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Rescaled Whittaker driven stochastic differential equations converge to the additive stochastic heat equation*

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Abstract

We study some linear SDEs arising from the two-dimensional q-Whittaker driven particle system on the torus as $q \to 1$. The main result proves that the SDEs along certain characteristics converge to the additive stochastic heat equation. Extensions for the SDEs with generalized coefficients and in other spatial dimensions are also obtained. Our proof views the limiting process after recentering as a process of the convolution of a space-time white noise and the Fourier transform of the heat kernel. Accordingly we turn to similar space-time stochastic integrals defined by the SDEs, but now the convolution and the Fourier transform are broken. To obtain tightness of these induced integrals, we bound the oscillations of complex exponentials arising from divergence of the characteristics, with two methods of different nature.

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Contents

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1 Introduction

The two-dimensional q-Whittaker driven particle system on the torus [6] has an interpretation as a discrete $(2+1)$-dimensional surface growth model. It is proven in [4] that the fluctuations of the particle locations satisfy a system of linear SDEs in the limit $q \to 1$. Our main objective in this paper is to prove the weak convergence of these SDEs to the additive stochastic heat equation. In this way, a full picture for the fluctuations of the Whittaker driven particle system follows and it holds at the process level.

For $q \in [0,1)$, the q-Whittaker driven particle system in [6] carries interlaced particles occupying different vertices of a periodic lattice in two dimensions. These particles jump to the right in the lattice according to rates defined by q and distances to certain nearest particles. The interlacing of the particles stays the same over time. Vertices occupied by the particles can be mapped to certain edges in perfect matchings of a periodized hexagonal lattice. These edges induce a certain height function for a discrete surface. Hence, the particle system can be seen as a discrete surface growth model.

Through this correspondence, the particle system is believed to belong to the anisotropic Kardar–Parisi–Zhang class described as follows. In general, the height function $H(x,t)$ of a surface growth model is expected to satisfy the following stochastic partial differential equation (SPDE) [14]:

$$\frac{\partial H}{\partial t}(x,t) = \nu \Delta H(x,t) + \langle \nabla H, \Lambda \nabla H \rangle (x,t) + \sigma \dot{W}(x,t), \quad x \in \mathbb{R}^2. \tag{1.1}$$

Here, Δ is the Laplacian in x and $\dot{W}$ is a space-time white noise. The scalars ν, σ and the 2×2 matrix Λ are physical parameters associated to the surface. The anisotropic class consists of growth models where signs of the eigenvalues of Λ differ [23]. It is predicted by Wolf [25] for this class that the expected noise in the height function should behave like the expected noise in the Edwards–Wilkinson equation which is also called the additive stochastic heat equation. (The additive stochastic heat equation is the SPDE in (1.1) without the nonlinear term $\langle \nabla H, \Lambda \nabla H \rangle$.) See [22] for a broad discussion of Wolf's prediction and the recent mathematical progress. For physics introductions to surface growth models, see the lecture by Kardar [13] and the monograph by Barabási and Stanley [1]. Walsh's lecture notes [24, Chapter 5] give a solution theory of the additive stochastic heat equation.

The first instance of Wolf's prediction is obtained in [4] by taking an iterated limit of the q-Whittaker driven particle system. The first limit is as mentioned above, showing that the fluctuations of the particle locations converge to the solution $\{\xi_t(x); x \in \mathcal{R}\}$ of a system of linear SDEs as $q \to 1$. Here, $\mathcal{R}$ is a finite quotient group in $\mathbb{Z}^2$ and describes the locations of the particles in a reduced manner. See [4, Theorem 1] or Example 2.2 for these SDEs, and they are called the **Whittaker driven SDEs** in this paper.

Our main interest arises from the other limits in [4] in proving Wolf's prediction for the particle system, which we now recall. The Whittaker driven SDEs are defined with the drift vector $A\xi$ for $\xi \in \mathbb{R}^{\mathcal{R}}$ and a constant matrix A. A crucial observation made in [4] is that the discrete Fourier transform of A:

$$\widehat{A}(k) \stackrel{\mathrm{def}}{=} \sum_{x\in\mathcal{R}} A_{x,0}e^{-\mathtt{i}\langle x,k\rangle}, \quad k \in \mathbb{R}^2, \tag{1.2}$$

satisfies

$$\widehat{A}(k) = -\mathtt{i}\langle k, U\rangle + \frac{\langle k, Qk\rangle}{2} + \mathcal{O}(|k|^3), \quad k \to 0, \tag{1.3}$$

for a real vector U and a real strictly negative definite matrix Q. In terms of Fourier multipliers, a Laplacian defined by the quadratic form in (1.3) is thus hidden in the drift vector of the Whittaker driven SDEs. Let the Whittaker driven SDEs be subject to the following spatial mesh points at time $\delta^{-1}t$:

$$\lfloor \delta^{-1}Ut + \delta^{-1/2}(-Q)^{1/2}z\rfloor, \quad z \in \mathbb{R}^2. \tag{1.4}$$

Upon passing $\mathcal{R} \nearrow \mathbb{Z}^2$ and then $\delta \to 0+$, the SDEs are proven in [4] to converge to the additive stochastic heat equation in terms of the correlation structure. It is pointed out in [4] that only a few properties of the matrix A (Assumption 2.4) are needed to obtain this convergence.

The main theorem of this paper (Theorem 3.1) proves convergence of the distribution-valued processes X^δ which are defined by the $(\mathcal{R} \nearrow \mathbb{Z}^2)$-limit of the Whittaker driven SDEs subject to the generalized matrices A mentioned above. In other dimensions d, X^δ defined by some generalized $\mathbb{Z}^d$-indexed processes also converge to the additive stochastic heat equation. The limiting scheme is for $\delta \to 0+$ and in the path space of continuous, distribution-valued functions. The definition of X^δ in any case incorporates discrete characteristics as in (1.4) and the Edwards–Wilkinson growth exponent $(d-2)/4$. The SDEs from [4] and the present result give a proof of the Edwards–Wilkinson fluctuation in the Whittaker driven particle system, now at the process level.

The proof of the theorem focuses on the stochastic integral part Z^δ of X^δ. We view its convergence as $\delta \to 0+$ as convergence of integrands of space-time stochastic integrals and do not use the limit of covariance functions from [4]. This viewpoint begins with the fact that the corresponding part of the solution of the additive stochastic heat equation can be written as a stochastic convolution of the space-time white noise and the Fourier transform of the heat kernel by Itô's and Plancherel's isometries. We find that Z^δ satisfies a similar form, the difference being that its integrand shows truncation of spatial domain and discretization from the characteristics. This representation provides an alternative explanation of the emergence of the additive stochastic heat equation. The choice of the characteristics in (1.4) also arises naturally from the usual diffusive scaling of space and time (see the discussion before Proposition 4.2). Nevertheless, the discrete characteristics diverge and break the convolution and the Fourier transform to the effect of inducing new oscillations from complex exponentials. These properties make it the central question whether the regularity of Z^δ will be lost in the limit.

We obtain the tightness of Z^δ by two different methods (Sections 4.1.3 and 4.1.4). The first method proves a uniform Hölder condition of the covariance functions since Z^δ are Gaussian processes. The other one generalizes the factorization method (cf. [8]), viewing Z^δ as stochastic integrals that approximate stochastic convolutions. Either method relies on a semi-discrete integration by parts to obtain precise decay rates of the broken Fourier transforms in the stochastic integral representation of Z^δ (Section 4.1.1).

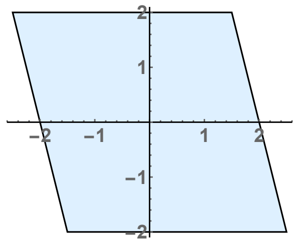

Figure 1: A set representation of $\mathcal{R}_m$ with $m = 4$ and $m_2 = 1$. The vertices are $(-\frac{3}{2}, -2), (\frac{5}{2}, -2), (\frac{3}{2}, 2), (-\frac{5}{2}, 2)$.

Organization of this paper. In Section 2, we discuss the explicit solutions of the Whittaker driven SDEs and their limit as $\mathcal{R} \nearrow \mathbb{Z}^2$. In Section 3, we set up notations for the main theorem (Theorem 3.1) of this paper and outline its proof. The proof is divided into Sections 4 and 5. As we need more complicated notations after Section 2, the reader can find a list of frequent notations for Sections 3–5 in Section 6.

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2 Fourier representations of the Whittaker driven SDEs

In this section, we specify the Whittaker driven SDEs and their generalizations. Then as in [4] we represent these SDEs in terms of Fourier transforms and derive their infinite volume limits.

First, we recall the discrete quotient groups that label coordinates in the SDEs. Given two positive integers m_2 and m such that $m_2/m \in (0, 1)$, $\mathcal{R}_m$ is defined to be the quotient group $\mathbb{Z}^2/\!\sim$, where the equivalence relation $\sim$ is given by

$$x \sim y \Longleftrightarrow x + (j_1 m, j_2 m) = y + (j_2 m_2, 0) \quad \text{for some } j_1, j_2 \in \mathbb{Z} \tag{2.1}$$

[4, Remark 1]. The quotient group $\mathbb{Z}^2/\!\sim$ can be identified as a discrete parallelogram subject to periodic boundary conditions. Whenever $\mathcal{R}_m$ is used as a set, we always refer to the discrete parallelogram defined by (2.2) below. See Figure 1 for an example of this set representation of $\mathcal{R}_m$.

Proposition 2.1. *The quotient group $\mathbb{Z}^2/\!\sim$ is isomorphic to the quotient group defined as the discrete parallelogram*

$$\left\{(x_1, x_2) \in \mathbb{Z}^2 \,\middle|\, -\frac{m}{2} \le x_2 < \frac{m}{2}, -\frac{m}{2} - \frac{m_2}{m} x_2 \le x_1 < \frac{m}{2} - \frac{m_2}{m} x_2 \right\} \tag{2.2}$$

subject to the pasting rule "$\equiv$" as follows:

(1) *Points on the lower and upper edges are pasted together by the rule*

$$\left(x_1, -\frac{m}{2}\right) \equiv \left(x_1 - m_2, \frac{m}{2}\right), \quad \forall\, x_1 \in \left[-\frac{m}{2} + \frac{m_2}{2}, \frac{m}{2} + \frac{m_2}{2}\right) \cap \mathbb{Z},$$

which is along the direction defining the left and right edges.

(2) *Points on the left and right edges are pasted together horizontally.*

Proof. Write $\mathcal{P}_m$ for the discrete set defined in (2.2). For $x,y\in\mathcal{P}_m$, $x\sim y$ implies $x=y$ since, with respect to the notation in (2.1), $j_2=0$ by the assumption that $-m/2\le x_2,y_2<m/2$ and then $j_1=0$ for a similar reason. Also, any point in $\mathbb{Z}^2$ is $\sim$-equivalent to a point in $\mathcal{P}_m$ by the observation that the translated parallelograms $\mathcal{P}_m+(j_1m-j_2m_2,j_2m)$ for j_1,j_2 ranging over $\mathbb{Z}$ tile the whole space $\mathbb{Z}^2$. Hence, we conclude that there is a natural isomorphism between $\mathbb{Z}^2/{\sim}$ and $\mathcal{P}_m/{\equiv}$. ■

Given the set $\mathcal{R}_m$, consider the following system of linear SDEs:

$$\mathrm{d}\xi_t^m(x)=\sum_{y\in\mathcal{R}_m}A_{x,y}\xi_t^m(y)\mathrm{d}t+\sqrt{v}\mathrm{d}W_t(x),\quad x\in\mathcal{R}_m. \tag{2.3}$$

Here, $A_{x,y}\in\mathbb{R}$ and $v\in(0,\infty)$ are constant, and $\{W(x);x\in\mathcal{R}_m\}$ is an m^2-dimensional standard Brownian motion.

Example 2.2 (Whittaker driven SDEs). In [4], the SDEs derived from the Whittaker driven particle system on $\mathcal{R}_m$ are defined by (2.3) with the following coefficients:

$$v=\frac{(1-e^{-D})(1-e^{-B})}{1-e^{-C}}$$

and

$$A_{x,y}=\begin{cases}\dfrac{e^{-D}(1-e^{-B})}{1-e^{-C}}-\dfrac{e^{-C}(1-e^{-D})(1-e^{-B})}{(1-e^{-C})^2}-\dfrac{e^{-B}(1-e^{-D})}{1-e^{-C}}, & y=x,\\[2mm] -\dfrac{e^{-D}(1-e^{-B})}{1-e^{-C}} & y=x+(-1,0),\\[2mm] \dfrac{e^{-C}(1-e^{-D})(1-e^{-B})}{(1-e^{-C})^2} & y=x+(0,-1),\\[2mm] \dfrac{e^{-B}(1-e^{-D})}{1-e^{-C}} & y=x+(1,-1),\\[2mm] 0, & \text{otherwise,}\end{cases}$$

for constants $D\in(0,\infty),C\in(0,D),$ and $B=D-C$ with $C/D=m_2/m$. ■

In the sequel, we work with more general matrices A satisfying only Assumption 2.3 and Assumption 2.4 stated below. See [4, Section 4, especially Theorem 2 and Remark 5] for these assumptions. In Remark 2.5, we will recall the reason why these assumptions are satisfied by the matrix in Example 2.2.

Assumption 2.3. For $d\ge1$, $\widehat{A}(k):\mathbb{R}^d\to\mathbb{C}$ satisfies the following conditions:

(1) $\widehat{A}(k)$ is 2π-periodic, is in $\mathscr{C}^\infty(\mathbb{R}^d)$, and satisfies $\overline{\widehat{A}(k)}=\widehat{A}(-k)$.

(2) $\widehat{A}(0)=0$ with $\mathtt{i}\nabla\widehat{A}(0)\in\mathbb{R}^d$.

(3) The function

$$R(k)\stackrel{\mathrm{def}}{=}\widehat{A}(k)+\widehat{A}(-k)=2\mathrm{Re}\,\widehat{A}(k),\quad k\in\mathbb{R}^d, \tag{2.4}$$

satisfies

$$R(k)=Q(k)+\mathcal{O}(|k|^3),\quad k\to0, \tag{2.5}$$

with $Q(k)=\langle k,Qk\rangle$ for a real strictly negative definite matrix Q.

(4) $R(k)$ defined by (2.4) is nonpositive and its only zero in $\mathbb{T}^d$ is $k=0$. ■

Here in (4) and throughout this paper, $\mathbb{T}^d=[-\pi,\pi]^d$ is a set and no periodic boundary conditions are implicitly imposed.

Assumption 2.4. A is a matrix indexed by $\mathbb{Z}^2\times\mathbb{Z}^2$ such that, for some integer $m_0\geq 2$, A restricted to $\mathcal{R}_m\times\mathcal{R}_m$ satisfies the following conditions every $m\geq m_0$:

(1) Translation invariance holds on the quotient group $\mathcal{R}_m$:

$$A_{x,y}=A_{x+z,y+z},\quad \forall\, x,y,z\in\mathcal{R}_m. \tag{2.6}$$

(2) The discrete Fourier transform $\widehat{A}(k)$ defined by (1.2) with $\mathcal{R}=\mathcal{R}_m$ satisfies Assumption 2.3 with $d=2$.

Moreover, these discrete Fourier transforms on $\mathcal{R}_m\times\mathcal{R}_m$ are independent of m for $m\geq m_0$. ■

Notice that the constancy of the Fourier transforms in Assumption 2.4 is equivalent to the finite support property of $x\mapsto A_{x,0}$ on $\mathbb{Z}^2$.

Remark 2.5. The matrix A in Example 2.2 satisfies the translation invariance in Assumption 2.4. By this property and the group property $\mathcal{R}_m=-\mathcal{R}_m$, $\widehat{A}(k)$ and $R(k)$ defined by (1.2) and (2.4) take the following simple forms: for all $k\in\mathbb{R}^2$,

$$\begin{aligned}\widehat{A}(k)&=\sum_{x\in\mathcal{R}_m}A_{0,x}e^{\mathtt{i}\langle x,k\rangle}\\&=A_{0,0}+A_{0,(1,-1)}e^{\mathtt{i}(k_1-k_2)}+A_{0,(0,-1)}e^{-\mathtt{i}k_2}+A_{0,(-1,0)}e^{-\mathtt{i}k_1},\\R(k)&=A_{0,0}+A_{0,(1,-1)}\cos(k_1-k_2)+A_{0,(0,-1)}\cos(k_2)+A_{0,(-1,0)}\cos(k_1).\end{aligned}$$

These explicit forms can be used to verify the rest of Assumption 2.4. See [4, Proposition 2 and Appendix B] for the details. ■

Assumption 2.4 is in force in the rest of this section.

Recall that the explicit solution to the linear system in (2.3) is given by

$$\xi_t^m(x)=\sum_{y\in\mathcal{R}_m}e^{tA}(x,y)\xi_0^m(y)+\sum_{y\in\mathcal{R}_m}\sqrt{v}\int_0^t e^{(t-s)A}(x,y)\mathrm{d}W_s(y),\quad\forall\, x\in\mathcal{R}_m \tag{2.7}$$

(cf. [12, Eq.(6.6) in Section 5.6]). Here in (2.7), e^{tA} is understood to be the usual matrix exponential of the sub-matrix of A restricted to $\mathcal{R}_m\times\mathcal{R}_m$. Note that even in the Whittaker driven case (Example 2.2), A is not a generator matrix. So not all of the entries of these matrix exponentials are nonnegative. We decompose the Gaussian process ξ^m into

$$\xi^m=\eta^m+\zeta^m, \tag{2.8}$$

where $\eta_t^m(x)$ and $\zeta_t^m(x)$ are defined by the first and second sums in (2.7), respectively. We say that η^m is the **deterministic part** of ξ^m and ζ^m is the **stochastic part**.

