

# A conformal Skorokhod embedding

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## Abstract

Start a planar Brownian motion and let it run until it hits some given barrier. We show that the barrier may be crafted so that the  $x$  coordinate at the hitting time has any prescribed centered distribution with finite variance. This provides a new, complex-analytic proof of the Skorokhod embedding theorem. Our method is constructive and can give an explicit description of the barrier.

**Keywords:** Skorokhod embedding; conformal mapping; domain-free Riemann mapping.

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## 1 Introduction

The Skorokhod embedding problem asks the following: Given a Brownian motion  $X_t$  and a probability distribution  $\mu$  with zero expectation and finite variance, find a stopping time  $T$  so that  $X_T \sim \mu$  and  $\mathbb{E}T < \infty$ .

There have been numerous solutions to this formulation of the problem and to several variations and generalizations over the years. See Obłój's extensive survey [10] for a detailed account of the problem, its characteristics and applications.

One important solution is given by Root [11], who sets  $T$  as the first time that the graph  $(t, X_t)$  of the Brownian motion hits some barrier  $\Omega \subseteq \mathbb{R}^+ \times \mathbb{R}$ . Root's solution does not use any additional randomness. Finding out the barrier  $\Omega$ , however, is often a difficult task, and not many explicit solutions are known (see [5] for constructions relying on solutions to PDEs).

Here, we present a new solution to the Skorokhod embedding problem. Our method is similar to Root's, in that the stopping time is the first hitting time of some barrier by a Brownian motion. The method requires additional randomness in the form of another independent Brownian motion, but can offer tractable analytic expressions for calculating the shape of the barrier explicitly.

For a domain  $\Omega \subseteq \mathbb{R}^2$  and a planar Brownian motion  $X_t = (X_t^{(1)}, X_t^{(2)})$  with  $X_0 \in \Omega$ , let  $T(X_t, \Omega)$  be the first time that  $X_t$  exits the domain  $\Omega$ ,

$$T(X_t, \Omega) = \inf \{t > 0 \mid X_t \notin \Omega\}.$$

In this paper, we prove the following result:

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**Theorem 1.1.** *Let  $\mu$  be a probability distribution on  $\mathbb{R}$  with zero expectation and finite variance. There exists a simply connected domain  $\Omega \subseteq \mathbb{R}^2$  containing the origin such that if  $Y_t$  is a standard planar Brownian, then  $Y_{T(Y_t, \Omega)}^{(1)}$  has distribution  $\mu$ .*

The proof of Theorem 1.1 is given in the next section. Section 3 gives properties and examples of  $\Omega$ . Finally, in Section 4 we exhibit some open questions.

## 2 Proof of Theorem 1.1

*Proof.* We identify  $\mathbb{R}^2$  with the complex plane  $\mathbb{C}$ . Denote the open unit disc by

$$D = \{z \in \mathbb{C} \mid |z| < 1\}$$

and the unit circumference by

$$\partial D = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Denote by  $F : \mathbb{R} \rightarrow [0, 1]$  the cumulative distribution function of  $\mu$ , and its generalized inverse by

$$G : (0, 1) \rightarrow \mathbb{R} \quad G(y) = \inf \{x \in \mathbb{R} \mid F(x) \geq y\}.$$

Observe that both  $F$  and  $G$  are monotone increasing. The generalized inverse  $G$  is useful due to the following elementary theorem, which can be found e.g in [3, Chapter II.2]:

**Theorem 2.1** (Inverse sampling method). *Let  $F$  be a distribution function on  $\mathbb{R}$  with generalized inverse*

$$G(y) = \inf \{x \in \mathbb{R} \mid F(x) \geq y\}.$$

*If  $U$  is a uniform random variable on  $[0, 1]$ , then  $G(U)$  has cumulative distribution function  $F$ .*

*Proof.* The proof is a one line calculation, using the fact that  $F$  is increasing:

$$\mathbb{P}[G(U) \leq x] = \mathbb{P}[\inf\{y \in \mathbb{R} \mid F(y) \geq U\} \leq x] = \mathbb{P}[U \leq F(x)] = F(x). \quad \square$$

