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Nonlinear randomized urn models: a stochastic approximation viewpoint

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Abstract

This paper extends the link between stochastic approximation (SA) theory and randomized urn models developed in [32], and their applications to clinical trials introduced in [2, 3, 4]. We no longer assume that the drawing rule is uniform among the balls of the urn (which contains d colors), but can be reinforced by a function f. This is a way to model risk aversion. Firstly, by considering that f is concave or convex and by reformulating the dynamics of the urn composition as an SA algorithm with remainder, we derive the $a.s.$ convergence and the asymptotic normality (Central Limit Theorem, CLT) of the normalized procedure by calling upon the so-called ODE and SDE methods. An in depth analysis of the case $d = 2$ exhibits two different behaviors: a single equilibrium point when f is concave, and, when f is convex, a transition phase from a single attracting equilibrium to a system with two attracting and one repulsive equilibrium points. The last setting is solved using results on non-convergence toward noisy and noiseless "traps" in order to deduce the $a.s.$ convergence toward one of the attracting points. Secondly, the special case of a Pólya urn (when the addition rule is the I_d matrix) is analyzed, still using result from SA theory about "traps". Finally, these results are used to solve another urn model with a more natural nonlinear drawing rule and we conclude by an example of application to optimal asset allocation in Finance.

Keywords: stochastic approximation; extended Pólya urn models; reinforcement; non-homogeneous generating matrix; strong consistency; asymptotic normality; bandit algorithms.
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1 Introduction

In this paper, we introduce and study in depth a class of generalized Pólya urns (with d colors) characterized by their nonlinear drawing rules. These models appear as a

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generalization of randomized urn models originally devised for clinical trials which take into account the risk aversion attitude of the prescriber of the treatment. Randomized urn models have been extensively investigated by various authors (see [2, 3, 4]) during the last twenty years based on *ad hoc* martingale arguments to solve $a.s.$ convergence as well as its rate of convergence. In a recent paper [32] (see also [33, 40]), we revisited, unified and often extended these results by showing that they can be established by relying on the main results of Stochastic Approximation (SA) theory, especially $a.s.$ convergence and weak convergence rate (Central Limit Theorem (CLT)). In a few words, SA theory is devoted to the search of the zero(s) of a function $h : \mathbb{R}^d \to \mathbb{R}$ whose values $h(\theta)$ cannot be computed directly at a reasonable cost but which admits a probabilistic representation of the form $h(\theta) = \mathbb{E}\, H(\theta, Z)$ where the Borel function $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^q$ can be easily computed and the random variable Z can be simulated at a low computational cost. One defines a recursive procedure based on the function H as follows

$$\theta_{n+1} = \theta_n - \gamma_{n+1}\big(H(\theta_n, Z_{n+1}) + r_{n+1}\big), \quad n \geq 0,\ \theta_0 \in \mathbb{R}^d, \tag{1.1}$$

where $(\gamma_n)_{n\geq 1}$ is a sequence of steps decreasing $a.s.$ to zero at appropriate rates, the sequence $(r_n)_{n\geq 1}$ is a sequence of remainder terms and the sequence $(Z_n)_{n\geq 1}$ is i.i.d. with the same distribution as Z. This procedure can be rewritten as a perturbation of the "regular" deterministic zero search procedure by a martingale increment term since it also reads

$$\theta_{n+1} = \theta_n - \gamma_{n+1}\big(h(\theta_n) + \Delta M_{n+1} + r_{n+1}\big), \quad n \geq 0,\ \theta_0 \in \mathbb{R}^d, \tag{1.2}$$

where

$$\begin{aligned}\Delta M_{n+1} &= H(\theta_n, Z_{n+1}) - \mathbb{E}\,[H(\theta_n, Z_{n+1}) \,|\, \mathcal{F}_n^Z] = H(\theta_n, Z_{n+1}) - \big[\mathbb{E}\, H(\theta, Z)\big]_{|\theta=\theta_n} \\ &= H(\theta_n, Z_{n+1}) - h(\theta_n)\end{aligned}$$

is a martingale increment (with respect to the natural filtration $\mathcal{F}^Z = \sigma(\theta_0, Z_1, \ldots, Z_n)$, $n \geq 1$). The basic paradigm of SA is that, under appropriate assumptions, the sequence converges $a.s.$ to a zero θ^* of the mean field function h. Finding out the assumptions on h, H, and establishing its rate of convergence is the main purpose of SA. Useful results – a toolbox – are available in the Appendix.

Although the analysis of these urn models with nonlinear drawing rules is more demanding than those with linear drawing rules, this SA "toolbox" turns out to be still very efficient (see also [5] and the references therein). SA deals with the asymptotic behavior of zero search stochastic recursive procedures and goes back to the seminal paper by Robbins & Monro in the 1950's. Since then, this theory has been developed extensively by many authors (see [27, 28, 9, 17, 18] and the references therein for an overview and historical notes) and has been applied in various directions (Automatic Control, Mathematical Psychology, Artificial Neural Networks, Statistics, Stochastic Control, Numerical Probability, etc.).

Considering nonlinear drawing rules leads, once the evolution of the urn composition is written as a recursive stochastic algorithm of the form (1.1), to consider situations where the *mean field function* h of the procedure has several zeros, also called *equilibrium points* in SA theory. Such an equilibrium point represents in practice a potential asymptotic composition of the urn. In the linear drawing rule setting, it turned out that this asymptotic urn composition is unique (see [32]).

Among these multiple equilibrium points, some are local attractors (or "targets"), but others are "parasitic" ones (repeller, saddle points, etc.) and $a.s.$ cannot appear as an asymptotic urn composition of the procedure. These terms should be understood in the sense of the Ordinary Differential Equation (ODE) attached to the mean field function h, namely $\dot{\theta} = -h(\theta)$.

This second aspect, in SA, called the ODE *method* is the deterministic side of the theory, essentially borrowed from perturbed dynamical system theory (see [36, 6, 19] among many others) and allows to improve the results of martingale methods. This approach which analyzes the *asymptotic pseudo-trajectories* of the ODE_h $\dot{y} = -h(y)$ and their shadowing properties to follow the terminology developed in [8, 6], often appears as a second “refined” step in the analysis. As for the rate, the analogy between the CLT for SA and urn models is striking. The CLT for SA, which itself relies on the CLT for arrays of martingale increments, is clearly a strikingly efficient shortcut to establish the rate of convergence of urn models (see [11, 32, 40]). In the analysis of this new model of skewed urns presented in this paper, we are facing settings with – a priori – multiple targets for the urn composition (by target we mean zero of ODE_h). To eliminate those which are parasitic, we take advantage on a third aspect of SA theory, not yet called upon so far to our best knowledge. These are results which ensure the $a.s.$ non-convergence of a stochastic algorithm toward a noisy equilibrium point (called “traps” in [12], see also [37, 35]), but also, in some situations, the $a.s.$ non-convergence toward noiseless equilibrium points (see [31, 29]).

Let us be more precise on the urn model under consideration in this paper. We consider an urn containing balls of (at most) d different types (or colors). All random variables involved in the model are supposed to be defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Denote by $Y_0 = (Y_0^i)_{i=1,\dots,d} \in \mathbb{R}_+^d \setminus \{0\}$ the initial composition of the urn, where Y_0^i is the number of balls of type $i \in \{1, \dots, d\}$ (of course a more natural, though not mandatory, assumption would be $Y_0 \in \mathbb{N}^d \setminus \{0\}$). The urn composition at draw n is denoted by $Y_n = (Y_n^i)_{i=1,\dots,d}$. At the n^{th} stage, one draws randomly (according to a law defined further on) a ball from the urn with instantaneous replacement. If the drawn ball is of type j, then the urn composition is updated by adding D_n^{ij} balls of type i, for every $i \in \{1, \dots, d\}$. The procedure is then iterated. The urn composition at stage n, modeled by an $\mathbb{R}^d$-valued vector Y_n, satisfies the following recursive updating rule between times n and $n+1$:

$$Y_{n+1} = Y_n + D_{n+1} X_{n+1}, \quad n \geq 0, \quad Y_0 \in \mathbb{R}_+^d \setminus \{0\}, \tag{1.3}$$

where $D_n = (D_n^{ij})_{1 \leq i,j \leq d}$ is the *addition rule matrix* and $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \to \{e^1, \cdots, e^d\}$ models the type of the drawn ball at time n ($\{e^1, \cdots, e^d\}$ denotes the canonical basis of $\mathbb{R}^d$ with e^j standing for type j). We assume that there is no extinction *i.e.* $Y_n \in \mathbb{R}_+^d \setminus \{0\}$ $a.s.$ for every $n \geq 1$: This is always the case if all the entries D_n^{ij} are $a.s.$ non-negative (see [32]). The filtration of the model is defined by $\mathcal{F}_n = \sigma(Y_0, X_k, D_k, 1 \leq k \leq n)$, $n \geq 0$.

The *generating matrices* are defined as the $\mathcal{F}_n$-compensator of the additions rule sequence *i.e.*

$$H_n = \left[\mathbb{E}\left(D_n^{ij} \,|\, \mathcal{F}_{n-1}\right)\right]_{1 \leq i,j \leq d}, \; n \geq 1. \tag{1.4}$$

We assume that the sequence of generating matrices converges $a.s.$ toward a limiting generating matrix denoted by H.

Moreover, we make the assumption that D_n and X_n are conditionally independent given $\mathcal{F}_{n-1}$ (see $(\mathbf{A2})$ further on). Such a drawing procedure can be performed by using an exogenous i.i.d. sequence $(U_n)_{n \geq 1}$ of random variables with uniform distribution on the unit interval, independent of the sequence $(D_n)_{n \geq 1}$, to simulate the above conditional probabilities.

In this paper, we study two different drawing rules, both associated to a *skewing function* $f : \mathbb{R}_+ \to \mathbb{R}_+$. We first and mostly investigate the asymptotic behavior of the following skewed drawing rules of the form

$$\forall i \in \{1,\dots,d\},\quad \mathbb{P}\big(X_{n+1}=e^i\big|\mathcal{F}_n\big)=\frac{f\big(Y_n^i/(n+\mathbf{w}(Y_0))\big)}{\sum_{j=1}^d f\big(Y_n^j/(n+\mathbf{w}\big(Y_0\big))\big)},\quad n\geq 0, \tag{1.5}$$

where $\mathbf{w}(y)=y^1+\cdots+y^d$ denotes the *weight* of a vector $y=(y^1,\dots,y^d)^t\in\mathbb{R}_+^d$ and the function

$$f:\mathbb{R}_+\to\mathbb{R}_+ \text{ is non-decreasing with } f(0)=0 \text{ and } f(1)=1. \tag{1.6}$$

The function f usually satisfies an *additional convexity or concavity property.*

This drawing rule is called *empirical frequency based normalized f-skewed drawing rule.* It is is nonlinear if and only if $f\neq \mathrm{Id}_{\mathbb{R}_+}$. When $f=\mathrm{Id}_{\mathbb{R}_+}$, we retrieve the linear drawing rule based on the regular empirical frequency of the types in the urn (see [2, 32] among others). Most part of the paper (Sections 2, 3, 4, 5) are devoted to the study of this randomized urn model.

We will see at the end of the paper that results obtained for this family of empirical frequency-based drawing rules allows us to elucidate a second – somewhat more natural – way to skew the drawing, called *distribution based normalized f-skewed rule.* It consists in changing the conditional distribution of the random variable X_{n+1} based this time on the *number* of balls of each type in the urn, namely

$$\forall i \in \{1,\dots,d\},\quad \mathbb{P}\big(X_{n+1}=e^i\,\big|\,\mathcal{F}_n\big)=\frac{f(Y_n^i)}{\sum_{j=1}^d f(Y_n^j)},\quad n\geq 0, \tag{1.7}$$

where the function

$$f:\mathbb{R}_+\to\mathbb{R}_+ \text{ is non-decreasing, with } f(0)=0 \text{ and is regularly varying with index } \alpha>0 \tag{1.8}$$

(*i.e.* for every $t>0$, $\frac{f(tx)}{f(x)}\underset{x\to+\infty}{\longrightarrow}t^\alpha$).

Let us remark that, when $f=\mathrm{Id}_{\mathbb{R}_+}$, both updating rules (1.5) and (1.7) coincide. In fact, the normalized f-skewed distribution drawing rule appears as a by-product of the first one (see Section 6.1) by noting that, if f is bounded on every interval $(0,M]$ and regularly varying with index $\alpha>0$, then $\frac{f(tx)}{f(x)}\underset{x\to\infty}{\longrightarrow}t^\alpha$ uniformly in t on every interval $(0,T]$, $0<T<+\infty$ (see Theorem 1.5.2 p.22 in [10]). Thus, if $\frac{Y_n}{n+\sum_{i=1}^d Y_0^i}$ lies in a compact set, then

$$\max_{1\leq i\leq d}\left|\frac{f(Y_n^i)}{f(n+\sum_{i=1}^d Y_0^i)}-\left(\frac{Y_n}{n+\sum_{i=1}^d Y_0^i}\right)^\alpha\right|\underset{n\to+\infty}{\longrightarrow}0.$$

Then, we conclude by applying the result related to the f-skewed empirical frequency based drawing rule to the functions $x\mapsto x^\alpha$, $\alpha>0$.

Given the generality of the model it turns out to be out of reach to entirely solve the asymptotic behavior of this skewed urn model in terms of convergence of the urn composition. We only obtained partial results of variable generality according to the convexity/concavity of the skewing function f. Thus, in the convex setting, we showed that even with two urns, a bifurcation phenomenon occurs. The main $a.s.$ convergence results are the following:

- If the skewing function f is strictly concave and if the generating matrix H is bi-stochastic, then (see Proposition 2.14), the urn composition satisfies

$$\widetilde{Y}_n=\frac{Y_n}{n+\mathbf{w}(Y_0)}\underset{n\to+\infty}{\longrightarrow}\mathbf{1}/d.$$

- If the skewing function f is convex, H is bi-stochastic (resp. irreducible) with $\mu_{\max}$ its eigenvalue with the second highest real part satisfying $\Re e(\mu_{\max}) > \frac{df(1/d)}{f'(1/d)}$ (resp. < 1) and if $y(d) = \mathbf{1}/d$ is a "noisy enough" zero of h, then (see Proposition 2.15), then

$$\mathbb{P}\Big(\widetilde{Y}_n \sim \frac{Y_n}{n + \mathbf{w}(Y_0)} \underset{n\to+\infty}{\longrightarrow} y(d)\Big) = 0.$$

- If $d = 2$ and f is convex, then (see Theorem 3.10), $\frac{Y_n}{\mathbf{w}(Y_n)}$ converges $a.s.$ toward one of the stable zeros of the mean function h which can be 1, 2 or even 3, depending on the urn selection probabilities.

The interpretation of the skewing functions in terms of risk aversion is the following in the first model: when f is concave, which corresponds to a *superlinear* reinforcement, the resulting strategy is risk averse and we will see that it tends to equalize the asymptotic urn composition whatever the (irreducible) generating matrix H is. As a consequence, it has a diversifying effect. Conversely a convex skewing function, which corresponds to a *sublinear reinforcement*, has a reinforcement effect and tends to amplify the effect of the limiting generating matrix H. This is illustrated by the results established infection of the two-urn model (see Section 3)) and the application to adaptive asset allocation in Section 6.2.

In this paper we both randomize a single urn in the sense that the addition rule matrix (defined by (1.4)) itself can be random (as introduced in [25, 3, 4] and studied with SA theory in [32]) and investigate wide non-parametric classes of convex and concave drawing rules (see (1.5) and (1.7)). $A.s.$ convergence of the urn composition and that of the drawing rule are proved as well as their convergence rate, either in a weak or strong sense, depending on the structure of the updating rules. In the particular case of Pólya's urns (*i.e.* when the addition rule matrix D_n is equal to identity), but implemented here with a convex skewed drawing rule (for generalized Pólya urn, see for example [24, 41, 16]), "noiseless traps" may appear (*i.e.* unstable equilibria, noiseless since they lie at the boundary of the state space). To determine whether they are parasitic, we develop a dedicated approach, close in spirit to that introduced in [31] and [29] on the analysis of adaptive bandit algorithms where the authors establish a kind of "oracle" inequality relying on a specified martingale (see Lemma 5.4 further on). These results highlight the efficiency of SA theory, even in presence of noiseless repulsive equilibrium points.

Recently, a system of Pólya's urns with graph based interactions and a "power drawing" rule has also been investigated using SA techniques in [7, 15, 39] and the $a.s.$ convergence of the normalized urn composition is established.

Generalized Pólya Urn models (GPU) have been widely studied in the literature with different points of view: "pure" martingale method (see $e.g.$ [21]), algebraic approach (see $e.g.$ [38]), reinforcement process (see $e.g.$ [38]), branching process (see $e.g.$ [25]), stochastic approximation (see for example [5, 34]), contraction method (see [26]). These models also have applications to many areas: Biology, random walks and clinical trials, statistics and learning, computer science, psychology, economics or finance for instance (see [41]).

In these adaptive models, the key point is the updating rules of the urn composition after each drawing given here by (1.5) and (1.7). Basically, we show that (a normalized version of) this urn composition can be formulated as a classical recursive stochastic algorithm with step $\gamma_n = \frac{1}{n+\mathbf{w}(Y_0)}$ where $\mathbf{w}(Y_0)$ denotes the number of balls in the urn at time 0. Doing so, we will be in position to first establish the $a.s.$ convergence of the procedure by calling upon the so-called Ordinary Differential Equation method

(ODE method) toward a finite set of equilibrium points (but usually not reduced to a single point). As a second step, we rely on $a.s.$ non-convergence results toward traps (see [12, 17]) and on $a.s.$ convergence in presence of multiple targets (see [8, 17, 19]). As a third step, we entirely elucidate the rate of convergence (namely a weak rate through a CLT or an $a.s.$ rate) by using the Stochastic Differential Equation method (SDE method, see $e.g.$ [18, 9]). The three main theoretical results from SA are recalled in a self-contained form in the Appendix. Proofs of such results can be found in classical textbooks on SA ([9, 17, 18, 28]). As for the CLT, they go back to [27] and [11], see also [40] more recent results. We will verify once again how powerful these general theorems are to solve such questions, sparing tedious computations and repetitive proofs.

Among many fields of application, we present in Section 6 an adaptive asset allocation procedure based on a reinforcement principle relying on non-linear randomized urns, illustrated by a first numerical test. One may also consider a similar procedure as a strategy to update the composition of a portfolio or even a whole fund, based on the (recent) past performances of the assets.

The paper is organized as follows. Section 2 presents the framework of skewed randomized urn models with the required assumptions on both the addition rule matrices and the generating matrices. After rewriting the dynamics of the urn composition as an SA procedure in Section 2.2, we analyze in Section 2.4 the equilibrium points and their stability for the associated ODE when the f-skewed drawing rule is convex/concave. An in-depth analysis of the 2-color urn is carried out in Section 3. We exhibit several kinds of behaviors: When f is concave, there is always a unique stable equilibrium point and when f is convex three generic situations may occur with one, tow or three equilibrium points (one being parasitic in the last two settings). By calling upon SA result on traps, we prove the $a.s.$ convergence towards one of the attracting equilibrium points; then we derive from the SDE method all the possible rates of convergence. In Section 5, we study the case of Pólya urns – urn dynamics whose addition rule matrix equals to identity – updated by a skewed drawing rule. We rely on methods borrowed from the analysis of adaptive bandit algorithms to prove the convergence towards the “targeted urn composition” and the non-convergence towards traps. Finally, in Section 6, we first transfer our results to the second type of drawing rule and we conclude by an application to portfolio allocation.

Notations. For $u = (u^i)_{i=1,\dots,d} \in \mathbb{R}^d$, $(\cdot\,|\,\cdot)$ denote the Euclidean inner product and $\|u\|$ its related norm, $\mathbf{w}(u) = \sum_{k=1}^d u^k$ denotes its “weight”, u^t denotes its transpose, $u \otimes v = [u^i v^j]_{i,j=1,\dots,d}$; $|\!|\!|A|\!|\!|$ denotes the operator norm of the matrix $A \in \mathcal{M}_{d,q}(\mathbb{R})$ with d rows and q columns with respect to the two canonical Euclidean norms. When $d=q$, $\mathrm{Sp}(A)$ denotes the set of eigenvalues of A. $\mathbf{1}=(1\cdots 1)^t$ denotes the unit column vector in $\mathbb{R}^d$, I_d denotes the $d\times d$ identity matrix, $\mathrm{diag}(u) = [\delta_{ij}u_i]_{1\le i,j\le d}$, where δ_{ij} stands for the Kronecker symbol and $\mathcal{S}_d = \left\{u \in \mathbb{R}_+^d : \sum_{i=1}^d u^i = 1\right\}$ denotes the canonical simplex. $\mathcal{E}_{d,h} = \{y \in \mathcal{S}_d : h(y) = 0\}$ denotes the set of zeros of a function h (defined on a neighborhood of $\mathcal{S}_d$) called equilibrium points in reference to the ODE $\dot y = -h(y)$. $h_{|A}$ denotes the restriction of a function h on a subset A of its definition set.

2 Skewed randomized urn models

2.1 Main assumptions and definitions

With the notations and definitions described in the introduction, we are in position to formulate the main assumptions needed to establish the $a.s.$ convergence of the urn composition.

$$(\mathbf{A1}) \equiv \begin{cases} (i) \ \textit{Addition rule matrix: } \text{For every } n \geq 1, \text{ the matrix } D_n \ a.s. \text{ has non-negative entries.} \\ (ii) \ \textit{Generating matrix: } \text{For every } n \geq 1, \text{ the generating matrix} \\ \qquad H_n = (H_n^{ij})_{1 \leq i,j \leq d} \ a.s. \text{ satisfies:} \quad \forall\, j \in \{1, \dots, d\}, \quad \sum_{i=1}^{d} H_n^{ij} = c > 0. \\ (iii) \ \textit{Starting value: } \text{The starting urn composition vector } Y_0 \in \mathbb{R}_+^d \setminus \{0\}. \end{cases}$$

The constant c is known as the *balance* of the urn. In fact, we may assume without loss of generality, up to a renormalization of Y_n, that $c = 1$. As a matter of fact, we set for every $n \geq 0$, $Y_n^c = \frac{Y_n}{c}$ and $D_{n+1}^c = \frac{D_{n+1}}{c}$ where (Y_n, X_n, D_n) satisfies (1.3), then the couple $(Y_n^c, X_n, D_n^c)_{n \geq 1}$, still satisfies the dynamics (1.3), namely

$$Y_{n+1}^c = Y_n^c + D_{n+1}^c X_{n+1}, \quad n \geq 0, \quad Y_0^c \in \mathbb{R}_+^d \setminus \{0\},$$

whereas $H_n^c = \mathbb{E}[D_n^c | \mathcal{F}_{n-1}]$ satisfies now $(\mathbf{A1})$-(iii) with $c = 1$ *i.e.* H_n^c is *co-stochastic* (in the sense that its transpose is a stochastic matrix). Assumptions $(\mathbf{A1})$-(i)&(iii) combined with the drawing rule (1.3) ensure that $Y_n \in \mathbb{R}_+^d \setminus \{0\}$, for every $n \geq 0$.

From now on, throughout the paper, we will consider this normalized balanced version, still designated by Y_n and D_n for convenience ($i.e.$ we drop the superscript c).

$$(\mathbf{A2}) \equiv \begin{cases} (i) \ \text{The addition rule } D_n \text{ and the drawing procedure } X_n \text{ are conditionally independent given } \mathcal{F}_{n-1}. \\ (ii) \quad \forall j \in \{1, \dots, d\}, \quad \sup_{n \geq 1} \mathbb{E}\left[\left\| D_n^{\cdot j} \right\|^2 \mid \mathcal{F}_{n-1} \right] < +\infty \quad a.s. \\ \qquad \iff \forall\, i, j \in \{1, \dots, d\}, \quad \sup_{n \geq 1} \mathbb{E}\left[(D_n^{ij})^2 \mid \mathcal{F}_{n-1} \right] < +\infty \ a.s. \end{cases}$$

where $D_n^{\cdot j} = (D_n^{ij})_{i=1,\dots,d}$ (column vector).

$(\mathbf{A3})$ There exists an irreducible $d \times d$ matrix H (with non-negative entries) such that

$$H_n \xrightarrow[n \to +\infty]{a.s.} H \quad \text{and} \quad \sum_{n \geq 1} \| H_n - H \|^2 < +\infty \quad a.s. \tag{2.1}$$

H is called the *limiting generating matrix*.

The central object of interest of this paper will be the *(quasi-)normalized composition* of the urn at time n defined by

$$\widetilde{Y}_n = \frac{Y_n}{n + \mathbf{w}(Y_0)} \tag{2.2}$$

(where the weight function is defined in the introduction). The reason for introducing such a renormalization factor is that, as established further on in the next subsection,

$$\forall n \geq 0, \quad \mathbb{E}\left[\mathbf{w}(Y_n) \right] = n + \mathbf{w}(Y_0).$$

Then, one may guess that $\widetilde{Y}_n$ is close to the simplex $\mathcal{S}_d = \left\{ u \in \mathbb{R}_+^d : \sum_{i=1}^d u^i = 1 \right\}$ and will possibly asymptotically lie in it. Even note that when the matrices D_n are themselves co-stochastic, $\widetilde{Y}_n$ is $\mathcal{S}_d$-valued. Therefore, this is a natural deterministic way to normalize the urn composition vector.

Definition 2.1 (Skewing functions and skewed drawing rules)**.** (a) *A function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying*

$$f \textit{ non-decreasing, } \text{convex or concave}, \ f(0) = 0 \textit{ and } f(1) = 1 \textit{ and } \{f > 0\} = (0, +\infty) \tag{2.3}$$

is called a skewing function.

(b) *Assuming* $(\mathbf{A1})$-(i)&(iii)*, the* f-skewed drawing rule $(X_n)_{n\geq 1}$ *induced by a skewing function* f *is defined by*

$$\forall\, i \in \{1,\ldots,d\}, \quad \mathbb{P}\big(X_{n+1} = e^i \big| \mathcal{F}_n\big) = \frac{f(\widetilde{Y}_n^i)}{\sum_{j=1}^d f(\widetilde{Y}_n^j)}, \quad n \geq 0, \tag{2.4}$$

where $\widetilde{Y}_n$ *is defined by* (2.2)*, and* $(Y_n, X_n, D_n)_{n\geq 1}$ *satisfies* (1.3).

