omega_{r,t}) \text{ and } \Delta_N(\chi_{t,s}\circ\chi_{s,r},\chi_{t,r}) \leq \beta_\chi\varpi(\omega_{r,t}), \tag{4.5}$$

for some constant $\beta_\chi, \beta_\phi \geq 0$.

Remark 4.7. In Hypothesis 4.6, the role of ψ and χ are not exchangeable: ψ is assumed to be Lipschitz, there is no such requirement on χ. The reason of this dissymmetry lies in (4.8).

Remark 4.8. If ψ is a Lipschitz almost flow and χ is an almost flow, then (ψ,χ) satisfies Hypothesis 4.6 for any partition π. The condition (4.4) is a particular case of (2.6).

We choose λ and T so that

$$\frac{1}{1-\log_2(\varkappa)} < \lambda < 1 \text{ and } 3\delta_T + \delta_T^3 < 2\frac{1-\varkappa_\lambda}{\varkappa_\lambda}.$$

We define (recall that $\Theta(\pi)$ is given by (4.2)),

$$\rho_T := \varpi(\omega_{0,T})^{1-\lambda},$$

$$\gamma(\pi) := \sup_{d_\pi(s,s')=1}\frac{\Delta_N(\psi,\chi)}{\varpi^\lambda(\omega_{s,s'})} \leq \Delta_{N,\varpi^\lambda}(\psi,\chi)\Theta(\pi),$$

$$\beta(\pi) := (2+3\delta_T+\delta_T^2)(\beta_\psi+\beta_\chi)\rho_T + (1+\delta_T)^2\gamma(\pi) \geq \gamma(\pi),$$

$$\text{and } L(\pi) := \frac{2\beta(\pi)}{2-\varkappa_\lambda(2+3\delta_T+\delta_T^2)} \geq \gamma(\pi).$$

Here, $\Theta(\pi)$ and thus $\gamma(\pi)$ converge to zero when the mesh of π tends to zero.

Lemma 4.9. *Let $\phi, \chi \in \mathcal{F}(\mathrm{V})$ and π be satisfying Hypothesis 4.6. With the above choice of λ and T, it holds that*

$$d(\phi_{t,r}(a),\chi_{t,r}(a)) \leq L(\pi\cup\{t,r\})N(a)\varpi^\lambda(\omega_{r,t}),\ \forall (r,t) \in \mathbb{T}^2_+. \tag{4.6}$$

Proof. We set $F_{r,t} := \Delta_N(\psi_{t,r},\chi_{t,r})$, where Δ_N is defined in Notation 1.3. From Definition 1.4, $F_{r,t} \leq \Delta_{N,\varpi}(\psi,\chi)\varpi(\omega_{r,t}) < +\infty$ since $\psi \sim \chi$.

In particular, for $(r,s,t) \in \pi^3_+$, with (4.4) in Hypothesis 4.6,

$$d(\psi_{t,s}\circ\chi_{s,r}(a),\chi_{t,s}\circ\chi_{s,r}(a)) \leq F_{s,t}N(\chi_{s,r}(a)) \leq (1+\delta_T)N(a)F_{s,t}. \tag{4.7}$$

For any $(r,s,t)\in\pi_3^+$, the fact that ϕ, χ are almost flow combined with (4.3)-(4.5) and (4.7) imply that for any $a\in\mathrm{V}$,

$$\begin{aligned}
&d(\psi_{t,r}(a),\chi_{t,r}(a))\\
&\leq d(\psi_{t,s}\circ\psi_{s,r}(a),\psi_{t,s}\circ\chi_{s,r}(a))+d(\psi_{t,s}\circ\chi_{s,r}(a),\chi_{t,s}\circ\chi_{s,r}(a))+(\beta_\phi+\beta_\chi)N(a)\varpi(\omega_{r,t})\\
&\qquad\leq(1+\delta_T)N(a)F_{r,s}+(1+\delta_T)N(a)F_{s,t}+(\beta_\chi+\beta_\phi)N(a)\varpi(\omega_{r,t}).
\end{aligned}\tag{4.8}$$

Thus, dividing by $N(a)$,

$$F_{r,t}\leq(1+\delta_T)(F_{r,s}+F_{s,t})+(\beta_\chi+\beta_\psi)\rho_T\varpi^\lambda(\omega_{r,t}).\tag{4.9}$$

We proceed by induction. Our hypothesis is that

$$F_{r,t}\leq L(\pi)\varpi^\lambda(\omega_{r,t}),\forall(r,t)\in\pi_2^+,\text{ at distance at most }m.\tag{4.10}$$

When $m=0$, $F_{r,r}=0$ since $\psi_{r,r}(a)=\chi_{r,r}(a)=a$ for any $a\in\mathrm{V}$. Thus (2.7) is true for $m=0$. When $m=1$, r and t are successive points. From the very definition of $\gamma(\pi)$,

$$F_{r,t}\leq\gamma(\pi)\varpi^\lambda(\omega_{r,t}).\tag{4.11}$$

The induction hypothesis (2.7) is true for $m=1$ since $L(\pi)\geq\gamma(\pi)$.

