Electronic Journal of Probability

Electron. J. Probab. **24** (2019), no. 19, 1–38.
ISSN: 1083-6489 https://doi.org/10.1214/19-EJP262

Scaling limits of population and evolution processes in random environment

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Abstract

We propose a general method for investigating scaling limits of finite dimensional Markov chains to diffusions with jumps. The results of tightness, identification and convergence in law are based on the convergence of suitable characteristics of the chain transition. We apply these results to population processes recursively defined as sums of independent random variables. Two main applications are developed. First, we extend the Wright-Fisher model to independent and identically distributed random environments and show its convergence, under a large population assumption, to a Wright-Fisher diffusion in random environment. Second, we obtain the convergence in law of generalized Galton-Watson processes with interaction in random environment to solutions of stochastic differential equations with jumps.

Keywords: tightness; diffusions with jumps; characteristics; semimartingales; Galton-Watson process; Wright-Fisher process; random environment.
AMS MSC 2010: 60J27; 60J75; 60F15; 60F05; 60F10; 92D25.
Submitted to EJP on February 19, 2018, final version accepted on January 2, 2019.

1 Introduction

This work is a contribution to the study of scaling limits of discrete population models. The parameter $N \in \mathbb{N}$ scales the population sizes. The population processes $(Z_n^N : n \in \mathbb{N})$ are $\mathbb{N}^d$-valued Markov chains inductively defined by

$$Z_{n+1}^N = \sum_{j=1}^{F_N(Z_n^N)} L_{j,n}^N(Z_n^N, E_n^N),$$

where F_N is a function giving the number of individual events. For each z, e, N, $(L_{i,n}^N(z,e) : i, n \geq 1)$ is a family of independent identically distributed random variables and E_n^N is a $\mathbb{R}^d$-random variable describing the environment at generation n. This

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class of processes includes well known processes in population dynamics and population genetics. In particular, Galton-Watson processes correspond to the case when $F_N(z) = z$ and $L_{i,n}^N = L^N$, i.e it does not depend on (z, e), while Wright-Fisher processes are obtained when $F_N(z) = N$ and $L_{i,n}^N(z, e)$ are Bernoulli random variables with parameter z/N. More generally, these population models can also take into account the effect of random environment and include many additional ecological forces such as competition, cooperation and sexual reproduction.

We are interested in the convergence of the sequence of processes $(Z_{[v_N t]}^N/N : t \geq 0)$, as N tends to infinity, v_N being a time scale tending to infinity with N. We provide a unified framework adapted to population models and characterize this convergence through the asymptotic properties relying on v_N, F_N and L^N. Many works have been devoted to the approximation of Markov processes. They are essentially based on tightness arguments and identification of the martingale problem, see for example [12, 19]. Unfortunately, this general method does not satisfactorily apply to our framework since the required assumptions are difficult to check. Applying for instance this method to the classical Galton-Watson framework seems to lead to moment assumptions. However, it is well known from the works of Lamperti [24, 25] and Grimvall [16] that the finite dimensional convergence of the renormalized processes $(Z_{[v_N t]}^N/N : t \geq 0)$ with a time scale $v_N \to \infty$ is equivalent to the convergence of a characteristic triplet associated with (v_N, L^N) when N tends to infinity. In this case, the sequence of processes $(Z_{[v_N t]}^N/N : t \geq 0)$ converges as $N \to \infty$ to a Continuous State Branching Process (CSBP) defined as the unique strong solution of a Stochastic Differential Equation (SDE). The parameters of this SDE are given by the limiting characteristic triplet of (v_N, L^N). Note that the proof is based on the branching property, using either the Laplace exponent [16], or the relation with the convergence of the associated random walk to a spectrally positive Lévy process via a Lamperti time change (cf. [25] [8]). Lamperti also introduced a powerful transform in the stable framework, see e.g. [25] and [29] and [5]. Other time changes have been successfully used to obtain scaling limits of discrete processes, in particular for some diffusion approximations, see for instance [21] for branching processes in random environment, [9] for branching processes with immigration and [33] for controlled branching processes, amongst others. Such time change techniques seem essentially restricted to branching processes or stable processes or diffusion approximations. In our work, we are interested in the convergence in law of discrete Markov processes $(Z^N)_N$ which do not enjoy the branching property and may jump in the limit. The limiting processes may even be explosive and are not necessarily stable.

It is well known that the law of the process $(Z_{[v_N .]}^N/N)$ is determined by its initial law and the family of functions

$$x \to \mathcal{G}_x^N(H) = v_N \mathbb{E}\left(H(Z_1^N/N - x) | Z_0^N = Nx\right)$$

for H continuous and bounded on $\mathbb{R}^d$. Moreover, the asymptotic behavior of $\mathcal{G}_x^N(H)$ as $N \to \infty$ for a large enough class of functions H captures the convergence of the processes. In such discrete cases, Jacod and Shiryaev in [19, II.3, IX] prove that the tightness and the identification are deduced from the convergence as $N \to \infty$ of the characteristics of the semimartingales

$$\sum_{i \leq [v_N t]} \mathcal{G}_{Z_i^N/N}^N(H)$$

defined for certain functions H: a truncation (vector) function and its squares and a determining class of smooth functions vanishing in a neighborhood of 0. The convergence of Markov chains to Lévy driven SDEs proved in [20] essentially uses such strategy. Unfortunately, this strategy is difficult to apply in our framework, even for Galton-Watson processes. This is why we prove that the functions H can be chosen differently, belonging to some (rich enough) functional space $\mathcal{H}$, dense in the set of regular functions vanishing at zero for a norm equivalent to

$$\|H\| = \sup_{u\in(\mathbb{R}^d)^*} \frac{|H(u)|}{1\wedge |u|^2}.$$

The choice of the space $\mathcal{H}$ depends on the assumptions on the model. In our applications, we exploit the independence property of the variables $(L^N_{i,.}(.,.) : i \geq 1)$. The characterization of the law by the Laplace exponent is then used at the level of conditional increments, which is well adapted to the sum of independent non-negative random variables.

Our main motivations were the famous frameworks of population genetics and population dynamics. The efficiency of our method can be seen in the generalizations obtained for the approximation of Wright-Fisher and Galton-Watson chains. We first study a Wright-Fisher model with selection in a random environment impacting the selective advantage. The environments are assumed to be independent and identically distributed and the associated random walk converges to a Lévy process. We obtain the convergence of the joint law of the processes and random walks, by using the functional space

$$\mathcal{H} = \{(u,w) \in [-1,1]\times(-1,\infty) \to 1-e^{-ku-\ell w}\,;\, k,\ell \geq 0\}.$$

We thus derive a diffusion with jumps in random environment, which generalizes the Wright-Fisher diffusion with selection and takes into account small random fluctuations and punctual dramatic advantages in the selective effects.

The second application focuses on generalized Galton-Watson processes with reproduction law that is both density and environment dependent. We obtain a result of convergence in law to the so called *continuous state branching process with interaction in Lévy environment* henceforth called BPILE (introduced in [31, 17]). These processes have unbounded characteristics and the result is deduced from the convergence of the compactified processes

$$\exp(-Z^N_k/N).$$

To deal with the joint laws of the latter and the environment random walk, we use the space of functions from $[-1,1]\times(-1,\infty)$ to $\mathbb{R}$ defined by

$$\mathcal{H} = \{(v,w) \to v^k \exp(-\ell w) : k \geq 1, \ell \geq 0\} \cup \{(v,w) \to 1-\exp(-\ell w) : \ell \geq 1\}.$$

Our results extend the criterion for the convergence of a sequence of Galton-Watson processes as well as the results we know in random environment [21, 4] or with interactions [11, 32]. They are further applied to Galton-Watson processes with cooperation and to branching processes with logistic growth in random environment.

The paper is organized as follows. In Section 2, we give general results for the tightness, the identification and the convergence in law of a scaled Markov process to a diffusion with jumps in $\mathbb{R}^d$. The functional space $\mathcal{H}$ is introduced in Section 2.1. Tightness and identification results are stated in Section 2.2 by assuming the uniform convergence and boundedness of characteristics $\mathcal{G}^N_.(H)$ for any $H \in \mathcal{H}$. Convergence requires an additional uniqueness assumption, obtained from pathwise uniqueness in the applications, using standard techniques for non-negative SDE [18, 14]. Proofs of these general statements are given in Section 2.3. In Section 3, we apply our method to a Wright-Fisher model with selection in a random environment. We obtain in a suitable scaling limit a Wright-Fisher diffusion in random environment for which we prove uniqueness of solution. In Section 4 (Sections 4.1, 4.2, 4.3), we apply our method to Galton-Watson processes with reproduction law both density dependent and environment dependent. Section 4.4 is devoted to explosive CSBP with interaction in random environment. In particular, we consider Galton-Watson processes with cooperative effects. Section 4.5 is dedicated to the conservative case and an application to Galton-Watson processes with logistic competition and small environmental fluctuations is studied. Finally, we expect the method to be applied in various contexts, in particular for structured populations models with sexual reproduction, competition or cooperation, see Section 5.

Notation. For $x \in \mathbb{R}^d$, we denote by $|x|$ the euclidian norm of x. If $A \subset \mathbb{R}^d$, $\overline{A}$ is the closure of A in $\mathbb{R}^d$.

The functional norms are denoted by $\|.\|$. In particular the sup norm of a bounded function f on a set $\mathcal{U}$ is denoted by $\|f\|_{\mathcal{U},\infty}$. The sets $C_b(\mathcal{U},\mathbb{R})$ and $C_c(\mathcal{U},\mathbb{R})$ denote the spaces of continuous real functions defined on $\mathcal{U}$ respectively bounded and with compact support.

As usual, we write $h(u) = o(g(u))$ (resp. $h(u) \sim g(u)$) when $h(u)/g(u)$ tends to 0 (resp. to 1) as u tends to 0. Id denotes the identity function.

For any $\mathcal{U}$ subset of $\mathbb{R}^d$ containing a neighborhood of 0, we define $\mathcal{U}^*$ as $\mathcal{U} \setminus \{0\}$.

2 A criterion for tightness and convergence in law

Let $\mathcal{X}$ be a Borel subset of $\mathbb{R}^d$ and $\mathcal{U}$ be a closed subset of $\mathbb{R}^d$ containing a neighborhood of 0.

Let us introduce a scaling parameter $N \geq 1$. For any N, we consider a discrete time $\mathcal{X}$-valued Markov chain $(X_k^N : k \in \mathbb{N})$ satisfying for any $k \geq 0$,

$$\mathcal{L}(X_{k+1}^N \,|\, X_k^N = x) = \mathcal{L}(F_x^N),$$

where for any $N \in \mathbb{N}$, $(F_x^N, x \in \mathcal{X})$ denotes a measurable family of $\mathcal{X}$-valued random variables such that for any $x \in \mathcal{X}$, the random variable $F_x^N - x$ takes values in $\mathcal{U}$.

The natural filtration of the process X^N is denoted by $(\mathcal{F}_k^N)_k$. Note that the increments $X_{k+1}^N - X_k^N$ take values in $\mathcal{U}$.

Our aim is the characterization of the convergence in law of the sequence of processes $(X_{[v_N .]}^N, N \in \mathbb{N})$, where $(v_N)_N$ is a given sequence of positive real numbers going to infinity when N tends to infinity. It is based on the criteria for tightness and identification of semimartingales by use of characteristics given in [19, IX], which consists in studying the asymptotic behavior of

$$\mathcal{G}_x^N(H) = v_N\, \mathbb{E}\big(H(F_x^N - x)\big) = v_N\, \mathbb{E}\big(H(X_{k+1}^N - X_k^N) \,|\, X_k^N = x\big), \tag{2.1}$$

for real valued bounded measurable functions H defined on $\mathcal{U}$.

Hypothesis (H0) *We first assume that the family of random variables $(F_x^N)_{N,x}$ satisfies*

$$\lim_{b\to\infty} \sup_{x\in\mathcal{X}, N\in\mathbb{N}^*} \mathcal{G}_x^N\big(\mathbb{1}_{B(0,b)^c}\big) = 0.$$

This hypothesis avoids to get infinite jumps in the limit. We will see in the examples that this condition affects both the population and the environment dynamics.

Under **(H0)**, we will prove that the study of (2.1) can be reduced to a rich enough and tractable subclass $\mathcal{H}$ of functions H. The choice of $\mathcal{H}$ depends on the particular models and is illustrated in the examples.

2.1 Specific and truncation functions

We consider a closed subset $\mathcal{U}$ of $\mathbb{R}^d$ containing a neighborhood of 0 and introduce the functional space

$$C^2_{b,0}=C^2_{b,0}(\mathcal{U},\mathbb{R})=\left\{H\in C_b(\mathcal{U},\mathbb{R}):H(u)=\sum_{i=1}^d \alpha_i u_i+\sum_{i,j=1}^d \beta_{i,j}u_iu_j+o(|u|^2),\ \alpha_i,\beta_{i,j}\in\mathbb{R}\right\}.$$

The functions of $C^2_{b,0}$ can be decomposed in a similar way with respect to any smooth function which behaves like the identity at 0, as stated in the next lemma. The proof uses the uniqueness of the second order Taylor expansion in a neighborhood of 0.

Lemma 2.1. Let $f=(f^1,\dots,f^d)\in(C^2_{b,0})^d$ such that $f^i(u)=u_i(1+o(|u|))$ for $i=1,\dots,d$. For any $H\in C^2_{b,0}$, there exists a unique decomposition of the form

$$H=\sum_{i=1}^d \alpha_i^f(H)f^i+\sum_{i,j=1}^d \beta^f_{i,j}(H)f^if^j+\overline{H}^f,$$

where $\overline{H}^f=o(|f|^2)$ is a continuous and bounded function and $\alpha_i^f(H)$, $\beta^f_{i,j}(H)$, $i,j=1\cdots d$ are real coefficients and β^f is a symmetric matrix.

We introduce

- the **specific function** h which satisfies

$$\begin{aligned}&h=(h^1,\cdots,h^d)\in(C^2_{b,0})^d\ ;\ h^i(u)=u_i(1+o(u))\ ;\\&h^i(u)\neq 0 \text{ for } u\neq 0\quad (i=1,\dots,d).\end{aligned}\tag{2.2}$$

- the **truncation function** h_0, as defined in [19] :

$$h_0=(h_0^1,\cdots,h_0^d)\in C_b(\mathcal{U},\mathbb{R}^d),\quad h_0(u)=u \text{ in a neighborhood of } 0.\tag{2.3}$$

  Obviously, $h_0^ih_0^j\in C^2_{b,0}$ for any $i,j=1,\dots,d$.

Note that in general a specific function is not a truncation function since it may not coincide with the identity function in a neighborhood of 0. Its choice will be driven by the processes we are considering. We will give different choices of functions h in the next sections, for instance $h(x)=1-\exp(-x)$ on $[-1,\infty)$ when $d=1$. These specific functions will play a crucial role in the whole paper.

2.2 General statements

We introduce a functional space $\mathcal{H}$ containing the coordinates of the specific function h and their square products and which “generates” the continuous functions with compact support in $\mathcal{U}$ in the sense described below. The space $\mathcal{H}$ will be a convergence determining class.

Hypotheses (H1) There exists a functional space $\mathcal{H}$ such that

1. *$\mathcal{H}$ is a subset of $C^2_{b,0}$ and $h^i,h^ih^j\in Vect(\mathcal{H})$ for $i,j=1,\dots,d$.*
2. *For any $g\in C_c(\mathcal{U},\mathbb{R})$ with $g(0)=0$, there exists a sequence $(g_n)_n\in C^2_{b,0}$ such that $\lim_{n\to\infty}\|g-g_n\|_{\infty,\mathcal{U}}=0$ and $|h|^2\,g_n\in Vect(\mathcal{H})$.*
3. *There exists a family of real numbers $(\mathcal{G}_x(H);x\in\mathcal{X},H\in\mathcal{H})$ such that for any $H\in\mathcal{H}$,*

$$(i)\qquad \lim_{N\to\infty}\sup_{x\in\mathcal{X}}\left|\mathcal{G}_x^N(H)-\mathcal{G}_x(H)\right|=0.$$

$$(ii)\qquad \sup_{x\in\mathcal{X}}|\mathcal{G}_x(H)|<+\infty.$$

Remark 2.2. In the examples of the next sections, **(H1.2)** is proved with the use of the locally compact version of the Stone-Weierstrass Theorem. We refer to the Appendix for a precise statement.

Contrary to the "convergence determining class" of [19], the functions of $\mathcal{H}$ will not be vanishing (or $o(u^2)$) in a neighborhood of 0.

Hypothesis (**H1.3**) implies that the map $x \in \mathcal{X} \to \mathcal{G}_x(H)$ is measurable and bounded for any $H \in \mathcal{H}$.

We first obtain a tightness result based on the space $\mathcal{H}$ of test functions.

Theorem 2.3. Assume that the sequence $(X_0^N)_N$ is tight in $\overline{\mathcal{X}}$ and that **(H0)** and **(H1)** hold. Then the sequence of processes $(X^N_{[v_N .]}, N \in \mathbb{N})$ is tight in $\mathbb{D}([0,\infty), \overline{\mathcal{X}})$.

The next hypothesis **(H2)** in addition to **(H1)** is sufficient to get the identification of the limiting values by their semimartingale characteristics, and then their representation as solutions of a stochastic differential equation.

Hypotheses (H2)

1. *For any $H \in \mathcal{H}$, the map $x \in \mathcal{X} \to \mathcal{G}_x(H)$ is continuous and extendable by continuity to $\overline{\mathcal{X}}$.*
2. *For any $x \in \overline{\mathcal{X}}$ and any $H \in \mathcal{H}$,*

$$\mathcal{G}_x(H) = \sum_{i=1}^d \alpha_i^{h_0}(H) b_i(x) + \sum_{i,j=1}^d \beta_{i,j}^{h_0}(H) c_{i,j}(x) + \int_V \overline{H}^{h_0}(K(x,v))\mu(dv), \tag{2.4}$$

where

i) $\alpha_i^{h_0}$, $\beta_{i,j}^{h_0}$ and $\overline{H}^{h_0}$ have been defined in Lemma 2.1,

ii) b_i and $\sigma_{i,j}$ are measurable functions defined on $\overline{\mathcal{X}}$,

iii) V is a Polish space, μ is a σ-finite positive measure on V, K is a function from $\overline{\mathcal{X}} \times V$ with values in $\mathcal{U}$, $\int_V 1 \wedge |K(.,v)|^2 \mu(dv) < +\infty$ and

$$c_{i,j}(x) = \sum_{k=1}^d \sigma_{i,k}(x)\sigma_{j,k}(x) + \int_V (h_0^i h_0^j)(K(x,v))\mu(dv).$$

The elements (b, σ, V, μ, K) will be specified in the applications.

