

## Small-time fluctuations for the bridge in a model class of hypoelliptic diffusions of weak Hörmander type\*

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### Abstract

We study the small-time asymptotics for hypoelliptic diffusion processes conditioned by their initial and final positions, in a model class of diffusions satisfying a weak Hörmander condition where the diffusivity is constant and the drift is linear. We show that, while the diffusion bridge can exhibit a blow-up behaviour in the small time limit, we can still make sense of suitably rescaled fluctuations which converge weakly. We explicitly describe the limit fluctuation process in terms of quantities associated to the unconditioned diffusion. In the discussion of examples, we also find an expression for the bridge from 0 to 0 in time 1 of an iterated Kolmogorov diffusion.

**Keywords:** hypoelliptic diffusions; small-time asymptotics; iterated Kolmogorov diffusion.

**AMS MSC 2010:** 35H10; 60F05; 60J60.

Submitted to EJP on August 15, 2018, final version accepted on February 4, 2019.

Supersedes arXiv:1808.04350v1.

## 1 Introduction

The small-time asymptotics for hypoelliptic diffusion processes can depend crucially on the drift term. For instance, Ben Arous and Léandre [6, 7] showed that an interaction of the flow of the drift vector field with the heat diffusion can lead to an exponential decay of the heat kernel on the diagonal. The current paper discusses and illustrates the effects the drift term can have on the small-time fluctuations for hypoelliptic diffusion bridges.

Bailleul, Mesnager and Norris [2] study the small-time asymptotics of sub-Riemannian diffusion bridges outside the cut locus. Their analysis was extended by us to the diagonal, cf. [9], to describe the asymptotics of sub-Riemannian diffusion loops. Both works are concerned with hypoelliptic diffusion processes whose associated generators satisfy the so-called strong Hörmander condition and where the drift vector fields are nice enough to not affect the small-time asymptotics. In continuation of this work, we would like to analyse the small-time asymptotics for hypoelliptic diffusion bridges, where one assumes

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a weak Hörmander condition only. As a first step towards this goal, we determine the small-time bridge fluctuations for a model class of hypoelliptic diffusions satisfying a weak Hörmander condition, and we contrast our results with [2] and [9].

We consider the same model class of hypoelliptic diffusions for which Barilari and Paoli [4] describe the small-time heat kernel expansion on the diagonal and give a geometric characterisation of the coefficients in terms of curvature-like invariants. The corresponding model class of hypoelliptic operators already features in the pioneering work of Hörmander [10], and Lanconelli and Polidoro [14] study a notion of principal part as well as the invariance with respect to suitable groups of translations and dilations for this class of operators.

Fix  $d, m \in \mathbb{N}$ . Let  $A$  be a  $d \times d$  matrix and  $B$  be a  $d \times m$  matrix such that there exists  $N \in \mathbb{N}$  with

$$\text{rank} [B, AB, A^2B, \dots, A^{N-1}B] = d, \tag{1.1}$$

where  $[B, AB, A^2B, \dots, A^{N-1}B]$  is the matrix formed by the columns of the matrices  $B, AB, A^2B, \dots, A^{N-1}B$ . Let  $n$  denote the minimal  $N$  satisfying (1.1). We study the diffusion process whose generator  $\mathcal{L}$  is the second order differential operator on  $\mathbb{R}^d$  given by

$$\mathcal{L} = \sum_{j=1}^d (Ax)_j \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{j,k=1}^d (BB^*)_{jk} \frac{\partial^2}{\partial x_j \partial x_k}. \tag{1.2}$$

For the linear vector field  $X_0$  and the constant vector fields  $X_1, \dots, X_m$  on  $\mathbb{R}^d$  defined by

$$X_0 = \sum_{j,k=1}^d A_{jk} x_k \frac{\partial}{\partial x_j} \quad \text{and} \quad X_i = \sum_{j=1}^d B_{ji} \frac{\partial}{\partial x_j} \quad \text{for } i \in \{1, \dots, m\},$$

the operator  $\mathcal{L}$  rewrites as

$$\mathcal{L} = X_0 + \frac{1}{2} \sum_{i=1}^m X_i^2.$$

We further note that, for  $i \in \{1, \dots, m\}$  and  $k \in \mathbb{N}$ ,

$$(\text{ad}_{X_0})^k (X_i) = \sum_{j=1}^d (-1)^k (A^k B)_{ji} \frac{\partial}{\partial x_j}, \tag{1.3}$$

where  $\text{ad}_{X_0}(Y) = [X_0, Y]$ . Hence, putting condition (1.1) on the matrices  $A$  and  $B$  ensures that any operator of the form (1.2) satisfies a weak Hörmander condition. In control theory, condition (1.1) is also known as the Kalman rank condition, cf. [11, Section 2.3]. As remarked in [14], it is indeed of interest to study the operators of the form (1.2) and its associated hypoelliptic diffusions because they arise when linearising the Fokker-Planck equation. Moreover, this model class contains some strongly degenerate operators, see Section 4.4.

In the analysis of the small-time fluctuations for the corresponding hypoelliptic diffusion bridges, it is of advantage that a diffusion process with generator of the form (1.2) is always Gaussian and in particular, that its bridge processes can be written down explicitly. Additionally, unlike [2], we do not come across any cut locus phenomena for this class of diffusions. Fix  $x \in \mathbb{R}^d$  and let  $\varepsilon > 0$ . There exists a diffusion process  $(x_t^\varepsilon)_{t \in [0,1]}$  starting from  $x$  and having generator  $\varepsilon \mathcal{L}$ . For  $y \in \mathbb{R}^d$ , let  $(z_t^\varepsilon(y))_{t \in [0,1]}$  be the process obtained by conditioning  $(x_t^\varepsilon)_{t \in [0,1]}$  on  $x_1^\varepsilon = y$ . An explicit expression for the bridge process  $(z_t^\varepsilon(y))_{t \in [0,1]}$  is given in Lemma 2.2. We consider these diffusion bridges in the limit  $\varepsilon \rightarrow 0$ .

Using the notion of the matrix exponential of a square matrix, we set, for  $t \in [0, 1]$ ,

$$\Gamma_t^\varepsilon = \int_0^t e^{-\varepsilon s A} B B^* e^{-\varepsilon s A^*} ds . \tag{1.4}$$

According to [14, Proposition A.1], the Kalman rank condition (1.1) implies that the square matrix  $\Gamma_t^\varepsilon$  is invertible for all  $t \in (0, 1]$ . Let  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$  be the deterministic path in  $\mathbb{R}^d$  defined by

$$\phi_t^\varepsilon(y) = e^{\varepsilon t A} x + e^{\varepsilon t A} \Gamma_t^\varepsilon (\Gamma_1^\varepsilon)^{-1} (e^{-\varepsilon A} y - x) . \tag{1.5}$$

We see that this path describes the leading order behaviour of the diffusion bridge  $(z_t^\varepsilon(y))_{t \in [0,1]}$  as  $\varepsilon \rightarrow 0$ . Set  $\Omega^{0,0} = \{\omega \in C([0, 1], \mathbb{R}^d) : \omega_0 = 0, \omega_1 = 0\}$ .

**Theorem 1.1.** *For all  $x, y \in \mathbb{R}^d$ , the processes  $(z_t^\varepsilon(y) - \phi_t^\varepsilon(y))_{t \in [0,1]}$  converge weakly as  $\varepsilon \rightarrow 0$  to the zero process on the set of continuous loops  $\Omega^{0,0}$ .*

In our discussion of examples in Section 4, we observe that the path  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$  can exhibit a blow-up behaviour in the limit  $\varepsilon \rightarrow 0$ . Hence, this path compensates for any blow-up occurring in the process  $(z_t^\varepsilon(y))_{t \in [0,1]}$ . Remark 3.6 shows that for any hypoelliptic diffusion in our model class which does not satisfy the strong Hörmander condition, there exist  $x, y \in \mathbb{R}^d$  such that  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$  blows up as  $\varepsilon \rightarrow 0$ . In comparison to the law of large number type theorem [2, Theorem 1.1] for sub-Riemannian diffusion bridges, we note that in the weak Hörmander setting the minimal-like path  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$  depends on  $\varepsilon > 0$ . However, as in [2, Section 2], the path  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$  can still be obtained as projection of a solution to an appropriate Hamiltonian system. Let us consider the Hamiltonian  $\mathcal{H}^\varepsilon : T^*\mathbb{R}^d \rightarrow \mathbb{R}$  given by

$$\mathcal{H}^\varepsilon(q, p) = \varepsilon p^* A q + \frac{1}{2} p^* B B^* p .$$

The description in [4, Section 2] implies that  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$  is the projection onto  $\mathbb{R}^d$  of the unique solution in  $T^*\mathbb{R}^d$  to the Hamiltonian equations associated with  $\mathcal{H}^\varepsilon$  subject to starting in  $T_x^*\mathbb{R}^d$  at time 0 and ending in  $T_y^*\mathbb{R}^d$  at time 1.

