Electron. Commun. Probab. **24** (2019), no. 72, 1–12.
https://doi.org/10.1214/19-ECP274
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

One-ended spanning trees in amenable unimodular graphs

Ádám Timár*

Abstract

We prove that every amenable one-ended Cayley graph has an invariant one-ended spanning tree. More generally, for any one-ended amenable unimodular random graph we construct a factor of iid percolation (jointly unimodular subgraph) that is almost surely a one-ended spanning tree. In [2] and [1] similar claims were proved, but the resulting spanning tree had 1 or 2 ends, and one had no control of which of these two options would be the case.

Keywords: invariant spanning tree; unimodular random graph; factor of iid, one-ended.
AMS MSC 2010: 60C05.
Submitted to ECP on June 13, 2018, final version accepted on October 31, 2019.

1 Introduction

Every unimodular amenable graph G (see Definition 2.2 for the definition of amenability) allows a percolation (random subgraph whose distribution is jointly unimodular with G) that is almost surely a spanning tree with 1 or 2 ends; see Theorem 8.9 in [1]. We strengthen this by showing that if G is amenable and one-ended then it has a one-ended spanning tree percolation, and this can be constructed as a factor of iid (fiid). The extra property that it is an fiid was already implicit in [2] for the quasi-transitive case, but that argument does not extend directly to more general unimodular random graphs. The real novelty of the present contribution is that we do not have to allow 2-ended trees. Our original motivation was [9], where it was crucial that the spanning forest is an fiid and has 1 end. See Definition 2.1 for the definition of amenability for unimodular random graphs.

Theorem 1.1. *Let G be an ergodic amenable unimodular random graph that has one end almost surely. Then there is a factor of iid spanning tree of G that has one end almost surely.*

In [2], Benjamini, Lyons, Peres and Schramm proved that a quasi-transitive unimodular graph is amenable if and only if it has an invariant spanning tree with at most 2 ends (Theorem 5.3). Note that a quasi-transitive amenable graph can only have 1 or 2 ends; and also that if it has 2 ends then all its invariant spanning trees are 2-ended. Our result can hence be thought of as a strengthening of the characterization in [2]:

Corollary 1.2. *A quasi-transitive unimodular graph is amenable and has 1 end if and only if it has an invariant spanning tree with 1 end.*

*Alfréd Rényi Institute of Mathematics, Hungary. E-mail: madaramit@gmail.com

Let $\mathcal{G}_*$ be the collection of all locally finite graphs with a special vertex (the "root"), up to isomorphism. Similarly, let $\mathcal{G}_{**}$ be the collection of all locally finite graphs with an ordered pair of vertices, up to isomorphism. By a slight abuse of notation, we will sometimes refer to elements of $\mathcal{G}_*$ in the form (G, o), and sometimes in the form G (hiding the root in notation). A unimodular random graph is a probability measure μ on $\mathcal{G}_*$ which satisfies the so-called Mass Transport Principle (MTP), namely, that

$$\int \sum_{x \in V(G)} f(G, o, x) d\mu((G, o)) = \int \sum_{x \in V(G)} f(G, x, o) d\mu((G, o)) \tag{1.1}$$

for every Borel $f : \mathcal{G}_{**} \to [0, \infty]$. The function $f(G, o, x)$ is usually referred to by saying "o sends mass $f(G, o, x)$ to x", and then (1.1) claims that the expected total mass sent out must equal the expected total mass received. We assume that the reader is familiar with these notions. When talking about a unimodular random graph, we sometimes mean the probability measure μ, sometimes the sampled random rooted graph (G, o), or just G; this is standard and will not cause ambiguity. A subgraph H of the rooted graph $(G, o) \in \mathcal{G}_*$ is a factor of iid (fiid), if it can be constructed as a measurable function from iid Lebesgue($[0, 1]$) labellings of $V(G)$; in other words, if one can tell the edges of H incident to o up to arbitrary precision from the labels in a large enough neighborhood of o. Similarly, one can define fiid functions from (G, o), fiid partitions, etc.. Along our fiid constructions, we will have to make some local choices, such as choosing a subgraph of a certain property out of finitely many possibilities but otherwise arbitrarily. To make the final result a fiid, these choices have to be made using some previously fixed local rule using the iid labels. We will skip the details of such choices, which are straightforward. We also mention the following standard fact. The Lebesgue($[0, 1]$) label on a vertex can be used to define countably many Lebesgue($[0, 1]$) labels on it, and in such a way that the final collection of all the labels over the vertices is also iid. Namely, if $0.\alpha_1\alpha_2\alpha_3\ldots$ is the decimal expansion of the label of a vertex x, let its i'th new label be $0.\alpha_{b_i(1)}\alpha_{b_i(2)}\alpha_{b_i(3)}\ldots$, where $b_i(k) = 2^i(2k-1)+1$. Now, suppose we have already defined some structure on (G, o) as a fiid, and now we want to define some other fiid structure, independently from the previous one. One can always do so, by assuming that there are more unused iid labels, which is possible by the above. We will use this fact repeatedly, without explicit mention.

2 Hyperfinite partions as fiid

The aim of this section is to prove Lemma 2.3, a factor of iid version of hyperfiniteness. We had a direct proof for the lemma, but Russ Lyons suggested to us that it follows by an argument similar to the proof of Theorem 8.5 in [1]. We decided to follow this track, and rely more heavily on existing theory but obtain a short proof. After reading a draft of this section, Tom Hutchcroft shared with us a direct proof for the lemma, which is shorter than our original one and does not use Borel equivalence relations. We decided to include both his proof and the other one. The reader familiar with the various equivalent definitions of amenability for unimodular random graphs and their relationship with graphings, or only interested in the probabilistic proof, is advised to go directly to Lemma 2.3 and its proof. Before proceeding to the lemma, we review measurable equivalence relations, invariant probability measures of such equivalence relations, and graphings. We recall their connections to unimodular random graphs. Then we define amenability and hyperfiniteness (two equivalent notions), both for graphings and for unimodular random graphs. This was needed for a complete proof, because of slight inconsistencies in the literature that we are using.