Theorem 2.4. If the sequence $(X_0^N)_N$ is tight in $\overline{\mathcal{X}}$ and **(H0)**, **(H1)**, **(H2)** hold then any limiting value of $(X^N_{[v_N .]}, N \in \mathbb{N})$ is a semimartingale solution of the stochastic differential system

$$\begin{aligned} X_t \quad &= \quad X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s + \int_0^t \int_V h_0(K(X_{s-},v))\tilde{N}(ds,dv) \\ &\qquad + \int_0^t \int_V (Id - h_0)(K(X_{s-},v))N(ds,dv), \end{aligned} \tag{2.5}$$

where $X_0 \in \overline{\mathcal{X}}$ and B is a d-dimensional Brownian motion and N is a Poisson point measure on $\mathbb{R}_+ \times V$ with intensity $ds\mu(dv)$. Moreover X_0, B , N are independent and $\tilde{N}$ is the compensated martingale measure of N.

To obtain the convergence in law of the sequence of processes $(X^N_{[v_N .]}, N \in \mathbb{N})$ in $\mathbb{D}([0,\infty), \overline{\mathcal{X}})$, we need

Hypothesis (H3) *The law of the initial condition $X_0 \in \overline{\mathcal{X}}$ being given, the uniqueness in law of the solution of (2.5) holds in $\mathbb{D}([0,\infty), \overline{\mathcal{X}})$.*

We are now in position to state the convergence result.

Theorem 2.5. Assume that the sequence $(X_0^N)_N$ converges in law in $\overline{\mathcal{X}}$ to X_0 and that **(H0)**, **(H1)**, **(H2)** and **(H3)** hold. Then the sequence of processes $(X_{[v_N \cdot]}^N)_N$ converges in law in $\mathbb{D}([0,\infty),\overline{\mathcal{X}})$ to the solution of (2.5).

2.3 Proofs

From now on, we assume that hypotheses **(H0)** and **(H1)** hold. We recall that $\mathcal{U}^* = \mathcal{U} \setminus \{0\}$.

In the proofs, we use the space $\mathcal{R}_b$ of continuous and bounded functions which are small enough close to 0 :

$$\mathcal{R}_b = \{H \in C_b(\mathcal{U},\mathbb{R}),\ H(u) = o(|u|^2)\}.$$

Using Lemma 2.1 and **(H1.1)**, we have

$$C^2_{b,0} = Vect(\mathcal{H}) + \mathcal{R}_b. \tag{2.6}$$

We work with the norm

$$\|H\|_h = \sup_{u\in\mathcal{U}^*} \frac{|H(u)|}{|h(u)|^2},$$

defined for $H \in \mathcal{R}_b$ such that $\sup_{u\in\mathcal{U}^*} |H(u)|/|h(u)|^2 < +\infty$. In that case, the positivity and linearity of $\mathcal{G}_x^N$ for all $x \in \mathcal{X}$ and $N \geq 1$ imply that

$$|\mathcal{G}_x^N(H)| \leq \mathcal{G}_x^N(|h|^2)\,\|H\|_h \leq \alpha(|h|^2)\,\|H\|_h, \tag{2.7}$$

where $\alpha(|h|^2) = \sup_{N,x\in\mathcal{X}} |\mathcal{G}_x^N(|h|^2)| < \infty$ by $(\mathbf{H1.3})$ since $|h|^2 \in Vect(\mathcal{H})$ by $(\mathbf{H1.1})$.

2.3.1 Proof of Theorem 2.3

We first extend the assumptions **(H1.3)** to $C^2_{b,0}$ in order to prove the tightness. We note that **(H1.3i)** and **(H1.3ii)** extend immediately to $H \in Vect(\mathcal{H})$ by linearity of $H \to \mathcal{G}_x^N(H)$ for any $x \in \mathcal{X}$ and $N \geq 1$.

Lemma 2.6. For any $x \in \mathcal{X}$, there exists a linear extension of $\mathcal{G}_x$ to $C^2_{b,0}$ such that $(\mathbf{H1.3})$ hold for any $H \in C^2_{b,0}$.

As a consequence, writing $\alpha(H) = \sup_{N,x\in\mathcal{X}} |\mathcal{G}_x^N(H)|$, for any $H \in C^2_{b,0}$,

$$\sup_{x\in\mathcal{X}} |\mathcal{G}_x(H)| \leq \sup_{N,x\in\mathcal{X}} |\mathcal{G}_x^N(H)| = \alpha(H) < +\infty. \tag{2.8}$$

Proof. Using (2.6) and linearity, we only have to prove the extension to $\mathcal{R}_b$. Let us first prove the result for the compactly supported functions of $\mathcal{R}_b$. We consider $H \in \mathcal{R}_b$ with compact support and show that the sequence $(\mathcal{G}_x^N(H))_N$ converges when N tends to infinity. The function $H/|h|^2$ defined on $\mathcal{U}^*$ can be extended to a continuous function g on $\mathcal{U}$ with compact support and $g(0) = 0$. Then by $(\mathbf{H1.2})$, there exists a sequence $(g_n)_n$ of functions of $C^2_{b,0}$ uniformly converging to g and such that $H_n = |h|^2 g_n \in Vect(\mathcal{H})$. Since $H = |h|^2 g$, $\|H_n - H\|_h \to 0$ when $n \to \infty$. Moreover the sequence $\big(\mathcal{G}_\cdot^N(H_n)\big)_N$ converges to $\mathcal{G}_\cdot(H_n)$ when N tends to infinity for any fixed n and uniformly on $\mathcal{X}$. Let us now consider two integers m and n. Equation (2.7) tells us that

$$\sup_{N,x} \left|\mathcal{G}_x^N(H_m) - \mathcal{G}_x^N(H_n)\right| \leq\ \alpha(|h|^2)\,\|H_m - H_n\|_h$$

and letting N go to infinity, we obtain that $(\mathcal{G}_x(H_n))_n$ is a Cauchy sequence. Then it converges to a limit denoted by $\mathcal{G}_x(H)$, which satisfies $\sup_{\mathcal{X}} |\mathcal{G}_\cdot(H)| < \infty$. Moreover

$$|\mathcal{G}_x^N(H) - \mathcal{G}_x(H)| \leq \left|\mathcal{G}_x^N(H) - \mathcal{G}_x^N(H_n)\right| + \left|\mathcal{G}_x^N(H_n) - \mathcal{G}_x(H_n)\right| + \left|\mathcal{G}_x(H_n) - \mathcal{G}_x(H)\right|.$$

Since $\left|\mathcal{G}_x^N(H)-\mathcal{G}_x^N(H_n)\right| \leq \alpha(|h|^2)\,\|g-g_n\|_\infty$, an appropriate choice of n and then of N allows us to upper bound the left hand side by any $\epsilon>0$ and this ensures

$$\sup_{x\in\mathcal{X}} |\mathcal{G}_x^N(H)-\mathcal{G}_x(H)| \overset{N\to\infty}{\longrightarrow} 0.$$

Let us now consider $H \in \mathcal{R}_b$. We introduce a non-decreasing sequence $(\varphi_n)_n \in C^2(\mathbb{R}^d,[0,1])$ such that

$$\varphi_n(x) = \begin{cases} 1 & \text{on } B(0,n) \\ 0 & \text{on } B(0,n+1)^c. \end{cases}$$

For $x\in\mathcal{X}$ and $N\geq 1$,

$$|\mathcal{G}_x^N(H\varphi_m)-\mathcal{G}_x^N(H\varphi_n)| \leq \|H\|_\infty\,\mathcal{G}_x^N(\mathbb{1}_{B(0,n)^c}) \leq \|H\|_\infty\,C_n, \qquad m\geq n\geq N\geq 1$$

where $C_n\to 0$ as $n\to\infty$ by **(H0)**. Letting N tend to infinity, we obtain that for any $x\in\mathcal{X}$, the sequence $(\mathcal{G}_x(H\varphi_n))_n$ is Cauchy and converges to some real number $\mathcal{G}_x(H)$. Moreover $|\mathcal{G}_x(H)-\mathcal{G}_x(H\varphi_n)|\leq C_n\|H\|_\infty$. It follows that for any $H\in\mathcal{R}_b$,

$$\begin{aligned} |\mathcal{G}_x^N(H)-\mathcal{G}_x(H)| &\leq |\mathcal{G}_x^N(H)-\mathcal{G}_x^N(H\varphi_n)|+|\mathcal{G}_x^N(H\varphi_n)-\mathcal{G}_x(H\varphi_n)|+|\mathcal{G}_x(H\varphi_n)-\mathcal{G}_x(H)| \\ &\leq 2C_n\|H\|_\infty+|\mathcal{G}_x^N(H\varphi_n)-\mathcal{G}_x(H\varphi_n)| \end{aligned}$$

As $H\varphi_n\in\mathcal{R}_b$ and has compact support, $\mathcal{G}_.^N(H\varphi_n)-\mathcal{G}_.(H\varphi_n)$ and then $\mathcal{G}_.^N(H)-\mathcal{G}_.(H)$ tend to 0 as N tends to infinity uniformly on $\mathcal{X}$. It proves (**H1.3**) and (2.8). □

We now prove that a σ-finite measure can be associated to $\mathcal{G}_x$ for each $x\in\mathcal{X}$. It describes the jumps of the limiting process.

Lemma 2.7. There exists a family of σ-finite measures $(\mu_x : x\in\mathcal{X})$ on $\mathcal{U}^*$ such that for any $x\in\mathcal{X}$ and $H\in\mathcal{R}_b$,

$$\mathcal{G}_x(H)=\int_{\mathcal{U}^*} H(u)\mu_x(du). \tag{2.9}$$

For any $x\in\mathcal{X}$, $\mathcal{G}_x$ is then extended by (2.9) to any measurable and bounded function H on $(\mathbb{R}^d)^*$ such that $H(u)=o(|u|^2)$. Moreover

$$\lim_{b\to\infty}\sup_{x\in\mathcal{X}} |\mathcal{G}_x(\mathbb{1}_{B(0,b)^c})|=0. \tag{2.10}$$

Proof. For any $x\in\mathbb{R}^d$ and $H\in C_c(\mathcal{U}^*,\mathbb{R})$, the map $H\to\mathcal{G}_x(H)$ is a positive linear operator. Adding that $\mathcal{U}^*$ is locally compact, Riesz Theorem leads to the existence of a σ-finite measure μ_x on $\mathcal{U}^*$ such that for any $H\in C_c(\mathcal{U}^*,\mathbb{R})$, $\mathcal{G}_x(H)=\int_{\mathcal{U}^*}H(u)\mu_x(du)$. The extension of this identity to any $H\in\mathcal{R}_b$ follows again from an approximation procedure, using φ_n defined in the proof of Lemma 2.6. Indeed, on the one hand monotone convergence ensures that $\int_{\mathcal{U}^*}H\varphi_n\mu_x$ goes to $\int_{\mathcal{U}^*}H\mu_x$. On the other hand, $|\mathcal{G}_x(H\varphi_n)-\mathcal{G}_x(H)|\leq C_n\|H\|_\infty$ goes to 0. Finally (2.10) comes from **(H0)** with a monotone approximation of $\mathbb{1}_{B(0,b)^c}$ by elements of $\mathcal{R}_b$ and the convergence of $\mathcal{G}^N$ to $\mathcal{G}$. □

We now prove the convergence of conditional increments functionals, defined for any function $H\in C^0_{b,2}$ and $t>0$ by

$$\phi_t^N(H)=\sum_{k=1}^{[v_Nt]}\mathbb{E}\big(H(X_k^N-X_{k-1}^N)\,|\,\mathcal{F}_{k-1}^N\big)=\frac{1}{v_N}\sum_{k=1}^{[v_Nt]}\mathcal{G}^N_{X^N_{k-1}}(H), \tag{2.11}$$

where the last identity follows from the Markov property.

Proposition 2.8. For any function $H \in C^0_{b,2}$ and $t > 0$,

$$\lim_{N\to\infty} \sup_{t\le T} \left| \phi^N_t(H) - \int_0^t \mathcal{G}_{X^N_{[v_N s]}}(H)\, ds \right| = 0 \qquad \text{a.s.}$$

Proof. Using (2.8), we have

$$\frac{1}{v_N}\sum_{k=1}^{[v_N t]} \mathcal{G}^N_{X^N_{k-1}}(H) = \int_0^t \mathcal{G}^N_{X^N_{[v_N s]}}(H)ds - \int_{\frac{[v_N t]}{v_N}}^t \mathcal{G}^N_{X^N_{[v_N s]}}(H)ds = \int_0^t \mathcal{G}^N_{X^N_{[v_N s]}}(H)ds + \mathcal{O}\left(\frac{\alpha(H)}{v_N}\right).$$

Then

$$\sup_{t\le T} \left| \phi^N_t(H) - \int_0^t \mathcal{G}_{X^N_{[v_N s]}}(H)ds \right| \le T \sup_{x\in\mathcal{X}} |\mathcal{G}^N_x(H) - \mathcal{G}_x(H)| + \mathcal{O}\left(\frac{\alpha(H)}{v_N}\right)$$

and the conclusion follows from (**H1.3i**), which holds for H thanks to Lemma 2.6. □

We define on the canonical space $\mathbb{D}([0,\infty),\mathcal{X})$ a triplet which characterizes the limiting values of the sequence $(X^N_{[v_N\cdot]}, N \in \mathbb{N})$. Using the measurability and boundedness of $x \to \mathcal{G}_x(f)$ for $x \in \mathcal{X}$ and $f \in C^2_{b,0}$ and the truncation function h_0 introduced in (2.3), we define for any $\omega = (\omega_s, s \ge 0) \in \mathbb{D}([0,\infty),\mathcal{X})$ the functionals

$$\left.\begin{aligned} B_t(\omega) &= \int_0^t \Big(\mathcal{G}_{\omega_s}(h^1_0), \cdots, \mathcal{G}_{\omega_s}(h^d_0)\Big)\, ds, \\ \widetilde{C}^{ij}_t(\omega) &= \int_0^t \mathcal{G}_{\omega_s}(h^i_0 h^j_0)\, ds, \\ \nu_t(\omega, H) &= \int_0^t \mathcal{G}_{\omega_s}(H\mathbb{1}_{\mathcal{U}})\, ds = \int_0^t \int_{\mathcal{U}^*} H(u)\mu_{\omega_s}(du)\, ds \end{aligned}\right\} \tag{2.12}$$

for any $H \in C_b(\mathbb{R}^d, \mathbb{R})$ such that $H(u) = o(|u|^2)$. The last identity comes from (2.9).

As in Chapters II. 2 & 3 in [19] adapted to the state space $\overline{\mathcal{X}}$ (instead of $\mathbb{R}^d$), the characteristic triplet associated with the semimartingale X^N is given for $i, j \in \{1, \dots, d\}$ by

$$\left.\begin{aligned} B^N_t &= \sum_{k\le[v_N t]} \mathbb{E}(h_0(U^N_k)|\mathcal{F}^N_{k-1}) = (\phi^N_t(h^1_0), \cdots, \phi^N_t(h^d_0)) \\ \widetilde{C}^{N,ij}_t &= \sum_{k\le[v_N t]} \Big(\mathbb{E}(h^i_0(U^N_k)h^j_0(U^N_k)|\mathcal{F}^N_{k-1}) - \mathbb{E}(h^i_0(U^N_k)|\mathcal{F}^N_{k-1})\mathbb{E}(h^j_0(U^N_k)|\mathcal{F}^N_{k-1})\Big) \\ \phi^N_t(H) &= \sum_{k\le[v_N t]} \mathbb{E}(H(U^N_k)|\mathcal{F}^N_{k-1}), \end{aligned}\right\} \tag{2.13}$$

where $U^N_k = X^N_k - X^N_{k-1}$ and H is a continuous bounded function on $\mathbb{R}^d$ vanishing in a neighborhood of 0. Proposition 2.8 implies the convergence of the characteristics, as stated in the next proposition.

Proposition 2.9. For any $T > 0$ and any $i, j = 1, \cdots, d$ and any $H \in C_b(\mathcal{U}, \mathbb{R})$ equal to 0 in some neighborhood of 0, we have the following almost-sure convergences

$$\sup_{t\le T} \left| B^{N,i}_t - B^i_t \circ X^N_{[v_N\cdot]} \right| \overset{N\to\infty}{\longrightarrow} 0; \tag{2.14}$$

$$\sup_{t\le T} \left| \widetilde{C}^{N,ij}_t - \widetilde{C}^{ij}_t \circ X^N_{[v_N\cdot]} \right| \overset{N\to\infty}{\longrightarrow} 0; \tag{2.15}$$

$$\sup_{t\le T} \left| \phi^N_t(H) - \nu_t(X^N_{[v_N\cdot]}, H) \right| \overset{N\to\infty}{\longrightarrow} 0. \tag{2.16}$$

Proof. From Proposition 2.8, we immediately obtain the first and last convergences and

$$\sup_{t\leq T}\left|\phi_t^N(h_0^i h_0^j)-\widetilde{C}_t^{ij}\circ X^N_{[v_N.]}\right|\to_{N\to\infty} 0\quad\text{a.s.}$$

So it remains to replace $\phi_t^N(h_0^i h_0^j)$ by $\widetilde{C}_t^{N,ij}$. We have

$$|\mathbb{E}(h_0^k(U_k^N)|\mathcal{F}_{k-1}^N)|\leq\frac{1}{v_N}\sup_{N,x\in\mathcal{X}}\left|\mathcal{G}_x^N(h_0^k)\right|\leq\alpha(h_0^k).\frac{1}{v_N}$$

and $\alpha(h_0^k)<\infty$ from (2.8). Hence the second term in $\widetilde{C}_t^{N,ij}$ tends to 0 as $N\to\infty$, which yields the result. □

We are now in position to provide a proof of Theorem 2.3. In order to apply Theorem 3.9 IX p543 in [19] and get the tightness, we need to check the strong majoration hypothesis and the condition on big jumps required in its statement.

First, if $H\in C^0_{b,2}$ then $\mathcal{G}_. H$ is bounded and there exists a positive constant A such that for any $\omega\in\mathbb{D}([0,\infty),\mathcal{X})$,

$$\sum_{i=1}^d\mathrm{Var}(B^i(\omega))_t+\sum_{i,j=1}^d\tilde{C}_t^{ij}(\omega)+\nu_t(w,r)\leq A\,t,\tag{2.17}$$

where $\mathrm{Var}(X)_t$ denotes the total variation of X on $[0,t]$ and $r(u):=|u|^2\wedge 1$.

Second, to control the big jumps, we use the fact that $\nu_t(.,\mathbb{1}_{B(0,b)^c})\leq t\,\|\mathcal{G}_.(\mathbb{1}_{B(0,b)^c})\|_\infty$, which tends to 0 as b tends to infinity from (2.10). We thus obtain

$$\lim_{b\uparrow\infty}\sup_{w\in\mathbb{D}([0,\infty),\mathcal{X})}\nu_t(w,\mathbb{1}_{B(0,b)^c})=0.\tag{2.18}$$

The tightness of $(X^N_{[v_N.]},N\in\mathbb{N})$ follows from (2.14)-(2.18) and from the tightness of the initial condition, by an application of the forementioned theorem in [19].