Theorem 1.1 is a consequence of our study of the small-time fluctuations for the bridge  $(z_t^\varepsilon(y))_{t \in [0,1]}$ . To state our fluctuation result, cf. Theorem 1.2, we first introduce a basis for  $\mathbb{R}^d$  which simplifies the analysis, also see [4] and [14]. For  $k \in \{1, \dots, n\}$ , set

$$E_k = \text{span} \{ A^l B v : v \in \mathbb{R}^m, 0 \leq l \leq k - 1 \} , \tag{1.6}$$

that is,  $E_k$  is the subspace of  $\mathbb{R}^d$  defined by the columns of the matrices  $A^l B$  for  $l \in \{0, \dots, k - 1\}$ . By condition (1.1) and the minimality of  $n$ , we know both that  $E_n = \mathbb{R}^d$  and that  $E_{n-1}$  is a strict subset of  $\mathbb{R}^d$ . Set  $d_k = \dim E_k$ . Since  $\{E_k\}_{1 \leq k \leq n}$  is an increasing filtration of subspaces of  $\mathbb{R}^d$ , we can and do choose an orthonormal basis  $\{e_1, \dots, e_d\}$  of  $\mathbb{R}^d$  such that  $\{e_1, \dots, e_{d_k}\}$  is a basis of  $E_k$ . For  $r \in \mathbb{R}$ , define

$$U^\varepsilon(r) = e^{\varepsilon r A} B . \tag{1.7}$$

As detailed in Lemma 3.1, in the limit  $\varepsilon \rightarrow 0$  and in our chosen basis,  $U^\varepsilon(r)$  takes the form

$$U^\varepsilon(r) = \begin{pmatrix} u_1 \\ \varepsilon^r u_2 \\ \vdots \\ \varepsilon^{n-1} r^{n-1} u_n \end{pmatrix} + \begin{pmatrix} O(\varepsilon) \\ O(\varepsilon^2) \\ \vdots \\ O(\varepsilon^n) \end{pmatrix} ,$$

where  $u_k$  is a  $(d_k - d_{k-1}) \times m$  matrix with constant entries. Here we use the convention that  $d_0 = 0$ . Let  $D_\varepsilon$  and  $J_t$  be the  $d \times d$  diagonal matrices whose  $j^{\text{th}}$  diagonal element, for  $d_{k-1} < j \leq d_k$ , equals  $\varepsilon^{k-1}$  and  $t^{k-1/2}$ , respectively. The natural rescaled fluctuation process to study is  $(F_t^\varepsilon)_{t \in [0,1]}$  given by

$$F_t^\varepsilon = \varepsilon^{-1/2} D_\varepsilon^{-1} (z_t^\varepsilon(y) - \phi_t^\varepsilon(y)) , \tag{1.8}$$

where we show that the fluctuations indeed neither depend on  $x \in \mathbb{R}^d$  nor on  $y \in \mathbb{R}^d$ . As in [9] and due to (1.3), the orders of  $\varepsilon$  which we rescale the fluctuations by are determined in terms of a filtration induced by the commutator brackets of the vector fields  $X_0, X_1, \dots, X_m$ . To describe the limit fluctuation process, we set, for  $r \in \mathbb{R}$ ,

$$\hat{U}(r) = \begin{pmatrix} u_1 \\ ru_2 \\ \vdots \\ r^{n-1}u_n \end{pmatrix}, \tag{1.9}$$

and further introduce the  $d \times d$  matrix  $V$  which is an  $n \times n$  block matrix whose  $(k, l)^{\text{th}}$  block element  $V_{kl}$  is the  $(d_k - d_{k-1}) \times (d_l - d_{l-1})$  matrix given by

$$V_{kl} = (-1)^{l+1} u_k u_l^* \frac{(k-1)! (l-1)!}{(k+l-1)!}. \tag{1.10}$$

As established in Lemma 3.3, the matrix  $V$  is invertible. This allows us to describe the small-time fluctuations for the bridge process  $(z_t^\varepsilon(y))_{t \in [0,1]}$  as follows.

**Theorem 1.2.** *Let  $(W_t)_{t \in [0,1]}$  be a standard Brownian motion in  $\mathbb{R}^m$ . In the chosen basis of  $\mathbb{R}^d$ , let  $(F_t)_{t \in [0,1]}$  be the process defined by*

$$F_t = \int_0^t \hat{U}(t-s) dW_s - J_t V J_t V^{-1} \int_0^1 \hat{U}(1-s) dW_s.$$

*Then, for all  $x, y \in \mathbb{R}^d$ , the rescaled fluctuations  $(F_t^\varepsilon)_{t \in [0,1]}$  converge weakly to  $(F_t)_{t \in [0,1]}$  as  $\varepsilon \rightarrow 0$ .*

It is of interest by itself that after compensating for a blow-up in the process  $(z_t^\varepsilon(y))_{t \in [0,1]}$  through the path  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$ , the small-time fluctuations do not exhibit any further blow-ups as  $\varepsilon \rightarrow 0$ . Moreover, the example discussed in Section 4.2 demonstrates that, while the bridge processes and the rescaled fluctuations can always be computed explicitly due to the Gaussian nature of the considered diffusion, Theorem 1.2 indeed simplifies the determination of the small-time fluctuations for the bridge.

We observe that since  $D_\varepsilon, J_t, \hat{U}(r)$  and  $V$  are uniquely determined in terms of  $n \in \mathbb{N}$  and  $u_1, \dots, u_n$ , processes which give rise to the same  $n \in \mathbb{N}$  and  $u_1, \dots, u_n$  for the same orthonormal basis of  $\mathbb{R}^d$  exhibit the same small-time fluctuations for the bridge, according to Theorem 1.2. A formulation of this property in terms of the generator  $\mathcal{L}$  is given in Remark 3.4. It is similar to [9] where, in a suitable chart, the small-time fluctuations for sub-Riemannian diffusion loops only depend on the nilpotent approximations of the vector fields  $X_1, \dots, X_m$ . In addition, as shown in [4], the matrices  $u_1, \dots, u_n$  also characterise the leading-order term in the small-time heat kernel expansion on the diagonal, see Remark 3.5.

The paper is organised as follows. In Section 2, we discuss in more detail the hypoelliptic diffusions in our model class, and we derive an expression for the associated bridge processes. The small-time analysis, which leads to the proofs of Theorem 1.1 and Theorem 1.2, is then performed in Section 3. We close by presenting a collection of examples in Section 4 to illustrate our results. As part of the discussions in Section 4.4, we find an explicit expression for the bridge from 0 to 0 in time 1 of an iterated Kolmogorov diffusion.

## 2 Diffusion bridge in the model class

We analyse the diffusion processes whose generators are of the form (1.2) for matrices  $A$  and  $B$  satisfying condition (1.1). We further derive explicit expressions for the associ-

ated bridge processes. Let  $(W_t)_{t \in [0,1]}$  be a standard Brownian motion in  $\mathbb{R}^m$ , which we assume is realised as the coordinate process on the path space  $\{w \in C([0,1], \mathbb{R}^m) : w_0 = 0\}$  under Wiener measure  $\mathbb{P}$ . Fix  $x \in \mathbb{R}^d$ . For  $\varepsilon > 0$ , let  $(x_t^\varepsilon)_{t \in [0,1]}$  be the unique strong solution to the Itô stochastic differential equation in  $\mathbb{R}^d$

$$dx_t^\varepsilon = \varepsilon Ax_t^\varepsilon dt + \sqrt{\varepsilon} B dW_t, \quad x_0^\varepsilon = x.$$

We note that the process  $(x_t^\varepsilon)_{t \in [0,1]}$  has generator  $\varepsilon \mathcal{L}$ , where  $\mathcal{L}$  is given by (1.2). From the discussions in the Introduction, we know that operators of this form satisfy a weak Hörmander condition and hence, that  $(x_t^\varepsilon)_{t \in [0,1]}$  is a hypoelliptic diffusion. It has the explicit expression

$$x_t^\varepsilon = e^{\varepsilon t A} x + e^{\varepsilon t A} \int_0^t e^{-\varepsilon s A} \sqrt{\varepsilon} B dW_s, \tag{2.1}$$

as can be checked by direct computation. We see that  $(x_t^\varepsilon)_{t \in [0,1]}$  is a Gaussian process with

$$\mathbb{E}[x_t^\varepsilon] = e^{\varepsilon t A} x \tag{2.2}$$

and whose covariance structure is given as follows in terms of  $\Gamma_t^\varepsilon$  defined by (1.4).

**Lemma 2.1.** *For  $t_1, t_2 \in [0, 1]$  with  $t_1 \leq t_2$ , we have*

$$\text{cov}(x_{t_1}^\varepsilon, x_{t_2}^\varepsilon) = \varepsilon e^{\varepsilon t_1 A} \Gamma_{t_1}^\varepsilon e^{\varepsilon t_2 A^*}.$$

*Proof.* Using the expression (2.1), the property (2.2) and the Itô isometry, we obtain

$$\begin{aligned} \text{cov}(x_{t_1}^\varepsilon, x_{t_2}^\varepsilon) &= \mathbb{E} \left[ (x_{t_1}^\varepsilon - \mathbb{E}[x_{t_1}^\varepsilon]) (x_{t_2}^\varepsilon - \mathbb{E}[x_{t_2}^\varepsilon])^* \right] \\ &= \mathbb{E} \left[ e^{\varepsilon t_1 A} \int_0^{t_1} e^{-\varepsilon s A} \sqrt{\varepsilon} B dW_s \left( \int_0^{t_2} e^{-\varepsilon s A} \sqrt{\varepsilon} B dW_s \right)^* e^{\varepsilon t_2 A^*} \right] \\ &= \varepsilon e^{\varepsilon t_1 A} \left( \int_0^{t_1} e^{-\varepsilon s A} B B^* e^{-\varepsilon s A^*} ds \right) e^{\varepsilon t_2 A^*} \\ &= \varepsilon e^{\varepsilon t_1 A} \Gamma_{t_1}^\varepsilon e^{\varepsilon t_2 A^*}, \end{aligned}$$

as claimed. □

With the covariance structure for the Gaussian process  $(x_t^\varepsilon)_{t \in [0,1]}$  at hand, we can find an explicit expression for the corresponding bridge processes. The derivation relies on the fact that Gaussian random variables are independent if and only if they are uncorrelated.