Assume that the induction hypothesis is true at some level $m\geq1$. Let $(r,s,t)\in\pi_3^+$ with $r<s<t$ and $d_\pi(r,t)=m+1$. Let s' be such that s and s' are successive points in π (possibly, $s'=t$). Clearly, $d_\pi(r,s)\leq m$ and $d_\pi(s',t)\leq m$. Using (4.9) to decompose $F_{s,t}$ using s' and using (4.11),

$$\begin{aligned}
F_{r,t}\leq(1+\delta_T)F_{r,s}+(1+\delta_T)^2F_{s',t}+(1+\delta_T)^2\gamma(\pi)\varpi^\lambda(\omega_{s,s'})\\
+(2+3\delta_T+\delta_T^2)(\beta_\psi+\beta_\chi)\rho_T\varpi^\lambda(\omega_{r,t})\\
\leq(1+\delta_T)F_{r,s}+(1+\delta_T)^2F_{s',t}+\beta(\pi)\varpi^\lambda(\omega_{r,t}).
\end{aligned}$$

With the induction hypothesis, since r and s (resp. s' and t) are at distance at most m,

$$F_{r,t}\leq L(\pi)(1+\delta_T)\varpi^\lambda(\omega_{r,s})+L(\pi)(1+\delta_T)^2\varpi^\lambda(\omega_{s',t})+\beta(\pi)\varpi^\lambda(\omega_{r,t}).$$

Choosing s and s' to satisfy (2.12), our choice of $L(\pi)$ and (1.6) imply that

$$F_{r,t}\leq\left(L(\pi)\frac{2+3\delta_T+\delta_T^2}{2}\varkappa_\lambda+\beta(\pi)\right)\varpi^\lambda(\omega_{r,t})\leq L(\pi)\varpi^\lambda(\omega_{r,t}).$$

The induction hypothesis (4.10) is then true at level $m+1$, and then whatever the distance between the points of the partition.

Finally, (4.6) is obtained by replacing π by $\pi\cup\{r,t\}$. □

Proof Theorem 4.2. Let σ and π be two partitions with $\pi\subset\sigma$. We set $\psi:=\phi^\sigma$ and $\chi:=\phi^\pi$.

With Theorem 2.5,

$$\Delta_{N,\varpi}(\phi^\sigma,\phi^\pi)\leq\Delta_{N,\varpi}(\phi^\sigma,\phi)+\Delta_{N,\varpi}(\phi^\pi,\phi)\leq2L_T.$$

With (4.1), $\Delta_{N,\varpi^\lambda}(\psi,\chi)\leq2L_T\rho_T$, so that $\{\Delta_{N,\varpi^\lambda}(\psi,\chi)\}_{\pi,\sigma}$ is bounded.

Again with Theorem 2.5, (ψ,χ) satisfies Hypothesis 4.6 for the subdivision π (up to changing δ_T) with $\beta_\psi=\beta_\chi=0$.

Hence, $L(\pi)=C\gamma(\pi)$ where

$$C:=\frac{2(1+\delta_T)^2}{(2-\varkappa_\lambda(2+3\delta_T+\delta_T^2)}.\tag{4.12}$$

We may then rewrite (4.6) as

$$d(\phi^{\sigma}_{t,r}(a), \phi^{\pi}_{t,r}(a)) \leq C\gamma(\pi \cup \{r,t\})N(a)\varpi^{\lambda}(\omega_{r,t}). \tag{4.13}$$

Since $\gamma(\pi)$ decreases to 0 as $|\pi|$ decreases to 0 and $|\pi \cup \{r,t\}| \leq |\pi|$, it is easily shown that $\{\phi^{\pi}_{t,s}\}_{\pi}$ forms a Cauchy net with respect to the nested partitions. Then, it does converges to a limit $\zeta_{s,t}(a)$. By Theorem 4.9 and the continuity of N, $N(\zeta_{s,r}) \leq K_T N(a)$. From the UL condition, $a \mapsto \zeta_{t,s}(a)$ is Lipschitz continuous with $\|\zeta_{t,s}\|_{\text{Lip}} \leq 1 + \delta_T$.

Moreover ζ does not depend on the subdivision π. Indeed, if $\tilde{\pi}$ is another subdivision, we obtain with (4.13), that $\{\phi^{\tilde{\pi}}\}_{\tilde{\pi}}$ converges to ζ when $|\tilde{\pi}| \to 0$.