2.3.2 Proofs of Theorems 2.4 and 2.5

Let us now assume the additional Hypothesis **(H2)**. We wish to identify the limiting values of $(X^N_{[v_N.]},N\in\mathbb{N})$ as solutions of the stochastic differential system (2.5). We first need to extend continuously the limiting characteristic triplet to the boundary.

Lemma 2.10. (i) For any $H\in\mathcal{R}_b$, the map $x\in\mathcal{X}\to\mathcal{G}_x(H)$ is continuous and extendable by continuity to $\overline{\mathcal{X}}$. Moreover

$$\sup_{x\in\overline{\mathcal{X}}}|\mathcal{G}_x(H)|\leq\alpha(H)<+\infty.\tag{2.19}$$

(ii) For any $H\in\mathcal{R}_b$ and $x\in\overline{\mathcal{X}}$,

$$\mathcal{G}_x(H)=\int_V H(K(x,v))\mu(dv)\tag{2.20}$$

and where k,μ are defined in $(\mathbf{H2})$ and $\int_V 1\wedge|K(.,v)|^2\mu(dv)$ is bounded on $\overline{\mathcal{X}}$.

Proof. Let $H\in\mathcal{R}_b$. Using the sequences φ_n and $(H_n)_n$ defined in the proof of Lemma 2.6 and approximating $\varphi_n H$ for $\|\,.\,\|_h$ by $H_n\in Vect(\mathcal{H})\cap\mathcal{R}_b$ as in the proof of Lemma 2.7, we obtain

$$\sup_{x\in\mathcal{X},N\geq 1}\left|\mathcal{G}_x^N(H)-\mathcal{G}_x^N(H_n)\right|\leq\|\,H\|_\infty C_n+\|\,\varphi_n H-H_n\,\|_h\,\alpha(|h|^2),$$

which tends to 0 as $n \to \infty$. Letting $N \to \infty$ ensures that $\mathcal{G}_. H_n$ converges uniformly to $\mathcal{G}_. H$ as $n \to \infty$. Combining this with (**H2.1**) applied to H_n, we deduce that $\mathcal{G}_. H$ is continuous on $\mathcal{X}$ and extendable by continuity to $\overline{\mathcal{X}}$. Moreover (2.8) yields (2.19) by continuity, which proves (i).

For (ii), we first consider $H \in Vect(\mathcal{H}) \cap \mathcal{R}_b$. Then $\alpha^{h_0}(H) = \beta^{h_0}(H) = 0$, $\overline{H}^{h_0} = H$ and (**H2.2**) ensures that (2.20) holds for H. Let us now extend this identity to $H \in \mathcal{R}_b$ with compact support. We note that $H = |h|^2 g$ with $g \in C_c(\mathcal{U}, \mathbb{R})$. By (**H1.2**), the function g is uniformly approximated by a sequence g_n such that $|h|^2 g_n \in Vect(\mathcal{H}) \cap \mathcal{R}_b$. The identity (2.4) implies that (2.20) holds for any $|h|^2 g_n$ and

$$\forall x \in \overline{\mathcal{X}}, \quad \mathcal{G}_x(|h|^2 g_n) = \int_V (|h|^2 g_n)(K(x,v))\mu(dv).$$

We let n tend to infinity in both terms using (2.8) and the assumption $\int_V 1 \wedge |K(x,v)|^2 \mu(dv) < +\infty$. The extension to $\mathcal{R}_b$ follows again from a monotone approximation by the compactly supported functions $H\varphi_n$, which ends the proof. □

This lemma allows us to extend the definitions of the characteristics and the identities of (2.12) to any $w \in \mathbb{D}([0,\infty), \overline{\mathcal{X}})$. Moreover (i) ensures that $w \in \mathbb{D}([0,\infty), \overline{\mathcal{X}}) \to (B_t(\omega), \widetilde{C}_t(\omega), \nu_t(\omega, H))$ is continuous and that the dominations (2.17) and (2.18) extend from $\mathcal{X}$ to $\overline{\mathcal{X}}$. We can then apply [19, Theorem 2.11, chapter IX, p530] on the closed set $\overline{\mathcal{X}}$ for the identification. We obtain that any limiting value of the law of $(X^N_{[v_N.]})_N$ is a solution of the martingale problem on the canonical space $\mathbb{D}([0,\infty), \overline{\mathcal{X}})$ with characteristic triplet (B, C, ν), where

$$C_t^{ij} = \widetilde{C}_t^{ij} - \nu_t(., h_0^i h_0^j).$$

Finally, using **(H2.2)** for $H \in \{h_0^i, h_0^i h_0^j\}$ and (2.20), the characteristics in (2.12) can be written as

$$\begin{aligned} B_t(w) &= \int_0^t b(w_s) ds \\ C_t^{ij}(w) &= \int_0^t \left(\sum_{k=1}^d \sigma_{i,k}(w_s)\sigma_{j,k}(w_s) \right) ds \\ \nu_t(w, H) &= \int_0^t \int_V H(K(w_s, v))\mu(dv) ds, \end{aligned}$$

for any $w \in \mathbb{D}([0,\infty), \overline{\mathcal{X}})$. By [19, Chapter III, Theorem 2.26 p157], the set of solutions of the martingale problem with characteristic triplet (B, C, ν) coincides with the set of weak solutions of the stochastic differential equation (2.5). The proof of Theorem 2.4 is now complete.

To conclude the proof of the convergence, we remark that uniqueness hypothesis (**H3**) guarantees (iii) in [19] Theorem 3.21, chapter IX, p.546]. The other points $(i - vi)$ of this theorem have been checked above and Theorem 2.5 follows.

3 Wright-Fisher process with selection in Lévy environment

3.1 The discrete model

Let us consider the framework of the Wright-Fisher model: at each generation, the alleles of a fixed size population are sampled from the previous generation. We consider a population of N individuals characterized by some allele. The number of individuals carrying this allele is a process $(Z_k^N, k \in \mathbb{N})$ whose dynamics depends on the environment. When $N \geq 1$ is fixed, we consider the coupled process describing the discrete time dynamics of the population process and the environment process. It is recursively defined for $k \geq 0$ by

$$\begin{cases} Z^N_{k+1} = \sum_{i=1}^N \mathcal{E}^N_{k,i}(Z^N_k/N, E^N_k), \\ \\ S^N_{k+1} = S^N_k + E^N_k, \end{cases} \tag{3.1}$$

and $S^N_0 = 0, Z^N_0 = [NZ_0]$, $Z_0 \in [0,1]$ is a finite random variable, $(E^N_k)_k$ are independent and identically distributed with values in $(-1,+\infty)$ and the family of random variables $\big((\mathcal{E}^N_{k,i}(z,w),(z,w) \in [0,1]\times(-1,\infty)); k \geq 1, i \geq 1\big)$ are independent. Moreover for each $(z,w) \in [0,1]\times(-1,\infty)$, the random variables $\big(\mathcal{E}^N_{k,i}(z,w); k \geq 1, i \geq 1\big)$ are identically distributed as a Bernoulli random variable $\mathcal{E}^N(z,w)$ defined by

$$\mathbb{P}(\mathcal{E}^N(z,w) = 1) = p(z,w)\ ;\ \mathbb{P}(\mathcal{E}^N(z,w) = 0) = 1 - p(z,w).$$

We also assume that Z_0, $\big((\mathcal{E}^N_{k,i}(z,w),(z,w) \in [0,1]\times(-1,\infty)); k \geq 1, i \geq 1\big)$ and $(E^N_k, k \geq 0)$ are independent.

Moreover p is a C^3-function from $[0,1]\times(-1,\infty)$ to $[0,1]$ verifying $p(z,0) = z$ for any $z \in [0,1]$. A main example, developed in Section 3.4, is given by $p(z,w) = z(1+w)/(z(1+w)+1-z)$ and extends the classical Wright Fisher model with rare selection to random environments.

Following [19] [chap.VII Corollary 3.6,p.415], we state an assumption for the random walk $S^N_{[N.]}$ to converge in law to a Lévy process with characteristics $(\alpha_E, \beta_E, \nu_E)$. Let us consider a truncation function h_E defined on $(-1,+\infty)$, i.e. continuous and bounded and satisfying $h_E(w) = w$ in a neighborhood of 0. For convenience, we also assume that $h_E(w) \neq 0$ for any $w \neq 0$.

Assumption A. There exist $\alpha_E \in \mathbb{R}$, $\sigma_E \geq 0$ and a measure ν_E on $(-1,+\infty)$ satisfying $\int_{(-1,+\infty)}(w^2 \wedge 1)\nu_E(dw) < +\infty$ such that

$$\lim_{N\to\infty} N\,\mathbb{E}(h_E(E^N)) = \alpha_E\ ;\ \lim_{N\to\infty} N\,\mathbb{E}(h^2_E(E^N)) = \beta_E = \sigma^2_E + \int_{(-1,\infty)} h^2_E(w)\nu_E(dw),$$

$$\lim_{N\to\infty} N\,\mathbb{E}(f(E^N)) = \int_{(-1,\infty)} f(w)\nu_E(dw),$$

for any f vanishing in a neighborhood of 0, continuous and bounded.

The small fluctuations of the environment are given by σ_E, while the dramatic events are given by the jump measure ν_E. Negative jumps will correspond to dramatic disadvantages of allele A and an usual set of selection coefficient is $(-1,\infty)$, as illustrated in Section 3.4.

The limiting environment process Y can thus be defined by

$$\begin{aligned} Y_t = \alpha_E t + \int_0^t \sigma_E dB^E_s + \int_0^t \int_{(-1,+\infty)} h_E(w)\widetilde{N}^E(ds,dw) \\ + \int_0^t \int_{(-1,+\infty)} (w - h_E(w))N^E(ds,dw), \end{aligned} \tag{3.2}$$

where B^E is a Brownian motion and N^E is a Poisson point measure on $\mathbb{R}_+ \times (-1,+\infty)$ independent of B^E with intensity measure ν_E. By construction, this Lévy process has jumps larger than -1.

Let us first prove a consequence of Assumption **A** which will be needed in the proof of the next theorem.

Lemma 3.1. Let $g \in C^3([0,1] \times (-1,\infty), \mathbb{R})$ bounded and satisfying $g(z,0) = 0$ for any $z \in [0,1]$. Then, under Assumption $\mathbf{A}$,

$$N\mathbb{E}(g(z,E^N)) \overset{N\to\infty}{\longrightarrow} \mathcal{B}_z(g),$$

uniformly for $z \in [0,1]$, with

$$\mathcal{B}_z(g) = \alpha_E \frac{\partial g}{\partial w}(z,0) + \frac{\beta_E}{2} \frac{\partial^2 g}{\partial w^2}(z,0) + \int_{(-1,\infty)} \widehat{g}(z,w)\nu_E(dw)$$

and $\widehat{g}(z,w) = g(z,w) - h_E(w)\frac{\partial g}{\partial w}(z,0) - \frac{h_E(w)^2}{2}\frac{\partial^2 g}{\partial w^2}(z,0)$.

Proof. Indeed, we can decompose $N\mathbb{E}(g(z,E^N))$ as follows

$$N\mathbb{E}(g(z,E^N)) = \frac{\partial g}{\partial w}(z,0)\, N\mathbb{E}(h_E(E^N)) + \frac{1}{2}\frac{\partial^2 g}{\partial w^2}(z,0)\, N\mathbb{E}(h_E(E^N)^2) + N\mathbb{E}(\widehat{g}(z,E^N)).$$

The first two terms converge uniformly as $N \to \infty$ by a direct application of Assumption $\mathbf{A}$. Moreover the last part of Assumption $\mathbf{A}$ can be extended to any continuous function $f(w) = o(w^2)$ using a monotone approximation of f by functions vanishing in a neighborhood of 0. Then the last term converges for fixed z and it remains to prove that the convergence is uniform on $[0,1]$. First, let us consider a compact subset $K = [0,1] \times [-1+\varepsilon_0, A]$ of $[0,1] \times (-1,\infty)$. As g is $C^3([0,1] \times (-1,\infty), \mathbb{R})$, the function

$$(z,w) \to \frac{\widehat{g}(z,w)}{h_E(w)^2} = \frac{g(z,w) - h_E(w)\frac{\partial g}{\partial w}(z,0)}{h_E(w)^2} - \frac{1}{2}\frac{\partial^2 g}{\partial w^2}(z,0)$$

and its first derivative with respect to z are well defined on $[0,1] \times (-1,\infty) \setminus [0,1] \times \{0\}$ and extendable by continuity to $[0,1] \times (-1,\infty)$. Thus the derivative of $\widehat{g}(z,w)/h_E(w)^2$ with respect to z is bounded on K. As $(N\mathbb{E}(h_E(E^N)^2))_N$ is bounded by the second part of Assumption $\mathbf{A}$, there exists $C > 0$ such that for any $N \geq 1$,

$$\left| N\mathbb{E}\left(\widehat{g}(z,E^N) 1_{E^N \in [-1+\varepsilon_0, A]}\right) - N\mathbb{E}\left(\widehat{g}(z',E^N) 1_{E^N \in [-1+\varepsilon_0, A]}\right) \right| \leq C|z - z'|.$$

Moreover, since all functions involved in the definition of $\widehat{g}$ are bounded, there exists $C' > 0$ such that

$$\left| N\,\mathbb{E}\left(|\widehat{g}(z,E^N)| 1_{E^N \notin [-1+\varepsilon_0, A]}\right) \right| \leq C' N\,\mathbb{P}(E^N \notin [-1+\varepsilon_0, A])$$

and by the last part of Assumption $\mathbf{A}$,

$$\lim_{\varepsilon_0 \to 0, A \to \infty} \sup_N N\,\mathbb{P}(E^N \notin [-1+\varepsilon_0, A]) = \lim_{\varepsilon_0 \to 0, A \to \infty} \nu_E((-1,-1+\varepsilon_0) \cup (A,\infty)) = 0.$$

Combining the last two inequalities, we obtain that the family of functions $(N\mathbb{E}(\widehat{g}(.,E^N)))_N$ is uniformly equicontinuous on $[0,1]$ and the convergence is uniform by Ascoli Theorem. □

We can now generalize the classical convergence in law to the Wright-Fisher diffusion with selection to i.i.d. environments.

3.2 Tightness and identification

We are interested in the asymptotic behavior of the Markov chain

$$X_k^N = \left(\frac{Z_k^N}{N}, S_k^N\right), \quad k \in \mathbb{N}$$

when N tends to infinity. This process takes values in $\mathcal{X} = [0,1] \times \mathbb{R}$.

For the statement, we introduce the drift coefficient inherited from the fluctuations of the environment:

$$b_1(z) = \alpha_{E} \frac{\partial p}{\partial w}(z,0) + \frac{\sigma_{E}}{2}\frac{\partial^2 p}{\partial w^2}(z,0) + \int_{(-1,\infty)} \left(p(z,w) - z - h_{E}(w)\frac{\partial p}{\partial w}(z,0)\right)\nu_{E}(dw).$$

Theorem 3.2. Under Assumption **A**, the sequence of processes $\left(Z^N_{[N.]}/N, S^N_{[N.]}\right)_N$ is tight in $\mathbb{D}([0,\infty),[0,1]\times\mathbb{R})$ and any limiting value of this sequence is solution of the following stochastic differential equation

$$\begin{aligned} Z_t &= Z_0 + \int_0^t b_1(Z_s)ds + \int_0^t \sqrt{Z_s(1-Z_s)}dB_s^{D} + \sigma_{E}\int_0^t \frac{\partial p}{\partial w}(Z_s,0)\,dB_s^{E} \\ &\quad + \int_{(-1,\infty)} (p(Z_{t-},w) - Z_{t-})\widetilde{N}(dt,dw); \qquad (3.3) \\ Y_t &= \alpha_{E} t + \sigma_{E} B_t^{E} + \int_0^t \int_{(-1,\infty)} h_{E}(w)\widetilde{N}(dt,dw) + \int_0^t\int_{(-1,\infty)} (w - h_{E}(w))N(dt,dw), \end{aligned}$$

where B^D and B^E are Brownian motions; N is a Poisson point measure on $\mathbb{R}_+\times(-1,\infty)$ with intensity $dt\nu_{E}(dw)$ and $\widetilde{N}$ is the compensated martingale measure of N; Z_0, B^D, B^E and N are independent.

Proof. We apply our results to the Markov chain $X_k^N = \left((\frac{Z_k^N}{N}, S_k^N), k\in\mathbb{N}\right)$.
Let $x=(z,y)\in\mathcal{X}$, we set $F_x^N = F^N_{(z,y)} = \left(\frac{1}{N}\sum_{i=1}^N \mathcal{E}_i(z,E^N), y+E^N\right)$ and we have

$$F_x^N - x = \left(\frac{1}{N}\sum_{i=1}^N (\mathcal{E}_i(z,E^N) - z), E^N\right). \qquad (3.4)$$

The state space of the random variables $F_x^N - x$ is $\mathcal{U} = [-1,1]\times(-1,+\infty)$.

We first prove that **(H0)**, **(H1)** and **(H2)** are satisfied with $v_N = N$.

(i) Let us first check **(H0)**. We take $b>0$ and consider

$$\mathcal{G}_x^N(\mathbb{1}_{\mathcal{B}(0,b)^c}) = N\,\mathbb{E}(\mathbb{1}_{\mathcal{B}(0,b)^c}(F_x^N - x)).$$

Then

$$N\,\mathbb{E}(\mathbb{1}_{\mathcal{B}(0,b)^c}(F_x^N - x)) \quad\leq\quad N\,\mathbb{P}\left(\frac{1}{N}\Big|\sum_{i=1}^N(\mathcal{E}_i(z,E^N)-z)\Big| > b/\sqrt{2}\right) + N\,\mathbb{P}\left(|E^N| > b/\sqrt{2}\right).$$

We observe that $\frac{1}{N}\big|\sum_{i=1}^N(\mathcal{E}_i(z,E^N)-z)\big| \leq 1$ a.s. Moreover the last part of Assumption **A** ensures that

$$\limsup_{N\to\infty} N\,\mathbb{P}(|E^N| > b/\sqrt{2}) \leq \nu[b/\sqrt{2}-1,\infty),$$

which tends to 0 as $b\to+\infty$. Then $\sup_{N,x\in[0,1]\times(-1,\infty)}\mathcal{G}_x^N(\mathbb{1}_{\mathcal{B}(0,b)^c})$ tends to 0 and **(H0)** is satisfied.