Of course, such a drawing rule is really *skewed* only if $f \not\equiv \mathrm{Id}_{\mathbb{R}_+}$. Note that this definition is consistent under $(\mathbf{A1})$-(i)&(iii) since $Y_n \in \mathbb{R}_+^d \setminus \{0\}$, for every $n \geq 0$ so that $\widetilde{Y}_n \in \mathbb{R}_+^d \setminus \{0\}$.

We will extensively use that a skewing function f always satisfies that the function $\xi \in \mathbb{R}_+ \mapsto \frac{f(\xi)}{\xi}$ is monotonic (non-decreasing if f is convex, non-increasing if f is concave) and strictly monotonic if the concavity/convexity of f is itself strict.

2.2 Representation as a stochastic algorithm

The starting point, like in [32], is to reformulate the dynamics (1.3)-(1.5) as a recursive stochastic algorithm in order to take advantage of classical results from SA theory to elucidate the asymptotic properties ($a.s.$ convergence) of both the urn composition Y_n and the ball drawing rule X_n. To do so, we start from (1.3) with $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$. For every $n \geq 0$, we note that

$$Y_{n+1} = Y_n + D_{n+1}X_{n+1} = Y_n + \mathbb{E}\left[D_{n+1}X_{n+1} \,|\, \mathcal{F}_n\right] + \Delta M_{n+1}, \tag{2.5}$$

where

$$\Delta M_{n+1} := D_{n+1}X_{n+1} - \mathbb{E}\left[D_{n+1}X_{n+1} \,|\, \mathcal{F}_n\right] \tag{2.6}$$

is an $\mathcal{F}_n$-local martingale increment (integrability follows from $(\mathbf{A2})$-(ii)). By the definition (1.4) of the generating matrix H_n, we have, owing to the conditional independence assumption $(\mathbf{A2})$-(i),

$$\begin{aligned}\mathbb{E}\left[D_{n+1}X_{n+1} \,|\, \mathcal{F}_n\right] &= \sum_{i=1}^d \mathbb{E}\big[D_{n+1}\mathbb{1}_{\{X_{n+1}=e^i\}} \,|\, \mathcal{F}_n\big]e^i = \sum_{i=1}^d \mathbb{E}\left[D_{n+1} \,|\, \mathcal{F}_n\right]\mathbb{P}\big(X_{n+1} = e^i \,|\, \mathcal{F}_n\big)e^i \\ &= H_{n+1}\sum_{i=1}^d \frac{f(\widetilde{Y}_n^i)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}e^i = H_{n+1}\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\end{aligned}$$

where, for every $y = (y^1,\ldots,y^d)^t \in \mathbb{R}_+^d \setminus \{0\}$,

$$\widetilde{f}\big((y^1,\ldots,y^d)^t\big) = \big(f(y^i)\big)_{1\leq i\leq d} \in \mathbb{R}_+^d \setminus \{0\} \tag{2.7}$$

is a column vector, so that

$$Y_{n+1} = Y_n + H_{n+1}\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} + \Delta M_{n+1}. \tag{2.8}$$

Now we can derive a stochastic approximation for the normalized urn composition $\widetilde{Y}_n = \dfrac{Y_n}{n + \mathbf{w}(Y_0)}$, $n \geq 0$. First, we have, for every $n \geq 0$,

$$\frac{Y_{n+1}}{n{+}1{+}\mathbf{w}(Y_0)} = \frac{Y_n}{n{+}\mathbf{w}(Y_0)} + \frac{1}{n{+}1{+}\mathbf{w}(Y_0)}\left(H_{n+1}\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} - \frac{Y_n}{n{+}\mathbf{w}(Y_0)}\right) + \frac{\Delta M_{n+1}}{n{+}1{+}\mathbf{w}(Y_0)}. \tag{2.9}$$

Consequently, the sequence $(\widetilde{Y}_n)_{n\geq 0}$, satisfies the canonical recursive stochastic approximation procedure starting from $\widetilde{Y}_0 \in \mathbb{R}_+^d \setminus \{0\}$,

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n + \frac{1}{n+1+\mathbf{w}(Y_0)}\left(H_{n+1}\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} - \widetilde{Y}_n\right) + \frac{1}{n+1+\mathbf{w}(Y_0)}\Delta M_{n+1} \tag{2.10}$$

or, equivalently,

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \gamma_{n+1} h(\widetilde{Y}_n) + \gamma_{n+1}\left(\Delta M_{n+1} + r_{n+1}\right) \tag{2.11}$$

where the *mean field* $h : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d$ of the procedure is defined by

$$h(y) := y - H\frac{\widetilde{f}(y)}{\mathbf{w}\big(\widetilde{f}(y)\big)}, \tag{2.12}$$

satisfying

$$\mathbb{E}\left[\widetilde{Y}_n - H\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\,\middle|\,\widetilde{Y}_n = y\right] = y - H\frac{\widetilde{f}(y)}{\mathbf{w}\big(\widetilde{f}(y)\big)}.$$

The sequence $\gamma_n := \frac{1}{n+\mathbf{w}(Y_0)}$, $n \geq 1$, is its step parameter and

$$r_{n+1} := (H_{n+1} - H)\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \quad \text{is an } \mathcal{F}_n\text{-adapted remainder term.} \tag{2.13}$$

Equation (2.11) shows that the urn dynamics can be viewed as a recursive zero search algorithm suggesting to use the SA toolbox to elucidate its asymptotics.

2.3 Boundedness of the normalized urn composition

Our first task is to establish the $a.s.$ boundedness of the sequence $(\widetilde{Y}_n)_{n\geq 0}$. By summing up the components of $\widetilde{Y}_n$ in (2.8), we obtain

$$\mathbf{w}(Y_{n+1}) = \mathbf{w}(Y_n) + \frac{\mathbf{w}(H_{n+1}\widetilde{f}(\widetilde{Y}_n))}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} + \mathbf{w}(\Delta M_{n+1}).$$

Using that the transpose of the generating matrix H_{n+1} is a stochastic matrix by $(\mathbf{A1})$-(ii) (with $c = 1$), we obtain

$$\begin{aligned}\mathbf{w}(H_{n+1}\widetilde{f}(\widetilde{Y}_n)) &= \sum_{i=1}^{d}\big(H_{n+1}\widetilde{f}(\widetilde{Y}_n)\big)_i = \sum_{i=1}^{d}\sum_{j=1}^{d} H_{n+1}^{ij} f(\widetilde{Y}_n^j) = \sum_{j=1}^{d}\left(\sum_{i=1}^{d} H_{n+1}^{ij}\right) f(\widetilde{Y}_n^j)\\ &= \mathbf{w}(\widetilde{f}(\widetilde{Y}_n)).\end{aligned} \tag{2.14}$$

Consequently, for every $n \geq 0$,

$$\mathbf{w}(Y_{n+1}) = \mathbf{w}(Y_n) + 1 + \mathbf{w}(\Delta M_{n+1}). \tag{2.15}$$

Let $N_0 = 0$ and $N_n := \sum_{k=1}^{n} X_k$, $n \geq 1$, denote the number of times each type of ball was drawn between draws 1 and n. For every $n \geq 0$,

$$\begin{aligned} N_{n+1} &= N_n + X_{n+1} = N_n + \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} + \Delta\widetilde{M}_{n+1},\\ \text{where} \qquad \Delta\widetilde{M}_{n+1} &:= X_{n+1} - \mathbb{E}\left[X_{n+1} \,|\, \mathcal{F}_n\right] = X_{n+1} - \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\end{aligned}$$

is an $\mathcal{F}_n$-martingale increment. Thus, $\widetilde{N}_n := \dfrac{N_n}{n}$ satisfies, still for every $n \geq 0$,

$$\widetilde{N}_{n+1} = \widetilde{N}_n - \frac{1}{n+1}\left(\widetilde{N}_n - \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right) + \frac{1}{n+1}\Delta\widetilde{M}_{n+1}.$$

Proposition 2.2. *Let $(Y_n)_{n\geq 0}$ be the urn composition sequence defined by (1.3)-(1.5).*
(a) *Under the assumptions **(A1)** and **(A2)**,*

$$\frac{\mathbf{w}(Y_n)}{n+\mathbf{w}(Y_0)} \xrightarrow[n\to+\infty]{a.s.} 1.$$

(b) *If the addition rule matrices D_n themselves are co-stochastic, then $\mathbf{w}(Y_n) = n + \mathbf{w}(Y_0)$, and the sequence $(\widetilde{Y}_n)_{n\geq 0}$ lives in the simplex $\mathcal{S}_d$.*

Proof. (a) We derive from the identity

$$D_{n+1}X_{n+1} = \sum_{j=1}^{d} D_{n+1}^{\cdot j}\mathbb{1}_{\{X_{n+1}=e^j\}}, \quad n\geq 0,$$

that

$$\|D_{n+1}X_{n+1}\|^2 = \sum_{j=1}^{d}\left\|D_{n+1}^{\cdot j}\right\|^2 \mathbb{1}_{\{X_{n+1}=e^j\}}.$$

Hence, owing to $(\mathbf{A2})$-(ii),

$$\begin{aligned}\mathbb{E}\left[\|D_{n+1}X_{n+1}\|^2 \mid \mathcal{F}_n\right] &= \sum_{j=1}^{d}\mathbb{E}\left[\left\|D_{n+1}^{\cdot j}\right\|^2 \mid \mathcal{F}_n\right]\mathbb{P}\left(X_{n+1}=e^j \mid \mathcal{F}_n\right)\\ &\leq \sup_{n\geq 0}\sup_{1\leq j\leq d}\mathbb{E}\left[\left\|D_{n+1}^{\cdot j}\right\|^2 \mid \mathcal{F}_n\right] < +\infty \quad a.s.\end{aligned}$$

Consequently $\sup_{n\geq 1}\mathbb{E}\left[\|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n\right] < +\infty$ $a.s.$ and thanks to the strong law of large numbers for conditionally L^2-bounded local martingale increments, we have $\frac{M_n}{n} \underset{n\to+\infty}{\longrightarrow} 0$ $a.s.$ Finally, it follows from (2.15) that

$$\frac{\mathbf{w}(Y_n)}{n+\mathbf{w}(Y_0)} = 1 + \frac{\mathbf{w}(M_n)}{n+\mathbf{w}(Y_0)} \longrightarrow 1 \quad a.s. \quad n\to+\infty.$$

(b) In this case $\mathbf{w}(M_n)=0$, consequently for every $n\geq 0$, $\mathbf{w}(\widetilde{Y}_n)=1$. □

2.4 Existence of equilibrium points

As written in (2.11), the urn dynamics appears as a recursive zero search algorithm with mean field $h:\mathbb{R}^d\setminus\{0\}\to\mathbb{R}^d$ whose potential limiting points all lie in the canonical simplex $\mathcal{S}_d$ defined by

$$\mathcal{S}_d = \mathbf{w}^{-1}\{1\} = \left\{y\in\mathbb{R}_+^d \mid \mathbf{w}(y)=1\right\}.$$

More generally, since the components of $\widetilde{Y}_n = \frac{Y_n}{n+\mathbf{w}(Y_0)}$ are non-negative by construction and – under $(\mathbf{A1})$-$(\mathbf{A2})$ – $\mathbf{w}(\widetilde{Y}_n) = \frac{\mathbf{w}(Y_n)}{n+\mathbf{w}(Y_0)} \underset{n\to+\infty}{\longrightarrow} 1$ $a.s.$, it is clear that $\mathbb{P}(d\omega)$-$a.s.$, the sequence $(\widetilde{Y}_n(\omega))_{n\geq 0}$ is bounded and that the set $\mathcal{Y}_\infty(\omega)$ of its limiting values lies in the simplex $\mathcal{S}_d$. Consequently, we search $\mathcal{S}_d$-valued equilibrium points *i.e.* points $y\in\mathcal{S}_d$ such that $h(y)=0$ where h is given by (2.12).

Throughout this section f denotes a skewing function in the sense of (2.3) and H is a *deterministic* (co-stochastic) matrix, intended to be a limiting generating matrix as soon as it is irreducible.

Proposition 2.3. *Assume the skewing function f satisfies* (2.3).

(a) *Let H be a deterministic co-stochastic matrix. The function $\varphi_{_H} : [0,1]^d \setminus \{0\} \to \mathcal{S}_d$ defined by $\varphi_{_H}(y) = H\frac{\widetilde{f}(y)}{\mathbf{w}\big(\widetilde{f}(y)\big)}$ has at least one fixed point. The mean field h defined by* (2.12) *has at least one zero y^* and $\{h=0\} \subset \mathcal{S}_d$. If H is bi-stochastic, then $y(d) := \frac{1}{d}\mathbf{1}$ is always such a zero of h.*

(b) *If, furthermore, for every $i, j \in \{1,\ldots,d\}$, $H^{ij} > 0$, then for every zero y^* of h in $\mathcal{S}_d$,*

$$\forall\, i \in \{1,\ldots,d\}, \quad \min_{1\le j\le d} H^{ij} \le y^{*,i} \le \max_{1\le j\le d} H^{ij}.$$

Proof. (a) The function $\varphi_{_H}$ is well-defined on $[0,1]^d\setminus\{0\}$ since $\mathbf{w}\big(\widetilde{f}(y)\big) > 0$ on $[0,1]^d\setminus\{0\}$ owing to the fact that $f>0$ on $(0,1)$. The function $\varphi : y \mapsto \frac{\widetilde{f}(y)}{\mathbf{w}(\widetilde{f}(y))}$ clearly maps $[0,1]^d\setminus\{0\}$ into $\mathcal{S}_d$. So does $\varphi_{_H}$ since H is co-stochastic and subsequently maps $\mathcal{S}_d$ into $\mathcal{S}_d$. Since $h(y) = y - \varphi_{_H}(y)$ by (2.12), it follows that $\{h=0\} \subset \mathcal{S}_d$.

Now, both functions $y \mapsto \widetilde{f}(y)$ and $y \mapsto \mathbf{w}\big(\widetilde{f}(y)\big)$ are continuous on $\mathcal{S}_d$. Moreover $\mathbf{w}\big(\widetilde{f}(y)\big) \ge f\left(\frac{1}{d}\right) > 0$ since, for any $y = (y^1,\ldots,y^d)^t \in \mathcal{S}_d$, there exists $i_0 \in \{1,\ldots,d\}$ such that $y^{i_0} \ge \frac{1}{d}$ so that $\mathbf{w}\big(\widetilde{f}(y)\big) \ge f(y^{i_0}) \ge f\left(\frac{1}{d}\right) > 0$. Therefore, $\varphi_{_H}$ is continuous and maps $\mathcal{S}_d$ into $\mathcal{S}_d$. Then, by Brouwer's Theorem, φ_H has at least one fixed point *i.e.* $\{h=0\} \neq \varnothing$. The last claim is obvious since $\varphi(y(d)) = y(d)$ and $H\mathbf{1} = \mathbf{1}$.

(b) Let $i \in \{1,\ldots,d\}$. It follows from the identity $\displaystyle\sum_{j=1}^d H^{ij}\frac{f(y^{*,j})}{\mathbf{w}(\widetilde{f}(y^*))} = y^{*,i}$, that $\displaystyle\min_{1\le j\le d} H^{ij} \le y^{*i} \le \max_{1\le j\le d} H^{ij}$ since f is non-negative. ☐

Proposition 2.4 (Bi-stochastic case)**.** *Let $\mathcal{E}_{d,h} := \{h=0\} = \big\{y \in \mathbb{R}^d_+ \setminus \{0\} : h(y) = 0\big\} \subset \mathcal{S}_d$ denote the (non-empty) set of equilibrium points.*

(a) *If H is bi-stochastic, then $y(d) := \frac{1}{d}\mathbf{1} \in \mathcal{E}_{d,h}$.*

(b) *If $H = I_d$, then $\big\{\widetilde{e}_{_I},\, I \subset \{1,\ldots,d\}, I \neq \varnothing\big\} \subset \mathcal{E}_{d,h}$, where $\widetilde{e}_{_I} = \frac{1}{|I|}\sum_{i\in I} e^i$ with $(e^i)_{1\le i\le d}$ the canonical basis of $\mathbb{R}^d$.*

(c) *If $H = I_d$ and f is a strictly convex or strictly concave skewing function, then*

$$\mathcal{E}_{d,h} = \big\{\widetilde{e}_{_I}, I \subset \{1,\ldots,d\}, I \neq \varnothing\big\}.$$

(d) *If H is bi-stochastic and irreducible, then $\mathcal{E}_{d,h} \subset \overset{\circ}{\mathcal{S}}_d = \Big\{y \in (0,1)^d : \sum_{i=1}^d y^i = 1\Big\}$.*

(e) *If H is bi-stochastic, irreducible and f is strictly concave, then $\mathcal{E}_{d,h} = \big\{\frac{1}{d}\mathbf{1}\big\}$.*

Proof. (a) Set $y(d) = \frac{1}{d}\mathbf{1} \in \mathcal{S}_d$. Thus, $\frac{f\left(\frac{1}{d}\right)}{d\cdot f\left(\frac{1}{d}\right)} = \frac{1}{d}$ since $f\left(\frac{1}{d}\right) > 0$ and, consequently, $\displaystyle\sum_{j=1}^d H^{ij}\frac{1}{d} = 1 \times \frac{1}{d}$ for every $i \in \{1,\ldots,d\}$, therefore $h\big(y(d)\big) = y(d) - y(d) = 0$.

(b) Let $I \neq \varnothing$. Then

$$f(\widetilde{e}^{\,i}_{_I}) = \begin{cases} 0 & \text{if} \quad i \notin I \\ f\left(\frac{1}{|I|}\right) > 0 & \text{if} \quad i \in I \end{cases}.$$

Hence, $\mathbf{w}(\widetilde{f}(\widetilde{e}_{_I})) = |I| f\left(\frac{1}{|I|}\right)$ as well. Consequently $h(\widetilde{e}_{_I}) = 0$.

(c) Let $y^* \in \mathcal{E}_{d,h}$. Assume that there exists y^{*,i_0}, y^{*,i_1} such that $0 < y^{*,i_0} < y^{*,i_1} \le 1$. Then $y^{*,i_0} = \frac{f(y^{*,i_0})}{\mathbf{w}(\widetilde{f}(y^*))}$ and $y^{*,i_1} = \frac{f(y^{*,i_1})}{\mathbf{w}(\widetilde{f}(y^*))}$ so that $\frac{f(y^{*,i_0})}{y^{*,i_0}} = \frac{f(y^{*,i_1})}{y^{*,i_1}} = \mathbf{w}(\widetilde{f}(y^*))$. Now, if f or $-f$

is strictly convex, then the function $\xi\in\mathbb{R}_+\mapsto\frac{f(\xi)}{\xi}$ is strictly monotonic since $f(0)=0$. This yields a contradiction.

As a consequence, there exists $\xi_0\in(0,1]$ such that $y^{*,i}\in\{0,\xi_0\}$, $i=1,\dots,d$. Consequently, if $I_{\xi_0}=\{i: y^{*,i}=\xi_0\}$, $\mathbf{w}(\widetilde{f}(y^*))=|I_{\xi_0}|f(\xi_0)$ and, for every $i\in I_{\xi_0}$, $\frac{f(y^{*,i})}{\mathbf{w}(\widetilde{f}(y^*))}=\frac{f(\xi_0)}{|I_{\xi_0}|f(\xi_0)}=\frac{1}{|I_{\xi_0}|}$, *i.e.* $h(y^*)=\frac{1}{|I_{\xi_0}|}\sum_{i\in I_{\xi_0}}e^i$. Hence, $y^*=\frac{1}{|I_{\xi_0}|}\sum_{i\in I_{\xi_0}}e^i$ so that $\xi_0=1$ and $y^*\in\big\{\tilde{e}_I,\, I\subset\{1,\dots,d\},\, I\neq\varnothing\big\}$.

(d) Let i_0 be such that $y^{*,i_0}=\min_i y^{*,i}$. If $y^*\notin\mathring{\mathcal{S}}_d$, then $y^{*,i_0}=0$ so that

$$\sum_{j=1}^d H^{i_0j}\frac{f(y^{*,j})}{\mathbf{w}(\widetilde{f}(y^*))}=0$$

and $i_0\in I_0^*=\{i: y^{*,i}=0\}\neq\varnothing$. Let $I^*_{>0}=I\setminus I^*_0=\{j: y^{*,j}>0\}\neq\varnothing$ since $y^*\in\mathcal{S}_d$. Let us show by induction that $I^*_{>0}=\{j: y^{*,j}>0\}$ and $I^*_0=\{j=y^{*,i}_j=0\}$ are not connected by any power of H which will contradict the irreducibility of H.

Let $i_0\in I^*_0$ and $j_0\in I^*_{>0}$, then the above equality implies $H^{i_0j_0}=0$ since $f(y^{*,j_0})>0$. Now assume that $H^k_{ij}=0$ for every $(i,j)\in I^*_0\times I^*_{>0}$. Then

$$H^k_{i_0j_0}=\sum_{\ell=1}^d H_{i_0\ell}H^{k-1}_{\ell j_0}=\sum_{\ell\in I^*_{>0}}H_{i_0\ell}H^{k-1}_{\ell j_0}+\sum_{\ell\in I^*_0}H_{i_0\ell}H^{k-1}_{\ell j_0}=\sum_{\ell\in I^*_0}H_{i_0\ell}H^{k-1}_{\ell j_0}=0.$$

This contradicts the irreducibility of H since i_0 and j_0 are not connected through a power of H.

(e) We know that $\frac{1}{d}\mathbf{1}\in\mathcal{E}_{d,h}$ and that, any $y^*\in\mathcal{E}_{d,h}$, $\min_i y^{*,i}>0$. Let i_0 be such that $y^{*,i_0}=\min_i y^{*,i}$. Then

$$\sum_{j=1}^d H^{i_0j}f(y^{*,j})\geq\sum_{j=1}^d H^{i_0j}f(y^{*,i_0})=f(y^{*,i_0})$$

since H is stochastic. On the other hand, using the concavity of f, we derive that

$$\forall\, y\in\mathcal{S}_d,\quad \mathbf{w}\big(\widetilde{f}(y)\big)=d\sum_{j=1}^d\frac{1}{d}f(y_j)\leq d\cdot f\Big(\frac{1}{d}\sum_{j=1}^d y_j\Big)=d\cdot f\Big(\frac{1}{d}\Big).$$

As a consequence

$$y^{*,i_0}=\frac{\sum_{j=1}^d H^{i_0j}f(y^{*,j})}{\mathbf{w}(\widetilde{f}(y^*))}\geq\frac{f(y^{*,i_0})}{d\cdot f\left(\frac{1}{d}\right)},$$

which can be rewritten as $\frac{f(y^{*,i_0})}{y^{*,i_0}}\leq\frac{f(1/d)}{1/d}$. As $\xi\mapsto\frac{f(\xi)}{\xi}$ is (strictly) decreasing since f is strictly concave and $f(0)=0$, it implies that $y^{*,i_0}\geq\frac{1}{d}$ which in turn implies that $y^*=\frac{1}{d}\mathbf{1}$ since $y^*\in\mathcal{S}_d$. □

Remark 2.5. When H is not bi-stochastic but simply co-stochastic, we have no closed form for an $\mathring{\mathcal{S}}_d$-valued equilibrium and we could not manage to establish uniqueness even if f is (strictly) concave.

Note that, as ($\widetilde{f}$ and $\mathbf{w}$ are defined on $[0,1]^d\setminus\{0\}$, $\widetilde{f}\big([0,1]^d\setminus\{0\}\big)\subset[0,1]^d\setminus\{0\}$ and $\mathbf{w}\big(\widetilde{f}(y)\big)>0$ on $[0,1]^d\setminus\{0\}$ since $f>0$ on $(0,1)$. Hence on may define the function

$$\varphi: y\mapsto\frac{\widetilde{f}(y)}{\mathbf{w}(\widetilde{f}(y))}\quad\text{on}\quad[0,1]^d\setminus\{0\}.\tag{2.16}$$

Lemma 2.6. *If f is differentiable on in the neighbourhood of $\frac{1}{d}$, then the functions $\widetilde{f}$, $\boldsymbol{w}\circ\widetilde{f}$ and φ defined on $[0,1]^d\setminus\{0\}$ are bounded on $(0,1]^d$ and the last two functions are differentiable in the neighbourhood of $y(d)=\frac{1}{d}\mathbf{1}$. Moreover, $\varphi(y(d))=y(d)$ and the Jacobian of φ at $y(d)$ is given by*

$$J_\varphi(y(d))=\begin{pmatrix} a & b & \dots & \dots & b\\ b & a & b & \dots & b\\ \vdots & b & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & b\\ b & b & \dots & b & a\end{pmatrix} \text{ with } a=\frac{f'(1/d)(d-1)}{d^2f(1/d)} \text{ and } b=-\frac{f'(1/d)}{d^2f(1/d)}=-\frac{a}{d-1}.$$

As a symmetric matrix, $J_\varphi(y(d))$ is diagonalizable in the orthogonal group $\mathcal{O}(d,\mathbb{R})$ with eigenvalues 0 (associated to the eigenvector $\mathbf{1}$) and $\frac{f'(1/d)}{d\,f(1/d)}$ with eigenspace the hyperplane $\mathbf{1}^\perp=\Big\{u\in\mathbb{R}^d:\sum_{i=1}^d u^i=0\Big\}$. Equivalently, if $\mathrm{Proj}^\perp_{\mathbf{1}^\perp}$ denotes the orthogonal projection on $\mathbf{1}^\perp$, one has

$$J_\varphi(y(d))=\frac{f'(1/d)}{d\,f(1/d)}\cdot\mathrm{Proj}^\perp_{\mathbf{1}^\perp}. \tag{2.17}$$

Proof. If $y\in(0,1)^d$ and f is differentiable in the neighborhood of $y^1,\dots,y^d$, then φ is differentiable at y and

$$\frac{\partial\varphi^i}{\partial y^j}(y)=-\frac{f(y^i)f'(y^j)}{\mathbf{w}\big(\widetilde{f}(y)\big)^2}+\delta_{ij}\frac{f'(y^i)}{\mathbf{w}\big(\widetilde{f}(y)\big)},\ 1\le i,j\le d. \tag{2.18}$$

The form of $J_\varphi(y(d))$ follows. Elementary and classical computations show that, for every real numbers a, b,

$$\det\begin{pmatrix} a & b & \dots & \dots & b\\ b & a & b & \dots & b\\ \vdots & b & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & b\\ b & b & \dots & b & a\end{pmatrix}=(a-b)^{d-1}(a+b(d-1)),$$

which in turn implies that the characteristic polynomial of $J_\varphi(y(d))$ is given by $(a-b-\lambda)^{d-1}(a+b(d-1)-\lambda)$ so that the eigenvalues are

$$\lambda_0=a+b(d-1)=0 \text{ (order 1)} \quad\text{ and }\quad \lambda_1=a-b=\frac{f'(1/d)}{df(1/d)} \text{ (order } d-1).$$

The eigenspace associated to λ_0 is clearly $\mathbb{R}\cdot\mathbf{1}$ and that associated to λ_1 is $\mathbf{1}^\perp=\Big\{u\in\mathbb{R}^d:\sum_{i=1}^d u^i=0\Big\}$, so that $J_\varphi(y(d))_{|\mathbf{1}^\perp}=\lambda_1 I_{d|\mathbf{1}^\perp}$ and $J_\varphi(y(d))=\frac{f'(1/d)}{d\,f(1/d)}\cdot\mathrm{Proj}^\perp_{\mathbf{1}^\perp}$. □

To go beyond and provide convergence results, especially in the bi-stochastic case, we will rely on an important tool in SA theory, the ODE method. This method makes the connection between the asymptotic behaviour of the stochastic algorithm with mean field h with that of $ODE_h\equiv\dot{\theta}=-h(\theta)$ (for the results on this theory, we refer to the Appendix).