Suppose we have a Borel measurable equivalence relation $\mathcal{R}$ defined on a standard Borel space X, and that its is countable (i.e., all classes are countable). Suppose there is a Borel probability measure ν on X such that $\mathcal{R}$ preserves ν, meaning that for every Borel function $f : X^2 \to [0, \infty]$,

$$\int_x \sum_{x\mathcal{R}y} f(x,y) d\nu(x) = \int_x \sum_{x\mathcal{R}y} f(y,x) d\nu(x). \tag{2.1}$$

This form of the definition is from Example 9.9 in [1], and its advantage is its similarity to (1.1). Definition 2.1 and Corollary 1 in [4] show the equivalence of this definition to the more standard definitions (such as the one in [6] and its further equivalents in Proposition 16.1 of [6]). A graphing of a measurable equivalence relation is a Borel measurable subset of the pairs in relation, defining a graph such that its connected components coincide with the equivalence classes of the equivalence relation. In our context there will always be some probability measure ν on X, and we require a graphing to induce the equivalence classes only up to ν-measure 0.

Consider the topological space $X := \mathcal{G}_*$, with the usual topology, where two elements are close if large neighborhoods of their roots are rooted isomorphic. Define a Borel equivalence relation $\mathcal{R}$ on X by claiming two elements equivalent if one arises from the other by moving the root. Define a corresponding graphing: let two points be adjacent, if one arises from the other by moving the root along an edge. Call the set of all edges in this graphing E. Now, if there is a probability measure μ on $\mathcal{G}_*$, it will automatically define a probability measure ν on X. (The only difference between the two measures is how we think about them: μ samples a rooted graph while ν samples a point of a graphing.) Suppose now that μ-almost every graph has no rooted isomorphism but the trivial one, and suppose that μ is unimodular. The unimodularity condition (1.1) transforms into (2.1) of ν, and this shows that $\mathcal{R}$ is ν-preserving. (The assumption on the lack of rooted isomorphisms cannot be completely omitted for this conclusion: see Exercise 18.47 in [7].) Furthermore, if we sample a random point of X using ν, and look at its connected component in (X, E), the resulting rooted graph has distribution given by μ. (Here the assumption on the almost sure lack of isomorphisms is necessary. For an illustration, consider μ to be supported on the single rooted graph $(\mathbb{Z}, 0)$. Then ν assigns measure 1 to a singleton whose component in (X, E) is itself with a loop-edge.)

Part (I) of the next definition defines amenability of unimodular random graphs differently from [1]. However, the two are equivalent, as shown in Theorem 8.5 in [1], based on work by Kaimanovich [5] and Connes, Feldman, and Weiss [3]. The definition of [1] makes the importance of isoperimetry more explicit. The reason we prefer the present definition is that it is formulated essentially the same way as the amenability of a measurable equivalence relation (see (II), and Section 9 in [6]). A key characteristic of this definition is that amenability of the unimodular random graph is witnessed by objects (here: functions) that depend deterministically on the graph and are *not using extra randomness*. This is not the case in other definitions such as (i) in Definition 2.2.

Definition 2.1. *Amenability*

(I) A unimodular probability measure μ on $\mathcal{G}_*$ is called amenable if there are Borel functions $\lambda_n : \mathcal{G}_{**} \to [0, 1]$ such that for μ almost every (G, o), we have

$$||\lambda_n(G, o, .)||_1 = 1$$

and

$$\lim_{n\to\infty} \sum_{x\sim o} ||\lambda_n(G, o, .) - \lambda_n(G, x, .)||_1 = 0.$$

(II) Given (X, E, ν), where E is a Borel equivalence relation and ν is a probability measure on X, we call E amenable if there are Borel functions $\lambda_n : E \to [0, 1]$ such that

for ν-almost every x and y with xEy,

$$||\lambda_n(x,.)||_1 = 1$$

and

$$\lim_{n\to\infty} ||\lambda_n(x,.) - \lambda_n(y,.)||_1 = 0.$$

The next definition, hyperfiniteness, is known to be equivalent to amenability both in the case of unimodular random graphs and in the case of measurable equivalence relations. See [6] and [1] for references.

Definition 2.2. *Hyperfiniteness*

(i) A unimodular random graph G is called hyperfinite if there exists a sequence of random subgraphs $G_n \subset G$ such that $(G, G_1, G_2, \dots)$ is jointly unimodular, every component of G_n is finite almost surely, $G_1 \subset G_2 \subset \dots$, and $\cup_n G_n = G$.

(ii) A Borel equivalence relation E on X is called hyperfinite if there is a sequence F_n, $n = 1, 2, \dots$, of finite Borel subequivalence relations of E, such that $F_1 \subset F_2 \subset \dots$, and $\cup_n F_n = E$. A Borel subequivalence relation is finite if all its classes are finite.

One could have defined $\mathcal{G}_*$ as the family of all rooted *decorated* graphs up to rooted isomorphism, where by a decoration we mean a map from the vertices of a graph to $[0,1]$ here, and rooted isomorphisms are defined to preserve the rooted graph structure and the decorations. The correspondance with graphings, and all the above definitions and facts remain valid for this generalization. To avoid too much formalism, we do not introduce new notation, but it will always be clear from the context whether a $G \in \mathcal{G}_*$ is a graph or a decorated graph.

One way to phrase Definition 2.2 is to say that a unimodular random graph is hyperfinite if there is a unimodular edge percolation on it with all finite components and an arbitrarily small portion of edges deleted. In Lemma 2.3 we show that such a percolation can be constructed as a *factor of iid*. A direct adjustment of the earlier proof for the quasi-transitive case (Theorem 5.1 in [2]) is not possible, for the following reason. In [2] some fixed finite subgraph F of G is translated by all automorphisms, creating an invariant collection of subgraphs (jointly unimodular with G). But the set of all translates of some fixed set F need not be jointly unimodular with G if G is not quasi-transitive.

Lemma 2.3. *Let G be a unimodular random graph. Then G is amenable if and only if there exist factor of iid subgraphs Γ_n in G such that all components of Γ_n are finite almost surely, $\Gamma_1 \subset \Gamma_2 \subset \dots$, and $\cup \Gamma_n = G$.*

1st proof. The "if" direction of the lemma is clear from the equivalence of hyperfiniteness and amenability.