(ii) We define the function h on $\mathcal{U}$ by

$$h(u,w) = (1-e^{-u}, 1-e^{-w}).$$

The space $\mathcal{H}$ is the subset of real functions on $\mathcal{U}$ defined as

$$\mathcal{H} = \{(u,w) \in \mathcal{U} \to H_{k,\ell}(u,w)\,;\, k,\ell \geq 0\}, \quad \text{with} \;\; H_{k,\ell}(u,w) = 1 - e^{-ku-\ell w}.$$

We can apply the local Stone-Weierstrass Theorem to the algebra $Vect(\mathcal{H}) \cap C_0(\mathcal{U}^*)$, $\mathcal{U}^* = \mathcal{U} \setminus \{0,0\}$ being a locally compact Hausdorff space (see Appendix 6.4). This algebra in dense in $C_0(\mathcal{U}^*)$ and then any function in $C_c(\mathcal{U})$ vanishing at zero is the uniform limit of elements of $Vect(\mathcal{H})$. Moreover $Vect(\mathcal{H})$ is stable by multiplication by $|h|^2$. We deduce that $(\mathbf{H1.2})$ is satisfied, while $(\mathbf{H1.1})$ is obvious.

Let us now prove that $(\mathbf{H1.3})$ is satisfied. We need to study the limit of $\mathcal{G}^N_x(H_{k,\ell})$ as N tends to infinity. Recall that $\mathcal{G}^N_x(H_{k,\ell}) = N\,\mathbb{E}(H_{k,\ell}(F^N_x - x))$ with $x=(z,y)$ and $F^N_x - x$ given by (3.4). We have

$$\begin{aligned}
\mathcal{G}^N_x(H_{k,\ell}) &= N\,\mathbb{E}\Big(1 - e^{-\frac{k}{N}\sum_{i=1}^N(\mathcal{E}_i(z,E^N)-z)}e^{-\ell E^N}\Big)\\
&= N\Big(1 - \mathbb{E}\Big(\mathbb{E}\Big[e^{-\frac{k}{N}(\mathcal{E}(z,E^N)-z)}\,|\,E^N\Big]^N e^{-\ell E^N}\Big)\Big)\\
&= N\Big(1 - \mathbb{E}\Big(\Big[e^{-\frac{k}{N}(1-z)}p(z,E^N) + e^{\frac{k}{N}z}(1-p(z,E^N))\Big]^N e^{-\ell E^N}\Big)\Big).
\end{aligned}$$

The following Taylor expansion gives

$$\log\left(e^{-\frac{k}{N}(1-z)}p + e^{\frac{k}{N}z}(1-p)\right) = \frac{k}{N}(z-p) + \frac{k^2}{2N^2}p(1-p) + \mathcal{O}(1/N^3),$$

with $N^3\mathcal{O}(1/N^3)$ bounded uniformly in $p,z\in[0,1]$. Then we obtain

$$\begin{aligned}
\mathcal{G}^N_x(H_{k,\ell}) &= N\mathbb{E}\Big(1 - e^{k(z-p(z,E^N))-\ell E^N}.e^{\frac{k^2}{2N}p(z,E^N)(1-p(z,E^N))}.e^{\mathcal{O}(1/N^2)}\Big)\\
&= N\,\mathbb{E}\Big(1 - [(1-A_{k,\ell})(1+B_{k,N})(1+R_{k,N})]\,(z,E^N)\Big),
\end{aligned}$$

where $N^2R_{k,N}(z,w)$ is uniformly bounded for $z\in[0,1], w\in(-1,\infty)$ and $N\geq 1$ and

$$A_{k,\ell}(z,w) = 1 - \exp\big(-k(p(z,w)-z)-\ell w\big); \quad B_{k,N}(z,w) = \frac{k^2}{2N}\,p(z,w)(1-p(z,w)) + \mathcal{O}\big(\frac{1}{N^2}\big).$$

By expansion, we deduce that

$$\begin{aligned}
\mathcal{G}^N_x(H_{k,\ell}) &= N\mathbb{E}\big(A_{k,\ell}(z,E^N)\big)\Big(1+\mathcal{O}\big(1/N\big)\Big)\\
&\qquad - \frac{k^2}{2}\mathbb{E}\Big(p(z,E^N)(1-p(z,E^N))\Big) + \mathcal{O}(1/N). \qquad (3.5)
\end{aligned}$$

Using Lemma 3.1 both for $(z,w)\to A_{k,\ell}(z,w)$ and $(z,w)\to p(z,w)(1-p(z,w)) - z(1-z)$, we obtain from (3.5) that

$$\mathcal{G}^N_x(H_{k,\ell}) \overset{N\to\infty}{\longrightarrow} \mathcal{G}_x(H_{k,\ell}) = \mathcal{B}_z(A_{k,\ell}) - \frac{k^2}{2}z(1-z),$$

uniformly for $x=(z,y)\in[0,1]\times\mathbb{R}$. Then **(H1.3i)** is satisfied and for any $x\in\mathcal{X}$,

$$\mathcal{G}_x(H_{k,\ell}) = \alpha_E\frac{\partial A_{k,\ell}}{\partial w}(z,0) + \frac{\beta_E}{2}\frac{\partial^2 A_{k,\ell}}{\partial w^2}(z,0) + \int_{(-1,\infty)}\widehat{A_{k,\ell}}(z,w)\nu_E(dw) - \frac{k^2}{2}z(1-z) \qquad (3.6)$$

with

$$\frac{\partial A_{k,\ell}}{\partial w}(z,0) = k\frac{\partial p}{\partial w}(z,0) + \ell, \qquad \frac{\partial^2 A_{k,\ell}}{\partial w^2}(z,0) = k\frac{\partial^2 p}{\partial w^2}(z,0) - \big(k\frac{\partial p}{\partial w}(z,0) + \ell\big)^2$$

and

$$\widehat{A_{k,\ell}}(z,w) = A_{k,\ell}(z,w) - h_E(w)\frac{\partial A_{k,\ell}}{\partial w}(z,0) - \frac{h_E^2(w)}{2}\frac{\partial^2 A_{k,\ell}}{\partial w^2}(z,0).$$

The assumptions on the function p allow us to conclude that **(H1.3ii)** is also satisfied.

(iii) We now check (**H2**). The continuity of $\mathcal{G}_.$ on $[0,1]\times\mathbb{R}$ comes from the regularity of p, from the integrability assumption on ν_E and from Lebesgue's Theorem (by Assumption **A**).

Now, let us introduce the truncation function defined on $\mathcal{U} = [-1,1]\times(-1,\infty)$ by

$$h_0(u,w) = (u, h_E(w)).$$

With the notation of Section 2, recall that $h_0^1(u,w) = u$ and $h_0^2(u,w) = h_E(w)$.

With the notation of Lemma 2.1 we have

$$\alpha_1^{h_0}(H_{k,\ell}) = k,\ \alpha_2^{h_0}(H_{k,\ell}) = \ell,\ \beta_{11}^{h_0}(H_{k,\ell}) = -\frac{k^2}{2},$$

$$\beta_{12}^{h_0}(H_{k,\ell}) = \beta_{21}^{h_0}(H_{k,\ell}) = -\frac{k\ell}{2},\ \beta_{22}^{h_0}(H_{k,\ell}) = -\frac{\ell^2}{2}.$$

Moreover, setting $K(z,w) = (p(z,w)-z, w)$, we note that $A_{k,\ell} = H_{k,\ell}\circ K$ and

$$\widehat{A_{k,\ell}}(z,w) \quad = \quad \overline{H_{k,\ell}}^{h_0}(K(x,w)) + kp_1(z,w) - \frac{k^2}{2}p_2(z,w) - k\ell h_E(w)q(z,w),$$

with $p_1(z,w) = p(z,w) - z - h_E(w)\frac{\partial p}{\partial w}(z,0) - \frac{h_E(w)^2}{2}\frac{\partial^2 p}{\partial w^2}(z,0)$,
$p_2(z,w) = (p(z,w)-z)^2 - h_E^2(w)\Big(\frac{\partial p}{\partial w}(z,0)\Big)^2$ and $q(z,w) = p(z,w) - z - h_E(w)\frac{\partial p}{\partial w}(z,0)$.

Now we set $V = (-1,\infty)$ and choose $\mu = \nu_E$ and for $x = (z,y)\in[0,1]\times\mathbb{R}$, we define

$$b_1(x) = b_1(z) = \alpha_E\frac{\partial p}{\partial w}(z,0) + \frac{\beta_E}{2}\frac{\partial^2 p}{\partial w^2}(z,0) + \int_V p_1(z,w)\nu_E(dw) \quad ; \quad b_2(x) = \alpha_E$$

$$\sigma_{1,1}(x) = \sqrt{z(1-z)} \quad ; \quad \sigma_{2,2}(x) = \sigma_E \quad ; \quad \sigma_{2,1}(x) = 0 \quad ; \quad \sigma_{1,2}(x) = \sigma_E\frac{\partial p}{\partial w}(z,0).$$

Then (3.6) can be written as

$$\mathcal{G}_x(H_{k,\ell}) = kb_1(x) + \ell b_2(x) - \frac{k^2}{2}c_{11}(x) - \frac{\ell^2}{2}c_{22}(x) - k\ell c_{12}(x) + \int_V \overline{H_{k,\ell}}^{h_0}(K(z,w))\nu_E(dw),$$

where, recalling that $\beta_E = \sigma_E^2 + \int_{(-1,\infty)} h_E^2(w)\nu_E(dw)$,

$$\begin{aligned} c_{11}(x) &= z(1-z) + \beta_E\left(\frac{\partial p}{\partial w}(z,0)\right)^2 + \int_V p_2(z,w)\nu_E(dw) \\ &= \sigma_{1,1}^2(x) + \sigma_{1,2}^2(x) + \int_V \big(h_0^1(K(z,w))\big)^2\mu(dw), \\ c_{22}(x) &= \beta_E = \sigma_{2,2}(x)^2 + \int_V \big(h_0^2(K(z,w))\big)^2\mu(dw), \\ c_{12}(x) &= \beta_E\frac{\partial p}{\partial w}(z,0) + \int_{(-1,\infty)} h_E(w)q(z,w)\nu_E(dw) \\ &= \sigma_{12}(x)\sigma_{2,2}(x) + \int_V h_0^1h_0^2(K(z,w))\mu(dw). \end{aligned}$$

Thus (**H2**) holds for any $H = H_{k,\ell}\in\mathcal{H}$.

We can now apply Theorems 2.3 and 2.4 for tightness and identification and conclude. □

3.3 Pathwise uniqueness and convergence in law

To get the uniqueness for (3.3), we will use the pathwise uniqueness result from Li-Pu [28].

Corollary 3.3. Let us assume that Assumption $\mathbf{A}$ holds and that the function $z \to p(z,w)$ is non-decreasing for any $w \in (-1,+\infty)$. Then the sequence of processes $\left(Z^N_{[N.]}/N, S^N_{[N.]}\right)_N$ converges in law in $\mathbb{D}([0,\infty),[0,1]\times\mathbb{R})$ to the unique strong solution (Z,Y) of (3.3).

The monotonicity assumption on p is natural regarding the model since the more individuals carry an allele in a generation, the more this allele should be carried in the next generation.

Proof. In order to apply Theorem 2.5, let us first show that **(H3)** holds. The pathwise uniqueness of the process Y is well known. Let us focus on the first equation of (3.3) and prove the pathwise uniqueness of the process Z.
First, we rewrite the SDE for Z as

$$Z_t \;=\; Z_0 + \int_0^t \widetilde{b_1}(Z_s)ds + \int_0^t \sqrt{Z_s(1-Z_s)}dB^D_s + \sigma_E \int_0^t \frac{\partial p}{\partial w}(Z_s,0)\,dB^E_s$$
$$\int_{(-1,\infty)\setminus[-1/2,1]}(p(Z_{t-},w)-Z_{t-})N(dt,dw) + \int_{[-1/2,1]}(p(Z_{t-},w)-Z_{t-})\widetilde{N}(dt,dw)$$

with

$$\widetilde{b}_1(z) \;=\; \left(\alpha_E - \int_{(-1,\infty)\setminus[-1/2,1]} h_E(w)\nu_E(dw)\right)\frac{\partial p}{\partial w}(z,0) + \frac{\sigma_E}{2}\frac{\partial^2 p}{\partial w^2}(z,0)$$
$$+\int_{[-1/2,1]}\left(p(z,w)-z-h_E(w)\frac{\partial p}{\partial w}(z,0)\right)\nu_E(dw).$$

We are in the conditions of application of Theorem 3.2 in [28]. Indeed, we observe first that $\widetilde{b_1}$ is Lipschitz since $p \in C^3([0,1],(-1,\infty))$ and

$$\sup_{w\in[-1/2,1],z\in[0,1]}\left|\frac{\partial}{\partial z}\left\{p(z,w)-z-h_E(w)\frac{\partial p}{\partial w}(z,0)\right\}\right|/w^2<\infty.$$

We remark also that the Brownian part of (3.3) writes

$$\sqrt{Z_t(1-Z_t)}dB^D_t + \sigma_E\frac{\partial p}{\partial w}(Z_t,0)\,dB^E_t = \sqrt{Z_t(1-Z_t)+\sigma_E^2\left(\frac{\partial p}{\partial w}(Z_t,0)\right)^2}dW_t = \sigma(Z_t)dW_t,$$

with W Brownian motion since B^D and B^E are two independent Brownian motions. We easily prove that for any $z_1,z_2\in[0,1]$, $|\sigma(z_1)-\sigma(z_2)|^2 \le L|z_1-z_2|$ for some constant $L>0$.

Finally $\nu_E((-1,\infty)\setminus[-1/2,1]) < \infty$ and $z\in[0,1] \to (p(z,w)-z)/w$ is uniformly Lipschitz for $w\in[-1,2,1]$ since its first derivative is bounded, so there exists $L>0$ such that

$$\int_{(-1,\infty)\setminus[-1/2,1]}([p(z_1,w)-z_1]-[p(z_2,w)-z_2])^2\nu_E(dw)\le L|z_1-z_2|$$

for any $z_1,z_2\in[0,1]$. Then all the required assumptions for [28] Theorem 3.2 are satisfied and we get the pathwise uniqueness of the solution of (3.3). ☐

3.4 Example

We consider the following main example

$$p(z,w) = \frac{z(1+w)}{z(1+w)+1-z}, \tag{3.7}$$

where the environment w acts as the selection factor. By construction, this selection coefficient w is larger than -1. The particular case when the environment is non-random, i.e. $E_k^N = s/N$ a.s. for some real number $s \in (-1,+\infty)$, yields the classical Wright-Fisher process with weak selection. It is well known that in this case, the processes $(Z^N_{[N.]})_N$ converge in law to the Wright-Fisher diffusion with selection coefficient s whose equation is given by $dZ_t = \sqrt{Z_t(1-Z_t)}dB_t + sZ_t(1-Z_t)dt$. Here we generalize this result for random independent identically distributed environments.

First, we observe that

$$\frac{\partial p}{\partial w}(z,0) = z(1-z)\ ;\ \frac{\partial^2 p}{\partial w^2}(z,0) = -2z^2(1-z).$$

and

$$b_1(z) = \alpha_E z(1-z) - \sigma_E z^2(1-z) + \int_{(-1,\infty)} \Big(\frac{wz(1-z)}{zw+1} - h_E(w)z(1-z)\Big)\nu_E(dw). \tag{3.8}$$

Under Assumption **A**, we can apply Corollary 3.3 to obtain the proposition stated below.

Proposition 3.4. The sequence of processes $\left(Z^N_{[N.]}/N, S^N_{[N.]}\right)_N$ converges in $\mathbb{D}([0,\infty), [0,1]\times\mathbb{R})$ and the limit of the first coordinate is the unique strong solution Z of

$$\begin{aligned} Z_t \;\; = \;\; & Z_0 + \int_0^t b_1(Z_s)ds + \int_0^t \sqrt{Z_s(1-Z_s)}dB_s^D + \sigma_E \int_0^t Z_s(1-Z_s)dB_s^E \\ & + \int_0^t \int_{(-1,+\infty)} \frac{wZ_{s-}(1-Z_{s-})}{1+wZ_{s-}} \widetilde{N}(ds,dw). \end{aligned} \tag{3.9}$$

In particular if $\sigma_E = 0$ and $\nu_E = 0$, we recover the classical Wright-Fisher diffusion with deterministic selection α_E. This extension allows us to consider small random fluctuations (asymptotically Brownian) and punctual dramatic advantage of the selective effects.

4 Continuous state branching process with interaction in Lévy environment

In this section, we are interested in approximations of large population dynamics with random environment and interaction. We generalize in different directions the classical convergence of Galton-Watson processes to *Continous State Branching processes* (CSBP), see for example [16, 24, 8]. We focus on models where the environment and the interaction mainly affect the mean of the reproduction law and thus modify the drift term of the CSBP by addition of stochastic and nonlinear terms. Our method based on Section 2 allows us to obtain new statements both for convergence of discrete population models and for existence of solutions of SDE with jumps, as can be seen in the following theorems. In particular we obtain a discrete population model approximating the so-called *CSBP with interaction in Lévy environment* (BPILE) for large populations.

The CSBPs in random environment or with interaction have recently been subject of great attention. We refer to [31] for existence of the solution of the associated SDE under general assumptions, [4, 3, 29] for approximations and study of some classes of CSBP in random environment (without interaction), [2, 11, 27] for CSBP with interaction in continuous time (without random environment) and [33] for diffusion approximations.

4.1 The discrete model

Let us now describe our framework. The population size is scaled by the integer $N \geq 1$. As in Section 3, we introduce for any N a sequence of independent identically distributed real-valued random variables $(E_k^N)_{k\geq 0}$ with same law as E^N. The asymptotic behavior of $(E_k^N)_{k\geq 0}$ is similar to the one in Section 3 (Assumption **A**). Nevertheless, the scaling parameter is no longer N but can be any sequence $(v_N)_N$ tending to infinity with N. As in the previous section, h_E denotes a truncation function defined on $(-1, +\infty)$.

Assumption A1. Let us consider $\alpha_E \in \mathbb{R}$, $\sigma_E \in [0, \infty)$ and ν_E a measure on $(-1, \infty)$ such that

$$\int_{(-1,\infty)} (1 \wedge w^2)\ \nu_E(dw) < \infty. \tag{4.1}$$

Writing $\beta_E = \sigma_E^2 + \int_{(-1,\infty)} h_E^2(w)\nu_E(dw)$, we assume that

$$\lim_{N\to\infty} v_N\, \mathbb{E}(h_E(E^N)) = \alpha_E;\ \lim_{N\to\infty} v_N\, \mathbb{E}(h_E^2(E^N)) = \beta_E;$$

$$\lim_{N\to\infty} v_N\, \mathbb{E}(f(E^N)) = \int_{(-1,\infty)} f(w)\, \nu_E(dw),$$

for any f vanishing in a neighborhood of zero.