Finally, if if $\{\pi_k\}_{k\geq 0}$ is a family of nested partitions, and $(r,s,t) \in \mathbb{T}^+_3$,

$$\phi^{\pi_k \cup \{s\}}_{t,r} = \phi^{\pi_k}_{t,s} \circ \phi^{\pi_k}_{s,r}.$$

Because $|\pi_k \cup \{s\}| \leq |\pi_k|$ and (4.13), $\phi^{\pi_k \cup \{s\}}$ converges to ζ when $k \to +\infty$. Moreover, for any $a \in \mathrm{V}$,

$$\begin{aligned}
&d(\zeta_{t,s} \circ \zeta_{s,r}(a), \phi^{\pi_k}_{t,s} \circ \phi^{\pi_k}_{s,r}(a)) \\
&\leq d(\zeta_{t,s} \circ \zeta_{s,r}(a), \phi^{\pi_k}_{t,s} \circ \zeta_{s,r}(a)) + d(\phi^{\pi_k}_{t,s} \circ \zeta_{s,r}(a), \phi^{\pi_k}_{t,s} \circ \phi^{\pi_k}_{s,r}(a)) \\
&\leq C\gamma(\pi_k \cup \{r,t\})N(\zeta_{s,r}(a))\varpi^{\lambda}(\omega_{r,t}) + (1+\delta_T)d(\zeta_{s,r}(a), \phi^{\pi_k}_{s,r}(a)) \\
&\leq C\gamma(\pi_k \cup \{r,t\})(1+K_T)N(a)\varpi(\omega_{r,t}),
\end{aligned}$$

because $N(\zeta_{s,r}(a)) \leq K_T N(a)$. So, $\{\phi^{\pi_k}_{t,s} \circ \phi^{\pi_k}_{s,r}\}_{\pi_k}$ converges uniformly to $\zeta_{t,s} \circ \zeta_{s,r}$ when $m \to +\infty$. Then, the flow property $\zeta_{t,s} \circ \zeta_{s,r} = \zeta_{t,r}$ holds. □

Proof Theorem 4.3. For a partition π, the pair (ψ, ϕ^{π}) satisfies Hypothesis 4.6 for the subdivision π with $\beta_{\psi} = \beta_{\chi} = 0$. As in the proof of Theorem 4.2 (we have assumed for convenience that $\Delta_{N,\varpi}(\phi, \psi) \leq L_T$),

$$\Delta_{N,\varpi^{\lambda}}(\psi, \phi^{\pi}) \leq C\gamma(\pi) \leq 2CL_T\rho_T\Theta(\pi)$$

for C given by (4.12). This proves the result. □

Proof Theorem 4.5. For any partition π, ψ and χ satisfy Hypothesis 4.6 with $\beta_{\psi} = \beta_{\chi} = 0$. Thus,

$$\Delta_{N,\varpi^{\lambda}}(\psi, \chi) \leq C\gamma(\pi)$$

with C given by (4.12). As $\gamma(\pi)$ decreases to 0 when $|\pi|$ decreases to 0, we obtain that $\psi = \chi$. □

Corollary 4.10. *Let ψ and χ be two almost flows with $\psi \sim \chi$ and ψ be Lipschitz. Then for T small enough (in function of some $\lambda < 1$, $\varkappa$ and δ)*

$$\Delta_{N,\varpi}(\psi, \chi) \leq \frac{2(2 + 3\delta_T + \delta_T^2)(\beta_{\psi} + \beta_{\chi})}{2 - \varkappa_{\lambda}(2 + 3\delta_T + \delta_T^2)}.$$

Proof. With Remark 4.8, (ψ, χ) satisfies Hypothesis 4.6. Letting the mesh of the partition decreasing to 0 as the in proof of Theorem 4.5, and then letting λ increasing to 1 leads to the result. □

5 Perturbations

In this section, we consider the construction of an almost flow by perturbations of existing ones. We assume that V is a Banach space.

Let $\phi \in \mathcal{F}(\mathrm{V})$ be an almost flow with respect to a function N_γ such that $N_\gamma(a) \geq N_\gamma(0) \geq 1$.

Notation 5.1. *For $\phi \in \mathcal{F}(\mathrm{V})$ when V is a Banach space, we write*

$$\phi_{t,s,r}(a) := \phi_{t,s}(\phi_{s,r}(a)) - \phi_{t,r}(a).$$

Definition 5.2. *Let $\epsilon \in \mathcal{F}(\mathrm{V})$ such that for any $(s,t) \in \mathbb{T}_2^+$, $a, b \in \mathrm{V}$,*

$$\epsilon_{t,t} \equiv 0, \tag{5.1}$$

$$|\epsilon_{t,s}(a)| \leq \lambda N_\gamma(a)\varpi(\omega_{s,t}), \tag{5.2}$$

$$|\epsilon_{t,s}(b) - \epsilon_{t,s}(a)| \leq \eta(\omega_{s,t})|b-a|^\gamma, \tag{5.3}$$

for some $\lambda \geq 0$. We say that ϵ is a perturbation.

Proposition 5.3. *If $\phi \in \mathcal{F}(\mathrm{V})$ is an almost flow and $\epsilon \in \mathcal{F}(\mathrm{V})$ is a perturbation, then $\psi := \phi + \epsilon$ is an almost flow. Besides, $\psi \sim \phi$.*

Proof. Let $(r,s,t) \in \mathbb{T}_3^+$ and $a, b \in \mathrm{V}$. From (5.1), (2.1) is satisfied. With $\delta'_T := \delta_T + \lambda\varpi(\omega_{0,T})$, (2.2) is also true. In addition, with (5.3),

$$|\psi_{t,s}(b) - \psi_{t,s}(a)| \leq (1+\delta_T)|b-a| + 2\eta(\omega_{s,t})|b-a|^\gamma.$$

Thus, ψ satisfies (2.3).