**Lemma 2.2.** *For  $t \in [0, 1]$ , set*

$$\alpha_t^\varepsilon = e^{\varepsilon t A} \Gamma_t^\varepsilon (\Gamma_1^\varepsilon)^{-1} e^{-\varepsilon A}. \tag{2.3}$$

*Then, for  $y \in \mathbb{R}^d$ , the stochastic process  $(z_t^\varepsilon(y))_{t \in [0,1]}$  in  $\mathbb{R}^d$  given by*

$$z_t^\varepsilon(y) = x_t^\varepsilon - \alpha_t^\varepsilon (x_1^\varepsilon - y)$$

*has the same law as the process  $(x_t^\varepsilon)_{t \in [0,1]}$  conditioned on  $x_1^\varepsilon = y$ .*

*Proof.* For all  $t \in [0, 1]$ , we can write

$$x_t^\varepsilon = z_t^\varepsilon(0) + \alpha_t^\varepsilon x_1^\varepsilon. \tag{2.4}$$

Applying Lemma 2.1, we compute that

$$\begin{aligned} \text{cov}(z_t^\varepsilon(0), x_1^\varepsilon) &= \text{cov}(x_t^\varepsilon - \alpha_t^\varepsilon x_1^\varepsilon, x_1^\varepsilon) = \text{cov}(x_t^\varepsilon, x_1^\varepsilon) - \alpha_t^\varepsilon \text{cov}(x_1^\varepsilon, x_1^\varepsilon) \\ &= \varepsilon e^{\varepsilon t A} \Gamma_t^\varepsilon e^{\varepsilon A^*} - \varepsilon \alpha_t^\varepsilon e^{\varepsilon A} \Gamma_1^\varepsilon e^{\varepsilon A^*} = 0. \end{aligned}$$

Since  $(z_t^\varepsilon(0))_{t \in [0,1]}$  and  $x_1^\varepsilon$  are both Gaussian, their vanishing covariance implies that  $(z_t^\varepsilon(0))_{t \in [0,1]}$  and  $x_1^\varepsilon$  are independent. Thus, from the representation (2.4) it follows that the bridge obtained by conditioning the process  $(x_t^\varepsilon)_{t \in [0,1]}$  on  $x_1^\varepsilon = y$  can be expressed, at time  $t \in [0, 1]$ , as

$$z_t^\varepsilon(0) + \alpha_t^\varepsilon y,$$

which equals  $z_t^\varepsilon(y)$ . □

We observe that the path  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$  in  $\mathbb{R}^d$  defined by (1.5) rewrites as

$$\phi_t^\varepsilon(y) = e^{\varepsilon t A} x + \alpha_t^\varepsilon (y - e^{\varepsilon A} x). \tag{2.5}$$

Hence, the expression (2.1) and Lemma 2.2 imply

$$z_t^\varepsilon(y) - \phi_t^\varepsilon(y) = e^{\varepsilon t A} \int_0^t e^{-\varepsilon s A} \sqrt{\varepsilon} B dW_s - \alpha_t^\varepsilon \left( e^{\varepsilon A} \int_0^1 e^{-\varepsilon s A} \sqrt{\varepsilon} B dW_s \right). \tag{2.6}$$

The analysis of this expression in the limit  $\varepsilon \rightarrow 0$  is performed in the next section.

### 3 Small-time analysis for the model diffusion bridge

We study the dependence of  $e^{\varepsilon r A} B$  and  $\alpha_t^\varepsilon$  given by (2.3) on  $\varepsilon \rightarrow 0$  and then use the expression (2.6) to give the proofs of Theorem 1.1 and Theorem 1.2. Recall from (1.7) that, for  $r \in \mathbb{R}$ , we define

$$U^\varepsilon(r) = e^{\varepsilon r A} B.$$

In a suitable basis,  $U^\varepsilon(r)$  takes the following form.

**Lemma 3.1.** *Let  $\{e_1, \dots, e_d\}$  be an orthonormal basis of  $\mathbb{R}^d$  such that  $\{e_1, \dots, e_{d_k}\}$  is a basis of the subspace  $E_k$  given by (1.6), for  $k \in \{1, \dots, n\}$ . In such a basis,  $U^\varepsilon(r)$  has the form, as  $\varepsilon \rightarrow 0$ ,*

$$U^\varepsilon(r) = \begin{pmatrix} u_1 \\ \varepsilon r u_2 \\ \vdots \\ \varepsilon^{n-1} r^{n-1} u_n \end{pmatrix} + \begin{pmatrix} O(\varepsilon) \\ O(\varepsilon^2) \\ \vdots \\ O(\varepsilon^n) \end{pmatrix}, \tag{3.1}$$

uniformly in  $r$  on compact intervals, where  $u_k$  is a  $(d_k - d_{k-1}) \times m$  matrix with constant entries.

*Proof.* Write  $\langle \cdot, \cdot \rangle$  for the standard inner product on  $\mathbb{R}^d$ . Since  $E_k$  is the subspace of  $\mathbb{R}^d$  spanned by the columns of  $A^l B$  for  $l \in \{0, \dots, k-1\}$ , these columns can be written as a linear combination of the vectors  $e_1, \dots, e_{d_k}$ . It follows that, for  $j \in \{d_k + 1, \dots, d\}$  and for all  $v \in \mathbb{R}^m$ ,

$$\langle e_j, A^l B v \rangle = 0 \quad \text{for } l \in \{0, \dots, k-1\}. \tag{3.2}$$

Due to the properties of the matrix exponential, we have, as  $\varepsilon \rightarrow 0$ ,

$$e^{\varepsilon r A} = \sum_{l=0}^{k-1} \frac{1}{l!} (\varepsilon r A)^l + \frac{1}{k!} (\varepsilon r A)^k + O(\varepsilon^{k+1}),$$

uniformly in  $r \in \mathbb{R}$  on compact intervals. By using the relation (3.2) we obtain that, for all  $j \in \{d_k + 1, \dots, d\}$  and all  $v \in \mathbb{R}^m$ ,

$$\begin{aligned} \langle e_j, U^\varepsilon(r)v \rangle &= \langle e_j, e^{\varepsilon r A} B v \rangle = \sum_{l=0}^{k-1} \frac{\varepsilon^l r^l}{l!} \langle e_j, A^l B v \rangle + \frac{\varepsilon^k r^k}{k!} \langle e_j, A^k B v \rangle + O(\varepsilon^{k+1}) \\ &= \frac{\varepsilon^k r^k}{k!} \langle e_j, A^k B v \rangle + O(\varepsilon^{k+1}), \end{aligned} \tag{3.3}$$

uniformly in  $r$  on compact intervals. This establishes  $U^\varepsilon(r)$  is indeed of the form (3.1). □

We work in such an orthonormal basis of  $\mathbb{R}^d$  which respects the filtration of subspaces  $\{E_k\}_{1 \leq k \leq n}$  for the remainder of the section. According to Lemma 3.1, for the rescaling matrix  $D_\varepsilon$  and for  $\hat{U}(r)$  defined by (1.9), we have

$$U^\varepsilon(r) = D_\varepsilon \left( \hat{U}(r) + O(\varepsilon) \right), \tag{3.4}$$

uniformly in  $r \in \mathbb{R}$  on compact intervals. We deduce that, uniformly in  $t \in [0, 1]$ ,

$$\begin{aligned} e^{\varepsilon t A} \Gamma_t^\varepsilon &= e^{\varepsilon t A} \int_0^t e^{-\varepsilon s A} B B^* e^{-\varepsilon s A^*} ds = \int_0^t U^\varepsilon(t-s) U^\varepsilon(-s)^* ds \\ &= D_\varepsilon \left( \int_0^t \hat{U}(t-s) \hat{U}(-s)^* ds + O(\varepsilon) \right) D_\varepsilon. \end{aligned} \tag{3.5}$$

We use the following lemma to obtain a concise expression of  $\int_0^t \hat{U}(t-s) \hat{U}(-s)^* ds$ , for  $t \in [0, 1]$ , in terms of  $u_1, \dots, u_n$ .

**Lemma 3.2.** *For  $k, l \in \mathbb{N}$  and for all  $t \in [0, 1]$ , we have*

$$\int_0^t (t-s)^{k-1} (-s)^{l-1} ds = (-1)^{l-1} \frac{(k-1)! (l-1)!}{(k+l-1)!} t^{k+l-1}.$$

Making use of the beta function, which for  $a, b \in \mathbb{R}$  with  $a, b > 0$  is defined by

$$B(a, b) = \int_0^1 s^{a-1} (1-s)^{b-1} ds,$$

the above lemma follows from the relationship

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{3.6}$$

between the beta function and the gamma function after a change of variable. For a proof of (3.6), see [1, pp. 18–19]. For completeness, we present an alternative proof of Lemma 3.2 which relies on the assumption that  $k, l \in \mathbb{N}$ .