We also consider the associated random walk defined by

$$S_0^N = 0, \qquad S_{k+1}^N = S_k^N + E_k^N \quad (k \geq 0).$$

We recall (as in Section 3) that **A1** is equivalent to the convergence of the random walk $S_{[v_N.]}^N$ to the Lévy process Y with characteristics $(\alpha_E, \beta_E, \nu_E)$ defined in (3.2). We reduce the set of jumps to $(-1, \infty)$ to avoid degenerated cases when a catastrophe below -1 could kill all the population in one generation.

Let us fix N. We assume that given a population size n and an environment w, each individual reproduces independently at generation k with the same reproduction law $L^N(n, w)$.We thus introduce random variables $Z_0 \geq 0$ and $L_{i,k}^N(n, w)$ such that the family of random variables $(Z_0, (L_{i,k}^N(n, w), n \in \mathbb{N}, w \in (-1, +\infty)), E_j^N; i, k \in \mathbb{N}^*, j \in \mathbb{N})$ is independent and for each $n \in \mathbb{N}, w \in (-1, +\infty)$, the random variables $L_{i,k}^N(n, w)$ are all distributed as $L^N(n, w)$ for $i, k \geq 1$. We also assume that the function $L_{i,k}^N$ defined on $\Omega \times \mathbb{N} \times (-1, +\infty)$ endowed by the product σ-field is measurable.

The population size Z_k^N at generation k is recursively defined as follows,

$$Z_0^N = [NZ_0], \qquad Z_{k+1}^N = \sum_{i=1}^{Z_k^N} L_{i,k}^N(Z_k^N, E_k^N) \quad \forall k \geq 0. \tag{4.2}$$

To investigate the convergence in law of the process $\Big(\big(\frac{1}{N} Z_{[v_N t]}^N, S_{[v_N t]}^N\big), t \in [0, \infty)\Big)$, we cannot apply directly our general result to $Z_{[v_N.]}^N$. Indeed, the (associated) characteristics of the first component are not bounded. Moreover, scaling limits of Z^N can lead to explosive processes, as already happens in the Galton-Watson case. Therefore, we first study the convergence of the process

$$X_k^N = \big(\exp(-Z_k^N/N), S_k^N\big) \qquad (k \in \mathbb{N}) \tag{4.3}$$

in $\mathbb{D}(\mathbb{R}_+, [0, 1] \times \mathbb{R})$ where the state space of the first coordinate has been compactified. Following the notation of Section 2, we introduce for $x = (\exp(-z), y) \in (0, 1] \times \mathbb{R}$ the quantity

$$F_x^N = \Big(\exp\Big(-\frac{1}{N}\sum_{i=1}^{[Nz]}\big(L_i^N([Nz],E^N)-1\big)-z\Big), y+E^N\Big), \tag{4.4}$$

and observe that for any $z\in\mathbb{N}/N$, conditionally on $X_k^N=(\exp(-z),y)$, the random variable X_{k+1}^N is distributed as F_x^N.

To apply the theoretical framework developed in Section 2, we define $\chi=(0,1]\times\mathbb{R}$, $\mathcal{U}=[-1,1]\times(-1,\infty)$ and for $(u,w)\in\mathcal{U}$,

$$h(u)=h(v,w)=(v,1-\exp(-w)), \qquad h_0(u)=h_0(v,w)=(v,h_E(w)) \tag{4.5}$$

respectively as the specific function and the truncation function. We choose the functional space $\mathcal{H}$ defined by

$$\mathcal{H}=\{H_{k,\ell}:k\geq 1,\ell\geq 0\}\cup\{H_\ell:\ell\geq 1\},$$

where for any $u=(v,w)\in\mathcal{U}$,

$$H_{k,\ell}(u)=v^k\exp(-\ell w) \quad \text{and} \quad H_\ell(u)=1-\exp(-\ell w).$$

The fact that $\mathcal{H}$ satisfies $(\mathbf{H1})$ is a consequence of the local Stone-Weierstrass Theorem on $[-1,1]\times[-1,\infty)\setminus\{(0,0)\}$ (cf. Appendix 6.4). For any $k,\ell\geq 0$ and $x=(\exp(-z),y)\in(0,1]\times\mathbb{R}$, we have

$$\begin{aligned}\mathcal{G}_x^N(H_{k,\ell}) &= v_N\mathbb{E}\Big(H_{k,\ell}\Big(\exp\Big(-\frac{1}{N}\sum_{i=1}^{[Nz]}(L_i^N([Nz],E^N)-1)-z\Big)-\exp(-z),E^N\Big)\Big)\\ &= e^{-kz}v_N\mathbb{E}\left(\left(e^{-\frac{1}{N}\sum_{i=1}^{[Nz]}(L_i^N([Nz],E^N)-1)}-1\right)^k e^{-\ell E^N}\right).\end{aligned}$$

Let us set

$$P_k^N(z,w)=\mathbb{E}\left(e^{-\frac{k}{N}(L^N([Nz],w)-1)}\right)^{[Nz]}-1 \tag{4.6}$$

and

$$A_{j,\ell}^N(z)=v_N\mathbb{E}\left(P_j^N(z,E^N)e^{-\ell E^N}\right). \tag{4.7}$$

The presence of the term -1 in (4.6) may look strange at first glance, but it ensures that $P_k^N\to 0$ as $N\to\infty$. Using the binomial expansion and by independence of the reproduction random variables conditionally on E^N, we obtain that

$$\mathcal{G}_x^N(H_{k,\ell})=e^{-kz}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}A_{j,\ell}^N(z) \tag{4.8}$$

for $k\geq 1$, since $\sum_{j=0}^k\binom{k}{j}(-1)^{j-k}=0$. We also obtain that for $\ell\geq 1$,

$$\mathcal{G}_x^N(H_\ell)=v_N\mathbb{E}(1-\exp(-\ell E^N)).$$

The convergence of $A_{j,\ell}^N$ characterizes the effect of the reproduction law on the population dynamics, including density dependence and random environment. The uniform convergence and boundedness of $\exp(-kz)A_{j,\ell}^N(z)$ will ensure the tightness of $X_{[v_N\cdot]}^N$ by Theorem 2.3. The continuity of the limiting functions will be checked for the identification of the characteristic triplet. Finally, the representation of the limiting semimartingales as solutions of a stochastic differential equation and its associated uniqueness will give the convergence (Theorem 2.5).

Remark 4.1. In the case of Galton-Watson processes, $E^N = 0, L^N(z,w) = L^N$ and $A^N_{j,\ell}(z)$ becomes

$$A^N_j(z) = v_N P^N_j(z,0) = v_N\left(\mathbb{E}\left(e^{-\frac{j}{N}(L^N-1)}\right)^{[Nz]} - 1\right).$$

We observe then that $A^N_j(z) \sim v_N[Nz]\mathbb{E}(1-\exp(-j(L^N-1)/N))$ as $N\to\infty$. It can easily be proved that the uniform convergence of $e^{-jz}A^N_j(z)$ is equivalent to the convergence of $v_N N\mathbb{E}(g((L^N-1)/N))$ for any g in a set containing a truncation function, its square and regular functions null in a neighborhood of zero. Thus this uniform convergence is equivalent to the classical necessary and sufficient condition for convergence in law of Galton-Watson processes [16, 4].

In the next section, we generalize this criterium to reproduction random variables depending on the population size and the environment.

4.2 Tightness

We first prove the tightness of $\left(\frac{1}{N}Z^N_{[v_N.]}, S^N_{[v_N.]}\right)_N$ in $\mathbb{D}(\mathbb{R}_+, [0,\infty]\times\mathbb{R})$ by assuming the uniform convergence of the characteristics.

Assumption A1'. Let the characteristics $A^N_{j,\ell}$ be defined in (4.7). For any $1\le j\le k$ and $\ell\ge 0$, there exists a bounded function $A_{j,k,\ell}$ such that

$$e^{-kz}A^N_{j,\ell}(z) \overset{N\to\infty}{\longrightarrow} A_{j,k,\ell}(z)$$

uniformly for $z\ge 0$.

Then we state a tightness criterion for the original scaled process in the state space $[0,\infty]\times\mathbb{R}$ endowed with a distance d which makes it compact and then Polish, say $d(z_1,z_2) = |\exp(-z_1)-\exp(-z_2)|$ for $z_1,z_2\in[0,\infty]$ with the convention $\exp(-\infty)=0$.

Theorem 4.2. Under Assumptions **A1** and **A1'**, the sequence of processes

$$\left(\left(\frac{1}{N}Z^N_{[v_N t]}, S^N_{[v_N t]}\right), t\in[0,\infty)\right)$$

is tight in $\mathbb{D}(\mathbb{R}_+, [0,\infty]\times\mathbb{R})$.

Proof. Let us prove the tightness of $(X^N_{[v_N.]})_N$ in $\mathbb{D}(\mathbb{R}_+, [0,1]\times\mathbb{R})$. For $\ell\ge 1$, it follows from Assumption **A1** that

$$v_N\,\mathbb{E}\left(1-e^{-\ell E^N}\right) \overset{N\to\infty}{\longrightarrow} \gamma^E_\ell = \alpha_E\, z - \frac{1}{2}\sigma_E^2\, z^2 + \int_{(-1,+\infty)}\left(1-e^{-zw}-zh_E(w)\right)\nu_E(dw), \quad (4.9)$$

since $1-e^{-\ell w} = \ell h_E(w) - \frac{1}{2}\ell^2 h_E^2(w) + \kappa(w)$, where $\kappa(w) = o(w^2)$ is continuous bounded.

Then we can define $\mathcal{G}_.$ on H_ℓ for $\ell\ge 1$ as

$$\mathcal{G}_x(H_\ell) = \gamma^E_\ell, \quad (4.10)$$

for any $x\in\mathcal{X} = (0,1]\times\mathbb{R}$. Let us now define $\mathcal{G}_.$ for $H_{k,\ell}\in\mathcal{H}$ and $k\ge 1, \ell\ge 0$. We set

$$\mathcal{G}_x(H_{k,\ell}) = \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}A_{j,k,\ell}(z). \quad (4.11)$$

for $x = (e^{-z}, y)\in\mathcal{X}$. Using Assumption **A1'** and (4.8), we obtain that

$$\lim_{N\to\infty}\sup_{x\in\mathcal{X}}\left|\mathcal{G}^N_x(H) - \mathcal{G}_x(H)\right| = 0$$

for any $H\in\mathcal{H}$. Moreover $\mathcal{G}_.(H)$ is bounded by **A1'** and Hypothesis **(H1.3)** is satisfied. The tightness of $(X^N_{[v_N.]})_N$ in $\mathbb{D}(\mathbb{R}_+, [0,1]\times\mathbb{R})$ is then a consequence of Theorem 2.3 and yields the result. ☐

4.3 Identification

Now, we identify the limiting values of $(X^N_{[v_N.]})_N$ as diffusions with jumps. We are interested in models where the environment and the interaction affect the mean reproduction law.

We introduce a truncation function h_D on the state space $(0,+\infty)$, parameters $\alpha_D \in \mathbb{R}$ and $\sigma_D \geq 0$ and a σ-finite measure ν_D on $(0,+\infty)$ such that

$$\int_0^\infty (1 \wedge z^2)\nu_D(dz) < +\infty. \tag{4.12}$$

We also consider a locally Lipschitz function g defined on $\mathbb{R}^+$ such that

$$e^{-z} z\, g(z) \overset{z\to\infty}{\longrightarrow} 0. \tag{4.13}$$

The function g models the interaction between individuals. In the applications to population dynamics, the most relevant functions will be polynomials.

We provide now the scaling assumption on the reproduction random variable L^N so that the limiting values of Z^N/N can be identified to a BPILE. This assumption will become more explicit and natural through the identification and examples of the next sections.

Assumption A2. Setting for $z \geq 0$,

$$\begin{aligned} \gamma^D_z &= \alpha_D\, z - \frac{1}{2}\,\sigma_D^2\, z^2 + \int_{(0,+\infty)} \big(1 - e^{-zr} - z h_D(r)\big)\nu_D(dr), \\ \gamma^E_z &= \alpha_E\, z - \frac{1}{2}\,\sigma_E^2\, z^2 + \int_{(-1,+\infty)} \big(1 - e^{-zw} - z h_E(w)\big)\nu_E(dw), \end{aligned}$$

we assume that for any $1 \leq j \leq k$ and $\ell \geq 0$,

$$\sup_{z\geq 0} e^{-kz}\big|A^N_{j,\ell}(z) \ + \ jz\, g(z) + \gamma^D_j\, z + \gamma^E_{jz+\ell} - \gamma^E_\ell\big| \overset{N\to\infty}{\longrightarrow} 0, \tag{4.14}$$

where $A^N_{j,\ell}$ has been defined in (4.7).

Remark 4.3. (i) In Appendix 6.1, we provide an explicit construction of a family of random variables $L^N(z,e)$ satisfying **A2**, in the case $\beta_D = 0$.
(ii) We believe that the pointwise convergence induced by **A2** is actually necessary for the convergence of the process Z^N/N to a BPILE. It does not seem sufficient in general since some integration argument is involved. Uniformity in **A2** provides a sufficient condition. It can be proved for many classes of reproduction laws via uniform continuity, using monotonicity or convexity arguments or boundedness of derivative on compact sets, see the examples below.
(iii) Finally, let us remark that we only need to prove the previous convergence for $z \in \mathbb{N}/N$ in **A2**, using the definition of $A^N_{j,\ell}(z)$ and the uniform continuity of the limit. It will be more convenient for examples.

We observe that under Assumption **A2**, Assumption **A1'** is satisfied with

$$A_{j,k,\ell}(z) = e^{-kz}\left(-jz\, g(z) - \gamma^D_j\, z + \gamma^E_\ell - \gamma^E_{jz+\ell}\right).$$

Indeed, this expression is bounded using (4.13) and the boundedness of $\exp(-kz)\gamma^E_{jz+\ell}$, since $|\gamma^E_{jz+\ell}| \leq C_{\ell,j}(z + z^2 + e^{jz/2}z^2\beta_E + e^{jz}\nu_E(-1,-1/2))$ for $j \leq k$ and $k \geq 1$.

Let us then observe from the proof of Theorem 4.2 that $(X^N_{[v_N.]})_N$ is tight in $\mathbb{D}(\mathbb{R}_+, [0,1]\times\mathbb{R})$. Moreover we can simplify the expression (4.11) of the limiting characteristic $\mathcal{G}_x$, which writes

$$\mathcal{G}_x(H_{k,\ell}) = e^{-kz}\sum_{j=0}^k \binom{k}{j}(-1)^{k-j}\left(-jz\, g(z) - \gamma^D_j\, z + \gamma^E_\ell - \gamma^E_{jz+\ell}\right) \tag{4.15}$$

for $x=(e^{-z},y)$. For that purpose, we denote

$$f_z(u)=1-e^{-zu}$$

and observe that

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}j = \delta_{1,k} \tag{4.16}$$

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}j^2 = 2\delta_{2,k}+\delta_{1,k} \tag{4.17}$$

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}f_j(u) = (-1)^{k+1}f_1(u)^k \tag{4.18}$$

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}f_{jz+\ell}(u) = (-1)^{k+1}e^{-\ell u}f_z(u)^k. \tag{4.19}$$

For $k\geq 3$, it follows from (4.18) and (4.19) and straightforward computation that

$$\mathcal{G}_x(H_{k,\ell})=(-1)^k e^{-kz}\left(\int_{(-1,+\infty)}e^{-\ell w}(f_z(w))^k\nu_E(dw)+z\int_{(0,+\infty)}(f_1(r))^k\nu_D(dr)\right). \tag{4.20}$$

For $k=2$, computation using (4.17) leads to

$$\mathcal{G}_x(H_{2,\ell}) = e^{-2z}\Big\{z^2\beta_E+\int_{(-1,+\infty)}\big(e^{-\ell w}(f_z(w))^2-z^2h_E^2(w)\big)\nu_E(dw)+z\beta_D + z\int_{(0,+\infty)}\big(f_1^2(r)-h_D^2(r)\big)\nu_D(dr)\Big\}. \tag{4.21}$$

Similarly (4.16) implies that

$$\mathcal{G}_x(H_{1,\ell})=e^{-z}\Big\{\gamma_\ell^E-\gamma_{z+\ell}^E-zg(z)-z\gamma_1^D\Big\}. \tag{4.22}$$

To identify the limiting SDE, we have to find the drift and variance terms and the jump measures in (2.5), from the expressions (4.10), (4.20), (4.21) and (4.22).

We first remark that for $k\geq 3,\ell\geq 0$, $H_{k,\ell}=\overline{H_{k,\ell}}$ with the notation introduced in Lemma 2.1. We work by identification for $x=(e^{-z},y)\in(0,1]\times\mathbb{R}$ using (4.20). We thus define the measure μ on $V=[0,+\infty)\times\mathbb{R}$ by

$$\mu(d\theta,dr)=\mathbb{1}_{\theta\leq 1,r>-1}\,d\theta\,\nu_E(dr)+\mathbb{1}_{\theta>1,r>0}\,d\theta\,\nu_D(dr), \tag{4.23}$$

and the image function $K=(K_1,K_2)$ by

$$K_1(x,\theta,r)=-e^{-z}.\Big(f_z(r)\,\mathbb{1}_{\theta\leq 1}+f_1(r)\,\mathbb{1}_{1<\theta\leq 1+z}\Big)\ ;\ K_2(x,\theta,r)=r\,\mathbb{1}_{\theta\leq 1}. \tag{4.24}$$

Then $H_{k,\ell}$ satisfies (**H2.2**) for $k\geq 3,\ell\geq 0$.