To show (2.4), we write

$$\begin{aligned}\psi_{t,s,r}(a) = \phi_{t,s}\circ\psi_{s,r}(a) + \epsilon_{t,s}\circ\psi_{s,r}(a) - \phi_{t,r}(a) - \epsilon_{t,s}(a)\\ = \underbrace{\phi_{t,s,r}(a)}_{\mathrm{I}_{r,s,t}} + \underbrace{\phi_{t,s}\circ(\phi_{s,r}+\epsilon_{s,r})(a) - \phi_{t,s}\circ\phi_{s,r}(a)}_{\mathrm{II}_{r,s,t}}\\ + \underbrace{\epsilon_{t,s}\circ(\phi_{s,r}(a)+\epsilon_{s,r}(a)) - \epsilon_{t,s}\circ\phi_{s,r}(a)}_{\mathrm{III}_{r,s,t}} + \underbrace{\epsilon_{t,s}\circ\phi_{s,r}(a) - \epsilon_{t,s}(a)}_{\mathrm{IV}_{r,s,t}}.\end{aligned} \tag{5.4}$$

We control the first term with (2.4), $|\mathrm{I}_{r,s,t}| \leq N_\gamma(a)\varpi(\omega_{r,t})$. For the second one, we use (1.7), (2.3) and (5.2),

$$\begin{aligned}|\mathrm{II}_{r,s,t}| &\leq (1+\delta_T)|\epsilon_{s,r}(a)| + \eta(\omega_{s,r})|\epsilon_{s,r}|^\gamma\\ &\leq (1+\delta_T)\lambda N_\gamma(a)\varpi(\omega_{r,s}) + \eta(\omega_{s,t})\lambda^\gamma N_\gamma^\gamma(a)\varpi(\omega_{r,s})^\gamma\\ &\leq [1+(1+\lambda^\gamma)\delta_T]N_\gamma(a)\varpi(\omega_{r,t}),\end{aligned}$$

because $N_\gamma(a) \geq 1$ implies that $N_\gamma^\gamma(a) \leq N_\gamma(a)$ for $\gamma \in (0,1)$.

With (5.2) and (5.3), we obtain for the third term,

$$|\mathrm{III}_{r,s,t}| \leq \eta(\omega_{s,t})|\epsilon_{s,r}(a)|^\gamma \leq \lambda^\gamma N_\gamma(a)^\gamma\eta(\omega_{s,t})\varpi(\omega_{r,s})^\gamma \leq \lambda^\gamma N_\gamma(a)\delta_T\varpi(\omega_{s,t}),$$

where the last inequality comes from (1.7). And for the last term, we use (2.6) and (5.2),

$$\begin{aligned}|\mathrm{IV}_{r,s,t}| \leq |\epsilon_{t,s}\circ\phi_{s,r}(a)| + |\epsilon_{t,s}(a)| &\leq (N_\gamma(\phi_{s,r}) + N_\gamma(a))\lambda\varpi(\omega_{s,t})\\ &\leq (K_T+1)N_\gamma(a)\lambda\varpi(\omega_{s,t}).\end{aligned}$$

Thus, combining estimations for each four terms of (5.4), we obtain (2.4) which proves that ψ is an almost flow.

Besides,

$$|\psi_{t,s}(a) - \phi_{t,s}(a)| = |\epsilon_{t,s}(a)| \leq \lambda N_\gamma(a)\varpi(\omega_{s,t}),$$

which proves that $\psi \sim \phi$ and concludes the proof. □

6 Applications

On this section, we show that our framework covers former different sewing lemmas.

6.1 The additive sewing lemma

The additive sewing lemma is the key to construct the Young integral [28] and the rough integral [22, 26].

We consider that V is a Banach space with a norm $|\cdot|$. The distance d is $d(a,b) := |b-a|$.

Definition 6.1 (Almost additive functional)**.** *A family* $\{\alpha_{s,t}\}_{(s,t)\in\mathbb{T}_2^+}$ *is an* almost additive functional *if*

$$\alpha_{r,s,t} := \alpha_{r,s} + \alpha_{s,t} - \alpha_{r,t} \text{ satisfies } |\alpha_{r,s,t}| \leq \varpi(\omega_{r,t}),\ \forall (r,s,t) \in \mathbb{T}_3^+.$$

It is an additive functional *if* $\alpha_{r,s,t} = 0$ *for any* $(r,s,t) \in \mathbb{T}_3^+$*.*

Proposition 6.2 (The additive sewing lemma [19, 25])**.** *If* $\{\alpha_{s,t}\}_{(s,t)\in\mathbb{T}_2^+}$ *is an almost additive functional with* $|\alpha_{s,t}| \leq \delta_T$*, there exists an additive functional* $\{\gamma_{s,t}\}_{(s,t)\in\mathbb{T}_2^+}$ *which is unique in the sense that for any constant* $C \geq 0$ *and any additive functional* $\{\beta_{s,t}\}_{(s,t)\in\mathbb{T}_2^+}$*,* $|\beta_{s,t} - \alpha_{s,t}| \leq C\varpi(\omega_{s,t})$ *implies that* $\beta = \gamma$*.*