We may assume that G is almost surely infinite, because the claim is trivial for finite graphs. Consider the unimodular measure μ on $\mathcal{G}_*$ that generates G. Let $\epsilon > 0$ be arbitrary and $\lambda_\epsilon := \lambda_n$, where λ_n is as in (I) of Definition 2.1 with n large enough that with μ-probability at least $1-\epsilon$, $\sum_{x\sim o} ||\lambda_n(G,o,.) - \lambda_n(G,x,.)||_1 < \epsilon$. Consider the decoration of the vertices of $G \in \mathcal{G}_*$ by iid Bernoulli labels, and call the resulting joint distribution μ^+. The random decorated graph has μ^+-almost surely no nontrivial symmetry. Hence the corresponding probability measure ν on the vertex set of the respective graphing is such that the decorated component of a ν-random point of the graphing has distribution μ^+, and hence its (undecorated) component has distribution μ.

We can extend the definition of λ_ϵ to any iid-decorated copy of G, by simply ignoring the decoration. Hence λ_ϵ is defined for any point of the graphing (and the measurable equivalence relation that it generates). Since $\epsilon > 0$ was arbitrary, this shows that the measurable equivalence relation is amenable in the sense of (II) in Definition 2.1. This implies hyperfiniteness (Definition 2.2, (ii)), as shown by Connes, Feldman, and Weiss [3] (see also Theorem 10.1 in [6]). So let F_n be as in Definition 2.2. The (rooted) component of a ν-random point in the graphing is partitioned into finite classes by F_n. We know that this rooted component has distribution μ, and so the partition makes sense on the original unimodular random graph. Let Γ_n be the subgraph of edges whose endpoints are in the same partition class. The properties $\Gamma_1 \subset \Gamma_2 \subset \ldots$, and $\cup\Gamma_n = G$ hold because of analogous properties of the F_n. Finally, since F_n was measurable with respect to the iid decoration, Γ_n is a factor of iid of the corresponding unimodular random graph G. $\square$

2nd proof (Tom Hutchcroft, personal communication). To see that hyperfiniteness implies that G is amenable, we can take $\lambda_n(G,x,y) = \mathbb{P}(x$ and y are in the same component of $\Gamma_n)$.

For the other direction, suppose that G is amenable. From now on we fix the instance G and hide the dependence on it in notation. Consider a family of functions $\lambda_n(G,.,.) = \lambda_n(.,.)$ as in Definition 2.1 (I). So the L^1 distance between $\lambda_n(x,.)$ and $\lambda_n(y,.)$ tends to zero as n tends to infinity. We will use iid labels to sample a point $v_{n,x}$ in such a way that $v_{n,x} = v_{n,y}$ with high probability as n tends to infinity. This way we can partition the graph into sets $(x : v_{n,x} = u)_{u\in V(G)}$, which are finite by the MTP, and which will a.s. exhaust the graph.

We will choose $\delta(n)$ very small as a function of n. For each vertex v take a sequence of iid uniform $[0,1]$ random variables $U_{v,1}, U_{v,2}, \ldots$. For each vertex u let $v_{n,u}$ be the vertex maximizing the quantity

$$\max\{U_{v,1}, \ldots, U_{v,\lfloor \lambda_n(u,v)/\delta(n)\rfloor}\}.$$

Let $\ell_n(v) = \lfloor \lambda_n(o,v)/\delta(n)\rfloor$ and $\epsilon_n(v) = \lambda_n(o,v)/\delta(n) - \ell_n(v)$ the fractional part.

Define $S(n) := \{v : \lambda_n(o,v) > 2^n\delta(n)\}$. Then for every $v \in S(n)$

$$\ell_n(v) \geq (1-2^{-n})\lambda_n(o,v)/\delta(n). \tag{2.2}$$

Choose $\delta(n)$ small enough so that $\sum_{v\in S(n)} \lambda_n(o,v) > 1 - 3^{-n}$. Let $\kappa_n(o,.)$ be the probability measure corresponding to $v_{n,o}$. The L^1 distance between $\kappa_n(o,.)$ and $\lambda_n(o,.)$ is

$$\begin{aligned}
&\sum_{x\in V(G)} \Big| \frac{\ell_n(x)}{\sum_{v\in V(G)} \ell_n(v)} - \lambda_n(o,x) \Big| \leq \\
&\leq \sum_{x\in S(n)} \Big| \frac{\ell_n(x)}{\sum_{v\in V(G)} \ell_n(v)} - \lambda_n(o,x) \Big| + \sum_{x\notin S(n)} \lambda_n(o,x) + \frac{\ell_n(x)}{\sum_{v\in V(G)} \ell_n(v)} \\
&\leq \sum_{x\in S(n)} \Big| \frac{\ell_n(x)\big(1-\delta(n)\sum_{v\in V(G)}\ell_n(v)\big)}{\sum_{v\in V(G)} \ell_n(v)} - \epsilon_n(x)\delta(n) \Big| + 3^{-n} + \sum_{x\notin S(n)} \frac{\ell_n(x)}{\sum_{v\in S(n)} \ell_n(v)} \\
&\leq \sum_{x\in S(n)} \Big| \frac{\ell_n(x)\big(1-\delta(n)\sum_{v\in V(G)}\ell_n(v)\big)}{\sum_{v\in V(G)} \ell_n(v)} \Big| + \sum_{x\in S(n)} \epsilon_n(x)\delta(n) + 3^{-n} + \sum_{x\notin S(n)} \frac{\ell_n(x)}{\sum_{v\in S(n)} \ell_n(v)}.
\end{aligned} \tag{2.3}$$

The first term here is $1 - \delta(n)\sum_{x\in S(n)} \ell_n(x) = 1 - \sum_{x\in S(n)} \lambda_n(o,x) + \epsilon_n(x)\delta(n) \leq 3^{-n} + |S(n)|\delta(n) \leq 3^{-n} + 2^{-n}$. The second term is bounded from above by $\delta(n)|S(o,n)| \leq 2^{-n}$, and the last term by $\frac{3^{-n}/\delta}{\sum_{v\in S(n)}(1-2^{-n})\lambda_n(x)/\delta(n)} \leq \frac{3^{-n}}{(1-2^{-n})(1-3^{-n})} \leq 2\cdot 3^{-n}$, using (2.2).