Proposition 2.7. (a) *From any $\xi\in\mathcal{S}_d$, there exists at least one $\mathcal{S}_d$-valued solution $(y(\xi,t))_{t\in\mathbb{R}_+}$ to ODE_h starting from ξ.*
(b) *Assume that the skewing function f is Lipschitz continuous on any closed interval of $(0,1]$. Then, from any $\xi\in\overset{\circ}{\mathcal{S}}_d=\mathcal{S}_d\cap(0,1]^d$, there exists a unique solution to ODE_h starting from ξ, denoted $(\Phi(\xi,t))_{t\in\mathbb{R}_+}$ is $\overset{\circ}{\mathcal{S}}_d$-valued.*

Remark 2.8. If f is concave, then it is Lipschitz continuous on $[\eta_0, 1]$ with Lipschitz coefficient $f'_\ell(\eta_0) \le 1$ for every $\eta_0 \in (0,1)$.

Proof. (a) The function h is clearly continuous on the compact set $\mathcal{S}_d \to \mathbf{1}^\perp = \{\sum_{i=1}^d u^i = 0\}$ since it does not contain 0. Hence it is bounded. Let us show that it takes values in $\mathbf{1}^\perp$. By mimicking (2.14), we prove that, for any co-stochastic matrix (here H) $\mathbf{w}(H\widetilde{f}(y)) = \mathbf{w}\big(\widetilde{f}(y)\big)$ for any $y \in \mathcal{S}_d$. Consequently, $\mathbf{w}\big(h(y)\big) = 0$ *i.e.* $h(y) \in \mathbf{1}^\perp$. Then, note that the Euler scheme of ODE_h with step $\frac{1}{n}$ starting from $\xi \in \mathcal{S}_d$ defined by

$$\bar{y}^n_{t_{\frac{k+1}{n}}} = \bar{y}^n_{\frac{k}{n}} - \tfrac{1}{n} h(\bar{y}^n_{\frac{k}{n}}) = (1 - \tfrac{1}{n})\bar{y}^n_{\frac{k}{n}} + \tfrac{1}{n} H \frac{\widetilde{f}(\bar{y}^n_{\frac{k}{n}})}{\mathbf{w}(\widetilde{f}(\bar{y}^n_{\frac{k}{n}}))}$$

is $\mathcal{S}_d$-valued. Indeed, $\mathbf{w}\big(h(\bar{y}^n_{\frac{k}{n}})\big) = 0$ so that, by linearity, $\mathbf{w}(\bar{y}^n_{\frac{k+1}{n}}) = \mathbf{w}(\bar{y}^n_{\frac{k}{n}}) = \mathbf{w}(\bar{y}^n_{\frac{0}{n}}) = \mathbf{w}(\xi) = 1$ is constant equal to 1. Moreover, one shows by induction thanks to the right hand side of the above right equality that all components of $\bar{y}^n_{\frac{k}{n}}$ are non-negative since those of $\xi \in \mathcal{S}_d$ are so. Then the interpolated functions defined by

$$\bar{y}^n_t = \bar{y}^n_0 - \int_0^t h\big(\bar{y}^n_{\underline{s}}\big) ds, \; t \ge 0, \quad \text{with} \quad \underline{s} = \tfrac{k}{n}, \; s \in [\tfrac{k}{n}, \tfrac{k+1}{n})$$

are $\mathcal{S}_d$-valued since the simplex is convex, K-Lipschitz continuous with $K = K(h) = \sup_{y \in \mathcal{S}_d} |h(y)|$. Consequently, they are relatively compact for the convergence on compact sets of $\mathbb{R}_+$ owing to the Arzela-Ascoli Theorem. Any limiting function y^∞ of $(\bar{y}^n)_{n \ge 1}$ is $\mathcal{S}_d$-valued and solution to ODE_h. On the other hand, the Euler scheme converges owing to Peano's theorem to a solution of ODE_h.

(b) Let $\eta_0 \in (0, \frac{1}{d}]$. The function f is Lipschitz continuous on $[\eta_0, 1]$. Hence the (canonical extensions of) the functions $\widetilde{f}$ and $\mathbf{w}(\widetilde{f})$ on $[\eta_0, 1]^d$ are Lipschitz as well. Both are also bounded. Moreover $\mathbf{w}\big(\widetilde{f}(y)\big) \ge f\left(\frac{1}{d}\right) > 0$ because for any $y = (y^1, \ldots, y^d)^t \in \mathcal{S}_d$, there exists $i_0 \in \{1, \ldots, d\}$ such that $y^{i_0} \ge \frac{1}{d}$ so that $\mathbf{w}\big(\widetilde{f}(y)\big) \ge f(y^{i_0}) \ge f\left(\frac{1}{d}\right) > 0$. As a consequence, $\mathbf{w}(\widetilde{f}) \ge \frac{1}{2} f\big(\frac{1}{d}\big)$ on a neigbourhood of $\mathcal{S}_d$ in $[0,1]^d$. Therefore, the functions $\varphi_{_H} : y \mapsto H \frac{\widetilde{f}(y)}{\mathbf{w}(\widetilde{f}(y))}$ from $\mathcal{S}_d \cap [\eta_0, 1]^d$ to $\mathcal{S}_d$ and $h = I_d - \varphi_{_H}$ from $\mathcal{S}_d \cap [\eta_0, 1]^d$ to $\{\sum_{i=1}^d u^i = 0\}$ are Lipschitz continuous too in that neighbourhood. Since this holds for every small enough η_0, this guarantees the existence of an $\mathcal{S}_d$-valued flow for ODE_h on $\mathcal{S}_d \cap (0,1]^d = \overset{\circ}{\mathcal{S}}_d$. $\square$

Definition 2.9 (Stable equilibrium point)**.** *An element y^* of $\mathcal{S}_d$ is a* stable equilibrium *for $ODE_h \equiv \dot{y} = -h(y)$ on $\mathcal{S}_d$ if there is a (compact) neighborhood $\mathcal{K}^*$ of y^* in $\mathcal{S}_d$ such that*

$$\lim_{t \to +\infty} \sup \Big\{ \big|y(\xi, t) - y^*\big|, \, \xi \in \mathcal{K}^*, \; y(\xi, \cdot) \; \mathcal{S}_d\text{-valued solution to } ODE_h, \; y(\xi, 0) = \xi \Big\} = 0$$

with a slight abuse of notation since possibly several solutions may start from $\xi \in \partial\mathcal{S}_d$. See also Theorem A.2$(b)$.

Remark 2.10. • If the $\mathcal{S}_d$-valued flow of ODE_h is well-defined in the neighborhood of y^* in $\mathcal{S}_d$, this boils down to showing that $y(\xi, t)$ is $\mathcal{S}_d$-valued and converges to y^* as $t \to +\infty$, uniformly with respect to ξ in a (compact) neighborhood of y^* in $\mathcal{S}_d$.

• Since h is differentiable, an equilibrium y^* is attractive if all the eigenvalues of $J_h(y^*)_{|\mathbf{1}^\perp} = \big(I_d - HJ_\varphi(y^*)\big)_{|\mathbf{1}^\perp}$ have (strictly) positive real parts, see [6]. If one of these eigenvalues has a negative real part, then the equilibrium is *unstable* and, if all eigenvalues have negative real parts, then the equilibrium is called a *repeller*.

We recall the notation $y(d) = \frac{\mathbf{1}}{d}$ introduced in Proposition 2.4. It is a zero of h when H is bi-stochastic.

Proposition 2.11. *Assume H is a bi-stochastic matrix. Then $H\mathbf{1}^\perp \subset \mathbf{1}^\perp$. Let $\lambda_1 = \frac{f'(1/d)}{d\cdot f(1/d)}$ and let $\mu_{\max}$ be the eigenvalue of the restriction $H_{|\mathbf{1}^\perp}$ with the highest real part.*

(a) *If $\Re e(\mu_{\max}) < \frac{1}{\lambda_1}$, then $y(d)$ is a stable equilibrium of $ODE_h \equiv \dot y = -h(y)$. Thus, if*

$$\big(\lambda_1 \le 1 \ \text{ and } 1 \notin \mathrm{Sp}(H_{|\mathbf{1}^\perp})\big) \text{ or } (\lambda_1 < 1),$$

then $y(d)$ is a stable equilibrium.

In particular the above left condition is satisfied if H is irreducible and f is concave, whereas the right one is always fulfilled if f is strictly concave over $(0, 1/d)$.

(b) *If $\Re e(\mu_{\max}) > \frac{1}{\lambda_1}$, then $y(d)$ is unstable (it is even a repeller when $H = I_d$). Note that if f is convex (resp. strictly over $(0, \frac{1}{d})$), then $\lambda_1 \ge 1$ (resp. > 1).*

Remark 2.12. • If $\lambda_1 > 1$ ($e.g.$ if f is strictly convex), then $\frac{1}{\lambda_1} < 1$ and the two opposite situations $\Re e(\mu_{\max}) < \frac{1}{\lambda_1}$ and $\Re e(\mu_{\max}) > \frac{1}{\lambda_1}$ may occur a priori *i.e.* $y(d)$ may switch from uniform stability to instability (see Section 3 for the two-type case: $d = 2$).

• Note that if f is strictly convex and $1 \in \mathrm{Sp}(H_{|\mathbf{1}^\perp})$, then $y(d)$ is always unstable.

Proof. (a) It follows from Lemma 2.6 that $J_\varphi(y(d)) = \lambda_1 \mathrm{Proj}^\perp_{\mathbf{1}^\perp}$. Hence

$$J_h\big(y(d)\big) = I_d - \lambda_1 H \mathrm{Proj}^\perp_{\mathbf{1}^\perp}. \tag{2.19}$$

Hence the restriction of $J_h\big(y(d)\big)_{|\mathbf{1}^\perp}$ takes values in $\mathbf{1}^\perp$ since $H^\perp \subset \mathbf{1}^\perp$. Consequently, $J_h\big(y(d)\big)_{|\mathbf{1}^\perp} = \big(I_d - \lambda_1 H\big)_{|\mathbf{1}^\perp}$. It follows that $\mathrm{Sp}\big(J_h\big(y(d)\big)_{|\mathbf{1}^\perp}\big) = \big\{1 - \lambda_1\mu, \mu \in \mathrm{Sp}(H_{|\mathbf{1}^\perp})\big\} \subset \mathbb{C}$.

Every $\mu \in \mathrm{Sp}(H)$ satisfies $|\mu| \le 1$ since H is bi-stochastic. If $\lambda_1 < 1$, then $|\lambda_1 \mu_{\max}| < 1$ and consequently $\Re e(1 - \lambda_1\mu_{\max}) > 0$ which ensures that $y(d)$ is attracting. The other case follows likewise.

When H is irreducible, the eigenvalue 1 is simple owing to the Perron-Frobenius Theorem so that $1 \notin \mathrm{Sp}(H_{|\mathbf{1}^\perp})$.

(b) If f is convex (and $f > 0$ on $(0,1)$), then $0 < f(1/d) \le f'(1/d)/d$ since $f(0) = 0$ so that $\lambda_1 \ge 1$. This inequality is strict if f is strictly convex over $(0, \frac{1}{d})$. ☐

Proposition 2.13. (a) *If H is bi-stochastic and irreducible and f is strictly concave, then ODE_h has at least one solution starting from every $\xi \in \mathcal{S}_d$. The flow of $ODE_h \equiv \dot y = -h(y)$, denoted by $\big(y(\xi,t)\big)_{\xi \in \mathring{\mathcal{S}}_d}$, $t \ge 0$, exists on $\mathring{\mathcal{S}}_d$ and is $\mathcal{S}_d$-valued. Furthermore, still with $y(d) = \frac{\mathbf{1}}{d}$,*

$$\lim_{t\to+\infty} \sup\Big\{\Big|y(\xi,t) - y(d)\Big|,\ \xi \in \mathcal{S}_d,\ y(\xi,.) \text{ solution to } ODE_h,\ y(\xi,0) = \xi\Big\} = 0.$$

(b) *If H is simply bi-stochastic (and possibly not irreducible), then the flow, denoted by $y(\xi,t)$, converges toward $y(d)$ for every $\xi \in \mathring{\mathcal{S}}_d$.*

Proof. (a) First, H being bi-stochastic and irreducible, it follows from Proposition 2.4 that $\mathcal{E}_{d,h} = \{y(d)\}$. Let us denote by $y(\xi,t)$ an $\mathcal{S}_d$-valued solution starting from $\xi \in \mathcal{S}_d$ and by $y^i(\xi,t)$, $i = 1,\dots,d$ its components.

Let $\xi \in \mathcal{S}_d \setminus \{y(d)\}$ and let $i(t)$ be the right continuous function such that $y^{i(t)}(\xi,t) = \min_j y^j(\xi,t) \in [0, \frac{1}{d}]$. We know that $y^{i(0)}(\xi,0) = \xi^{i(0)} < \frac{1}{d}$ since $\xi \ne y(d)$. Note that the function $g_i : t \mapsto y^{i(t)}(\xi,t)$ is continuous and right differentiable with a right derivative given by

$$(g_i)'_r(\xi,t) = \sum_{j=1}^{d} H^{i(t)j} \frac{f(y^j(\xi,t))}{\mathbf{w}(\widetilde{f}(y(\xi,t)))} - y^{i(t)}(\xi,t) \geq 1 \times \frac{f(y^{i(t)}(\xi,t))}{\mathbf{w}(\widetilde{f}(y(\xi,t)))} - y^{i(t)}(\xi,t).$$

Note that $\mathbf{w}(\widetilde{f}(y(\xi,t))) \leq d \cdot f\left(\frac{1}{d}\right)$ since f is concave. It follows that

$$(g_i)'_r(\xi,t) \geq \frac{f(y^{i(t)}(\xi,t))}{d \cdot f\left(\frac{1}{d}\right)} - y^{i(t)}(\xi,t) \geq 0, \tag{2.20}$$

since $u \mapsto \frac{f(u)}{u}$ is non-increasing (f is concave and $f(0)=0$) and $y^{i(t)}(\xi,t) \leq \frac{1}{d}$ for every $t \geq 0$.

If $\xi^{i(0)} = 0$, then $y^{i(t)}(\xi,t) \geq 0$, for every $t \geq 0$. Assume there exists $\varepsilon_0 > 0$ such that $y^{i(t)}(\xi,t) = 0$ for every $t \in (0,\varepsilon_0]$, then one derives from the integrated form of ODE_h that

$$\int_0^{\varepsilon_0} \left(\sum_{j=1}^{d} H^{i(t)j} \frac{f(y^j(\xi,s))}{\mathbf{w}(\widetilde{f}(y(\xi,s)))} \right) ds = 0,$$

so that, as the above integrand is non-negative and continuous, the function $s \mapsto \sum_{j=1}^{d} H^{i(s)j} \frac{f(y^j(\xi,s))}{\mathbf{w}(\widetilde{f}(y(\xi,s)))} \equiv 0$ on $[0,\varepsilon_0]$. Let $I_0 = \{i \text{ s.t. } y^i(\xi,\varepsilon_0) = 0,\, 1 \leq i \leq d\}$ and $I_1 = I_0^c$. Then, it follows that $i(\varepsilon_0) \in I_0$ and $H^{i(\varepsilon_0)j} = 0$, for every $j \in I_1$, which is not empty since $\sum_j y^j(\xi,t) = 1$. Hence, reasoning like in Proposition 2.4(d), $i(\varepsilon_0)$ (and more generally any element of I_0) and I_1 are not H-connected which contradicts the irreducibility. Consequently, $y^{i(t)}(\xi,t) > 0$, $t \in (0,\varepsilon_0]$, *i.e.* $y(\xi,t)$ lives in $\mathring{\mathcal{S}}_d$ and, consequently, lives in $\mathring{\mathcal{S}}_d$ for every $t \geq 0$ by monotonicity of $y^{i(t)}(\xi,t)$.

If $\xi^{i(0)} > 0$, the same conclusion follows simply from the monotonicity of $y^{i(t)}(\xi,t)$.

Finally, for every $t > 0$, $y^{i(t)}(\xi,t)$ is positive and increasing. We can integrate (2.20) between a fixed $\varepsilon > 0$ and $t > \varepsilon$ as follows

$$-\log\left(\frac{d}{\xi^{i(0)}(\varepsilon)}\right) \geq \log\left(\frac{y^{i(t)}(\xi,t)}{\xi^{i(0)}(\varepsilon)}\right) \geq \int_{\varepsilon}^{t} \left(\underbrace{\frac{f(y^{i(s)}(\xi,s))}{y^{i(s)}(\xi,s)} - \frac{f(1/d)}{1/d}}_{\geq 0} \right) ds \geq 0$$

and the latter integral is increasing in t as long as $y^{i(t)}(\xi,t) < \frac{1}{d}$. As the left hand side of the above string of inequalities is finite, it follows that $\int_0^{+\infty} \left(\frac{f(y^{i(s)}(\xi,s))}{y^{i(s)}(\xi,s)} - \frac{f(1/d)}{1/d} \right) ds < +\infty$. Then, combining that $u \mapsto \frac{f(u)}{u}$ is decreasing (by strict concavity) on $(0,+\infty)$ with the fact that $y^{i(t)}(\xi,t)$ is increasing and positive, we derive that the function $t \mapsto \frac{f(y^{i(t)}(\xi,t))}{y^{i(t)}(\xi,s)} - \frac{f(1/d)}{1/d}$ is decreasing. The finiteness of the above integral implies that $\dfrac{f(1/d)}{1/d} - \dfrac{f(y^{i(t)}(\xi,t))}{y^{i(t)}(\xi,t)} \to 0$ as $t \to +\infty$ or, equivalently, that

$$y^{i(t)}(\xi,t) = \min_{1 \leq i \leq d} y^i(\xi,t) \to \frac{1}{d} \quad \text{as} \quad t \to +\infty.$$

The convergence of every component $y^i(\xi,t)$ follows since $y(\xi,t) \in \mathcal{S}_d$. It remains to prove that the convergence holds uniformly in the starting value (the existence of the flow *i.e* uniqueness starting from the boundary of the simplex is not mandatory). First note that, h being bounded, the family of all possible solutions $((y(\xi,t)_{t\ge 0})_{\xi\in\mathcal{S}_d})$ of ODE_h are $\|h\|_\infty$-Lipschitz continuous since

$$y(\xi,t) = \xi - \int_0^t h\big(y(\xi,s)\big)ds.$$

Assume there exists $\xi_n \to \xi_\infty$ and $t_n \to +\infty$ such that $\big|y(\xi_n,t_n) - y(d)\big| \ge \varepsilon_0 \ge 0$. By Proposition 2.6, $\frac{\mathbf{1}}{d}$ is an attractor of ODE_h, hence there exists $\eta_0 > 0$ such that $\sup_{|\xi - y(d)|\le\eta_0}\big|y(\xi,t) - y(d)\big| \le \varepsilon_0$, $t \ge t_0$ (the flow does exist in the neighbourhood of $y(d)$ so that $y(\xi,\cdot)$ is unique). Consequently, for every $t \in [0, t_n - t_0]$ (at least for n large enough), $|y(\xi_n,t) - y(d)| > \eta_0$. By Ascoli's Theorem, up to an extraction, one may assume that $y(\xi_n,\cdot)$ converges on compact sets of $\mathbb{R}_+$ toward a solution $y(\xi_\infty,\cdot)$ of ODE_h since h is continuous. Then, letting n go to infinity shows that this solution satisfies $|y(\xi_\infty,t) - y(d)| \ge \eta_0$ for every $t \ge 0$. This contradicts the fact that any solution starting from $\mathcal{S}_d$ converges toward $y(d)$. We can conclude that all the solutions of ODE_h starting from $\mathcal{S}_d$ converge uniformly toward $y(d)$.

(b) is obvious, given the above proof since no component is 0 at $t = 0$. ☐

Application (I): $A.s.$ **convergence of the algorithm when** f **is concave.** When f is concave and H is bi-stochastic, the mean point $y(d) = \frac{\mathbf{1}}{d}$ of the simplex is the unique $a.s.$ target of the urn composition.

Proposition 2.14 (When f is concave, $y(d)$ is the target)**.** *If H is (deterministic and) bi-stochastic, irreducible and f is a strictly concave skewing function, then*

$$\widetilde{Y}_n \xrightarrow[n\to+\infty]{a.s.} y(d) = \frac{\boldsymbol{1}}{d}.$$

Proof. It follows from Proposition 2.13(a) that the flow of ODE_h uniformly converges to $y(d)$. Hence, Theorem A.2(b) from the Appendix implies that the set Θ^∞ of the limiting values of $(\widetilde{Y}_n)_{n\ge 0}$ is $a.s.$ is reduced to $\{\frac{1}{d}\mathbf{1}\}$ which completes the proof. ☐

Application (II): The mean point $y(d)$ **may be a noisy trap when** f **is convex.** We now give a (partial) result in the convex setting (a more precise one is provided in Section 5 devoted to randomized Pólya's urns, that is when $H = I_d$). We keep the notations introduced Propositions 2.11: $\lambda_1 = \frac{f'(1/d)}{df(1/d)}$ and $\mu_{\max}$ the eigenvalues of $H_{|\mathbf{1}^\perp}$ with the highest real part.

As H is (bi-)stochastic, $\Re e(\mu_{\max}) \le 1$. If H is also irreducible, by the Perron-Frobenius Theorem, we know that $\Re e(\mu_{\max}) < 1$ since the eigenspace of the eigenvalue 1 is one-dimensional, hence equal to $\mathbb{R}\cdot\mathbf{1}$. On the other hand, when f is convex then $\lambda_1 \ge 1$ and if f is strictly convex, on $[0,\frac{1}{d}]$, then $\lambda_1 > 1$. Consequently there exist matrices H for which the inequality $\Re e(\mu_{\max}) > \frac{1}{\lambda_1}$ is satisfied, implying by Proposition 2.11 that, in that case, $y(d)$ is unstable for ODE_h. We show that $y(d)$ may be a noisy trap.

Proposition 2.15 (When $y(d)$ becomes a noisy trap)**.** *Assume that* ***(A1)*** *and* ***(A3)*** *are in force and that the following slightly reinforced version of* ***(A2)*** *holds for some $\delta > 0$*

$$(\mathbf{A2})_{2+\delta} \;\equiv\; \forall i,j \in \{1,\dots,d\}, \quad \sup_{n\ge 1}\mathbb{E}\left[\|D_n^{ij}\|^{2+\delta}\,|\,\mathcal{F}_{n-1}\right] < +\infty \quad a.s.$$

Assume that he skewing unction f is convex and H is a bi-stochastic matrix satisfying $\Re e(\mu_{\max}) > \frac{1}{\lambda_1}$ so that $y(d)$ is unstable.
If H is invertible or symmetric, then $\mathbb{P}\big(\widetilde{Y}_n \to y(d)\big) = 0$.

Proof. We want to apply Theorem A.3 from the Appendix. The remainder term (2.13) and the martingale increment fulfill condition (A.3), owing to $(\mathbf{A2})_{2+\delta}$ and $(\mathbf{A3})$, once noted as concerns the remainder, that $0 \le \frac{\widetilde{f}}{\mathbf{w}(f)} \le \frac{d}{f(1/d)}$.