To apply Assumption 2.4, we turn to the Fourier transform of ξ^m. Define

$$f_k(x)\stackrel{\rm def}{=}\frac{1}{m}e^{-\mathtt{i}\langle k,x\rangle} \tag{2.9}$$

and

$$\widehat{\xi}(k) \stackrel{\rm def}{=} \sum_{x\in\mathcal{R}_m} \xi(x) f_k(x), \quad \xi \in \mathbb{C}^{\mathcal{R}_m}. \tag{2.10}$$

Then the translation invariance in Assumption 2.4 and the definition (1.2) of $\widehat{A}(k)$ imply that for any analytic function F, the usual multiplier formula holds:

$$\widehat{F(A)\xi}\,(k) = F\big(\widehat{A}(k)\big)\widehat{\xi}(k), \quad \forall\, k \in \mathbb{R}^2. \tag{2.11}$$

The processes η^m and ζ^m can be represented by their Fourier transforms $\widehat{\eta^m}(k)$ and $\widehat{\zeta^m}(k)$. For this purpose, it is enough to require that k be points ranging over the following set:

$$\mathcal{K}_m \stackrel{\rm def}{=} \left\{ \left(\frac{2\pi}{m}r_1, \frac{2\pi}{m}\Big(\frac{m_2}{m}r_1 + r_2\Big)\right) \middle| r_1, r_2 \in \mathbb{Z}, -\frac{m}{2} \leq r_1, r_2 < \frac{m}{2} \right\}. \tag{2.12}$$

The additional properties that we need are summarized in Lemma 2.6 below (see [21, Chapter 1] or [4, Section 3.1]). For any subset E of $\mathbb{Z}^2$, write

$$\langle \phi_1, \phi_2 \rangle_E = \sum_{x\in E} \phi(x)\overline{\phi_2(x)}. \tag{2.13}$$

Lemma 2.6. *Let f_k and $\mathcal{K}_m$ be defined by (2.9) and (2.12), respectively.*

(1) *For any $k \in \mathcal{K}_m$, f_k is well-defined on the quotient group $\mathcal{R}_m$.*

(2) *$\{f_k; k \in \mathcal{K}_m\}$ is an orthonormal basis of $(\mathbb{C}^{\mathcal{R}_m}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{R}_m})$.*

(3) *The following inversion formula holds:*

$$\xi(x) = \sum_{k\in\mathcal{K}_m} \widehat{\xi}(k)\overline{f_k(x)}, \quad \forall\, x \in \mathcal{R}_m. \tag{2.14}$$

Corollary 2.7. *With respect to the decomposition in (2.8), it holds that*

$$\eta_t^m(x) = \sum_{k\in\mathcal{K}_m} e^{t\widehat{A}(k)}\widehat{\xi_0^m}(k)\overline{f_k(x)}, \tag{2.15}$$

$$\zeta_t^m(x) = \sqrt{v} \sum_{k\in\mathcal{K}_m} \int_0^t e^{(t-s)\widehat{A}(k)} \mathrm{d}\widehat{W}_s(k)\overline{f_k(x)} \tag{2.16}$$

for all $x \in \mathcal{R}_m$, where $\{\widehat{W}(k); k \in \mathcal{K}_m\}$ is an m^2-dimensional complex-valued centered Brownian motion defined by

$$\widehat{W}_t(k) \stackrel{\rm def}{=} \sum_{y\in\mathcal{R}_m} W_t(y) f_k(y). \tag{2.17}$$

Proof. Since $\eta_t^m = e^{tA}\xi_0^m$ by definition, (2.15) follows from (2.11) and (2.14). Similarly, we obtain from the definition of ζ^m that

$$\begin{aligned}
\zeta_t^m(x) &= \sqrt{v} \sum_{y\in\mathcal{R}_m} \int_0^t \sum_{k\in\mathcal{K}_m} e^{(t-s)\widehat{A}(k)}\widehat{\mathbb{1}_y}(k)\overline{f_k(x)}\mathrm{d}W_s(y) \\
&= \sqrt{v} \sum_{k\in\mathcal{K}_m} \int_0^t e^{(t-s)\widehat{A}(k)}\mathrm{d}\widehat{W}_s(k)\overline{f_k(x)},
\end{aligned}$$

which is (2.16). ■

The next result uses the above Fourier characterizations to define the limit of ξ^m as $m \to \infty$. See also [4, (6.4) and (6.8)] for the following limiting mean and covariance functions. To state the result, we introduce some notation. For all $m \geq m_0$, we extend ξ^m to the whole space $\mathbb{Z}^2$ by setting

$$\xi^m(x) \equiv 0, \quad \forall\, x \in \mathcal{R}_m^{\complement}.$$

The same extension applies to η^m and ζ^m. Also, we write

$$\mathcal{K}_\infty(\overline{m}) \stackrel{\text{def}}{=} \left\{(k_1,k_2) \in \mathbb{R}^2 \middle| -\pi \leq k_1 \leq \pi, -\pi \leq k_2 - \overline{m}k_1 \leq \pi\right\} \tag{2.18}$$

and $\mathrm{Cov}[X;Y] = \mathbb{E}[X\overline{Y}] - \mathbb{E}[X]\mathbb{E}[\overline{Y}]$ for complex-valued random variables X and Y.

Proposition 2.8. *Assume that* (1) *the m_2's defining $\mathcal{R}_m$'s are chosen so that $\lim_{m\to\infty} m_2/m = \overline{m}$ and* (2) *the initial conditions ξ_0^m satisfy*

$$\sup_{m\in\mathbb{N}} \sup_{k'\in\mathcal{K}_m} |m\widehat{\xi_0^m}(k')| < \infty, \quad \textit{and for some } \widehat{\mu} \in \mathscr{C}(\mathbb{R}^2),\ \lim_{m\to\infty} m\widehat{\xi_0^m}(k_m) = \widehat{\mu}(k) \tag{2.19}$$

for all $k \in \mathcal{K}_\infty(\overline{m})$ and sequences (k_m) such that $k_m \in \mathcal{K}_m$ and $k_m \to k$.

Under these assumptions, the sequence $\{\xi^m(x); x \in \mathbb{Z}^2\}$ converges in distribution in $C(\mathbb{R}_+,\mathbb{R})^{\mathbb{Z}^2}$ to a Gaussian process $\xi^\infty = \{\xi^\infty(x); x \in \mathbb{Z}^2\}$ characterized by the following equations: for all $0 \leq s \leq t < \infty$ and $x, y \in \mathbb{Z}^2$,

$$\mathbb{E}[\xi_t^\infty(x)] = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \mathrm{d}k e^{t\widehat{A}(k)} e^{\mathtt{i}\langle k,x\rangle}\widehat{\mu}(k), \tag{2.20}$$

$$\mathrm{Cov}[\xi_s^\infty(x);\xi_t^\infty(y)] = \frac{\upsilon}{(2\pi)^2}\int_0^s \mathrm{d}r \int_{\mathbb{T}^2} \mathrm{d}k e^{(s-r)\widehat{A}(k)} e^{\mathtt{i}\langle k,x\rangle} e^{(t-r)\widehat{A}(-k)} e^{-\mathtt{i}\langle k,y\rangle}. \tag{2.21}$$

Moreover, by (2.20) and (2.21), ξ^∞ admits an extension, still denoted by ξ^∞, which is a jointly continuous real-valued Gaussian process indexed by $\mathbb{R}_+ \times \mathbb{R}^2$.

Proof. We compute the mean function and covariance function of ξ^m in the limit $m \to \infty$ first. By (2.15), (2.19) and dominated convergence,

$$\begin{aligned}\lim_{m\to\infty} \eta_t^m(x) &= \frac{1}{(2\pi)^2}\int_{\mathcal{K}_\infty(\overline{m})} \mathrm{d}k e^{t\widehat{A}(k)}\widehat{\mu}(k) e^{\mathtt{i}\langle k,x\rangle} \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \mathrm{d}k e^{t\widehat{A}(k)}\widehat{\mu}(k) e^{\mathtt{i}\langle k,x\rangle},\end{aligned} \tag{2.22}$$

where the last equality follows from the 2π-periodicity of the integrand. ($\mathcal{K}_\infty(\overline{m})$ can be read as the limiting parallelogram of $\mathcal{K}_m$ in $\mathbb{R}^2$ as $m \to \infty$.) For the covariance function, first notice that by Lemma 2.6 (2), the complex-valued Brownian motion in (2.17) satisfies

$$\mathrm{Cov}[\widehat{W}_s(k);\widehat{W}_t(k')] = \delta_{k=k'}s, \quad \forall\, 0 \leq s \leq t < \infty,\ k,k' \in \mathcal{K}_m. \tag{2.23}$$

Hence, for any $x, y \in \mathbb{Z}^2$ and m large such that $x, y \in \mathcal{R}_m$, (2.16) gives

$$\begin{aligned}&\mathrm{Cov}[\xi_s^m(x);\xi_t^m(y)] \\ &= \frac{\upsilon}{m^2}\mathbb{E}\left[\sum_{k,k'\in\mathcal{K}_m}\int_0^s e^{(s-r)\widehat{A}(k)} e^{\mathtt{i}\langle k,x\rangle}\mathrm{d}\widehat{W}_r(k) \times \int_0^t e^{(t-r)\widehat{A}(-k')} e^{-\mathtt{i}\langle k',y\rangle}\mathrm{d}\overline{\widehat{W}_r(k')}\right]\end{aligned}$$

$$= \frac{v}{m^2}\sum_{k\in\mathcal{K}_m}\int_0^s \mathrm{d}r e^{(s-r)\widehat{A}(k)}e^{\mathtt{i}\langle k,x\rangle}e^{(t-r)\widehat{A}(-k)}e^{-\mathtt{i}\langle k,y\rangle} \tag{2.24}$$

$$\xrightarrow[m\to\infty]{} \frac{v}{(2\pi)^2}\int_0^s \mathrm{d}r\int_{\mathbb{T}^2}\mathrm{d}k e^{(s-r)\widehat{A}(k)}e^{\mathtt{i}\langle k,x\rangle}e^{(t-r)\widehat{A}(-k)}e^{-\mathtt{i}\langle k,y\rangle} \tag{2.25}$$

by dominated convergence and the 2π-periodicity of the integrand as above in (2.22).

Next, we show the weakly relative compactness of the sequence of laws of ξ^m. For any $0\le s\le t\le T$ and $x,y\in\mathbb{Z}^2$, (2.24) gives

$$\begin{aligned}
&\mathbb{E}\big[\big|\zeta_t^m(y)-\zeta_s^m(x)\big|^2\big]\\
&= \frac{v}{m^2}\sum_{k\in\mathcal{K}_m}\left(\int_0^s \mathrm{d}r\big|e^{(t-r)\widehat{A}(k)}e^{\mathtt{i}\langle k,y\rangle}-e^{(s-r)\widehat{A}(k)}e^{\mathtt{i}\langle k,x\rangle}\big|^2+\int_s^t \mathrm{d}r e^{(t-r)\widehat{A}(k)}e^{(t-r)\widehat{A}(-k)}\right)\\
&\le \frac{v}{m^2}\sum_{k\in\mathcal{K}_m}\left[\int_0^s \mathrm{d}r\Big(2\big|e^{(t-r)\widehat{A}(k)}-e^{(s-r)\widehat{A}(k)}\big|^2+2\big|e^{(s-r)\widehat{A}(k)}\big(e^{\mathtt{i}\langle k,y\rangle}-e^{\mathtt{i}\langle k,x\rangle}\big)\big|^2\Big)\right.\\
&\qquad\left.+\int_s^t \mathrm{d}r e^{(t-r)\widehat{A}(k)}e^{(t-r)\widehat{A}(-k)}\right]\\
&\le \frac{v}{m^2}\sum_{k\in\mathcal{K}_m}\Big(2s|\widehat{A}(k)|^2|t-s|^2+2s|y-x|^2+|t-s|\Big),
\end{aligned} \tag{2.26}$$

where the last inequality uses the following inequality:

$$|e^{z_1}-e^{z_2}|\le \max\{|e^{z_1}|,|e^{z_2}|\}\cdot|z_1-z_2|,\quad \forall\, z_1,z_2\in\mathbb{C}, \tag{2.27}$$

and Assumption 2.3 (4). Now we use (2.26) with $x=y$, Assumption 2.3 (1), and the fact that the fourth moment of a centered, real-valued Gaussian with variance σ^2 is given by $3\sigma^4$. Hence, for any $T\in(0,\infty)$, we can find $C_{2.28}$ and $\varepsilon>0$ such that

$$\sup_{m\in\mathbb{N}}\mathbb{E}\big[\,|\zeta_t^m(x)-\zeta_s^m(x)|^4\,\big]\le C_{2.28}|t-s|^{1+\varepsilon},\quad \forall\, 0\le s\le t\le T,\quad x\in\mathbb{Z}^2. \tag{2.28}$$

A calculation similar to (2.26) shows the equicontinuity of $\{\eta^m(x);m\in\mathbb{N}\}$ for all $x\in\mathbb{Z}^2$. Hence, by Kolmogorov's criterion [20, Theorem XIII.1.8], we obtain the weakly relative compactness of the sequence of laws of $\xi^m(x)$ for every fixed $x\in\mathbb{Z}^2$. By [10, Proposition 3.2.4], an extension to the sequence of laws of ξ^m applies. Then (2.20) and (2.21) follow from (2.22) and (2.25), respectively.

Next, we show that ξ^∞ admits an extension to a jointly continuous Gaussian process. The extension after recentering, called ζ^∞, can be obtained from the standard reproducing kernel argument for the following family of functions in $L_2(\mathbb{R}_+\times\mathbb{T}^2,\mathrm{d}r\mathrm{d}k)$:

$$(r,k)\longmapsto \frac{\sqrt{v}}{2\pi}\mathbb{1}_{[0,s]}(r)e^{(s-r)\widehat{A}(k)}e^{\mathtt{i}\langle k,x\rangle},\quad (s,x)\in\mathbb{R}_+\times\mathbb{R}^2. \tag{2.29}$$

A jointly continuous modification of ζ^∞ follows since an argument similar to (2.26) gives

$$\begin{aligned}
&\mathbb{E}\big[\big|\zeta_t^\infty(y)-\zeta_s^\infty(x)\big|^2\big]\\
&\le \frac{v}{(2\pi)^2}\int_{\mathbb{T}^2}\mathrm{d}k\,\Big(2s|\widehat{A}(k)|^2|t-s|^2+2s|y-x|^2+|t-s|\Big)
\end{aligned} \tag{2.30}$$

for all $0\le s\le t<\infty$ and $x,y\in\mathbb{R}^2$. Hence, an analogue of (2.28) holds. The proof is complete. ■

3 The main theorem

3.1 Setup

Fix $v \in (0,\infty)$ and $d \geq 1$. Let $\widehat{A}$ satisfy Assumption 2.3 and $\mu \in \ell_1(\mathbb{Z}^d)$ with Fourier transform

$$\widehat{\mu}(k) = \sum_{x\in\mathbb{Z}^d} \mu(x) e^{-\mathtt{i}\langle k,x\rangle} \in \mathscr{C}(\mathbb{T}^d). \tag{3.1}$$

We consider a real-valued Gaussian process $\{\xi^\infty_t(x); x \in \mathbb{Z}^d\}$ with initial condition μ. Its mean function and covariance function are given by the following generalizations of (2.20) and (2.21): For all $0 \leq s \leq t < \infty$ and $x, y \in \mathbb{Z}^d$,

$$\mathbb{E}[\xi^\infty_t(x)] = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \mathrm{d}k e^{t\widehat{A}(k)} e^{\mathtt{i}\langle k,x\rangle}\widehat{\mu}(k), \tag{3.2}$$

$$\mathrm{Cov}[\xi^\infty_s(x);\xi^\infty_t(y)] = \frac{v}{(2\pi)^d}\int_0^s \mathrm{d}r \int_{\mathbb{T}^d}\mathrm{d}k e^{(s-r)\widehat{A}(k)} e^{\mathtt{i}\langle k,x\rangle} e^{(t-r)\widehat{A}(-k)} e^{-\mathtt{i}\langle k,y\rangle}. \tag{3.3}$$

The proof of Proposition 2.8 shows that ξ^∞ exists and allows for a jointly continuous version on $\mathbb{R}_+ \times \mathbb{R}^d$.

To state the following theorem, we define

$$U = \mathtt{i}\nabla\widehat{A}(0) \tag{3.4}$$

and let V be the square root of $-Q^{-1}$ so that

$$Q = -(V^{-1})^2. \tag{3.5}$$

Here, $U \in \mathbb{R}^d$ by Assumption 2.3 (2) and Q is real and strictly negative definite by Assumption 2.3 (3). Also, write $\mathcal{S}(\mathbb{R}^d)$ for the space of real-valued Schwartz functions on $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{R}^d)$ for the space of bounded linear functionals over $\mathbb{R}$ on $\mathcal{S}(\mathbb{R}^d)$ [19, Section V.3]. By convention, $\mathcal{S}'(\mathbb{R}^d)$ is equipped with the weak topology. Given $\{\mu^\delta\}_{\delta\in(0,1]} \subset \ell^1(\mathbb{Z}^d)$, consider $\xi^{\infty,\delta}$ satisfying (3.2) and (3.3) with ξ^∞ and μ replaced by $\xi^{\infty,\delta}$ and μ^δ, respectively. Then we define $\mathcal{S}'(\mathbb{R}^d)$-valued processes X^δ by

$$X^\delta_t(\phi) \stackrel{\mathrm{def}}{=} \delta^{-\frac{d-2}{4}}\int_{\mathbb{R}^d}\mathrm{d}z\, \xi^{\infty,\delta}_{\delta^{-1}t}(\lfloor \delta^{-1}Ut + \delta^{-1/2}V^{-1}z\rfloor)\phi(z), \quad \phi \in \mathcal{S}(\mathbb{R}^d). \tag{3.6}$$

The growth exponent $\frac{d-2}{4}$ of the Edwards–Wilkinson equation is applied in (3.6) [1, (5.16)]. Our goal is to prove the convergence of X^δ as $\delta \to 0+$ to the solution of a stochastic heat equation under suitable assumptions on the initial conditions $\xi^{\infty,\delta}_0$. The main result is stated in the following theorem.

Theorem 3.1 (Main theorem). *Let $v \in (0,\infty)$ and $d \geq 1$. Let $\widehat{A}$ satisfy Assumption 2.3 and $\{\mu^\delta\}_{\delta\in(0,1]} \subset \ell_1(\mathbb{Z}^d)$ satisfy*

$$\delta^{\frac{d+2}{4}} \sum_{y\in\delta^{1/2}V\mathbb{Z}^d} \mu^\delta(\delta^{-1/2}V^{-1}y)\phi(y) \xrightarrow[\delta\to 0+]{} \mu^0(\phi), \quad \forall\, \phi \in \mathcal{S}(\mathbb{R}^d), \tag{3.7}$$

for some $\mu^0 \in \mathcal{S}'(\mathbb{R}^d)$. If $\xi^{\infty,\delta}$ are continuous Gaussian processes subject to $\xi^{\infty,\delta}_0 = \mu^\delta$, (3.2) and (3.3), then the processes X^δ defined by (3.6) satisfy

$$X^\delta \xrightarrow[\delta\to 0+]{(\mathrm{d})} X^0 \quad \textit{in} \quad C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d)).$$

Here, the limiting process X^0 is the pathwise unique solution to the following additive stochastic heat equation:

$$\frac{\partial X^0}{\partial t} = \frac{\Delta X^0}{2} + \sqrt{v|\det(V)|}\,\dot{W}, \quad X_0^0 = |\det(V)|\mu^0, \tag{3.8}$$

subject to a $(d+1)$-dimensional space-time white noise $\dot{W}$ on $\mathbb{R}_+ \times \mathbb{R}^d$.

Remark 3.2. (1) See [2, 3, 5] for rescaled limits of related growth models in $(2+1)$ dimensions and [16, 11] for results in higher dimensions.

(2) Pathwise explicit solutions for general additive stochastic heat equations can be found in [24, Theorem 5.1 on page 342]. See also [15] for uniqueness theorems of general stochastic equations.

