

Spectral conditions for equivalence of Gaussian random fields with stationary increments*

Abolfazl Safikhani[†] Yimin Xiao[‡]

Abstract

This paper studies the problem of equivalence of Gaussian measures induced by Gaussian random fields (GRFs) with stationary increments and proves a sufficient condition for the equivalence in terms of the behavior of the spectral measures at infinity. The main results extend those of Stein (2004), Van Zanten (2007, 2008) and are applicable to a rich family of nonstationary space-time models with possible anisotropy behavior.

Keywords: Gaussian random fields; equivalence of measures; nonstationary; spectral density; spatio-temporal models.

AMS MSC 2010: Primary 60G60, Secondary 62H11.

Submitted to EJP on July 21, 2018, final version accepted on January 22, 2019.

Supersedes arXiv:1806.04181.

1 Introduction

Space-time models have become increasingly popular in scientific studies such as geology, climatology, geophysics, environmental and atmospheric sciences, etc ([7], [9] and [30]). Gaussian random fields (GRFs) are ubiquitous in space-time modeling due to the prevalence of central limit theorems and the mathematical/computational amenability of the multivariate normal distributions. Most of the parametric models proposed in the literature are GRFs with specific parametric covariance structure (see [8], [14], [32], and [15] for rich families of space-time covariance functions). One of the main objectives in statistics then is to find consistent estimates for the parameters, and finally use them for prediction of the underlying random field at unobserved locations. Given a parametric family of Gaussian random fields, an important question is to determine whether all the parameters are consistently estimable. First step to answer this question demands an

*The research is supported in part by NSF grants DMS-1612885 and DMS-1607089.

[†]Columbia University, Department of Statistics, New York, United States of America. E-mail: as5012@columbia.edu

[‡]Michigan State University, Department of Statistics and Probability, Michigan, United States of America. E-mail: xiao@stt.msu.edu

investigation on the equivalence or singularity of the corresponding Gaussian measures induced by this family of GRFs on their space of sample functions, since if two sets of parameters in the Gaussian models give equivalent Gaussian distributions, then it is impossible to find consistent estimators for these parameters involved regardless of the method chosen for estimation (see for example [40] for a discussion on inconsistent estimation in Matérn covariance functions under the framework of fixed domain asymptotics). Another application of equivalence of Gaussian measures comes from covariance structure misspecification, and its effect on spatial interpolation (see [30]). Therefore, finding explicit conditions for deciding whether two Gaussian random fields induce equivalent Gaussian distributions on their spaces of sample functions has direct impact in evaluating the prediction error in interpolation of spatial data, and thus proving asymptotically optimal prediction under misspecified covariance structure. There are other applications of equivalence and perpendicularity of GRFs in spatial modeling. For example, we refer to [12], [18], [27], [26], and [4] for the application in covariance tapering.

Equivalence of Gaussian measures is a classical problem in probability theory that has been studied since the 1950's. We refer to the books [13], [17], [38], [21] and references therein for systematic accounts. Necessary and sufficient conditions for the equivalence of Gaussian measures induced by stationary Gaussian processes in terms of their mean and covariance functions are given in [17]. Their extensions to stationary isotropic Gaussian random fields are proved in [28] and [38].

However, investigation on the equivalence of nonstationary GRFs has been limited to some special cases. For instance, we refer to [6] on mixed fractional Brownian motion, [3] on Volterra processes, [29] on Gaussian random fields that are equivalent to fractional Brownian sheets, [31] on a family of intrinsic random functions with power law generalized covariance functions (including fractional Brownian fields), and [33, 34] and [36] on certain Gaussian processes/fields with stationary increments. Also, there are some recent results on equivalence of measures for multivariate GRFs [25], for GRFs on spheres [1], and for self-similar GRFs [20].

Our work is mainly motivated by [31] and by [33, 34] where explicit sufficient conditions for the equivalence of Gaussian processes with stationary increments in terms of their spectral densities similar to the criterion in [17] for the stationary case have been established. The main purpose of this paper is to extend their results to the setting of Gaussian random fields with stationary increments which may have different regularities in each direction. Besides of theoretical interest, our results are applicable to anisotropic nonstationary space-time Gaussian models.

The rest of the paper is organized as follows. We start Section 2 by introducing some useful Hilbert spaces connected to the frequency domain, and study their structure. In Section 3, we state the main results of the paper, which are sufficient conditions for equivalence of GRFs with stationary increments using the tail behavior of their spectral densities. In the last section, we apply the main results (Theorems 3.5 and 3.6) to a rich family of anisotropic nonstationary spatio-temporal Gaussian models.

2 Preliminary

Let $X = \{X_t : t \in \mathbb{R}^d\}$ be a centered GRF with stationary increments. The covariance structure of X is fully described by [39]. For simplicity, we assume that $X(0) = 0$ and the covariance function of X can be written as

$$C(t, s) = \mathbf{E}(X_t X_s) = \int_{\mathbb{R}^d} (e^{i\langle t, \lambda \rangle} - 1)(e^{-i\langle s, \lambda \rangle} - 1) F(d\lambda), \quad (2.1)$$

where F is a non-negative symmetric measure on $\mathbb{R}^d \setminus \{0\}$, called the spectral measure of X , that satisfies

$$\int_{\mathbb{R}^d} \frac{|\lambda|^2}{1 + |\lambda|^2} F(d\lambda) < \infty. \quad (2.2)$$

It follows from (2.1) that X has the following spectral representation:

$$X(t) \stackrel{d}{=} \int_{\mathbb{R}^d} (e^{i\langle t, \lambda \rangle} - 1) W(d\lambda), \quad (2.3)$$

where W is a complex-valued Gaussian random measure with mean 0 and control measure F .

If the spectral measure F is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d , we will call its Radon-Nikodym derivative, denoted by $f(\lambda)$, the spectral density of X . We will give conditions for the equivalence of GRFs with stationary increments in terms of their spectral densities, but first, we recall the definition of equivalence of GRFs.

Definition 2.1. For a fixed set $D \subseteq \mathbb{R}^d$, we call two GRFs $X = \{X_t : t \in \mathbb{R}^d\}$, $Y = \{Y_t : t \in \mathbb{R}^d\}$ equivalent on D if they induce equivalent measures¹ on the measurable space $(\mathbb{R}^D, \mathcal{B}(\mathbb{R}^D))$, in which $\mathcal{B}(\mathbb{R}^D)$ is the σ -field generated by the cylinder subsets² of \mathbb{R}^D . Moreover, we call X and Y locally equivalent if they are equivalent on all bounded subsets of \mathbb{R}^d .

The spectral representation (2.1) makes an important bridge between the problem of equivalence of GRFs and the description of the space generated by the linear combinations of the kernel functions. For that purpose, in this section we define for a fixed and bounded set $D \subseteq \mathbb{R}^d$, an incomplete Hilbert space $\mathcal{L}_D^e = \text{span} \{e_t(\lambda) := e^{i\langle t, \lambda \rangle} - 1 : t \in D\}$ with the inner product

$$\langle e_t, e_s \rangle_F = \int_{\mathbb{R}^d} e_t(\lambda) \overline{e_s(\lambda)} F(d\lambda).$$

We denote the closure of \mathcal{L}_D^e in $L^2(F)$ by $\mathcal{L}_D(F)$. Also, for $T > 0$, denote by $\Pi_T = [-T, T]^d$ the cube with side $2T$.

Observe that the functions in \mathcal{L}_D^e are entire functions defined on \mathbb{C}^d (see [24] for definition and more properties), and they are of finite exponential type. Recall that an entire function φ on \mathbb{C}^d is called of finite exponential type if

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \max_{\|z\|=r} \log |\varphi(z)| < \infty.$$

However, in general, the elements in the completed Hilbert space, $\mathcal{L}_D(F)$, may not have the same properties as the functions in \mathcal{L}_D^e . This problem is discussed in details in [22, 23]. In this paper, we assume that the spectral measure F has a density function $f(\lambda)$ that satisfies the following condition:

- (C1) There exist constants $c, k, \eta > 0$ such that $f(\lambda) \geq \frac{c}{|\lambda|^\eta}$ for all $\lambda \in \mathbb{R}^d$ with $|\lambda| > k$.

This assumption on the spectral density will imply the elements in $\mathcal{L}_D(F)$ to be entire functions of finite exponential type. These properties enable us to apply the Paley-Wiener type theorems to get nice description of the elements in the Hilbert space $\mathcal{L}_D(F)$ for $D = \Pi_T$.

¹Two measures defined on the same measurable space are called equivalent if they are mutually absolutely continuous with respect to each other.

²A cylinder subset of \mathbb{R}^D is a set of the form $\{f \in \mathbb{R}^D : f(t^1) \in B_1, \dots, f(t^n) \in B_n\}$, where $t^1, \dots, t^n \in D$ and B_1, \dots, B_n are Borel sets in \mathbb{R} .

The next two lemmas will prove these statements. The following lemma is taken from [35] and we state it here again for completeness.