*Proof of Lemma 3.2.* We prove the identity by induction over  $k \in \mathbb{N}$  with  $l \in \mathbb{N}$  fixed. For  $k = 1$ , we compute

$$\int_0^t (-s)^{l-1} ds = \frac{(-1)^{l-1} t^l}{l} = \frac{(-1)^{l-1} (l-1)!}{l!} t^l,$$

which settles the base case for all  $t \in [0, 1]$ . To establish the induction step, consider the functions  $f_k, g_k : [0, 1] \rightarrow \mathbb{R}$  defined by

$$f_k(t) = \int_0^t (t-s)^{k-1} (-s)^{l-1} ds \quad \text{and} \quad g_k(t) = (-1)^{l-1} \frac{(k-1)! (l-1)!}{(k+l-1)!} t^{k+l-1}.$$

We have

$$\frac{d}{dt} f_k(t) = (k-1) \int_0^t (t-s)^{k-2} (-s)^{l-1} ds$$

as well as

$$\frac{d}{dt} g_k(t) = (-1)^{l-1} \frac{(k-1)! (l-1)!}{(k+l-2)!} t^{k+l-2}.$$

The induction hypothesis implies that

$$\frac{d}{dt} f_k(t) = (k-1) f_{k-1} = (k-1) g_{k-1} = \frac{d}{dt} g_k(t) \tag{3.7}$$

for all  $t \in [0, 1]$ . Due to  $f_k(0) = 0 = g_k(0)$ , the result follows upon integrating (3.7).  $\square$

For  $t \in [0, 1]$ , the matrix  $\int_0^t \hat{U}(t-s)\hat{U}(-s)^* ds$  is an  $n \times n$  block matrix whose  $(k, l)^{\text{th}}$  block element is the  $(d_k - d_{k-1}) \times (d_l - d_{l-1})$  matrix

$$u_k u_l^* \int_0^t (t-s)^{k-1} (-s)^{l-1} ds .$$

Using Lemma 3.2 we deduce that, with the  $n \times n$  block matrix  $V$  defined by (1.10) and the rescaling matrix  $J_t$ ,

$$\int_0^t \hat{U}(t-s)\hat{U}(-s)^* ds = J_t V J_t . \tag{3.8}$$

Following on from (3.5), we end up with the expression

$$e^{\varepsilon t A} \Gamma_t^\varepsilon = D_\varepsilon \left( \int_0^t \hat{U}(t-s)\hat{U}(-s)^* ds + O(\varepsilon) \right) D_\varepsilon = D_\varepsilon J_t (V + O(\varepsilon)) J_t D_\varepsilon , \tag{3.9}$$

uniformly in  $t \in [0, 1]$ . To use (3.9) to obtain an alternative expression for  $\alpha_t^\varepsilon$ , we first show that the square matrix  $V$  is invertible.

**Lemma 3.3.** *The  $n \times n$  block matrix  $V$  whose  $(k, l)^{\text{th}}$  block element is given by (1.10) is invertible.*

*Proof.* As shown in [14, Proposition 2.1], in our chosen basis of  $\mathbb{R}^d$ , the matrix  $A$  takes the form of an  $n \times n$  block matrix whose  $(k, l)^{\text{th}}$  block element, for  $k, l \in \{1, \dots, n\}$ , is a  $(d_k - d_{k-1}) \times (d_l - d_{l-1})$  matrix, where all the blocks with  $k \geq l + 2$  vanish. Let  $\hat{A}$  be an  $n \times n$  block matrix of the same block structure. We set its block elements to zero unless  $k = l + 1$ , in which case we set that block element to equal the  $(k, l)^{\text{th}}$  block element of  $A$ . By definition of the subspace  $E_1$  of  $\mathbb{R}^d$ , we further observe that in our chosen basis, for all  $j \in \{d_1 + 1, \dots, d\}$ , the  $j^{\text{th}}$  row of  $B$  vanishes.

For  $l \in \{1, \dots, n - 1\}$ , let  $A_l$  denote the  $(l + 1, l)^{\text{th}}$  block element of the matrix  $A$  and let  $B_1$  be the  $d_1 \times m$  matrix obtained by considering the first  $d_1$  rows of  $B$  only. For  $k \in \{1, \dots, n\}$ , we set

$$\hat{E}_k = \text{span} \left\{ \hat{A}^l B v : v \in \mathbb{R}^m, 0 \leq l \leq k - 1 \right\} .$$

By construction of  $\hat{A}$ , the  $d \times m$  matrix  $\hat{A}^l B$  is an  $n \times 1$  block matrix, whose  $(k, 1)^{\text{th}}$  block element is a  $(d_k - d_{k-1}) \times m$  matrix, which vanishes unless  $k = l + 1$ , in which case it equals  $A_l \cdots A_1 B_1$ . From this form it follows that, in the chosen basis  $\{e_1, \dots, e_d\}$  of  $\mathbb{R}^d$ , we have, for all  $l \in \{0, \dots, n - 1\}$  and all  $v \in \mathbb{R}^m$ ,

$$\langle e_j, \hat{A}^l B v \rangle = 0 \quad \text{unless } j \in \{d_l + 1, \dots, d_{l+1}\} . \tag{3.10}$$

Moreover, for  $l \geq n$ , we obtain  $\hat{A}^l B = 0$ , which implies that, for  $r \in \mathbb{R}$ ,

$$e^{\varepsilon r \hat{A}} B = \sum_{l=0}^{n-1} \frac{\varepsilon^l r^l}{l!} \hat{A}^l B . \tag{3.11}$$

Combining (3.10) and (3.11) yields, for all  $v \in \mathbb{R}^m$ ,

$$\langle e_j, e^{\varepsilon r \hat{A}} B v \rangle = \frac{\varepsilon^l r^l}{l!} \langle e_j, \hat{A}^l B v \rangle \quad \text{for } j \text{ with } d_l < j \leq d_{l+1} . \tag{3.12}$$

After understanding  $A^l B$  as an  $n \times 1$  block matrix of the same structure as the matrix  $\hat{A}^l B$ , we further see that the  $(l + 1, 1)^{\text{th}}$  block element of  $A^l B$  also equals  $A_l \cdots A_1 B_1$ .



This is a consequence of the observation that a block element in  $A$  with  $k \geq l + 2$  vanishes. In particular, for  $v \in \mathbb{R}^m$  and  $j$  with  $d_l < j \leq d_{l+1}$ , we have

$$\langle e_j, \hat{A}^l Bv \rangle = \langle e_j, A^l Bv \rangle ,$$

and (3.3) together with (3.12) implies that, for  $r \in \mathbb{R}$ ,

$$\hat{U}(r) = e^{r\hat{A}} B .$$

Using (3.8), we conclude that

$$V = \int_0^1 \hat{U}(1-s)\hat{U}(-s)^* ds = e^{\hat{A}} \int_0^1 e^{-s\hat{A}} BB^* e^{-s\hat{A}^*} ds .$$

Our discussion above shows that  $E_k = \hat{E}_k$ , for all  $k \in \{1, \dots, n\}$ , and especially  $\hat{E}_n = \mathbb{R}^d$ . Therefore, the matrices  $\hat{A}$  and  $B$  satisfy the Kalman rank condition, which ensures that

$$\int_0^1 e^{-s\hat{A}} BB^* e^{-s\hat{A}^*} ds$$

is invertible. Since  $e^{\hat{A}}$  has the matrix inverse  $e^{-\hat{A}}$ , the invertibility of  $V$  follows. □

For completeness, we note that Lemma 3.3 implies that, for all  $k \in \{1, \dots, n\}$ , the matrix  $u_k$  has maximal rank. If it did not then, since  $d_k - d_{k-1} \leq m$  by construction, its rows would be linearly dependent leading to  $V$  having a collection of linearly dependent rows, which is not possible.

**Remark 3.4.** Let  $\hat{A}$  be the  $d \times d$  matrix constructed from the matrix  $A$  as in the previous proof, and let  $\hat{\mathcal{L}}$  be the operator on  $\mathbb{R}^d$  given by

$$\hat{\mathcal{L}} = \sum_{j=1}^d (\hat{A}x)_j \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{j,k=1}^d (BB^*)_{jk} \frac{\partial^2}{\partial x_j \partial x_k} .$$

In [14], the operator  $\hat{\mathcal{L}} - \frac{\partial}{\partial t}$  is called the principal part of  $\mathcal{L} - \frac{\partial}{\partial t}$ , and it is shown that the fundamental solution with pole at zero of  $\mathcal{L} - \frac{\partial}{\partial t}$  can be controlled in terms of the fundamental solution with pole at zero of  $\hat{\mathcal{L}} - \frac{\partial}{\partial t}$ , cf. [14, Theorem 3.1]. Similarly, let us call  $\hat{\mathcal{L}}$  the principal part of  $\mathcal{L}$ .

In our model class of hypoelliptic diffusions the small-time fluctuations for the bridge are given by Theorem 1.2 in terms of  $D_\varepsilon, J_t, \hat{U}(r)$  and  $V$ , which due to the proof of Lemma 3.3 can be uniquely determined from  $\hat{A}$  and  $B$ . Therefore, the small-time fluctuations for the bridge are fully governed by the principal part  $\hat{\mathcal{L}}$  of the generator  $\mathcal{L}$ . A similar property was observed in [9].

**Remark 3.5.** Consider the  $d \times d$  matrix

$$Z = \int_0^1 \hat{U}(-s)\hat{U}(-s)^* ds ,$$

which is the  $n \times n$  block matrix whose  $(k, l)^{\text{th}}$  block element  $Z_{kl}$  is the  $(d_k - d_{k-1}) \times (d_l - d_{l-1})$  matrix given by

$$Z_{kl} = \frac{(-1)^{k+l} u_k u_l^*}{k + l - 1} ,$$

for  $k, l \in \{1, \dots, n\}$ , and set

$$\mathcal{N} = \sum_{k=1}^n (2k - 1) (d_k - d_{k-1}) .$$

Barilari and Paoli [4, Theorem 1.1 and Theorem 1.3] show that the small-time heat kernel expansion on the diagonal at  $x \in \mathbb{R}^d$  has polynomial decay if  $Ax \in E_1$  with leading-order term

$$\frac{\varepsilon^{-N/2}}{(2\pi)^{d/2} (\det Z)^{1/2}} \quad \text{as } \varepsilon \rightarrow 0,$$

and has exponential decay to zero otherwise. If  $Ax = 0$ , they further provide a characterisation of all the coefficients in the small-time heat kernel expansion which is analogous to the one obtained for Riemannian manifolds.