Moreover it is easy to find b_2 and $\sigma_{2,2}$ so that H_ℓ satisfies (**H2.2**) for $\ell\geq 1$ using that

$$H_\ell(u)=\ell h_E(w)-\frac{\ell^2}{2}h_E(w)^2+\overline{H_\ell}(u),\qquad \overline{H_\ell}(v,w)=f_\ell(w)-\ell h_E(w)-\frac{\ell^2}{2}h_E(w)^2$$

for $u=(v,w)$ and (4.10). Indeed, by identification and from (2.4), we set for $x=(e^{-z},y)$

$$b_2(x)=\alpha_E\ ;\ \sigma_{2,2}(x)=\sigma_E. \tag{4.25}$$

Let us now consider the functions $H_{2,\ell}$ ($\ell \geq 0$). Note that for $u = (v, w)$, we have

$$H_{2,\ell}(u) = v^2 e^{-\ell w} = h_{D}^2(v) + \overline{H_{2,\ell}}(u), \qquad \overline{H_{2,\ell}}(u) = v^2(e^{-\ell w} - 1) + v^2 - h_{D}^2(v).$$

The fact that (**H2.2**) is satisfied for $H_{2,\ell}$ comes from (4.21) for the left hand side and for the right hand it is given by a direct computation of

$$\sigma_{1,1}(x)^2 + \sigma_{1,2}(x)^2 + \int_V K_1^2(x,\theta,r)\mu(d\theta,dr) + \int_V \overline{H_{2,\ell}}(K(x,\theta,r))\mu(d\theta,dr),$$

where K is defined from (4.24). Using $\overline{H_{2,\ell}}(K(x,\theta,r)) = K_1(x,\theta,r)^2(e^{-\ell K_2(x,\theta,r)} - 1)$, the condition writes for $x = (e^{-z}, y)$,

$$\sigma_{1,1}(x)^2 + \sigma_{1,2}(x)^2 = e^{-2z}(z\sigma_{D}^2 + z^2\sigma_{E}^2).$$

It remains to check **(H2.2)** for $H_{1,\ell}$, with

$$H_{1,\ell}(u) = ve^{-\ell w} = v(1 - \ell h_{E}(w)) + \overline{H_{1,\ell}}(u),$$

where $\overline{H_{1,\ell}}(u) = v(\ell h_{E}(w) - f_\ell(w)) = o(|u|^2)$. Using (4.22), we have

$$\begin{aligned}\mathcal{G}_x(H_{1,\ell}) \quad &= \quad e^{-z}\bigg(- \alpha_{E} z - z\gamma_1^{D} - zg(z) + \frac{z^2}{2}\sigma_{E}^2 + \ell z \sigma_{E}^2 \\ &\qquad + \int_{(-1,+\infty)} f_z f_\ell(w)\nu_{E}(dw) + \int_{(-1,+\infty)} (zh_{E}(w) - f_z(w))\nu_{E}(dw)\bigg).\end{aligned}$$

As a conclusion, both sides of (2.4) coincide for $H \in \mathcal{H}$ by setting for any $x = (e^{-z}, y) \in (0,1] \times \mathbb{R}$,

$$b_1(x) = e^{-z}\bigg(- \alpha_{E} z - z\gamma_1^{D} - zg(z) + \frac{z^2}{2}\sigma_{E}^2 + \int_{(-1,+\infty)} (zh_{E}(w) - f_z(w))\nu_{E}(dw)\bigg) \tag{4.26}$$

and $b_2(x) = \alpha_{E}$ and K, μ defined by (4.24) and (4.23) and

$$\sigma_{1,1}(x) = -\sqrt{z}\sigma_{D} e^{-z} \ ; \ \sigma_{1,2}(x) = -z\sigma_{E} e^{-z} \ ; \ \sigma_{2,1}(x) = 0 \ ; \ \sigma_{2,2}(x) = \sigma_{E}, \tag{4.27}$$

and for any $x \in \{0\} \times \mathbb{R}$ and $(\theta, r) \in V$,

$$b(x) = (0, \alpha_{E}), \quad \sigma_{11}(x) = \sigma_{21}(x) = \sigma_{12}(x) = 0, \quad \sigma_{22}(x) = \sigma_{E}, \tag{4.28}$$

$$K_1(x,\theta,r) = 0, \quad K_2(x,\theta,r) = r\,\mathbb{1}_{\theta \leq 1}. \tag{4.29}$$

The general identification result for the exponential transformation of the processes can then be stated as follows, with $h_0(v,w) = (v, h_{E}(w))$.

Theorem 4.4. Under Assumptions **A1** and **A2**, the sequence of processes

$$\left(\left(\exp\left(-\frac{1}{N} Z^N_{[v_N t]}\right), S^N_{[v_N t]}\right) : t \in [0,\infty)\right)$$

is tight in $\mathbb{D}([0,\infty), [0,1] \times \mathbb{R})$ and any limiting value $X \in \mathbb{D}([0,\infty), [0,1] \times \mathbb{R})$ is a weak solution of the following two-dimensional stochastic differential equation

$$\begin{aligned} X_t \quad &= \quad X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s + \int_0^t \int_V h_0(K(X_{s-}, v))\tilde{N}(ds,dv) \\ &\qquad + \int_0^t \int_V (Id - h_0)(K(X_{s-}, v))N(ds,dv), \end{aligned} \tag{4.30}$$

where $X_0 = (\exp(-Z_0), 0)$, N is Poisson point measure with intensity $ds\mu(dv)$ on $\mathbb{R}^+ \times V = [0,+\infty)^2 \times \mathbb{R}$ and B is a two-dimensional Brownian motion and Z_0, B, N are independent. The function $b = (b_1, b_2)$, the matrix σ, the measure μ and the image function K have been defined in (4.23)-(4.29).

Proof of Theorem 4.4. We already know that **(H1)** is a consequence of **A2**. Let us check that **(H2)** is satisfied. We first prove the continuity **(H2.1)** of $x \to \mathcal{G}_x(H)$ for any $H \in \mathcal{H}$ and its extension to $\overline{\mathcal{X}}$. Recalling (4.15), we need to prove that $z \in [0,\infty) \to \gamma^E_\ell - \gamma^E_{jz+\ell}$ is continuous and $\exp(-jz)(\gamma^E_\ell - \gamma^E_{jz+\ell}) \to 0$ as $z \to \infty$. Indeed, the continuity can be obtained from the bound $|1 - e^{-(jz+\ell)w} - (jz+\ell)h_E(w)| \le C(1 \wedge w^2)$ for any $z \in [z_0, z_1] \subset [0,\infty)$, while the limit as $z \to \infty$ can be proved using Lemma 6.2 in Appendix and $\nu_E(-1,-1+\varepsilon) \to 0$ as $\varepsilon \to 0$. That allows us to prove that **(H2.1)** is satisfied.

Our choice of parameters in (4.23)- (4.29) ensures that (**H2.2**) is satisfied for any H_ℓ and $H_{k,\ell}$. Applying Theorem 2.4 to X^N allows us to conclude. □

Let us now write explicitly the stochastic differential equation (4.30) for $X_t = (X^1_t, Y_t)$:

$$\begin{aligned}
dX^1_t &= X^1_t \log X^1_t \left(\alpha_E + \frac{\sigma_E^2}{2} \log X^1_t + g(-\log X^1_t) + \alpha_D - \frac{\sigma_D^2}{2} \right) dt \\
&\quad - X^1_t \Bigg(\int_{(-1,+\infty)} (1 - e^{w \log(X^1_t)} + \log X^1_t \, h_E(w)) \nu_E(dw) \\
&\qquad\qquad - \log X^1_t \int_{(0,+\infty)} (1 - e^{-r} - h_D(r)) \nu_D(dr) \Bigg) dt \\
&\quad + \sigma_E X^1_t \log X^1_t dB^E_t - \sigma_D X^1_t \sqrt{-\log X^1_t} dB^D_t \\
&\qquad - \int_{(-1,+\infty)} X^1_{t-} (1 - e^{w \log(X^1_t)}) \widetilde{N}^E(dt, dw) \\
&\qquad - \int_{(0,+\infty)^2} \mathbb{1}_{\theta \le -\log X^1_{t-}} X^1_{t-} (1 - e^{-r}) \widetilde{N}^D(dt, d\theta, dr) \\
dY_t &= \alpha_E dt + \sigma_E dB^E_t + \int_{(-1,+\infty)} h_E(w) \widetilde{N}^E(dt, dw) + \int_{(-1,+\infty)} (w - h_E(w)) N^E(dt, dw),
\end{aligned}$$

where B^E and B^D are Brownian motions, N^D and N^E are Poisson Point measures respectively on $[0,\infty) \times (0,\infty)$ and on $[0,\infty) \times (-1,\infty)$ with intensity $dt\nu_D(du)$ and $dt\nu_E(dw)$ and Z_0, B^E, B^D, N^D and N^E are independent.

Using Itô's formula (see [18]), a straightforward computation leads to the equation satisfied by $Z_t = -\log X^1_t$. More precisely, we define the explosion time T_{exp} by

$$T_{exp} = \lim_{\varepsilon \to 0+} \inf\{t \ge 0; X^1_t \le \varepsilon\} = \lim_{a \to +\infty} \inf\{t \ge 0; Z_t \ge a\} \in [0, +\infty].$$

We obtain

$$\begin{aligned}
Z_t &= Z_0 + \alpha_D \int_0^t Z_s ds + \int_0^t Z_{s-} dY_s + \int_0^t Z_s g(Z_s) ds + \sigma_D \int_0^t \sqrt{Z_s} dB^D_s \qquad (4.31) \\
&\quad + \int_0^t \int_{(0,+\infty)^2} \mathbb{1}_{\theta \le Z_{s-}} h_D(r) \widetilde{N}^D(ds, d\theta, dr) \\
&\quad + \int_0^t \int_{(0,+\infty)^2} \mathbb{1}_{\theta \le Z_{s-}} (r - h_D(r)) N^D(dt, d\theta, dr).
\end{aligned}$$

on the time interval $[0, T_{exp})$ and $Z_t = +\infty$ for $t \ge T_{exp}$.

When $T_{exp} = +\infty$ almost surely, the process is said to be *conservative* (or non-explosive). Grey's condition gives a criteria for CSBP, which has been recently extended to CSBP in random Lévy environment in [17].

We have thus proved the tightness of the process and identified the limiting values of $(X^N_{[v_N \cdot]})_N$ as weak solutions of a SDE. Uniqueness of the SDE (4.31) (Hypothesis **H3**) has to be proven to conclude convergence. From the pioneering works of Yamada and

Watanabe, several results have been obtained for pathwise uniqueness relaxing the Lipschitz conditions on coefficients. In particular, general results for positive processes with jumps have been obtained in [14, 28] and used in random environment, see in particular [31]. This technique allows us to conclude strong uniqueness before explosion. Here, the process may explode in finite time, which is already the case for classical CSBP and in our framework, explosion can also be due to cooperation or random environment. This leads us to consider two cases. In the first case, we obtain a convergence in law on the state space $[0,\infty]$ under an additional regularity assumption on the drift term close to infinity. This result extends the classical criterion for convergence of Galton-Watson processes, adding both random environment and interaction. In the second case, we obtain the convergence of $Z^N_{[v_N.]}$ in $[0,\infty)$ when the limiting values of the sequence of processes are non-explosive. We observe that it also extends results of [4] to Lévy environment with infinite variation and of [11] by relaxing moment assumptions for interaction.
The pathwise uniqueness of the SDE allows us to capture limiting processes where infinity is either absorbing or non-accessible. Other situations are interesting, where infinity is regular and uniqueness in law could be invoked. In particular, we refer to [13] for a criterion for reflection at infinity of CSBP with quadratic competition and [22] and [5] for similar issues.

4.4 Explosive CSBP with interaction and random environment

In this section, the process may be non-conservative, i.e. T_{exp} may be finite. In order to obtain the strong uniqueness and following [14, 29], we consider the following assumption concerning the regularity of the drift term.

Assumption A3. There exist continuous functions r, b_r and b_d such that for any $z\in[0,\infty)$,

$$e^{-z}\left(zg(z)-\frac{\sigma_E^2}{2}z^2+\int_{[-1/2,1]}(1-e^{-zw}-zh_E(w))\nu_E(dw)\right)=b_r(z)+b_d(z), \qquad (4.32)$$

with r non-negative, non-decreasing and concave, $\int_0^\cdot 1/r(z)dz=\infty$, $|b_r(-\log(u))-b_r(-\log(u'))|\leq r(|u-u'|)$ for any $u,u'\in(0,1]$ and b_d non-increasing.

Theorem 4.5. We assume that **A1**, **A2** and **A3** hold.

Then there exists a unique strong solution $(Z,Y)\in\mathbb{D}([0,\infty),[0,\infty]\times\mathbb{R})$ of (3.2) and (4.31) and

$$\left(\left(\frac{1}{N}Z^N_{[v_Nt]},S^N_{[v_Nt]}\right):t\in[0,\infty)\right)\Rightarrow((Z_t,Y_t):t\in[0,+\infty))$$

in $\mathbb{D}([0,\infty),[0,\infty]\times\mathbb{R})$, where $[0,\infty]$ is endowed with $d(z_1,z_2)=|\exp(-z_1)-\exp(-z_2)|$.

Proof of Theorem 4.5. We first remark that the convergence in law of $(X^N_{[v_N.]})_N$ in $\mathbb{D}([0,\infty),[0,1]\times\mathbb{R})$ implies the weak convergence of $(Z^N_{[v_N.]}/N,S^N_{[v_N.]})$ to $(-\log(X^1),Y)$ in $\mathbb{D}([0,\infty),[0,\infty]\times\mathbb{R})$, where $[0,\infty]$ is endowed with d and $-\log(0)=\infty$.

We recall from the previous section that X^N satisfies ($\mathbf{H1}$) and ($\mathbf{H2}$). To apply Theorem 2.5, it remains to check that X defined in (4.30) is unique in law.

Let us prove that under **A3**, pathwise uniqueness holds for X in $\mathbb{D}([0,T],[0,1]\times\mathbb{R})$. First, the second component Y of X is a Lévy process and the pathwise uniqueness is well known. Second, the equation for the first component X^1 writes

$$\begin{aligned}X^1_t &= X^1_0+\int_0^t\widetilde{b}_1(X^1_s)ds+\int_0^t\sigma(X^1_s)dW_s+\int_0^t\int_{V\setminus V_0}K^1(X^1_{s-},v)N(ds,dv)\\ &\quad+\int_0^t\int_{V_0}K^1(X^1_{s-},v)\widetilde{N}(ds,dv)\end{aligned}$$

where $\sigma(u) = \sqrt{\sigma_{1,1}(u)^2 + \sigma_{1,2}(u)^2}$ and W is a Brownian motion independent of X_0^1 and of the Poisson point measure N. The set V_0 is defined as $V_0 = [0,1]\times[-1/2,1]\cup(1,\infty]\times(0,\infty)$. For $x_1 = \exp(-z)$ and $\rho = \alpha_E + \gamma_1^D - \int_{(-1,\infty)-[-1/2,1]} h_E \nu_E$,

$$\widetilde{b}_1(x_1) = e^{-z}\bigg(- z\rho - zg(z) + \frac{z^2}{2}\sigma_E^2 + \int_{[-1/2,1]} (zh_E(w) - f_z(w))\nu_E(dw)\bigg).$$

We first observe that $\mu(V \setminus V_0) < \infty$. Moreover, combining (4.26) and (4.32), we have

$$\widetilde{b}_1(x_1) = x_1 \log(x_1)\rho - b_r(-\log(x_1)) - b_d(-\log(x_1)) = \widetilde{b}_r(x_1) + \widetilde{b}_d(x_1),$$

where $\widetilde{b}_d = -b_d(-\log .)$ is non-decreasing and $\widetilde{b}_r$ satisfies $|\widetilde{b}_r(x_1) - \widetilde{b}_r(\widetilde{x}_1)| \leq \widetilde{r}(|x_1 - \widetilde{x}_1|)$ for $x_1, \widetilde{x}_1 \in [0,1]$, with $\int_0^{\cdot} 1/\widetilde{r}(z)dz = \infty$ and $\widetilde{r}$ non-decreasing and concave. Indeed using Lemma 6.3 in Appendix, one can take $\widetilde{r}(y) = r(y)+Cy+C_1 r_1(y)$, with $r_1(x_1) = -x_1\log(x_1)$ and C, C_1 well chosen.

Then we easily check that σ^2 is Lipschitz continuous and $|\sigma(y) - \sigma(y')|^2 \leq |\sigma(y)^2 - \sigma(y')^2|$ and $y \to y + K^1(y,v)$ is non-decreasing. Finally,

$$\begin{aligned}\int_{V_0} (K^1(y,v) - K^1(y',v))^2 \mu(dv) &= g_1(y,y')\int_{\mathbb{R}^+} (e^{-r}-1)^2 \nu_D(dr) \\ &\quad + \int_{[-1/2,1]} (g_2(y,w) - g_2(y',w))^2 \nu_E(dw),\end{aligned}$$

where for any $y, y' \in (0,1]$,

$$g_1(y,y') = \min(-\log(y), -\log(y'))(y-y')^2 + \min(y,y')^2|\log(y) - \log(y')| \tag{4.33}$$

(with a null extension at 0) and

$$g_2(y,w) = u(e^{\log(y)w} - 1). \tag{4.34}$$

Using now Lemma 6.4 in Appendix and the integrability assumptions on ν_D and ν_E, there exists $L > 0$ such that

$$\int_{V_0} (K^1(y,v) - K^1(y',v))^2 \mu(dv) \leq L|y - y'|.$$

Then we can apply Theorem 3.2 in [28] and conclude by observing that $X_t = 0$ for $t \geq T_{exp}$ by pathwise uniqueness. □

Recently, Pardoux and Dramé [11] have proven the convergence of some continuous time and discrete space processes to CSBP with interaction. Here we relax their conservative assumption and extend to random environments and to general classes of reproduction laws, in a discrete time setting.

Application to Galton-Watson processes with cooperative effects. Note that Theorem 4.5 allows us to recover the convergence in law of the Galton-Watson processes $(\widehat{Z}^N_{[v_N\cdot]})_N$ defined as in (4.2) with the reproduction laws $L^N \in \mathbb{N}$ satisfying:

$$\lim_{N\to\infty} v_N N\,\mathbb{E}(h_D((L^N-1)/N))) = \alpha_D; \qquad \lim_{N\to\infty} v_N N\,\mathbb{E}(h_D^2((L^N-1)/N)) = \beta_D;$$
$$\lim_{N\to\infty} v_N N\,\mathbb{E}(f((L^N-1)/N)) = \int_0^\infty f(v)\nu_D(dv), \tag{4.35}$$

for any continuous bounded function f vanishing in a neighborhood of 0, where h_D is a truncation function, $\alpha_D \in \mathbb{R}$, $\int_{(0,\infty)} (1 \wedge v^2)\,\nu_D(dv) < \infty$, $\beta_D = \sigma_D^2 + \int_{(0,\infty)} h_D^2\,\nu_D$ and $\sigma_D \geq 0$.

The limiting process is the (possibly explosive) CSBP with characteristics $(\alpha_D, \beta_D, \nu_D)$ solution of the stochastic differential equation

$$\begin{aligned}\widehat{Z}_t &= Z_0 + \alpha_D \int_0^t \widehat{Z}_s ds + \sigma_D \int_0^t \sqrt{\widehat{Z}_s} dB^D_s + \\ &\quad + \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \le \widehat{Z}_{s-}} h_D(r) \widetilde{N}^D(ds, d\theta, dr) \\ &\quad + \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \le \widehat{Z}_{s-}} (r - h_D(r)) N^D(dt, d\theta, dr),\end{aligned} \tag{4.36}$$

where N^D is a Poisson measure with intensity $dtd\theta\nu_D(dr)$.