$\triangleright$ H *symmetric.* We will firstly focus on the symmetric case (the invertible case relies on the same formal proof applied to any nonzero vector v with $\mathbf{w}(v) = 0$). We have to check that the Jacobian J_h is locally Lipschitz continuous in the neighbourhood of $y(d)$. Note that $h = I_d - H\varphi$ where φ is defined in (2.16). This follows from the expression (2.18) for $\frac{\partial \varphi_i}{\partial x_j}$ having in mind that f' is non-decreasing by convexity of f. Then, we have to check Assumption (A.4). The Jacobian $J_h\big(y(d)\big) = I_d - \lambda_1 H \mathrm{Proj}^{\perp}_{\mathbf{1}^\perp}$ (see (2.19)) is self-adjoint since H is symmetric and commutes with $\mathrm{Proj}^{\perp}_{\mathbf{1}^\perp}$. Hence, its restriction to $\mathbf{1}^\perp$ leaves $\mathbf{1}^\perp$ stable and is also self-adjoint and its eigenspaces are orthogonal in $\mathbf{1}^\perp$. In particular, the stable and unstable subspaces K_+ and K_- as defined in Theorem A.3 are orthogonal and $\Delta M^{rep}_{n+1} = \mathrm{Proj}^{\perp}_{K_-}(\Delta M_{n+1})$. Consequently, it suffices to show that, for a unitary eigenvector vector $v_\mu \in \mathbf{1}^\perp$ attached to an eigenvalue $\mu > \frac{1}{\lambda_1}$ (hence lying in K_-)

$$\liminf_n \mathbb{E}\left[\|\Delta M^{rep}_{n+1}\|^2 \,|\, \mathcal{F}_n\right] \ge \liminf_n \mathbb{E}\left[(\Delta M_{n+1}|v_\mu)^2 \,|\, \mathcal{F}_n\right] > 0 \ a.s.$$

Elementary computations show (keeping in mind that $u \otimes v = [u^i v^j]_{1\le i,j\le d}$) that

$$\mathbb{E}\left[(\Delta M_{n+1}|v_\mu)^2 \,|\, \mathcal{F}_n\right] = v_\mu^t \left[\mathbb{E}\left[D_{n+1}\mathrm{diag}(\varphi_{_H}(\widetilde{Y}_n))D^t_{n+1} \,|\, \mathcal{F}_n\right] - 2\varphi_{_H}(\widetilde{Y}_n) \otimes \widetilde{Y}_n + \widetilde{Y}_n^{\otimes 2}\right] v_\mu.$$

On the event $\big\{\widetilde{Y}_n \to y(d)\big\}$, we derive, owing to Assumption $(\mathbf{A2})$, the continuity of $\varphi_{_H}$ and $\varphi_{_H}(y(d)) = y(d)$, that

$$\mathbb{E}\left[(\Delta M_{n+1}|v_\mu)^2 \,|\, \mathcal{F}_n\right] = v_\mu^t \mathbb{E}\left[D_{n+1}\mathrm{diag}\big(y(d)\big)D^t_{n+1} \,|\, \mathcal{F}_n\right] v_\mu - (v_\mu|y(d))^2 + o(1).$$

Now, $\mathrm{diag}\big(y(d)\big) = \frac{1}{d} I_d$, and $(v_\mu|y(d))^2 = 0$ since $v_\mu \in \mathbf{1}^\perp$. Consequently,

$$\liminf_n \mathbb{E}\left[(\Delta M_{n+1}|v_\mu)^2 \,|\, \mathcal{F}_n\right] = \frac{1}{d} \liminf_n \mathbb{E}\left[\|D^t_{n+1} v_\mu\|^2 \,|\, \mathcal{F}_n\right].$$

Finally, we not that, owing to Jensen's Inequality,

$$\mathbb{E}\left[\|D^t_{n+1} v_\mu\|^2 \,|\, \mathcal{F}_n\right] \ge \|H^t_n v_\mu\|^2 \xrightarrow{a.s.} \|H^t v_\mu\|^2 = \|H v_\mu\|^2 = \mu^2 > 0.$$

$\triangleright$ H *is invertible (and bi-stochastic).* We know that $J_h\big(y(d)\big)$ leaves $\mathbf{1}^\perp$ stable and we consider its restriction to it in the rest of the proof. If we consider again the stable and unstable subspaces K_+ and K_- (in $\mathbf{1}^\perp$) associated to $J_h\big(y(d)\big)$, as defined in Theorem A.3 and $\pi^{rep} : \mathbf{1}^\perp \to K_-$ the projection onto K_- alongside K_+, then for every unitary vector v such that $\mathbf{w}(v) = 0$, one has

$$|\Delta M^{rep}_{n+1}|^2 \ge \big|(\pi^{rep}\Delta M^{rep}_{n+1} \,|\, v)\big|^2 = \big|(\pi^{rep,t} v \,|\, \Delta M^{rep}_{n+1})\big|^2.$$

Following the lines of the proof of claim (a) with $\pi^{rep,t} v$ instead of v_μ yields

$$\liminf_n \mathbb{E}\left[|\Delta M^{rep}_{n+1}|^2 \,|\, \mathcal{F}_n\right] \ge \frac{1}{d}\|H\pi^{rep,t} v\|^2.$$

As H is invertible, the right hand side of the inequality is always positive for some v since $\pi^{rep,t}$ is not identically 0. □

Example 2.16. • If we set $f(u) = u^\alpha$, $\alpha > 1$, $d = 2$ and $D_n = H = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$, $p \in (0,1)$, then f is a (strictly) convex skewing function and H is a bi-stochastic matrix.

Note that $\lambda_1 = f'(1/2)/(2f(1/2)) = \alpha$ and $\mu_{\max} = 2p-1$. Hence, if $p > \frac{1}{2}+\frac{1}{2\alpha}$, then $y(2) = (\frac{1}{2},\frac{1}{2})^t$ is unstable. Moreover as $H\begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix} = (2p-1)\begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}$, then $v_{\mu_{\max}} = \begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}$ and $\liminf_n \mathbb{E}\big[(\Delta M^{rep}_{n+1})^2 \,|\, \mathcal{F}_n\big] \geq \frac{1}{2}(2p-1)^2 > 0$.

• If we keep $f(u)=u^\alpha$, $\alpha>1$, with $d=3$ and $D_n=H=\begin{pmatrix}p & a & 1-p-a\\ 1-p-a & p & a\\ a & 1-p-A & p\end{pmatrix}$, $p,a\in(0,1)$, then f is a (strictly) convex skewing function and H is an invertible bi-stochastic matrix. Note that $\lambda_1 = f'(1/3)/(3f(1/3)) = \alpha$ and $\Re e(\mu_{\max}) = \frac{3p-1}{2}$. Hence, if $p > \frac{2+\alpha}{3\alpha}$, then $y(3) = (\frac{1}{3},\frac{1}{3},\frac{1}{3})^t$ is unstable.

3 Bi-dimensional non linear randomized urn model

When the skewing function f is convex the situation becomes much more involved: thus, if H is bi-stochastic and irreducible, then $y(d)$ is an equilibrium of h and $\mathcal{E}_{d,h}\subset \mathring{\mathcal{S}}_d$ (cf. Proposition 2.4(d)), but $\mathcal{E}_{d,h}$ is not reduced to $y(d)$ (which is a unstable for ODE_h when $H=I_d$).

To start elucidating this case, we limit ourselves in this paper to a two-type urn ($d=2$) and an irreducible matrix H. We will see, as expected, that the asymptotic behavior of the urn is much more involved since a phase transition appears.

In such a simplified setting, the irreducible co-stochastic generating matrix H can be written as follows

$$H = \begin{pmatrix}p_1 & 1-p_2\\ 1-p_1 & p_2\end{pmatrix},\qquad 0<p_i<1,\quad i=1,2,$$

and the mean function h of the model is still given by (2.12).

In this section, we assume for simplicity that the skewing function f *is differentiable on* $(0,1)$, so that the mean field h is differentiable too on $\mathring{\mathcal{S}}_2$. Note that, as soon as f is convex or compact, f' can be extended on $[0,1]$ into a monotonic $\bar{\mathbb{R}}_+$-valued function. We saw that analyzing the $a.s.$ convergence properties essentially boils down to elucidate the behavior of ODE_h on $\mathcal{S}_2$ which in turn can be reduced to a one dimension differential system since the simplex $\mathcal{S}_2$ can be parametrized by $(u,1-u)$, $u\in[0,1]$, and h is $\mathbf{1}^\perp = \{z : z^1+z^2=0\}$-valued on $\mathring{\mathcal{S}}_2$ so that $h^2 = 1-h^1$. Thus, the asymptotic analysis of ODE_h is equivalent to that of

$$ODE_{h_0} \equiv \dot{u} = -h_0(u) \quad \text{with} \quad h_0(u) = h^1(u,1-u),\quad u\in[0,1].$$

Elementary computations show that, for every $u\in[0,1]$,

$$h_0(u) = u - \frac{p_1 f(u) + (1-p_2)f(1-u)}{f(u)+f(1-u)} \tag{3.1}$$

and

$$h_0'(u) = 1-(p_1+p_2-1)\frac{f'(u)f(1-u)+f(u)f'(1-u)}{\big(f(u)+f(1-u)\big)^2}. \tag{3.2}$$

We will now determine the combinatorics and the nature of the equilibrium points depending on the parameters p_1 and p_2.

Note that H is bi-stochastic if and only if $p_1=p_2$. Note also that $h_0\big(\frac{1}{2}\big) = \dfrac{p_2-p_1}{2}$ whatever the skewing function f is.

3.1 Counting equilibrium points

It follows from (3.1) that the equation $h_0(u) = 0$ reads

$$(p_1 - u)f(u) + (1 - p_2 - u)f(1 - u) = 0. \tag{3.3}$$

As preliminary remarks, note that:

– if $p_1 = 1 - p_2$, then $u^* = p_1$ is the unique solution of (3.3) because $f > 0$ on $(0, 1]$ and $h_0'(u^*) = 1 > 0$ so that u^* is a stable equilibrium.

– For every $u \in [0, 1]$,

$$h_0(u) + h_0(1 - u) = p_2 - p_1. \tag{3.4}$$

– Hence, h_0' is symmetric and h_0'' is antisymmetric w.r.t. $\frac{1}{2}$. In particular $h_0''\left(\frac{1}{2}\right) = 0$.

– When the generating matrix H is bi-stochastic (if and only if $p_1 = p_2$), then h_0 itself is symmetric w.r.t. $\frac{1}{2}$ and $u^* = \frac{1}{2}$ is always solution to (3.3). Thus, if $h_0'\left(\frac{1}{2}\right) = 0$, as $h_0''\left(\frac{1}{2}\right) = 0$, the status of the equilibrium u^* requires to investigate higher order (see Example in Section 3.2 devoted to the bi-stochastic case).

– Since $h_0(0) = -(1 - p_2) < 0$ and $h_0(1) = 1 - p_1 > 0$, h_0 always has (as expected) at least one zero on $(0, 1)$.

Let u^* be a solution of (3.3). From Proposition 2.3(b), we have that any zero u^* of h_0 lies in $I^* := [p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2)]$ (or in its interior $\overset{\circ}{I^*}$).

Proposition 3.1. *Let p_1, $p_2 \in (0, 1)$ and let f be a skewing function.*

(a) *If $p_1 + p_2 \leq 1$, then h_0 has a unique zero, lying in I^* (and in $\overset{\circ}{I^*}$ if $p_1 + p_2 < 1$).*

(b) *If $p_1 + p_2 > 1$ and f is concave, then h_0 has a unique zero u^*, lying in $\overset{\circ}{I^*}$.*

(c) *If $p_1 + p_2 > 1$ and f is convex, then h_0 has at least one zero, lying in $\overset{\circ}{I^*}$. If, furthermore, the function $\bar{f}(u) = \frac{f(u)}{f(u)+f(1-u)}$ is convex on $[0, 1/2]$, then h_0 may have one, two or three zeros, all lying in $\overset{\circ}{I^*}$. However, when $f'(1) \leq \frac{2f(1/2)}{p_1+p_2-1}$, then h_0 is increasing and h_0 subsequently has a unique zero u^*, lying in $\overset{\circ}{I^*}$.*

Proof. (a) As f is non-decreasing and non-negative, $h_0' > 0$ when $p_1 + p_2 \leq 1$, therefore h_0 is increasing with $h_0(0) = p_2 - 1 < 0$ and $h_0(1) = 1 - p_1 > 0$, so h_0 has a unique zero lying in I^*.

(b) Assume that $p_1 + p_2 > 1$. Then it is obvious that p_1 and $1 - p_2$ are not solutions of (3.3) which can be rewritten as

$$g_1(u) = g_2(u) \quad \text{on} \quad \mathcal{J} = [0, 1] \setminus \{p_1, 1 - p_2\},$$

where

$$g_1(u) = \frac{f(1 - u)}{p_1 - u}, \quad u \in [0, 1] \setminus \{p_1\} \quad \text{and} \quad g_2(u) = \frac{f(u)}{u - 1 + p_2}, \quad u \in [0, 1] \setminus \{1 - p_2\}. \tag{3.5}$$

Let us compute the first derivative of these functions: We obtain, for $u \in \mathcal{J}$,

$$g_1'(u) = \frac{f(1 - u) - f'(1 - u)(p_1 - u)}{(p_1 - u)^2} \quad \text{and} \quad g_2'(u) = \frac{f'(u)(u - 1 + p_2) - f(u)}{(u - 1 + p_2)^2}.$$

As f is concave non-negative and $f(0) = 0$, $f(u) - uf'(u) \geq 0$, $u \in [0, 1]$. Let us show that, as $0 < p_1, p_2 < 1$, $g_1' > 0$ and $g_2' < 0$ on I^*. In fact,

$$g_2'(u) = \frac{uf'(u) - f(u) - (1 - p_2)f'(u)}{(u - 1 + p_2)^2} \leq 0 \quad \text{since} \quad f'(u) \geq 0.$$

If $uf'(u) - f(u) - (1-p_2)f'(u) = 0$, then $(1-p_2)f'(u) \le 0$ so that $f'(u) = 0$ since $p_2 < 1$. Hence so that $f \equiv 0$ on $[0,u]$ which contradicts the positivity of f on $(0,1)$ as a skewing function. This shows that $g_2' < 0$. One shows likewise that $g_1' > 0$. Hence (3.3) has a unique solution $u^* \in \overset{\circ}{I^*}$.

(c) If $p_1 + p_2 > 1$ and f is convex, then $f'(0)$ exists in $\mathbb{R}_+$ and $f'(0) \le f(1) - f(0) = 1$. Note that $h_0(1/2) = \frac{p_2 - p_1}{2}$ which sign depends on $p_2 - p_1$. Moreover, one checks that $h_0'(u) = 1 - (p_1 + p_2 - 1)\bar{f}'(u)$ by (3.2) so that h_0' is non-increasing on $[0,1/2]$ owing to the convexity of $\bar{f}$. Hence h_0 is concave on $[0,1/2]$.

(To help understanding the following reasoning, see Figure 1-$(right)$ where several situations are illustrated).

$\triangleright$ If $p_2 > p_1$, then $h_0(1/2) > 0$ and $h_0'(0) = 1 - (p_1+p_2-1)f'(0) \ge (1-p_1) + (1-p_2) > 0$. Thus h_0' has at most one zero $u_0 \in (0,1/2)$. If such is the case, h_0 attains at u_0 its maximum $h_0(u_0) \ge h_0(1/2) > 0$ on $[0,1/2]$ and h_0 has a unique zero u_1^* (lying in $(0,u_0) \cap I^* = (1-p_2, u_0)$) on $[0,1/2]$ with $h_0'(u_1^*) > 0$. Otherwise h_0 is increasing and has again only one zero $u_1^* \in (1-p_2, 1/2)$ with $h_0'(u_0) > 0$. On $[1/2,1]$, using the symmetry (3.4) of h_0 w.r.t. $1/2$, we derive that h_0 has no zero on $(1/2,1)$ if u_0 does not exist and that, otherwise, their number depends on the sign of $h_0(1-u_0) = p_2 - p_1 - h_0(u_0)$. If $h_0(1-u_0) > 0$ there is again no zero on $(1/2,1)$, if $h_0(1-u_0) = 0$, $u_2^* = 1-u_0$ is the only zero in $(1/2,1) \cap I^* = (1/2, p_1)$ (with $h_0'(u_2^*) = 0$) and, finally, if $h_0(1-u_0) < 0$, there are two additional zeros u_2^* and u_3^* in $(1/2,p_1)$ such that $h_0'(u_2^*) < 0$ and $h_0'(u_3^*) > 0$.

$\triangleright$ If $p_2 < p_1$, one can make the same reasoning as above with the function $u \mapsto -h_0(1-u)$. This yields to the following three possible situations: one zero $u_1^* \in (1/2, p_1)$ with $h_0'(u_1^*) > 0$; two zeros $u_1^* \in (1-p_2, 1/2)$ with $h_0'(u_1^*) = 0$ and $u_2^* \in (1/2, p_1)$ with $h_0'(u_2^*) > 0$; three zeros u_1^*, $u_2^* \in (1-p_2, 1/2)$ with $h_0'(u_1^*) > 0$, $h_0'(u_2^*) < 0$ and $u_3^* \in (1/2, p_1)$ with $h_0'(u_3^*) > 0$.

Finally, if $\bar{f}'(1) < (p_1 + p_2 - 1)^{-1}$ on $(0,1)$, then h_0 is increasing on $[0,1]$ and has a unique zero. As f is convex, $f'(u) \le f'(1)$, $u \in [0,1]$, and $f(u) + f(1-u) \ge 2f(1/2)$, $u \in [0,1]$. Thus $\bar{f}'(u) \le f'(1)/(2f(1/2))$ and we obtain the required condition. $\square$

Example 3.2. of phase transition (I): Convex power skewing function. Let us illustrate the strictly convex setting by considering the family of strictly convex functions $f(u) = f_\alpha(u) = u^\alpha$, $\alpha > 1$. Thus the functions $\bar{f}_\alpha(u) = \frac{u^\alpha}{u^\alpha + (1-u)^\alpha}$ are convex on $[0,1/2]$ ([1]). We also assume $p_1 + p_2 > 1$. Then Equation $h_0(u) = 0$ (see (3.3)) can be rewritten as

$$\varphi_1(u) = \varphi_2(u) \quad \text{on} \quad I^*,$$

where

$$\varphi_1(u) = (p_1 - u)f(u) = (p_1 - u)u^\alpha \quad \text{and} \quad \varphi_2(u) = (u - 1 + p_2)f(1-u) = (u - 1 + p_2)(1-u)^\alpha.$$

Then, the first derivatives of these functions read

$$\varphi_1'(u) = \big(\alpha p_1 - (\alpha+1)u\big)u^{\alpha-1} \quad \text{and} \quad \varphi_2'(u) = \big((\alpha+1)(1-u) - \alpha p_2\big)(1-u)^{\alpha-1}.$$

Each of these two derivatives φ_i' has only one zero lying in I^* which are the maximum of each function φ_i, $i = 1,2$: $u_1 = \frac{\alpha p_1}{\alpha+1}$ for φ_1' and $u_2 = 1 - \frac{\alpha p_2}{\alpha+1}$ for φ_2'. As $p_1 + p_2 > 1$, we have that $1 - \frac{\alpha p_2}{\alpha+1} < \frac{\alpha p_1}{\alpha+1}$. Besides, we have $\varphi_1(1-p_2) = (p_1+p_2-1)(1-p_2)^\alpha > 0$, $\varphi_1(p_1) = 0$, $\varphi_2(1-p_2) = 0$ and $\varphi_2(p_1) = (p_1+p_2-1)(1-p_1)^\alpha > 0$. Therefore, we have three possibilities (of course consistent with claim (c) of the above proposition):

[1] The second derivative reads, if we set $\varphi_\alpha(u) = f_\alpha(u) + f_\alpha(1-u)$,

$$\bar{f}_\alpha''(u) = (\alpha-1)(1-2u)\varphi_\alpha^{-2}(u) + 2\alpha\big(u(1-u)\big)^{\alpha-1}\big((1-u)^{\alpha-1} - u^{\alpha-1}\big)\varphi_\alpha^{-3}(u) \ge 0 \quad \text{on} \quad [0,1/2].$$

- If $\varphi_1\left(1-\frac{\alpha p_2}{\alpha+1}\right) > \varphi_2\left(1-\frac{\alpha p_2}{\alpha+1}\right)$, then (3.3) has a unique solution lying in I^*.
- If $\varphi_1\left(1-\frac{\alpha p_2}{\alpha+1}\right) = \varphi_2\left(1-\frac{\alpha p_2}{\alpha+1}\right)$, then (3.3) has two solutions lying in I^*.
- If $\varphi_1\left(1-\frac{\alpha p_2}{\alpha+1}\right) < \varphi_2\left(1-\frac{\alpha p_2}{\alpha+1}\right)$, then (3.3) has three solutions lying in I^*.

In this parametric setting, as $f'(1) = \alpha$, it follows from Proposition 3.1 that the simpler criterion for the existence of a unique attractive equilibrium reads

$$1 < p_1 + p_2 \leq 1 + \frac{2^{1-\alpha}}{\alpha}.$$

To exhibit a transition phase with three equilibrium points, this condition need to be violated. However it is not a sufficient condition. This is illustrated by Figure 1 below: Choose p_1 and p_2 such that $p_1 + p_2 > 1$. The critical value α_0 is clearly strictly larger than the barrier suggested by the criterion. For example, for $p_1 = 0.7$ and $p_2 = 0.75$, $\alpha_0 \approx 3.09 > \min\{\alpha > 1 : p_1 + p_2 > 1 + \frac{2^{1-\alpha}}{\alpha}\} \approx 1.54$.

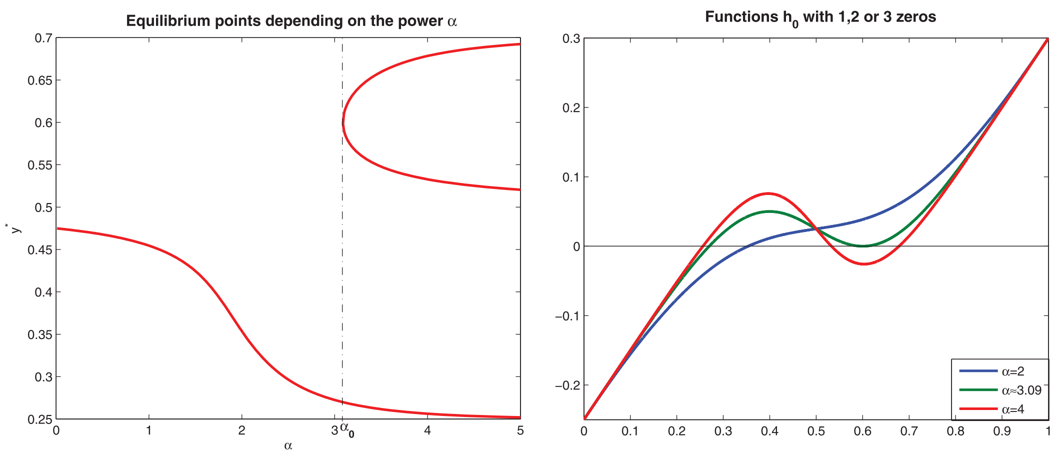

Figure 1: *Equilibrium points for $f(u) = u^\alpha$ depending on $\alpha \in (0, 5]$ with $p_1 = 0.7$ and $p_2 = 0.75$. Left: Bifurcation appears at $\alpha_0 \approx 3.09$. Right: Examples of function h_0 with one zero for $\alpha = 2$, two zeros for $\alpha \approx 3.09$, three zeros for $\alpha = 4$.*

Example 3.3. with f convex on $[0,1]$, $\bar{f}$ not convex on $[0,1/2]$, $p_1 + p_2 > 1$ (I). Let $t_0 \in (0, \frac{1}{2})$ and $\beta \geq 1$. For every positive real constants $\lambda > 0$, we define

$$f(t) = f_{\lambda,t_0}(t) = \frac{t + \lambda(t + t_0 - 1)^\beta \mathbf{1}_{\{1-t_0 \leq t \leq 1\}}}{1 + \lambda t_0^\beta}, \quad t \in [0,1]. \tag{3.6}$$

The function f is clearly continuous, positive on $(0,1]$, convex as sum of two convex functions and one checks it is differentiable on $[0,1]$. On the other hand, $\bar{f}(0) = 0$ and $\bar{f}(t) = t$ for every $t \in [t_0, 1-t_0]$ so that $\bar{f}' \equiv 1$ on $[t_0, 1-t_0]$. Hence, there exists $\xi_0 \in (0, t_0)$ such that $\bar{f}'(\xi_0) = \frac{\bar{f}(t_0) - \bar{f}(0)}{t_0} = 1$. If $\bar{f}$ were convex on $[0, \frac{1}{2}]$, then f' would be non-decreasing, hence constant on $[\xi_0, \frac{1}{2}]$ since $\bar{f}'(1/2) = 1$. This is impossible since $\bar{f}$ is not affine on any such intervals $[\xi, t_0]$ so that $\bar{f}$ is not convex on $[0, 1/2]$.

▷ *Non-symmetric case ($p_1 \neq p_2$).* We also assume in this setting that

$$0 < p_2 \leq p_1, \;\; p_1 + p_2 > 1, \;\; 2(1-p_2) < \frac{1-p_1}{1-p_1+1-p_2} \quad \text{and} \quad t_0 \in \left(2(1-p_2), \frac{1-p_1}{1-p_1+1-p_2}\right). \tag{3.7}$$

As $p_1 + p_2 > 1$, one may rewrite (3.1) as

$$h_0(u) = u - (1-p_2) - (p_1 + p_2 - 1)\bar{f}(u)$$

so that

$$h_0(t_0) = t_0(2-(p_1+p_2)) - (1-p_2) < p_2 - p_1 \leq 0.$$

Hence $h_0(t_0) < 0$. Moreover, since $t_0/2 > 1-p_2$,

$$h_0(t_0/2) = t_0/2 - (1-p_2) - (p_1+p_2-1)\bar{f}_{\lambda,t_0}(t_0/2) \quad \text{and} \quad \lim_{\lambda\to+\infty} f_{\lambda,t_0}(t_0/2) = 0,$$

we derive that there exists $\lambda_0 = \lambda(t_0,p_1,p_2) > 0$ such that, for every $\lambda > \lambda_0$, $h(t_0/2) > 0$.

Using the closed form $\bar{f}(t) = \frac{t}{1+(\lambda/c)(t_0-t)^\beta}$, $t \in [0,t_0]$, for $\bar{f}$, one shows after elementary though tedious computations that for large enough λ, h_0' has only one zero in $(0,t_0)$ so that h_0 is unimodal on $(0,t_0)$ (increasing then decreasing on $(0,t_0)$). Combining all these facts proves that h_0 has exactly two zeros u_1^* and u_2^* in $(1-p_2,t_0)$ for large enough λ.

Now, by the symmetry equation (3.4), we know that h_0 is convex on $[1-t_0,1]$ and $h_0(1-t_0) = p_2 - p_1 - h_0(t_0)$. In particular, $h_0(1-t_0) > 0$ if $t_0 \in (0, \frac{1-p_1}{1-p_1+1-p_2})$. In that case, h_0 has a third zero u_3^* on $(t_0, 1-t_0)$ by intermediate value theorem. Finally, as $h_0(1-t_0/2) = p_2 - p_1 - h_0(t_0/2) < 0$ and $h_0(1) > 0$, h_0 has two further zeros u_4^* and u_5^* on $(1-t_0,p_1)$.