(3) Suppose that $\mu^\delta(x) \equiv \delta^{\frac{d-2}{4}}\psi(\delta^{1/2}x)$ for some $\psi \in \mathcal{S}(\mathbb{R}^d)$. Then the convergence in (3.7) holds with

$$\begin{aligned}\delta^{\frac{d+2}{4}} \sum_{y\in\delta^{1/2}V\mathbb{Z}^d} \mu^\delta(\delta^{-1/2}V^{-1}y)\phi(y) &= \sum_{y\in\delta^{1/2}V\mathbb{Z}^d} \delta^{d/2}\psi(V^{-1}y)\phi(y) \\ &\xrightarrow[\delta\to 0+]{} \frac{1}{|\det(V)|}\int_{\mathbb{R}^d}\psi(V^{-1}y)\phi(y)\mathrm{d}y. \qquad \blacksquare\end{aligned}$$

3.2 Outline of the proof

For the proof of Theorem 3.1, we decompose the Gaussian process $\xi^{\infty,\delta}$ according to its deterministic part and stochastic part as in Section 2:

$$\xi_t^{\infty,\delta}(x) = \eta_t^{\infty,\delta}(x) + \zeta_t^{\infty,\delta}(x). \tag{3.9}$$

That is, $\eta_t^{\infty,\delta}(x)$ is the mean function of $\xi_t^{\infty,\delta}(x)$ in (3.2) and $\zeta^{\infty,\delta}$ is a centered Gaussian process with a covariance function given by (3.3). The analogous decomposition of $X^\delta(\phi)$ is defined by:

$$X_t^\delta(\phi) = Y_t^\delta(\phi) + Z_t^\delta(\phi), \quad \phi \in \mathcal{S}(\mathbb{R}^d),$$

where

$$Y_t^\delta(\phi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \mathrm{d}z\,\eta^{\infty,\delta}_{\delta^{-1}t}(\lfloor \delta^{-1}Ut + \delta^{-1/2}V^{-1}z\rfloor)\phi(z), \tag{3.10}$$

$$Z_t^\delta(\phi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \mathrm{d}z\,\zeta^{\infty,\delta}_{\delta^{-1}t}(\lfloor \delta^{-1}Ut + \delta^{-1/2}V^{-1}z\rfloor)\phi(z). \tag{3.11}$$

Also, to lighten notation, we use the following notation from now on: for $0^{-1/2}\mathbb{T}^d = \mathbb{R}^d$ and any $\delta \in [0,1]$,

$$\begin{aligned}\int_0^t\int_{\delta^{-1/2}\mathbb{T}^d} \Phi(r,k)\bullet\mathbb{W}(\mathrm{d}r,\mathrm{d}k) \stackrel{\text{def}}{=} &\int_0^t\int_{\delta^{-1/2}\mathbb{T}^d} \operatorname{Re}\Phi(r,k)W^1(\mathrm{d}r,\mathrm{d}k) \\ &+ \int_0^t\int_{\delta^{-1/2}\mathbb{T}^d} \operatorname{Im}\Phi(r,k)W^2(\mathrm{d}r,\mathrm{d}k).\end{aligned} \tag{3.12}$$

Here, W^1 and W^2 are independent copies of a space-time white noise. The covariance measure of W^j is given by $\mathrm{d}r\mathrm{d}k$:

$$\mathbb{E}\big[W_s^j(\phi_1)W_t^j(\phi_2)\big] = \min\{s,t\}\langle\phi_1,\phi_2\rangle_{L_2(\mathbb{R}^d,\mathrm{d}k)}.$$

In Section 4, we show that the weak limit of the family of laws $\{Z^\delta\}_{\delta\in(0,1]}$ as $\delta \to 0+$ is given by the law of a $C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$-valued random element Z^0 solving the following additive stochastic heat equation:

$$Z_t^0(\phi) = \int_0^t Z_s^0\left(\frac{\Delta\phi}{2}\right)\mathrm{d}s + \sqrt{v|\det(V)|}\int_0^t\int_{\mathbb{R}^d}\phi(k)W(\mathrm{d}r,\mathrm{d}k). \tag{3.13}$$

In the preliminary steps of the proof, we explain the choice of the characteristics in (3.11) (Proposition 4.2). The discussion also shows that $Z^\delta(\phi)$ admits the following stochastic integral representation (Proposition 4.2):

$$\sqrt{v}\int_0^t \int_{\delta^{-1/2}\mathbb{T}^d} \Phi_t^\delta(r,k) \bullet \mathbb{W}(\mathrm{d}r,\mathrm{d}k), \tag{3.14}$$

where

$$\begin{aligned} \Phi_t^\delta(r,k) &= e^{\delta^{-1}(t-r)[\widehat{A}(\delta^{1/2}k)+\mathtt{i}\langle\delta^{1/2}k,U\rangle]} \\ &\quad\times \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}z\phi(z)e^{\mathtt{i}\langle\delta^{1/2}k,\lfloor\delta^{-1}Ut+\delta^{-1/2}V^{-1}z\rfloor\rangle-\mathtt{i}\langle\delta^{1/2}k,\delta^{-1}Ut\rangle}. \end{aligned} \tag{3.15}$$

Our proof of the convergence of Z^δ is based on (3.14). Assumption 2.3 implies

$$\delta^{-1}[\widehat{A}(\pm\delta^{1/2}k) \pm \mathtt{i}\langle\delta^{1/2}k,U\rangle] = \frac{Q(k)}{2} + \mathcal{O}(\delta^{1/2}|k|^3), \tag{3.16}$$

and so (3.15) shows that

$$\lim_{\delta\to0+}\Phi_t^\delta(r,k) = \Phi_t^0(r,k) \stackrel{\mathrm{def}}{=} e^{(t-r)Q(k)/2}\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}z\phi(z)e^{\mathtt{i}\langle k,V^{-1}z\rangle}. \tag{3.17}$$

Hence, passing $\delta\to0+$ in (3.14) yields the following stochastic convolution:

$$Z_t^0(\phi) \stackrel{\mathrm{def}}{=} \sqrt{v}\int_0^t\int_{\mathbb{R}^d} e^{(t-r)Q(k)/2}\mathcal{F}\phi_V(k) \bullet \mathbb{W}(\mathrm{d}r,\mathrm{d}k), \tag{3.18}$$

where

$$\mathcal{F}\phi(k) \stackrel{\mathrm{def}}{=} \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}z\phi(z)e^{\mathtt{i}\langle k,z\rangle}. \tag{3.19}$$

In Section 4.1.5, we prove (3.13) by Itô's and Plancherel's isometries as mentioned in Section 1.

The main argument of the proof is to show that Z^δ converges in distribution to Z^0 as processes taking values in $\mathcal{S}'(\mathbb{R}^d)$. We use Mitoma's conditions [17, Theorem 3.1] for tightness of probability measures on $C(\mathbb{R}_+,\mathcal{S}'(\mathbb{R}^d))$. This amounts to verifying the tightness of the family of probability measures of $\{Z^\delta(\phi)\}_{\delta\in(0,1]}$ for any fixed $\phi\in\mathcal{S}(\mathbb{R}^d)$. We decompose the integrands in (3.14) to handle the approximation errors from (3.16) and the following integrals in (3.15):

$$\int_{\mathbb{R}^d}\mathrm{d}z\phi(z)e^{\mathtt{i}\langle\delta^{1/2}k,\lfloor\delta^{-1}Ut+\delta^{-1/2}V^{-1}z\rfloor\rangle-\mathtt{i}\langle\delta^{1/2}k,\delta^{-1}Ut\rangle}. \tag{3.20}$$

Then we proceed as discussed at the end of Section 1.

The main result of Section 5 (Proposition 5.1) shows that Y^δ converges to Y^0 as $\mathcal{S}'(\mathbb{R}^d)$-valued continuous processes. This limit Y^0 satisfies

$$Y_t^0(\phi) = |\det(V)|\mu^0(\phi) + \int_0^t Y_s^0\left(\frac{\Delta\phi}{2}\right)\mathrm{d}s. \tag{3.21}$$

In summary, writing $\xrightarrow[\delta\to0+]{(\mathrm{d})}$ for convergence in distribution as $\delta\to0+$, we have

$$X^\delta = Y^\delta + Z^\delta \xrightarrow[\delta\to0+]{(\mathrm{d})} Y^0 + Z^0 = X^0.$$

Moreover, we obtain from (3.13) and (3.21) that X^0 solves the additive stochastic heat equation defined in (3.8).

4 Convergence of the stochastic parts

In this section, we prove the convergence of Z^δ defined in (3.11). First, we start with some preliminary results.

Proposition 4.1. *For any $\delta \in (0,1]$, the stochastic part $\zeta^{\infty,\delta}$ in (3.9) continuously extended to $\mathbb{R}_+ \times \mathbb{R}^d$ satisfies the following growth bounds:*

$$\mathbb{E}\left[\sup_{x\in\mathbb{Z}^d} \frac{1}{1+\|x\|_\infty^{2r}} \sup_{t\in[0,T]} \sup_{y\in x+[0,1)^d} \left|\zeta_t^{\infty,\delta}(y)\right|^{2r}\right] < \infty, \quad \forall\, r \in (d/2,\infty). \tag{4.1}$$

Hence, for every $\delta \in (0,1]$, Z^δ takes values in $D(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$ almost surely.

Proof. We partition $\mathbb{Z}^d\setminus\{0\}$ according to the level sets $E_n = \{x \in \mathbb{Z}^d; 2^{n-1} \le \|x\|_\infty < 2^n\}$ for $n \ge 1$. Since $|\{x \in \mathbb{Z}^d; \|x\|_\infty = n\}| \le C_d n^{d-1}$ for some constant C_d depending only on the dimension d, we have $|E_n| \le \sum_{j=2^{n-1}}^{2^n-1} C_d j^{d-1} \le C_d \cdot 2^{nd}$. It follows that for $r > d/2$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{x\in\mathbb{Z}^d} \frac{1}{1+\|x\|_\infty^{2r}} \sup_{t\in[0,T]} \sup_{y\in x+[0,1)^d} \left|\zeta_t^{\infty,\delta}(y)\right|^{2r}\right]\\
&\le \mathbb{E}\left[\sup_{t\in[0,T]} \sup_{y\in[0,1)^d} |\zeta_t^{\infty,\delta}(y)|^{2r}\right] + \sum_{n=1}^\infty \mathbb{E}\left[\sup_{x\in E_n} \frac{1}{1+\|x\|_\infty^{2r}} \sup_{t\in[0,T]} \sup_{y\in x+[0,1)^d} |\zeta_t^{\infty,\delta}(y)|^{2r}\right]\\
&\le \left(1 + \sum_{n=1}^\infty \frac{C_d\cdot 2^{nd}}{1+2^{2r(n-1)}}\right) \mathbb{E}\left[\sup_{t\in[0,T]} \sup_{y\in[0,1)^d} |\zeta_t^{\infty,\delta}(y)|^{2r}\right] < \infty.
\end{aligned} \tag{4.2}$$

Here, in the second inequality, we use the spatial translation invariance of $\zeta^{\infty,\delta}$ from (3.3). By an analogue of (2.30), the Gaussian property of $\zeta^{\infty,\delta}$ and Kolmogorov's criterion for continuity [20, Theorem I.2.1], we deduce that the expectation in (4.2) is finite. We have proved (4.1). The required property of Z^δ then follows from the almost surely polynomial growth of $\zeta^{\infty,\delta}$ implied by (4.1). ∎

Next, we explain the choice of the characteristics in (3.6). Under the diffusive scaling $(\delta^{-1/2}k, \delta^{-1}r)$ of space and time, we consider the covariance function of the process $\zeta^{\infty,\delta}$ defined by (3.3): for any $a, b \in \mathbb{Z}^d$,

$$\begin{aligned}
&\mathbb{E}\big[\zeta^{\infty,\delta}_{\delta^{-1}s}(a)\zeta^{\infty,\delta}_{\delta^{-1}t}(b)\big]\\
&= \frac{v}{(2\pi)^d}\int_0^{\delta^{-1}s} \mathrm{d}r \int_{\mathbb{T}^d} \mathrm{d}k e^{\delta^{-1}(s-\delta r)\widehat{A}(k)} e^{\mathtt{i}\langle k,a\rangle} e^{\delta^{-1}(t-\delta r)\widehat{A}(-k)} e^{-\mathtt{i}\langle k,b\rangle}\\
&= \frac{v\delta^{\frac{d-2}{2}}}{(2\pi)^d}\int_0^s \mathrm{d}r' \int_{\delta^{-1/2}\mathbb{T}^d} \mathrm{d}k' e^{\delta^{-1}(s-r')\widehat{A}(\delta^{1/2}k')} e^{\mathtt{i}\langle\delta^{1/2}k',a\rangle}\\
&\quad\times e^{\delta^{-1}(t-r')\widehat{A}(-\delta^{1/2}k')} e^{-\mathtt{i}\langle\delta^{1/2}k',b\rangle}
\end{aligned}$$

by changing variables to $\delta^{1/2}k' = k$ and $\delta^{-1}r' = r$. To elicit a Laplacian via the quadratic form $\langle k, Qk\rangle/2$ in (3.16), we write the last equality as

$$\begin{aligned}
&\mathbb{E}\big[\zeta^{\infty,\delta}_{\delta^{-1}s}(a)\zeta^{\infty,\delta}_{\delta^{-1}t}(b)\big]\\
&= \frac{v\delta^{\frac{d-2}{2}}}{(2\pi)^d}\int_0^s \mathrm{d}r' \int_{\delta^{-1/2}\mathbb{T}^d} \mathrm{d}k' e^{\delta^{-1}(s-r')[\widehat{A}(\delta^{1/2}k')+\mathtt{i}\langle\delta^{1/2}k',U\rangle]} e^{\mathtt{i}\langle\delta^{1/2}k',a\rangle-\delta^{-1}s\mathtt{i}\langle\delta^{1/2}k',U\rangle}\\
&\quad\times e^{\delta^{-1}(t-r')[\widehat{A}(-\delta^{1/2}k')-\mathtt{i}\langle\delta^{1/2}k',U\rangle]} e^{-\mathtt{i}\langle\delta^{1/2}k',b\rangle+\delta^{-1}t\mathtt{i}\langle\delta^{1/2}k',U\rangle}.
\end{aligned} \tag{4.3}$$

Note that an additional factor $\zeta\overline{\zeta}$ for $\zeta = e^{-\delta^{-1}r'\mathtt{i}\langle\delta^{1/2}k',U\rangle}$ is introduced.

To get a nontrivial limit from (4.3), it is necessary to remove the scaling factor $\delta^{\frac{d-2}{2}}$. Also, we need to choose $a,b\in\mathbb{Z}^d$ to remove the large terms $\delta^{-1}s\mathtt{i}\langle\delta^{1/2}k',U\rangle$ and $\delta^{-1}t\mathtt{i}\langle\delta^{1/2}k',U\rangle$ in the exponents. The chosen a,b need to induce the Edwards–Wilkinson limit (recall (3.17)). Taking all these into account, the reader can see that one choice of (a,b) is

$$a=\lfloor\delta^{-1}Us+\delta^{-1/2}V^{-1}z\rfloor,\quad b=\lfloor\delta^{-1}Ut+\delta^{-1/2}V^{-1}z'\rfloor.$$

This leads to the definition of Z^δ in (3.11).

From (4.3), we get

$$\mathbb{E}\big[Z_s^\delta(\phi)Z_t^\delta(\phi)\big]=\int_{\mathbb{R}^d}\mathrm{d}z\phi(z)\int_{\mathbb{R}^d}\mathrm{d}z'\phi(z')\kappa_{s,t}^\delta(z,z'),\tag{4.4}$$

where

$$\begin{aligned}\kappa_{s,t}^\delta(z,z')=\frac{v}{(2\pi)^d}\int_0^s\mathrm{d}r'\int_{\delta^{-1/2}\mathbb{T}^d}&\mathrm{d}k'e^{\delta^{-1}(s-r')[\widehat{A}(\delta^{1/2}k')+\mathtt{i}\langle\delta^{1/2}k',U\rangle]}\\ &\times e^{\mathtt{i}\langle\delta^{1/2}k',\lfloor\delta^{-1}Us+\delta^{-1/2}V^{-1}z\rfloor\rangle-\delta^{-1}s\mathtt{i}\langle\delta^{1/2}k',U\rangle}\\ &\times e^{\delta^{-1}(t-r')[\widehat{A}(-\delta^{1/2}k')-\mathtt{i}\langle\delta^{1/2}k',U\rangle]}\\ &\times e^{-\mathtt{i}\langle\delta^{1/2}k',\lfloor\delta^{-1}Ut+\delta^{-1/2}V^{-1}z'\rfloor\rangle+\delta^{-1}t\mathtt{i}\langle\delta^{1/2}k',U\rangle}.\end{aligned}$$

For our purpose below, it is more convenient to rewrite (4.4) by the following change-of-variable operator T_V on $\mathcal{S}(\mathbb{R}^d)$:

$$\phi_V(z)=T_V\phi(z)\stackrel{\mathrm{def}}{=}|\det(V)|\phi(Vz)\in\mathcal{S}(\mathbb{R}^d).\tag{4.5}$$

We summarize this discussion in the following proposition.

Proposition 4.2. *For any fixed $\phi\in\mathcal{S}(\mathbb{R}^d)$ and $\delta\in(0,1]$, $Z^\delta(\phi)$ defined by (3.11) has the same law as the process*

$$\widetilde{Z}_t^\delta(\phi)=\sqrt{v}\int_0^t\int_{\delta^{-1/2}\mathbb{T}^d}e^{\delta^{-1}(t-r)[\widehat{A}(\delta^{1/2}k)+\mathtt{i}\langle\delta^{1/2}k,U\rangle]}\phi_t^\delta(k)\bullet\mathbb{W}(\mathrm{d}r,\mathrm{d}k)\tag{4.6}$$

in $D(\mathbb{R}_+,\mathbb{R})$. Here, ϕ_t^δ is a transformation of ϕ given by

$$\phi_t^\delta(k)\stackrel{\mathrm{def}}{=}\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}z\phi_V(z)e^{\mathtt{i}\langle\delta^{1/2}k,\lfloor\delta^{-1}Ut+\delta^{-1/2}z\rfloor\rangle-\mathtt{i}\langle\delta^{1/2}k,\delta^{-1}Ut\rangle},\tag{4.7}$$

where ϕ_V is defined in (4.5).

Proof. By (4.4) and (4.5), the Gaussian processes $Z^\delta(\phi)$ and $\widetilde{Z}^\delta(\phi)$ have the same covariance function. Since they have càdlàg paths, they have the same law in $D(\mathbb{R}_+,\mathbb{R})$. ■

Henceforth, we identify $Z^\delta(\phi)$ with the stochastic integral defined in (4.6).

4.1 Tightness

4.1.1 A semi-discrete integration by parts formula

We write

$$\mathbb{S}_\delta(k)\stackrel{\mathrm{def}}{=}\frac{e^{\mathtt{i}\delta^{1/2}k/2}\big(e^{\mathtt{i}\delta^{1/2}k/2}-e^{-\mathtt{i}\delta^{1/2}k/2}\big)}{\mathtt{i}\delta^{1/2}},\quad k\in\delta^{-1/2}\mathbb{T},\ \delta\in(0,1].\tag{4.8}$$

This sine-like function $\mathbb{S}_\delta$ will be used repeatedly in the rest of this paper, along with the following two properties:

$$\begin{cases} \dfrac{2}{\pi}|k| \leq |\mathbb{S}_\delta(k)| \leq |k|, \quad \forall\, k \in \delta^{-1/2}\mathbb{T}; \\ \lim\limits_{\delta\to 0+} \mathbb{S}_\delta(k) = k, \quad \forall\, k \in \mathbb{R}. \end{cases} \tag{4.9}$$

The first property in (4.9) follows from Jordan's inequality.

Proposition 4.3. *For any $f \in \ell_1(\mathbb{Z})$, $n \in \mathbb{Z}_+$, $\delta \in (0,1]$ and $k_1 \in \delta^{-1/2}\mathbb{T} \setminus \{0\}$, we have*

$$\sum_{x_1\in\mathbb{Z}} e^{\mathrm{i}\delta^{1/2}k_1x_1} f(x_1) = \frac{(-1)^n}{\big(\mathrm{i}\mathbb{S}_\delta(k_1)\big)^n} \sum_{x_1\in\mathbb{Z}} e^{\mathrm{i}\delta^{1/2}k_1x_1} \nabla_\delta^n f(x_1), \tag{4.10}$$

where $\mathbb{S}_\delta$ is defined in (4.8) and ∇_δ is the ordinary (backward) difference operator defined by

$$\nabla_\delta f(x_1) = \frac{f(x_1) - f(x_1-1)}{\delta^{1/2}}. \tag{4.11}$$

Proof. It suffices to prove (4.10) for $n = 1$. The case of general n follows from iteration.