We now proceed with our analysis to find an alternative expression for  $\alpha_t^\varepsilon$ . Since the set of invertible matrices is open, Lemma 3.3 shows that, for  $\varepsilon > 0$  sufficiently small, the inverse of  $V + O(\varepsilon)$  exists. It satisfies

$$(V + O(\varepsilon))^{-1} = V^{-1} + O(\varepsilon).$$

From (3.9) and as  $J_1$  equals the identity matrix, it follows that

$$(\Gamma_1^\varepsilon)^{-1} e^{-\varepsilon A} = D_\varepsilon^{-1} (V^{-1} + O(\varepsilon)) D_\varepsilon^{-1},$$

which yields

$$\alpha_t^\varepsilon = e^{\varepsilon t A} \Gamma_t^\varepsilon (\Gamma_1^\varepsilon)^{-1} e^{-\varepsilon A} = D_\varepsilon (J_t V J_t V^{-1} + O(\varepsilon)) D_\varepsilon^{-1}, \tag{3.13}$$

uniformly in  $t \in [0, 1]$ . The two estimates (3.4) and (3.13) are the essential ingredients for proving Theorem 1.1 and Theorem 1.2.

*Proof of Theorem 1.2.* Using (1.7), we can rewrite (2.6) as

$$z_t^\varepsilon(y) - \phi_t^\varepsilon(y) = \sqrt{\varepsilon} \left( \int_0^t U^\varepsilon(t-s) dW_s - \alpha_t^\varepsilon \int_0^1 U^\varepsilon(1-s) dW_s \right), \tag{3.14}$$

and therefore,

$$F_t^\varepsilon = \varepsilon^{-1/2} D_\varepsilon^{-1} (z_t^\varepsilon(y) - \phi_t^\varepsilon(y)) = \int_0^t D_\varepsilon^{-1} U^\varepsilon(t-s) dW_s - D_\varepsilon^{-1} \alpha_t^\varepsilon D_\varepsilon \int_0^1 D_\varepsilon^{-1} U^\varepsilon(1-s) dW_s.$$

The estimate (3.4) gives

$$\sup_{r \in [0,1]} \left\| D_\varepsilon^{-1} U^\varepsilon(r) - \hat{U}(r) \right\| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

whereas (3.13) implies that

$$\sup_{t \in [0,1]} \left\| D_\varepsilon^{-1} \alpha_t^\varepsilon D_\varepsilon - J_t V J_t V^{-1} \right\| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Hence, the covariances of the mean-zero Gaussian processes  $(F_t^\varepsilon)_{t \in [0,1]}$  converge uniformly as  $\varepsilon \rightarrow 0$  to the covariance of the mean-zero Gaussian process  $(F_t)_{t \in [0,1]}$  given by

$$F_t = \int_0^t \hat{U}(t-s) dW_s - J_t V J_t V^{-1} \int_0^1 \hat{U}(1-s) dW_s.$$

From [13, Section 3], it follows that the rescaled fluctuations  $(F_t^\varepsilon)_{t \in [0,1]}$  indeed converge weakly to  $(F_t)_{t \in [0,1]}$  as  $\varepsilon \rightarrow 0$ . □

*Proof of Theorem 1.1.* Since the rescaled fluctuations  $(F_t^\varepsilon)_{t \in [0,1]}$  defined by

$$F_t^\varepsilon = \varepsilon^{-1/2} D_\varepsilon^{-1} (z_t^\varepsilon(y) - \phi_t^\varepsilon(y))$$

converge weakly as  $\varepsilon \rightarrow 0$  to a well-defined limit process, the non-rescaled processes  $(z_t^\varepsilon(y) - \phi_t^\varepsilon(y))_{t \in [0,1]}$  converge weakly as  $\varepsilon \rightarrow 0$  to the zero process on the set  $\Omega^{0,0}$ . □

The next remark shows that if  $\text{rank } B < d$  there exist  $x, y \in \mathbb{R}^d$  such that the path  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$  exhibits a blow-up behaviour in the limit  $\varepsilon \rightarrow 0$ . As  $\text{rank } B < d$  holds for those hypoelliptic diffusions in our model class which do not satisfy the strong Hörmander condition, any such diffusion features bridge processes which blow up in the small time limit. Specific examples illustrating this phenomena are given in Section 4.

**Remark 3.6.** Suppose  $\text{rank } B < d$ , that is, we have  $n \geq 2$ . In this setting, the rows of  $U^\varepsilon(r)$  have to be linearly dependent as  $e^{\varepsilon r A}$  is invertible for  $\varepsilon > 0$  and  $r \in \mathbb{R}$ . Moreover, for all  $k \in \{1, \dots, n\}$ , the rows of  $u_k$  are linearly independent because  $u_k$  has maximal rank and  $d_k - d_{k-1} \leq m$ . Hence, due to (1.10), the  $n \times n$  block matrix  $V$  has off-diagonal block elements which are non-zero.

We further argue that  $J_t V J_t V^{-1}$  considered as an  $n \times n$  block matrix cannot be lower triangular for all  $t \in [0, 1]$ . If it was then, as a consequence of the entries of  $J_t V J_t V^{-1}$  being polynomials in  $t$ , all coefficients in the strictly upper diagonal blocks would vanish. In particular, for all  $k, l \in \{1, \dots, n\}$  with  $k < l$ , we would have

$$V_{kk} (V^{-1})_{kl} = 0 .$$

However, for all  $k \in \{1, \dots, n\}$ , the  $(d_k - d_{k-1}) \times (d_k - d_{k-1})$  matrix  $V_{kk}$  is invertible because  $u_k u_k^*$  has the same rank as the matrix  $u_k$  of maximal rank. Thus,  $V^{-1}$  would be a lower triangular  $n \times n$  block matrix which together with  $V_{kl}^* = (-1)^{k+l} V_{lk}$  would contradict the observation that  $V$  has non-zero off-diagonal block elements.

Let  $x \in \mathbb{R}^d$  be an equilibrium point for the drift, that is, it satisfies  $Ax = 0$ . Then  $e^{\varepsilon r A} x = x$  for all  $r \in \mathbb{R}$ , and the expression (2.5) for the path  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$  as well as the estimate (3.13) yield

$$\phi_t^\varepsilon(y) = x + D_\varepsilon (J_t V J_t V^{-1} + O(\varepsilon)) D_\varepsilon^{-1} (y - x) ,$$

uniformly in  $t \in [0, 1]$ . We see if  $y - x \in E_1$  then the path  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$  remains finite as  $\varepsilon \rightarrow 0$ . On the other hand, since  $J_t V J_t V^{-1}$  cannot be a lower triangular block matrix for all  $t \in [0, 1]$  and due to  $x = 0$  always being an equilibrium point for the drift, if  $\text{rank } B < d$ , we can find  $x, y \in \mathbb{R}^d$  such that  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$  exhibits a blow-up behaviour in the limit  $\varepsilon \rightarrow 0$ .

Before we move on to a discussion of four examples in the following section, we make an observation regarding the process

$$\left( \int_0^t \hat{U}(t-s) dW_s \right)_{t \in [0,1]} . \tag{3.15}$$

By integration by parts, we have that, for  $k \in \mathbb{N}$ ,

$$\int_0^t (t-s)^k dW_s = k! \int_0^t \int_0^{s_k} \dots \int_0^{s_2} W_{s_1} ds_1 \dots ds_k .$$

Thus, the process (3.15) can be expressed solely in terms of the matrices  $u_1, \dots, u_n$  and an iterated Kolmogorov diffusion, that is, a standard Brownian motion together with a finite number of its iterated time integrals. Since the iterated Kolmogorov diffusion arises as a canonical example, we determine its small-time fluctuations for the bridge in Section 4.4. The Kolmogorov diffusion is discussed separately as a first example in Section 4.1 as it already exhibits interesting features.

## 4 Illustrating examples

We discuss four examples which illustrate different aspects of Theorem 1.1 and Theorem 1.2.