This theorem can be used to show that  $G$  is in  $L^2(0, 1)$ , despite the fact that it may diverge at 0 and 1: If  $U$  is a uniform random variable on  $[0, 1]$  then  $G(U)$  distributes as  $\mu$ , and so

$$\int_0^1 G^2(y) dy = \text{Var}\mu < \infty.$$

Define the function  $\varphi : (-\pi, \pi) \setminus \{0\} \rightarrow \mathbb{R}$  by

$$\varphi(\theta) = G\left(\frac{|\theta|}{\pi}\right). \quad (2.1)$$

The function  $\varphi$  may be viewed as a periodic function on  $\mathbb{R}$  with period  $2\pi$ , with the possibility of it diverging at integer multiples of  $\pi$ . It is an even function in  $L^2(-\pi, \pi)$ , and thus has a Fourier series representation containing only cosines. Denote the  $n$ -th Fourier coefficient of  $\varphi$  by  $\hat{\varphi}(n)$ , and recall that by Carleson's theorem [2] for an  $L^2(-\pi, \pi)$  function, the Fourier representation  $\sum_{n=0}^{\infty} \hat{\varphi}(n) \cos(\theta n)$  agrees with  $\varphi$  almost everywhere.

Let  $\psi : D \rightarrow \mathbb{C}$  be the complex function defined by

$$\psi(z) = \sum_{n=0}^{\infty} \hat{\varphi}(n) z^n. \quad (2.2)$$

The function  $\psi$  may not necessarily be defined for points on the unit circle itself, but its real part is well-defined for almost all such points. Indeed, let  $z = e^{i\theta}$  with  $\theta \in [-\pi, \pi]$ , then

$$\begin{aligned} \operatorname{Re}\psi(z) &= \operatorname{Re}\psi(e^{i\theta}) \\ &= \operatorname{Re} \sum_{n=0}^{\infty} \hat{\varphi}(n) e^{i\theta n} \\ &= \sum_{n=0}^{\infty} \hat{\varphi}(n) \cos \theta n. \end{aligned}$$

So on the unit circle, the real part of  $\psi(e^{i\theta})$  agrees with  $\varphi(\theta)$  almost everywhere. Further, if the imaginary part of  $\psi(e^{i\theta})$  exists then it is given from the real part by replacing all cosines with sines in the series expansion of  $\psi$ :

$$\operatorname{Im}\psi(e^{i\theta}) = \sum_{n=1}^{\infty} \hat{\varphi}(n) \sin \theta n.$$

Alternatively, the imaginary part is obtained from the real part by the Hilbert transform operator  $H$ , which, for a function  $u : \mathbb{R} \rightarrow \mathbb{R}$ , is given by

$$(Hu)(x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{u(\tau)}{x - \tau} d\tau, \tag{2.3}$$

where  $PV$  is Cauchy's principal value, defined as

$$PV \int_{-\infty}^{\infty} \frac{u(\tau)}{x - \tau} d\tau := \lim_{R \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \int_{-R}^{x-\varepsilon} \frac{u(\tau)}{x - \tau} d\tau + \int_{x+\varepsilon}^R \frac{u(\tau)}{x - \tau} d\tau.$$

One way to see that the imaginary part is the Hilbert transform of the real part is to note that the Hilbert transform is linear and maps the function  $\cos nx$  to  $\sin nx$  for positive integers  $n$ . For a comprehensive source on the Hilbert transform, see [7, 8].

Since  $\lim_{n \rightarrow \infty} \hat{\varphi}(n) = 0$  by the Riemann-Lebesgue lemma, the function  $\psi$  is analytic (and non constant) in the open disc  $D$ , and so the image  $\Omega = \psi(D)$  is some connected domain in  $\mathbb{C}$ . As all the coefficients of  $\psi$  are real, this domain is symmetric to conjugation:  $\psi(\bar{z}) = \overline{\psi(z)}$ , i.e it is symmetric to reflection about the  $x$  axis.

**Proposition 2.2.**  $\psi$  is one-to-one in the unit disc  $D$ .

*Proof.* The proof relies on the following theorem:

**Theorem 2.3** (Chapter VIII.3 in [4]). Let  $\{f_k(z)\}_{k=1}^{\infty}$  be a sequence of one-to-one analytic functions on a domain  $D$  that converges uniformly on every compact subset of  $D$  to a function  $f$ . Then  $f$  is either one-to-one or constant.