Now, summation by parts gives

$$\begin{aligned} &\sum_{x_1\in\mathbb{Z}} e^{\mathrm{i}\delta^{1/2}k_1x_1} f(x_1) \\ &= \lim_{N\to\infty} \sum_{x_1=-N}^{N} e^{\mathrm{i}\delta^{1/2}k_1x_1} f(N) - \sum_{x_1=-N}^{N-1} \sum_{m=-N}^{x_1} e^{\mathrm{i}\delta^{1/2}k_1m} [f(x_1+1) - f(x_1)]. \end{aligned}$$

Since $k_1 \in \delta^{-1/2}\mathbb{T} \setminus \{0\}$, we have

$$\sum_{m=-N}^{x_1} e^{\mathrm{i}\delta^{1/2}k_1m} = \frac{e^{\mathrm{i}\delta^{1/2}k_1(x_1+1)} - e^{-\mathrm{i}\delta^{1/2}k_1N}}{e^{\mathrm{i}\delta^{1/2}k_1} - 1}.$$

Then by a telescoping sum argument and the assumption that $f \in \ell_1(\mathbb{Z})$, we get from the last two equalities that

$$\begin{aligned} &\sum_{x_1\in\mathbb{Z}} e^{\mathrm{i}\delta^{1/2}k_1x_1} f(x_1) \\ &= -\sum_{x_1=-\infty}^{\infty} \left(\frac{e^{\mathrm{i}\delta^{1/2}k_1(x_1+1)}}{e^{\mathrm{i}\delta^{1/2}k_1} - 1} \right) [f(x_1+1) - f(x_1)] \\ &= \frac{-1}{e^{\mathrm{i}\delta^{1/2}k_1/2}\big(e^{\mathrm{i}\delta^{1/2}k_1/2} - e^{-\mathrm{i}\delta^{1/2}k_1/2}\big)\delta^{-1/2}} \sum_{x_1=-\infty}^{\infty} e^{\mathrm{i}\delta^{1/2}k_1x_1} \frac{f(x_1) - f(x_1-1)}{\delta^{1/2}}. \end{aligned}$$

Applying the notations $\mathbb{S}_\delta$ and ∇_δ to the last equality proves (4.10) for $n = 1$. This completes the proof. ■

To state the next result, we introduce a few more notations. First, $\lfloor z_j \rfloor_{\delta,t,j}$ denotes the nearest point in $\delta^{1/2}\mathbb{Z} - \delta^{-1/2}U_j t$ to the left of $z_j \in \mathbb{R}$ and

$$\lfloor z \rfloor_{\delta,t} \stackrel{\text{def}}{=} (\lfloor z_1 \rfloor_{\delta,t,1}, \lfloor z_2 \rfloor_{\delta,t,2}, \cdots, \lfloor z_d \rfloor_{\delta,t,d}), \quad z = (z_1, z_2, \cdots, z_d) \in \mathbb{R}^d. \tag{4.12}$$

We also write

$$\lfloor z \rfloor_{\delta,t,j} \stackrel{\text{def}}{=} \lfloor z_j \rfloor_{\delta,t,j}.$$

Then the following inequalities hold:

$$0 \leq z_j - \lfloor z_j \rfloor_{\delta,j,t} < \delta^{1/2}, \quad \forall\, z_j \in \mathbb{R},\ \delta \in (0,1],\ 1 \leq j \leq d,\ t \in \mathbb{R}_+. \tag{4.13}$$

Also, we define a partial difference operator $\nabla_{\delta,1}$ by

$$\nabla_{\delta,1}\phi(z) \stackrel{\rm def}{=} \frac{\phi(z_1, z_2, \cdots, z_d) - \phi(z_1 - \delta^{1/2}, z_2, \cdots, z_d)}{\delta^{1/2}}. \tag{4.14}$$

The operators $\nabla_{\delta,j}$ for $2 \leq j \leq d$ are similarly defined. Note that in contrast to ∇_δ defined in (4.11), a scaling of space by $\delta^{1/2}$ is now in the definitions of the $\nabla_{\delta,j}$'s.

Proposition 4.4. *Let $\delta \in (0,1]$, $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $1 \leq j \leq d$. Then for all $n \in \mathbb{Z}_+$, multi-indices $\alpha \in \mathbb{Z}_+^d$, and $k \in \delta^{-1/2}\mathbb{T}^d$ with $k_j \neq 0$ when $n > 0$, it holds that*

$$\begin{aligned}
&\frac{\partial^\alpha}{\partial k^\alpha}\int_{\mathbb{R}^d} \mathrm{d}z e^{\mathtt{i}\langle \delta^{1/2}k, \lfloor \delta^{-1}Ut + \delta^{-1/2}z\rfloor\rangle - \mathtt{i}\langle \delta^{1/2}k, \delta^{-1}Ut\rangle}\phi(z) \\
&= \frac{(-1)^n \mathtt{i}^{|\alpha|}}{\big(\mathtt{i}\mathbb{S}_\delta(k_j)\big)^n}\int_{\mathbb{R}^d} \mathrm{d}z e^{\mathtt{i}\langle k, \lfloor z\rfloor_{\delta,t}\rangle}\nabla^n_{\delta,j}\big(\lfloor\cdot\rfloor^\alpha_{\delta,t}\phi\big)(z),
\end{aligned} \tag{4.15}$$

where $|\alpha| = \sum_{j=1}^d \alpha_j$ and $z^\alpha = \prod_{j=1}^d z_j^{\alpha_j}$ for all $z \in \mathbb{R}^d$.

Proof. The integral on the left-hand side of (4.15) can be written as

$$\begin{aligned}
&\frac{\partial^\alpha}{\partial k^\alpha}\int_{\mathbb{R}^d} \mathrm{d}z e^{\mathtt{i}\langle \delta^{1/2}k, \lfloor \delta^{-1}Ut + \delta^{-1/2}z\rfloor\rangle - \mathtt{i}\langle \delta^{1/2}k, \delta^{-1}Ut\rangle}\phi(z) \\
&= \mathtt{i}^{|\alpha|}\delta^{|\alpha|/2}\int_{\mathbb{R}^d} \mathrm{d}z e^{\mathtt{i}\langle \delta^{1/2}k, \lfloor \delta^{-1}Ut + \delta^{-1/2}z\rfloor\rangle - \mathtt{i}\langle \delta^{1/2}k, \delta^{-1}Ut\rangle} \\
&\quad \times \big(\lfloor \delta^{-1}Ut + \delta^{-1/2}z\rfloor - \delta^{-1}Ut\big)^\alpha \phi(z).
\end{aligned} \tag{4.16}$$

Below we prove the required formula (4.15) for $j = 1$ in the form (4.16).

Now, we partition $\mathbb{R}^d$ by the semi-closed cubes $Q^\delta_{\delta^{1/2}x - \delta^{-1/2}Ut}$ for x ranging over $\mathbb{Z}^d$, where

$$Q^\delta_y = [y, y + \delta^{1/2}) \stackrel{\rm def}{=} \prod_{j=1}^d [y_j, y_j + \delta^{1/2}), \quad y \in \mathbb{R}^d. \tag{4.17}$$

These cubes $Q^\delta_{\delta^{1/2}x - \delta^{-1/2}Ut}$ are chosen such that

$$\lfloor \delta^{-1}Ut + \delta^{-1/2}z\rfloor = x, \quad \forall\, z \in Q^\delta_{\delta^{1/2}x - \delta^{-1/2}Ut},\ x \in \mathbb{Z}^d.$$

Then by the foregoing display, the right-hand side of (4.16) can be written as

$$\begin{aligned}
&\mathtt{i}^{|\alpha|}\delta^{|\alpha|/2}\int_{\mathbb{R}^d} \mathrm{d}z e^{\mathtt{i}\langle \delta^{1/2}k, \lfloor \delta^{-1}Ut + \delta^{-1/2}z\rfloor\rangle - \mathtt{i}\langle \delta^{1/2}k, \delta^{-1}Ut\rangle}\big(\lfloor \delta^{-1}Ut + \delta^{-1/2}z\rfloor - \delta^{-1}Ut\big)^\alpha \phi(z) \\
&= \mathtt{i}^{|\alpha|}\delta^{|\alpha|/2}\sum_{x\in\mathbb{Z}^d} e^{\mathtt{i}\langle \delta^{1/2}k, x\rangle}\int_{Q^\delta_{\delta^{1/2}x - \delta^{-1/2}Ut}} \mathrm{d}z e^{-\mathtt{i}\langle \delta^{1/2}k, \delta^{-1}Ut\rangle}(x - \delta^{-1}Ut)^\alpha \phi(z) \\
&= \mathtt{i}^{|\alpha|}\delta^{|\alpha|/2}\sum_{x_1=-\infty}^{\infty} e^{\mathtt{i}\delta^{1/2}k_1(x_1 - \delta^{-1}U_1t)}\Phi_\delta(x_1),
\end{aligned} \tag{4.18}$$

where

$$
\begin{aligned}
\Phi_\delta(x_1) \stackrel{\text{def}}{=} & \int_{\delta^{1/2}x_1-\delta^{-1/2}U_1t}^{\delta^{1/2}x_1-\delta^{-1/2}U_1t+\delta^{1/2}} \mathrm{d}z_1 \\
& \sum_{x_2\in\mathbb{Z}} e^{\mathtt{i}\delta^{1/2}k_2(x_2-\delta^{-1}U_2t)} \int_{\delta^{1/2}x_2-\delta^{-1/2}U_2t}^{\delta^{1/2}x_2-\delta^{-1/2}U_2t+\delta^{1/2}} \mathrm{d}z_2 \cdots \\
& \sum_{x_d\in\mathbb{Z}} e^{\mathtt{i}\delta^{1/2}k_d(x_d-\delta^{-1}U_dt)} \int_{\delta^{1/2}x_d-\delta^{-1/2}U_dt}^{\delta^{1/2}x_d-\delta^{-1/2}U_dt+\delta^{1/2}} \mathrm{d}z_d(x-\delta^{-1}Ut)^\alpha\phi(z).
\end{aligned}
\tag{4.19}
$$

By Proposition 4.3, (4.16) and (4.18), we get

$$
\begin{aligned}
&\frac{\partial^\alpha}{\partial k^\alpha}\int_{\mathbb{R}^d}\mathrm{d}z e^{\mathtt{i}\langle\delta^{1/2}k,\lfloor\delta^{-1}Ut+\delta^{-1/2}z\rfloor\rangle-\mathtt{i}\langle\delta^{1/2}k,\delta^{-1}Ut\rangle}\phi(z)\\
&=\frac{(-1)^n\mathtt{i}^{|\alpha|}\delta^{|\alpha|/2}}{\big(\mathtt{i}\mathbb{S}_\delta(k_1)\big)^n}\sum_{x_1=-\infty}^{\infty} e^{\mathtt{i}\delta^{1/2}k_1(x_1-\delta^{-1}U_1t)}\nabla_\delta^n\Phi_\delta(x_1),\quad\forall\, n\in\mathbb{Z}_+.
\end{aligned}
\tag{4.20}
$$

Our next step is to rewrite the last sum as an integral. We claim that, for all $n\in\mathbb{Z}_+$,

$$
\sum_{x_1=-\infty}^{\infty} e^{\mathtt{i}\delta^{1/2}k_1(x_1-\delta^{-1}U_1t)}\nabla_\delta^n\Phi_\delta(x_1)=\delta^{-|\alpha|/2}\int_{\mathbb{R}^d}\mathrm{d}z e^{\mathtt{i}\langle k,\lfloor z\rceil_{\delta,t}\rangle}\nabla_{\delta,1}^n(\lfloor\cdot\rceil_{\delta,t}^\alpha\phi)(z),
\tag{4.21}
$$

where $\lfloor\cdot\rceil_{\delta,t}$ and $\nabla_{\delta,1}$ are defined in (4.12) and (4.14), respectively.

We first show by an induction on n that

$$
\begin{aligned}
\nabla_\delta^n\Phi_\delta(x_1)=&\delta^{-|\alpha|/2}\int_{y_1}^{y_1+\delta^{1/2}}\mathrm{d}z_1\int_{\mathbb{R}}\mathrm{d}z_2e^{\mathtt{i}k_2\lfloor z_2\rceil_{\delta,t,2}}\cdots\\
&\times\int_{\mathbb{R}}\mathrm{d}z_de^{\mathtt{i}k_d\lfloor z_d\rceil_{\delta,t,d}}\nabla_{\delta,1}^n\big(\lfloor\cdot\rceil_{\delta,t}^\alpha\phi\big)(z),\quad\forall\, n\in\mathbb{Z}_+,
\end{aligned}
\tag{4.22}
$$

where the following change of variables for $x\in\mathbb{Z}^d$ is in use:

$$
y=\delta^{1/2}x-\delta^{-1/2}Ut\in\delta^{1/2}\mathbb{Z}^d-\delta^{-1/2}Ut.
\tag{4.23}
$$

First, (4.21) for $n=0$ follows immediately from the definition (4.19) of Φ_δ:

$$
\begin{aligned}
\Phi_\delta(x_1)=\delta^{-|\alpha|/2}&\int_{y_1}^{y_1+\delta^{1/2}}\mathrm{d}z_1\sum_{y_2\in\delta^{1/2}\mathbb{Z}-\delta^{-1/2}U_2t}e^{\mathtt{i}k_2y_2}\int_{y_2}^{y_2+\delta^{1/2}}\mathrm{d}z_2\cdots\\
&\sum_{y_d\in\delta^{1/2}\mathbb{Z}-\delta^{-1/2}U_dt}e^{\mathtt{i}k_dy_d}\int_{y_d}^{y_d+\delta^{1/2}}\mathrm{d}z_dy^\alpha\phi(z)\\
=\delta^{-|\alpha|/2}&\int_{y_1}^{y_1+\delta^{1/2}}\mathrm{d}z_1\int_{\mathbb{R}}\mathrm{d}z_2e^{\mathtt{i}k_2\lfloor z_2\rceil_{\delta,t,2}}\cdots\int_{\mathbb{R}}\mathrm{d}z_de^{\mathtt{i}k_d\lfloor z_d\rceil_{\delta,t,d}}\lfloor z\rceil_{\delta,t}^\alpha\phi(z),
\end{aligned}
\tag{4.24}
$$

where the last equality uses the definition in (4.12). In general, if (4.22) holds for some $n\in\mathbb{Z}_+$, we write

$$
\begin{aligned}
&\nabla_\delta^{n+1}\Phi(x_1)\\
&=\frac{\nabla_\delta^n\Phi_\delta(x_1)-\nabla_\delta^n\Phi_\delta(x_1-1)}{\delta^{1/2}}\\
&=\frac{\delta^{-|\alpha|/2}}{\delta^{1/2}}\int_{y_1}^{y_1+\delta^{1/2}}\mathrm{d}z_1\int_{\mathbb{R}}\mathrm{d}z_2e^{\mathtt{i}k_2\lfloor z_2\rceil_{\delta,t,2}}\cdots\int_{\mathbb{R}}\mathrm{d}z_de^{\mathtt{i}k_d\lfloor z_d\rceil_{\delta,t,d}}\nabla_{\delta,1}^n\big(\lfloor\cdot\rceil_{\delta,t}^\alpha\phi\big)(z_1,z_2)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\delta^{-|\alpha|/2}}{\delta^{1/2}}\int_{y_1}^{y_1+\delta^{1/2}}\mathrm{d}z_1\int_{\mathbb{R}}\mathrm{d}z_2 e^{\mathrm{i}k_2\lfloor z_2\rfloor_{\delta,t,2}}\cdots\int_{\mathbb{R}}\mathrm{d}z_d e^{\mathrm{i}k_d\lfloor z_d\rfloor_{\delta,t,d}}\nabla^n_{\delta,1}\big(\lfloor\cdot\rfloor^{\alpha}_{\delta,t}\phi\big)(z_1-\delta^{1/2},z_2)\\
&=\delta^{-|\alpha|/2}\int_{y_1}^{y_1+\delta^{1/2}}\mathrm{d}z_1\int_{\mathbb{R}}\mathrm{d}z_2 e^{\mathrm{i}k_2\lfloor z_2\rfloor_{\delta,t,2}}\cdots\int_{\mathbb{R}}\mathrm{d}z_d e^{\mathrm{i}k_d\lfloor z_d\rfloor_{\delta,t,d}}\nabla^{n+1}_{\delta,1}\big(\lfloor\cdot\rfloor^{\alpha_2}_{\delta,t}\phi\big)(z),
\end{aligned}
$$

which gives (4.22) for n replaced by $n+1$. Hence, by mathematical induction, (4.22) holds for all $n\in\mathbb{Z}_+$.

In summary, from (4.22) and the definition in (4.12), we get

$$
\begin{aligned}
&\sum_{x_1=-\infty}^{\infty}e^{\mathrm{i}\delta^{1/2}k_1(x_1-\delta^{-1}U_1t)}\nabla^n_{\delta}\Phi_\delta(x_1)\\
&=\delta^{-|\alpha|/2}\sum_{y_1\in\delta^{1/2}\mathbb{Z}-\delta^{-1/2}U_1t}\int_{y_1}^{y_1+\delta^{1/2}}\mathrm{d}z_1 e^{\mathrm{i}k_1\lfloor z_1\rfloor_{\delta,t,1}}\\
&\quad\times\int_{\mathbb{R}}\mathrm{d}z_2 e^{\mathrm{i}k_2\lfloor z_2\rfloor_{\delta,t,2}}\cdots\int_{\mathbb{R}}\mathrm{d}z_d e^{\mathrm{i}k_d\lfloor z_d\rfloor_{\delta,t,d}}\nabla^n_{\delta,1}\big(\lfloor\cdot\rfloor^{\alpha}_{\delta,t}\phi\big)(z)\\
&=\delta^{-|\alpha|/2}\int_{\mathbb{R}^d}\mathrm{d}z e^{\mathrm{i}\langle k,\lfloor z\rfloor_{\delta,t}\rangle}\nabla^n_{\delta,1}\big(\lfloor\cdot\rfloor^{\alpha}_{\delta,t}\phi\big)(z),
\end{aligned}
$$

which gives the required identity in (4.21). The proof of (4.15) with $j=1$ is complete upon combining (4.20) and (4.21). ■

4.1.2 Decomposition

Now we introduce decompositions of $Z^\delta(\phi)$ which will be used for the rest of Section 4. We start with the following representations of the function ϕ^δ_t defined by (4.7). Note that these representations show the precise decay rate of ϕ^δ_t.