**4.1 Kolmogorov diffusion**

The Kolmogorov diffusion, named after Kolmogorov [12], is the simplest example of a stochastic process which satisfies a weak Hörmander condition but not the strong Hörmander condition. It is the diffusion  $(x_t)_{t \in [0,1]}$  in  $\mathbb{R}^2$  which pairs a standard Brownian motion  $(W_t)_{t \in [0,1]}$  in  $\mathbb{R}$  with its time integral, that is,

$$x_t = \left( W_t, \int_0^t W_s \, ds \right) .$$

It is the unique strong solution to the stochastic differential equation

$$\begin{aligned} d(x_t)_1 &= dW_t , \\ d(x_t)_2 &= (x_t)_1 \, dt , \end{aligned}$$

subject to  $x_0 = 0$ . This process falls into our model class of hypoelliptic diffusions by taking

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

which corresponds to the operator

$$\mathcal{L} = x_1 \frac{\partial}{\partial x_2} + \frac{1}{2} \left( \frac{\partial}{\partial x_1} \right)^2$$

on  $\mathbb{R}^2$ . The Kalman rank condition (1.1) is satisfied because

$$AB = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

implies  $E_2 = \mathbb{R}^2$ . We first use Lemma 2.2 to determine the expressions for the associated diffusion bridges in small time to then explicitly see that Theorem 1.1 and Theorem 1.2 hold. For  $\varepsilon > 0$ , the rescaled Kolmogorov diffusion  $(x_t^\varepsilon)_{t \in [0,1]}$  with generator  $\varepsilon \mathcal{L}$  is given by

$$x_t^\varepsilon = \left( \varepsilon^{1/2} W_t, \varepsilon^{3/2} \int_0^t W_s \, ds \right) .$$

Since  $A^2 = 0$ , we obtain, for  $r \in \mathbb{R}$ ,

$$e^{\varepsilon r A} = I + \varepsilon r A = \begin{pmatrix} 1 & 0 \\ \varepsilon r & 1 \end{pmatrix} .$$

We further compute, for  $t \in [0, 1]$ ,

$$\Gamma_t^\varepsilon = \int_0^t e^{-\varepsilon s A} B B^* e^{-\varepsilon s A^*} \, ds = \begin{pmatrix} t & -\frac{1}{2} \varepsilon t^2 \\ -\frac{1}{2} \varepsilon t^2 & \frac{1}{3} \varepsilon^2 t^3 \end{pmatrix} .$$

It follows that

$$e^{\varepsilon t A} \Gamma_t^\varepsilon = \begin{pmatrix} t & -\frac{1}{2} \varepsilon t^2 \\ \frac{1}{2} \varepsilon t^2 & -\frac{1}{6} \varepsilon^2 t^3 \end{pmatrix} \quad \text{and} \quad (\Gamma_1^\varepsilon)^{-1} e^{-\varepsilon A} = \begin{pmatrix} -2 & 6\varepsilon^{-1} \\ -6\varepsilon^{-1} & 12\varepsilon^{-2} \end{pmatrix} ,$$

which implies

$$\alpha_t^\varepsilon = e^{\varepsilon t A} \Gamma_t^\varepsilon (\Gamma_1^\varepsilon)^{-1} e^{-\varepsilon A} = \begin{pmatrix} 3t^2 - 2t & (6t - 6t^2) \varepsilon^{-1} \\ (t^3 - t^2) \varepsilon & 3t^2 - 2t^3 \end{pmatrix} .$$

Thus, by Lemma 2.2 and for  $y = (a, b) \in \mathbb{R}^2$ , the process  $(z_t^\varepsilon(a, b))_{t \in [0,1]}$  in  $\mathbb{R}^2$  given by

$$\begin{aligned} (z_t^\varepsilon(a, b))_1 &= (3t^2 - 2t) a + (6t - 6t^2) \frac{b}{\varepsilon} \\ &\quad + \varepsilon^{1/2} \left( W_t - (3t^2 - 2t) W_1 - (6t - 6t^2) \int_0^1 W_s ds \right), \\ (z_t^\varepsilon(a, b))_2 &= (t^3 - t^2) a\varepsilon + (3t^2 - 2t^3) b \\ &\quad + \varepsilon^{3/2} \left( \int_0^t W_s ds - (t^3 - t^2) W_1 - (3t^2 - 2t^3) \int_0^1 W_s ds \right) \end{aligned}$$

has the same law as the rescaled Kolmogorov diffusion  $(x_t^\varepsilon)_{t \in [0,1]}$  conditioned to have  $x_1^\varepsilon = (a, b)$ . From the explicit expression, it follows that the processes

$$\left( (z_t^\varepsilon(a, b))_1 - (3t^2 - 2t) a - (6t - 6t^2) \frac{b}{\varepsilon}, (z_t^\varepsilon(a, b))_2 - (t^3 - t^2) a\varepsilon - (3t^2 - 2t^3) b \right)_{t \in [0,1]} \tag{4.1}$$

converge weakly as  $\varepsilon \rightarrow 0$  to the zero process. This is consistent with Theorem 1.1 because for the Kolmogorov diffusion starting from  $x = 0$ , we have

$$\phi_t^\varepsilon(y) = \alpha_t^\varepsilon y = \left( (3t^2 - 2t) a + (6t - 6t^2) \frac{b}{\varepsilon}, (t^3 - t^2) a\varepsilon + (3t^2 - 2t^3) b \right).$$

We note that while the path  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$  is well-defined for each  $\varepsilon > 0$ , its first component blows up as  $\varepsilon \rightarrow 0$ , unless  $b = 0$ . From the above expression for a Kolmogorov bridge from 0 to  $(a, b)$  in small time, we further see that rescaling the processes (4.1) by  $\varepsilon^{1/2}$  in the first component and by  $\varepsilon^{3/2}$  in the second component leads to the fluctuation process which, at  $t \in [0, 1]$ , has first component

$$W_t - (3t^2 - 2t) W_1 - (6t - 6t^2) \int_0^1 W_s ds$$

and second component

$$\int_0^t W_s ds - (t^3 - t^2) W_1 - (3t^2 - 2t^3) \int_0^1 W_s ds.$$

Since this coincides with the expression for  $z_t^1(0)$ , the resulting limit fluctuations are equal in law to a Kolmogorov bridge from 0 to 0 in time 1. Below we conclude that this is also what is given to us by Theorem 1.2. For  $r \in \mathbb{R}$ , we have

$$U^\varepsilon(r) = e^{\varepsilon r A} B = \begin{pmatrix} 1 \\ \varepsilon r \end{pmatrix},$$

which is of the form (3.1) with  $u_1 = u_2 = 1$ . In particular, we already work in a suitable basis. The rescaling map  $D_\varepsilon$  is then

$$D_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}. \tag{4.2}$$

Since we consider the rescaled fluctuations  $(F_t^\varepsilon)_{t \in [0,1]}$  defined by (1.8) this corresponds to rescaling the first component by  $\varepsilon^{1/2}$  and the second component by  $\varepsilon^{3/2}$ , as above. We further obtain that

$$J = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{3/2} \end{pmatrix}, \quad \hat{U}(r) = \begin{pmatrix} 1 \\ r \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{6} \end{pmatrix}. \tag{4.3}$$

By integration by parts, we have

$$\int_0^t \hat{U}(t-s) dW_s = \left( W_t, \int_0^t (t-s) dW_s \right) = \left( W_t, \int_0^t W_s ds \right). \tag{4.4}$$

This together with the computation

$$J_t V J_t V^{-1} = \begin{pmatrix} t & -\frac{1}{2}t^2 \\ \frac{1}{2}t^2 & -\frac{1}{6}t^3 \end{pmatrix} \begin{pmatrix} -2 & 6 \\ -6 & 12 \end{pmatrix} = \begin{pmatrix} 3t^2 - 2t & 6t - 6t^2 \\ t^3 - t^2 & 3t^2 - 2t^3 \end{pmatrix} \tag{4.5}$$

shows that Theorem 1.2 indeed yields the same small-time fluctuations for a Kolmogorov bridge as derived above. Irrespective of the initial and final positions, the small-time fluctuations are equal in law to a Kolmogorov bridge from 0 to 0 in time 1.

#### 4.2 Ornstein-Uhlenbeck process paired with its area

Performing the small-time analysis for the bridge of an Ornstein-Uhlenbeck process paired with its area demonstrates that Theorem 1.2 can greatly simplify the determination of the small-time fluctuations for the bridge. Let  $(W_t)_{t \in [0,1]}$  be a standard Brownian motion in  $\mathbb{R}$  and fix  $x \in \mathbb{R}^2$ . We consider the diffusion  $(x_t)_{t \in [0,1]}$  in  $\mathbb{R}^2$  which is the unique strong solution to the stochastic differential equation

$$\begin{aligned} d(x_t)_1 &= -(x_t)_1 dt + dW_t, \\ d(x_t)_2 &= (x_t)_1 dt, \end{aligned}$$

subject to the initial condition  $x_0 = x$ . This corresponds to the choice

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

in our model class of diffusion processes. The matrices  $A$  and  $B$  satisfy condition (1.1) since

$$\text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = \mathbb{R}^2.$$

In the following, we first use Lemma 2.2 to find explicit expressions for the corresponding bridge processes in small time to then determine the small-time fluctuations for the bridge by hand, before we show that Theorem 1.2 greatly simplifies the analysis. Using  $A^k = (-1)^{k-1} A$  for  $k \in \mathbb{N}$ , we compute, for  $\varepsilon > 0$  and  $r \in \mathbb{R}$ ,

$$e^{\varepsilon r A} = \begin{pmatrix} e^{-\varepsilon r} & 0 \\ 1 - e^{-\varepsilon r} & 1 \end{pmatrix}.$$

It follows that, for  $t \in [0, 1]$ ,

$$\Gamma_t^\varepsilon = \begin{pmatrix} \frac{1}{2\varepsilon} (e^{2\varepsilon t} - 1) & -\frac{1}{2\varepsilon} (e^{\varepsilon t} - 1)^2 \\ -\frac{1}{2\varepsilon} (e^{\varepsilon t} - 1)^2 & \frac{1}{2\varepsilon} (e^{2\varepsilon t} - 4e^{\varepsilon t} + 2\varepsilon t + 3) \end{pmatrix},$$

which yields

$$e^{\varepsilon t A} \Gamma_t^\varepsilon = \begin{pmatrix} \frac{1}{2\varepsilon} (e^{\varepsilon t} - e^{-\varepsilon t}) & -\frac{1}{2\varepsilon} (e^{\varepsilon t} + e^{-\varepsilon t} - 2) \\ \frac{1}{2\varepsilon} (e^{\varepsilon t} + e^{-\varepsilon t} - 2) & -\frac{1}{2\varepsilon} (e^{\varepsilon t} - e^{-\varepsilon t} - 2\varepsilon t) \end{pmatrix}.$$