Lemma 2.2. *Suppose that the spectral density f satisfies (C1). Then for fixed $T > 0$, there exists positive constants C and M such that for all functions ϕ of the form*

$$\phi(\lambda) = \sum_{k=1}^n a_k (e^{i\langle t^k, \lambda \rangle} - 1), \quad (2.4)$$

where $a_k \in \mathbb{R}$ and $t^k \in \Pi_T$, we have for all $z \in \mathbb{C}^d$

$$|\phi(z)| \leq C \|\phi\|_F \exp\{M|z|\}. \quad (2.5)$$

Moreover, for fixed $C_1 > 0$, there exists a positive constant C_2 such that for all functions of the form (2.4), we have

$$|\phi(z)| \leq C_2 |z| \|\phi\|_F \quad (2.6)$$

for all $|z| \leq C_1$.

One can use (2.5) to define the limiting functions in $\mathcal{L}_{\Pi_T}(F)$ in such a way that they also satisfy both (2.5) and (2.6). We will prove this in the next Lemma.

Lemma 2.3. *Suppose that the spectral density f satisfies (C1). Then, for each $T > 0$, the space $\mathcal{L}_{\Pi_T}(F)$ consists of the restriction to \mathbb{R}^d of entire functions on \mathbb{C}^d of finite exponential type. Moreover, (2.6) holds for all functions $\phi \in \mathcal{L}_{\Pi_T}(F)$.*

Proof. The idea of the proof is similar to [23], p. 304. Take a sequence $\phi_n \in \mathcal{L}_{\Pi_T}^e$, such that $\|\phi_n - \phi\|_F \rightarrow 0$ for some $\phi \in \mathcal{L}_{\Pi_T}(F)$. Then, it is a Cauchy sequence in $L^2(F)$, and using (2.5), we get

$$|\phi_n(z) - \phi_m(z)| \leq C \|\phi_n - \phi_m\|_F \exp\{M|z|\} \quad (2.7)$$

This means for each fixed $z \in \mathbb{C}^d$, the sequence $\{\phi_n(z), n \geq 1\}$ is a Cauchy sequence in \mathbb{C} . So, it is convergent and, moreover, the convergence is locally uniform. Denote the limit by $\tilde{\phi}(z)$. Now, since limit in $L^2(F)$ sense implies the almost everywhere convergence for a subsequence, $\phi = \tilde{\phi}$ a.e. with respect to F . From now on, we will take $\tilde{\phi}$ as our favorite version of the limits of functions in $\mathcal{L}_{\Pi_T}^e$. Therefore, the elements in the space $\mathcal{L}_{\Pi_T}(F)$, are not only the $L^2(F)$ limits of functions in $\mathcal{L}_{\Pi_T}^e$, but also the pointwise limits as well. Thus, both (2.5) and (2.6) are true for all the elements in $\mathcal{L}_{\Pi_T}(F)$. The only thing left to prove is that these functions $\tilde{\phi}$ are entire functions on \mathbb{C}^d . But this is true since any element of the space $\mathcal{L}_{\Pi_T}(F)$ is the locally uniform limit of functions of the form (2.4) which are obviously entire functions, and thus, they are entire functions as well (This is called the Weierstrass Theorem; see, for example, Proposition 2.8 on p. 52 of [11]). \square

This lemma shows that if the spectral density satisfies the assumption (C1), we can complete the space $\mathcal{L}_{\Pi_T}^e$ in such a way that the resulting functions are locally uniform limits of entire functions, and hence, they are entire functions of finite exponential type. Furthermore, since (2.5) is true for all the elements in the Hilbert space $\mathcal{L}_{\Pi_T}(F)$, we can see that the point evaluators, i.e. the functionals on $\mathcal{L}_{\Pi_T}(F)$ of the form $\phi \mapsto \phi(z)$ for each fixed $z \in \mathbb{C}^d$ are bounded operators. Now, we can apply the Riesz Representation Theorem (See [16], Theorem 3. p. 31) to prove that the space $\mathcal{L}_{\Pi_T}(F)$ is a Reproducing Kernel Hilbert Space (RKHS) in the sense of [2]. This means that there exists a function $K_T(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ such that (i) $K_T(\omega, \cdot) \in \mathcal{L}_{\Pi_T}(F)$ for all $\omega \in \mathbb{R}^d$, and (ii) for every $\phi \in \mathcal{L}_{\Pi_T}(F)$ and $\omega \in \mathbb{R}^d$, we have the following kernel property

$$\langle \phi, K_T(\omega, \cdot) \rangle_F = \int_{\mathbb{R}^d} \phi(\lambda) \overline{K_T(\omega, \lambda)} F(d\lambda) = \phi(\omega). \quad (2.8)$$

Also, it is worthwhile to mention that the set of all functions $\{K_T(\omega, \cdot) : \omega \in \mathbb{R}^d\}$ is dense in $\mathcal{L}_{\Pi_T}(F)$ (To see this, note that if $\phi \in \mathcal{L}_{\Pi_T}(F)$ is orthogonal to $K_T(\omega, \cdot)$ for all $\omega \in \mathbb{R}^d$, then $\phi(\omega) = \langle \phi, K_T(\omega, \cdot) \rangle_F = 0$, which implies $\phi = 0$). Furthermore, for all $\psi \in L^2(F)$, the function

$$\omega \mapsto \langle \psi, K_T(\omega, \cdot) \rangle_F \tag{2.9}$$

is the orthogonal projection of ψ on $\mathcal{L}_{\Pi_T}(F)$ (See the proof in [2], p. 345). We denote this projection by $\pi_{\mathcal{L}_{\Pi_T}(F)}\psi$.

Finding explicit forms of the reproducing kernels is not an easy job. However, in order to prove the results in Section 3, we only need to establish upper bounds for the growth rate of the diagonal elements of the reproducing kernels at origin and also at infinity. The following proposition proves an important growth rate for the diagonal elements in the reproducing kernels.

Proposition 2.4. *Suppose that the spectral density $f(\lambda)$ of F satisfies (C1). Then, for fixed $T > 0$ and $C_1 > 0$, there exists a positive constant C_2 such that*

$$K_T(\omega, \omega) \leq C_2 |\omega|^2 \tag{2.10}$$

for all $\omega \in \mathbb{R}^d$ with $|\omega| < C_1$.

Proof. Since for any fixed $\omega \in \mathbb{R}^d$, $K_T(\omega, \cdot) \in \mathcal{L}_{\Pi_T}(F)$, we can apply Lemma 2.3 to these functions. It follows from (2.6) that

$$|K_T(\omega, \lambda)| \leq C_2 |\lambda| \|K_T(\omega, \cdot)\|_F = C_2 |\lambda| (K_T(\omega, \omega))^{1/2}$$

for all $\omega \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}^d$ with $|\lambda| < C_1$. By taking $\lambda = \omega$, we obtain the desired result. \square

We also need to define another Hilbert space based on the tensor product of the elements in $\mathcal{L}_{\Pi_T}(F)$. For this purpose, first we define $\mathcal{L}_{\Pi_T}^e \otimes \mathcal{L}_{\Pi_T}^e$ to be the span of functions $(e_t \otimes e_s)(\omega, \lambda) := e_t(\omega) \overline{e_s(\lambda)}$ with $t, s \in \Pi_T$. Now, denote by $\mathcal{L}_{\Pi_T}(F) \otimes \mathcal{L}_{\Pi_T}(F)$ the closure in $L^2(F \otimes F)$ of the space $\mathcal{L}_{\Pi_T}^e \otimes \mathcal{L}_{\Pi_T}^e$. According to Theorem 1 on p. 361 of [2], the new Hilbert space $\mathcal{L}_{\Pi_T}(F) \otimes \mathcal{L}_{\Pi_T}(F)$ is also a RKHS with reproducing kernel

$$((\omega_1, \lambda_1), (\omega_2, \lambda_2)) \mapsto K_T(\omega_1, \omega_2) \overline{K_T(\lambda_1, \lambda_2)}. \tag{2.11}$$

This implies that for $\psi \in \mathcal{L}_{\Pi_T}(F) \otimes \mathcal{L}_{\Pi_T}(F)$,

$$\langle \psi, K_T(\omega, \cdot) \otimes K_T(\lambda, \cdot) \rangle_{F \otimes F} = \psi(\omega, \lambda). \tag{2.12}$$

We finish this section by a lemma stating that the norm of the elements in spaces $\mathcal{L}_{\Pi_T}(F)$ depends essentially on the tail behavior of the spectral measure F . To this end, we define the following notation. For two functions $f, g : \mathbb{R}^d \mapsto \mathbb{R}$, we write $f(\lambda) \asymp g(\lambda)$ as $|\lambda| \rightarrow \infty$, if the ratio $f(\lambda)/g(\lambda)$ is bounded from below and above by positive and finite constants which are independent of λ as $|\lambda| \rightarrow \infty$.

Lemma 2.5. *Suppose f_0 and f_1 are two spectral densities satisfying the condition (C1), and further $f_0(\lambda) \asymp f_1(\lambda)$ as $|\lambda| \rightarrow \infty$. Then, $\mathcal{L}_{\Pi_T}(f_0) = \mathcal{L}_{\Pi_T}(f_1)$, and further there exist positive constants C_3 and C_4 such that*

$$C_3 \|\phi\|_{f_1} \leq \|\phi\|_{f_0} \leq C_4 \|\phi\|_{f_1}$$

for all $\phi \in \mathcal{L}_{\Pi_T}(f_0)$.