We have just seen that $\kappa_n(o,.)$ and $\lambda_n(o,.)$ are close to each other if n is large enough. For every neighbor x of o we know that $\lambda_n(G,o,.)$ and $\lambda_n(G,x,.)$ are close, by assumption. Hence $\kappa_n(o,.)$ and $\kappa_n(x,.)$ are also close, and this is what we wanted to prove. $\square$

3 One-ended tree from sparse connected subgraphs

Lemma 3.1. *Let $G=(G,o)$ be an ergodic amenable unimodular random graph. Suppose that there exists a factor of iid sequence (H_n) of connected subgraphs of G such that $\mathbb{P}(o\in H_n)\to 0$. Then G has a one-ended factor of iid spanning tree.*

Proof. By switching to a subsequence if necessary, we may assume that $\mathbb{P}(o\in H_n)<2^{-n}$. We may also assume that $H_{n+1}\subset H_n$, as we explain next. First note that for every $\epsilon>0$ one can modify every H_n in a fiid way to get an H'_n, and in such a way that $H_n\subset H'_n$, H'_n is invariant, connected, $\mathbb{P}(o\in H'_n)<2^{-n}(1+\epsilon)$, and $H'_n\cap H'_{n+1}\neq\emptyset$. Namely, suppose that the distance between H_n and H_{n+1} is k. If $k=0$, choose $H'_n=H_n$. Otherwise, for every point of H_n at distance k from H_{n+1}, fix a path of length k between this point and H_{n+1}, and select it with probability $\epsilon/(k+1)$. Add all the selected paths to H_n to obtain H'_n. There must be infinitely many points in H_n at distance k from H_{n+1} by the MTP, so H'_n in fact intersects $H_{n+1}\subset H'_{n+1}$ almost surely. Define now $H''_n:=\cup_{i=n}^{\infty}H'_n$. It is connected, $H''_{n+1}\subset H''_n$, and $\mathbb{P}(o\in H''_n)<2^{-n+1}(1+\epsilon)$, as we wanted. So we will assume that $H_{n+1}\subset H_n$.

Apply Lemma 2.3 and let $G_n=\Gamma_n$ from there.

Denote by x a uniformly chosen neighbor of o. If H is a subgraph of G, v and w two vertices, we let $v\leftrightarrow_H w$ stand for the event that v and w are in the same component of H. For an arbitrary forest $\mathcal{F}$ and vertex v, let $\mathcal{F}(v)$ be the component of v in $\mathcal{F}$.

We will define fiid spanning forests F_n of G that all have only finite components, (G,F_n) is jointly unimodular, and the limit of F_n as n goes to infinity will be the tree in the claim. Let $H_0:=G$ and $F_0:=(V(G),\emptyset)$.

Let $k(n)$ be a strictly increasing sequence of positive integers, to be defined later, with $k(0)=0$. Suppose recursively that F_n has been defined, all its edges are in $G\setminus H_{k(n)}$, and every component of it is adjacent to $H_{k(n)}$. Suppose further that

$$\mathbb{P}(x\leftrightarrow_{F_n} o)\geq 1-2^{-n}. \tag{3.1}$$

The recursive assumptions trivially hold for $n=0$.

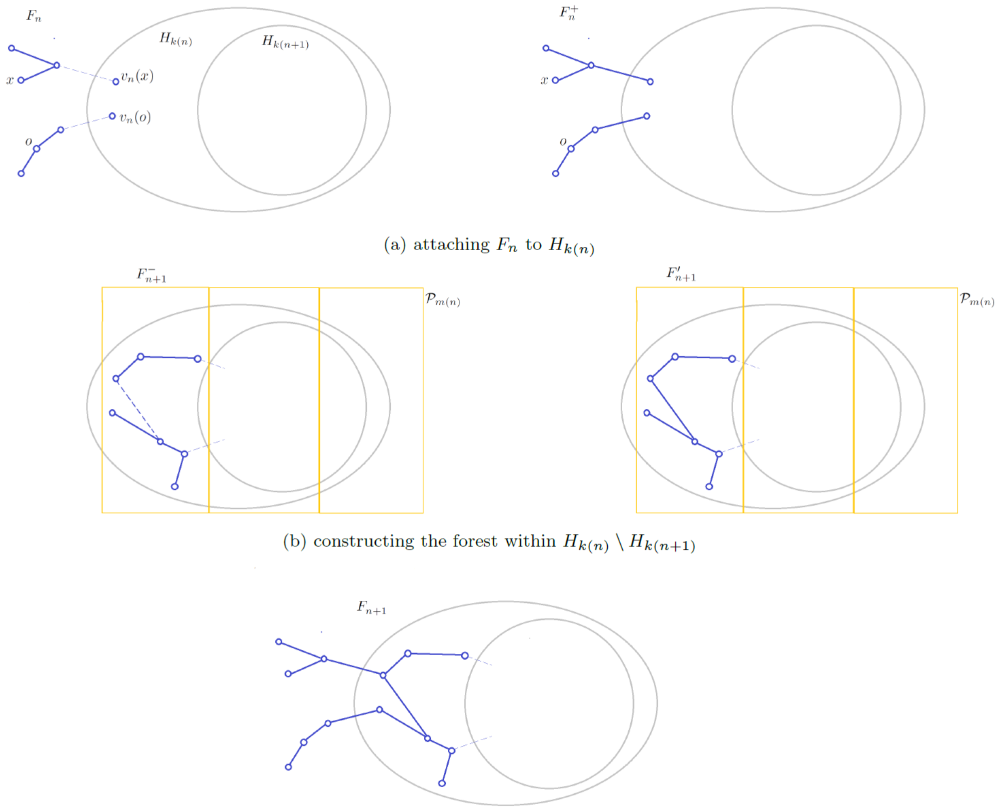
Figure 1 illustrates the steps of the construction that are explained next.

For every component C of F_n, let $v(C)$ be a vertex of $H_{k(n)}$ that is adjacent to C. If there are multiple vertices that could be picked, we choose one according to some rule that will make the result an fiid. We will do so in every later choice in the proof, without explicitely mentioning. Define F_n^+ as the union of F_n and one edge of the form $\{v(C),u\}$ for every component C of F_n, where $u\in C$. Let $v_n(x)$ (respectively $v_n(o)$) be equal to $v(C_x)$ (resp. $v(C_o)$), where C_x is the component of x in F_n.