As a conclusion, for the above function f, under the Condition (3.7) there are five equilibrum points for this urn model (see Figure 2-($left$)) for λ large enough. Such a condition is fulfilled $e.g.$ by choosing

$$t_0 = 1/4,\ \beta = 2,\ p_1 = 0.95,\ p_2 = 0.90.$$

$\triangleright$ *Focus on the case* $p_1 = p_2 = p \in (3/4,1)$ *(bi-stochastic matrix* H*).* We temporarily denote for convenience $h_{0,\lambda}$ the function h_0 associated to the skewing function f_{λ,t_0} defined in (3.6). It follows from the previous case that $\sup_{t\in[0,t_0]} h_{0,\lambda}(t) \geq h_{0,\lambda}(t_0) \to t_0/2 - (1-p) > 0$ as $\lambda \to +\infty$ whereas, for $\lambda = 0$, $\bar{f}(t) = t$ so that $h_{0,\lambda}(t) = (2t-1)(1-p)$ and $\sup_{t\in[0,t_0]} h_{0,\lambda}(t) = (2t_0-1)(1-p) < 0$. Consequently there exists $\lambda_0 > 0$ such that $\sup_{t\in[0,t_0]} h_{0,\lambda}(t) = 0$ by continuity of $h_{0,\lambda}(t)$ in (λ,t). As $\lambda \to h_{0,\lambda}(t)$ is increasing we derive (see Figure 2-($right$)) that:

– if $0 < \lambda < \lambda_0$, h_0 has only one zero, namely $u_1^* = \frac{1}{2}$ by symmetry, with $h_0'(1/2) > 0$.– if $\lambda = \lambda_0$, h_0 has three zeros: $u_1^* \in (1-p,t_0)$ with $h_0'(u_3^*) = 0$, $u_2^* = 1/2$ with $h_0'(u_2^*) = 0$ and $u_3^* = 1-u_1^*$ with $h_0'(u_3^*) = 0$. – if $\lambda > \lambda_0$, th_0 has five zeros: u_1^*, $u_2^* \in (1-p,t_0)$, $u_3^* = 1/2$ and $u_4^* = 1-u_2^*$, $u_5^* = 1-u_1^*$ with $h_0'(u_{2i-1}^*) > 0$, $i = 1,2,3$ and $h_0'(u_{2i}^*) < 0$, $i = 1,2$.

The status of the corresponding equilibrium points of the urn model is discussed in the next section.

3.2 Stability of equilibrium points of ODE_{h_0}

Once elucidated the existence and the number of equilibrium points u^* of ODE_{h_0}, we have to determine their nature *i.e.* the sign of $h_0'(u^*)$.

Let $u^* \in \{h_0 = 0\} \subset I^*$. We deduce from (3.1) that the condition $h_0(u^*) = 0$ reads

$$(u-p_1)f(u) + (u-1+p_2)f(1-u) = 0.$$

Plugging this equality into the expression (3.2) for $h_0'(u^*)$ shows that, for such equilibrium points u^*,

$$h_0'(u^*) = 1 - \frac{f'(u^*)(p_1-u^*) + f'(1-u^*)(u^*-1+p_2)}{f(u^*)+f(1-u^*)}. \tag{3.8}$$

Indeed, when we compute h_0', we have

$$h_0'(u) = 1 - \frac{p_1 f'(u) - (1-p_2)f'(1-u)}{f(u)+f(1-u)} + \frac{f'(u)-f'(1-u)}{f(u)+f(1-u)} \times \frac{p_1 f(u) + (1-p_2)f(1-u)}{f(u)+f(1-u)}.$$

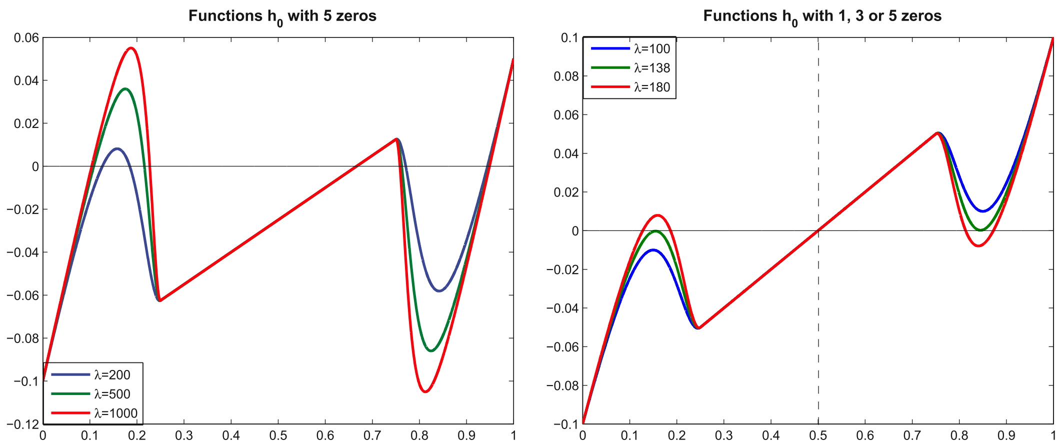

Figure 2: *Examples of function* h_0 *with* $\bar{f}$ *non-convex*: f defined by (3.6) with $t_0 = 1/4$ and $\beta = 2$. *Left*: 5 equilibrium points for $p_1 = 0.95$, $p_2 = 0.9$ and $\lambda = 200, 500, 1000$. *Right*: $p_1 = p_2 = p = 0.9$, one equilibrium point for $\lambda = 100$, two for $\lambda = 138$ and three for $\lambda = 180$.

As $h_0(u^*) = 0$, we use $\frac{p_1 f(u^*)+(1-p_2)f(1-u^*)}{f(u^*)+f(1-u^*)} = u^*$ to derive (3.8).

The function h_0 has at most three zeros owing to Proposition 3.1. Therefore, each such zero u^* of h_0 has three possible "status" (see $e.g.$ Figure 1-($right$)):

- If there exists $\varepsilon_0 > 0$ such that the following two restrictions of the function h_0 satisfy $h_{0|(u^*-\varepsilon_0,u^*)} < 0$ and $h_{0|(u^*,u^*+\varepsilon_0)} > 0$, then u^* is a *stable* equilibrium point. This is the case $e.g.$ if $h_0'(u^*) > 0$ or, more generally, if the first non zero derivative at u^* has an odd order and is positive.
- If there exists $\varepsilon_0 > 0$ such that $h_{0|(u^*-\varepsilon_0,u^*)} > 0$ and $h_{0|(u^*,u^*+\varepsilon_0)} < 0$, then u^* is an *unstable* equilibrium point (or a repeller). This is the case $e.g.$ if $h_0'(u^*) < 0$ or if the first non-zero derivative at u^* has an odd order and is negative.
- If h_0 has a constant sign over an interval $(u^* - \varepsilon_0, u^* + \varepsilon_0)$, for some $\varepsilon_0 > 0$, then u^* is a *semi-stable equilibrium point* (one-sided stable and one-sided unstable ([2])). This is the case if $h_0'(u^*) = 0$ and the first non-zero derivative at u^* occurs at an even order. The simplest case is when $h_0''(u^*)$ exists and is non-zero.

Then it is an elementary exercise to derive the following proposition, if one keeps in mind that $h_0(0) = -(1-p_2) < 0$ and $h_0(1) = 1 - p_1 > 0$.

Proposition 3.4. *Let* $p_1, p_2 \in (0,1)$. *Let* f *be a skewing function. Assume that* f *satisfies one of the additional assumptions in claims* (a)-(b)-(c) *of Proposition 3.1. Then*

(i) *If* h_0 *has a unique equilibrium point, then it is stable.*

(ii) *If* h_0 *has two equilibrium points, then one is stable and one is semi-stable.*

(iii) *If* h_0 *has three equilibrium points, then the lowest and the highest ones are stable and the one in the middle is unstable.*

When f *is concave, we have* (a)-(i); *when* f *and* $\bar{f}$ *are convex,* (a)-(iii).

Remark 3.5. From a heuristic viewpoint? stable equilibrium points are those obtained from zeros of h_0 occurring when h_0 crosses the abscissa axis in an ascending sense.

[2] We mean by *right semi-stable* that the solution of $u(u_0,t)$ of ODE_h starting from u_0 satisfies $u(u_0,t) \to u^*$ as $t \to +\infty$ for $u_0 > u^*$ in the right neighbourhood of u^* and $\liminf_{t\to+\infty} |u(u_0,t) - u^*| > 0$ if $u_0 > u^*$ in the left (resp. right) neighbourhood of u^*. *Left semi-stability* is defined accordingly and *one-sided semi-stable* corresponds to any of these two behaviours.

Example 3.6. of phase transition (II). We still consider the family $f(u) = u^\alpha$, $\alpha > 1$. The "bifurcation" has been established in part I. Now we want to elucidate what happens at α_0. The critical two equilibrium case occurs at $\alpha_0 \approx 3.09$. By continuity of $(\alpha, u) \mapsto u^\alpha$ and monotony in α and u, it is clear that at least one – in fact exactly one – of the two equilibrium points u^* satisfies $h_0(u^*) = 0$ and that h_0 does not change its sign around u^* so that $h_0'(u^*) = 0$. To determine its status we have to look at the sign of the second derivative $h_0''(u^*)$ (see Figure 3). In fact elementary computations show that, for any $u \in [0,1]$ such that $h_0(u) = h_0'(u) = 0$,

$$h_0''(u) = (1-p_1-p_2)\alpha u^{\alpha-2}(1-u)^{\alpha-2} \\ \times \frac{(\alpha-1)(1-2u)(u^\alpha+(1-u)^\alpha) - 2\alpha u(1-u)(u^{\alpha-1}-(1-u)^{\alpha-1})}{(u^\alpha+(1-u)^\alpha)^3} > 0.$$

Hence, u^* is semi-stable (stable on the right and unstable on the left).

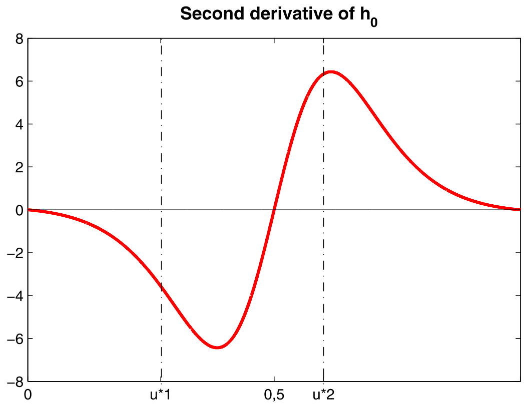

Figure 3: Second derivative of h_0 for $f(u) = u^{3.09}$ with $p_1 = 0.7$ and $p_2 = 0.75$.

Example 3.7. with f convex on $[0,1]$, $\bar{f}$ not convex on $[0,1/2]$, $p_1 + p_2 > 1$, $0 < p_1, p_2 < 1$ (II). Under the assumption (3.7), one has:
– In case of 5 equilibrium points $(u_i^*)_{1/\le i\le 5}$, u_{2i-1}^*, $i = 1,2,3$ are stable, u_{2i}^*, $i = 1,2$ are unstable.
– In case of 3 equilibrium points $(u_i^*)_{1/\le i\le 3}$ when $p_1 = p_2 > 3/4$, u_{2i-1}^*, $i = 1,2$, are semi-stable, $u_2^* = 1/2$, $i = 1,2$ are unstable.

It remains to show that the algorithm does not converge towards the repulsive equilibrium point, denoted by $\hat{u}$ in what follows. To show that there is an excitation in the repulsive direction, we have to prove that assumption (A.4) holds (see Theorem A.3 in the Appendix).

Proposition 3.8. *Assume that (**A1**), (**A2**) and (**A3**) hold. Let $\hat{u}$ be an unstable equilibrium point for h_0 satisfying $h_0'(\hat{u}) < 0$. Then*

$$\mathbb{P}\big(\widetilde{Y}_n \to (\hat{u}, 1-\hat{u})\big) = 0.$$

Proof. From the proof of Proposition 2.15, we know that the remainder term and the martingale increment satisfy respectively (A.3) (with $\delta = 0$). Then we rely on the remark that follows Theorem A.3. It follows from (3.2) that h_0' is locally Lipschitz continuous. We know that $\{\widetilde{Y}_n \to (\hat{u}, 1-\hat{u})\} = \{\widetilde{Y}_n^1 \to \hat{u}\}$, so we may focus on the first component. We rely on Theorem A.3 in the Appendix and adopt its notation, namely

$$\Delta M_{n+1}^{rep} = \Delta M_{n+1}^1.$$

Using Assumption **(A1)**, we obtain

$$\Delta M^{rep}_{n+1} = D^{11}_{n+1}X^1_{n+1} + D^{12}_{n+1}X^2_{n+1} - \frac{H^{11}_{n+1}f(\widetilde{Y}^1_n) + H^{12}_{n+1}f(\widetilde{Y}^2_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}.$$

Therefore

$$\begin{aligned}
\mathbb{E}\left[\left\|\Delta M^{rep}_{n+1}\right\| \big| \mathcal{F}_n\right] &= \mathbb{E}\left[\left|\Delta M^{1}_{n+1}\right| \big| \mathcal{F}_n\right] \\
&= \frac{f(\widetilde{Y}^1_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\mathbb{E}\left[\left|D^{11}_{n+1} - \frac{H^{11}_{n+1}f(\widetilde{Y}^1_n) + H^{12}_{n+1}f(\widetilde{Y}^2_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right| \Big| \mathcal{F}_n\right] \\
&\quad + \frac{f(\widetilde{Y}^2_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\mathbb{E}\left[\left|D^{12}_{n+1} - \frac{H^{11}_{n+1}f(\widetilde{Y}^1_n) + H^{12}_{n+1}f(\widetilde{Y}^2_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right| \Big| \mathcal{F}_n\right].
\end{aligned}$$

By Jensen's inequality applied to both conditional expectations in the right hand side, we obtain

$$\begin{aligned}
\mathbb{E}\left[\left\|\Delta M^{rep}_{n+1}\right\| \Big| \mathcal{F}_n\right] &\geq \frac{f(\widetilde{Y}^1_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\left|H^{11}_{n+1} - \frac{H^{11}_{n+1}f(\widetilde{Y}^1_n) + H^{12}_{n+1}f(\widetilde{Y}^2_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right| \\
&\quad + \frac{f(\widetilde{Y}^2_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\left|H^{12}_{n+1} - \frac{H^{11}_{n+1}f(\widetilde{Y}^1_n) + H^{12}_{n+1}f(\widetilde{Y}^2_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right|.
\end{aligned}$$

Owing to **(A5)**$_v$, $H^{ij}_{n+1} \xrightarrow[n\to+\infty]{a.s.} H^{ij}$, where $H^{ii} = p_i$ and $H^{ij} = 1 - p_j$, $1 \leq i,j \leq 2$. Furthermore, on $\hat{\mathcal{Y}}_\infty = \left\{\omega : \widetilde{Y}^1_n(\omega) \to \hat{u}\right\}$,

$$\frac{H^{11}_{n+1}f(\widetilde{Y}^1_n) + H^{12}_{n+1}f(\widetilde{Y}^2_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \xrightarrow[n\to+\infty]{a.s.} \hat{u}.$$

Consequently,

$$\frac{f(\widetilde{Y}^1_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\left|H^{11}_{n+1} - \frac{H^{11}_{n+1}f(\widetilde{Y}^1_n) + H^{12}_{n+1}f(\widetilde{Y}^2_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right| \xrightarrow[n\to+\infty]{a.s.} \frac{f(\hat{u})}{f(\hat{u}) + f(1-\hat{u}))}\left|p_1 - \hat{u}\right| > 0$$

and

$$\frac{f(\widetilde{Y}^2_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\left|H^{12}_{n+1} - \frac{H^{11}_{n+1}f(\widetilde{Y}^1_n) + H^{12}_{n+1}f(\widetilde{Y}^2_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right| \xrightarrow[n\to+\infty]{a.s.} \frac{f(\hat{u})}{f(\hat{u}) + f(1-\hat{u}))}\left|1 - p_2 - \hat{u}\right| > 0$$

since $\hat{u} \in \overset{\circ}{I}{}^*$. Thus (A.4) is satisfied and by applying Theorem A.3 in the Appendix, $\mathbb{P}(\hat{\mathcal{Y}}_\infty) = 0$. ☐

Example 3.9. (bi-stochastic generating matrix H). If $p_1 = p_2 = p \in (0,1)$ then (H is bi-stochastic) $(\frac{1}{2}; \frac{1}{2})$ is an equilibrium and $h_0'(1/2) = 1 - \frac{f'(1/2)}{2f(1/2)}(2p-1)$.

– If f is convex and $2p - 1 \leq 0$, then $h_0'\left(\frac{1}{2}\right) > 0$ and the equilibrium is unique (since h_0' is increasing).

– If f is convex and $2p-1 > 0$, then we have three possibilities:

- If $p > \frac{1}{2} + \frac{f(1/2)}{f'(1/2)}$, then $h_0'(1/2) < 0$, so that $\mathbb{P}(\hat{\mathcal{Y}}_\infty) = 0$.
- If $p = \frac{1}{2} + \frac{f(1/2)}{f'(1/2)}$, then $h_0'(1/2) = 0$. But $h_0''(1/2) = 0$, so we need to investigate higher order.
- If $p < \frac{1}{2} + \frac{f(1/2)}{f'(1/2)}$, then $h_0'(1/2) > 0$, so $1/2$ is stable.

If $f(u) = u^\alpha$, the first case also reads $\alpha > \frac{1}{2p-1}$. For the second case, higher order analysis leads to $h_0^{(3)}(1/2) > 0$, so that the equilibrium is stable.

3.3 $A.s.$ convergence

Theorem 3.10. *Let $(Y_n)_{n\geq 0}$ be the urn composition sequence defined by (1.3)-(1.5). Under the assumptions **(A1)**, **(A2)**, **(A3)** and the empirical frequency based f-skewed drawing rule,*

(a) $\dfrac{Y_n}{\mathbf{w}(Y_n)} \xrightarrow[n\to+\infty]{a.s.} (U^*, 1-U^*)$ *where U^* is a random variable taking values in $\{u^* :$ $h_0(u^*) = 0$ and u^* is not unstable for $ODE_{h_0}\}$.*

(b) *The proportions of drawing each type of ball satisfies*

$$\widetilde{N}_n = \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow[n\to+\infty]{a.s.} \begin{pmatrix} \dfrac{f(U^*)}{f(U^*)+f(1-U^*))} \\ \\ \dfrac{f(1-U^*)}{f(U^*)+f(1-U^*))} \end{pmatrix}.$$

Proof. First, we will prove that $(a) \Rightarrow (b)$, then (a).

$(a) \Rightarrow (b)$. For every $n \geq 1$, we have

$$\mathbb{E}\left[X_n \,|\, \mathcal{F}_{n-1}\right] = \sum_{i=1}^{2} \frac{f(\widetilde{Y}^i_{n-1})}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_{n-1}))} e^i = \frac{\widetilde{f}(\widetilde{Y}_{n-1})}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_{n-1}))}$$

and, by construction $\|X_n\|^2 = 1$, so that $\mathbb{E}\left[\|X_n\|^2 \,|\, \mathcal{F}_{n-1}\right] = 1$. Hence, the martingale

$$\widetilde{M}_n = \sum_{k=1}^{n} \frac{X_k - \mathbb{E}\left[X_k \,|\, \mathcal{F}_{k-1}\right]}{k} \xrightarrow[n\to+\infty]{a.s.\,\&\,L^2} \widetilde{M}_\infty \in L^2.$$

Finally, it follows from the Kronecker Lemma that

$$\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{n}\sum_{k=1}^{n} \frac{\widetilde{f}(\widetilde{Y}_{k-1})}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_{k-1}))} \xrightarrow[n\to+\infty]{a.s.} 0.$$

This proves the announced implication owing to the Césaro Lemma since $\frac{\widetilde{f}}{\mathbf{w}(\widetilde{f})}$ is continuous at every point of $\mathcal{S}_2$.

(a) The algorithm is bounded by construction since it is $[0,1]$-valued. Assumption **(A2)** implies that $\sup_{n\geq 0} \mathbb{E}\left[\|\Delta M_{n+1}\|^2 \,|\, \mathcal{F}_n\right] < +\infty$ $a.s.$ and Assumption **(A3)** implies that $r_n \xrightarrow[n\to+\infty]{a.s.} 0$. The set $\{h = 0\}$ is finite hence $\widetilde{Y}_n$ $a.s.$ converges toward a zero of h (see Theorem A.2(c) in the appendix). Moreover, it follows from Proposition 3.4 devoted to attractiveness that this zero cannot be a repulsive (a point at which h_0' is negative). □

4 Weak rate of convergence

To establish a CLT for the sequence $(\widetilde{Y}_n)_{n\geq 0}$ on a convergence event $\{\widetilde{Y}_n \to y^*\}$, we need to make the following additional assumptions:

(A4) The addition rules D_n $a.s.$ satisfy on the event $\{\widetilde{Y}_n \to y^*\}$

$$\forall k \in \{1,\dots,d\}, \quad \begin{cases} (i) & \sup_{n\geq 1} \mathbb{E}\left[\|D_n^{\cdot k}\|^{2+\delta} \,|\, \mathcal{F}_{n-1}\right] \leq \kappa < +\infty \quad \text{for a } \delta > 0, \\ (ii) & \mathbb{E}\left[D_n^{\cdot k}(D_n^{\cdot k})^t \,|\, \mathcal{F}_{n-1}\right] \xrightarrow[n\to+\infty]{} C^k_{y^*}, \end{cases}$$

where $C^k_{y^*} = (C^k_{y^*,ij})_{1\leq i,j\leq d}$, $k = 1,\dots,d$, are $d\times d$ positive definite matrices.

Note that **(A4)** $\Rightarrow$**(A2)** since $\mathbb{E}\left[\|D_n^{\cdot k}\|^2 \,|\, \mathcal{F}_{n-1}\right] \le \left(\mathbb{E}\left[\|D_n^{\cdot k}\|^{2+\delta} \,|\, \mathcal{F}_{n-1}\right]\right)^{\frac{2}{2+\delta}}$.

Also note that, if the matrices D_n are *deterministic* (hence co-stochastic) then $C^k_{y^*} = \lim_n D_n^{\cdot k} \otimes D_n^{\cdot k}$ so that $\mathbf{1}^t C^k_{y^*}\mathbf{1} = 1$ for every $k \in \{1,\dots,d\}$.

Let $v = (v_n)_{n\ge 1}$ be a sequence of positive real numbers.

(A5)$_v$ The matrices H_n and H satisfy on the event $\{\widetilde{Y}_n \to y^*\}$

$$n\, v_n\, \mathbb{E}\left[\|\!|H_n - H|\!\|^2\right] \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.1}$$

To establish the weak rate of convergence we cannot restrict ourselves to the simplex as we essentially did for the convergence result since we must take into account the rate of convergence of the algorithm $\widetilde{Y}_n$ toward the simplex which is itself non trivial in general. Actually, we saw in (2.15) that

$$\mathbf{w}(\widetilde{Y}_n) - 1 = \frac{\mathbf{w}(M_n)}{n + \mathbf{w}(Y_0)}.$$

Elementary computations show, under Assumptions **(A1)**, **(A3)**, **(A4)**-(i) and if $\widetilde{Y}_n \to y^*$, that

$$\mathbb{E}\left[\mathbf{w}(M_n)^2 \,|\, \mathcal{F}_{n-1}\right] \to \sigma^2(y^*) = \mathbf{1}^t\, \frac{\sum_{k=1}^d f(y^k_*)C^k_{y^*}}{\mathbf{w}(\widetilde{f}(y^*))}\mathbf{1} - 1.$$

Note that, if the matrices D_n are deterministic then $\sigma^2(y^*) = 0$ which is expected since in that case $\widetilde{Y}_n$ is $\mathcal{S}_d$-valued. Owing to Condition **(A4)**, Lindeberg's *CLT* for arrays of martingale applies to $\left(\frac{\mathbf{w}(M_\ell)}{\sqrt{n}}\right)_{1\le \ell\le n}$ (see *e.g.* Corollary 3.1, p.58 in [22]), if one keeps in mind that or that **(A4)**-(ii) implies the usual condition as a straightforward application of Markov inequality). Consequently

$$\sqrt{n}\Big(\mathbf{w}(\widetilde{Y}_n) - 1\Big) = \frac{n}{n+\mathbf{w}(Y_0)}\frac{\mathbf{w}(M_n)}{\sqrt{n}} \overset{\mathcal{L}_{stably}}{\longrightarrow} \mathcal{N}\big(0;\sigma^*(y^*)\big) \text{ on the event } \{\widetilde{Y}_n \to y^*\}.$$

4.1 Strictly concave case with irreducible bi-stochastic limiting generating matrix

Assume that H is bi-stochastic and irreducible and that the skewing function is strictly concave. We know from Proposition 2.14 that $\widetilde{Y}_n \to y(d) = \frac{1}{d}$ $a.s.$ On the other hand we know from (2.17) in Lemma 2.6 and (the proof of) Proposition 2.11 (see (2.19)) that

$$J_h\big(y(d)\big) = I_d - \lambda_1 H \mathrm{Proj}^{\perp}_{\mathbf{1}^\perp} \quad \text{still with} \quad \lambda_1 = \frac{f'(1/d)}{d\cdot f(1/d)} < 1.$$

As H is (co-)stochastic, we know that 1 is its eigenvalue with the highest real part so that the eigenvalue of $J_h\big(y(d)\big)$ with the lowest real part is $1-\lambda_1 > 0$.

The Central Limit Theorem stated in the proposition below holds "stably" $i.e.$ for the stable weak convergence in the following sense: $Z_n \overset{\mathcal{L}_{stably}}{\longrightarrow} Z_\infty$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in a Polish space (E,d) if, for every $f \in \mathcal{C}(E,\mathbb{R})$ and every $A \in \mathcal{A}$, $\mathbb{E}\,\mathbf{1}_A f(Z_n) \to \mathbb{E}\,\mathbf{1}_A f(Z_\infty)$ as $n \to +\infty$ (see [23] for more insight on stable convergence).