Proof. Suppose $g \in \mathcal{L}_{\Pi_T}(f_0)$. There exists functions g_n of the form (2.4) such that $\|g - g_n\|_{f_0} \rightarrow 0$ as $n \rightarrow \infty$. Now, using Lemma 2.3, we get

$$\begin{aligned} \|g - g_n\|_{f_1}^2 &= \int_{\mathbb{R}^d} |g(\lambda) - g_n(\lambda)|^2 f_1(\lambda) d\lambda \\ &\leq C_2 \|g - g_n\|_{f_0}^2 \int_{|\lambda| < C_1} |\lambda|^2 f_1(\lambda) d\lambda \\ &\quad + C \|g - g_n\|_{f_0}^2 \end{aligned} \quad (2.13)$$

which implies $g \in \mathcal{L}_{\Pi_T}(f_1)$. Thus $\mathcal{L}_{\Pi_T}(f_0) \subseteq \mathcal{L}_{\Pi_T}(f_1)$. Similarly we have $\mathcal{L}_{\Pi_T}(f_1) \subseteq \mathcal{L}_{\Pi_T}(f_0)$. Finally one can see that (2.13) holds if we replace $g - g_n$ by $\phi \in \mathcal{L}_{\Pi_T}(f_0)$. This leads to the second part of the lemma. \square

3 Main results

In this section, we study the equivalence of GRFs with stationary increments, and clarify its connection to the Hilbert spaces constructed in Section 2. In particular, the role of the reproducing kernels of the RKHS $\mathcal{L}_{\Pi_T}(F)$ will be emphasized. We start this section by an extension of Theorem 5 on p. 84 in [17] (Theorem 1 on p. 149 in [38]) for stationary Gaussian processes (fields) to Gaussian random fields with stationary increments. Some extensions of the criteria for equivalence of stationary Gaussian processes (fields) have also been obtained by [33, 34] and [36]. The following theorem is an extension of Theorems 3.3.9 and 3.3.10 in [36] and also Theorem 4.3 in [33].

Theorem 3.1. Two centered GRFs with stationary increments and spectral measures F_0 and F_1 are equivalent on D if and only if:

- (i) $\|\phi\|_{F_0} \asymp \|\phi\|_{F_1}, \forall \phi \in \mathcal{L}_D^e$, and
- (ii) There exists a function $\psi \in \mathcal{L}_D(F_0) \otimes \mathcal{L}_D(F_0)$ such that for all $t, s \in D$

$$\langle e_t, e_s \rangle_{F_0} - \langle e_t, e_s \rangle_{F_1} = \langle \psi, e_t \otimes e_s \rangle_{F_0 \otimes F_0}. \quad (3.1)$$

Proof. The proof is essentially a reconstruction of the proof of the Theorem 5, p. 84 of [17], and is given here for the sake of completeness. The starting point is however, Theorem 4 on p. 80 of the same reference. The proof there can be adapted to our context with little change since it only involves the “entropy distance” between the Gaussian measures, and thus is true for general GRFs (See also [5], Theorem 4.1, 4.4 pp. 180-185). After doing so, we get that two GRFs with stationary increments and spectral measures F_0 and F_1 are equivalent on $D \subseteq \mathbb{R}^d$, if and only if

$$\|\phi\|_{F_0} \asymp \|\phi\|_{F_1}, \quad \forall \phi \in \mathcal{L}_D^e,$$

and $\Delta = I - A^*A$ is a Hilbert-Schmidt operator in $\mathcal{L}_D(F_0)$, in which I is the identity operator on $\mathcal{L}_D(F_0)$, and $A : \mathcal{L}_D(F_0) \mapsto \mathcal{L}_D(F_1)$ with $A\phi = \phi$ for all $\phi \in \mathcal{L}_D(F_0)$. Now, since Δ is a self-adjoint operator, if it is also a Hilbert-Schmidt operator, by the Spectral Theorem (See [10], Corollary 5 p. 905), we can conclude that there exists an orthonormal basis for $\mathcal{L}_D(F_0)$ consisting of the eigenvectors of Δ , denoting them by $\phi_j, j = 1, 2, \dots$, with corresponding eigenvalues $\lambda_j, j = 1, 2, \dots$ with $\sum_j \lambda_j^2 < \infty$. Note that we can write $\sum_j \lambda_j^2 = \sum_{j,k} \langle \Delta \phi_j, \phi_k \rangle_{F_0}^2$. The square root of this quantity is called the Hilbert-Schmidt norm. This norm doesn't depend on the choice of the orthonormal basis (See [10], Lemma 2, p. 1010). Therefore, we can rephrase Theorem 4 in the following form: two GRFs with stationary increments and spectral measures F_0 and F_1 are equivalent on

D , if and only if, $\|\phi\|_{F_0} \asymp \|\phi\|_{F_1}$, $\forall \phi \in \mathcal{L}_D^e$, and $\sum_{j,k} \langle \Delta\phi_j, \phi_k \rangle_{F_0}^2 < \infty$ for any orthonormal basis for $\mathcal{L}_D(F_0)$.

Now, take an arbitrary orthonormal basis for $\mathcal{L}_D(F_0)$, ϕ_1, ϕ_2, \dots , and suppose $\sum_{j,k} \langle \Delta\phi_j, \phi_k \rangle_{F_0}^2 < \infty$. Define $\psi_0(\omega, \lambda) = \sum_{j,k} \langle \Delta\phi_j, \phi_k \rangle_{F_0} \phi_j(\omega) \phi_k(\lambda)$. We can see that $\|\psi_0\|_{F_0 \otimes F_0}^2 = \sum_{j,k} \langle \Delta\phi_j, \phi_k \rangle_{F_0}^2 < \infty$, and thus by the form of ψ_0 , it's clear that it belongs to $\mathcal{L}_D(F_0) \otimes \mathcal{L}_D(F_0)$. Also, observe that

$$\begin{aligned} \langle \psi_0, \phi_j \otimes \phi_k \rangle_{F_0 \otimes F_0} &= \langle \Delta\phi_j, \phi_k \rangle_{F_0} \\ &= \langle (I - A^*A)\phi_j, \phi_k \rangle_{F_0} \\ &= \langle \phi_j, \phi_k \rangle_{F_0} - \langle \phi_j, \phi_k \rangle_{F_1}. \end{aligned}$$

This shows that (3.1) holds for orthonormal basis of the space $\mathcal{L}_D(F_0)$. Therefore, by continuity of inner product (3.1) will be true for all the elements in $\mathcal{L}_D(F_0)$, especially for e_t and e_s when $t, s \in D$.

Conversely, suppose there exists a function $\psi_0 \in \mathcal{L}_D(F_0) \otimes \mathcal{L}_D(F_0)$, such that $\langle \phi_j, \phi_k \rangle_{F_0} - \langle \phi_j, \phi_k \rangle_{F_1} = \langle \psi_0, \phi_j \otimes \phi_k \rangle_{F_0 \otimes F_0}$ for an orthonormal basis ϕ_j 's for $\mathcal{L}_D(F_0)$. Then, we have

$$\begin{aligned} \sum_{j,k} \langle \Delta\phi_j, \phi_k \rangle_{F_0}^2 &= \sum_{j,k} \left(\langle \phi_j, \phi_k \rangle_{F_0} - \langle \phi_j, \phi_k \rangle_{F_1} \right)^2 \\ &= \sum_{j,k} \langle \psi_0, \phi_j \otimes \phi_k \rangle_{F_0 \otimes F_0}^2 \\ &\leq \|\psi_0\|_{F_0 \otimes F_0}^2 < \infty. \end{aligned}$$

This completes the proof. \square

Theorem 3.1 is stated in a general form for GRFs with stationary increment, with no restriction on their spectral measures. However, verifying the second condition in this theorem, which involves finding a function $\psi \in \mathcal{L}_D(F_0) \otimes \mathcal{L}_D(F_0)$ with the property (3.1), seems to be hard. If we put the condition (C1) on one of the spectral measures (say, F_0), we get the following theorem using the reproducing kernels of $\mathcal{L}_{\Pi_T}(F_0)$. In fact, this theorem clarifies what must be the function ψ in Theorem 3.1.

Theorem 3.2. Two centered GRFs with stationary increments, and spectral measures F_0 and F_1 , with F_0 satisfying assumption (C1), are equivalent on Π_T for some $T > 0$ if and only if:

- (i) $\|\phi\|_{F_0} \asymp \|\phi\|_{F_1}$, $\forall \phi \in \mathcal{L}_{\Pi_T}^e$,
- (ii) $\psi(\omega, \lambda) = K_T^0(\omega, \lambda) - \int_{\mathbb{R}^d} K_T^0(\omega, \gamma) \overline{K_T^0(\lambda, \gamma)} F_1(d\gamma) \in \mathcal{L}_{\Pi_T}(F_0) \otimes \mathcal{L}_{\Pi_T}(F_0)$, where $K_T^0(\cdot, \cdot)$ are the reproducing kernels of the space $\mathcal{L}_{\Pi_T}(F_0)$.