Let $\partial^{\mathrm{up}}H_{k(n)}$ be the set of vertices in $H_{k(n)}\setminus H_{k(n+1)}$ that are adjacent to $H_{k(n+1)}$. Grow a forest within $H_{k(n)}\setminus H_{k(n+1)}$ starting from $\partial^{\mathrm{up}}H_{k(n)}$ iteratively as follows. As $i=0,1,\ldots$, consider the set U_i of vertices at distance i from $\partial^{\mathrm{up}}H_{k(n)}$ (so $U_0=\partial^{\mathrm{up}}H_{k(n)}$), and pick a randomly chosen edge between each vertex in U_i and some vertex in U_{i-1}. As $i\to\infty$, we end up with a forest F^-_{n+1} in $H_{k(n)}\setminus H_{k(n+1)}$, which has the property that each of its components contains a unique point of $\partial^{\mathrm{up}}H_{k(n)}$ (by the connectedness of $H_{k(n)}$), and consequently, each component if finite (by the MTP).

Let $A(n,m)$ be the event that there is a path with consecutive vertices $p_1,\ldots,p_\ell$, between $v_n(x)$ and $v_n(o)$ ($p_1=v_n(x),p_\ell=v_n(o)$), with the $F^-_{n+1}(p_i)$ all fully contained in the same component of G_m. By definition of G_m, we have that $\lim_{m\to\infty}\mathbb{P}(v_n(x)\leftrightarrow_{H_{k(n)}\setminus H_{k(n+1)}} v_n(o);A(n,m))=\mathbb{P}(v_n(x)\leftrightarrow_{H_{k(n)}\setminus H_{k(n+1)}} v_n(o))$. This latter probability is arbitrary close to 1 if $k(n)$ is fixed and $k(n+1)$ is large enough, because of the assumptions given for H_n. So let $k(n+1)$ be chosen such that $\mathbb{P}(v_n(x)\leftrightarrow_{H_{k(n)}\setminus H_{k(n+1)}} v_n(o))>1-2^{-n-2}$. Choose $m(n)$ large enough so that

$$\mathbb{P}(v_n(x)\leftrightarrow_{H_{k(n)}\setminus H_{k(n+1)}} v_n(o);A(n,m))\geq 1-2^{-n-1}. \tag{3.2}$$



(a) attaching F_n to $H_{k(n)}$

(b) constructing the forest within $H_{k(n)} \setminus H_{k(n+1)}$

(c) the union of the two provides F_{n+1}

Figure 1: The construction of F_{n+1} from F_n. Dashed lines are in $E(G)$, but not in the graph at display.

For each component K of $G_{m(n)}$ consider the set of components of F_{n+1}^- that lie entirely in K, and add a maximal number of edges to them (following some otherwise arbitrary rule) so that the result is still cycle-free. Call the resulting forest F'_{n+1} (so F'_{n+1} is F^-_{n+1} with all these added edges). Then, by (3.2), $\mathbb{P}(x \leftrightarrow_{F'_{n+1} \cup F_n^+} o) = \mathbb{P}(v_n(x) \leftrightarrow_{F'_{n+1}} v_n(o)) \geq 1 - 2^{-n-1}$. Finally, define F_{n+1} as $F'_{n+1} \cup F_n^+$. By construction, the recursive assumptions are satisfied by F_{n+1}.

Let F be the limit of the increasing sequence F_n. It is clearly a forest, and by (3.1), F is a spanning tree. To see that F has one end, pick an arbitrary vertex v, and let $n \in \{0, 1, \dots\}$ be such that $v \in H_{k(n)} \setminus H_{k(n+1)}$. If C is the component of v in F_n, then v is in a finite component of $F \setminus \{v(C)\}$, hence v is separated from infinity by one point, as we wanted. □

4 Constructing sparse connected subgraphs

In what follows we are going to construct a sequence of fiid connected subgraphs of H_n with marginals tending to 0, as in Lemma 3.1. This will then establish Theorem 1.1.

From now on, *intervals always mean discrete intervals*, e.g. $[a, b]$ with $a, b \in \mathbb{Z}$ is the set $\{a, a+1, \dots, b\}$. An interval may only consist of 1 point. Given a set of intervals, it will automatically define an *interval graph*, as the graph whose vertices are the given intervals, and two are adjacent if they intersect. By a slight sloppiness, we will refer to the graph induced by a set $\mathcal{I}$ of intervals by the same notation $\mathcal{I}$.

Lemma 4.1. *Let $a, b \in \mathbb{Z}$, and let $\mathcal{I}$ be a connected interval graph of intervals in $[a, b]$. Suppose that both a and b are contained in some interval in $\mathcal{I}$. Then there is some $\mathcal{I}' \subset \mathcal{I}$ such that the graph induced by $\mathcal{I}'$ is connected, and every integer of $[a, b]$ is contained in exactly 1 or 2 elements of $\mathcal{I}'$.*

Denote the minimal length of an interval in $\mathcal{I}$ by Δ, and suppose that $\Delta \geq 2$. Fix $\delta \leq \lfloor \Delta/2 \rfloor$. Define $O = \delta\mathbb{Z}$. Then there is a map ι from the set of endpoints $\mathcal{V}(\mathcal{I})$ of $\mathcal{I}$ to O that has the following properties:

1. *$|x - \iota(x)| \leq 2\delta$ for every $x \in \mathcal{V}(\mathcal{I})$.*
2. *If $x \leq y$, $x, y \in \mathcal{V}(\mathcal{I})$, then $\iota(x) \leq \iota(y)$. In particular, the interval graph defined by $\mathcal{I}'' := \{[\iota(a), \iota(b)] : [a, b] \in \mathcal{I}'\}$ is such that ι maps adjacent (intersecting) intervals in $\mathcal{I}'$ to adjacent intervals in $\mathcal{I}''$.*
3. *Every point of $[a, b]$ is contained in at most two elements of $\mathcal{I}''$.*

Proof. Choose a path $I_0, \ldots, I_m$ (with $I_i \cap I_{i+1} \neq \emptyset$) in the interval graph $\mathcal{I}$ with the property that $a \in I_0$, $b \in I_m$ (this latter we refer to by saying that the path bridges a and b), and make the choice so that m is minimal. By assumption, every point $k \in [a, b]$ is contained in some I_i. Suppose now that for some $k \in [a, b]$ there exist three distinct intervals that contain k. Then one can choose two of these three such that their union contains the third one (choose one with the leftmost and one with the rightmost endpoint). But then this third one could be dropped from $I_0, \ldots, I_b$, and one would still be left with a connected graph (and a path that bridges a and b in it), contradicting the minimality of m. Hence $\mathcal{I}' := \{I_0, \ldots, I_m\}$ satisfies the first assertion.