Proposition 4.1 (Weak rate)**.** *Assume that the skewed drawing rule f is strictly concave and H is irreducible and bi-stochastic. Assume **(A1)**, **(A3)**, **(A4)** hold.*

(a) *If $f'(1/d) < \frac{d}{2}f(1/d)$ and **(A5)**$_v$ holds with $v_n = 1$, then*

$$\sqrt{n}\big(\widetilde{Y}_n - y(d)\big) \overset{\mathcal{L}_{stably}}{\longrightarrow} \mathcal{N}\big(0;\Sigma^*\big) \quad \textit{with} \quad \Sigma^* = \int_0^{+\infty} e^{-u(J_h(y^*)-\frac{I_2}{2})^t}\Gamma^* e^{-u(J_h(y^*)-\frac{I_2}{2})}du$$

with

$$\Gamma^* = \frac{\sum_{k=1}^d f(y^{*,k}) C^k_{y^*}}{\boldsymbol{w}\big(\widetilde{f}(y^*)\big)} - y^*(y^*)^t. \tag{4.2}$$

(b) *If* $f'(1/d) = \frac{d}{2} f(1/d)$ *and* **(A5)**$_v$ *holds with* $v_n = \log n$*, then*

$$\sqrt{\frac{n}{\log n}}\big(\widetilde{Y}_n - y(d)\big) \xrightarrow{\mathcal{L}_{stably}} \mathcal{N}\big(0; \Sigma^*\big)$$

with

$$\Sigma^* = \lim_n \frac{1}{n} \int_0^n e^{-u\left(J_h(y^*) - \frac{I_d}{2}\right)^t} \Gamma^* e^{-u\left(J_h(y^*) - \frac{I_d}{2}\right)} du. \tag{4.3}$$

(c) *If* $f'(1/d) > \frac{d}{2} f(1/d)$ *and* **(A5)**$_v$ *holds with* $v_n = n^{1-2\lambda_1+\eta}$ *for some* $\eta > 0$*, then*

$n^{1-\lambda_1}\big(\widetilde{Y}_n - y(d)\big)$ *converges* $a.s.$ *toward a finite random variable.*

Proof. First note that $\widetilde{Y}_n \to y(d)$ $a.s.$ under the assumptions owing to Proposition 2.14. We will check the three assumptions of Theorem A.5 (CLT for SA algorithms) recalled in the Appendix. The parameter Λ which rules the regime of the rate is given here by $\Lambda = 1 - \lambda_1$ which justifies the above three cases. Secondly Assumption **(A4)** ensures that Condition (A.6) is satisfied since

$$\sup_{n\geq 1} \mathbb{E}\left[\|\Delta M_n\|^{2+\delta} \mid \mathcal{F}_{n-1}\right] < +\infty \quad a.s. \quad \text{and} \quad \mathbb{E}\left[\Delta M_n \Delta M_n^t \mid \mathcal{F}_{n-1}\right] \xrightarrow[n\to+\infty]{a.s.} \Gamma^*.$$

To be more precise on the convergence on the right-hand side, we have

$$\begin{aligned}
\mathbb{E}\left[\Delta M_{n+1}\Delta M_{n+1}^t \mid \mathcal{F}_n\right] &= \sum_{k=1}^d \mathbb{P}(X_{n+1} = e^k \mid \mathcal{F}_n)\Big(\mathbb{E}\left[D^{\cdot k}_{n+1}(D^{\cdot k}_{n+1})^t \mid \mathcal{F}_n\right] \\
&\quad - \mathbb{E}\left[D_{n+1}X_{n+1} \mid \mathcal{F}_n\right]\mathbb{E}\left[D_{n+1}X_{n+1} \mid \mathcal{F}_n\right]^t\Big) \\
&= \sum_{k=1}^d \frac{f(\widetilde{Y}^k_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\mathbb{E}\left(D^{\cdot k}_{n+1}(D^{\cdot k}_{n+1})^t \mid \mathcal{F}_n\right) \\
&\qquad - \left(H_{n+1}\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right)\left(H_{n+1}\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right)^t \\
&\xrightarrow[n\to+\infty]{a.s.} \&\Gamma^* = \frac{\sum_{k=1}^d f(y^{*k})C^k_{y^*}}{\mathbf{w}(\widetilde{f}(y^*))} - y^*(y^*)^t.
\end{aligned}$$

Finally, using **(A5)**$_v$ with the appropriate sequence $(v_n)_{n\geq 1}$, one proves in the three cases that the remainder sequence $(r_n)_{n\geq 1}$ defined by (2.13) satisfies (A.7). This follows from the fact that $\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}$ is bounded. □

4.2 Back to the "convex" bi-dimensional case

In this section, we elucidate the rate of convergence of the algorithm, mostly in the weak sense. To this end, it is more convenient to deal with its original 2-dimensional form.

Theorem 4.2 (Weak rate)**.** *Assume* ***(A1)****,* ***(A3)****,* ***(A4)*** *hold. Every equilibrium point* $y^* \in \mathcal{S}_2$ *is of the form* $y^* = (u^*, 1-u^*)$ *where* u^* *is solution to* (3.3) *and lies in* $I^* = [p_1 \wedge (1-p_2), p_1 \vee (1-p_2)]$*, and*

$$Sp\big(J_h(y^*)\big) = \big\{1, 1-\rho^*\big\},$$

where

$$\rho^* = \rho(y^*) = \frac{f'(u^*)(p_1 - u^*) + f'(1-u^*)(u^* - 1 + p_2)}{f(u^*) + f(1-u^*)}.$$

(a) *If* $p_1 + p_2 \le 1$ *and* ***(A5)***$_v$ *holds with* $v_n = 1$, $n \ge 1$, *then* y^* *is unique, stable,* $Y_n \to y^*$ $a.s.$ *and*

$$\sqrt{n}\left(\widetilde{Y}_n - y^*\right) \xrightarrow[n\to+\infty]{\mathcal{L}_{stably}} \mathcal{N}\left(0, \Sigma^*\right) \quad \textit{with} \quad \Sigma^* = \int_0^{+\infty} e^{-u(J_h(y^*) - \frac{I_2}{2})^t}\Gamma^* e^{-u(J_h(y^*) - \frac{I_2}{2})} du$$

$$\textit{and} \quad \Gamma^* = \frac{f(u^*)C^1_{y^*} + f(1-u^*)C^2_{y^*}}{\boldsymbol{w}(\widetilde{f}(y^*))} - y^*(y^*)^t.$$

(b) *If* $p_1 + p_2 > 1$, *we have three possible rates of convergence on an event* $\{\widetilde{Y}_n \to y^*\}$, *where* $y^* \in \mathcal{E}_{2,h}$ *is not unstable, depending on* $\rho^* = \lambda(y^*)$:

(i) *If* $0 < \rho^* < \frac{1}{2}$ *and* ***(A5)***$_v$ *holds with* $v_n = 1$, $n \ge 1$, *then*

$$\sqrt{n}\left(\widetilde{Y}_n - y^*\right) \xrightarrow[n\to+\infty]{\mathcal{L}_{stably}} \mathcal{N}\left(0, \Sigma^*\right) \quad \textit{on} \quad \{\widetilde{Y}_n \to y^*\},$$

where Σ^* *is formally defined like in item* (a).

(ii) *If* $\rho^* = \frac{1}{2}$ *and* ***(A5)***$_v$ *holds with* $v_n = \log n$, $n \ge 1$, *then*

$$\sqrt{\frac{n}{\log n}}\left(\widetilde{Y}_n - y^*\right) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}\left(0, \Sigma^*\right) \quad \textit{where } \Sigma^* \textit{ is given by (4.3).}$$

(iii) *If* $\frac{1}{2} < \rho^* < 1$ *and* ***(A5)***$_v$ *holds with* $v_n = n^{1-2\rho^*+\eta}$, $\eta > 0$, *then* $n^{1-\rho^*}\left(\widetilde{Y}_n - y^*\right) \to \Upsilon$ $a.s.$ *converges as* $n \to +\infty$ *where* Υ *is a finite random variable.*

Remark 4.3. • The condition $0 < \rho^* < \frac{1}{2}$ is satisfied as soon as

$$\begin{cases} \dfrac{(f'(1-p_1) + f'(1-p_2))(p_1 + p_2 - 1)}{f(1-p_1) + f(1-p_2)} < \dfrac{1}{2} & \text{if } f \text{ is concave,} \\[2ex] \dfrac{(f'(p_1) + f'(p_2))(p_1 + p_2 - 1)}{f(1-p_1) + f(1-p_2)} < \dfrac{1}{2} & \text{if } f \text{ is convex,} \end{cases}$$

by using the monotonicity of f and f' and that $u^* \in (1 - p_2, p_1)$.

• If $f(y) = y$, then y^* is unique, is given by

$$y^* = \left(\frac{1-p_2}{2-p_1-p_2}, \frac{1-p_1}{2-p_1-p_2}\right) \quad \text{and} \quad \rho^* = p_1 + p_2 - 1.$$

Thus, if $p_1 + p_2 < \frac{3}{2}$, then the recursive procedure (2.11) satisfies a regular CLT; if $p_1 + p_2 = \frac{3}{2}$, (2.11) satisfies Theorem 4.2(b)-(ii); and if $p_1 + p_2 > \frac{3}{2}$, (2.11) admits an $a.s.$-rate of convergence.

• In [13, 14], the properties of the random variable Υ are deeply investigated in the more standard framework of Pólya's urn with deterministic addition rule matrix. It is shown to be solution to a smoothing equation obtained by a smart decomposition of the urn into canonical components. Thus, it is proved that its distribution is characterized by its moments. It is clear that such results are out of reach of standard SA techniques although it would be challenging to check whether similar results about Υ in the randomized and nonlinear framework are true.

Proof. We again rely on Theorem A.5 from the Appendix. Elementary though tedious preliminary computations yield the following formula for the Jacobian $J_h(y)$ of h at $y\in\mathbb{R}^d_+\setminus\{0\}$. We obtain

$$J_h(y)=\begin{pmatrix} 1+\frac{f'(y^1)}{f(y^1)+f(y^2)}\left(\frac{p_1f(y^1)+(1-p_2)f(y^2)}{f(y^1)+f(y^2)}-p_1\right) & \frac{f'(y^2)}{f(y^1)+f(y^2)}\left(\frac{p_1f(y^1)+(1-p_2)f(y^2)}{f(y^1)+f(y^2)}-(1-p_2)\right)\\ \frac{f'(y^1)}{f(y^1)+f(y^2)}\left(\frac{(1-p_1)f(y^1)+p_2f(y^2)}{f(y^1)+f(y^2)}-(1-p_1)\right) & 1+\frac{f'(y^2)}{f(y^1)+f(y^2)}\left(\frac{(1-p_1)f(y^1)+p_2f(y^2)}{f(y^1)+f(y^2)}-p_2\right)\end{pmatrix}.$$

As all equilibrium points y^* lie in the simplex $\mathcal{S}_2$, we have $y^{*2}=1-y^{*1}$. Combined with the constraint $h(y^*)=0$ (see (3.3)), we finally obtain the following formula only true at equilibrium points:

$$J_h(y^*)=\begin{pmatrix} 1+\frac{f'(y^{*1})}{f(y^{*1})+f(1-y^{*1})}\left(y^{*1}-p_1\right) & \frac{f'(1-y^{*1})}{f(y^{*1})+f(1-y^{*1})}\left(y^{*1}-(1-p_2)\right)\\ \frac{f'(y^{*1})}{f(y^{*1})+f(1-y^{*1})}\left(p_1-y^{*1}\right) & 1+\frac{f'(1-y^{*1})}{f(y^{*1})+f(1-y^{*1})}\left(1-p_2-y^{*1}\right)\end{pmatrix}.$$

Then, one easily checks that the spectrum of $J_h(y^*)$ is real given by

$$\mathrm{Sp}\left(J_h(y^*)\right)=\{1,1-\rho^*\}\,.$$

(a) When $p_1\le 1-p_2$, we know from Proposition 3.1(a) that the equilibrium point $y^*=(u^*,1-u^*)$ is unique and u^* lies in I^*. Moreover, we know from Theorem 3.10(a) that $Y_n\to y^*$ $a.s.$ Hence $\rho^*\le 0$ so that 1 is the lowest eigenvalue of $J_h(y^*)$. Consequently, we are in the "regular" case of the CLT for SA (Theorem A.5(a) in the Appendix) since $1>\frac{1}{2}$.

Then, following the lines of the proof of Proposition 4.1, we check that Assumption **(A4)** ensures that Condition (A.6) is satisfied with $\Gamma^*=\dfrac{f(y^{*1})C^1_{y^*}+f(1-y^{*1})C^2_{y^*}}{\mathbf{w}(\widetilde{f}(y^*))}-y^*(y^*)^t$. Finally, using **(A5)**$_v$, the remainder sequence $(r_n)_{n\ge1}$ defined by (2.13) satisfies (A.7) since $\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}$ is bounded.

(b) If $p_1+p_2>1$, then $\rho^*>0$ which explains the three cases. The rest of the proof is the same as above, given the convergence event $\{\widetilde{Y}_n\to y^*\}$. ☐

5 Pólya urn with concave reinforced drawing rule: a bandit approach

By *Pólya urn*, we mean in this section that the matrices D_n involved in the drawing rule all satisfy $D_n=I_d$, $n\ge1$. Moreover we assume that the drawing rue is still skewed following (2.4) where the function f is concave/convex and that the initial urn composition vector $Y_0\in\mathbb{R}^d_+\setminus\{0\}$. Note that when $f(u)=u$, then the urn dynamics is that of a regular Pólya urn with d colors.

In such a framework, $H=H_n=I_d$, $n\ge1$, therefore H is no more irreducible and we cannot use the results proved in Sections 2 and 3.

We still normalize Y_n by setting $\widetilde{Y}_n:=\frac{Y_n}{n+\mathbf{w}(Y_0)}$, $n\ge0$. The sequence $(\widetilde{Y}_n)_{n\ge0}$ satisfies the following recursive stochastic algorithm (obvious consequence of (2.10))

$$\widetilde{Y}_{n+1}=\widetilde{Y}_n-\frac{1}{n+1+\mathbf{w}(Y_0)}\left(\widetilde{Y}_n-\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}\left(\widetilde{f}(\widetilde{Y}_n)\right)}\right)+\frac{1}{n+1+\mathbf{w}(Y_0)}\Delta M_{n+1},\quad n\ge1,\quad (5.1)$$

where

$$\Delta M_{n+1}:=X_{n+1}-\mathbb{E}\left[X_{n+1}\,|\,\mathcal{F}_n\right]\quad (5.2)$$

is a true $(\mathcal{F}_n)_{n\geq 0}$ martingale increment. Let us remark that, in this setting,

$$\frac{\mathbf{w}(Y_n)}{n+\mathbf{w}(Y_0)} = \frac{n+\mathbf{w}(Y_0)}{n+\mathbf{w}(Y_0)} = 1, \quad n \geq 0,$$

so that the sequence

$$\widetilde{Y}_n \in \mathcal{S}_d, \quad \text{for every } n \geq 0,$$

since it has non-negative components.

The special case of a linear drawing rule $f(y) = y$ is entirely elucidated by the celebrated Athreya-Karlin theorem recalled below for completeness.

Theorem 5.1 (Athreya-Karlin's Theorem, see [1])**.** *Let $(Y_n)_{n\geq 0}$ be the urn composition sequence defined by (1.3) and (1.5) with $D_n = I_d$, $n \geq 1$, and a linear drawing rule ($i.e.$ (2.4) with $f(u) = u$). Then, if $Y_0 \in \mathbb{R}^d \setminus \{0\}$ is deterministic, there exists a random vector $\widetilde{Y}_\infty$ having values in the simplex $\mathcal{S}_d$ such that*

$$\widetilde{Y}_n = \frac{Y_n}{\boldsymbol{w}(Y_n)} \xrightarrow[n\to+\infty]{a.s.} \widetilde{Y}_\infty \quad a.s.$$

Furthermore,

(i) $\widetilde{Y}_\infty$ has a Dirichlet distribution with parameter Y_0.

(ii) In particular, if $d = 2$, $\widetilde{Y}^1_\infty$ has a Beta distribution with parameters Y_0^1 and Y_0^2 (in particular, $\widetilde{Y}^1_\infty$ has a uniform distribution on $[0,1]$ if $Y_0^1 = Y_0^2 = 1$).

Now, we investigate the case $f \neq \mathrm{Id}_{\mathbb{R}_+}$ by borrowing tools to adaptive bandit models analysis (see [31, 29, 30]). First note that, as $\widetilde{Y}_n$ lives in the simplex $\mathcal{S}_d$, *the function f only needs to be defined on* $[0,1]$. Moreover, we will no longer ask f to be convex or concave on $(0,1)$ but require existence and finiteness of the derivatives at 0 and 1.

We will have to deal in this section with *noiseless traps*, that is parasitic equilibrium points long on the boundary of the simplex $\mathcal{S}_d$. The term noiseless refers to the fact that the martingale increment in the canonical decomposition (2.11) of the algorithm vanishes at such point, otherwise $\widetilde{Y}_n$ could exit the simplex. However there are examples of recursive algorithms converging toward such a noiseless equilibrium point: this is the case of the so-called *bandit algorithm* investigated in [31] and [29]. To elucidate how the algorithm behaves in the neighbourhood of such points, we need to have more precise information how the skewing function f itself behaves near 0 and 1. This study will require new tools, still based on a martingale approach but of a different nature of what we used so far in a somewhat hidden by calling upon classical results of SA theory.

To this end we make the following assumption on the function f:

f is continuous, non-decreasing, $f(0)=0$, $f(1)=1$, $f>0$ on $(0,1]$, with finite right derivative at 0 and left derivative at 1.

By a slight abuse of notation, we will denote $f'(0)$ and $f'(1)$ these two derivatives. A typical example could be $f(u) = 4\left(u - \frac{1}{2}\right)^3 + \frac{1}{2}$, $u \in [0,1]$.

Theorem 5.2. (a) *Let $I \subsetneq \{1,\ldots,d\}$ be non-empty. If f satisfies $f'(0) > |I| f\big(\frac{1}{|I|}\big)$, then, for every deterministic initial value such that $Y_0^j > 0$ for some $j \notin I$,*

$$\mathbb{P}\left(\widetilde{Y}_\infty = \widetilde{e}_I\right) = 0.$$

(b) *If $d = 2$ and the right second derivative of f at 0 exists, the above conclusion still* *holds if* $f'(0) = 1$ *and* $f'(1) + \frac{f''(0)}{2} > 1$.

(c) *If f is strictly concave then* $\mathcal{E}_{d,h} = \{\widetilde{e}_I, I \subset \{1,\dots,d\}, I \neq \varnothing\}$ *and, for every starting value* $Y_0 \in (0,+\infty)^d$,

$$\widetilde{Y}_n \xrightarrow{a.s.} \widetilde{e}_{\{1,\dots,d\}} = y(d) \ \ \textit{as} \ \ n \to +\infty.$$

Remark 5.3. • In claim (b), if $f(0) = 0$, $f(1) = 1$, $f'_r(0) = 1$ and f is convex or concave, then $f = \mathrm{Id}$, so this case is interesting only out of the concave/convex framework.

• If $Y_0^i = 0$ for some $i \in \{1,\dots,d\}$, then, as proved below, $Y_n^i = 0$ for every $n \geq 0$. So, as soon as $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$, one may apply the above result (c) to the urn restricted to $I' = \{i \in I,\ Y_0^i > 0\}$ to prove that $\widetilde{Y}_n \to \widetilde{e}_{I'}$ as $n \to +\infty$.

Proof. (a)-(b) It follows from (1.3) and the fact that $D_n \equiv I_d$ that, if $Y_0^i = 0$ for some i, then $Y_n^i = 0$ for every instant $n \geq 0$. So, up to a reduction of the dimension d, we may always assume that $Y_0^i > 0$, $i = 1,\dots,d$. As a consequence we may assume that, for every $n \geq 0$, $\min_i \widetilde{Y}_n^i > 0$. Our aim is to prove that $\mathbb{P}\left(\widetilde{Y}_\infty^j = 0\right) = 0$, for every $j \notin I$. To this end, we will show that $\{\widetilde{Y}_\infty^j = 0\} \subset \{\widetilde{L}_\infty = 0\}$ where $\widetilde{L}_\infty$ is the terminal value of a non-negative martingale. Then we will apply an "oracle" inequality to this martingale. Without loss of generality, we may assume that, up to a permutation, $1 \notin I$ and $j = 1$ in what follows.

Step 1: First, we define the function $\widetilde{h}$ by

$$\widetilde{h}(y) = 1 - \frac{f(y^1)}{y^1 \mathbf{w}(\tilde{f}(y))} \mathbf{1}_{\{y^1 \neq 0\}},\ y \in \mathcal{S}_d,$$

which satisfies $\widetilde{h}(y) < 1$, for every $y \in \mathcal{S}_d \setminus \{y : y^1 = 0\}$.

Starting from the dynamics of $\widetilde{Y}_n^1$ given by (5.1), we have, for every $n \geq 0$,

$$\begin{aligned}
\widetilde{Y}_{n+1}^1 &= \widetilde{Y}_n^1 - \frac{1}{n+1+\mathbf{w}(Y_0)} \left(\widetilde{Y}_n^1 - \frac{f(\widetilde{Y}_n^1)}{\mathbf{w}\big(\widetilde{f}(\widetilde{Y}_n)\big)} \right) + \frac{1}{n+1+\mathbf{w}(Y_0)} \Delta M_{n+1}^1 \\
&= \widetilde{Y}_n^1 \left(1 - \frac{1}{n+1+\mathbf{w}(Y_0)} \widetilde{h}\big(\widetilde{Y}_n^1\big) \right) + \frac{1}{n+1+\mathbf{w}(Y_0)} \Delta M_{n+1}^1.
\end{aligned}$$

We derive that the (non-negative) sequence

$$\widetilde{L}_n := \frac{\widetilde{Y}_n^1}{\prod_{k=1}^n \left(1 - \frac{1}{k+\mathbf{w}(Y_0)} \widetilde{h}(\widetilde{Y}_{k-1}^1)\right)}, \quad n \geq 0, \tag{5.3}$$

is a non-negative martingale satisfying the recursive equation $\widetilde{L}_0 = \widetilde{Y}_0^1$ and

$$\widetilde{L}_{n+1} = \widetilde{L}_n + \frac{1}{n+1+\mathbf{w}(Y_0)} \frac{\Delta M_{n+1}^1}{\prod_{k=1}^{n+1} \left(1 - \frac{1}{k+\mathbf{w}(Y_0)} \widetilde{h}(\widetilde{Y}_{k-1}^1)\right)}, \quad n \geq 0.$$

– Case (a) If $f'(0) > |I| f\big(\frac{1}{|I|}\big)$, then $\widetilde{h}(y) \longrightarrow \kappa := 1 - \frac{f'(0)}{|I| f\left(\frac{1}{|I|}\right)} < 0$ as $y \to \widetilde{e}_I$. Therefore, one has on the event $\left\{\widetilde{Y}_n \to \widetilde{e}_I\right\}$, $\widetilde{Y}_n^1 \to 0$ so that $\widetilde{h}\left(\widetilde{Y}_{n-1}^1\right) \overset{a.s.}{\sim} \kappa < 0$. This in turn implies, still on this event, that $\prod_{k=1}^n \left(1 - \frac{1}{k+\mathbf{w}(Y_0)} \widetilde{h}(\widetilde{Y}_{k-1}^1)\right) \xrightarrow{a.s.} +\infty$. It follows from its definition in (5.3) that $\widetilde{L}_n \xrightarrow{a.s.} 0$ on $\left\{\widetilde{Y}_n \to \widetilde{e}_I\right\}$ since $0 \leq \widetilde{Y}_n^1 \leq \frac{\mathbf{w}(Y_n)}{n+\mathbf{w}(Y_0)} \xrightarrow{a.s.} 1$. Consequently, we get

$$\left\{\widetilde{Y}_n \to \widetilde{e}_I\right\} \subset \left\{\widetilde{L}_n \to 0\right\}.$$

– Case (b) (Case $d=2$). In that case the function $\widetilde{h}$ can be viewed as a function of y^1 on $[0,1]$ since $y^1+y^2=1$ on $\mathcal{S}_2$. As $f'(0)=1$ and $f''(0)$ exists, a second order Taylor expansion shows that $\widetilde{h}(y)=y^1\big((1-f'(1)-\frac{f''(0)}{2})+O(y^1)\big)$ as $y^1\to 0$ so that $\widetilde{h}(y)<0$ for y in the neighborhood of $\widetilde{e}_{\{2\}}$. Hence, with in mind $\widetilde{e}_{\{2\}}=e^2$, we still have $\Big\{\widetilde{Y}_n\to e^2\Big\}=\Big\{\widetilde{Y}^1_n\to 0\Big\}\subset\Big\{\widetilde{L}_n\to 0\Big\}$.

Step 2 *Oracle inequality*. The end of the proof is based on the following (short) "oracle" lemma already used in [31] in a somewhat hidden manner (see the proof of Lemma 1), reproduced and proved here for the reader's convenience.