Proof. First, assume that the measures induced by them are equivalent. Then, by Theorem 3.1, there exists a function $\psi \in \mathcal{L}_{\Pi_T}(F_0) \otimes \mathcal{L}_{\Pi_T}(F_0)$ such that (3.1) holds. Now, because of bilinearity and continuity of inner product together with the fact that $\mathcal{L}_{\Pi_T}(F_0) = \mathcal{L}_{\Pi_T}(F_1)$, we get

$$\langle \phi_1, \phi_2 \rangle_{F_0} - \langle \phi_1, \phi_2 \rangle_{F_1} = \langle \psi, \phi_1 \otimes \phi_2 \rangle_{F_0 \otimes F_0} \quad (3.2)$$

for all $\phi_1, \phi_2 \in \mathcal{L}_{\Pi_T}(F_0) = \mathcal{L}_{\Pi_T}(F_1)$. Now, simply choose for fixed $\omega, \lambda \in \mathbb{R}^d$, $\phi_1(\gamma) = K_T^0(\omega, \gamma)$ and $\phi_2(\gamma) = K_T^0(\lambda, \gamma)$, and replace them in (3.2) to get

$$\psi(\omega, \lambda) = K_T^0(\omega, \lambda) - \int_{\mathbb{R}^d} K_T^0(\omega, \gamma) \overline{K_T^0(\lambda, \gamma)} F_1(d\gamma).$$

Conversely, since $\psi \in \mathcal{L}_{\Pi_T}(F_0) \otimes \mathcal{L}_{\Pi_T}(F_0)$, by the reproducing kernel property we get $\psi(\omega, \lambda) = \langle \psi, K_T^0(\omega, \cdot) \otimes K_T^0(\lambda, \cdot) \rangle_{F_0 \otimes F_0}$. Also, note that by the form of ψ , we have $\psi(\omega, \lambda) = \langle K_T^0(\omega, \cdot), K_T^0(\lambda, \cdot) \rangle_{F_0} - \langle K_T^0(\omega, \cdot), K_T^0(\lambda, \cdot) \rangle_{F_1}$. Combining them together, we get

$$\langle K_T^0(\omega, \cdot), K_T^0(\lambda, \cdot) \rangle_{F_0} - \langle K_T^0(\omega, \cdot), K_T^0(\lambda, \cdot) \rangle_{F_1} = \langle \psi, K_T^0(\omega, \cdot) \otimes K_T^0(\lambda, \cdot) \rangle_{F_0 \otimes F_0}. \quad (3.3)$$

Now, since the $\text{span}\{K_T^0(\omega, \cdot); \omega \in \mathbb{R}^d\}$ is dense in $\mathcal{L}_{\Pi_T}(F_0)$ ($= \mathcal{L}_{\Pi_T}(F_1)$), Equality (3.3) holds true for all the elements in $\mathcal{L}_{\Pi_T}(F_0)$. \square

Checking the first assumption in Theorem 3.2 may not be easy in general since we need to compare the norms of all the elements in the space $\mathcal{L}_{\Pi_T}^e$ under two different measures. For that purpose, in the following, we will find equivalent conditions which may be easier to verify in application.

It is well known (See [10], p. 1009) that for $\psi \in L^2(F \otimes F)$, a Hilbert-Schmidt operator on $L^2(F)$ can be defined as follows

$$(\psi\phi)(\omega) = \int_{\mathbb{R}^d} \psi(\omega, \lambda)\phi(\lambda) F(d\lambda) \quad (3.4)$$

for every $\phi \in L^2(F)$. If we use specifically the ψ in Theorem 3.2, and restrict the domain to $\mathcal{L}_{\Pi_T}(F)$, we will have again a Hilbert-Schmidt operator on $\mathcal{L}_{\Pi_T}(F)$. Note that the image of the operator ψ is in fact inside the $\mathcal{L}_{\Pi_T}(F)$. To prove this, observe that for $\phi \in \mathcal{L}_{\Pi_T}(F)$,

$$\begin{aligned} (\pi_{\mathcal{L}_{\Pi_T}}(\psi\phi))(\omega) &= \int_{\mathbb{R}^d} (\psi\phi)(x)\overline{K_T(\omega, x)} F(dx) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(y)\psi(x, y)\overline{K_T(\omega, x)} F(dx)F(dy) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(y)K_T(x, \gamma)\overline{K_T(y, \gamma)}\overline{K_T(\omega, x)} F(dx)F(dy) \\ &\quad \times (F(d\gamma) - F_1(d\gamma)) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(y)K_T(\omega, \gamma)\overline{K_T(y, \gamma)} (F(d\gamma) - F_1(d\gamma))F(dy) \\ &= \int_{\mathbb{R}^d} \phi(y)\psi(\omega, y) F(dy) = (\psi\phi)(\omega). \end{aligned}$$

This argument shows that $\psi\phi \in \mathcal{L}_{\Pi_T}(F)$ for any $\phi \in \mathcal{L}_{\Pi_T}(F)$. Also, observe that since $\psi(\omega, \lambda) = \overline{\psi(\lambda, \omega)}$, the operator ψ is self-adjoint. This fact together with compactness of this operator (Since ψ is a Hilbert-Schmidt operator, it is already compact, see [10], p. 1009) enable us to use the Spectral Theorem for compact normal operators (See [10], Corollary 5, p. 905), which we will use in the proof of the next theorem. In fact, the next theorem is an extension of Theorem 4.3 in [33] to the setting of random fields and shows that the first condition in Theorem 3.2 can be replaced by $1 \notin \sigma(\psi)$, where $\sigma(\psi)$ is the spectrum of the operator ψ . Recall that $\sigma(\psi)$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - \psi$ is not an invertible operator where I is the identity operator (cf. [10], p. 902).

Theorem 3.3. Two GRFs with stationary increments and spectral measures F_0 and F_1 with F_0 satisfying the condition (C1), are equivalent on Π_T if and only if the function defined by

$$\psi(\omega, \lambda) = K_T^0(\omega, \lambda) - \int_{\mathbb{R}^d} K_T^0(\omega, \gamma)\overline{K_T^0(\lambda, \gamma)} F_1(d\gamma) \quad (3.5)$$

belongs to $\mathcal{L}_{\Pi_T}(F_0) \otimes \mathcal{L}_{\Pi_T}(F_0)$, and $1 \notin \sigma(\psi)$.

Proof. From (3.2), by putting $\phi_1 = \phi_2 = \phi$, and the definition of the operator ψ in (3.4), we get

$$\frac{\|\phi\|_{F_1}^2}{\|\phi\|_{F_0}^2} = 1 - \frac{\langle \psi\phi, \phi \rangle_{F_0}}{\|\phi\|_{F_0}^2} \quad (3.6)$$

for all $\phi \in \mathcal{L}_{\Pi_T}(F_0)$. This simply implies that first $\sigma(\psi) \subseteq (-\infty, 1]$, and second there exists a finite positive constant C such that $\|\phi\|_{F_1} \leq C\|\phi\|_{F_0}$ for all $\phi \in \mathcal{L}_{\Pi_T}^e$ since ψ is a bounded operator. This fact shows that proving $\psi \in L^2(F_0 \otimes F_0)$ is helping us to verify half of what we need in the first condition of Theorem 3.2 as well. What remains is to show that $1 \notin \sigma(\psi)$ if and only if there exists a positive constant c such that $\|\phi\|_{F_0} \leq c\|\phi\|_{F_1}$ for all $\phi \in \mathcal{L}_{\Pi_T}^e$.

First, suppose that $\|\phi\|_{F_0} \leq c\|\phi\|_{F_1}$ for some $c > 0$. If $1 \in \sigma(\psi)$, it means that there exists $\phi \in \mathcal{L}_{\Pi_T}(F_0)$ with $\|\phi\|_{F_0} = 1$ such that $\psi\phi = \phi$. Putting it in (3.6), we get $\|\phi\|_{F_1} = 0$ which is contradiction. Conversely, suppose $1 \notin \sigma(\psi)$, and also there exists a sequence $\phi_n \in \mathcal{L}_{\Pi_T}^e$ such that $\|\phi_n\|_{F_0} = 1$ for all $n \geq 1$, and $\|\phi_n\|_{F_1} \rightarrow 0$ as $n \rightarrow \infty$. Since ψ is a self-adjoint compact operator, by Corollary 5 on p. 905 in [10], there exists a countable orthonormal basis for $\mathcal{L}_{\Pi_T}(F_0)$ consisting of eigenvectors of ψ , denoting them by $g_j, j = 1, 2, \dots$ with corresponding eigenvalues λ_j . Now, each ϕ_n has the representation $\phi_n = \sum_{j=1}^{\infty} a_{nj}g_j$ for $a_{nj} \in \mathbb{R}$. Putting this sequence (3.6), we get that $\langle \psi\phi_n, \phi_n \rangle_{F_0} \rightarrow 1$ which means $\sum_{j=1}^{\infty} a_{nj}^2\lambda_j \rightarrow 1$ as $n \rightarrow \infty$. Now, since $1 = \|\phi_n\|_{F_0}^2 = \sum_{j=1}^{\infty} a_{nj}^2$, we can rewrite the above equation as $0 \leq \sum_{j=1}^{\infty} a_{nj}^2(1 - \lambda_j) \rightarrow 0$ (This quantity is non-negative since all the eigenvalues are bounded above by 1). Since $1 \notin \sigma(\psi)$, and $\{\lambda_j, j = 1, 2, \dots\}$ has no accumulation points in \mathbb{C} except possibly 0 (See [10], Corollary 5, p. 905), there exists $\epsilon > 0$ such that $\sup\{\lambda_j, j \geq 1\} = 1 - \epsilon$. However, this implies that $\sum_{j=1}^{\infty} a_{nj}^2(1 - \lambda_j) \geq \epsilon \sum_{j=1}^{\infty} a_{nj}^2 = \epsilon$ for all $n \in \mathbb{N}$, which is contradiction by the fact that this sequence must go to 0 when n goes to ∞ . This completes the proof. \square