Still using that the $\mathcal{I}'$ we defined is a minimal path, one can check the following. See Figure 2 for the pattern of the intervals and the naming introduced in the present paragraph. Denote the endpoints of I_0 by x_0 and x_2, with $x_0 < x_2$. Denote the endpoints of I_m by x_{2m-1} and x_{2m+1}, where $x_{2m-1} < x_{2m+1}$. Finally, for $0 < k < m$, let the endpoints of I_k be x_{2k-1} and x_{2k+2}, where $x_{2k-1} < x_{2k+2}$. Then $x_{2k-1} \leq x_{2k}$, because I_{k-1} intersects I_k, and the latter is closer to b than the former. Similarly, for $k \geq 1$ we have $x_{2k} < x_{2k+1}$, because $I_{k-1} \cap I_{k+1} = \emptyset$ (by the assumption that every point of $[a, b]$ is in at most two of the intervals).

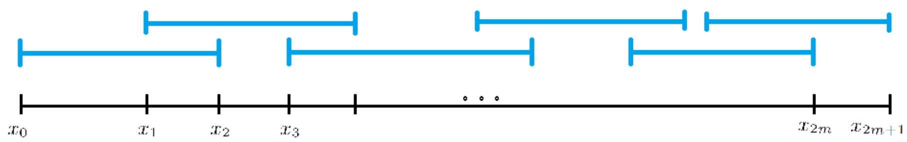


Figure 2: Intervals representing a path in the interval graph. The lowest line represents the underlying set $[a, b]$.

To construct ι, do the following. Define a map ι' first, for $x \in \mathcal{V}(\mathcal{I})$, by letting $\iota'(x)$ be the point of O closest to x (in case of a tie, decide arbitrarily). Vertex x_{2k-1} and x_{2k+2} are always at least $\Delta \geq 2\delta$ apart from each other, hence they cannot be mapped to the same point or to neighbors in $\delta\mathbb{Z}$. Therefore at most 3 points can be mapped to the same point by ι', and if 3 points are mapped to the same $v \in O$, then no point is mapped to $v + \delta$ or $v - \delta$. Suppose that 3 vertices are mapped to some $v \in O$, that is, $\iota'(x_i) = \iota'(x_{i+1}) = \iota'(x_{i+2})$. Then, if $x_{i+2} > \iota'(x_{i+2})$, define $\iota(x_{i+2}) = \iota'(x_{i+2}) + \delta$, and $\iota(x_i) = \iota(x_{i+1}) = \iota'(x_{i+1})$. Otherwise we have $x_i < x_{i+1} < x_{i+2} \leq \iota'(x_{i+2})$. In this case define $\iota(x_i) = \iota'(x_i) - \delta$, and $\iota(x_{i+1}) = \iota(x_{i+2}) = \iota'(x_{i+2})$. It is easy to check that ι satisfies the requirements. $\square$

Lemma 4.2. *Let G and B be random graphs, B being a biinfinite path, and suppose that (G,B) is jointly unimodular, $V(G)=V(B)$ and $E(B)\subset E(G)$. Suppose further that G has only one end. Let x_n and x_{-n} be the two vertices whose distance from the root is n in B. Then*

$$\lim_{n\to\infty}\mathbb{E}(\mathrm{dist}_G(x_{-n},x_n)/2n)\to 0.$$

Proof. There are two graph isomorphisms from B to $\mathbb{Z}$ that take the root to 0, pick one of the two randomly with probability 1/2 and fix is, for simpler reference. Through this isomorphism, we can refer to the points of B as integers; we will use B and $\mathbb{Z}$ interchangeably. This way, to every edge $e=\{k,\ell\}\in E(G)$, we can assign the interval $I(e)=[k,\ell]$, which can be thought of as the unique path in B between the endpoints of e. We refer to $\ell-k$ as the length of the edge e. Note that by subadditivity the limit exists, $\lim\mathbb{E}(\frac{\mathrm{dist}_G(-n,n)}{2n})=\inf\mathbb{E}(\frac{\mathrm{dist}_G(-n,n)}{2n})$. We need to prove that this number is 0.

Suppose to the contrary, that $\lim\mathbb{E}(\frac{\mathrm{dist}_G(0,n)}{n})=\lim\mathbb{E}(\frac{\mathrm{dist}_G(-n,n)}{2n})=c>0$. Let $c<c'<16c/15$. Let d be a positive integer such that $\mathbb{E}(\mathrm{dist}_G(o,n))<c'n)$ for every $n\geq d$. Pick some $D>(64d+32)/3c'(>2d)$.

Because of the one-endedness of G, for any point $x\in\mathbb{Z}$, there are infinitely many intervals $I(e)$, $e\in E(G)$, that contain x. In other words, there are infinitely many edges whose endpoints belong to different components of $B\setminus\{x\}$. Hence we can choose some number D' with the property that $\mathbb{P}\big(o\in I(e)$ for some $I(e)$ with $D\leq|I(e)|<D'\big)>1-c'/32$. By unimodularity, we have the same probability if we replace o by any given $x\in\mathbb{Z}$. As we have just set,

$$\frac{1}{2N+1}\mathbb{E}(|\{x\in[-N,N],\,x\in I(e)\text{ for some }I(e)\text{ with }D\leq|I(e)|<D'\}|)>1-\frac{c'}{32}.$$