Lemma 4.5. *For $m\in\mathbb{N}$, let $\{\Gamma_1,\cdots,\Gamma_m\}$ be a partition of $\mathbb{R}^d$ by Borel subsets, $(n_1,\cdots,n_m)\in\mathbb{Z}_+^m$, and $(j_1,\cdots,j_m)\in\{1,2\}^m$ such that $k_{j_\ell}\neq 0$ for all $k=(k_1,k_2,\cdots,k_d)\in\Gamma_\ell$ whenever $n_\ell>0$. Then for any $\delta\in(0,1]$ and $t\in\mathbb{R}_+$, the function ϕ^δ_t defined on $\delta^{-1/2}\mathbb{T}^d$ by (4.7) can be written as*

$$
\phi^\delta_t(k)=\frac{1}{(2\pi)^{d/2}}\sum_{\ell=1}^{m}\mathbb{1}_{\Gamma_\ell}(k)\int_{\mathbb{R}^d}\mathrm{d}z\frac{(-1)^{n_\ell}e^{\mathrm{i}\langle k,\lfloor z\rfloor_{\delta,t}\rangle}}{\big(\mathrm{i}\mathbb{S}_\delta(k_{j_\ell})\big)^{n_\ell}}\nabla^{n_\ell}_{\delta,j_\ell}\phi_V(z) \tag{4.25}
$$

$$
\begin{aligned}
&=\sum_{x\in\mathbb{Z}^d}\int_{\delta^{1/2}x_1-\delta^{-1/2}U_1t}^{\delta^{1/2}x_1-\delta^{-1/2}U_1t+\delta^{1/2}}\mathrm{d}z_1\cdots\int_{\delta^{1/2}x_d-\delta^{-1/2}U_dt}^{\delta^{1/2}x_d-\delta^{-1/2}U_dt+\delta^{1/2}}\mathrm{d}z_d\\
&\quad\left(\sum_{\ell=1}^{m}\mathbb{1}_{\Gamma_\ell}(k)\phi^{\delta,n_\ell}_{\delta^{1/2}x-\delta^{-1/2}Ut,z,j_\ell}(k)\right),
\end{aligned}
\tag{4.26}
$$

where ϕ_V and $\mathbb{S}_\delta$ are defined in (4.5) and (4.8), respectively, and

$$
\phi^{\delta,n}_{y,z,j}(k)\stackrel{\mathrm{def}}{=}\frac{1}{(2\pi)^{d/2}}\frac{(-1)^n e^{\mathrm{i}\langle k,y\rangle}}{\big(\mathrm{i}\mathbb{S}_\delta(k_j)\big)^n}\nabla^n_{\delta,j}\phi_V(z). \tag{4.27}
$$

Proof. For all $n\in\mathbb{Z}_+$ and $k=(k_1,k_2,\cdots,k_d)\in\delta^{-1/2}\mathbb{T}^d$ with $k_j\neq 0$ if $n>0$, the first integral in the definition (4.7) of $\phi^\delta_t(k)$ can be written as

$$
\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}z e^{\mathrm{i}\langle\delta^{1/2}k,\lfloor\delta^{-1}Ut+\delta^{-1/2}z\rfloor\rangle-\mathrm{i}\langle\delta^{1/2}k,\delta^{-1}Ut\rangle}\phi_V(z)
$$

$$= \frac{1}{(2\pi)^{d/2}} \frac{(-1)^n}{\big(\mathtt{i}\mathbb{S}_\delta(k_j)\big)^n} \int_{\mathbb{R}^d} \mathrm{d}z e^{\mathtt{i}\langle k, \lfloor z \rfloor_{\delta,t}\rangle} \nabla^n_{\delta,j} \phi_V(z) \tag{4.28}$$

by (4.15) with $\alpha = (0, 0, \cdots, 0)$. It follows that

$$\begin{aligned} \phi^\delta_t(k) &= \int_{\mathbb{R}^d} \mathrm{d}z \frac{1}{(2\pi)^{d/2}} \frac{(-1)^n e^{\mathtt{i}\langle k, \lfloor z \rfloor_{\delta,t}\rangle}}{\big(\mathtt{i}\mathbb{S}_\delta(k_j)\big)^n} \nabla^n_{\delta,j} \phi_V(z) \\ &= \int_{\mathbb{R}^d} \mathrm{d}z \sum_{y \in \delta^{1/2}\mathbb{Z}^d - \delta^{-1/2}Ut} \mathbb{1}_{Q^\delta_y}(z) \phi^{\delta,n}_{y,z,j}(k), \end{aligned}$$

where Q^δ_y and $\phi^{\delta,n}_{y,z,j}(k)$ are defined by (4.17) and (4.27), respectively. The last display is enough for both (4.25) and (4.26). ■

Assumption 4.6. Set $\Gamma_1 = [-1,1]^d$, $j_1 = 1$, $n_1 = 0$ and $n_2 = \cdots = n_m = d + 10$. Fix a choice of (possibly infinite) rectangles $\Gamma_2, \cdots, \Gamma_m$ and $j_2, \cdots, j_m \in \{1, 2, \cdots, d\}^m$ for some $m \geq 2$ such that $k = (k_1, k_2, \cdots, k_d) \mapsto |k_{j_\ell}|$ is bounded away from zero on Γ_ℓ, for all $2 \leq \ell \leq m$, and $\{\Gamma_1, \cdots, \Gamma_m\}$ is a partition of $\mathbb{R}^d$.

For every $\delta \in (0, 1]$, we decompose the function ϕ^δ_t, defined by (4.7), according to (4.25) as follows:

$$\phi^\delta_t(k) = \phi^{\delta,1}(k) + \phi^{\delta,2}_t(k), \quad k \in \delta^{-1/2}\mathbb{T}^d, \tag{4.29}$$

where

$$\phi^{\delta,1}(k) = \frac{1}{(2\pi)^{d/2}} \sum_{\ell=1}^m \mathbb{1}_{\Gamma_\ell}(k) \int_{\mathbb{R}^d} \mathrm{d}z \frac{(-1)^{n_\ell} e^{\mathtt{i}\langle k, z\rangle}}{\big(\mathtt{i}\mathbb{S}_\delta(k_{j_\ell})\big)^{n_\ell}} \nabla^{n_\ell}_{\delta,j_\ell} \phi_V(z), \tag{4.30}$$

$$\phi^{\delta,2}_t(k) = \frac{1}{(2\pi)^{d/2}} \sum_{\ell=1}^m \mathbb{1}_{\Gamma_\ell}(k) \int_{\mathbb{R}^d} \mathrm{d}z \frac{(-1)^{n_\ell}\big(e^{\mathtt{i}\langle k, \lfloor z \rfloor_{\delta,t}\rangle} - e^{\mathtt{i}\langle k, z\rangle}\big)}{\big(\mathtt{i}\mathbb{S}_\delta(k_{j_\ell})\big)^{n_\ell}} \nabla^{n_\ell}_{\delta,j_\ell} \phi_V(z). \tag{4.31}$$

■

We also set

$$S(k) \overset{\mathrm{def}}{=} \widehat{A}(k) + \mathtt{i}\langle k, U\rangle - \frac{Q(k)}{2} \tag{4.32}$$

and

$$\begin{aligned} \phi^{\delta,3}_t(r,k) &= e^{-(t-r)Q(k)/2}\Big(e^{\delta^{-1}(t-r)[\widehat{A}(\delta^{1/2}k) + \mathtt{i}\langle \delta^{1/2}k, U\rangle]} - e^{(t-r)Q(k)/2}\Big)\phi^\delta_t(k) \\ &= \phi^\delta_t(k)\big(e^{\delta^{-1}(t-r)S(\delta^{1/2}k)} - 1\big). \end{aligned} \tag{4.33}$$

Remark 4.7. (1) We stress that the function $\phi^{\delta,1}$ defined in (4.30) does not depend on t. The modified time-dependent floor function $\lfloor \cdot \rfloor_{\delta,t}$ is used only in $\phi^{\delta,2}_t$.

(2) Under Assumption 4.6,

$$\max\left\{ \sup_{t\in\mathbb{R}_+} \sup_{\delta\in(0,1]} |\phi^\delta_t(k)|,\ \sup_{\delta\in(0,1]} |\phi^{\delta,1}(k)|,\ \sup_{t\in\mathbb{R}_+} \sup_{\delta\in(0,1]} |\phi^{\delta,2}_t(k)| \right\} \leq \frac{C_{4.34}}{1 + |k|^{d+10}} \tag{4.34}$$

by (4.9) and the choice of n_ℓ. ■

With the definitions of $\phi^{\delta,1}, \phi^{\delta,2}_t$ and $\phi^{\delta,3}_t$ in (4.30), (4.31) and (4.33), we define

$$I^{\delta,1}_t(r,k) = e^{(t-r)Q(k)/2}\phi^{\delta,1}(k), \tag{4.35}$$

$$I_t^{\delta,2}(r,k) = e^{(t-r)Q(k)/2}\phi_t^{\delta,2}(k), \tag{4.36}$$
$$I_t^{\delta,3}(r,k) = e^{(t-r)Q(k)/2}\phi_t^{\delta,3}(r,k). \tag{4.37}$$

Then we decompose $Z^\delta(\phi)$ by

$$Z_t^\delta(\phi) = Z_t^{\delta,1}(\phi) + Z_t^{\delta,2}(\phi) + Z_t^{\delta,3}(\phi), \tag{4.38}$$

using the notation in (3.12) and the representation in (4.6), where

$$Z_t^{\delta,j}(\phi) = \sqrt{v}\int_0^t \int_{\delta^{-1/2}\mathbb{T}^d} I_t^{\delta,j}(r,k) \bullet \mathbb{W}(\mathrm{d}r,\mathrm{d}k), \quad 1 \le j \le 3. \tag{4.39}$$

Proposition 4.8. *Recall the process Z^0 defined in (3.18). Then $Z^{\delta,1}(\phi)$ converges in distribution to $Z^0(\phi)$ in $C(\mathbb{R}_+,\mathbb{R})$ as $\delta \to 0+$.*

Proof. For any $0 \le s \le t \le T$,

$$\begin{aligned}
&\mathbb{E}\left[\left|Z_s^{\delta,1}(\phi) - Z_t^{\delta,1}(\phi)\right|^2\right] \\
&= v\int_0^s \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}k \left|e^{(s-r)Q(k)/2}\phi^{\delta,1}(k) - e^{(t-r)Q(k)/2}\phi^{\delta,1}(k)\right|^2 \\
&\quad + v\int_s^t \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}k \left|e^{(t-r)Q(k)/2}\phi^{\delta,1}(k)\right|^2 \\
&\le (t-s)^2 v \int_0^T \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}k e^{rQ(k)}|Q(k)/2|^2|\phi^{\delta,1}(k)|^2 + v(t-s)\int_{\mathbb{R}^d}\mathrm{d}k|\phi^{\delta,1}(k)|^2
\end{aligned} \tag{4.40}$$

by (2.27) and the nonpositivity of $Q(k)$ (Assumption 2.3 (3)). Hence, by Remark 4.7 (2), the proposition follows from the last inequality, Kolmogorov's criterion for weak compactness [20, Theorem XIII.1.8], (3.18) and [10, Theorem 3.7.8 (b)]. ■

4.1.3 Tightness by regularity of covariance functions

Recall that $Z^{\delta,j}(\phi)$ for $j = 2,3$ are defined in (4.38). In this section, we discuss the Hölder continuity of their covariance functions. We define metrics $\rho^{\delta,j}$ on $\mathbb{R}_+$ as follows: for $0 \le s \le t < \infty$,

$$\begin{aligned}
&\rho^{\delta,j}(s,t)^2 \\
&= \mathbb{E}\left[\left|Z_s^{\delta,j}(\phi) - Z_t^{\delta,j}(\phi)\right|^2\right] \\
&= v\int_0^s \mathrm{d}r \int_{\delta^{-1/2}\mathbb{T}^d} \mathrm{d}k|I_s^{\delta,j}(r,k) - I_t^{\delta,j}(r,k)|^2 + v\int_s^t \mathrm{d}r\int_{\delta^{-1/2}\mathbb{T}^d}\mathrm{d}k|I_t^{\delta,j}(r,k)|^2.
\end{aligned} \tag{4.41}$$

Observe that the following bounds for $Q(k)$, $R(k)$ and $S(k)$ (defined by (4.32)) follow from Assumption 2.3: For constants $C_{4.42} \in (0,1]$ and $C_{4.43} > 0$,

$$-C_{4.42}^{-1}|k|^2 \le \min\{Q(k),R(k)\} \le \max\{Q(k),R(k)\} \le -C_{4.42}|k|^2, \quad \forall\, k \in \mathbb{T}^d \tag{4.42}$$

and

$$|S(k)| \le C_{4.43}|k|^3, \quad \forall\, k \in \mathbb{T}^d. \tag{4.43}$$

Proposition 4.9. *For all $T \in (0,\infty)$, we can find $C_{4.44} > 0$ depending only on (ϕ, A, v, T) such that*

$$\sup_{\delta\in(0,1]} \max\left\{\rho^{\delta,2}(s,t)^2, \rho^{\delta,3}(s,t)^2\right\} \le C_{4.44}|s-t|, \quad \forall\, 0 \le s \le t \le T. \tag{4.44}$$

Proof. First, we observe a differentiation rule. For $x \in \mathbb{Z}^d$ and $1 \leq j \leq d$, write $y_j(a) = \delta^{1/2} x_j - \delta^{-1/2} U_j a$. Then for a $\mathscr{C}^1$-function $\Phi(b,z)$ on $\mathbb{R} \times \mathbb{R}^d$,

$$\frac{\mathrm{d}}{\mathrm{d}a} \int_{y_1(a)}^{y_1(a)+\delta^{1/2}} \mathrm{d}z_1 \int_{y_2(a)}^{y_2(a)+\delta^{1/2}} \mathrm{d}z_2 \cdots \int_{y_d(a)}^{y_d(a)+\delta^{1/2}} \mathrm{d}z_d \Phi(\delta^{-1/2}a, z)$$

can be written as the sum of $\alpha_1 + \alpha_2$. Here,

$$\begin{aligned}\alpha_1 = -\delta^{-1/2} \sum_{j=1}^d U_j & \int_{y_1(a)}^{y_1(a)+\delta^{1/2}} \mathrm{d}z_1 \cdots \\ & \times \int_{y_{j-1}(a)}^{y_{j-1}(a)+\delta^{1/2}} \mathrm{d}z_{j-1} \int_{y_{j+1}(a)}^{y_{j+1}(a)+\delta^{1/2}} \mathrm{d}z_{j+1} \cdots \int_{y_d(a)}^{y_d(a)+\delta^{1/2}} \mathrm{d}z_d \\ & \times \big[\Phi\big(\delta^{-1/2}a, z_1, \cdots, z_{j-1}, y_j(a)+\delta^{1/2}, z_{j+1}, \cdots, z_d\big) \\ & \quad - \Phi\big(\delta^{-1/2}a, z_1, \cdots, z_{j-1}, y_j(a), z_{j+1}, \cdots, z_d\big)\big]\end{aligned}$$

and

$$\alpha_2 = \delta^{-1/2} \int_{y_1(a)}^{y_1(a)+\delta^{1/2}} \mathrm{d}z_1 \int_{y_2(a)}^{y_2(a)+\delta^{1/2}} \mathrm{d}z_2 \cdots \int_{y_d(a)}^{y_d(a)+\delta^{1/2}} \mathrm{d}z_d \partial_b \Phi(\delta^{-1/2}a, z).$$

It follows that

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}a} \int_{y_1(a)}^{y_1(a)+\delta^{1/2}} \mathrm{d}z_1 \int_{y_2(a)}^{y_2(a)+\delta^{1/2}} \mathrm{d}z_2 \cdots \int_{y_d(a)}^{y_d(a)+\delta^{1/2}} \mathrm{d}z_d \Phi(\delta^{-1/2}a, z) \\ &= \delta^{-1/2} \int_{y_1(a)}^{y_1(a)+\delta^{1/2}} \mathrm{d}z_1 \int_{y_2(a)}^{y_2(a)+\delta^{1/2}} \mathrm{d}z_2 \cdots \int_{y_j(a)}^{y_d(a)+\delta^{1/2}} \mathrm{d}z_d \\ & \quad \times \left(-\sum_{j=1}^d U_j \frac{\partial}{\partial z_j} + \frac{\partial}{\partial b} \right) \Phi(\delta^{-1/2}a, z). \end{aligned} \tag{4.45}$$

We are ready to bound $\rho^{\delta,2}(s,t)^2$. Recall the definition (4.31) of $\phi_a^{\delta,2}$ and note that an expression similar to (4.26) applies to $\phi_a^{\delta,2}$. Hence, by (4.45), we get

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}a} I_a^{\delta,2}(r,k) \\ &= Q(k) e^{(a-r)Q(k)} \phi_a^{\delta,2}(k) \\ & \quad + e^{(a-r)Q(k)} \delta^{-1/2} \sum_{x \in \mathbb{Z}^d} \int_{\delta^{1/2}x_1 - \delta^{-1/2}U_1 t}^{\delta^{1/2}x_1 - \delta^{-1/2}U_1 t + \delta^{1/2}} \mathrm{d}z_1 \cdots \int_{\delta^{1/2}x_d - \delta^{-1/2}U_d t}^{\delta^{1/2}x_d - \delta^{-1/2}U_d t + \delta^{1/2}} \mathrm{d}z_d \\ & \quad \frac{1}{(2\pi)^{d/2}} \sum_{\ell=1}^m \mathbb{1}_{\Gamma_\ell}(k) \left(\frac{(-1)^{n_\ell} \big(\mathtt{i} \sum_{j=1}^d U_j k_j e^{\mathtt{i}\langle k,z \rangle} - \mathtt{i}\langle k,U \rangle e^{\mathtt{i}\langle k, \delta^{1/2}x - \delta^{-1/2}Ua \rangle} \big)}{\big(\mathtt{i}\mathbb{S}_\delta(k_{j_\ell})\big)^{n_\ell}} \nabla_{\delta,j_\ell}^{n_\ell} \phi_V(z) \right). \end{aligned}$$

Here, we can kill the factor $\delta^{-1/2}$ in the second term since for z such that $\delta^{1/2}x_j - \delta^{-1/2}U_j t \leq z_j < \delta^{1/2}x_j - \delta^{-1/2}U_j t + \delta^{1/2}$ for all $1 \leq j \leq d$,

$$\left| \mathtt{i} \sum_{j=1}^d U_j k_j e^{\mathtt{i}\langle k,z \rangle} - \mathtt{i}\langle k,U \rangle e^{\mathtt{i}\langle k, \delta^{1/2}x - \delta^{-1/2}Ua \rangle} \right| \leq |\langle k,U \rangle| \delta^{1/2}$$

by (2.27). Hence, by (4.41), (4.42) and Remark 4.7 (2),

$$\begin{aligned} \sup_{\delta \in (0,1]} \rho^{\delta,2}(s,t)^2 \leq v|s-t|^2 & \int_0^s \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}k \frac{C_{4.46}}{(1+|k|^{d+8})^2} \\ & + \int_s^t \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}k \frac{C_{4.46}}{(1+|k|^{d+10})^2}, \quad \forall\, 0 \leq s \leq t \leq T \end{aligned} \tag{4.46}$$

for some constant $C_{4.46}$ depending only on (ϕ, A, T).

To bound $\rho^{\delta,3}(s,t)^2$, we need more estimates. For the first integral in (4.41), observe that given $a \in [s,t]$, $r \in [0,s]$ and functions $A_\delta(k)$, $B_\delta(k)$ and $f_\delta(a,k)$, we have

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}a}\left[(e^{\delta^{1/2}(a-r)A_\delta(k)} - e^{\delta^{1/2}(a-r)B_\delta(k)})f_\delta(a,k)\right]\right| \\
&\leq \left(\left|\delta^{1/2}A_\delta(k)e^{\delta^{1/2}(a-r)A_\delta(k)}\right| + \left|\delta^{1/2}B_\delta(k)e^{\delta^{1/2}(a-r)B_\delta(k)}\right|\right)\left|f_\delta(a,k)\right| \\
&\quad + \max\left\{|e^{\delta^{1/2}(a-r)A_\delta(k)}|, |e^{\delta^{1/2}(a-r)B_\delta(k)}|\right\} \\
&\quad \times (a-r)|A_\delta(k) - B_\delta(k)| \cdot |\delta^{1/2}f_\delta'(a,k)|,
\end{aligned} \tag{4.47}$$

where $f_\delta'(a,k) = (\partial/\partial a)f_\delta(a,k)$. Then we apply (4.47) with the following choice of functions in $k \in \delta^{-1/2}\mathbb{T}^d$:

$$\begin{aligned}
A_\delta(k) &= \delta^{-3/2}[\widehat{A}(-\delta^{1/2}k) - \mathtt{i}\langle\delta^{1/2}k, U\rangle], \\
B_\delta(k) &= \delta^{-3/2}Q(\delta^{1/2}k)/2, \\
f_\delta(a,k) &= \phi_a^\delta(k).
\end{aligned} \tag{4.48}$$

Here, $\operatorname{Re}\delta^{1/2}A_\delta(k) = \delta^{-1}R(-\delta^{1/2}k)/2$ can be bounded by (4.42) and (4.43): for all $k \in \delta^{-1/2}\mathbb{T}^d$,

$$-C_{4.42}^{-1}|k|^2 \leq \operatorname{Re}\delta^{1/2}A_\delta(k) \leq -C_{4.42}|k|^2, \tag{4.49}$$

$$|A_\delta(k)| \leq C_{4.50}\big(\delta^{-1}|\delta^{1/2}k|^2 + \delta^{-1}|\delta^{1/2}k|^3\big) = C_{4.50}\big(|k|^2 + \delta^{1/2}|k|^3\big), \tag{4.50}$$

where $C_{4.50} = \max\{C_{4.42}^{-1}, C_{4.43}\}$ depends only on A. The same bound (4.49) is satisfied by $\delta^{1/2}B_\delta$. Also, (4.43) shows that

$$|A_\delta(k) - B_\delta(k)| \leq C_{4.43}|k|^3, \quad \forall\, k \in \delta^{-1/2}\mathbb{T}^d.$$

By Remark 4.7 (2), (4.45), and the choice of $f_\delta(a,k) = \phi_a^\delta(k)$ in (4.48) represented according to (4.26), we deduce that

$$\sup_{\delta\in(0,1]}\sup_{a\in[0,T]}|\delta^{1/2}f_\delta'(a,k)| \leq \frac{C_{4.51}}{1+|k|^{d+9}}, \quad \forall\, k \in \delta^{-1/2}\mathbb{T}^d, \tag{4.51}$$

for some constant $C_{4.51}$ depending only on (ϕ, A).