Remark 3.1. Notice that based on the proof of Theorem 3.3, we can change the first condition in Theorem 3.2 to

(i)' There exists a positive constant c such that $\|\phi\|_{F_0} \leq c\|\phi\|_{F_1}$ for all $\phi \in \mathcal{L}_{\Pi_T}^e$.

As Lemma 2.5 emphasizes that the behavior of the spectral measure at origin does not affect the structure of the space $\mathcal{L}_{\Pi_T}(F)$, one might expect the same formation in terms of the equivalence of Gaussian measures. The following theorem shows that changing the spectral measure on bounded subsets of \mathbb{R}^d will not affect the equivalence of the corresponding GRFs. In other words, for checking the equivalence of GRFs, only the behavior of their spectral measures at infinity is important.

Theorem 3.4. Suppose two GRFs with stationary increments have spectral measures F_0 and F_1 such that F_0 satisfies the condition (C1), and $F_0 = F_1$ on I^c , where I is a bounded subset of \mathbb{R}^d . Then, these two GRFs are locally equivalent.

Proof. Define $\tilde{F}_1(d\lambda) = 1_{I^c}(\lambda)F_0(d\lambda)$. First, we show that F_0 and \tilde{F}_1 will produce locally equivalent GRFs with stationary increments. For that, fix $T > 0$. We will investigate the equivalence of measures on Π_T . The function ψ appearing in Theorem 3.3 in this case is given by

$$\psi(\omega, \lambda) = \int_I K_T^0(\omega, \gamma) \overline{K_T^0(\lambda, \gamma)} F_0(d\gamma).$$

Notice that by the reproducing kernel property,

$$\psi(\omega, \lambda) = \pi_{\mathcal{L}_{\Pi_T}}(K_T^0(\omega, \cdot)1_I(\cdot))(\lambda) = \overline{\pi_{\mathcal{L}_{\Pi_T}}(K_T^0(\lambda, \cdot)1_I(\cdot))(\omega)} \quad (3.7)$$

The specific representation of the function $\psi(\omega, \lambda)$ in (3.7) helps us to show that $\psi(\omega, \lambda) \in \mathcal{L}_{\Pi_T}(F_0) \otimes \mathcal{L}_{\Pi_T}(F_0)$. The idea of the proof is to first show that $\psi(\omega, \lambda) \in L^2(F_0 \otimes F_0)$ and then, use the projection technique to further derive that $\psi(\omega, \lambda) \in \mathcal{L}_{\Pi_T}(F_0) \otimes \mathcal{L}_{\Pi_T}(F_0)$. Note that

$$\begin{aligned} \|\psi\|_{F_0 \otimes F_0}^2 &= \int_{\mathbb{R}^d} \|\pi_{\mathcal{L}_D(F_0)}(K_T^0(\omega, \cdot)1_I(\cdot))\|_{F_0}^2 F_0(d\omega) \\ &\leq \int_{\mathbb{R}^d} \|K_T^0(\omega, \cdot)1_I(\cdot)\|_{F_0}^2 F_0(d\omega) \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |K_T^0(\omega, \gamma)|^2 1_I(\gamma) F_0(d\gamma) \right) F_0(d\omega) \quad (3.8) \\ &= \int_{\mathbb{R}^d} 1_I(\gamma) K_T^0(\gamma, \gamma) F_0(d\gamma) \\ &\leq C \int_I |\gamma|^2 F_0(d\gamma) < +\infty. \end{aligned}$$

The second inequality in (3.8) is based on Proposition 2.4 and the fact that I is bounded. Now, we prove that the projection of $\psi(\omega, \lambda)$ to the space $\mathcal{L}_{\Pi_T}(F_0) \otimes \mathcal{L}_{\Pi_T}(F_0)$ is in fact itself. This verifies that $\psi(\omega, \lambda) \in \mathcal{L}_{\Pi_T}(F_0) \otimes \mathcal{L}_{\Pi_T}(F_0)$. To this end, observe that

$$\begin{aligned} (\pi_{\mathcal{L}_{\Pi_T}(F_0) \otimes \mathcal{L}_{\Pi_T}(F_0)}(\psi))(\omega, \lambda) &= \langle \psi, K_T^0(\omega, \cdot) \otimes K_T^0(\lambda, \cdot) \rangle_{F_0 \otimes F_0} \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(x, y) \overline{K_T^0(\omega, x)} \overline{K_T^0(\lambda, y)} F_0(dy) F_0(dx) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \pi_{\mathcal{L}_{\Pi_T}}(K_T^0(x, \cdot)1_I(\cdot)) (y) \overline{K_T^0(\omega, x)} \overline{K_T^0(\lambda, y)} \\ &\quad \times F_0(dy) F_0(dx) \\ &= \int_{\mathbb{R}^d} \overline{\pi_{\mathcal{L}_{\Pi_T}}(K_T^0(\lambda, \cdot)1_I(\cdot)) (x)} \overline{K_T^0(\omega, x)} F_0(dx) \\ &= \overline{\pi_{\mathcal{L}_{\Pi_T}}(K_T^0(\lambda, \cdot)1_I(\cdot)) (\omega)} \\ &= \psi(\omega, \lambda). \end{aligned}$$

This implies $\psi \in \mathcal{L}_{\Pi_T}(F_0) \otimes \mathcal{L}_{\Pi_T}(F_0)$.

It remains to show that $1 \notin \sigma(\psi)$. For that purpose, take an arbitrary $\phi \in \mathcal{L}_{\Pi_T}(F_0)$, and observe that (We use the fact that $K_T^0(\omega, \lambda) = \overline{K_T^0(\lambda, \omega)}$, see [2], p. 344.)

$$\begin{aligned} (\psi\phi)(\lambda) &= \int_{\mathbb{R}^d} \phi(\omega) \psi(\lambda, \omega) F_0(d\omega) \\ &= \int_{\mathbb{R}^d} \phi(\omega) \left(\int_I K_T^0(\lambda, \gamma) \overline{K_T^0(\omega, \gamma)} F_0(d\gamma) \right) F_0(d\omega) \\ &= \overline{\int_I \int_{\mathbb{R}^d} \overline{\phi(\omega)} \overline{K_T^0(\lambda, \gamma)} K_T^0(\gamma, \omega) F_0(d\omega) F_0(d\gamma)} \\ &= \overline{\int_I \overline{\phi(\gamma)} \overline{K_T^0(\lambda, \gamma)} F_0(d\gamma)} \\ &= \pi_{\mathcal{L}_{\Pi_T}(F_0)}(\overline{\phi}1_I)(\lambda). \end{aligned}$$

Therefore, if $\psi\phi = \phi$, it implies in particular that $\|\phi\|_{F_0} \leq \|\overline{\phi}1_I\|_{F_0}$. This means $\phi = 0$ almost everywhere with respect to F_0 in I^c . Hence, since ϕ is an entire function, this implies that $\phi = 0$. Thus, 1 cannot be in the spectrum of ψ .

So far, we proved GRFs with spectral measures F_0 and \tilde{F}_1 are locally equivalent, but since $F_0 = F_1$ on I , similarly, we can say that F_1 and \tilde{F}_1 produce locally equivalent GRFs. Putting these two together, we get the desired result. \square

Theorems 3.2 and 3.3 give necessary and sufficient conditions for equivalence of GRFs with stationary increments, but it might be difficult to verify the conditions in these theorems. In the literature, there are sufficient conditions for equivalence of certain GRFs in terms of their spectral densities. These conditions are easily verifiable once the two spectral densities are known. For example, we refer to [17], Theorem 17, p. 104, [28], Theorem 4, and [38], theorem 4, p. 156 for stationary GRFs; and to [33, 34], and [31] for some nonstationary cases. The following is the main theorem of this paper which gives an explicit sufficient condition in terms of the spectral measures for the equivalence of GRFs with stationary increments.