Lemma 5.4 (Oracle inequality, see [31]). *Let $(M_n)_{n\ge 0}$ be a positive L^2-bounded martingale with predictable quadratic variation process $(\langle M\rangle_n)_{n\ge 0}$ ([3]). Let $M_\infty=a.s.\text{-}\lim_n M_n$. Then*

$$\forall n\ge 0,\quad \mathbb{P}\big(M_\infty=0\,\big|\,\mathcal{F}_n\big)\le\frac{\mathbb{E}\left[\big\langle M\big\rangle_\infty-\big\langle M\big\rangle_n\,\big|\,\mathcal{F}_n\right]}{M_n^2}.$$

Lemma 5.4. It is sufficient to observe that, for every $n\ge 0$,

$$\begin{aligned}\mathbb{P}\big(M_\infty=0\,\big|\,\mathcal{F}_n\big)&=\frac{\mathbb{E}\left[\mathbb{1}_{\{M_\infty=0\}}M_n^2\Big|\mathcal{F}_n\right]}{M_n^2}\\&\le\frac{\mathbb{E}\left[(M_\infty-M_n)^2\Big|\mathcal{F}_n\right]}{M_n^2}=\frac{\mathbb{E}\left[\big\langle M\big\rangle_\infty-\big\langle M\big\rangle_n\,\big|\,\mathcal{F}_n\right]}{M_n^2}.\end{aligned}\qquad\square$$

First we note that

$$\mathbb{E}\left[(\Delta\widetilde{L}_{n+1})^2\,\big|\,\mathcal{F}_n\right]=\left(\frac{1}{n+1+\mathbf{w}(Y_0)}\right)^2\frac{\mathbb{E}\left[(\Delta M^1_{n+1})^2\,\big|\,\mathcal{F}_n\right]}{\left(\prod_{k=1}^{n+1}\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{k-1})}{k+\mathbf{W}(Y_0)}\right)\right)^2}$$

and

$$\mathbb{E}\left[\big(\Delta M^1_{n+1}\big)^2\,\big|\,\mathcal{F}_n\right]=\frac{f(\widetilde{Y}^1_n)\big(\mathbf{w}\big(\widetilde{f}(\widetilde{Y}_n)\big)-f(\widetilde{Y}^1_n)\big)}{\mathbf{w}\big(\widetilde{f}(\widetilde{Y}_n)\big)^2}\le\frac{d-1}{f(1/d)^2}$$

since $\mathbf{w}(f(y))\ge f\big(\max_i y_i\big)\ge f(1/d)$ and $\widetilde{Y}_n\in\mathcal{S}_d$. As a consequence

$$\mathbb{E}\left[(\Delta\widetilde{L}_{n+1})^2\,\big|\,\mathcal{F}_n\right]=\frac{1}{(n+1+\mathbf{w}(Y_0))^2\left(\prod_{k=1}^{n+1}\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{k-1})}{k+\mathbf{W}(Y_0)}\right)\right)^2}\frac{f(\widetilde{Y}^1_n)\big(\mathbf{w}\big(\widetilde{f}(\widetilde{Y}_n)\big)-f(\widetilde{Y}^1_n)\big)}{\Big(\mathbf{w}\big(\widetilde{f}(\widetilde{Y}_n)\big)\Big)^2}.$$

Then, applying Lemma 5.4 to the non-negative martingale $(\widetilde{L}_n)_{n\ge 1}$ yields

$$\begin{aligned}\mathbb{P}\big(\widetilde{L}_\infty=0\,\big|\,\mathcal{F}_n\big)&\le\frac{\mathbb{E}\left[\langle\widetilde{L}\rangle_\infty-\langle\widetilde{L}\rangle_n\,\big|\,\mathcal{F}_n\right]}{\widetilde{L}_n^2}\\&=\frac{1}{\widetilde{L}_n^2}\mathbb{E}\left[\sum_{k=n+1}^{\infty}\frac{F(\widetilde{Y}^1_{k-1})}{(k+\mathbf{w}(Y_0))^2\left(\prod_{\ell=1}^{k}\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{\ell-1})}{\ell+\mathbf{W}(Y_0)}\right)\right)^2}\,\Bigg|\,\mathcal{F}_n\right],\end{aligned}$$

[3] *i.e.* the ($a.s.$) unique predictable process null at 0 such that $(M_n^2-\langle M\rangle_n)_{n\ge 0}$ is martingale.

where the function F, defined by $F(y) := \frac{f(y^1)\big(\mathbf{w}(\tilde{f}(y))-f(y^1)\big)}{\big(\mathbf{w}(\tilde{f}(y))\big)^2}$, $y \in (0,1] \times [0,1]^{d-1}$, is clearly non-negative and bounded by $\kappa_d := (d-1)/f(1/d)^2$. Consequently,

$$\mathbb{P}\big(\widetilde{L}_\infty = 0 \,\big|\, \mathcal{F}_n\big) \leq \frac{\kappa_d}{\widetilde{L}_n^2} \sum_{k=n+1}^{\infty} \frac{1}{(k+\mathbf{w}(Y_0))^2} \mathbb{E}\left[\underbrace{\frac{\widetilde{Y}^1_{k-1}}{\prod_{\ell=1}^{k-1}\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{\ell-1})}{\ell+\mathbf{w}(Y_0)}\right)}}_{=\widetilde{L}_{k-1}} \frac{\left(1-\frac{1}{k+\mathbf{w}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{k-1})\right)^{-1}}{\prod_{\ell=1}^{k}\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{\ell-1})}{\ell+\mathbf{w}(Y_0)}\right)} \,\middle|\, \mathcal{F}_n \right].$$

Let $\widetilde{h}_+ := \max(\widetilde{h}, 0)$, so that $\widetilde{h} \leq \widetilde{h}_+ \leq \|\widetilde{h}_+\|_\infty$. First note that $\widetilde{h}_+(y) \leq 1 - \frac{f(y^1)}{d \cdot y^1} < 1$, for every $y \in \mathcal{S}_d \setminus \{y^1 = 0\}$ since $\mathbf{w}(\tilde{f}(y)) \leq d$ and we assumed $f(y^1) > 0$ on $(0,1]$. Moreover, $\limsup_{y^1 \to 0} \widetilde{h}(y) \leq 1 - f'(0)/d < 1$ since $f'(0) > 0$ by assumption. As a consequence $\|\widetilde{h}_+\|_\infty < 1$.

In 2-dimensions, under the additional assumption in the critical case ($f'(0) = 1$), the extension of the function h over $[0,1]$ is negative so that $\widetilde{h}_+ \equiv 0$.

Finally, we obtain

$$\left(1 - \frac{\widetilde{h}(\widetilde{Y}^1_{k-1})}{k+\mathbf{w}(Y_0)}\right)^{-1} \leq \left(1 - \frac{\|\widetilde{h}_+\|_\infty}{k+\mathbf{w}(Y_0)}\right)^{-1}, \quad k \geq 1.$$

Then, as $\mathbb{E}\left[\widetilde{L}_{k-1} \middle| \mathcal{F}_n\right] = \widetilde{L}_n$ for every $k \geq n+1$, since $\widetilde{M}$ is a $(\mathbb{P}, \mathcal{F}_n)$-martingale,

$$\begin{aligned}
&\mathbb{P}(\widetilde{L}_\infty = 0 \,|\, \mathcal{F}_n) \\
&\quad \leq \frac{\kappa_d}{\widetilde{L}_n^2} \sum_{k=n+1}^{\infty} \frac{1}{(k+\mathbf{w}(Y_0))^2} \frac{\widetilde{L}_n}{\left(1-\frac{\|\widetilde{h}_+\|_\infty}{k+\mathbf{w}(Y_0)}\right) \prod_{\ell=1}^{n}\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{\ell-1})}{\ell+\mathbf{w}(Y_0)}\right) \prod_{\ell=n+1}^{k}\left(1-\frac{\|\widetilde{h}_+\|_\infty}{\ell+\mathbf{w}(Y_0)}\right)} \\
&\quad = \frac{\kappa_d}{\widetilde{L}_n \prod_{\ell=1}^{n}\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{\ell-1})}{\ell+\mathbf{w}(Y_0)}\right)} \sum_{k=n+1}^{\infty} \frac{1}{\left(1-\frac{\|\widetilde{h}_+\|_\infty}{k+\mathbf{w}(Y_0)}\right)(k+\mathbf{w}(Y_0))^2 \prod_{\ell=n+1}^{k}\left(1-\frac{\|\widetilde{h}_+\|_\infty}{\ell+\mathbf{w}(Y_0)}\right)} \\
&\quad \leq \frac{\kappa_d}{C_0 \widetilde{Y}^1_n} \sum_{k=n+1}^{\infty} \frac{1}{(k+\mathbf{w}(Y_0))^2} \exp\left(-\sum_{\ell=n+1}^{k} \log\left(1-\frac{\|\widetilde{h}_+\|_\infty}{\ell+\mathbf{w}(Y_0)}\right)\right)
\end{aligned} \tag{5.4}$$

where $C_0 = \frac{\mathbf{w}(Y_0)}{1+\mathbf{w}(Y_0)} \in (0,1)$ since $1 - \frac{\|\widetilde{h}_+\|_\infty}{k+\mathbf{w}(Y_0)} > 1 - \frac{1}{k+\mathbf{w}(Y_0)} \geq C_0$.

Note that $\log(1-u) \geq -\frac{u}{1-u_0}$, $u \in (0, u_0)$. Applying this inequality with $u_0 = \frac{1}{n+1+\mathbf{w}(Y_0)}$ yields, for every $\ell \geq n+1$,

$$\log\left(1 - \frac{\|\widetilde{h}_+\|_\infty}{\ell+\mathbf{w}(Y_0)}\right) \geq -\left(1+\frac{1}{n+\mathbf{w}(Y_0)}\right)\frac{\|\widetilde{h}_+\|_\infty}{\ell+\mathbf{w}(Y_0)}.$$

Hence

$$\begin{aligned}
\sum_{\ell=n+1}^{k} \log\left(1-\frac{\|\widetilde{h}_+\|_\infty}{\ell+\mathbf{w}(Y_0)}\right) &\geq -\|\widetilde{h}_+\|_\infty \left(1+\frac{1}{n+\mathbf{w}(Y_0)}\right) \sum_{\ell=n+1}^{k} \frac{1}{\ell+\mathbf{w}(Y_0)} \\
&\geq -\|\widetilde{h}_+\|_\infty \left(1+\frac{1}{n+\mathbf{w}(Y_0)}\right) \int_n^k \frac{du}{u+\mathbf{w}(Y_0)} \\
&= -\|\widetilde{h}_+\|_\infty \left(1+\frac{1}{n+\mathbf{w}(Y_0)}\right) \log\left(\frac{k+\mathbf{w}(Y_0)}{n+\mathbf{w}(Y_0)}\right).
\end{aligned}$$

Plugging this inequality into (5.4) and using that $\widetilde{Y}_n^1 = \frac{Y_n^1}{n+\mathbf{w}(Y_0)}$,

$$\mathbb{P}\big(\widetilde{L}_\infty = 0 \,|\, \mathcal{F}_n\big) \le \frac{\kappa_d(n+\mathbf{w}(Y_0))}{C_0 Y_n^1} \sum_{k=n+1}^{+\infty} \frac{e^{(1+\frac{1}{n+\mathbf{w}(Y_0)})\|\widetilde{h}_+\|_\infty \log\left(\frac{k+\mathbf{w}(Y_0)}{n+\mathbf{w}(Y_0)}\right)}}{(k+\mathbf{w}(Y_0))^2}.$$

As $\|\widetilde{h}_+\|_\infty < 1$, there exists n_0 such that for every $n \ge n_0$, $1-(1+\frac{1}{n+\mathbf{w}(Y_0)})\|\widetilde{h}_+\|_\infty > 0$. Hence

$$\begin{aligned}
\mathbb{P}\big(\widetilde{L}_\infty = 0 \,|\, \mathcal{F}_n\big) &\le \frac{\kappa_d(n+\mathbf{w}(Y_0))}{C_0 Y_n^1} \sum_{k=n+1}^{\infty} \frac{(n+\mathbf{w}(Y_0))^{-(1+\frac{1}{n+\mathbf{w}(Y_0)})\|\widetilde{h}_+\|_\infty}}{(k+\mathbf{w}(Y_0))^{2-(1+\frac{1}{n+\mathbf{w}(Y_0)})\|\widetilde{h}_+\|_\infty}} \\
&= \frac{\kappa_d(n+\mathbf{w}(Y_0))^{1-(1+\frac{1}{n+\mathbf{w}(Y_0)})\|\widetilde{h}_+\|_\infty}}{C_0 Y_n^1} \sum_{k=n+1}^{\infty} \frac{1}{(k+\mathbf{w}(Y_0))^{2-(1+\frac{1}{n+\mathbf{w}(Y_0)})\|\widetilde{h}_+\|_\infty}} \\
&\le \frac{\kappa_d(n+\mathbf{w}(Y_0))^{1-(1+\frac{1}{n+\mathbf{w}(Y_0)})\|\widetilde{h}_+\|_\infty}}{C_0 Y_n^1} \int_n^{+\infty} \frac{du}{(u+\mathbf{w}(Y_0))^{2-(1+\frac{1}{n+\mathbf{w}(Y_0)})\|\widetilde{h}_+\|_\infty}} \\
&\le \frac{\kappa_d}{C_0 Y_n^1} \frac{1}{1-(1+\frac{1}{n+\mathbf{w}(Y_0)})\|\widetilde{h}_+\|_\infty}.
\end{aligned}$$

Now, it remains to prove that $Y_n^1 \xrightarrow{a.s.} +\infty$. Let $Y_\infty^1 := \lim_n Y_n^1$ (the components of Y_n are non-decreasing). One checks that

$$\{Y_\infty^1 < +\infty\} = \bigcup_{n\ge 0} \bigcap_{k>n} \left\{ U_k > \frac{f(\frac{Y_n^1}{k-1+\mathbf{w}(Y_0)})}{f(\frac{Y_n^1}{k-1+\mathbf{w}(Y_0)}) + f(1-\frac{Y_n^1}{k-1+\mathbf{w}(Y_0)})} \right\},$$

then, for every $n \ge 0$,

$$\mathbb{P}\left(Y_\infty^1 < +\infty \mid Y_n^1 = y\right) = \prod_{k>n} \left(1 - \frac{f(y/(k-1+\mathbf{w}(Y_0)))}{f(y/(k-1+\mathbf{w}(Y_0))) + f(1-y/(k-1+\mathbf{w}(Y_0)))}\right) = 0,$$

since $\sum_{k\ge 1} \frac{f(y/k+\mathbf{w}(Y_0))}{f(y/k+\mathbf{w}(Y_0)) + f(1-y/k+\mathbf{w}(Y_0))} = +\infty$ because $f'(0) > 0$. Therefore $Y_\infty^1 = +\infty$ $a.s.$

On the other hand, the closed martingale $\mathbb{P}\big(\widetilde{L}_\infty = 0 |\, \mathcal{F}_n\big) \to \mathbf{1}_{\{\widetilde{L}_\infty = 0\}}$ $a.s.$ and in L^1 so that

$$\mathbf{1}_{\{\widetilde{L}_\infty=0\}} = 0 \quad a.s. \quad \text{i.e.} \quad \mathbb{P}(\widetilde{L}_n \to 0) = \mathbb{P}(\widetilde{L}_\infty = 0) = 0$$

which in turn implies that $\mathbb{P}\left(\widetilde{Y}_\infty^1 = 0\right)$ since it was proved in Step 1 that $\{\widetilde{Y}_n^1 \to 0\} \subset \{\widetilde{L}_n \to 0\}$.

(c) The fact that $\mathcal{E}_{d,h} = \{\widetilde{e}_I, I \subset \{1,\dots,d\}, I \ne \varnothing\}$ follows from Proposition 2.4(c). As $Y_0^i > 0$, $i \in \{1,\dots,d\}$, we derive from what precedes that $\mathbb{P}\big(\widetilde{Y}_n \to \partial\mathcal{S}_d\big) = 0$. As a consequence, following Theorem A.2, $\mathbb{P}(d\omega)$-$a.s.$, the compact connected flow invariant set $\Theta^\infty(\omega)$ of limiting values of $(\widetilde{Y}_n(\omega))_{n\ge 0}$ is a minimal connected attractor of ODE_h in $\mathring{\mathcal{S}}_d$. We know from Proposition 2.11(b) that $y(d) = \frac{1}{d}\mathbf{1}$ is a uniformly attracting point ODE_h and from Proposition 2.13(b) that the flow of ODE_h $(y(y_0,t))_{t\ge 0, y_0 \in \mathring{\mathcal{S}}_d}$ converges toward $\frac{1}{d}\mathbf{1}$, so it converges uniformly with respect to $y_0 \in \Theta^\infty(\omega)$. Consequently, one concludes by Theorem A.2 that $\Theta^\infty(\omega) = \{y(d)\}$ (otherwise it would have an internal attractor). Hence $\widetilde{Y}_n \xrightarrow{a.s.} y(d)$. □

6 An extension and an example of application

6.1 Distribution based normalized f-skewed rule

Let us recall that this alternative distribution rule (1.7) reads

$$\forall\, i \in \{1,\dots,d\},\quad \mathbb{P}(X_{n+1}=e^i\,|\,\mathcal{F}_n)=\frac{f(Y_n^i)}{\sum_{j=1}^d f(Y_n^j)},\quad n\geq 0,$$

where f has regular variation with index $\alpha>0$ in the sense that for every $t>0$, $\dfrac{f(tx)}{f(x)}\underset{x\to\infty}{\longrightarrow} t^\alpha$ and f is bounded on each interval $(0,M]$. Then, by applying Theorem 1.5.2 p.22 in [10], $\dfrac{f(tx)}{f(x)}\underset{x\to\infty}{\longrightarrow} t^\alpha$ uniformly in t on each $(0,b]$, $0<b<\infty$.

We can reformulate the dynamics (1.3)-(1.7) into a recursive stochastic algorithm like in the Section 2.2, and we obtain the following recursive procedure satisfied by the sequence $(\widetilde{Y}_n)_{n\geq 0}$:

$$\widetilde{Y}_{n+1}=\widetilde{Y}_n-\gamma_{n+1}\left(\widetilde{Y}_n-H\frac{\widetilde{Y}_n^\alpha}{\mathbf{w}(\widetilde{Y}_n^\alpha)}\right)+\gamma_{n+1}\Big(\Delta M_{n+1}+\widehat{r}_{n+1}\Big) \tag{6.1}$$

with the step $\gamma_n=\frac{1}{n+\mathbf{w}(Y_0)}$, $\widetilde{Y}_n^\alpha=\Big((\widetilde{Y}_n^i)^\alpha\Big)_{1\leq i\leq d}$ and an $\mathcal{F}_n$-measurable remainder term given by

$$\widehat{r}_{n+1}:=H_{n+1}\frac{\widetilde{f}(Y_n)}{\mathbf{w}(\widetilde{f}(Y_n))}-H\frac{\widetilde{Y}_n^\alpha}{\mathbf{w}(\widetilde{Y}_n^\alpha)}. \tag{6.2}$$

Notice that, in the empirical frequency based normalized skewing case (defined by (1.5)), the remainder term was $r_{n+1}=(H_{n+1}-H)\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}$, therefore assumption **(A3)** implied directly that $r_n\xrightarrow[n\to+\infty]{a.s.}0$ since $\left\|\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right\|\leq\sqrt{d}$. Our aim in this section is simply to use the uniform convergence of the regular variation to prove that the successive required assumption on the remainder term are satisfied by $\widehat{r}_{n+1}$.

By the same arguments as those used in Section 2.2, we check that $\mathbf{w}(Y_n)$ satisfies (2.15). Moreover, the quantity $\widetilde{N}_n:=\frac{1}{n}\sum_{k=1}^n X_k$ can also by re-written as a stochastic recursive procedure as follows

$$\widetilde{N}_{n+1}=\widetilde{N}_n-\frac{1}{n+1}\left(\widetilde{N}_n-\frac{\widetilde{Y}_n^\alpha}{\mathbf{w}(\widetilde{Y}_n^\alpha)}\right)+\frac{1}{n+1}\left(\Delta\widetilde{M}_{n+1}+\widetilde{r}_{n+1}\right),$$

where $\widetilde{r}_{n+1}=\dfrac{\widetilde{f}(Y_n)}{\mathbf{w}(\widetilde{f}(Y_n))}-\dfrac{\widetilde{Y}_n^\alpha}{\mathbf{w}(\widetilde{Y}_n^\alpha)}$, thus $\widetilde{r}_{n+1}\in\mathcal{F}_n$.

Theorem 6.1 (Distribution based normalized f-skewed rule, $a.s.$ convergence)**.** *Let $d=2$. Assume that **(A1)**, **(A2)** and **(A3)** hold.*
(a) If $0<\alpha\leq 1$, then h has a unique zero $y^*\in I^*$ and

$$\frac{\mathbf{w}(Y_n)}{n+\mathbf{w}(Y_0)}\xrightarrow[n\to+\infty]{a.s.}1,\quad \frac{Y_n}{\mathbf{w}(Y_n)}\xrightarrow[n\to+\infty]{a.s.}y^*\quad\text{and}\quad \widetilde{N}_n\xrightarrow[n\to+\infty]{a.s.}\frac{(y^*)^\alpha}{\mathbf{w}((y^*)^\alpha)}.$$

(b) If $\alpha>1$, then either h has a unique zero $y^*\in I^*$ or ODE_h has two attracting equilibrium points in I^* (as we have established in Section 2.3). Thus, the stochastic recursive procedure $a.s.$ converges to one of the possible limit values.

Proof. By the same arguments like in Section 2.3, $\mathbf{w}(Y_n)$ satisfies (2.15), therefore Proposition 2.2 holds. Consequently, $\widetilde{Y}_n$ lies in a compact of $\mathbb{R}_+$, thus

$$\max_{1\leq i\leq d}\left|\frac{f(Y_n^i)}{f(n+\mathbf{w}(Y_0))}-\left(\frac{Y_n}{n+\mathbf{w}(Y_0)}\right)^\alpha\right|\underset{n\to+\infty}{\longrightarrow}0.$$

Set $a_n^i = \frac{f(Y_n^i)}{f(n+\mathbf{w}(Y_0))}$ and $b_n^i = (\widetilde{Y}_n^i)^\alpha$, $i \in \{1,\dots,d\}$. Then, for every $i \in \{1,\dots,d\}$,

$$\frac{a_n^i}{\mathbf{w}(a_n)} - \frac{b_n^i}{\mathbf{w}(b_n)} = \frac{a_n^i - b_n^i}{\mathbf{w}(b_n)} + \frac{a_n^i}{\mathbf{w}(a_n)}\left(1 - \frac{\mathbf{w}(a_n)}{\mathbf{w}(b_n)}\right).$$

But

$$\mathbf{w}(b_n) = \sum_{i=1}^d (\widetilde{Y}_n^i)^\alpha \geq \begin{cases} \left(\sum_{i=1}^d \widetilde{Y}_n^i\right)^\alpha = \mathbf{w}(\widetilde{Y}_n)^\alpha & \text{if} \quad \alpha \in [0,1] \\ d^{1-\alpha}\mathbf{w}(\widetilde{Y}_n)^\alpha & \text{if} \quad \alpha > 1 \end{cases},$$

therefore

$$\mathbf{w}(b_n) \geq \frac{\mathbf{w}(\widetilde{Y}_n)^\alpha}{d^{(\alpha-1)_+}} \underset{a.s.}{\sim} \frac{(n+\mathbf{w}(\widetilde{Y}_0))^\alpha}{d^{(\alpha-1)_+}}.$$

Consequently, for every $i \in \{1,\dots,d\}$,

$$\frac{a_n^i}{\mathbf{w}(a_n)} - \frac{b_n^i}{\mathbf{w}(b_n)} \leq \frac{\max_{1\leq i\leq d}|a_n^i - b_n^i| + \sum_{j=1}^d |a_n^j - b_n^j|}{\mathbf{w}(b_n)},$$

i.e.

$$\max_{1\leq i\leq d}\left|\frac{a_n^i}{\mathbf{w}(a_n)} - \frac{b_n^i}{\mathbf{w}(b_n)}\right| \leq \frac{d+1}{\mathbf{w}(b_n)} \max_{1\leq i\leq d}|a_n^i - b_n^i| \xrightarrow[n\to+\infty]{a.s.} 0.$$

Thus

$$|\widehat{r}_{n+1}| \leq \|H\| \max_{1\leq i\leq d}\left|\frac{a_n^i}{\mathbf{w}(a_n)} - \frac{b_n^i}{\mathbf{w}(b_n)}\right| + \|H_{n+1} - H\| \xrightarrow[n\to+\infty]{a.s.} 0,$$

and, in the same way, $\widetilde{r}_{n+1} \xrightarrow[n\to+\infty]{a.s.} 0$. Consequently claim (a) follows from Proposition 3.1(a) and Theorem 3.10.

We have to check the assumption on the remainder term to apply results on traps for SA (see Theorem A.3 in the Appendix). We have that

$$\max_{1\leq i\leq d}\left|\frac{a_n^i}{\mathbf{w}(a_n)} - \frac{b_n^i}{\mathbf{w}(b_n)}\right| \lesssim \frac{(d+1)d^{(\alpha-1)_+}}{(n+\mathbf{w}(Y_0))^\alpha} \max_{1\leq i\leq d}|a_n^i - b_n^i| = o(n^{-\alpha}) \tag{6.3}$$

where $a_n \lesssim b_n$ stands for $\limsup_n (a_n/b_n) \leq 1$. Since $\alpha > 1$, H is co-stochastic and the generating matrices satisfy assumption **(A3)**, we have

$$\sum_{n\geq 0} \|\widehat{r}_{n+1}\|^2 < +\infty.$$

(a) This claim follows from Proposition 3.1(b)&(c) and Theorem 3.10, namely h has a unique zero $y^* \in I^*$ or ODE_h has two attracting equilibrium points in I^*.
(b) As formerly mentioned, the function f is strictly convex when $f(y) = y^\alpha$, $\alpha > 1$, so that Proposition 3.4. □

To establish a CLT for the sequence $(\widetilde{Y}_n)_{n\geq 0}$ we need that the remainder term $(\widehat{r}_n)_{n\geq 1}$ satisfies (A.7). Then we will assume that the addition rule matrices $(D_n)_{n\geq 1}$ satisfy **(A1)**-(ii) to ensure that $(\widetilde{Y}_n)_{n\geq 0}$ lies in the simplex (which implies that the rate in (6.3) is no more $a.s.$) and we assume also that $\alpha > 1/2$.

Theorem 6.2 (Weak rate)**.** *Let $d = 2$. Assume that the index of regular variation $\alpha > 1/2$, that the addition rule matrices $(D_n)_{n\geq 1}$ satisfy* ***(A1)****-(ii),* ***(A3)****,* ***(A4)*** *We have*

$$\mathit{Sp}\left(J_h(y^*)\right) = \{1, 1-\rho^*\},$$

where

$$\rho^* = \frac{\alpha(y^{*1})^{\alpha-1}(p_1 - y^{*1}) + \alpha(1-y^{*1})^{\alpha-1}(y^{*1} - 1 + p_2)}{(y^{*1})^\alpha + f(1-y^{*1})^\alpha}.$$

(a) *If $p_1 + p_2 - 1 \le 0$, $v_n = 1$, $n \ge 1$, and **(A5)**$_v$ holds, then*

$$\sqrt{n}\left(\widetilde{Y}_n - y^*\right) \underset{n\to+\infty}{\overset{\mathcal{L}_{stably}}{\longrightarrow}} \mathcal{N}(0,\Sigma) \quad \textit{with} \quad \Sigma = \int_0^{+\infty} e^{u(J_h(y^*)-\frac{I}{2})}\Gamma e^{u(J_h(y^*)-\frac{I}{2})^t} du$$

$$\textit{and } \Gamma = \frac{(y^{*1})^\alpha C^1_{y^*} + (1-y^{*1})^\alpha C_{y^*} r}{\boldsymbol{w}((y^*)^\alpha)} - y^*(y^*)^t = a.s.\text{-}\lim_{n\to+\infty} \mathbb{E}\left[\Delta M_n \Delta M_n^t \,|\, \mathcal{F}_{n-1}\right]. \quad (6.4)$$

(b) *If $p_1 + p_2 - 1 > 0$, we have three possible rate of convergence depending on the second eigenvalue:*

*(i) If $0 < \rho^* < \frac{1}{2}$ and $v_n = 1$, $n \ge 1$, and **(A5)**$_v$ holds, then*

$$\sqrt{n}\left(\widetilde{Y}_n - y^*\right) \underset{n\to+\infty}{\overset{\mathcal{L}_{stably}}{\longrightarrow}} \mathcal{N}(0,\Sigma).$$

*(ii) If $\rho^* = \frac{1}{2}$ and $v_n = \log n$, $n \ge 1$, and **(A5)**$_v$ holds, then*

$$\sqrt{\frac{n}{\log n}}\left(\widetilde{Y}_n - y^*\right) \underset{n\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}(0,\Sigma) \ \textit{ where } \Sigma = \lim_n \frac{1}{n}\int_0^n e^{u(J_h(y^*)-\frac{I}{2})}\Gamma e^{u(J_h(y^*)-\frac{I}{2})^t} du.$$

*(iii) If $\frac{1}{2} < \rho^* < 1$ and $v_n = n^{1-2\rho^*+\eta}$, $\eta > 0$ and **(A5)**$_v$ holds, then $n^{1-\rho^*}\left(\widetilde{Y}_n - y^*\right)$ $a.s.$ converges as $n \to +\infty$ towards a positive finite random variable Υ.*

This result follows from Theorem A.5 and Theorem 4.2.