Proof. Clearly, $\phi_{t,s}(a) = a + \alpha_{s,t}$ is an almost flow which satisfies the UL condition. Hence the result. □

6.2 The multiplicative sewing lemma

Here we recover the results of [14, 19, 26]. We consider now that the metric space V has a monoid structure: there exists a product $ab \in \mathrm{V}$ of two elements $a, b \in \mathrm{V}$. We also assume that there exists a Lipschitz function $N : \mathrm{V} \to [1, +\infty)$ such that

$$d(ac, bc) \leq N(c)d(a,b) \text{ and } d(ca, cb) \leq N(c)d(a,b) \text{ for all } a, b, c \in \mathrm{V}.$$

Definition 6.3. *A family* $\{\alpha_{s,t}\}_{(s,t)\in\mathbb{T}_2^+}$ *is said to be an* almost multiplicative functional *if*

$$d(\alpha_{r,s}\alpha_{s,t}, \alpha_{r,t}) \leq \varpi(\omega_{r,t}),\ \forall (r,s,t) \in \mathbb{T}_+^3.$$

It is a multiplicative functional *if* $\alpha_{r,s}\alpha_{s,t} = \alpha_{r,t}$*.*

Proposition 6.4 (The multiplicative sewing lemma [19])**.** *If* $\{\alpha_{s,t}\}_{(s,t)\in\mathbb{T}_+^2}$ *is an* almost multiplicative functional *then there exists a unique multiplicative functional* $\{\gamma_{s,t}\}_{(s,t)\in\mathbb{T}_+^2}$ *such that any other multiplicative functional* $\{\gamma_{s,t}\}_{(s,t)\in\mathbb{T}_+^2}$ *such that* $d(\beta_{s,t}, \alpha_{s,t}) \leq C\varpi(\omega_{s,t})$ *for any* $(s,t) \in \mathbb{T}_+^2$ *satisfies* $\beta = \gamma$*.*

Proof. For this, it is sufficient to consider $\phi_{t,s}(a) = a\alpha_{s,t}$ which is an almost flow which satisfies the UL condition. □

Remark 6.5. Actually, as for the additive sewing lemma (which is itself a subcase of the multiplicative sewing lemma), we have a stronger statement: No (non-linear) flow satisfies $d(\psi_{t,s}(a), a\alpha_{s,t}) \leq C\varpi(\omega_{s,t})$ except $\{a \mapsto a\beta_{s,t}\}_{(s,t)\in\mathbb{T}_+^2}$ which is constructed as the limit of the products of the $\alpha_{s,t}$ over smaller and smaller intervals.

6.3 The multiplicative sewing lemma in a Banach algebra

Consider now that V has a Banach algebra structure with a norm $|\cdot|$ such that $|ab| \leq |a| \times |b|$ and a unit element 1 (the product of two elements is still denoted by ab).

A typical example is the Banach algebra of bounded operators over a Banach space X.

This situation fits in the multiplicative sewing lemma with $d(a,b) = |a-b|$ and $N(a) = |a|$, $a, b \in \mathrm{V}$. As seen in [14], we have many more properties: continuity, existence of an inverse, Dyson formula, Duhamel principle, ...

In particular, this framework is well suited for considering linear differential equations of type

$$y_{t,s} = a + \int_s^t y_{r,s}\,\mathrm{d}\alpha_r,\ a \in \mathrm{V},\ t \geq s \geq 0$$

for an operator valued path $\alpha : \mathbb{T}_2^+ \to \mathrm{V}$. If α is γ-Hölder with $\gamma > 1/2$, then $\phi_{t,s}(a) = a(1+\alpha_{s,t})$ defines an almost flow which satisfies the condition UL (at the price of imposing some conditions on α, this could be extended to $\gamma < 1/2$).

Defining an "affine flow" $\phi_{t,s}(a) = a(1+\alpha_{s,t}) + \beta_{s,t}$ where both α and β are γ-Hölder with $\gamma > 1/2$, the associated flow ψ is such that $\psi_{t,s}(a)$ is solution to the perturbed equation

$$\psi_{t,s}(a) = a + \int_s^t \psi_{r,s}(a)\,\mathrm{d}\alpha_r + \beta_{s,t}.$$

This gives an alternative construction to the one of [14] where a *backward integral* between β and α was defined in the style of the Duhamel formula. All these results are extended to the rough case $1/3 < \gamma \leq 1/2$.