As a new application of Theorem 4.5, we extend the convergence above by taking into account a cooperative effect. In this case, the interactions prevent the use of the classical generating function tool. The reproduction random variable $L^N(n)$ depends on the total population size n and we set

$$L^N(n) = L^N + \mathcal{E}^N(n), \tag{4.37}$$

where for each $n \geq 0$, $\mathcal{E}^N(n) \in \{0, 1\}$ is a Bernoulli random variable independent of L^N and

$$\mathbb{P}\left(\mathcal{E}^N(n) = 1\right) = \frac{g(n/N) \wedge v_N}{v_N} \tag{4.38}$$

for some function $g \in \mathcal{C}^1([0,\infty), [0,\infty))$. The process Z^N is defined as in (4.2) with this reproduction random variable $L^N(n)$.

We obtain the following convergence result.

Proposition 4.6. We assume that $v_N \to \infty$ and that (4.35), (4.37) and (4.38) hold. We also assume that $z \to \exp(-z)zg(z)$ is non-increasing for z large enough and goes to 0 as $z \to \infty$.

Then $(Z^N_{[v_N.]}/N : t \geq 0)$ converges in $\mathbb{D}([0,\infty), [0,\infty] \times \mathbb{R})$ to the unique strong solution Z of

$$\begin{aligned}Z_t &= Z_0 + \alpha_D \int_0^t Z_s ds + \int_0^t Z_s g(Z_s) ds + \sigma_D \int_0^t \sqrt{Z_s} dB^D_s \\ &\quad + \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \le Z_{s-}} h_D(z) \widetilde{N}^D(ds, dz, d\theta) \\ &\quad + \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \le Z_{s-}} (z - h_D(z)) N^D(dt, dz, d\theta)\end{aligned} \tag{4.39}$$

for $t < T_{exp}$ and $Z_t = +\infty$ for $t \geq T_{exp}$.

The monotonicity assumption on $z \to \exp(-z)zg(z)$ is chosen for sake of simplicity to obtain the pathwise uniqueness. It captures in particular simple cooperative functions as $g(z) = cz^\alpha$ $(c > 0, \alpha > 0)$ or $g(z) = c + b(1 - 1/(1+z))$ $(c \geq 0, b > 0)$.

We observe that the limiting process Z may be explosive, due to the heavy tails of the reproduction random variable L^N (i.e. the CSBP part is explosive) or due to cooperative effects (note for instance that $y'_t = y_t g(y_t)$ is explosive if $g(z) = z^\alpha$, $\alpha > 0$).

Finally, we add that extensions of the last convergence to random environments are possible in several ways, in particular catastrophes can be added and **A3** still holds. But if $\sigma_E > 0$, the function g has to compensate the quadratic term so that **A3** can be fulfilled. Otherwise, other arguments have to be invoked and one may expect to get uniqueness in law using quenched Laplace exponent (without interaction) or duality arguments.

Proof. Let us introduce

$$C_j^N(z) = v_N\left(\mathbb{E}\left(e^{-j\Sigma^N}\right)^{Nz}\left(\frac{g(z)\wedge v_N}{v_N}e^{-j/N}+\left(1-\frac{g(z)\wedge v_N}{v_N}\right)\right)^{Nz}-1\right). \quad (4.40)$$

By a Taylor expansion (developed in Appendix 6.2), one can prove that

$$\sup_{z;zN\in\mathbb{N}} e^{-kz}\left|C_j^N(z)+\ jz\,g(z)+\gamma_j^D\,z\right| \stackrel{N\to\infty}{\longrightarrow} 0. \quad (4.41)$$

Assumption **A2** is fulfilled for $z\in\frac{\mathbb{N}}{N}$, which is enough as commented in Remark 4.3, while **A1** is trivial (no random environment). As $g\in\mathcal{C}^1([0,\infty),[0,\infty))$ and $\exp(-z)zg(z)$ is non-increasing for z large enough and goes to 0 as z goes to infinity, there exist b_r and b_d such that

$$e^{-z}zg(z) = b_r(z)+b_d(z),$$

with b_d non-increasing and $b_r(-\log(u))$ Lipschitz continuous such that Assumption **A3** is fulfilled. Indeed there exists z_0 such that $z\to e^{-z}zg(z)$ is non increasing for $z\geq z_0$ and one can take $b_d(z)=e^{-z}zg(z)$ for $z\geq z_0$ and b_d constant for $z\leq z_0$, and $b_r(z)=e^{-z}zg(z)-b_d(z)$.

We conclude using Theorem 4.5. □

4.5 Conservative CSBP with interaction and random environment

We focus on the conservative case. Now $+\infty$ is not accessible and the pathwise uniqueness is obtained without Assumption **A3**.

Theorem 4.7. We assume that **A1** and **A2** hold and that any solution of (4.31) is conservative, i.e. $T_{exp}=+\infty$ a.s. Then there exists a unique strong solution $(Z,Y)\in\mathbb{D}([0,\infty),[0,\infty)\times\mathbb{R})$ of (3.2) and (4.31) and

$$\left(\left(\frac{1}{N}Z^N_{[v_Nt]},S^N_{[v_Nt]}\right):t\in[0,\infty)\right)\Rightarrow((Z_t,Y_t):t\in[0,\infty))$$

in $\mathbb{D}([0,\infty),[0,\infty)\times\mathbb{R})$.

Theorem 4.7 allows us to obtain various scaling limits to diffusions with jumps due either to the environment or to demographic stochasticity. The conditions for tightness and identification are very general. The conservativeness can be obtained by different methods as moment estimates or comparison with a conservative CSBP or conservative CSBP in random environment when the process is competitive or with bounded cooperation.

Proof. Using that $T_{exp}=+\infty$ a.s., one can check that pathwise uniqueness holds for (4.31). It can be achieved by using the pathwise uniqueness for Z obtained in [31] before T_{exp} or by adapting the proof of Theorem 4.5. We recall from Theorem 4.4 that weak existence also holds for (4.30) under **A1** and **A2**, so that both strong existence and weak uniqueness hold.

Then **(H3)** is fulfilled and we can apply Theorem 2.5 to X^N and get the weak convergence of $\left(\exp(-Z^N_{[v_N.]}/N),S^N_{[v_N.]}\right)$ to X in $\mathbb{D}([0,\infty),[0,1]\times\mathbb{R})$. Since $T_{exp}=+\infty$, the weak convergence of $(Z^N_{[v_N.]}/N,S^N_{[v_N.]})$ in $\mathbb{D}([0,\infty),[0,\infty)\times\mathbb{R})$ and the pathwise uniqueness of (Z,Y) follow, which ends up the proof. □

Application to logistic Feller diffusion in a Brownian environment. The next example illustrates the result. We consider a reproduction law which takes into account logistic competition and small fluctuations of the environment.

Corollary 4.8. Assume that $(E^N)_N$ are centered random variables such that $(\sqrt{N}E^N)_N$ is uniformly bounded and has variance σ_E^2. We define $L^N \in \{0,1,2\}$ for N large enough, $n \in \mathbb{N}$ and $e \in (-1,\infty)$ by

$$\mathbb{P}(L^N(n,e)=0)=\frac{1}{2}(\sigma_D^2-e+g_N(n/N)),\ \mathbb{P}(L^N(n,e)=2)=\frac{1}{2}(\sigma_D^2+e-g_N(n/N)),\tag{4.42}$$

where $\sigma_D \in (0,\sqrt{2})$, $g_N(z)=\alpha_D/N+c(z/N)\wedge(1/\sqrt{N})$ for $z\geq 0$ and $c\geq 0$ and $\alpha_D\in\mathbb{R}$. Then $(Z_t^N/N : t\in[0,\infty))$ converges in law in $\mathbb{D}([0,\infty),\mathbb{R}\times[0,\infty))$ to the unique strong solution Z of

$$Z_t = Z_0+\alpha_D\int_0^t Z_s ds - c\int_0^t Z_s^2 ds+\sigma_E\int_0^t Z_s dB_s^E+\sigma_D\int_0^t\sqrt{Z_s}dB_s^D,$$

where B^E and B^D are two independent Brownian motions.

Proof. Assumption **A1** holds with $v_N=N$, $\alpha_E=0$, $\nu_E=0$ and $\beta_E=\sigma_E^2$. Let us now prove that **A2** holds.
First, from (4.42), we get

$$\mathbb{E}\left(e^{-\frac{j}{N}(L^N(n,e)-1)}\right) = 1-\frac{j}{N}(e-g_N(n/N))+\frac{j^2}{2N^2}\sigma_D^2+o(1/N^2),$$

where $o(1/N^2)$ is uniform with respect to z and e. Then, for any $z\in\mathbb{N}/N$,

$$\begin{aligned} P_j^N(z,e) &= \mathbb{E}\left(e^{-\frac{j}{N}(L^N(Nz,e)-1)}\right)^{Nz}-1 = e^{Nz\left(-\frac{j}{N}(e+g_N(z))+\frac{j^2}{2N^2}\sigma_D^2+o(1/N^2)\right)}-1\\ &= -jz(e-g_N(z))+\frac{j^2}{2}z^2e^2+\frac{j^2z}{2N}\sigma_D^2+o(e^{jz}/N)\end{aligned}$$

by considering the cases $z\leq\sqrt{N}$ and $z\geq\sqrt{N}$. We obtain that for any $1\leq j\leq k$ and $\ell\geq 0$,

$$\begin{aligned} e^{-kz}N\mathbb{E}\big(P_j^N(z,E^N)\,e^{-\ell E^N}\big) = e^{-kz}\Bigg(&\left(jzNg_N(z)+\frac{j^2z}{2}\sigma_D^2\right)\mathbb{E}\left(e^{-\ell E^N}\right)\\ &-jzN\mathbb{E}\left(E^Ne^{-\ell E^N}\right)+\frac{j^2}{2}z^2N\mathbb{E}\left((E^N)^2e^{-\ell E^N}\right)\Bigg)+o(1).\end{aligned}$$

Finally, $\sqrt{N}E^N$ is centered, bounded with variance 1, so $\mathbb{E}\left(e^{-\ell E^N}\right)\to 1$ and $N\mathbb{E}(f(E^N))\to \sigma_E^2 f''(0)/2$ for $f\in C_0^{b,2}$ when N tends to infinity. In particular,

$$N\mathbb{E}(E^Ne^{-\ell E^N})\to -\ell\sigma_E^2,\qquad N\mathbb{E}((E^N)^2e^{-\ell E^N})\to\sigma_E^2.$$

Writing $g(z)=cz$ and using that $\gamma_j^D=j\alpha_D-\frac{j^2}{2}\sigma_D^2$ and $\gamma_v^E=\sigma_E^2v^2/2$, we get

$$\sup_{z\in\mathbb{N}/N} e^{-kz}\left|\mathcal{C}_{j,\ell}^N(z)+\gamma_{jz+\ell}^E-\gamma_\ell^E-jz\,g(z)+\gamma_j^Dz\right|\overset{N\to\infty}{\longrightarrow}0.$$

since $\gamma_{jz+\ell}^E-\gamma_\ell^E=\sigma_E^2(zj\ell+z^2j^2/2)$. We recall from Remark 4.2(iii) that this uniform convergence then holds for $z\geq 0$ and **A2** is satisfied.

Finally, a coupling with the Feller diffusion in Brownian environment ($c=0$, studied in [6]) allows us to prove that the process Z is conservative. The result is then an application of Theorem 4.7. □

5 Perspectives and multidimensional population models

The general results of Section 2 have been applied in the two previous sections to the Wright-Fisher processes in a Lévy environment and to the Galton-Watson processes with interaction in a Lévy environment with jumps larger than -1. These generalizations of historical population models were our initial motivation for this work. The results of Section 2 can be applied in other interesting contexts. We mention here some suggestions in these directions and ongoing works.

First, we could consider environments which are not independent and identically distributed or not restricted to $(-1,\infty)$.

This restriction to $(-1,\infty)$ allowed us to consider a functional space generated by the functions $\exp(-k.)$ $(k \geq 0)$ which are bounded on $(-1,\infty)$. To extend the results to random walks converging to Lévy processes with a jump measure ν on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1\wedge w^2)\nu_E(dw) < \infty$, one could consider the functional space of compactly supported functions

$$\mathcal{H} = \{(x,w) \to (1-e^{-kx})f(w) : k \geq 1, f \in C_c^\infty(\mathbb{R})\} \cup \{(x,w) \to f(w) : f \in C_c^\infty(\mathbb{R}), f(0) = 0\}$$

for studying Wright Fisher in a Lévy environment and

$$\mathcal{H} = \{(u,w) \to u^k f(w) : k \geq 1, f \in C_c^\infty(\mathbb{R})\} \cup \{(u,w) \to f(w) : f \in C_c^\infty(\mathbb{R}), f(0) = 0\}$$

for studying branching processes with interaction in random environment. Indeed these spaces satisfy **(H1.1,2)**. This would require to check that **(H1.3)** holds.

Such functional spaces could also help to study cases when the environment E_k^N depends on S_k^N and S^N converges to a diffusion with jumps.

Second, as explained in the introduction, we are more generally interested in k-type population models, where the population at generation n is described by a vector

$$Z_n^N = (Z_n^{1,N}, Z_n^{2,N}, \ldots, Z_n^{k,N}),$$

where $Z_n^{i,N}$ counts the number of individuals of type i in generation n. The following processes have attracted a lot of attention in population dynamics framework:

$$Z_{n+1}^{i,N} = \sum_{\alpha=1}^{k} \sum_{j=1}^{F_N^{(\alpha)}(Z_n^N)} L_{i,j,n}^{N,\alpha}(Z_n^N).$$

Such processes allow to model competition, prey-predators interactions, sexual reproduction, mutations .... Some examples have been well studied, as multitype branching processes, controlled branching processes or bisexual Galton-Watson processes, see e.g. respectively [30], [15] and [1].

One way to obtain the scaling limits is to consider the compactified proces

$$X^N = \big(\exp(-Z_n^{1,N}), \exp(-Z_n^{2,N}), \ldots, \exp(-Z_n^{k,N})\big)$$

and to use the functional space

$$\mathcal{H} = \big\{(u_1,\ldots,u_k) \to u^{i_1} \times u^{i_k} : (i_1,\ldots,i_k) \in \mathbb{N}^k \setminus (0,\ldots 0)\big\}.$$

Indeed $\mathcal{H}$ satisfies Assumption **(H1.1,2)** and the exponential transformation combined with this functional space may allow to exploit the independence structure of the model as for extended branching processes in Section 4. Some work will then be required to check that Assumption **(H1.3)** holds. Moreover uniqueness can be delicate. In an ongoing work, we consider bisexual Galton-Watson processes and their scaling limits to bisexual

CSBPs under general conditions. It is also worth noticing that in the scaling limits, the nonlinearity or the environment can impact the diffusion or jump terms, and not only the drift as for BPILE considered in Section 4. One could also prove limits to CSBP with Lévy environment, where the jump measure associated with the demographical stochasticity (large jumps coming from the offsprings of one single individual, at a rate proportional to the number of individuals) is impacted by the environment, see [4], [29] for an example.

Note also that one may want to go beyond the boundedness assumptions on the characteristics $\mathcal{G}^N$. This seems to be a challenging question but our approach may be extendable. Indeed, we obtain the boundedness assumptions in Section 4 by a compactification of the state space using the function $z \to \exp(-z)$, which allows us to consider explosive processes.

The last point to mention is that our criteria concern semimartingales in general. The Markov setting allows us to simplify the form of the characteristics $\mathcal{G}^N$ and to reduce the problem to analytical approximations, nevertheless we could try to work with non Markovian processes with similar techniques.

6 Appendix

6.1 General construction of a discrete random variable satisfying A2

We first consider the case $\sigma_D = 0$ and assume $E^N \in (-1 + 1/\sqrt{N}, \infty)$ for simplicity. We also introduce g_N which converges to g and such that

$$e^{-z} z g(z) \overset{z\to\infty}{\longrightarrow} 0, \qquad \sup_{z\geq 0} e^{-z} z |g_N(z) - g(z)| \overset{N\to\infty}{\longrightarrow} 0, \qquad \sup_{z\geq 0} \frac{|g_N(z)|}{N^{1/3}} < \infty. \tag{6.1}$$

One can take for instance $g_N(.) = g(.) \wedge N^{1/3}$. Let us define

$$m_N(n,e) = 1 + g_N(n/N)/N + \alpha_D/N + e$$

and observe that $m_N(n, E^N)$ is a.s. positive for N large enough. We consider the reproduction random variable $A^N(n,e) \in \{[m_N(n,e)], [m_N(n,e)] + 1\}$ defined by $\mathbb{E}(A^N(n,e)) = m_N(n,e)$, i.e.

$$\mathbb{P}(A^N(n,e) = [m_N(n,e)]) = p_N(n,e), \quad \mathbb{P}(A^N(n,e) = [m_N(n,e)] + 1) = 1 - p_N(n,e),$$

with $p_N(n,e) = [m_N(n,e)] + 1 - m_N(n,e)$. For the large reproductions events, we also introduce $\Sigma^N \in \mathbb{N}$ independent of $(A^N(n,e) : n \geq 0, e \in (-1,\infty))$ such that

$$\lim_{N\to\infty} N^2\, \mathbb{E}(h_D(\Sigma^N)) = 0; \;\; \lim_{N\to\infty} N^2\, \mathbb{E}(h_D^2(\Sigma^N)) = 0; \;\; \lim_{N\to\infty} N^2\, \mathbb{E}(f(\Sigma^N)) = \int_0^\infty f(v)\nu_D(dv)$$

for f continuous bounded and vanishing in a neighborhood of 0. The reproduction random variable L^N is then defined by

$$L^N(n,e) = A^N(n,e) + N\Sigma^N$$

for $n \in \mathbb{N}$ and $e > -1$ and writing $\kappa_j^N = -\log(1 - \mathbb{E}(f_j(\Sigma^N))$, we have for $z \in \mathbb{N}/N$,

$$\begin{aligned}
&\mathbb{E}\left(e^{-\frac{j}{N}(L^N(Nz,e)-1)}\right)\\
&\quad = \mathbb{E}\left(e^{-j\Sigma^N}\right)\mathbb{E}\left(e^{-\frac{j}{N}(A^N(Nz,e)-1)}\right)\\
&\quad = e^{-\kappa_j^N - je/N}\left(p(Nz,e)e^{-j([m_N(n,e)]-1-e)/N} + (1-p(Nz,e))e^{-j([m_N(n,e)]-e)/N}\right)\\
&\quad = e^{-\kappa_j^N - je/N}\left(1 - \frac{j}{N}\left(m_N(Nz,e) - e - 1\right) + \frac{\phi_N(z,e)}{N^2}\right)
\end{aligned}$$

where ϕ_N is bounded. By Taylor expansion, we obtain

$$\begin{aligned}\big|\phi_N(z,e)\big| &\leq c.\bigg(([m_N(Nz,e)]-e-g_N(z)/N-\alpha/N)(1-2([m_N(Nz,e)]-e))\\ &\qquad +([m_N(Nz,e)]-e)^2\bigg). \qquad (6.2)\end{aligned}$$

Moreover $m_N(Nz,e)-e-1=\mathcal{O}(N^{-2/3})$ uniformly for z,e and we obtain

$$\begin{aligned}P_j^N(z,e) &= e^{-zN\kappa_j^N-jez}\left(1-\frac{j}{N}\left(m_N(Nz,e)-e-1\right)+\frac{\phi_N(z,e)}{N^2}\right)^{Nz}-1\\ &= e^{-zN\kappa_j^N-jez-jz(m_N(Nz,e)-e-1)+z\psi_N(z,e)}-1\\ &= e^{-zN\kappa_j^N-zj(g_N(z)+\alpha_D)/N}.e^{-jze-z\psi_N(z,e)}-1,\end{aligned}$$

where $N\psi_N(z,e)$ is continuous bounded and $N|\psi_N(z,e)|\leq c(1/N^{4/3}+|\phi_N(z,e)|)$ and c is a constant which may change from line to line. Thus

$$\mathbb{E}\big(P_j^N(z,E^N)\,e^{-\ell E^N}\big)=A_1^N(z)+A_2^N(z)+A_1^N(z)A_2^N(z)+A_3^N,$$

where

$$A_1^N(z)=e^{-zN\kappa_j^N-zj(g_N(z)+\alpha_D)/N}-1,\quad A_2^N(z)=\mathbb{E}\left(e^{-(jz+\ell)E^N-z\psi_N(z,E^N)}\right)-1$$

and $A_3^N=\mathbb{E}(f_\ell(E^N))$. Assumption $\mathbf{A1}$ ensures that $v_NA_3^N$ converges to γ_ℓ^E when N tends to infinity (see (4.9) for details). To conclude and prove (4.14), we prove and combine the asymptotic results stated below.