A straightforward but elaborate calculation shows that, for  $t \in [0, 1]$ , the matrix  $\alpha_t^\varepsilon$  is given by

$$\begin{aligned} (\alpha_t^\varepsilon)_{11} &= \frac{(1 - e^{-\varepsilon t}) ((\varepsilon - 1) e^{\varepsilon(1+t)} + e^{\varepsilon t} + (\varepsilon + 1) e^\varepsilon - e^{2\varepsilon})}{(e^\varepsilon - 1) ((\varepsilon - 2) e^\varepsilon + \varepsilon + 2)}, \\ (\alpha_t^\varepsilon)_{12} &= \frac{e^{-\varepsilon} - e^{-\varepsilon(1-t)} - e^{-\varepsilon t} + 1}{(\varepsilon + 2) e^{-\varepsilon} + \varepsilon - 2}, \\ (\alpha_t^\varepsilon)_{21} &= \frac{e^{2\varepsilon} - 1 + (\varepsilon + 1) e^{\varepsilon(1-t)} + e^{\varepsilon t} + (\varepsilon - 1) e^{\varepsilon(1+t)} - \varepsilon t (e^\varepsilon - 1)^2 - 2\varepsilon e^\varepsilon - e^{\varepsilon(2-t)}}{(e^\varepsilon - 1) ((\varepsilon - 2) e^\varepsilon + \varepsilon + 2)}, \\ (\alpha_t^\varepsilon)_{22} &= \frac{e^{-\varepsilon t} - e^{-\varepsilon(1-t)} + (\varepsilon t + 1) e^{-\varepsilon} + \varepsilon t - 1}{(\varepsilon + 2) e^{-\varepsilon} + \varepsilon - 2}. \end{aligned}$$

By Lemma 2.2, this gives an explicit expression for the bridge of the Ornstein-Uhlenbeck process considered paired with its area. Repeatedly applying l'Hôpital's rule, we see that, as  $\varepsilon \rightarrow 0$ ,

$$\begin{aligned} (\alpha_t^\varepsilon)_{11} &= 3t^2 - 2t + O(\varepsilon^2), \\ (\alpha_t^\varepsilon)_{12} &= \frac{6t - 6t^2}{\varepsilon} + O(\varepsilon), \\ (\alpha_t^\varepsilon)_{21} &= (t^3 - t^2)\varepsilon + O(\varepsilon^3), \\ (\alpha_t^\varepsilon)_{22} &= 3t^2 - 2t^3 + O(\varepsilon^2), \end{aligned}$$

uniformly in  $t \in [0, 1]$ . A comparison to the expressions obtained in the small-time analysis for the Kolmogorov diffusion in Section 4.1 implies that the Ornstein-Uhlenbeck process paired with its area exhibits the same small-time fluctuations for the bridge as the Kolmogorov diffusion. This follows much more easily by applying Theorem 1.2. For  $r \in \mathbb{R}$ , we have

$$U^\varepsilon(r) = \begin{pmatrix} e^{-\varepsilon r} \\ 1 - e^{-\varepsilon r} \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon r \end{pmatrix} + \begin{pmatrix} O(\varepsilon) \\ O(\varepsilon^2) \end{pmatrix}.$$

We see that  $U^\varepsilon(r)$  is of the form (3.1) with  $u_1 = u_2 = 1$ . Hence, the rescaling matrix  $D_\varepsilon$  as well as  $J_t, \hat{U}(r)$  and  $V$  are again given by (4.2) as well as (4.3). Similarly, the quantities (4.4) and (4.5), which characterise the small-time fluctuations uniquely, remain unchanged. Therefore, as a result of giving rise to the same  $\hat{U}(r)$  for all  $r \in \mathbb{R}$ , the Kolmogorov diffusion and the Ornstein-Uhlenbeck process paired with its area exhibit the same small-time fluctuations for the bridge.

For  $x = 0$  there is another interesting observation we can make in regards to these two processes, which is a consequence of certain terms vanishing in the Laurent expansion of  $\alpha_t^\varepsilon$  in  $\varepsilon \rightarrow 0$ . If we consider the path  $(\psi_t^\varepsilon(y))_{t \in [0,1]}$  defined by, for  $y = (a, b)$ ,

$$\psi_t^\varepsilon(y) = \left( (3t^2 - 2t) a + (6t - 6t^2) \frac{b}{\varepsilon}, (t^3 - t^2) a\varepsilon + (3t^2 - 2t^3) b \right)$$

then this is sufficient to compensate for the blow-up behaviour in the bridge process  $(z_t^\varepsilon(y))_{t \in [0,1]}$  as  $\varepsilon \rightarrow 0$ , and the two processes

$$\left( \varepsilon^{-1/2} D_\varepsilon^{-1} (z_t^\varepsilon(y) - \phi_t^\varepsilon(y)) \right)_{t \in [0,1]} \quad \text{and} \quad \left( \varepsilon^{-1/2} D_\varepsilon^{-1} (z_t^\varepsilon(y) - \psi_t^\varepsilon(y)) \right)_{t \in [0,1]}$$

have the same limit process as  $\varepsilon \rightarrow 0$ . Since the approximate minimal-like path  $(\psi_t^\varepsilon(y))_{t \in [0,1]}$  for the current example with  $x = 0$  coincides with the minimal-like path for a Kolmogorov bridge from 0 to  $y$  in small time, not only the small-time fluctuations for the bridge but also a sufficiently good approximation of the minimal-like path is given in terms of the Kolmogorov diffusion. Though, as shown in the next example, the latter need not hold for two processes which admit the same  $n \in \mathbb{N}$  and  $u_1, \dots, u_n$ .

### 4.3 Compensating for blow-ups in the bridge process

While the small-time fluctuations for the bridge are uniquely determined in terms of the matrices  $u_1, \dots, u_n$ , we present an example which shows that knowledge of  $u_1, \dots, u_n$  is not sufficient to construct a path which approximates the minimal-like path well enough to recover the limit fluctuations as in the previous section. We consider the hypoelliptic diffusion corresponding to the matrices

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which satisfy condition (1.1). Using the eigendecomposition of  $A$ , we obtain, for  $\varepsilon > 0$  and  $r \in \mathbb{R}$ ,

$$e^{\varepsilon r A} = \begin{pmatrix} e^{-\varepsilon r} & 0 \\ \frac{1}{3}(e^{2\varepsilon r} - e^{-\varepsilon r}) & e^{2\varepsilon r} \end{pmatrix}$$

as well as

$$U^\varepsilon(r) = \begin{pmatrix} e^{-\varepsilon r} \\ \frac{1}{3}(e^{2\varepsilon r} - e^{-\varepsilon r}) \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon r \end{pmatrix} + \begin{pmatrix} O(\varepsilon) \\ O(\varepsilon^2) \end{pmatrix},$$

uniformly in  $r$  on compact intervals. Thus, as for the Ornstein-Uhlenbeck process paired with its area and the Kolmogorov diffusion,  $U^\varepsilon(r)$  is of the form (3.1) with  $u_1 = u_2 = 1$ . By Theorem 1.2, these three processes exhibit the same small-time fluctuations for the bridge. We further compute that, for  $t \in [0, 1]$ ,

$$e^{\varepsilon t A} \Gamma_t^\varepsilon = \begin{pmatrix} \frac{1}{2\varepsilon}(e^{\varepsilon t} - e^{-\varepsilon t}) & -\frac{1}{6\varepsilon}(2e^{-2\varepsilon t} - 3e^{-\varepsilon t} + e^{\varepsilon t}) \\ \frac{1}{6\varepsilon}(2e^{2\varepsilon t} - 3e^{\varepsilon t} + e^{-\varepsilon t}) & -\frac{1}{12\varepsilon}(e^{2\varepsilon t} - 2e^{\varepsilon t} + 2e^{-\varepsilon t} - e^{-2\varepsilon t}) \end{pmatrix},$$

which has the expansion

$$e^{\varepsilon t A} \Gamma_t^\varepsilon = \begin{pmatrix} t + \frac{1}{6}\varepsilon^2 t^3 + O(\varepsilon^4) & -\frac{1}{2}\varepsilon t^2 + \frac{1}{3}\varepsilon^2 t^3 + O(\varepsilon^3) \\ \frac{1}{2}\varepsilon t^2 + \frac{1}{3}\varepsilon^2 t^3 + O(\varepsilon^3) & -\frac{1}{6}\varepsilon^2 t^3 - \frac{1}{24}\varepsilon^4 t^5 + O(\varepsilon^6) \end{pmatrix},$$

uniformly in  $t \in [0, 1]$ . Setting

$$R = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$$

and with

$$D_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad J_t = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{3/2} \end{pmatrix} \quad \text{as well as} \quad V = \begin{pmatrix} 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{6} \end{pmatrix},$$

we have

$$e^{\varepsilon t A} \Gamma_t^\varepsilon = D_\varepsilon J_t (V + \varepsilon t R + O(\varepsilon^2)) J_t D_\varepsilon,$$

uniformly in  $t \in [0, 1]$ . Let  $I$  denote the  $2 \times 2$  identity matrix. Since  $V$  is invertible, we deduce that, for  $\varepsilon > 0$  sufficiently small,

$$(V + \varepsilon R + O(\varepsilon^2))^{-1} = (I + \varepsilon V^{-1}R + O(\varepsilon^2))^{-1} V^{-1} = V^{-1} - \varepsilon V^{-1}R V^{-1} + O(\varepsilon^2),$$

and therefore, due to  $J_1 = I$ ,

$$(\Gamma_1^\varepsilon)^{-1} e^{-\varepsilon A} = D_\varepsilon^{-1} (V^{-1} - \varepsilon V^{-1}R V^{-1} + O(\varepsilon^2)) D_\varepsilon^{-1}.$$