To bound the metric $\rho^{\delta,3}$ defined by (4.41), we apply (4.49), (4.50) and (4.51) to (4.47). Also, recall (4.41) and Remark 4.7 (2). It follows that

$$\begin{aligned}
\sup_{\delta\in(0,1]}\rho^{\delta,3}(s,t)^2 \leq v|s-t|^2 &\int_0^s \mathrm{d}r \int_{\mathbb{R}^d}\mathrm{d}k \frac{C_{4.52}}{(1+|k|^{d+6})^2} \\
&+ \int_s^t \mathrm{d}r \int_{\mathbb{R}^d}\mathrm{d}k \frac{C_{4.52}}{(1+|k|^{d+10})^2}, \quad \forall\, 0 \leq s \leq t \leq T,
\end{aligned} \tag{4.52}$$

for some constant $C_{4.52}$ depending only on (ϕ, A, T).

The required inequality in (4.44) for $\rho^{\delta,j}$ for $j = 2,3$ follows from (4.46) and (4.52). ■

Proposition 4.10. *For $j = 2,3$, the processes $Z^{\delta,j}(\phi)$ defined in (4.38) converge to zero in distribution in $C(\mathbb{R}_+, \mathbb{R})$ as $\delta \to 0+$.*

Proof. By dominated convergence, it follows from (2.27), (4.42) and (4.43) that $Z_t^{\delta,j}(\phi)$ converge to zero in $L_2(\mathbb{P})$ for all $t \in \mathbb{R}_+$. Then we use Kolmogorov's criterion [20, Theorem XIII.1.8], Proposition 4.9 and [10, Theorem 3.7.8 (b)]. ■

4.1.4 Tightness by approximate factorizations

Now we provide a different method to prove Proposition 4.10, aiming to show convergence to zero of expectations of the following form:

$$\mathbb{E}\left[\sup_{t\in[0,T]\cap\mathbb{Q}}\left|\int_0^t\int_{\delta^{-1/2}\mathbb{T}^d}e^{(t-r)Q(k)/2}v_t(r,k)W(\mathrm{d}r,\mathrm{d}k)\right|^p\right],\quad p\in[1,\infty),\tag{4.53}$$

where $W(\mathrm{d}r,\mathrm{d}k)$ is a space-time white noise on $\mathbb{R}_+\times\mathbb{R}^d$. This setup is chosen for the process $Z_t^{\delta,j}(\phi)$ defined by (4.39) in view of the explicit form of $I_t^{\delta,j}$, for $j=2,3$. Indeed, we can write

$$\begin{aligned}&I_t^{\delta,2}(r,k)\\&=e^{(t-r)Q(k)}\frac{1}{(2\pi)^{d/2}}\sum_{\ell=1}^m\frac{\mathbb{1}_{\Gamma_\ell}(k)(-1)^{n_\ell}}{\big(\mathtt{i}\mathbb{S}_\delta(k_{j_\ell})\big)^{n_\ell}}\int_{\mathbb{R}^d}\mathrm{d}ze^{\mathtt{i}\langle k,z\rangle}\nabla^{n_\ell}_{\delta,j_\ell}\phi_V(z)\big(e^{-\mathtt{i}\langle k,z-\lfloor z\rfloor_{\delta,t}\rangle}-1\big)\end{aligned}\tag{4.54}$$

by (4.31) and (4.36), and

$$I_t^{\delta,3}(r,k)=e^{(t-r)Q(k)/2}\phi_t^\delta(k)\big(e^{\delta^{-1}(t-r)S(\delta^{1/2}k)}-1\big)$$

by (4.33) and (4.37).

Our plan below can be outlined as follows. We use $e^a=\sum_{n=0}^\infty a^n/n!$ to expand the differences

$$e^{-\mathtt{i}\langle k,z-\lfloor z\rfloor_{\delta,t}\rangle}-1\quad\text{and}\quad e^{\delta^{-1}(t-r)S(\delta^{1/2}k)}-1$$

in $I_t^{\delta,j}$ for $j=2,3$. (These differences are "small" due to (4.13) and (4.43).) Since $[-\mathtt{i}\langle k,z-\lfloor z\rfloor_{\delta,t}\rangle]^n$ and $[\delta^{-1}(t-r)S(\delta^{1/2}k)]^n$ are polynomials in $(k,\lfloor z\rfloor_{\delta,t})$ or in $(t,r,S(\delta^{1/2}k))$, the expansions separate t and (r,k). This yields true stochastic convolutions without an interference of t from v_t in terms of the general form in (4.53). Hence, the factorization method [8, Section 5.3.1] can be generalized to accommodate the present setup to bound the suprema of those stochastic convolutions. If we undo the series expansions after using the factorization method, then the convergence to zero of $\sup_{t\in[0,T]}|Z_t^{\delta,j}(\phi)|$ is attributable to a limit of the following form: For a square-integrable function φ and a constant $C>0$,

$$\int_0^t\mathrm{d}r\int_{\delta^{-1/2}\mathbb{T}^d}\mathrm{d}ke^{-2a}e^{(t-r)Q(k)}\varphi(k)^2\big(e^{C(\delta^{1/2}|k|)^2}-1\big)\xrightarrow[\delta\to0]{}0\tag{4.55}$$

since Q is strictly negative definite by Assumption 2.3 (3). Here, the additional term e^{-2a} for some $a\in(0,\frac12)$ arises from the factorization method.

We carry out the above steps in the rest of Section 4.1.4. First, we adapt the factorization method to a Fourier setting as follows. We write $(q_t(w_1,w_2))_{t>0}$ for the transition densities of a d-dimensional Brownian motion with covariance matrix $-Q$. We also write $q_t(w)=q_t(0,w)$. Then for $a\in(0,\frac12)$ and Borel measurable functions $u(s,w_1):\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$ and $v(r,k):\mathbb{R}_+\times\delta^{-1/2}\mathbb{T}^d\to\mathbb{R}$, we define two integral operators J^{a-1} and J_{-a}:

$$J^{a-1}u(t)\stackrel{\text{def}}{=}\frac{\sin(\pi a)}{\pi}\int_0^t\mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}w_1(t-s)^{a-1}q_{t-s}(w_1)u(s,w_1),\tag{4.56}$$

$$J_{-a}v(s,w_1)\stackrel{\text{def}}{=}\int_0^s\int_{\delta^{-1/2}\mathbb{T}^d}(s-r)^{-a}e^{\mathtt{i}\langle k,w_1\rangle+(s-r)Q(k)/2}v(r,k)W(\mathrm{d}r,\mathrm{d}k)\tag{4.57}$$

$$\begin{aligned}&=\int_0^s\int_{\delta^{-1/2}\mathbb{T}^d}(s-r)^{-a}\left(\int_{\mathbb{R}^d}\mathrm{d}w_2q_{s-r}(w_1,w_2)e^{\mathtt{i}\langle k,w_2\rangle}\right)\\&\quad\times v(r,k)W(\mathrm{d}r,\mathrm{d}k).\end{aligned}\tag{4.58}$$

See [8] and [18, Appendix A] for analogous integral operators in the standard setting of the factorization method.

Lemma 4.11 (Factorization). *Let $a \in (0, \frac{1}{2})$ and $t \in (0, \infty)$. For $v(r,k)$ such that $\sup_{r\in[0,t]} |v(r,k)| \in L_2(\delta^{-1/2}\mathbb{T}^d, \mathrm{d}k)$, $J_{-a}v(s,w_1)$ is a well-defined integral a.s. for all (s,w_1) and, as a function of (s,w_1), is a.s. in $L_2([0,t]\times\mathbb{R}^d, \mathrm{d}r\mathrm{d}k)$. Moreover, we have*

$$J^{a-1}J_{-a}v(t) = \int_0^t \int_{\delta^{-1/2}\mathbb{T}^d} e^{(t-r)Q(k)/2} v(r,k) W(\mathrm{d}r, \mathrm{d}k). \tag{4.59}$$

Proof. By Itô's isometry, the integrability assumption on v, and the choice $a \in (0, \frac{1}{2})$, it holds that $\sup_{(s,w_1)\in[0,t]\times\mathbb{R}^d} \mathbb{E}[|J_{-a}v(s,w_1)|^2]$ is finite. Hence, the first two assertions hold. It remains to prove (4.59). By the definition of (q_t) and the Chapman–Kolmogorov equation, we can write

$$e^{(t-r)Q(k)/2} = \int_{\mathbb{R}^d} \mathrm{d}w q_{t-r}(w) e^{\mathrm{i}\langle k,w\rangle} \tag{4.60}$$

$$= \int_{\mathbb{R}^d} \mathrm{d}w_1 q_{t-s}(w_1) \int_{\mathbb{R}^d} \mathrm{d}w_2 q_{s-r}(w_1,w_2) e^{\mathrm{i}\langle k,w_2\rangle}, \quad \forall\, 0 < r < s < t. \tag{4.61}$$

Note that (4.60) gives

$$\begin{aligned} &\int_0^t \int_{\delta^{-1/2}\mathbb{T}^d} e^{(t-r)Q(k)/2} v(r,k) W(\mathrm{d}r, \mathrm{d}k) \\ &= \int_0^t \int_{\delta^{-1/2}\mathbb{T}^d} \left(\int_{\mathbb{R}^d} \mathrm{d}w e^{\mathrm{i}\langle k,w\rangle} q_{t-r}(w) \right) v(r,k) W(\mathrm{d}r, \mathrm{d}k). \end{aligned} \tag{4.62}$$

On the other hand, it follows from (4.56) and (4.58) that

$$\begin{aligned} J^{a-1}J_{-a}v(t) &= \frac{\sin(\pi a)}{\pi} \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}w_1 (t-s)^{a-1} q_{t-s}(w_1) \\ &\quad \times \int_0^s \int_{\delta^{-1/2}\mathbb{T}^d} (s-r)^{-a} \left(\int_{\mathbb{R}^d} \mathrm{d}w_2 q_{s-r}(w_1,w_2) e^{\mathrm{i}\langle k,w_2\rangle} \right) v(r,k) W(\mathrm{d}r, \mathrm{d}k) \\ &= \frac{\sin(\pi a)}{\pi} \int_0^t \int_{\delta^{-1/2}\mathbb{T}^d} \left(\int_r^t \mathrm{d}s (t-s)^{a-1} (s-r)^{-a} \right. \\ &\quad \left. \times \int_{\mathbb{R}^d} \mathrm{d}w_1 q_{t-s}(w_1) \int_{\mathbb{R}^d} \mathrm{d}w_2 q_{s-r}(w_1,w_2) e^{\mathrm{i}\langle k,w_2\rangle} \right) v(r,k) W(\mathrm{d}r, \mathrm{d}k) \\ &= \int_0^t \int_{\delta^{-1/2}\mathbb{T}^d} \left(\int_{\mathbb{R}^d} \mathrm{d}w e^{\mathrm{i}\langle k,w\rangle} q_{t-r}(w) \right) v(r,k) W(\mathrm{d}r, \mathrm{d}k), \end{aligned} \tag{4.63}$$

where the second equality follows from the stochastic Fubini theorem (see [24, Theorem 2.6 on page 296]) and the third equality follows from the identity:

$$\int_r^t \mathrm{d}s (t-s)^{\alpha-1} (s-r)^{-\alpha} = \frac{\pi}{\sin(\pi\alpha)}, \quad \forall\, 0 \le r \le t,\ \alpha \in (0,1)$$

and (4.61). The last term in (4.63) is the same as the right-hand side of (4.62). The required identity (4.59) is proved. ■

The next two lemmas give bounds for J^{a-1}.

Lemma 4.12. *Let $(a, p_\star)$ be such that*

$$0 < a < \frac{1}{2} \quad \text{and} \quad p_\star > \frac{1+d/2}{a} > 2. \tag{4.64}$$

For any $T, \lambda \in (0,\infty)$ *and Borel measurable function* $(s,w_1) \mapsto u(s,w_1)$*, we have*

$$|J^{a-1}u(t)|^{p_\star} \leq C_{4.65} \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}w_1 |u(s,w_1)|^{p_\star} e^{-\lambda|w_1|}, \quad \forall\, t \in [0,T], \tag{4.65}$$

where $C_{4.65}$ *depends only on* $(a,p_\star)$*,* Q*,* T *and* λ*. In particular, given a family* $(u_t)_{t\in[0,T]\cap\mathbb{Q}}$*,*

$$\sup_{t\in[0,T]\cap\mathbb{Q}} |J^{a-1}u_t(t)|^{p_\star} \leq C_{4.65} \int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}w_1 \sup_{t\in[s,T]\cap\mathbb{Q}} |u_t(s,w_1)|^{p_\star} e^{-\lambda|w_1|}. \tag{4.66}$$

Proof. The proof of (4.65) is the same as the proof of a similar bound in [18, Lemma A.3]. We include it here for completeness. Write C for a constant depending only on $(a,p_\star)$, Q, T and λ, which may change from line to line. By (4.56), it holds that

$$\begin{aligned}
&|J^{a-1}u(t)| \\
&\leq\ C \int_0^t \mathrm{d}s (t-s)^{a-1} \int_{\mathbb{R}^d} \mathrm{d}w_1 q_{t-s}(w_1)|u(s,w_1)| \\
&\leq\ C \int_0^t \mathrm{d}s (t-s)^{a-1} \left(\int_{\mathbb{R}^d} \mathrm{d}w_1 q_{t-s}(w_1) e^{\lambda|w_1|/2} \cdot |u(s,w_1)|^{p_\star/2} e^{-\lambda|w_1|/2} \right)^{2/p_\star} \\
&\leq\ C \int_0^t \mathrm{d}s (t-s)^{a-1} \left(\int_{\mathbb{R}^d} \mathrm{d}w_1 q_{t-s}(w_1)^2 e^{\lambda|w_1|} \right)^{1/p_\star} \left(\int_{\mathbb{R}^d} \mathrm{d}w_1 |u(s,w_1)|^{p_\star} e^{-\lambda|w_1|} \right)^{1/p_\star} \\
&\leq\ C \int_0^t \mathrm{d}s (t-s)^{a-1-\frac{d}{2p_\star}} \left(\int_{\mathbb{R}^d} \mathrm{d}w_1 |u(s,w_1)|^{p_\star} e^{-\lambda|w_1|} \right)^{1/p_\star} \\
&\leq\ C \left(\int_0^t \mathrm{d}s (t-s)^{\left(a-1-\frac{d}{2p_\star}\right)\frac{p_\star}{p_\star-1}} \right)^{\frac{p_\star-1}{p_\star}} \left(\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}w_1 |u(s,w_1)|^{p_\star} e^{-\lambda|w_1|} \right)^{1/p_\star} \\
&\leq\ C \left(\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}w_1 |u(s,w_1)|^{p_\star} e^{-\lambda|w_1|} \right)^{1/p_\star}.
\end{aligned}$$

Here, the second, the third and the fifth inequalities follow from Hölder's inequality. The last inequality uses (4.64) so that the first integral in the fifth inequality is finite. The last inequality proves (4.65). ■

We prove the required convergence of $Z^{\delta,2}(\phi)$ first. Now we use the particular form of $I_t^{\delta,2}$.

Lemma 4.13. *For any* $\delta \in (0,1]$*,* $p \in [1,\infty)$ *and* $T \in (0,\infty)$*, it holds that*

$$\begin{aligned}
&\sup_{w_1\in\mathbb{R}^d} \sup_{s\in[0,T]} \mathbb{E}\left[\sup_{t\in[s,T]\cap\mathbb{Q}} |J_{-a}\phi_t^{\delta,2}(s,w_1)|^p \right]^{1/p} \\
&\leq\ C_{4.67} \left(\int_0^T \mathrm{d}r \int_{\delta^{-1/2}\mathbb{T}^d} \mathrm{d}k \frac{r^{-2a} e^{rQ(k)} (e^{|\delta^{1/2}k|^2} - 1)}{1+|k|^{d+10}} \right)^{1/2}
\end{aligned} \tag{4.67}$$

for $C_{4.67}$ *depending only on* p*,* ϕ*,* d *and the parameters fixed in Assumption 4.6.*

Proof. Let $0 \leq s \leq t \leq T$, and recall the definition (4.31) of $\phi_t^{\delta,2}$. By the stochastic Fubini theorem [24, Theorem 2.6 on page 296], we can write

$$J_{-a}\phi_t^{\delta,2}(s,w_1) = \frac{1}{(2\pi)^{d/2}} \sum_{\ell=1}^m \int_{\mathbb{R}^d} \mathrm{d}z \nabla_{\delta,j_\ell}^{n_\ell} \phi_V(z)$$

$$\times\int_0^s\int_{\delta^{-1/2}\mathbb{T}^d}(s-r)^{-a}e^{(s-r)Q(k)/2}\frac{\mathbb{1}_{\Gamma_\ell}(k)(-1)^{n_\ell}}{\big(\mathtt{i}\mathbb{S}_\delta(k_{j_\ell})\big)^{n_\ell}}e^{\mathtt{i}\langle k,(w_1+z)\rangle}$$
$$\times\big(e^{-\mathtt{i}\langle k,z-\lfloor z\rfloor_{\delta,t}\rangle}-1\big)W(\mathrm{d}r,\mathrm{d}k). \tag{4.68}$$

To proceed, we write

$$\begin{aligned}
&e^{-\mathtt{i}\langle k,z-\lfloor z\rfloor_{\delta,t}\rangle}-1\\
&=\sum_{n=1}^{\infty}\frac{(-\mathtt{i})^n}{n!}\left(\sum_{j=1}^{d}k_j\big(z_j-\lfloor z_j\rfloor_{\delta,t,j}\big)\right)^n\\
&=\sum_{n=1}^{\infty}\frac{(-\mathtt{i})^n}{n!}\sum_{\boldsymbol{\iota}:\iota_1+\cdots+\iota_d=n}\binom{n}{\iota_1,\cdots,\iota_d}\prod_{j=1}^{d}\left(\frac{z_j-\lfloor z_j\rfloor_{\delta,t,j}}{\delta^{1/2}}\right)^{\iota_j}(\delta^{1/2}k_j)^{\iota_j}.
\end{aligned}\tag{4.69}$$