Theorem 3.5. Suppose that the spectral measure F_0 and F_1 have positive densities f_0 and f_1 with respect to the Lebesgue measure, and F_0 satisfies the condition (C1). If there exists a finite constant $C > 0$ such that $\|\phi\|_{F_0} \leq C\|\phi\|_{F_1}$ for all $\phi \in \mathcal{L}_{\Pi_T}^e$, and

$$\int_{|\lambda|>k} \left(\frac{f_1(\lambda) - f_0(\lambda)}{f_0(\lambda)} \right)^2 K_T^0(\lambda, \lambda) f_0(\lambda) d\lambda < \infty \quad (3.9)$$

for some $k > 0$, then GRFs with stationary increments and spectral measures F_0 and F_1 are equivalent on Π_T .

Proof. Applying Theorem 3.4, we can change the value of f_1 on any bounded set, without having any consequences on the equivalence. So, we assume here that $f_0 = f_1$ on $|\lambda| \leq k$. The function ψ in Theorem 3.3 here will be of the form

$$\begin{aligned} \psi(\omega, \lambda) &= \int_{\mathbb{R}^d} K_T^0(\omega, \gamma) \overline{K_T^0(\lambda, \gamma)} \left(f_0(\gamma) - f_1(\gamma) \right) d\gamma \\ &= \pi_{\mathcal{L}_{\Pi_T}(F_0)} \left(K_T^0(\omega, \cdot) \frac{f_0 - f_1}{f_0} \right) (\lambda). \end{aligned}$$

(Since $|K_T^0(\omega, \lambda)|^2 \leq K_T^0(\omega, \omega) K_T^0(\lambda, \lambda)$, (3.9) implies that $K_T^0(\omega, \cdot) \frac{f_0 - f_1}{f_0} \in L^2(F_0)$ for all $\omega \in \mathbb{R}^d$. Hence using the orthogonal projection is eligimate). Now, it follows that

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\psi(\omega, \lambda)|^2 F_0(d\lambda) F_0(d\omega) &= \int_{\mathbb{R}^d} \left\| \pi_{\mathcal{L}_D(F_0)} \left(K_T^0(\omega, \cdot) \frac{f_0 - f_1}{f_0} \right) \right\|_{F_0}^2 F_0(d\omega) \\ &\leq \int_{\mathbb{R}^d} \left\| K_T^0(\omega, \cdot) \frac{f_0 - f_1}{f_0} \right\|_{F_0}^2 F_0(d\omega) \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |K_T^0(\omega, \gamma)|^2 \left(\frac{f_0(\gamma) - f_1(\gamma)}{f_0(\gamma)} \right)^2 F_0(d\gamma) \right) F_0(d\omega) \\ &= \int_{\mathbb{R}^d} \left(\frac{f_0(\gamma) - f_1(\gamma)}{f_0(\gamma)} \right)^2 K_T^0(\gamma, \gamma) f_0(\gamma) d\gamma \\ &= \int_{|\gamma|>k} \left(\frac{f_1(\gamma) - f_0(\gamma)}{f_0(\gamma)} \right)^2 K_T^0(\gamma, \gamma) f_0(\gamma) d\gamma < \infty \end{aligned}$$

by the integrability assumption (3.9). Hence $\psi \in L^2(F_0 \otimes F_0)$.

Now, we apply similar arguments as in the proof of Theorem 3.4 to show that in fact $\psi \in \mathcal{L}_{\Pi_T}(F_0) \otimes \mathcal{L}_{\Pi_T}(F_0)$. To this end, observe that

$$\begin{aligned} (\pi_{\mathcal{L}_{\Pi_T}(F_0) \otimes \mathcal{L}_{\Pi_T}(F_0)}(\psi))(\omega, \lambda) &= \langle \psi, K_T^0(\omega, \cdot) \otimes K_T^0(\lambda, \cdot) \rangle_{F_0 \otimes F_0} \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(x, y) \overline{K_T^0(\omega, x) K_T^0(\lambda, y)} F_0(dy) F_0(dx) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \pi_{\mathcal{L}_{\Pi_T}(F_0)} \left(K_T^0(x, \cdot) \frac{f_0 - f_1}{f_0} \right) (y) \overline{K_T^0(\omega, x) K_T^0(\lambda, y)} \\ &\quad \times F_0(dy) F_0(dx) \\ &= \int_{\mathbb{R}^d} \pi_{\mathcal{L}_{\Pi_T}(F_0)} \left(K_T^0(\lambda, \cdot) \frac{f_0 - f_1}{f_0} \right) (x) \overline{K_T^0(\omega, x)} F_0(dx) \\ &= \pi_{\mathcal{L}_{\Pi_T}(F_0)} \left(K_T^0(\lambda, \cdot) \frac{f_0 - f_1}{f_0} \right) (\omega) \\ &= \pi_{\mathcal{L}_{\Pi_T}(F_0)} \left(K_T^0(\omega, \cdot) \frac{f_0 - f_1}{f_0} \right) (\lambda) = \psi(\omega, \lambda). \end{aligned}$$

This completes the proof. □

In (3.9), in addition to the behavior of the spectral densities at infinity, the growth rate of the diagonal elements of the reproducing kernels of the space $\mathcal{L}_{\Pi_T}(F_0)$ at infinity also plays an important role. Since finding explicit forms of reproducing kernels are difficult, we need at least to find upper bounds for the growth rate of the diagonal terms. The following condition on spectral density helps us to accomplish this task:

- (C2) For spectral density f , there exist an entire function ϕ_0 on \mathbb{C}^d of finite exponential type such that $f(\lambda) \asymp |\phi_0(\lambda)|^2$ as $|\lambda| \rightarrow \infty$ on \mathbb{R}^d .

The following lemma shows that under (C2) we have an upper bound for the behavior of the reproducing kernels on the diagonal at infinity.

Lemma 3.2. *Suppose f_0 is a spectral density such that it satisfies (C1) and (C2) for some entire function ϕ_0 . Then, for $T > 0$, there exists a finite constant $C > 0$ such that the reproducing kernel K_T^0 of $\mathcal{L}_{\Pi_T}(f_0)$ satisfies*

$$|K_T^0(\omega, \lambda)|^2 \leq C \frac{K_T^0(\omega, \omega)}{f_0(\lambda)}$$

for all $\omega, \lambda \in \mathbb{R}^d$ with $|\lambda|$ large enough. In particular,

$$|K_T^0(\lambda, \lambda)| \leq \frac{C}{f_0(\lambda)} \tag{3.10}$$

for all $\lambda \in \mathbb{R}^d$ with $|\lambda|$ large enough.

Proof. The idea of the proof is similar to the one in Lemma 3 in [34]. Put $f(\lambda) = |\phi_0(\lambda)|^2$. Since f and f_0 are comparable at ∞ , and f is bounded around 0, it is clear that f is satisfying both conditions (2.2) and (C1). This means we can define $\mathcal{L}_{\Pi_T}(f)$ the same way, and this space is also a RKHS. Consider an arbitrary orthonormal basis for this space, and denote them by $\psi_k, k = 1, 2, \dots$. Now, by Lemma 2.3, they are entire functions on \mathbb{C}^d with finite exponential type which doesn't depend on k . Further, we know $\psi_k \phi_0 \in L^2(\mathbb{R}^d)$ since

$$\int_{\mathbb{R}^d} |\psi_k(\lambda) \phi_0(\lambda)|^2 d\lambda = \int_{\mathbb{R}^d} |\psi_k(\lambda)|^2 f(\lambda) d\lambda = 1 < \infty.$$

Therefore, we can apply the Paley-Wiener Theorem ([24], Theorem 3.4.2, p. 171) to get $\psi_k \phi_0 = \widehat{g}_k$ for certain functions $g_k \in L^2(B)$ where B is a bounded subset of \mathbb{R}^d (Here \widehat{h} stands for the Fourier transform of h). By Parseval's identity, we can deduce that g_k 's are orthonormal in $L^2(B)$. It follows from Bessel's inequality that

$$\begin{aligned} \sum_k |\psi_k(\lambda)|^2 f(\lambda) &= \sum_k \left| \int_B e^{-i\langle t, \lambda \rangle} g_k(t) dt \right|^2 \\ &\leq \int_B |e^{i\langle t, \lambda \rangle}|^2 dt = m(B), \end{aligned}$$

where $m(B)$ is the Lebesgue measure of B . Therefore,

$$\sum_k |\psi_k(\lambda)|^2 \leq m(B)/f(\lambda) \leq C/f_0(\lambda)$$

for $\lambda \in \mathbb{R}^d$ with $|\lambda|$ large enough. Now, consider the reproducing kernels of $\mathcal{L}_{\Pi_T}(f_0)$, denoting them by $K_T^0(\omega, \cdot)$. Since $f(\lambda) \asymp f_0(\lambda)$ as $|\lambda| \rightarrow \infty$, by Lemma 2.5, $K_T^0(\omega, \cdot)$ belong to $\mathcal{L}_{\Pi_T}(f)$ as well for all $\omega \in \mathbb{R}^d$. Thus, we can expand it using the basis ψ_k , and get $K_T^0(\omega, \lambda) = \sum_k \langle K_T^0(\omega, \cdot), \psi_k \rangle_f \psi_k(\lambda)$, and then by Cauchy-Schwarz and Lemma 2.5, we get

$$\begin{aligned} |K_T^0(\omega, \lambda)|^2 &\leq \|K_T^0(\omega, \cdot)\|_f^2 \sum_k |\psi_k(\lambda)|^2 \\ &\leq c \|K_T^0(\omega, \cdot)\|_{f_0}^2 \sum_k |\psi_k(\lambda)|^2 \\ &= c K_T^0(\omega, \omega) \sum_k |\psi_k(\lambda)|^2, \end{aligned}$$

which makes the proof complete. \square

Theorem 3.5 in combination with Lemma 3.2 leads to an appealing result. If the relative difference between two spectral densities is square integrable at infinity, then the corresponding GRFs with stationary increments will be locally equivalent. We finish this section by proving this fact.