6.2 An application to finance: adaptive asset allocation

Such urn based recursive procedures can be applied to adaptive portfolio allocation by an asset manager or a trader, or to optimal split across liquidity pools. Indeed the first setting has already been done in [31] and successfully implemented with multi-armed bandit procedure. We develop in this section the adaptive portfolio allocation, but the optimal split across liquidity pools can be implemented in the same way, by considering that the different colors represent the different liquidity pools, and the trader want to optimally split a large volume of a single asset among the different possible destinations.

Imagine an asset manager who deals with a portfolio of d tradable assets. To optimize the yield of her portfolio, she can modify the proportions invested in each asset. She starts with the initial allocation vector Y_0. At stage n, she chooses a tradable asset according to the distribution (1.5) or (1.7) of X_n, then evaluates its performance over one time step and modifies the portfolio composition accordingly (most likely virtually) and proceeds. Thus the normalized urn composition $\widetilde{Y}_n$ represents the allocation vector among the assets and the addition rule matrices D_n model the successive reallocations depending on the past performances of the different assets. The evaluation of the asset performances can be carried out recursively with an estimator like with multi-arm clinical trials (see [4, 32]). In practice, it can be used to design the addition rule matrices D_n. For example, we may consider $(T_n^i)_{n\ge1}$, $i \in \{1,\dots,d\}$, a *success indicator*, namely d independent sequences of i.i.d. $\{0,1\}$-valued Bernoulli trials with respective parameter p_i (with convention $T_n^i = 1$ if the return of the i^{th} asset in the n^{th} reallocation is positive and $T_n^i = 0$ otherwise).

Let $N_n^i := \sum_{k=1}^n X_k^i$ be the number of times the i^{th} asset is selected among the first n stages with $N_0^i = 0$, $i \in \{1,\dots,d\}$, and let S_n be the d dimensional vector defined by

$$S_n^i = S_{n-1}^i + T_n^i X_n^i, \quad n \ge 1, \quad S_0^i = 1, \quad i \in \{1,\dots,d\},$$

denoting the number of successes of the i^{th} asset among these N_n^i reallocations. Define Π_n an estimator of the vector of success probabilities, namely $\Pi_n^i = \frac{S_n^i}{N_n^i}$, $i \in \{1,\dots,d\}$.

We can prove that $\Pi_n \xrightarrow[n\to+\infty]{a.s.} p := (p_1, \ldots, p_d)^t$ (see [4, 32]). Then we build the following addition rule matrices

$$D_{n+1} = \begin{pmatrix} T^1_{n+1} & \frac{\Pi^1_n(1-T^2_{n+1})}{\sum_{j\neq 2}\Pi^j_n} & \cdots & \frac{\Pi^1_n(1-T^d_{n+1})}{\sum_{j\neq d}\Pi^j_n} \\ \frac{\Pi^2_n(1-T^1_{n+1})}{\sum_{j\neq 1}\Pi^j_n} & T^2_{n+1} & \cdots & \frac{\Pi^2_n(1-T^d_{n+1})}{\sum_{j\neq d}\Pi^j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Pi^d_n(1-T^1_{n+1})}{\sum^d_{j\neq 1}\Pi^j_n} & \frac{\Pi^d_n(1-T^2_{n+1})}{\sum^d_{j\neq 2}\Pi^j_n} & \cdots & T^d_{n+1} \end{pmatrix}, \tag{6.5}$$

i.e. at stage $n+1$, if the return of the j^{th} asset is positive, then one ball of type j is added in the urn. Otherwise, $\frac{\Pi^i_n}{\sum_{k\neq j}\Pi^k_n}$ (virtual) balls of type i, $i \neq j$, are added. This addition rule matrix clearly satisfies **(A1)**-(i) and **(A2)**. Then, one easily checks that the generating matrices $H_n = \mathbb{E}\left[D_{n+1} \,|\, \mathcal{F}_n\right]$ satisfy **(A1)**-(ii) and, as soon as $Y_0 \in \mathbb{R}^d_+ \setminus \{0\}$, $H_n \xrightarrow{a.s.} H$ (see [4, 32]), where

$$H_{n+1} = \begin{pmatrix} p_1 & \frac{\Pi^1_n(1-p_2)}{\sum_{j\neq 2}\Pi^j_n} & \cdots & \frac{\Pi^1_n(1-p_d)}{\sum_{j\neq d}\Pi^j_n} \\ \frac{\Pi^2_n(1-p_1)}{\sum_{j\neq 1}\Pi^j_n} & p_2 & \cdots & \frac{\Pi^2_n(1-p_d)}{\sum_{j\neq d}\Pi^j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Pi^d_n(1-p_1)}{\sum_{j\neq 1}\Pi^j_n} & \frac{\Pi^d_n(1-p_2)}{\sum_{j\neq 2}\Pi^j_n} & \cdots & p_d \end{pmatrix}, \; H = \begin{pmatrix} p_1 & \frac{p_1(1-p_2)}{\sum_{j\neq 2}p_j} & \cdots & \frac{p_1(1-p_d)}{\sum_{j\neq d}p_j} \\ \frac{p_2(1-p_1)}{\sum_{j\neq 1}p_j} & p_2 & \cdots & \frac{p_2(1-p_d)}{\sum_{j\neq d}p_j} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p_d(1-p_1)}{\sum_{j\neq 1}p_j} & \frac{p_d(1-p_2)}{\sum_{j\neq 2}p_j} & \cdots & p_d \end{pmatrix}.$$

Let us remark that H is $\mathbb{R}$-diagonalizable since it is symmetric with respect to its invariant measure (see [33]). Therefore, the number of each asset in the portfolio Y_n follows the dynamics (1.3) and the distribution of the portfolio in each asset follows the dynamics (2.11) or (6.1) depending on the drawing rule.

Here the components of the limiting generating matrix H can be interpreted as constraints on the composition of the portfolio. Indeed, in presence of two assets (or colors), we prove that the first component of the allocation vector y^{*1} lies in I^* (see Proposition 3.1), therefore the portfolio will contain at least a proportion $p_1 \vee (1-p_2)$ and no more than $p_1 \wedge (1-p_2)$ of the first asset. Such rules may be prescribed by the regulation, the bank policy or the bank customer, and our approach is a natural way to have them satisfied (at least asymptotically).

The idea of reinforcing the drawing rule (instead of considering the uniform drawing) like in (1.5) or (1.7) can be interpreted as a way to take into account the risk aversion of the trader or the customer. Indeed, if f is concave the equilibrium point will be in the middle of the simplex (see Theorem 3.4 and Theorem 3.8), so the trader prefers to have diversification in her/his portfolio. On the contrary, if f is convex, the equilibrium points will lie on the boundary of the set of constraints induced by the limiting generating matrix H, so she/he prefers to take advantage of the most money-making asset (like in a "winner take all" or a "0-1" strategy).

Numerical experiments. We present some numerical experiments for the drawing rule defined by (1.5), firstly with a concave function $f: y \mapsto \sqrt{y}$ and secondly with a convex function $f: y \mapsto y^4$. Therefore we have a unique equilibrium point in the first setting and two attracting targets in the second framework. We consider an asset manager who deal with a portfolio of 2 tradable assets. We model the addition rule matrices like in the multi-arm clinical trials, namely D_n is defined by (6.5). We use the same success probabilities, namely $p_1 = 0.7$ and $p_2 = 0.75$, and the initial urn composition is chosen randomly in the simplex $\mathcal{S}_2$.

▷ *Convergence of the portfolio allocation with concave drawing rule.*

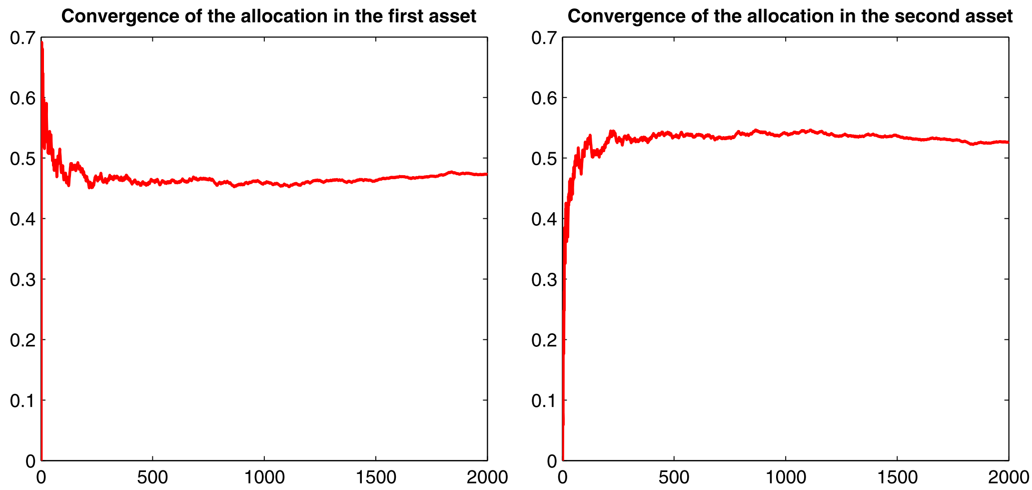

Figure 4: Convergence of $\widetilde{Y}_n$ toward y^* for $f(y) = \sqrt{y}$ with $p_1 = 0.7$ and $p_2 = 0.75$.

We have that $y^{*1} \in (0.25, 0.7)$ and y^{*1} and y^{*2} are close to $\frac{1}{2}$, so the portfolio is diversified because in this case the investor is risk adverse.

▷ *Convergence of the portfolio allocation with convex drawing rule.*

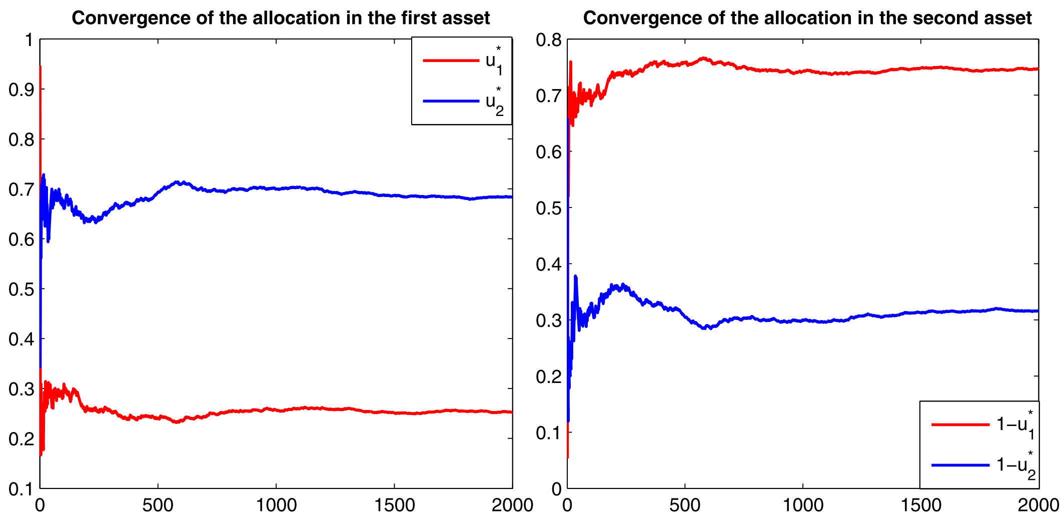

Figure 5: Convergence of $\widetilde{Y}_n$ toward y^* for $f(y) = y^4$ with $p_1 = 0.7$ and $p_2 = 0.75$. We have two possible equilibrium points: $1 - p_2 < u_1^* < 1/2$ (in red) and $1/2 < u_2^* < p_1$ (in blue).

In the convex framework, we have two possible strategies and they are close to the boundaries defined by regulation. Moreover the distribution of the portfolio between the two assets is more asymmetric, because the trader chooses to invest two times more in one asset than in the other.

The probability that our algorithm converges toward u_1^* (resp. u_2^*) depends on the initial value $\widetilde{Y}_0$. Approximations by Monte-Carlo simulations of these probabilities are presented in the following figure to illustrate this dependency on the initial urn composition.

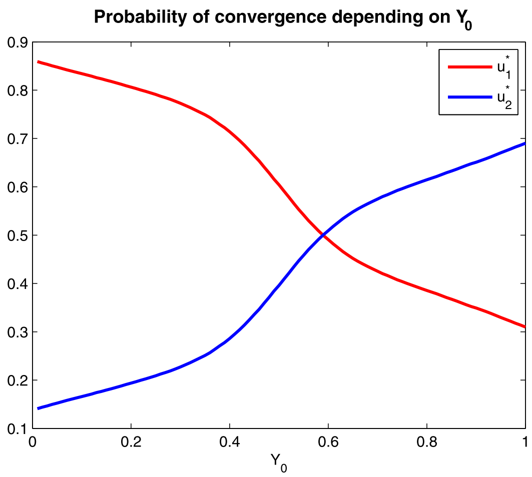

Figure 6: Estimation by MC of the probabilities for the algorithm to converge toward u_1^* (red) and u_2^* (blue) depending on $\widetilde{Y}_0$ for $f(y) = y^4$ with $p_1 = 0.7$ and $p_2 = 0.75$.

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Appendix

A Basic tools from Stochastic Approximation

Consider the following recursive procedure defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ having values in a convex set $C \subset \mathbb{R}^d$,

$$\forall\, n \geq 0, \quad \theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n) + \gamma_{n+1}\left(\Delta M_{n+1} + r_{n+1}\right), \tag{A.1}$$

where $(\gamma_n)_{n\geq 1}$ is a $(0, \bar{\gamma}]$-valued step sequence for some $\bar{\gamma} > 0$, $h : C \to \mathbb{R}^d$ is a continuous function with linear growth (the *mean field* of the algorithm) such that

$$(I_d - \gamma h)(C) \subset C \;\text{ for every } \quad \gamma \in (0, \bar{\gamma}], \tag{A.2}$$

and θ_0 is an $\mathcal{F}_0$-measurable finite random vector and, for every $n \geq 1$, ΔM_n is an $(\mathcal{F}_n)_n$-martingale increment and r_n is an $(\mathcal{F}_n)_n$-adapted remainder term.

Note that the assumptions of the theorems recalled below are possibly not minimal, but adapted to the problems we want to solve.

$\triangleright$ $A.s.$ **Convergence.** Let us introduce a few additional notions on differential systems. We consider the differential system $ODE_h \equiv \dot{x} = -h(x)$ associated to the (continuous) *mean field* $h : C \to \mathbb{R}^d$. We assume that this system has a C-valued *flow* ([4]) $\Phi(t, \xi)_{t\in\mathbb{R}_+, \xi\in C}$: For every $\xi \in C$, $(\Phi(t, \xi))_{t\geq 0}$ is the unique solution to ODE_h defined on the whole positive real line. This flow exists as soon as h is locally Lipschitz with linear growth.

Let K be a compact connected, flow invariant subset of C, *i.e.* such that $\Phi(t, K) \subset K$ for every $t \in \mathbb{R}_+$.

A non-empty subset $A \subset K$ is an *internal attractor* of K for ODE_h if

(i) $A \subsetneq K$,

(ii) $\exists\, \varepsilon_0 > 0$ such that $\displaystyle\sup_{x\in K, \mathrm{dist}(x,A)\leq\varepsilon_0} \mathrm{dist}\big(\Phi(t, x), A\big) \to 0$ as $t \to +\infty$.

[4] this is satisfied under the above stability condition (A.2).

A compact connected flow invariant set K is a *minimal attractor* for ODE_h if it contains no internal attractor. This terminology coming from dynamical systems may be misleading: Thus any equilibrium point of ODE_h (zero of h) is a minimal attractor by this definition, regardless of its stability (see Claim (b) in Theorem A.2 below, see also Definition 2.9).

Remark A.1. When the flow does not exist, the above definition should be understood as follows: One replaces the flow $\Phi(x,\cdot)$ by the family of all solutions of ODE_h starting from x at time 0 (whose existence follow from Peano's Theorem). For more details on this natural extension, we refer to [20] (see Appendix "the ODE method without flow"). Up to this extension, the theorem below remains true even when uniqueness of solutions of ODE_h fails.

Theorem A.2. *($A.s.$ convergence with ODE method, see $e.g.$ [9, 18, 28, 19, 6]). Assume that $h : C \to \mathbb{R}^d$ satisfies* (A.2) *and that ODE_h has a C-valued flow ($e.g.$ because h is a locally Lipschitz function with linear growth). Assume furthermore that*

$$r_n \xrightarrow[n\to+\infty]{a.s.} 0 \quad \textit{and} \quad \sup_{n\geq 0} \mathbb{E}\left[\|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n\right] < +\infty \quad a.s.,$$

and that $(\gamma_n)_{n\geq 1}$ is a positive sequence satisfying ($\gamma_n \in (0, \bar{\gamma}]$, $n \geq 1$) and

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \textit{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty.$$

On the event $A_\infty = \big\{\omega : (h(\theta_n(\omega)))_{n\geq 0}$ is bounded$\big\}$, $\mathbb{P}(d\omega)$-$a.s.$, the set $\Theta^\infty(\omega)$ of the limiting values of $(\theta_n(\omega)_{n\geq 0})$ as $n \to +\infty$ is a compact connected flow invariant minimal attractor for ODE_h (see Proposition 5.3 in Section 5.1 in [6]).

Furthermore:

(a) Equilibrium point(s) as limiting value(s). *If $\operatorname{dist}\big(\Phi(\theta_0, t), \{h = 0\}\big) \to 0$ as $t \to +\infty$, for every $\theta_0 \in \mathbb{R}^d$, then $\Theta^\infty(\omega) \cap \{h = 0\} \neq \varnothing$.*

(b) Single stable equilibrium point. *If $\{h = 0\} = \{\theta^*\}$ and $\Phi(\theta_0, t) \to \theta^*$ as $t \to +\infty$ locally uniformly in θ_0, then $\Theta^\infty(\omega) = \{\theta^*\}$ i.e. $\theta_n \xrightarrow{a.s.} \theta^*$ as $n \to +\infty$.*

(c) 1-dimensional setting. *If $d = 1$ and $\{h = 0\}$ is locally finite, then $\Theta^\infty(\omega) = \{\theta_\infty\} \subset \{h = 0\}$ i.e. $\theta_n \xrightarrow{a.s.} \theta_\infty \in \{h = 0\}$.*

A stochastic algorithm may $a.s.$ converge under the existence of multiple equilibrium points, typically stochastic gradient or pseudo-descents, but we do not need such results to solve the urn problems under consideration in this paper. We refer to [18, 28, 19, 6], among others. Note also that examples of situation (a) where the algorithm $a.s.$ *does not converge* are developed in [19], [20] or [6] (necessarily with $d \geq 2$ owing to Claim (c)).

$\triangleright$ **Traps (Unstable equilibrium point).** This second theorem deals with "traps" *i.e.* repulsive zeros of the mean function h. It shows that, provided such a trap is noisy enough, such a trap cannot be a limiting point of the algorithm.

Theorem A.3. *($A.s.$ non-convergence toward a noisy trap, see $e.g.$ [12, 17]). Assume that $z^* \in \mathbb{R}^d$ is a trap for the stochastic algorithm (A.1), i.e.*

(i) *$h(z^*) = 0$,*
(ii) *there exists a neighborhood $V(z^*)$ of z^* in which h is differentiable with a Lipschitz differential,*
(iii) *the eigenvalue of $M = J_h(z^*)$ with the lowest real part, denoted by $\lambda_{\min}$, satisfies $\Re e(\lambda_{\min}) < 0$.*

Assume furthermore that $a.s.$ *on* $\Gamma(z^*) = \{\theta_n \underset{n\to+\infty}{\longrightarrow} z^*\}$,

$$\sum_{n\geq 1} \|r_n\|^2 < +\infty \quad a.s. \quad \textit{and} \quad \exists\, \delta > 0 \textit{ such that } \limsup_n \mathbb{E}\left[\|\Delta M_{n+1}\|^{2+\delta} \mid \mathcal{F}_n\right] < +\infty \quad a.s. \tag{A.3}$$

Let $K_\pm = \mathrm{Ker}(P_\pm M)$ *where* P_- *(resp.* P_+*) is a polynomial with zeros having negative (resp. positive) real parts and* $P_- P_+ = P$*, minimal polynomial of* M *so that* $\mathbb{R}^d = K_+ \oplus K_-$*. Denoting by* ΔM^{rep}_{n+1} *the projection of* ΔM_{n+1} *on* K_- *alongside* K_+*, assume that,* $a.s.$ *on* $\Gamma(z^*)$*,*

$$\liminf_n \mathbb{E}\left[\left\|\Delta M^{rep}_{n+1}\right\|^2 \middle| \mathcal{F}_n\right] > 0. \tag{A.4}$$

Moreover, assume the positive sequence $(\gamma_n)_{n\geq 1}$ *satisfies*

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \textit{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty.$$

Then, $\mathbb{P}\big(\Gamma(z^*)\big) = 0$.

Remark A.4. If (A.3) holds with $\delta = 0$, then the conclusion is still true if (A.4) is replaced by the slightly more stringent condition $\liminf_n \mathbb{E}\left[\left\|\Delta M^{rep}_{n+1}\right\| \middle| \mathcal{F}_n\right] > 0\ a.s.$

$\triangleright$ **Rate(s) of convergence.** We will say that h is ϵ-differentiable ($\epsilon > 0$) at θ^* if

$$h(\theta) = h(\theta^*) + J_h(\theta^*)(\theta - \theta^*) + o(\|\theta - \theta^*\|^{1+\epsilon}) \quad \text{as} \quad \theta \to \theta^*.$$

Theorem A.5. *(Rate of convergence see [18] Theorem 3.III.14 p.131 (for the* CLT *see also* $e.g.$ *[9, 28])). Let* θ^* *be an equilibrium point of* $\{h = 0\}$ *and* $\{\theta_n \to \theta^*\}$ *the convergence event associated to* θ^* *(supposed to have a positive probability). Set the gain parameter sequence* $(\gamma_n)_{n\geq 1}$ *as follows*

$$\forall n \geq 1, \quad \gamma_n = \frac{1}{n}. \tag{A.5}$$

Assume that the function h *is differentiable at* θ^* *and all the eigenvalues of* $J_h(\theta^*)$ *have positive real parts. Assume that, for a real number* $\delta > 0$*,*

$$\sup_{n\geq 0} \mathbb{E}\left[\|\Delta M_{n+1}\|^{2+\delta} \mid \mathcal{F}_n\right] < +\infty\, a.s., \quad \mathbb{E}\left[\Delta M_{n+1} \Delta M_{n+1}^t \mid \mathcal{F}_n\right] \xrightarrow[n\to+\infty]{a.s.} \Gamma^* \quad \textit{on} \quad \{\theta_n \to \theta^*\}, \tag{A.6}$$

where $\Gamma^* \in \mathcal{S}^+(d, \mathbb{R})$ *(deterministic symmetric positive matrix) and for an* $\varepsilon > 0$ *and a positive sequence* $(v_n)_{n\geq 1}$ *(specified below),*

$$n\, v_n \mathbb{E}\left[\|r_{n+1}\|^2 \mathbb{1}_{\{\|\theta_n - \theta^*\| \leq \varepsilon\}}\right] \underset{n\to+\infty}{\longrightarrow} 0. \tag{A.7}$$

Let $\lambda_{\min}$ *denote the eigenvalue of* $J_h(\theta^*)$ *with the lowest real part and set* $\Lambda := \Re e(\lambda_{\min})$.

(a) *If* $\Lambda > \frac{1}{2}$ *and* $v_n = 1$*,* $n \geq 1$*, then, the weak convergence rate is ruled on the convergence event* $\{\theta_n \xrightarrow{a.s.} \theta^*\}$ *by the following Central Limit Theorem*

$$\sqrt{n}\,(\theta_n - \theta^*) \xrightarrow[n\to+\infty]{\mathcal{L}_{stably}} \mathcal{N}\left(0, \Sigma^*\right) \quad \textit{with} \quad \Sigma^* := \int_0^{+\infty} e^{-u\left(J_h(\theta^*)^t - \frac{I_d}{2}\right)} \Gamma^* e^{-u\left(J_h(\theta^*) - \frac{I_d}{2}\right)} du.$$

(b) *If* $\Lambda = \frac{1}{2}$*,* $v_n = \log n$*,* $n \geq 1$*, and* h *is* ϵ*-differentiable at* θ^**, then*

$$\sqrt{\frac{n}{\log n}}\,(\theta_n - \theta^*) \overset{\mathcal{L}_{stably}}{\underset{n\to+\infty}{\longrightarrow}} \mathcal{N}(0,\Sigma^*) \quad \textit{on} \quad \{\theta_n \to \theta^*\},$$

where $\Sigma^* = \lim_n \frac{1}{n}\int_0^n e^{-u\left(J_h(\theta^*)^t - \frac{I_d}{2}\right)}\Gamma e^{-u\left(J_h(\theta^*) - \frac{I_d}{2}\right)}du$*.*

(c) *If* $\Lambda \in \left(0,\frac{1}{2}\right)$*,* $v_n = n^{2\Lambda-1+\eta}$*,* $n \geq 1$*, for some* $\eta > 0$*, and* h *is* ϵ*-differentiable at* θ^**, for some* $\epsilon > 0$*, then* $n^{\Lambda}\,(\theta_n - \theta^*)$ *is* $a.s.$ *bounded on* $\{\theta_n \to \theta^*\}$ *as* $n \to +\infty$*.*

If, moreover, $\Lambda = \lambda_{\min}$ *(*$\lambda_{\min}$ *is real), then* $n^{\Lambda}\,(\theta_n - \theta^*)$ $a.s.$ *converges as* $n \to +\infty$ *toward a finite random variable.*