Let $\mathcal{E}_N$ be the collection of all edges $e=\{k,\ell\}\in E(G)$ of length at least D, such that $k,\ell\in[-N,N]$. Then,

$$\frac{1}{2N+1}\mathbb{E}(|\{x\in[-N,N],x\in I(e)\text{ for some }e\in\mathcal{E}_N\}|)\geq$$
$$\frac{1}{2N+1}\mathbb{E}(|\{x\in[-N+D',N-D'],\,x\in I(e)\text{ for some }I(e)\text{ with }D\leq|I(e)|<D'\}|)\geq$$
$$\frac{1}{2N+1}\mathbb{E}(|\{x\in[-N,N],\,x\in I(e)\text{ for some }I(e)\text{ with }D\leq|I(e)|<D'\}|)-\frac{2D'}{2N+1}\geq 1-\frac{c'}{16}$$

if N is large enough. Let $\mathbf{S}_0$ be $\{x\in[-N,N]:\ \text{for every }I\in\mathcal{E}_N,x\notin I\}$. The previous inequalities directly imply

$$\mathbb{E}(|\mathbf{S}_0|)\leq c'(2N+1)/16. \tag{4.1}$$

From now on, $\mathbf{I}$ will denote an arbitrary connected component of $[-N,N]\setminus\mathbf{S}_0$ in $\mathbb{Z}$. Let $\mathcal{E}(\mathbf{I})$ be the subset of edges in $\mathcal{E}_N$ both of whose endpoints are in $\mathbf{I}$. Now, one can apply Lemma 4.1, for $\mathbf{I}=[a,b]$, with $\{I(e):\,e\in\mathcal{E}(\mathbf{I})\}$ as $\mathcal{I}$, $D=\Delta$, and d as in the lemma. Let $\mathcal{I}'(\mathbf{I})=\mathcal{I}'$ and ι be as in Lemma 4.1. One of the implications of the lemma is that for every $k\in[-N,N]$, there is exactly 1 or 2 elements of $\mathcal{I}'$ that contain k. From this we have

$$|\mathcal{I}'|\leq 2|\mathbf{I}|/D, \tag{4.2}$$

because every interval in $\mathcal{I}'\subset\mathcal{I}$ has length at least D.

Let $P_2\subset\mathbf{I}$ be the set of those points that are contained in exactly two elements of $\mathcal{I}'$, and $P_1=\mathbf{I}\setminus P_2$ be the set of those that are contained in one. Now, let $\mathcal{S}_2$ be the set of maximal connected subintervals induced by P_2. Consider also the set of maximal connected subintervals induced by P_1, and partition it into two subsets, using the natural ordering on these intervals from left to right: let $\mathcal{S}_1$ be the subset of these intervals that

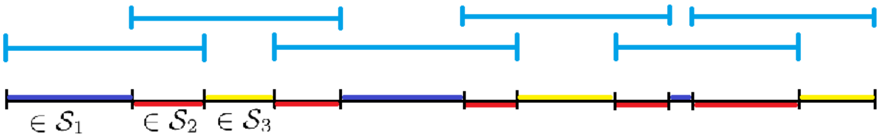


Figure 3: The subinterval partition $\mathcal{I}'$ of $\mathbf{I}$, and the categorization of its elements to classes $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$.

are at odd positions at this ordering, and $\mathcal{S}_3$ be the set of those that are at even positions. See Figure 3.

Denote $\mathcal{S} = \mathcal{S}(\mathbf{I}) := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. We have $|\mathcal{S}| \leq 2|\mathcal{I}'| \leq 4|\mathbf{I}|/D$ by (4.2).

If $I \subset [-N, N]$ is an arbitrary interval, let I_- be its left endpoint and I_+ be its right endpoint. Fix $j \in \{1,2,3\}$ for now. For every $I \in \mathcal{S}_j(\mathbf{I})$, pick a path P_I in G of minimal length between I_- and I_+. It is easy to check that the subgraph $\cup_{I\in\mathcal{S}_j} P_I \cup \{e :\ I(e) \in \mathcal{I}'\}$ of G is connected, and it contains the endpoints $\mathbf{I}_-$ and $\mathbf{I}_+$ of $\mathbf{I}$ (see Figure 3). Hence its total size is an upper bound on $\mathrm{dist}_G(\mathbf{I}_-, \mathbf{I}_+)$. We obtain that for every $j \in \{1,2,3\}$

$$\mathbb{E}(\mathrm{dist}_G(I_-, I_+)) \leq |\mathcal{I}'| + \cup_{I\in\mathcal{S}_j} |P_I| \leq 2|\mathbf{I}|/D + \mathbb{E}\big(\sum_{I\in\mathcal{S}_j} \mathrm{dist}_G(I_-, I_+)\big) \tag{4.3}$$

using (4.2). As $\mathbf{I}$ runs over all connected components of $[-N,N] \setminus \mathbf{S}_0$, one has

$$\mathbb{E}\big(\sum_{\mathbf{I}} \mathrm{dist}_G(\mathbf{I}_-, \mathbf{I}_+)\big) \leq \frac{1}{3} \sum_{\mathbf{I}} \sum_{j=1}^{3} \mathbb{E}\big(2|\mathbf{I}|/D + \sum_{I\in\mathcal{S}_j(\mathbf{I})} \mathrm{dist}_G(I_-, I_+)\big) = \tag{4.4}$$

$$\frac{4N+2}{3D} + \frac{1}{3}\Big(\mathbb{E}\big(\big(\sum_{I\in\mathcal{S}(\mathbf{I})} \mathrm{dist}_G(\iota(I_-), \iota(I_+))\big) + \sum_{\mathbf{I}} 2d|\mathcal{S}(\mathbf{I})|\big)\Big) \leq \tag{4.5}$$

$$\frac{4N+2}{3D} + \frac{2}{3}\sum_{i=0}^{\lfloor N/d\rfloor} \mathbb{E}\big(\mathrm{dist}_G(id, (i+1)d)\big) + \frac{1}{3}\mathbb{E}\big(\sum_{\mathbf{I}} 8d|\mathbf{I}|/D\big) \leq \frac{4N+2}{3D} + \frac{2c'N}{3} + \frac{16dN}{3D}, \tag{4.6}$$

where the last inequality follows by unimodularity (via $\mathbb{E}(\mathrm{dist}_G(id, (i+1)d)) = \mathbb{E}(\mathrm{dist}_G(0, d))$) and the definition of c', and the inequality before it uses Lemma 4.1. We conclude that

$$\mathbb{E}(\mathrm{dist}_G(-N, N)) \leq \mathbb{E}(|\mathbf{S}_0| + \sum_{\mathbf{I}} \mathrm{dist}_G(\mathbf{I}_-, \mathbf{I}_+)) \leq \frac{c'(2N+1)}{16} + \frac{4N+2}{3D} + \frac{2c'N}{3} + \frac{16dN}{3D}$$
$$\leq \frac{15c'N}{16}.$$

This holds for every large enough N, contradicting $c > 15c'/16$. □

Proof of Theorem 1.1. Let T_0 be a fiid spanning tree of G with one or two ends. Such a tree exists, as a generalization of Theorem 8.9 of [1] to the fiid setting, using Lemma 2.3.