Theorem 3.6. Suppose that the spectral measures F_0 and F_1 have positive densities f_0 and f_1 with respect to the Lebesgue measure, with f_0 satisfying (C1) and (C2) for some entire function ϕ_0 on \mathbb{C}^d of finite exponential type. If there exists a finite constant $k > 0$ such that

$$\int_{|\lambda| > k} \left(\frac{f_1(\lambda) - f_0(\lambda)}{f_0(\lambda)} \right)^2 d\lambda < \infty, \quad (3.11)$$

then GRFs with stationary increments having spectral measures F_0 and F_1 are locally equivalent.

Proof. Thanks to Lemma 3.2 and Theorem 3.5, it is sufficient to prove that $1 \notin \sigma(\psi)$. In spirit of Theorem 3.4, we can assume that $f_0 = f_1$ on $|\lambda| \leq k$. Now, take an arbitrary element $\phi \in \mathcal{L}_D(f_0)$, and observe that by using the multidimensional Paley-Wiener Theorem ([24], Theorem 3.4.2, p. 171), we derive that $\phi \phi_0$ is the inverse Fourier transform of a squared integrable function g with bounded support, B in \mathbb{R}^d . This implies that

$$\begin{aligned} |\phi(\lambda) \phi_0(\lambda)|^2 &= \left| \int_B e^{-i\langle \lambda, \gamma \rangle} g(\gamma) d\gamma \right|^2 \\ &\leq \int_B g^2(\lambda) d\lambda < \infty \end{aligned}$$

for all $\lambda \in \mathbb{R}^d$. This means $\phi\phi_0$ is bounded on \mathbb{R}^d . This fact together with (3.11) imply that $\phi \frac{f_0 - f_1}{f_0} \in L^2(f_0)$. Now, observe that

$$\begin{aligned} (\psi\phi)(\lambda) &= \int_{\mathbb{R}^d} \phi(\omega)\psi(\lambda, \omega) F_0(d\omega) \\ &= \int_{\mathbb{R}^d} \phi(\omega) \left(\int_{|\gamma|>k} K_T^0(\lambda, \gamma) \overline{K_T^0(\omega, \gamma)} \left(\frac{f_0(\gamma) - f_1(\gamma)}{f_0(\gamma)} \right) F_0(d\gamma) \right) F_0(d\omega) \\ &= \frac{\int_{|\gamma|>k} \int_{\mathbb{R}^d} \overline{\phi(\omega)} K_T^0(\lambda, \gamma) K_T^0(\gamma, \omega) \left(\frac{f_0(\gamma) - f_1(\gamma)}{f_0(\gamma)} \right) F_0(d\omega) F_0(d\gamma)}{\int_{|\gamma|>k} \overline{\phi(\gamma)} \left(\frac{f_0(\gamma) - f_1(\gamma)}{f_0(\gamma)} \right) \overline{K_T^0(\lambda, \gamma)} F_0(d\gamma)} \\ &= \pi_{\mathcal{L}_{\Pi_T}(F_0)} \left(\overline{\phi} \frac{f_0 - f_1}{f_0} 1_{\{|\gamma|>k\}} \right) (\lambda). \end{aligned}$$

Now, similar to the proof of Theorem 3.4, if $\psi\phi = \phi$, we get that $\|\phi\|_{F_0} \leq \left\| \overline{\phi} \frac{f_0 - f_1}{f_0} 1_{\{|\gamma|>k\}} \right\|_{F_0}$. Letting $k \rightarrow \infty$, by the Dominated Convergence Theorem, we get $\|\phi\|_{F_0} = 0$. This implies that $\phi = 0$. Thus, 1 cannot be in the spectrum of ψ and this concludes proof. \square

4 Application

In this section, we apply the results in Section 3 to some anisotropic GRFs with stationary increments. In particular, we consider GRFs with stationary increments and spectral density of the form

$$f(\lambda) = \frac{1}{\left(\sum_{j=1}^d |\lambda_j|^{\beta_j} \right)^\gamma}, \quad (4.1)$$

where $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d \setminus \{0\}$, $\beta_j > 0$ for all $j = 1, \dots, d$, and $\gamma > \sum_{j=1}^d \frac{1}{\beta_j}$. The latter condition guaranties the integrability condition in (2.2) for spectral measures (See Proposition 2.1 in [37]). Fractal and smoothness properties of this family of GRFs are discussed in [37]. Now we apply Theorem 3.6 to determine the equivalence of Gaussian measures induced by these GRFs.

To this end, first notice that (C1) is obviously satisfied for spectral densities of the form (4.1). The next lemma shows that these spectral densities also satisfy (C2).

Lemma 4.1. *Spectral density functions of the form (4.1) satisfy condition (C2).*

Proof. First of all, it is obvious that

$$\frac{1}{\left(\sum_{j=1}^d |\lambda_j|^{\beta_j} \right)^\gamma} \asymp \frac{1}{\left(1 + \sum_{j=1}^d |\lambda_j|^{\beta_j} \right)^\gamma},$$

as $|\lambda| \rightarrow \infty$. Therefore, it suffices to prove the lemma for functions of the form on the right hand side. Now, similar to the construction made in the proof of Lemma 2.3 in [19], we can find a function $\phi \in L^2(B)$ for some bounded subset $B \subseteq \mathbb{R}^d$ such that

$$\frac{1}{\left(1 + \sum_{j=1}^d |\lambda_j|^{\beta_j} \right)^\gamma} \asymp |\widehat{\phi}(\lambda)|^2,$$

as $|\lambda| \rightarrow \infty$, which $\widehat{\phi}$ is the Fourier transform of ϕ . By the Paley-Wiener Theorem ([24], Theorem 3.4.2. p. 171), $\widehat{\phi}$ is actually the restriction on \mathbb{R}^d of an entire function on \mathbb{C}^d with finite exponential type. This finishes the proof. \square

The next theorem proves that, under certain conditions, the mixture of spectral densities of the form (4.1) will be equivalent to the one with the lowest decay rate at infinity. Similar results of this type have been proved by [33] and [6] for linear combinations of independent fractional Brownian motions.

Theorem 4.1. Suppose X and Y are two independent centered GRFs with stationary increments with spectral densities of the form (4.1) with parameters (β_j, γ) and (β'_j, γ') , respectively. Then, if

$$\gamma' > \frac{1}{2} \sum_{j=1}^d \left(\frac{1}{\beta'_j} \right) + \max_{1 \leq j \leq d} \left\{ \frac{\beta_j}{\beta'_j} \right\} \gamma, \quad (4.2)$$

then X and $X + Y$ are locally equivalent.

Proof. Using Lemma 4.1 and Theorem 3.6, we only need to show

$$\int_{|\lambda| > 1} \frac{\left(\sum_{j=1}^d |\lambda_j|^{\beta_j} \right)^{2\gamma}}{\left(\sum_{j=1}^d |\lambda_j|^{\beta'_j} \right)^{2\gamma'}} d\lambda < \infty. \quad (4.3)$$

By using the inequality $(a + b)^p \leq 2^p(a^p + b^p)$, we can break the integral in (4.3) into d integrals. Thus, it's enough to show for each fixed $j = 1, \dots, d$

$$I_j := \int_{|\lambda| > 1} \frac{|\lambda_j|^{2\beta_j\gamma}}{\left(\sum_{k=1}^d |\lambda_k|^{\beta'_k} \right)^{2\gamma'}} d\lambda < \infty. \quad (4.4)$$

Since $|\lambda| > 1$, this implies that $|\lambda_k| > 1/\sqrt{d}$ for some $k \in \{1, \dots, d\}$. We distinguish two cases: Case I is when $k = j$, and Case II is when $k \neq j$. In both cases, we use the following fact that, for positive constants β and γ , and nonnegative constant b , there exists a finite positive constant c such that for all $a > 0$

$$\begin{aligned} \int_0^\infty \frac{x^b}{(a + x^\beta)^\gamma} dx &= a^{-(\gamma - \frac{1}{\beta} - \frac{b}{\beta})} \int_0^\infty \frac{y^b}{(1 + y^\beta)^\gamma} dy \\ &= \begin{cases} c a^{-(\gamma - \frac{1}{\beta} - \frac{b}{\beta})} & \text{if } \beta\gamma - b > 1, \\ \infty & \text{if } \beta\gamma - b \leq 1. \end{cases} \end{aligned} \quad (4.5)$$

First, let's consider case I. By applying (4.5) d times, we have

$$\begin{aligned} I_j &\leq \int_{\frac{1}{\sqrt{d}}}^\infty |\lambda_j|^{2\beta_j\gamma} \underbrace{\int_0^\infty \dots \int_0^\infty}_{d-1} \frac{d\lambda}{\left(\sum_{r=1}^d |\lambda_r|^{\beta'_r} \right)^{2\gamma'}} \\ &\leq c' \int_{\frac{1}{\sqrt{d}}}^\infty \frac{|\lambda_j|^{2\beta_j\gamma}}{\left(|\lambda_j|^{\beta'_j} \right)^{2\gamma' - \sum_{r \neq j} \frac{1}{\beta'_r}}} d\lambda_j < \infty \end{aligned}$$

since $\beta'_j \left(2\gamma' - \sum_{r \neq j} \frac{1}{\beta'_r} \right) > 2\beta_j\gamma + 1$ due to condition (4.2).