This implies that

$$\begin{aligned} \alpha_t^\varepsilon &= D_\varepsilon J_t (V + \varepsilon t R + O(\varepsilon^2)) J_t (V^{-1} - \varepsilon V^{-1}R V^{-1} + O(\varepsilon^2)) D_\varepsilon^{-1} \\ &= D_\varepsilon (J_t V J_t V^{-1} + \varepsilon t J_t R J_t V^{-1} - \varepsilon J_t V J_t V^{-1} R V^{-1} + O(\varepsilon^2)) D_\varepsilon^{-1}. \end{aligned}$$



We compute

$$J_t V J_t V^{-1} = \begin{pmatrix} 3t^2 - 2t & 6t - 6t^2 \\ t^3 - t^2 & 3t^2 - 2t^3 \end{pmatrix}$$

as well as

$$\varepsilon t J_t R J_t V^{-1} = \begin{pmatrix} -2\varepsilon t^3 & 4\varepsilon t^3 \\ -\frac{2}{3}\varepsilon t^3 & 2\varepsilon t^3 \end{pmatrix} \quad \text{and} \quad \varepsilon J_t V J_t V^{-1} R V^{-1} = \begin{pmatrix} -2\varepsilon t^2 & 4\varepsilon t \\ -\frac{2}{3}\varepsilon t^3 & 2\varepsilon t^2 \end{pmatrix},$$

which yields

$$\alpha_t^\varepsilon = \begin{pmatrix} 3t^2 - 2t + (2t^2 - 2t^3)\varepsilon + O(\varepsilon^2) & (6t - 6t^2)\varepsilon^{-1} + 4t^3 - 4t + O(\varepsilon) \\ (t^3 - t^2)\varepsilon + O(\varepsilon^3) & 3t^2 - 2t^3 + (2t^3 - 2t^2)\varepsilon + O(\varepsilon^2) \end{pmatrix}.$$

In particular, for  $y = (a, b)$ , we obtain

$$\alpha_t^\varepsilon y = \begin{pmatrix} 3t^2 - 2t & (6t - 6t^2)\varepsilon^{-1} \\ (t^3 - t^2)\varepsilon & 3t^2 - 2t^3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} (4t^3 - 4t)b \\ (2t^3 - 2t^2)b\varepsilon \end{pmatrix} + \begin{pmatrix} O(\varepsilon) \\ O(\varepsilon^2) \end{pmatrix}.$$

It follows that in our current example for an approximate minimal-like path to lead to well-defined small-time fluctuations for the bridge from  $x = 0$  to  $y = (a, b)$  with respect to the rescaling  $D_\varepsilon$ , we have to at least subtract the path which, at  $t \in [0, 1]$ , is given as

$$\left( (3t^2 - 2t)a + (6t - 6t^2)\frac{b}{\varepsilon} + (4t^3 - 4t)b, (t^3 - t^2)a\varepsilon + (3t^2 - 2t^3)b + (2t^3 - 2t^2)b\varepsilon \right).$$

This differs from the minimal-like path  $(\phi_t^\varepsilon(y))_{t \in [0,1]}$  for the Kolmogorov diffusion, and the approximate minimal-like path  $(\psi_t^\varepsilon(y))_{t \in [0,1]}$  found for the Ornstein-Uhlenbeck process paired with its area starting from 0.

#### 4.4 Iterated Kolmogorov diffusion

The diffusions studied in Section 4.2 and Section 4.3 both exhibit the same small-time fluctuations for the bridge as the Kolmogorov diffusion. Similarly, there is a family of diffusions which all have the same small-time fluctuations for the bridge as the iterated Kolmogorov diffusion, that is, a standard Brownian motion together with a finite number of its iterated time integrals. Banerjee and Kendall [3] study maximal and efficient couplings for iterated Kolmogorov diffusions, and Baudoin, Gordina and Mariano [5] obtain gradient bounds for this hypoelliptic diffusion. We close by explicitly determining the small-time fluctuations for the bridge of an iterated Kolmogorov diffusion. By the independence of the components of a Brownian motion in  $\mathbb{R}^m$ , it is sufficient to focus on a standard Brownian motion in  $\mathbb{R}$  and its iterated time integrals. In our model class, this diffusion corresponds to the choice of the  $d \times d$  matrix  $A$  and the  $d \times 1$  matrix  $B$ , understood as a column vector, whose entries are, for  $i, j \in \{1, \dots, d\}$ ,

$$A_{ij} = \begin{cases} 1 & \text{if } i = j + 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad B_i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}.$$

With  $\mathcal{L}$  on  $\mathbb{R}^d$  given by (1.2), the operator  $\mathcal{L} - \frac{\partial}{\partial t}$  is a strongly degenerate ultraparabolic operator. For  $k \in \mathbb{N}$ , we have

$$(A^k)_{ij} = \begin{cases} 1 & \text{if } i = j + k \\ 0 & \text{otherwise} \end{cases}.$$

This yields

$$\text{span} \{B, AB, \dots, A^{d-1}B\} = \mathbb{R}^d,$$

that is, the Kalman rank condition (1.1) is satisfied. Moreover, we obtain  $n = d$  since  $\mathbb{R}^d$  cannot be spanned by less than  $d$  vectors. From  $A^k = 0$  for  $k \geq d$ , it follows that, for  $r \in \mathbb{R}$ ,

$$e^{\varepsilon r A} = \sum_{k=0}^{d-1} \frac{\varepsilon^k r^k}{k!} A^k,$$

which implies

$$(U^\varepsilon(r))_i = (e^{\varepsilon r A} B)_i = \frac{\varepsilon^{i-1} r^{i-1}}{(i-1)!}.$$

Hence,  $U^\varepsilon(r)$  is of the form (3.1) with

$$u_i = \frac{1}{(i-1)!}.$$

In the current example, the matrices  $D_\varepsilon$  and  $J_t$  are the  $d \times d$  diagonal matrices, whose  $i^{\text{th}}$  diagonal element equals  $\varepsilon^{i-1}$  and  $t^{i-1/2}$ , respectively. We further see that  $V$  has the entries

$$V_{ij} = (-1)^{j+1} u_i u_j \frac{(i-1)!(j-1)!}{(i+j-1)!} = (-1)^{j+1} \frac{1}{(i+j-1)!}.$$

Let  $H$  be the  $d \times d$  Hankel matrix defined by

$$H_{ij} = \frac{1}{(i+j-1)!},$$

and let  $S$  be the  $d \times d$  diagonal matrix whose  $i^{\text{th}}$  diagonal element equals  $(-1)^{i+1}$ . Due to

$$(HS)_{ij} = \sum_{k=1}^d H_{ik} S_{kj} = H_{ij} S_{jj} = (-1)^{j+1} \frac{1}{(i+j-1)!} = V_{ij},$$

for all  $i, j \in \{1, \dots, d\}$ , we have  $V = HS$ . Since  $S^{-1} = S$ , it follows that

$$V^{-1} = SH^{-1}.$$

Using my formula for the inverse of the factorial Hankel matrix  $H$ , see [8, Theorem 1.1], we obtain

$$\begin{aligned} (V^{-1})_{ij} &= \sum_{k=1}^d S_{ik} (H^{-1})_{kj} = (-1)^{i+1} (H^{-1})_{ij} \\ &= (-1)^{d+j} (i-1)! j! \binom{d-1}{i-1} \binom{d+j-1}{j} \sum_{k=0}^{i-1} \binom{d-i+k}{j-1} \binom{d+k-1}{k}. \end{aligned}$$

We further compute that

$$\begin{aligned} (J_t V J_t V^{-1})_{ij} &= \sum_{l=1}^d t^{i-1/2} V_{il} t^{l-1/2} (V^{-1})_{lj} \\ &= \sum_{l=1}^d (-1)^{d+j+l+1} \frac{(l-1)! j!}{(i+l-1)!} \binom{d-1}{l-1} \binom{d+j-1}{j} \sum_{k=0}^{l-1} \binom{d-l+k}{j-1} \binom{d+k-1}{k} t^{i+l-1}. \end{aligned}$$

As  $\hat{U}(r)$  has the entries

$$\left(\hat{U}(r)\right)_i = r^{i-1}u_i = \frac{r^{i-1}}{(i-1)!},$$

we see, by integration by parts, that the process

$$\left(\int_0^t \hat{U}(t-s) dW_s\right)_{t \in [0,1]}$$

is again the iterated Kolmogorov diffusion. Using Theorem 1.2, this observation and the formula for  $J_t V J_t V^{-1}$  together give an explicit expression of the small-time fluctuations for the bridge of an iterated Kolmogorov diffusion. Moreover, since  $U^\varepsilon(r) = D_\varepsilon \hat{U}(r)$  for  $r \in \mathbb{R}$ , these small-time fluctuations are equal in law to the bridge from 0 to 0 in time 1 of an iterated Kolmogorov diffusion with the same number of iterated time integrals.

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