Lemma 6.1. For any $j\geq 1$,

$$\begin{aligned}&(i) \qquad \sup_{z\geq 0} e^{-jz}\big|NA_1^N(z)+z(\gamma_j^D+\alpha_D+g(z))\big| \overset{N\to\infty}{\longrightarrow} 0\\ &(ii) \qquad \sup_{z\geq 0} e^{-jz}\big|NA_2^N(z)+\gamma_{jz+\ell}^E\big| \overset{N\to\infty}{\longrightarrow} 0\\ &(iii) \qquad \sup_{z\geq 0} e^{-jz}\big|NA_1^N(z)A_2^N(z)\big| \overset{N\to\infty}{\longrightarrow} 0\end{aligned}$$

Proof. (i) First, by Taylor expansion and using that $g_N(z)/N^{1/3}$ is bounded, there exists $c>0$ such that for any $z\leq N^{2/3}$,

$$\begin{aligned}&e^{-jz}\big|Ne^{-zN\kappa_j^N-zj(g_N(z)+\alpha_D)/N}+z(\gamma_j^D+\alpha_D+g(z))\big|\\ &\qquad \leq c.e^{-jz}z\left(|N^2\kappa_j^N-\gamma_j^D|+|g_N(z)-g(z)|+N^{-1/3}\right).\end{aligned}$$

The right hand side goes to 0 uniformly as $N\to\infty$. Second,

$$e^{-jz}z(\gamma_j^D+\alpha_D+g(z)) \overset{z\to\infty}{\longrightarrow} 0, \qquad \sup_{z\geq N^{2/3},N\geq 1} e^{-jz}\big|Ne^{-zN\kappa_j^N-zj(g_N(z)+\alpha_D)/N}-1\big| \overset{A\to\infty}{\longrightarrow} 0,$$

since for $z\geq N^{2/3}$,

$$e^{-jz}\big|Ne^{-zN\kappa_j^N-zj(g_N(z)+\alpha_D)/N}-1\big|\leq e^{-jz}N(e^{zc/N^{2/3}}+1)\leq Ne^{-N^{1/3}.(1-c/N^{2/3})}+Ne^{-N^{1/3}},$$

which goes to 0. This proves (i).

Let us turn to (ii). We first prove the uniform convergence on compact sets using convexity and simple convergence. Indeed, recalling that $|N\psi_N(z,e)|\leq c(1/N^{4/3}+|\phi_N(z,e)|)$,

$$\big|A_2^N(z)+\mathbb{E}\left(f_{jz+\ell}(E^N)\right)\big|\leq\frac{c}{N}\left(1/N^{4/3}+\mathbb{E}(|\phi_N(z,E^N)|)\right)$$

for $z \in [0, A]$. By **A1**, E^N goes in probability to 0 and ϕ_N is bounded and $\phi_N(z, e) \to 0$ as $e \to 0$ uniformly with respect to $z \in [0, A]$ from (6.2). It turns out that $\sup_{z\in[0,A]} \mathbb{E}(|\phi_N(z, E^N)|) \to 0$ and

$$\sup_{z\in[0,A]} \left|NA_2^N(z) + N\mathbb{E}\left(f_{jz+\ell}(E^N)\right)\right| \stackrel{N\to\infty}{\longrightarrow} 0.$$

Moreover, for any $z \geq 0$, $N\,\mathbb{E}\left(f_{jz+\ell}(E^N)\right) \to \gamma^E_{jz+\ell}$ by Assumption **A1** (see again (4.9) for details) and the convergence is uniform on $[0, A]$ by convexity of $z \to N\mathbb{E}\left(f_{jz+\ell}(E^N)\right)$ and by continuity of $z \to \gamma^E_{jz+\ell}$ (third Dini's theorem). It proves (ii) on compacts sets. Let us now prove that $\sup_{z\geq A, N\geq 1} \exp(-jz)\left|NA_2^N(z)\right| \to 0$ as $A \to \infty$.

Let us fix $\varepsilon > 0$ and

$$A_2^N(z) = B_\varepsilon^N(z) + C_\varepsilon^N(z) \tag{6.3}$$

where

$$B_\varepsilon^N(z) = \mathbb{E}\left(1_{E^N\geq -1+\varepsilon} e^{-(jz+\ell)E^N - z\psi_N(z,E^N)}\right) - 1.$$

Recalling that $E^N \geq -1 + 1/\sqrt{N}$ and $N\psi_N$ bounded, we have

$$C_\varepsilon^N(z) = \mathbb{E}\left(1_{E^N<-1+\varepsilon} e^{-(jz+\ell)E^N - z\psi_N(z,E^N)}\right) \leq \mathbb{P}(E^N < -1+\varepsilon)e^{-(jz+\ell)(1-1/\sqrt{N}+c/N)}.$$

Thus, the last part of Assumption **A1** ensures that

$$\lim_{\varepsilon\to 0} \sup_{N\geq c^2, z\geq 0} e^{-jz} NC_\varepsilon^N(z) = \lim_{\varepsilon\to 0} \sup_{N\geq 1} N\mathbb{P}(E^N < -1+\varepsilon) = \lim_{\varepsilon\to 0} \nu_E(-1, -1+\varepsilon) = 0.$$

Writing

$$g_x(y) = f_x(y) - xh_E(y) = 1 - e^{-xy} - xh_E(y), \quad R_N(z, e) = e^{-(jz+\ell)e}\left(e^{-z\psi_N(z,e)} - 1\right),$$

we have

$$\begin{aligned} B_\varepsilon^N(z) \quad = \quad & \mathbb{P}(E^N < -1+\varepsilon) - (jz+\ell)\mathbb{E}\left(h_E(E^N)1_{E^N\geq -1+\varepsilon}\right) - \mathbb{E}(g_{jz+\ell}(E^N))1_{E^N\geq -1+\varepsilon}) \\ & + \mathbb{E}\left(R_N(z, E^N)1_{E^N\geq -1+\varepsilon}\right). \end{aligned}$$

First, we recall that $\sup_N \mathbb{P}(E^N < -1+\varepsilon) \to 0$ as $\varepsilon \to 0$ and $N\mathbb{E}\left(h_E(E^N)1_{E^N\geq -1+\varepsilon}\right)$ is bounded (actually convergent by Assumption **A1**). Second, we prove that

$$\sup_{z\geq A, N\geq 1} Ne^{-jz}\mathbb{E}\left(|g_{jz+\ell}(E^N)|1_{E^N\geq -1+\varepsilon}\right) \stackrel{A\to\infty}{\longrightarrow} 0$$

using that (see forthcoming Lemma 6.2 for details)

$$\sup_{y\geq -1+\varepsilon,\ y\neq 0} \frac{e^{-jz}}{(1-e^{-y})^2}|g_x(y)| \stackrel{x\to\infty}{\longrightarrow} 0$$

and that $N\mathbb{E}((1-\exp(-E^N))^2)$ is bounded from **A1**. Finally

$$e^{-jz}N|R_N(z, e)1_{e\geq -1+\varepsilon}| \leq N\exp(-(\varepsilon - 1/\sqrt{N})jz).\left|\exp(cz/N) - 1\right|$$

ensures that

$$\sup_{z\geq A, N} N\mathbb{E}\left(R_N(z, E^N)1_{E^N\geq -1+\varepsilon}\right) \stackrel{A\to\infty}{\longrightarrow} 0,$$

using again $|\exp(cz/N) - 1| \leq c'z/N$ for $z \leq N$, while the right hand is bounded by $z\exp(-\varepsilon jz/2)$ for $z \geq N$ and $N \geq 16/\varepsilon^2$. Combing these estimates in (6.3) yields

$\sup_{z\geq A, N\geq 1}\exp(-jz)\big|NA_2^N(z)\big| \to 0$ as $A\to\infty$ and ends the proof of (ii) by recalling that $\exp(-jz)\gamma^E_{jz+\ell}\to 0$ as $z\to\infty$.

We finally prove (iii). First,

$$\sup_{z\geq 0} e^{-jz/2\sqrt{N}}|A_1^N(z)| \overset{A\to\infty}{\longrightarrow} 0$$

using that $\exp(-jz/2\sqrt{N})|A_1^N(z)| \leq c\exp(-jz/2\sqrt{N})z/N^{2/3}$ for $z\leq N$ (and then one may use for N large that $z/N^{2/3}$ is small for $z\leq N^{7/12}$ and that $\exp(-z/2\sqrt{N})$ is small for $N^{7/12}\leq z\leq N$) and $\exp(-jz/2\sqrt{N})|A_1^N(z)|\leq \exp(-jz(1/2\sqrt{N}-c/N^{2/3}))$ for $z\geq N$. Second

$$\sup_{N\geq 1, z\in[0,\infty)} e^{-jz(1-1/2\sqrt{N})}N\mathbb{E}(A_2^N)<\infty,$$

since

$$\mathbb{E}(A_2^N)\leq \mathbb{P}(E^N\leq -1/2)e^{(jz+\ell)(1-1/\sqrt{N})+zc/N}+\mathbb{E}(A_2^N 1_{E^N>-1/2})$$

and $N\mathbb{P}(E^N\leq -1/2)$ is bounded from **A1** and $N\mathbb{E}(A_2^N 1_{E^N>-1/2})$ is bounded, following the point (ii) and using the following slight modification of Lemma 6.2

$$\sup_{y>-1/2,\ y\neq 0,\ N\geq N_0} \frac{e^{-x(1-2/\sqrt{N})}}{(1-e^{-y})^2}|g_x(y)| \overset{x\to\infty}{\longrightarrow} 0,$$

where N_0 is chosen such that $1-2/\sqrt{N_0}>1/2$. □

6.2 Taylor expansion for a Galton-Watson process with cooperation

Recalling (4.40) and (4.14),

$$\begin{aligned} C_j^N(z) &= v_N\left(\left(1-\gamma_j^{N,D}/Nv_N\right)^{Nz}\left(1-\frac{j}{N}\frac{g(z)\wedge v_N}{v_N}+\mathcal{O}\left(\frac{g(z)\wedge v_N}{N^2v_N}\right)\right)^{Nz}-1\right)\\ &= v_N\left(e^{-z\gamma_j^{N,D}/v_N+\mathcal{O}(z/Nv_N^2)-jz(1\wedge(g(z)/v_N))(1+\mathcal{O}(1/N))}-1\right)\end{aligned}$$

for any $z\in\mathbb{N}/N$. For z such that $z+g(z)z\leq v_N$, we have $z/v_N\leq 1$ and $1\wedge(g(z)/v_N)=g(z)/v_N$ for $z\geq 1$. We make a Taylor expansion and get

$$e^{-jz}\big|C_j^N(z)+\ z\gamma_j^{N,D}+jzg(z)\big|\leq e^{-jz}c\left(\frac{1}{N}+\frac{zg(z)}{N}\right)$$

for some constant $c>0$. We obtain

$$\sup_{z+g(z)z\leq v_N} e^{-jz}\big|C_j^N(z)+\ z\gamma_j^D+jzg(z)\big| \overset{N\to\infty}{\longrightarrow} 0.$$

To conclude, we observe that $\min\{z\ :\ z+g(z)z\geq v_N\}\to\infty$ as $N\to\infty$. Then $\sup_{z+g(z)z\geq v_N} e^{-jz}\big|z\gamma_j^{N,D}+jzg(z)\big|\to\infty$. Let us now prove that

$$\sup_{z+g(z)z\geq v_N} e^{-jz}v_N\big|C_j^N(z)\big| \overset{N\to\infty}{\longrightarrow} 0.$$

Indeed, $g\geq 0$ and either $z\geq v_N/2$ and

$$e^{-jz}v_N\big|C_j^N(z)\big|\leq e^{-jz}v_Ne^{zc/v_N}\leq 2ze^{-zj/2}$$

for N such that $j-c/v_N\geq j/2$ or $z\leq v_N/2$ and $v_N\leq 2zg(z)$ and there exists $c>0$ such that

$$e^{-jz}v_N\big|C_j^N(z)\big|\leq e^{-jz}2zg(z)\,e^c.$$

Recalling that $\min\{z;z+g(z)z\geq v_N\}\to\infty$ as $N\to\infty$ and that $zg(z)\exp(-z)\to 0$ as $z\to\infty$, we obtain the desired result.

6.3 Some technical results

Lemma 6.2. For $x > 0$, let us consider

$$g_x(y) = 1 - e^{-xy} - xh_E(y).$$

We have

$$\sup_{y>-1+\varepsilon,\ y\neq 0} \frac{e^{-x}}{(1-e^{-y})^2}|g_x(y)| \stackrel{x\to\infty}{\longrightarrow} 0.$$

Proof. Let $\mathcal{V}_0$ be an open bounded interval containing 0 such that $h_E(y) = y$ for $y \in \mathcal{V}_0$. There exists $C > 0$ such that for any $y \notin \mathcal{V}_0$,

$$\frac{|g_x(y)|}{(1-e^{-y})^2} \leq C\left(1 + x + e^{x(1-\varepsilon)}\right)$$

since h_E and $1/(1-\exp(-y))$ are bounded. The result follows on the complementary set of $\mathcal{V}_0$. Let us now consider $y \in \mathcal{V}_0$. Assuming $|xy| \leq 1$, we get $|g_x(y)| \leq C\,x^2y^2$ and we conclude using that $y/(1-\exp(-y))$ is bounded on $(-1,\infty)$.

If $y \in \mathcal{V}_0$ and $|xy| \geq 1$, we have

$$\frac{|g_x(y)|}{(1-e^{-y})^2} \leq C\left(\frac{|1-e^{-xy}|}{y^2} + \frac{x}{y}\right) \leq Cx^2\left(1 + e^{x(1-\varepsilon)}\right),$$

which ends the proof. □

Let us now prove the forthcoming inequality (6.4).

Lemma 6.3. Let $r_1(x) = -x\log(x)$. Then for any $x, x' \in [0,1]$,

$$|x\log(x) - x'\log(x')| \leq K\big(|x-x'| + r_1(|x-x'|)\big) \tag{6.4}$$

for some constant $K > 0$.

Proof. Let us first assume that $\min(x,x') \geq |x-x'|$. In this case, it is immediate that

$$|x\log x - x'\log x'| \leq |x-x'|(1 + \log(|x-x'|))$$

by the mean value theorem. We now assume that $0 \leq x \leq |x-x'| \leq x'$, which implies that $x' \leq 2|x-x'|$. We have

$$\begin{aligned} |x\log x - x'\log x'| &\leq |x\log(x/x') + (x-x')\log(x')| \\ &\leq |x\log(x/x')| + |\log(|x-x'|)|\,|x-x'| \\ &\leq x' - x + |\log(|x-x'|)|\,|x-x'|, \end{aligned}$$

using that $x/x' \in [0,1]$ and that the function $\alpha \in [0,1] \to \alpha\log\alpha$ is bounded by some constant C. We obtain that $|x\log x - x'\log x'| \leq 2C|x-x'| + |\log(|x-x'|)|\,|x-x'|$, which ends the proof. □

Lemma 6.4. Recall the definitions (4.33) and (4.34). For any $x_1, \widetilde{x}_1 \in [0,1]$,

$$g_1(x_1,\widetilde{x}_1) \leq L|x_1 - \widetilde{x_1}|$$

and for any $u \in [-1/2, 1]$,

$$(g_2(x_1,u) - g_2(\widetilde{x}_1,u))^2 \leq C|x_1 - \widetilde{x_1}|u^2.$$

Proof. For the first inequality, one can use that $-x\log x$ is bounded for the first term in (4.33) and the mean value theorem for the second one.

Concerning the second inequality, we use

$$(g_2(x_1,u)-g_2(\widetilde{x}_1,u))^2 \le \left|g_2(x_1,u)^2-g_2(\widetilde{x}_1,u)^2\right| \le \sup|(g_2^2(.,u))'||Z_s|$$

and

$$(g_2^2(.,u))'(x_1)=2x_1(e^{\log(x_1)u}-1)^2+ux_1e^{\log(x_1)u}2(e^{\log(x_1)u}-1).$$

The results then come from the inequality $|e^{\log(x_1)u}-1|\le|\log(x_1)|u$. □

6.4 Stone-Weierstrass theorem on locally compact space

We recall here the local version of Stone-Weierstrass Theorem and assume that the space X is a locally compact Hausdorff space.

Let $C_0(X,\mathbb{R})$ the space of real-valued continuous functions on X which vanish at infinity, i.e. given $\varepsilon>0$, there is a compact subset K such that $\|f(x)\|<\varepsilon$ whenever the point x lies outside K. In other words, the set $\{x,\|f(x)\|\ge\varepsilon\}$ is compact.

Let us consider a subalgebra A of $C_0(X,\mathbb{R})$. Then A is dense in $C_0(X,\mathbb{R})$ for the topology of uniform convergence if and only if it separates points and vanishes nowhere.

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Acknowledgments. This work was partially funded by the Chair "Modélisation Mathématique et Biodiversité" of VEOLIA-Ecole Polytechnique-MnHn-FX and by the ANR ABIM 16-CE40-0001.