Combining the last two displays gives the following equation where the series on the right-hand side converges absolutely in $L_2(\mathbb{P})$:

$$\begin{aligned}
&J_{-a}\phi_t^{\delta,2}(s,w_1)\\
&=\sum_{\ell=1}^{m}\sum_{n=1}^{\infty}\frac{(-\mathtt{i})^n}{n!}\sum_{\boldsymbol{\iota}:\iota_1+\cdots+\iota_d=n}\binom{n}{\iota_1,\cdots,\iota_d}\int_{\mathbb{R}^d}\mathrm{d}z\nabla^{n_\ell}_{\delta,j_\ell}\phi_V(z)\prod_{j=1}^{d}\left(\frac{z_j-\lfloor z_j\rfloor_{\delta,t,j}}{\delta^{1/2}}\right)^{\iota_j}\\
&\quad\times\int_0^s\int_{\delta^{-1/2}\mathbb{T}^d}(s-r)^{-a}e^{(s-r)Q(k)/2}\frac{\mathbb{1}_{\Gamma_\ell}(k)(-1)^{n_\ell}}{\big(\mathtt{i}\mathbb{S}_\delta(k_{j_\ell})\big)^{n_\ell}}e^{\mathtt{i}\langle k,(w_1+z)\rangle}\\
&\quad\times\prod_{j=1}^{d}(\delta^{1/2}k_j)^{\iota_j}W(\mathrm{d}r,\mathrm{d}k),
\end{aligned}\tag{4.70}$$

where $\boldsymbol{\iota}$ ranges over $\mathbb{Z}_+^d$. By the Minkowski inequality and the Burkholder–Davis–Gundy inequality [20, Theorem IV.4.1], we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[s,T]\cap\mathbb{Q}}|J_{-a}\phi_t^{\delta,2}(s,w_1)|^p\right]^{1/p}\\
&\le C_{4.71}\sum_{\ell=1}^{m}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\boldsymbol{\iota}:\iota_1+\cdots+\iota_d=n}\binom{n}{\iota_1,\cdots,\iota_d}\left(\int_{\mathbb{R}^d}\mathrm{d}z|\nabla^{n_\ell}_{\delta,j_\ell}\phi_V(z)|\right)\\
&\quad\times\left(\int_0^T\mathrm{d}r'\int_{\delta^{-1/2}\mathbb{T}^d}\mathrm{d}k(r')^{-2a}e^{r'Q(k)}\frac{\mathbb{1}_{\Gamma_\ell}(k)}{|\mathbb{S}_\delta(k_{j_\ell})|^{2n_\ell}}\prod_{j=1}^{d}(\delta^{1/2}k_j)^{2\iota_j}\right)^{1/2}.
\end{aligned}\tag{4.71}$$

In more detail, for (4.71), $C_{4.71}$ is a constant depending only on p, and we change variables by $r'=s-r$. Let $C_{4.71}$ absorb the finite constant

$$\sup_{\delta\in(0,1]}\sup_{\ell\in\{1,\cdots,m\}}\int_{\mathbb{R}^d}\mathrm{d}z|\nabla^{n_\ell}_{\delta,j_\ell}\phi_V(z)|.$$

Then the required inequality (4.67) follows from (4.71) upon applying the Cauchy–Schwarz inequality with respect to the counting measure $f\mapsto\sum_{\ell,n,\boldsymbol{\iota}}\frac{1}{n!}\binom{n}{\iota_1,\cdots,\iota_d}f(\ell,n,\boldsymbol{\iota})$ and then undoing series expansions. The proof is complete. ■

Proposition 4.14. *The laws of the processes $Z^{\delta,2}(\phi)$ defined in (4.38) converge weakly to zero in the space of probability measures on $C(\mathbb{R}_+,\mathbb{R})$ as $\delta\to0+$.*

Proof. Let $(a,p_\star)$ satisfy (4.64). It is enough to show that $\sup_{t\in[0,T]}|Z_t^{\delta,2}(\phi)|$ converges to zero in $L_{p_\star}(\mathbb{P})$ as $\delta\to0+$ for every $T\in(0,\infty)$.

Recalling (4.36), (4.38) and the notation $\iint \Phi \bullet d\mathbb{W}$ in (3.12), we can use Lemma 4.11 to rewrite the stochastic integral defining $Z_t^{\delta,2}(\phi)$ with respect to $\dot{W}^\ell$ as a linear combination of $J^{a-1}J_{-a}\phi_t^{\delta,2}(t)$ and $J^{a-1}J_{-a}\overline{\phi_t^{\delta,2}}(t)$ with $\dot{W}$ replaced by $\dot{W}^\ell$, for $\ell = 1,2$. Since $a \in (0, \frac{1}{2})$ by (4.64), it follows from Lemma 4.13 that, for any $p \in [1,\infty)$,

$$\lim_{\delta\to 0+} \sup_{w_1\in\mathbb{R}^d} \sup_{s\in[0,T]} \mathbb{E}\left[\sup_{t\in[s,T]\cap\mathbb{Q}} \left|J_{-a}\phi_t^{\delta,2}(s,w_1)\right|^p\right]^{1/p} = 0.$$

Then applying these two properties to Lemma 4.12 with $u_t(s,w_1)(\omega) \equiv J_{-a}\phi_t^{\delta,2}(s,w_1)(\omega)$, we obtain from dominated convergence that

$$\lim_{\delta\to 0+} \mathbb{E}\left[\sup_{t\in[0,T]\cap\mathbb{Q}} \left|J^{a-1}J_{-a}\phi_t^{\delta,2}(t)\right|^{p_\star}\right] = 0. \tag{4.72}$$

The same limit holds with $\phi_t^{\delta,2}$ replaced by $\overline{\phi_t^{\delta,2}}$ since $J_{-a}\overline{\phi_t^{\delta,2}}(s,-w_1) = \overline{J_{-a}\phi_t^{\delta,2}(s,w_1)}$.

Note that $Z^{\delta,2}(\phi)$ has continuous paths by (4.26), (4.31), (4.36) and (4.38). Then by the limits obtained in the previous paragraph, we deduce that, as $\delta \to 0+$, $\sup_{t\in[0,T]}|Z_t^{\delta,2}(\phi)|$ converges to zero in $L_{p_\star}(\mathbb{P})$, as required. ■

Almost the same argument can be used to prove the required convergence of $Z^{\delta,3}(\phi)$.

Proposition 4.15. *The laws of the processes $Z^{\delta,3}(\phi)$ defined in (4.38) converge weakly to zero in the space of probability measures on $C(\mathbb{R}_+,\mathbb{R})$ as $\delta \to 0+$.*

Proof. We prove an analogue of Lemma 4.13 for $\phi_t^{\delta,2}$ replaced by $\phi_t^{\delta,3}$. By (4.33),

$$\phi_t^{\delta,3}(r,k) = \phi_t^\delta(k)\sum_{n=1}^\infty \frac{1}{n!}\sum_{j=0}^n \binom{n}{j}(t-s)^{n-j}(s-r)^j[\delta^{-1}S(\delta^{1/2}k)]^n.$$

Recall the uniform bound of ϕ_t^δ in Remark 4.7. Then as a counterpart of (4.67), we have the following: for any $p \in [1,\infty)$ and $w_1 \in \mathbb{R}^d$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[s,T]} |J_{-a}\phi_t^{\delta,3}(s,w_1)|^p\right]^{1/p} \\
&\leq C_{4.73}\sum_{n=1}^\infty \frac{1}{n!}\sum_{j=0}^n \binom{n}{j} T^{n-j}\left(\int_0^T \mathrm{d}r \int_{\delta^{-1/2}\mathbb{T}^d} \mathrm{d}k r^{-2a}e^{rQ(k)}r^{2j}|\delta^{-1}S(\delta^{1/2}k)|^{2n}\right)^{1/2} \\
&\leq C_{4.73}\left(\int_0^T \mathrm{d}r\int_{\delta^{-1/2}\mathbb{T}^d}\mathrm{d}k\frac{r^{-2a}e^{rQ(k)}\left(e^{(T+T^2)|\delta^{-1}S(\delta^{1/2}k)|^2}-1\right)}{1+|k|^{d+10}}\right)^{1/2}
\end{aligned} \tag{4.73}$$

for a constant $C_{4.73}$ depending only on p, ϕ, d and the parameters fixed in Assumption 4.6. Thanks to (4.43), the last term tends to zero as $\delta \to 0+$. The rest follows similarly as in the proof of Proposition 4.14. ■

4.1.5 Characterization of limits

By Propositions 4.1, 4.8, 4.14 and 4.15 (or Proposition 4.10 in place of Propositions 4.14 and 4.15), we have verified conditions in Mitoma's theorem [17, Theorem 3.1] and proved that $\{Z^\delta\}_{\delta\in(0,1]}$ converge in distribution to Z^0 in $C(\mathbb{R}_+,\mathcal{S}'(\mathbb{R}^d))$. Recall that Z^0 is defined by (3.18). Our goal here is to show that Z^0 solves (3.13).

In the following, we apply Duhamel's principle to get a preliminary SPDE satisfied by Z^0. Then we apply Fourier inversions to transform the SPDE to (3.13).

Lemma 4.16. *Recall that we write $0^{-1/2}\mathbb{T}^d$ for $\mathbb{R}^d$. For any $\delta\in[0,1)$ and any bounded continuous complex-valued function ϕ defined on $\delta^{-1/2}\mathbb{T}^d$,*

$$Z_t(\phi)=\sqrt{v}\int_0^t\int_{\delta^{-1/2}\mathbb{T}^d}e^{(t-r)Q(k)/2}\phi(k)\bullet\mathbb{W}(\mathrm{d}r,\mathrm{d}k)$$

solves the following SPDE:

$$Z_t(\phi)=\int_0^t Z_r\left(\frac{Q\phi}{2}\right)\mathrm{d}r+\sqrt{v}\int_0^t\int_{\delta^{-1/2}\mathbb{T}^d}\phi(k)\bullet\mathbb{W}(\mathrm{d}r,\mathrm{d}k). \tag{4.74}$$

Proof. We write out the right-hand side of (4.74) and then use the stochastic Fubini theorem [24, Theorem 2.6 on page 296] in the second equality below. These give

$$\begin{aligned}
&\int_0^t Z_s\left(\frac{Q\phi}{2}\right)\mathrm{d}s+\sqrt{v}\int_0^t\int_{\delta^{-1/2}\mathbb{T}^d}\phi(k)\bullet\mathbb{W}(\mathrm{d}r,\mathrm{d}k)\\
&=\sqrt{v}\int_0^t\int_0^s\int_{\delta^{-1/2}\mathbb{T}^d}\frac{Q(k)}{2}e^{(s-r)Q(k)/2}\phi(k)\bullet\mathbb{W}(\mathrm{d}r,\mathrm{d}k)\mathrm{d}s+\sqrt{v}\int_0^t\int_{\delta^{-1/2}\mathbb{T}^d}\phi(k)\bullet\mathbb{W}(\mathrm{d}r,\mathrm{d}k)\\
&=\sqrt{v}\int_0^t\int_{\delta^{-1/2}\mathbb{T}^d}\int_r^t\frac{Q(k)}{2}e^{(s-r)Q(k)/2}\mathrm{d}s\phi(k)\bullet\mathbb{W}(\mathrm{d}r,\mathrm{d}k)+\sqrt{v}\int_0^t\int_{\delta^{-1/2}\mathbb{T}^d}\phi(k)\bullet\mathbb{W}(\mathrm{d}r,\mathrm{d}k)\\
&=\sqrt{v}\int_0^t\int_{\delta^{-1/2}\mathbb{T}^d}(e^{(t-r)Q(k)/2}-1)\phi(k)\bullet\mathbb{W}(\mathrm{d}r,\mathrm{d}k)+\sqrt{v}\int_0^t\int_{\delta^{-1/2}\mathbb{T}^d}\phi(k)\bullet\mathbb{W}(\mathrm{d}r,\mathrm{d}k)\\
&=\sqrt{v}\int_0^t\int_{\delta^{-1/2}\mathbb{T}^d}e^{(t-r)Q(k)/2}\phi(k)\bullet\mathbb{W}(\mathrm{d}r,\mathrm{d}k)\\
&=Z_t(\phi),
\end{aligned}$$

which is (4.74). ∎

Proposition 4.17. *The unique distributional limit Z^0 defined in (3.18) of Z^δ as $\delta\to0+$ solves the following SPDE: for some space-time white noise $W(\mathrm{d}r,\mathrm{d}k)$ with covariance measure $\mathrm{d}r\mathrm{d}k$ on $\mathbb{R}_+\times\mathbb{R}^d$,*

$$Z_t^0(\phi)=\int_0^t Z_s^0\left(\frac{\Delta\phi}{2}\right)\mathrm{d}s+\sqrt{v|\det(V)|}\int_0^t\int_{\mathbb{R}^d}\phi(k)W(\mathrm{d}r,\mathrm{d}k),\quad\phi\in\mathcal{S}(\mathbb{R}^d). \tag{4.75}$$

Proof. Recall ϕ_V and T_V defined in (4.5). We define Z by

$$Z_t(\mathcal{F}T_V\phi)\stackrel{\text{def}}{=}Z_t^0(\phi),\quad\phi\in\mathcal{S}(\mathbb{R}^d).$$

By the bijectivity of $\mathcal{F}$ and T_V on $\mathcal{S}(\mathbb{R}^d)$, Z has a domain given by $\mathcal{S}(\mathbb{R}^d)$ and is well-defined. Then Lemma 4.16 implies that

$$Z_t(\mathcal{F}T_V\phi)-\int_0^t Z_s\left(\frac{Q\mathcal{F}T_V\phi}{2}\right)\mathrm{d}s,\quad 0\le t<\infty, \tag{4.76}$$

is a continuous centered Gaussian process. Its covariance across times $0\le s\le t<\infty$ is given by

$$\begin{aligned}
sv\int_{\mathbb{R}^d}|\mathcal{F}T_V\phi(k)|^2\mathrm{d}k&=sv\int_{\mathbb{R}^d}\left|\mathcal{F}\phi(V^{-1}k)\right|^2\mathrm{d}k\\
&=sv|\det(V)|\int_{\mathbb{R}^d}|\mathcal{F}\phi(k')|^2\mathrm{d}k'\\
&=sv|\det(V)|\int_{\mathbb{R}^d}|\phi(k')|^2\mathrm{d}k'.
\end{aligned}$$

Here, the first two equalities follow from the change of variables $Vz' = z$ and $k' = V^{-1}k$, respectively, and the last equality follows from Plancherel's identity. Note that we use the normalization of Fourier transforms as in [19, Section IX.1]. To rewrite the Riemann-integral term in (4.76) in terms of ϕ, we recall $V = \sqrt{-Q^{-1}}$ and then change variables to get

$$\begin{aligned}\frac{Q(k)}{2}\mathcal{F}T_V\phi(k) &= \frac{-\langle V^{-1}k, V^{-1}k\rangle}{2}\mathcal{F}\phi(V^{-1}k)\\ &= \mathcal{F}\left(\frac{\Delta\phi}{2}\right)(V^{-1}k)\\ &= \mathcal{F}T_V\left(\frac{\Delta\phi}{2}\right)(k).\end{aligned}$$

From the last three displays, we deduce that, for a space-time white noise $\dot{W}$ with covariance measure $\mathrm{d}r\mathrm{d}k$,

$$\begin{aligned}\sqrt{v|\det(V)|}\int_0^t\int_{\mathbb{R}}\phi(k)W(\mathrm{d}r,\mathrm{d}k) &= Z_t(\mathcal{F}T_V\phi) - \int_0^t Z_s\left(\frac{Q\mathcal{F}T_V\phi}{2}\right)\mathrm{d}s\\ &= Z_t(\mathcal{F}T_V\phi) - \int_0^t Z_s\left(\mathcal{F}T_V\left(\frac{\Delta\phi}{2}\right)\right)\mathrm{d}s\\ &= Z_t^0(\phi) - \int_0^t Z_s^0\left(\frac{\Delta\phi}{2}\right)\mathrm{d}s,\end{aligned}$$

as required in (4.75). ■

5 Convergence of the deterministic parts

Recall that Y^δ is defined by (3.10). In this section, we prove the convergence of Y^δ as $\delta \to 0+$.

Proposition 5.1. *Let $\{\mu^\delta\}_{\delta\in(0,1]} \subset \ell_1(\mathbb{Z}^d)$ satisfy (3.7) and (P_t) denote the semigroup of the d-dimensional standard Brownian motion. Then Y^δ takes values in $C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$ for every $\delta \in (0,1]$ and*

$$Y^\delta \xrightarrow[\delta\to 0+]{} Y^0 \quad \text{in } C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d)), \tag{5.1}$$

where

$$Y_t^0(\phi) \stackrel{\text{def}}{=} |\det(V)|\mu^0(P_t\phi). \tag{5.2}$$

Proof. We divide the proof into the following steps.

Step 1. We compute the explicit form of $Y_t^\delta(\phi)$ first. We use (3.2) to write

$$\begin{aligned}&\eta^{\infty,\delta}_{\delta^{-1}t}\big(\lfloor\delta^{-1}Ut + \delta^{-1/2}V^{-1}z\rfloor\big)\\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\mathrm{d}k e^{\delta^{-1}t\widehat{A}(k)}e^{\mathtt{i}\langle k,\lfloor\delta^{-1}Ut+\delta^{-1/2}V^{-1}z\rfloor\rangle}\widehat{\mu^\delta}(k)\\ &= \sum_{x\in\mathbb{Z}^d}\mu^\delta(x)\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\mathrm{d}k e^{\delta^{-1}t\widehat{A}(k)}e^{-\mathtt{i}\langle k,x\rangle}e^{\mathtt{i}\langle k,\lfloor\delta^{-1}Ut+\delta^{-1/2}V^{-1}z\rfloor\rangle}\\ &= \delta^{\frac{d}{2}}\sum_{y\in\delta^{1/2}V\mathbb{Z}^d}\frac{\mu^\delta(\delta^{-1/2}V^{-1}y)}{(2\pi)^d}\int_{\delta^{-1/2}\mathbb{T}^d}\mathrm{d}k e^{\delta^{-1}t\widehat{A}(\delta^{1/2}k)}e^{-\mathtt{i}\langle k,V^{-1}y\rangle}e^{\mathtt{i}\langle\delta^{1/2}k,\lfloor\delta^{-1}Ut+\delta^{-1/2}V^{-1}z\rfloor\rangle},\end{aligned}$$

where we use (3.1) and the assumption that $\mu^\delta \in \ell_1(\mathbb{Z}^d)$ in the second equality. By the definition (3.10) of $Y^\delta(\phi)$ and the last equality, we can write

$$
\begin{aligned}
Y_t^\delta(\phi) = \delta^{\frac{d+2}{4}} &\sum_{y\in\delta^{1/2}V\mathbb{Z}^d} \mu^\delta(\delta^{-1/2}V^{-1}y)\\
&\times \int_{\delta^{-1/2}\mathbb{T}^d} \mathrm{d}k e^{\delta^{-1}t\widehat{A}(\delta^{1/2}k)+\mathtt{i}\langle \delta^{1/2}k,\delta^{-1}Ut\rangle} e^{-\mathtt{i}\langle k,V^{-1}y\rangle}\\
&\times \int_{\mathbb{R}^d}\mathrm{d}z\frac{1}{(2\pi)^d} e^{\mathtt{i}\langle \delta^{1/2}k,\lfloor \delta^{-1}Ut+\delta^{-1/2}V^{-1}z\rfloor\rangle-\mathtt{i}\langle \delta^{1/2}k,\delta^{-1}Ut\rangle}\phi(z),\quad \forall\ \phi\in\mathcal{S}(\mathbb{R}^d).
\end{aligned}\tag{5.3}
$$

By the weak topology on $\mathcal{S}'(\mathbb{R}^d)$, Lemma 4.5 and (4.42), Y^δ takes values in $C(\mathbb{R}_+,\mathcal{S}'(\mathbb{R}^d))$.