Example 6.6 (Lyons extension theorem)**.** With the tensor product $\otimes$ as product and a suitable norm, for any integer k, the tensor algebra

$$\mathrm{T}_k(\mathrm{X}) := \mathbb{R} \oplus \mathrm{X} \oplus \mathrm{X}^{\otimes 2} \oplus \cdots \oplus \mathrm{X}^{\otimes k}$$

is a Banach algebra. Chen series of iterated integrals (and then rough paths) take their values in some space $\mathrm{T}_k(\mathrm{X})$. The Lyons extension theorem states that any rough path $\mathbf{x}$ of finite p-variation with values in $\mathrm{T}_k(\mathrm{X})$ for some $k \geq \lfloor p \rfloor$ is uniquely extended to a rough path with values in $\mathrm{T}_\ell(\mathrm{X})$ for any $\ell \geq k$, which leads to the concept of *signature* [25, 26]. This follows $a \mapsto a \otimes \mathbf{x}_{s,t}$ as an almost flow which satisfies the UL condition (see also [19] and also [14]).

6.4 Rough differential equation

Now, we show that our construction is related to the one of A.M. Davie [16]. The main idea of Davie was to construct solutions as paths $y : \mathbb{T} \to \mathrm{V}$ that satisfies (3.1) for a suitable "algorithm" $\phi_{t,s}$ of the solution between s and t. Solutions passing through a at 0 are then constructed as limit of using $\{\phi^\pi_{t,0}(a)\}_\pi$ (See Proposition 2.11). The algorithm $\phi_{t,s}$ is given by a truncated Taylor expansion of the solution of (1.1). The number of terms to consider in the Taylor expansion depends directly on the regularity of x. In the Young case one term is needed whereas in the rough case two terms are required.

In this section, we show that the algorithms provided in [16] are almost flows under the same regularity on the vector field f and the path x. Not only we recover existence of D-solutions, but we also show that measurable flows exist when the vector fields f are $\mathcal{C}^1_\mathrm{b}$ (Young case) or $\mathcal{C}^2_\mathrm{b}$ (rough case) in situation in which non-uniqueness of solutions is known to hold, again due to [16], unless f is of class $\mathcal{C}^{1+\gamma}_\mathrm{b}$ (Young case) or $\mathcal{C}^{2+\gamma}_\mathrm{b}$ (rough case).

Here U and V are two Banach spaces, where we use the same notation $|\cdot|$ for their norms. We denote by $\mathcal{L}(\mathrm{U},\mathrm{V})$ the continuous linear maps from U to V. Let f be a map from V to $\mathcal{L}(\mathrm{U},\mathrm{V})$. If f is regular, we denote its Fréchet derivative in $a \in \mathrm{V}$, $\mathrm{d}f(a) \in \mathcal{L}(V, \mathcal{L}(U,V))$.

Moreover, for any $a \in \mathrm{W}$ and $(r,s,t) \in \mathbb{T}_3^+$, we set $\phi_{t,s,r}(a) := \phi_{t,s} \circ \phi_{s,r}(a) - \phi_{t,r}(a)$.

6.4.1 Almost flow in the Young case

Let $x : \mathbb{T} \to \mathrm{U}$ be a path of finite p-variation controlled by ω with $1 \leq p < 2$.

We define a family $(\phi_{t,s})_{(s,t)\in\mathbb{T}_2^+}$ in $\mathcal{F}(\mathrm{V})$ such that for all $a \in \mathrm{V}$ and $(s,t) \in \mathbb{T}_2^+$,

$$\phi_{t,s}(a) := a + f(a)x_{s,t}, \tag{6.1}$$

where $x_{s,t} := x_t - x_s$.

Proposition 6.7. *Assume that $f \in \mathcal{C}^\gamma(\mathrm{V}, \mathcal{L}(U, V))$, with $1 + \gamma > p$. Then ϕ is an almost flow.*

Proof. We check that assumptions of Definition 2.1 hold. Let (r, s, t) be in $\mathbb{T}_3^+$ and let a, b be in V. First, $\phi_{t,t}(a) = a$ because $x_{t,t} = 0$. Second,

$$|\phi_{t,s}(a) - a| \leq |f(a)| \cdot |x_{s,t}| \leq |f(a)| \cdot \|x\|_p \omega_{s,t}^{1/p},$$

which proves (2.2). Third,

$$|\phi_{t,s}(a) - \phi_{t,s}(b)| \leq |a - b| + |f(a) - f(b)||x_{s,t}| \leq |a - b| + \|f\|_\gamma \|x\|_p \omega_{s,t}^{1/p} |a - b|^\gamma,$$

which proves (2.3). It remains to prove (2.4). Since

$$\phi_{t,s,r}(a) = f(\phi_{s,r}(a))x_{s,t} - f(a)x_{s,t},$$

we obtain

$$\begin{aligned}|\phi_{t,s,r}(a)| \leq \|f\|_\gamma \|x\|_p \omega_{s,t}^{1/p} |\phi_{s,r}(a) - a|^\gamma &\leq \|f\|_\gamma |f(a)|^\gamma \|x\|_p^2 \omega_{r,t}^{(1+\gamma)/p} \\ &\leq \|f\|_\gamma (1 + |f(a)|) \|x\|_p^2 \omega_{r,t}^{(1+\gamma)/p}.\end{aligned}$$

Setting $\varpi(\omega_{r,t}) := \omega_{r,t}^{(1+\gamma)/p}$, $\eta(\omega_{s,t}) := \|f\|_\gamma \|x\|_p \omega_{s,t}^{1/p}$ and

$$N_\gamma(a) := (1 + |f(a)|) \left(\|x\|_p + \|f\|_\gamma \|x\|_p^2\right),$$

it proves that ϕ is an almost flow.