Next, we consider case II, where $k \neq j$. Similar to case I, we use (4.5) iteratively, but we take the integration in different order. Denote the integration in λ_i for $i \neq j, k$ by

$d\lambda_{\setminus j,k}$, and observe

$$\begin{aligned} I_j &\leq \int_{\frac{1}{\sqrt{d}}}^{\infty} d\lambda_k \underbrace{\int_0^{\infty} \dots \int_0^{\infty}}_{d-1} \frac{|\lambda_j|^{2\beta_j\gamma}}{\left(\sum_{r=1}^d |\lambda_r|^{\beta'_r}\right)^{2\gamma'}} d\lambda_j d\lambda_{\setminus j,k} \\ &\leq c \int_{\frac{1}{\sqrt{d}}}^{\infty} d\lambda_k \underbrace{\int_0^{\infty} \dots \int_0^{\infty}}_{d-2} \frac{1}{\left(\sum_{r=1}^d |\lambda_r|^{\beta'_r}\right)^{2\gamma' - \frac{1}{\beta'_j} - \frac{2\beta_j\gamma}{\beta'_j}}} d\lambda_{\setminus j,k} \\ &\leq c' \int_{\frac{1}{\sqrt{d}}}^{\infty} \frac{1}{\left(|\lambda_k|^{\beta'_k}\right)^{2\gamma' - \frac{2\beta_j\gamma}{\beta'_j} - \sum_{r \neq k} \frac{1}{\beta'_r}}} d\lambda_k < \infty, \end{aligned}$$

where the second and the third inequality follow since $2\gamma'\beta'_j > 2\beta_j\gamma + 1$ and $\beta'_k \left(2\gamma' - \frac{2\beta_j\gamma}{\beta'_j} - \sum_{r \neq k} \frac{1}{\beta'_r}\right) > 1$, respectively using the assumption (4.2). This finishes the proof. \square

Next, we consider a similar situation as in Theorem 4.1, but this time we put discrete spectral measure mixed with the ones of the form (4.1). For that purpose, consider discrete spectral measure of the form

$$F(\{-\gamma^n\}) = F(\{\gamma^n\}) = \alpha_n, \tag{4.6}$$

where $\gamma^n \in \mathbb{R}^d$, $\alpha_n \geq 0$, for $n \geq 1$, and $\sum_{n=1}^{\infty} \frac{|\gamma^n|^2}{1+|\gamma^n|^2} \alpha_n < \infty$. If $\{\gamma^n, n = 1, 2, \dots\}$ is a bounded subset of \mathbb{R}^d , then in view of Theorem 3.4, this spectral measure will not affect the equivalence of Gaussian measures. Therefore, we consider here only the case where $|\gamma^n| \rightarrow \infty$ as $n \rightarrow \infty$.

Theorem 4.2. Let X and Y be two independent centered GRFs with stationary increments with spectral measures F_X and F_Y . Suppose F_X has density with respect to Lebesgue measure on \mathbb{R}^d , denoted by f , which satisfies both conditions (C1) and (C2), and F_Y is a discrete measure of the form (4.6). Then, if

$$\sum_{n>N} \frac{\alpha_n}{f(\gamma^n)} < \infty, \tag{4.7}$$

for some $N \geq 1$, then X and $X + Y$ are locally equivalent.

Proof. First of all, using Theorem 3.4, we can assume $\alpha_n = 0$ for $n = 1, \dots, N$, without having any consequences on the equivalence of Gaussian measures. Second, observe that for all $\phi \in \mathcal{L}_{\Pi_T}^e$, $\|\phi\|_{F_X} \leq \|\phi\|_{F_X+F_Y}$, which by Remark 3.1, is equivalent to condition (i) in Theorem 3.2.

All we need to prove is then to show that the function ψ in Theorem 3.2 is in $\mathcal{L}_{\Pi_T}(f) \otimes \mathcal{L}_{\Pi_T}(f)$. For that, note that the function ψ can be written as

$$\psi(\omega, \gamma) = \sum_{n>N} \alpha_n K_T(\omega, \gamma^n) \overline{K_T(\lambda, \gamma^n)}. \tag{4.8}$$

Observe that functions of the form $K_T(\cdot, \gamma^n) \overline{K_T(\cdot, \gamma^n)}$ belong to the space $\mathcal{L}_{\Pi_T}^e \otimes \mathcal{L}_{\Pi_T}^e$ since $K_T(\cdot, \gamma^n)$'s are the reproducing kernel elements of $\mathcal{L}_{\Pi_T}^e$. Therefore, if we can show that $\|\psi\|_{f \otimes f}^2 < +\infty$, it implies that the function ψ defined in (4.8) is the $L^2(f \otimes f)$ limit of the partial sums and hence, $\psi \in \mathcal{L}_{\Pi_T}(f) \otimes \mathcal{L}_{\Pi_T}(f)$ since $\mathcal{L}_{\Pi_T}(f) \otimes \mathcal{L}_{\Pi_T}(f)$ is the closure

in $L^2(f \otimes f)$ of the space $\mathcal{L}_{\Pi_T}^e \otimes \mathcal{L}_{\Pi_T}^e$. To this end, observe that

$$\begin{aligned} \|\psi\|_{f \otimes f} &\leq \sum_{n>N} \|\alpha_n K_T(\cdot, \gamma^n) \overline{K_T(\cdot, \gamma^n)}\|_{f \otimes f} \\ &= \sum_{n>N} \alpha_n K_T(\gamma^n, \gamma^n) \\ &\leq C \sum_{n>N} \frac{\alpha_n}{f(\gamma^n)} < +\infty, \end{aligned}$$

by the assumption (4.7). The proof is complete. \square

Finally, we give another application of Theorem 3.6. We consider the spectral densities of the following form

$$f(\lambda) = \frac{1}{\left(\sum_{j=1}^d |\lambda_j|^{H_j}\right)^{Q+2}}, \quad (4.9)$$

where $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d \setminus \{0\}$, $0 < H_j < 1$ for all $j = 1, \dots, d$, and $Q = \sum_{j=1}^d \frac{1}{H_j}$. According to Remark 2.2 in [37], every positive function of the form (4.1) is comparable to a function of the form (4.9) as $|\lambda| \rightarrow \infty$. See [37] for the explicit relationship between the parameters $(\beta_1, \dots, \beta_d, \gamma)$ in (4.1) and (H_1, \dots, H_d) in (4.9). [37] proved that the smoothness and fractal properties of a Gaussian random field with spectral density (4.1) is characterized by the corresponding parameters (H_1, \dots, H_d) . The following theorem shows that a similar phenomenon occurs for equivalence of these Gaussian random fields.

Theorem 4.3. Suppose f_0 and f_1 are spectral densities of the form (4.9) with parameters H_j^0 and H_j^1 ($j = 1, \dots, d$), respectively. Then, GRFs with stationary increments and spectral densities f_0 and f_1 are locally equivalent if and only if $H_j^0 = H_j^1$ for all $j = 1, \dots, d$.

Proof. The sufficiency is obvious, so we only need to prove the necessity. Suppose for some $k \in \{1, \dots, d\}$, $H_k^0 < H_k^1$. By Lemma 3.2 in [37], there exist $c_1, c_2 > 0$, such that for all $t \in \mathbb{R}^d$

$$c_1 \sum_{j=1}^d |t|^{2H_j^i} \leq \|e_t\|_{f_i}^2 \leq c_2 \sum_{j=1}^d |t|^{2H_j^i}, \quad (4.10)$$

for $i = 0, 1$. If we simply choose $t \in \mathbb{R}^d$ with $t_k = l$, and $t_j = 0$ for $j \neq k$, we get

$$\frac{\|e_t\|_{f_1}^2}{\|e_t\|_{f_0}^2} \leq \frac{c_2}{c_1} l^{2(H_k^1 - H_k^0)} \rightarrow 0 \quad \text{as } l \rightarrow 0.$$

This violates the necessary condition for equivalence of Gaussian measures in Theorem 3.1. \square

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Acknowledgments. The research is supported in part by NSF grants DMS-1612885 and DMS-1607089.

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