Step 2. By Mitoma's theorem [17, Theorem 3.1], (5.1) follows if we show that for all $\phi\in\mathcal{S}(\mathbb{R}^d)$,

$$
Y_t^\delta(\phi)\xrightarrow[\delta\to 0+]{} Y_t^0(\phi)\quad \text{in } C(\mathbb{R}_+,\mathbb{R}),\tag{5.4}
$$

where Y^0 is defined in (5.2). To this end, we first show in Step 3 that the following convergence holds in $\mathcal{S}(\mathbb{R}^d)$ for functions of $\zeta=V^{-1}y$: for all $t\in\mathbb{R}_+$ and sequences $\mathbb{R}_+\ni t_\delta\to t$,

$$
\begin{aligned}
&\int_{\delta^{-1/2}\mathbb{T}^d}\mathrm{d}k e^{\delta^{-1}t_\delta\widehat{A}(\delta^{1/2}k)+\mathtt{i}\langle\delta^{1/2}k,\delta^{-1}Ut_\delta\rangle}e^{-\mathtt{i}\langle k,\zeta\rangle}\\
&\quad\times\int_{\mathbb{R}^d}\mathrm{d}z\frac{1}{(2\pi)^d}e^{\mathtt{i}\langle\delta^{1/2}k,\lfloor\delta^{-1}Ut_\delta+\delta^{-1/2}V^{-1}z\rfloor\rangle-\mathtt{i}\langle\delta^{1/2}k,\delta^{-1}Ut_\delta\rangle}\phi(z)\\
&\qquad\xrightarrow[\delta\to 0+]{}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}k e^{tQ(k)/2-\mathtt{i}\langle k,\zeta\rangle}\int_{\mathbb{R}^d}\mathrm{d}z e^{\mathtt{i}\langle k,V^{-1}z\rangle}\phi(z).
\end{aligned}\tag{5.5}
$$

In Step 4, we show that the limit of $Y_t^\delta(\phi)$ coincides with $Y_t^0(\phi)$ defined in (5.2). Then we put the results in Steps 3–4 together for the proof of (5.4) in Step 5.

Step 3. Observe that the integrands with respect to $\mathrm{d}k$ in (5.3) depend on ζ only through $e^{-\mathtt{i}\langle k,\zeta\rangle}$. Recall the notation ϕ_V in (4.5). Hence, (5.5) follows if we can show that for any multi-indices $\beta,\gamma\in\mathbb{Z}_+^d$,

$$
K^\delta(\zeta)\xrightarrow[\delta\to 0+]{} K^0(\zeta)\quad\text{uniformly in }\zeta\in\mathbb{R}^d,\tag{5.6}
$$

where

$$
\begin{aligned}
K^\delta(\zeta) &= \zeta^\beta\int_{\delta^{-1/2}\mathbb{T}^d}\mathrm{d}k\Bigg(k^\gamma e^{\delta^{-1}t_\delta\widehat{A}(\delta^{1/2}k)+\mathtt{i}\langle\delta^{1/2}k,\delta^{-1}Ut_\delta\rangle}\\
&\quad\times\int_{\mathbb{R}^d}\mathrm{d}z\frac{1}{(2\pi)^d}e^{\mathtt{i}\langle\delta^{1/2}k,\lfloor\delta^{-1}Ut_\delta+\delta^{-1/2}z\rfloor\rangle-\mathtt{i}\langle\delta^{1/2}k,\delta^{-1}Ut_\delta\rangle}\phi_V(z)\Bigg)e^{-\mathtt{i}\langle k,\zeta\rangle},\quad\delta\in(0,1],\\
K^0(\zeta) &= \zeta^\beta\int_{\mathbb{R}^d}\mathrm{d}k\Big(k^\gamma e^{tQ(k)/2}\int_{\mathbb{R}^d}\mathrm{d}z\frac{1}{(2\pi)^d}e^{\mathtt{i}\langle k,z\rangle}\phi_V(z)\Big)e^{-\mathtt{i}\langle k,\zeta\rangle}.
\end{aligned}
$$

In formulating these functions, we introduce $\zeta^\beta k^\gamma$ to (5.5) after a change of variable in z.

For the proof of (5.6), we handle the growth of ζ^β by integration by parts with respect to k_j multiple times, for all $j\in\{1,\cdots,d\}$. (The assumption $\widehat{A}\in\mathscr{C}^\infty(\mathbb{R}^d)$ from Assumption 2.3 (1) is now used.) This argument also uses the fact that whenever $\zeta_j\neq 0$, $\partial_{k_j}[e^{-\mathtt{i}\langle k,\zeta\rangle}/(-\mathtt{i}\zeta_j)]=e^{-\mathtt{i}\langle k,\zeta\rangle}$.

In more detail, we apply integration by parts to K^δ for any fixed $\delta\in[0,1]$ inductively as follows. Suppose that $(\beta_{i_1},\cdots,\beta_{i_\ell})$ are the nonzero entries of β for $i_1<i_2<\cdots<i_\ell$. We integrate by parts with respect to k_{i_1} first. This yields three terms, two of them being

boundary terms, and then the next stage applies integration by parts to each of the three terms in the following way. The two boundary terms are integrated by parts with respect to k_{i_2} and the remaining term is integrated by parts with respect to k_{i_1} again if $\beta_{i_1} > 1$ and with respect to k_{i_2} otherwise. In summary, this procedure focuses on exhausting ζ^β in the non-boundary terms by multiple integration by parts with respect to variables in the order $k_{i_1}, k_{i_2}, \cdots, k_{i_\ell}$.

In this way, we obtain the decomposition

$$K^\delta(\zeta) = \sum_{b\in\mathcal{B}} K^{\delta,b}(\zeta) + K^{\delta,c}(\zeta), \quad \delta \in [0,1].$$

Here, $K^{\delta,b}$, $b\in\mathcal{B}$, range over terms as the boundary terms in applying the above multiple integration by parts for $K^\delta(\zeta)$. The finite index set $\mathcal{B}$ has a size depending only on β. Note that $K^{0,b} = 0$ for all $b \in \mathcal{B}$ by (4.42). (For $\delta > 0$, it is possible that $K^{\delta,b}$ is a nonzero integral since the integration with respect to k_j in K^δ is only over a bounded domain. Therefore, the uniform convergence $K^\delta \to K^0$ as $\delta \to 0+$ follows if we show the uniform convergence of

$$K^{\delta,b} \to 0, \quad \forall\, b \in \mathcal{B}, \quad \text{and} \quad K^{\delta,c} \to K^{0,c} \quad \text{as } \delta \to 0+. \tag{5.7}$$

Thanks to domination from (4.42) and (4.43) in passing the limits, all the convergences in (5.7) can be obtained by the following bounds: For any $m,n \in \mathbb{Z}_+$ and $T \in (0,\infty)$,

$$\sup_{\alpha\in\mathbb{Z}_+^d:|\alpha|=m}\ \sup_{\delta\in(0,1]}\ \sup_{s\in[0,T]}\ \sup_{k\in\delta^{-1/2}\mathbb{T}^d} \left|\frac{\partial^\alpha}{\partial k^\alpha} e^{\delta^{-1}s\widehat{A}(\delta^{1/2}k)+\mathtt{i}\langle\delta^{1/2}k,\delta^{-1}Us\rangle}\right| < \infty \tag{5.8}$$

and

$$\begin{aligned}\sup_{\alpha:|\alpha|=m}\ \sup_{\delta\in(0,1]}\ \sup_{s\in[0,T]} &\left|\frac{\partial^\alpha}{\partial k^\alpha}\int_{\mathbb{R}^d}\mathrm{d}z\frac{1}{(2\pi)^d} e^{\mathtt{i}\langle\delta^{1/2}k,\lfloor\delta^{-1}Us+\delta^{-1/2}z\rfloor\rangle-\mathtt{i}\langle\delta^{1/2}k,\delta^{-1}Us\rangle}\phi_V(z)\right|\\ &\qquad\leq \frac{C_{5.9}}{1+|k|^n}, \quad \forall\, k\in\delta^{-1/2}\mathbb{T}^d,\end{aligned} \tag{5.9}$$

for a constant $C_{5.9} > 0$.

We prove (5.8) and (5.9) now. First, the Taylor expansion of $\widehat{A}(k) + \mathtt{i}\langle k, U\rangle$ around $k = 0$ has the lowest order term $\langle k, Qk\rangle/2$ by Assumption 2.3 (3). Hence, (5.8) follows upon using (4.42). To see (5.9), we apply the alternative forms (4.15) of the derivatives of the integrals in (5.9). Then we still need to bound $\nabla^n_{\delta,j}(\lfloor\cdot\rfloor^\alpha_{\delta,s}\phi_V)(z)$ as in (4.15) by an integrable function independent of α,δ,s. For this property, we first apply the following discrete Leibniz rule for $\nabla_{\delta,1}$ defined by (4.14):

$$\nabla^n_{\delta,1}(fg)(z) = \sum_{\ell=0}^n \binom{n}{\ell}\nabla^\ell_{\delta,1}(f)(z)\times\nabla^{n-\ell}_{\delta,1}(g)(z_1-\ell\delta^{1/2},z_2,\cdots,z_d), \quad \forall\, n\geq 1, \tag{5.10}$$

and its analogue for with respect to $z_2,\cdots,z_d$. By these rules, we can expand the partial difference $\nabla^n_{\delta,j}(\lfloor\cdot\rceil^\alpha_{\delta,s}\phi_V)$ as in (4.15) into sums of products of $\nabla^\ell_{\delta,j}\phi_V$ and $\nabla^{\ell_j}_{\delta,j}\lfloor\cdot\rceil_{\delta,s,j}$ for $1 \leq j \leq d$ and $0 \leq \ell, \ell_j \leq n$. Observe that the partial differences $\nabla^{\ell_2}_{\delta,j}\lfloor\cdot\rceil_{\delta,s,j} \equiv 1$ if $\ell_2 = 1$ by the definition (4.12) of $\lfloor\cdot\rceil_{\delta,s,j}$ and so $\equiv 0$ whenever $\ell_2 \geq 2$. We have proved (5.9) since $\lfloor z\rceil_{\delta,s,j}$ grows at most linearly in $|z|$ and the partial derivatives of ϕ_V of all fixed orders are rapidly decreasing.

Step 4. The limiting integral in (5.5) with respect to k over $\mathbb{R}^d$ can be simplified as follows: with the change of variables $k = Vj/\sqrt{t}$,

$$\begin{aligned}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}k e^{tQ(k)/2-\mathrm{i}\langle k,V^{-1}y\rangle+\mathrm{i}\langle k,V^{-1}z\rangle} &= \frac{|\det(V)|}{(2\pi)^d t^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}j e^{-|j|^2/2-\mathrm{i}\langle j,y/\sqrt{t}\rangle+\mathrm{i}\langle j,z/\sqrt{t}\rangle}\\ &= \frac{|\det(V)|}{(2\pi t)^{d/2}}\exp\left(-\frac{|y-z|^2}{2t}\right).\end{aligned}$$

It follows that

$$\int_{\mathbb{R}^d}\mathrm{d}z\phi(z)\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}k e^{tQ(k)/2-\mathrm{i}\langle V^{-1}y,k\rangle+\mathrm{i}\langle k,V^{-1}z\rangle} = |\det(V)|P_t\phi(y), \tag{5.11}$$

where (P_t) is the semigroup of the d-dimensional standard Brownian motion.

Step 5. Note that all the functionals in (3.7) are $\mathcal{S}'(\mathbb{R}^d)$-valued since $\mu^\delta \in \ell_1(\mathbb{Z}^d)$. Also, the convergence in (3.7) holds uniformly on compact subsets of $\mathcal{S}(\mathbb{R}^d)$. Indeed, the convergence in (3.7) is with respect to the weak topology of $\mathcal{S}'(\mathbb{R}^d)$ [19, Section V.3] so that [19, Theorems V.8 and V.9] apply. Hence, by (5.3), (5.5) and (5.11), we deduce the uniform convergence of $Y_t^\delta(\phi)$ to $|\det(V)|\mu^0(P_t\phi)$ on compacts in t. This completes the proof of (5.4). ■

6 List of frequent notations for Sections 3–5

$\mathcal{F}\phi(k)$: the Fourier transform of ϕ with a normalization defined in (3.19).
$I_t^{\delta,j}(r,k)$: the integrands of stochastic integrals defined in (4.35)– (4.37).
J^{a-1}: the integral operator defined in (4.56).
J_{-a}: the stochastic integral operator defined in (4.58).
Q: the 2×2 strictly negative definite matrix defined in Assumption 2.3 (3).
$Q(k)$: the function $\langle k, Qk\rangle$ defined in Assumption 2.3 (3).
$R(k)$: twice the real part of $\widehat{A}(k)$ defined in (2.4).
$S(k)$: a remainder function of $\widehat{A}(k)$ defined in (4.32).
$\mathbb{S}_\delta(k)$: the sine-like function defined in (4.8).
U: the two-dimensional real vector defined in (3.4).
V: the square root of $-Q^{-1}$ defined in (3.5).
X^δ: the rescaled $\mathcal{S}'(\mathbb{R}^d)$-valued process defined in (3.6).
Y^δ: the deterministic part of X^δ defined in (3.10).
Z^δ: the stochastic part of X^δ defined in (3.11).
$\nabla_{\delta,1}$: the partial difference operator defined in (4.14).
$\phi_t^\delta(k) = \phi^{\delta,1}(k) + \phi_t^{\delta,2}(k)$: the auxiliary function defined in (4.7) and decomposed in (4.29).
$\phi_t^{\delta,3}(r,k)$: the auxiliary function defined in (4.33).
$\phi_V(z) = T_V\phi(z)$: the change-of-variable transformation of ϕ defined in (4.5).
$\int\int \Phi(r,k)\bullet \mathbb{W}(\mathrm{d}r,\mathrm{d}k)$: the sum of stochastic integrals of $\mathrm{Re}\,\Phi$ and $\mathrm{Im}\,\Phi$ defined in (3.12).
$\lfloor z\rfloor_{\delta,t}$, $\lfloor z\rfloor_{\delta,t,j}$, $\lfloor z_j\rfloor_{\delta,t,j}$: the modified floor functions on rescaled lattices defined in (4.12).

References

[1] Barabási, A.-L. and Stanley, H. E.: *Fractal Concepts in Surface Growth*. Cambridge University Press, 1995. doi:10.1017/CBO9780511599798

[2] Borodin, A. and Ferrari, P. L.: Anisotropic growth of random surfaces in $2+1$ dimensions. *Communications in Mathematical Physics* **325**, (2014), 603–684. doi:10.1007/s00220-013-1823-x

[3] Borodin, A., Corwin, I. and Ferrari, P. L.: Anisotropic $(2+1)d$ growth and Gaussian limits of q-Whittaker processes. *Probability Theory and Related Fields* **172**, (2017), 245–321. doi:10.1007/s00440-017-0809-6

[4] Borodin, A., Corwin, I. and Toninelli, F. L.: Stochastic heat equation limit of a (2 + 1)d growth model. *Communications in Mathematical Physics* **350**, (2017), 957–984. doi:10.1007/s0022 MR-3607467

[5] Caravenna, F., Sun, R. and Zygouras, N.: Universality in marginally relevant disordered systems. *Annals of Applied Probability* **27**, (2017), 3050–3112. doi:10.1214/17-AAP1276 MR-3719953

[6] Corwin, I. and Toninelli, F. L.: Stationary measure of the driven two-dimensional q-Whittaker particle system on the torus. *Electronic Communications in Probability* **21**, (2016), paper no. 44. doi:10.1214/16-ECP4624

[7] Da Prato, G., Kwapien, S. and Zabczyk, J.: Regularity of solutions of linear stochastic equations in Hilbert spaces. *Stochastics* **23**, (1987), 1–23. doi:10.1080/17442508708833480

[8] Da Prato, G. and Zabczyk, J.: *Stochastic Equations in Infinite Dimensions* (Encyclopedia of Mathematics and its Applications). Cambridge University Press, 2014. doi:10.1017/CBO9781107295513

[9] Edwards, S. F. and Wilkinson, D. R.: The surface statistics of a granular aggregate. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* **381**, (1982), 17–31. doi: 10.1098/rspa.1982.0056 MR-0661715

[10] Ethier, S. N. and Kurtz, T. G.: *Markov Processes: Convergence and Characterization*. John Wiley & Sons, New Jersey, 1986. doi:10.1002/9780470316658

[11] Gu, Y., Ryzhik, L. and Zeitouni, O.: The Edwards–Wilkinson limit of the random heat equation in dimensions three and higher. *Communications in Mathematical Physics* **363**, (2018), 351–388. doi:10.1007/s00220-018-3202-0 MR-3851818

[12] Karatzas, I. and Shreve, S. E.: *Brownian Motion and Stochastic Calculus*, 2nd edition. Springer-Verlag, Berlin, 1998. doi:10.1007/978-1-4612-0949-2 MR-1121940

[13] Kardar, M.: Nonequilibrium dynamics of interfaces and lines. *Physics Reports* **301**, (1998), 85–112. doi:10.1016/S0370-1573(98)00007-6

[14] Kardar, M., Parisi, G., and Zhang, Y.-C.: Dynamic scaling of growing interfaces. *Physical Review Letters* **56**, (1986), 889–892. doi:10.1103/physrevlett.56.889

[15] Kurtz, T. G.: The Yamada-Watanabe-Engelbert theorem for general stochastic equations and inequalities. *Electronic Journal Probability* **12**, paper no. 33, (2007), 951–965. doi:10.1214/EJP.v12-431 MR-2336594

[16] Magnen, J. and Unterberger, J.: The scaling limit of the Kardar–Parisi–Zhang equation in space dimension 3 and higher. *Journal of Statistical Physics* **171**, (2018), 543–598. doi:10.1007/s10955-018-2014-0

[17] Mitoma, I.: Tightness of probabilities on $C([0,1];\mathscr{S}')$ and $D([0,1];\mathscr{S}')$. *Annals of Probability* **11**, (1983), 989–999. doi:10.1214/aop/1176993447

[18] Mytnik, L., Perkins, E. A. and Sturm, A.: On pathwise uniqueness for stochastic heat equations with non-Lipschitz coefficients. *Annals of Probability* **34**, (2006), 1910–1959. doi:10.1214/009117906000000331 MR-2271487

[19] Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics: Functional Analysis*. Academic Press, New York-London, 1980. MR-0493419

[20] Revuz, D. and Yor, M.: *Continuous Martingales and Brownian Motion*, 3rd corrected edition. Springer-Verlag, Berlin, 2005. doi:10.1007/978-3-662-06400-9

[21] Rudin, W.: *Fourier Analysis on Groups*. Interscience Tracts in Pure and Applied Mathematics, No. 12 Interscience Publishers (a division of John Wiley and Sons), New York-London, 1962. doi:10.1002/9781118165621

[22] Toninelli, F. L.: $(2+1)$-dimensional interface dynamics: mixing time, hydrodynamic limit and Anisotropic Kardar–Parisi–Zhang growth, arXiv:1711.05571.

[23] Villain, J.: Continuum models of crystal growth from atomic beams with and without desorption. *Journal de Physique I* **1**, (1991), 19–42. doi:10.1051/jp1:1991114

[24] Walsh, J. B.: *An Introduction to Stochastic Partial Differential Equations*. In: Hennequin P.L. (eds) École d'Été de Probabilités de Saint Flour XIV - 1984. Lecture Notes in Mathematics, vol 1180. Springer, Berlin, Heidelberg, 1983. doi:10.1007/BFb0074920

[25] Wolf, D. E.: Kinetic roughening of vicinal surfaces. *Physical Review Letters* **67**, (1991), 1783–1786. doi:10.1103/PhysRevLett.67.1783