This concludes the proof. □

Let ψ be a flow in the same galaxy as the almost flow ϕ. For any $a \in \mathrm{V}$ and any $(r, t) \in \mathbb{T}_+^2$, we set

$$y_t(r, a) := \psi_{t,r}(a) \text{ so that } y_r(r, a) = a.$$

Clearly, $(r, a) \mapsto (t \in [r, T] \mapsto y_t(r, a))$ is a family of continuous paths which satisfies

$$|y_t(r, a) - \phi_{t,r}(y_s(r, a))| \leq C N_\gamma(y_s(r, a)) \omega_{s,t}^{2/p},\ \forall (s, t) \in \mathbb{T}_+^2,\ \forall a \in \mathrm{V}$$

since $y_t(r, a) = \psi_{t,s}(y_s(r, a))$. Besides, $s \in [r, T] \mapsto N_\gamma(y_s(r, a))$ is bounded. Therefore, with our choice of the almost flow ϕ, $\psi_{\cdot,r}(a) = y(r, a)$ is a solution in the sense defined by A.M. Davie [16] for the Young differential equation $z_t = a + \int_r^t f(z_s)\,\mathrm{d}x_s$. Even if several solutions may exist for a given (r, a), the flow corresponds to a particular choice of a family of solutions which is constructed thanks to a selection principle. This family of solution is stable under splicing (see Definition 3.5).

Corollary 6.8. *We assume that V is a finite-dimensional vector space and f is in $\mathcal{C}_\mathrm{b}^1(\mathrm{V}, \mathcal{L}(\mathrm{U}, \mathrm{V}))$. Then there exists a flow $\psi \in \mathcal{F}(\mathrm{V})$ in the same galaxy as ϕ such that $\psi_{t,s}$ is Borel measurable for any $(s, t) \in \mathbb{T}_+^2$.*

Remark 6.9. When $f \in \mathcal{C}_\mathrm{b}^\gamma$, several D-solutions to the Young differential equation $y = a + \int_0^\cdot f(y_s(a))\,\mathrm{d}x_s$ may exist (Example 1 in [16]). Uniqueness arises when $f \in \mathcal{C}_\mathrm{b}^{1+\gamma}$ with $1 + \gamma > p$. Hence, a measurable flow may exist even when several D-solution may exist.

Proof. According to Proposition 6.7, ϕ is an almost flow. Here $\gamma = 1$, so ϕ is Lipschitz. Then, we conclude the proof in applying Theorem 3.10 to ϕ. □

6.4.2 Almost flow in the rough case

When the regularity of x is weaker than in the Young case, we need more terms in the Taylor expansion to obtain an almost flow.

Let $T_2(\mathrm{U}) := \mathbb{R} \oplus \mathrm{U} \oplus (\mathrm{U} \otimes \mathrm{U})$ be the truncated tensor algebra (with addition $+$ and tensor product $\otimes$). A distance is defined on the subset of elements of $T_2(\mathrm{U})$ on the form $a = 1 + a^1 + a^2$ with $a^i \in \mathrm{U}^{\otimes i}$ by $d(a,b) = |a^{-1} \otimes b|$ where $|\cdot|$ is a norm on $T_2(\mathrm{U})$ such that $|a \otimes b| \leq |a| \cdot |b|$ for any $a, b \in \mathrm{U}$.

Let $\mathbf{x} = (1, \mathbf{x}^1, \mathbf{x}^2)$ be a rough path with values in $T_2(\mathrm{U})$ of finite p-variation, $2 \leq p < 3$, controlled by ω (see *e.g.*, [20, 25] for a complete definition).

We define a family $(\phi_{t,s})_{(s,t)\in\mathbb{T}_2^+}$ in $\mathcal{F}(\mathrm{V})$ such that for all $a \in \mathrm{V}$ and $(s,t) \in \mathbb{T}_2^+$,

$$\phi_{t,s}(a) := a + f(a)\mathbf{x}^1_{s,t} + \mathrm{d}f(a) \cdot f(a)\mathbf{x}^2_{s,t}. \tag{6.2}$$

Proposition 6.10. *Assume that $f \in \mathcal{C}^{1+\gamma}_{\mathrm{b}}(\mathrm{V}, \mathcal{L}(\mathrm{U},\mathrm{V}))$, with $2 + \gamma > p$. Then ϕ is an almost flow.*

Proof. We check that the assumptions of Definition 2.1 hold. The proofs of (2.1), (2.2) and (2.3) are very similar to the ones in the proof of Proposition 6.7. The computation to show (2.4) is a bit more involved.