If T_0 has one end, then the claim is proved, so let us assume that it has 2 ends. Let B be the biinfinite path in T_0. To every vertex x in B, define B_x as the subgraph induced in T_0 by x and all vertices that are in a finite component of $T_0 \setminus \{x\}$. For every vertex $v \in V(G)$ define $b(v) \in V(B)$ to be the (unique) vertex such that $v \in B_{b(v)}$. We define a new unimodular graph B^+ on the vertex set of B, as a deteministic function of (G, T_0). For an edge $e = \{v, w\}$ in G, define $e^+ = \{b(v), b(w)\}$, and let $E(B^+) := \{e^+ :\ e \in E(G)\}$. (We keep only one copy of any collection of parallel edges in this notation; keeping all of them would not change the proof.) We will define an fiid sequence (K_n) of subgraphs of B^+ that satisfy the following:

1. K_n is connected;
2. $\lim_{n\to\infty} \mathbb{P}(o \in K_n) = 0$.

Once we have (K_n), we will define a sequence (H_n) of subgraphs of G, where $H_n := \cup_{x\in V(K_N)} B_x \cup \{e \in E(G) : e^+ \in K_N\}$. It is easy to check that if (K_n) satisfies conditions (1) and (2), then so does (H_n), and thus the theorem follows from Lemma 3.1. It remains to construct the K_n.

Fix n and consider Bernoulli(2^{-n}) percolation on $V(B)$ (which is an fiid process itself), independently from all other iid labels used so far. For every pair of open vertices x and y such that every vertex of B on the path between x and y is closed (including the case when x and y are neighbors), choose a connected finite subgraph $C_{x,y}$ of minimal size of B^+ that contains both x and y. Let K_n be the union of all these $C_{x,y}$. Then the K_n are connected. We will show that they also satisfy item 2.

As in the proof of Lemma 4.2, choose a random uniform isomorphism between B and $\mathbb{Z}$ that maps o to 0, just for the sake of simpler reference. When convenient, we will refer to the vertices of B as elements of $\mathbb{Z}$. For an arbitrary $x \in V(B)$, let x_+ be the smallest $x_+ > x$ that is open, and let x_- be the largest $x_- \leq x$ that is open. Let $\epsilon > 0$ be arbitrary. Choose M such that $\mathbb{E}(\mathrm{dist}_{B^+}(o, m)/m) \leq \epsilon/4$ for every $m \geq M$, and choose n_0 so that $\mathbb{P}(o_+ \leq M) = \mathbb{P}(1, \dots, M-1 \text{ are closed}) < \epsilon/4$ whenever $n \geq n_0$. An M with this first property exists by Lemma 4.2. Define $\mathcal{C} = \{C_{x,x_+} : x \text{ open}\}$. We have $K_n = \cup_{C\in\mathcal{C}} C$. Define the following mass transport: let o send mass $\frac{1}{o_+ - o_-}$ to every vertex of C_{o_-,o_+}. The expected mass received is $\sum_{i=1}^\infty i\mathbb{P}(o \text{ is in exactly } i \text{ elements of } \mathcal{C}) \geq \mathbb{P}(o \text{ is in some element of } \mathcal{C}) = \mathbb{P}(o \in K_n)$. The expected mass sent out is $\mathbb{E}(|C_{o_-,o_+}|/|o_+ - o_-|) \leq 2\mathbb{E}(|\mathrm{dist}_{B^+}(o, o_+)|/|o_+ - o|) = 2\sum_{j=1}^\infty \mathbb{P}(o_+ = j)\mathbb{E}(\mathrm{dist}_{B^+}(o, j)/j)$, using the independence of the percolation process. The first M terms of this sum are less than $\epsilon/4$, while the sum $\sum_{j=M+1}^\infty \mathbb{P}(o_+ = j)\mathbb{E}(\mathrm{dist}_{B^+}(o, j)/j)$ is also bounded by $\epsilon/4$. By the MTP (1.1), the expected mass sent out and the expected mass received are the same, hence we obtain that $\mathbb{P}(o \in K_n) < \epsilon$, as we wanted. □

References

[1] Aldous, D., Lyons, R. (2007) Processes on unimodular random networks *Electron. J. Probab.* **12**, 1454–1508. MR-2354165

[2] Benjamini, I., Lyons, R., Peres, Y., Schramm, O. (1999) Group-invariant percolation on graphs *Geom. Funct. Anal.* **9**, 29–66. MR-1675890

[3] Connes, A., Feldman, J., and Weiss, B. (1981) An amenable equivalence relation is generated by a single transformation, *Ergodic Theory Dynamical Systems* **1**, 431–450. MR-0662736

[4] Feldman, J., and Moore, C. (1977) Ergodic equivalence relations, cohomology, and von Neumann algebras I. *Trans. Amer. Math. Soc.*, **234** (2), 289–324. MR-0578656

[5] Kaimanovich, V.A. (1997) Amenability, hyperfiniteness, and isoperimetric inequalities. *C. R. Acad. Sci. Paris Sér. I Math.* **325**, 999–1004. MR-1485618

[6] Kechris, A.S. and Miller, B.D. (2004) *Topics in Orbit Equivalence*, volume 1852 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin. MR-2095154

[7] Lovász, L. (2012) *Large Networks and Graph Limits*, volume 60 of *Colloquium Publications*, Amer. Math. Soc., Providence, RI. MR-3012035

[8] Timár, Á. (2004) Tree and grid factors of general point processes, *Electronic Communications in Probability* **9**, 53–59. MR-2081459

[9] Timár, Á. (2017) A nonamenable "factor" of a euclidean space, preprint, arXiv:1712.08210.

Acknowledgments. I would like to thank the anonymous referee for valuable suggestions and corrections, and Tom Hutchcroft for letting me include his proof for Lemma 2.3. This research was partially supported by the Hungarian National Research, Development and Innovation Office, NKFIH grant K109684, grant LP 2016-5 of the Hungarian Academy of Sciences, and the ERC Consolidator Grant 772466 “NOISE”.