Indeed, for any $a \in \mathrm{V}$, $(r,s,t) \in \mathbb{T}_3^+$,

$$\begin{aligned}
\phi_{t,s,r}(a) =& - f(a)\mathbf{x}^1_{s,t} + f(\phi_{s,r}(a))\mathbf{x}^1_{s,t} - \mathrm{d}f(a) \cdot f(a)(\mathbf{x}^2_{s,t} + \mathbf{x}^1_{r,s} \otimes \mathbf{x}^1_{s,t}) \\
&+ \mathrm{d}f(\phi_{s,r}) \cdot f(\phi_{s,r}(a))\mathbf{x}^2_{s,t} \\
=&[f(\phi_{s,t}(a)) - f(a) - \mathrm{d}f(a) \cdot f(a)\mathbf{x}^1_{r,s}] \otimes \mathbf{x}^1_{s,t} \\
&+ [\mathrm{d}f(\phi_{s,r}(a)) \cdot f(\phi_{s,r}(a)) - \mathrm{d}f(a) \cdot f(a)]\mathbf{x}^2_{s,t} \\
=& \underbrace{f(\phi_{s,t}(a)) - f(a) - \mathrm{d}f(a) \cdot (\phi_{s,r}(a) - a)}_{\mathrm{I}_{r,s,t}} + \underbrace{\mathrm{d}f(a) \cdot f(a)\mathbf{x}^2_{r,s} \otimes \mathbf{x}^1_{s,t}}_{\mathrm{II}_{r,s,t}} \\
&+ \underbrace{[\mathrm{d}f(\phi_{s,t}(a)) \cdot f(\phi_{s,r}(a)) - \mathrm{d}f(a) \cdot f(a)]\mathbf{x}^2_{s,t}}_{\mathrm{III}_{r,s,t}}.
\end{aligned}$$

For the first term,

$$\begin{aligned}
|\mathrm{I}_{r,s,t}| &\leq \|\mathrm{d}f\|_\gamma \|\mathbf{x}^1\|_p \omega^{1/p}_{s,t} |\phi_{s,r}(a) - a|^{1+\gamma} \\
&\leq \|\mathrm{d}f\|_\gamma \|\mathbf{x}^1\|_p \omega^{1/p}_{s,t} [\|f\|_\infty \|\mathbf{x}^1\|_p + \|\mathrm{d}f \cdot f\|_\infty \|\mathbf{x}^2\|_{\frac{p}{2}} \omega^{1/p}_{0,T}]^{1+\gamma} \omega^{(1+\gamma)/p}_{r,s}.
\end{aligned}$$

For the two last terms,

$$|\mathrm{II}_{r,s,t}| \leq \|\mathrm{d}f \cdot f\|_\infty \|\mathbf{x}^1\|_p \|\mathbf{x}^2\|_{\frac{p}{2}} \omega^{3/p}_{r,t} \leq \|\mathrm{d}f \cdot f\|_\infty \|\mathbf{x}^1\|_p \|\mathbf{x}^2\|_{\frac{p}{2}} \omega^{(1-\gamma)/p}_{0,T} \omega^{(2+\gamma)/p}_{r,t}$$

and

$$|\mathrm{III}_{r,s,t}| \leq \|\mathbf{x}^2\|_{\frac{p}{2}} \omega^{2/p}_{r,t} [\|\mathrm{d}f\|_\gamma \|f\|_\infty |\phi_{s,r}(a) - a|^\gamma + \|\mathrm{d}f\|_\infty \|f\|_{\mathrm{Lip}} |\phi_{s,r}(a) - a|] \leq C\omega^{3/p}_{r,t},$$

where C is a constant which depends on f, $\mathrm{d}f$, ω, γ, $\mathbf{x}$. It proves that ϕ is a Lipschitz almost flow.

This concludes the proof. □

As for the Young case, any flow ψ in the same galaxy as the almost flow ϕ given by (6.2) gives rise to a family of solutions to the RDE $z_t = a + \int_0^t f(z_s)\,\mathrm{d}\mathbf{x}_s$.

Corollary 6.11. *We assume that V is a finite-dimensional vector space and f is in $\mathcal{C}^2_{\mathrm{b}}(\mathrm{V}, \mathcal{L}(\mathrm{U},\mathrm{V}))$. Then there exists a flow $\psi \in \mathcal{F}(\mathrm{V})$ in the same galaxy as ϕ such that $\psi_{t,s}$ is Borel measurable for any $(s,t) \in \mathbb{T}^2_+$.*

Remark 6.12. When $f \in \mathcal{C}^{1+\gamma}_{\mathrm{b}}$, several D-solutions to the RDE $y = a + \int_0^\cdot f(y_s(a))\,\mathrm{d}\mathbf{x}_s$ may exist (Example 2 in [16]). Uniqueness requires f to be $(2+\gamma)$-Hölder continuous with $2+\gamma > p$. Hence, Corollary 6.11 shows that a measurable flow exists even when several D-solutions may exist.

Proof. According to Proposition 6.10, ϕ is an almost flow. Here $\gamma = 1$, so ϕ is Lipschitz. Then, we conclude the proof in applying Theorem 3.10 to ϕ. □

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