All we have to do then is find a sequence of one-to-one functions  $\psi_k$  which converges to  $\psi$  uniformly on every compact subset of  $D$ . Let  $\{G_k : [0, 1] \rightarrow \mathbb{R}\}_{k=1}^{\infty}$  be a sequence of bounded, twice differentiable, strictly increasing functions satisfying

1.  $G_k \rightarrow G$  in  $L^1(0, 1)$ .
2. The first and second derivatives of  $G_k$  are in  $L^1(0, 1)$ .
3.  $G'_k(0) = G'_k(1) = 0$  and  $G''_k(0) = G''_k(1) = 0$ .

Such a sequence may be found, for example, by taking step-function approximations of  $G$  on increasingly finer partitions of  $[0, 1]$ , smoothly interpolating the jump discontinuities, and adding some small strictly increasing smooth function.

For each  $G_k$ , define the corresponding  $\varphi_k$  and  $\psi_k$  as in equations (2.1) and (2.2). By property (3), the left and right derivatives of  $\varphi_k$  agree at the points  $0, -\pi$  and  $\pi$ , and so  $\varphi_k$  is twice differentiable on  $[-\pi, \pi]$ . By property (2), the first and second derivatives of  $\varphi_k$  are both in  $L^1[-\pi, \pi]$ . Using the fact that if a function  $f$  is  $s$ -times differentiable and  $f^{(s)} \in L^1[-\pi, \pi]$  then  $|\hat{f}(n)| \leq \|f^{(s)}\|_1/n^s$ , we get that the Fourier coefficients  $\{\hat{\varphi}_k(n)\}_{n=0}^\infty$  are absolutely convergent, and so  $\psi_k$  can be extended to the closed disc  $\bar{D}$ . This allows us to look at the image of the unit circle  $\partial D$  under  $\psi_k$ .

Parameterize the circle by  $\gamma(\theta) = e^{i\theta}$  for  $\theta \in [-\pi, \pi]$ . As  $\theta$  increases from  $-\pi$  to  $0$ ,  $\operatorname{Re}\psi_k(\gamma(\theta)) = \varphi_k(\theta)$  strictly decreases from  $G_k(1)$  to  $G_k(0)$ , and as  $\theta$  increases from  $0$  to  $\pi$ ,  $\operatorname{Re}\psi_k(\gamma(\theta))$  strictly increases from  $G_k(0)$  to  $G_k(1)$ . Further, using the Hilbert transform equation (2.3) it can be directly calculated that  $\operatorname{Im}\psi_k(\gamma(\theta))$  is always positive for  $\theta \in (-\pi, 0)$  and always negative for  $\theta \in (0, \pi)$ . Thus, the image  $\psi_k(\partial D)$  is a simple loop. Since the preimage of  $\partial\psi_k(D)$  is a subset of  $\partial D$  (this is a consequence of the maximum principle: If an interior point  $z \in D$  were mapped to  $\partial\psi_k(D)$ , we could obtain a local maximum for  $|\psi_k|$ ), we have that  $\psi_k(\partial D) = \partial\psi_k(D)$ . So for every  $x \in \psi_k(D)$ ,  $\psi_k(\partial D)$  winds around  $x$  exactly once. By Cauchy's argument principle, this means that  $x$  has exactly one preimage in  $D$  under  $\psi_k$ , i.e  $\psi_k$  is one-to-one.

All that remains is to show that  $\{\psi_k\}_{k=1}^\infty$  converge uniformly to  $\psi$  on every compact subset  $A \subseteq D$ . To see this, note that  $A$  is contained in some closed disc of radius  $\rho < 1$ . Denoting  $z = re^{i\theta} \in A$  with  $0 \leq r \leq \rho$ , we have

$$\begin{aligned} |\psi(z) - \psi_k(z)| &= \left| \sum_{n=0}^\infty (\hat{\varphi}(n) - \hat{\varphi}_k(n)) r^n e^{in\theta} \right| \\ &\leq \sum_{n=0}^\infty |(\hat{\varphi}(n) - \hat{\varphi}_k(n)) r^n e^{in\theta}| \\ &\leq \sup_n |(\hat{\varphi}(n) - \hat{\varphi}_k(n))| \sum_{n=0}^\infty \rho^n. \end{aligned}$$

The sum  $\sum_{n=0}^\infty \rho^n$  converges, and the supremum converges to 0 since for each  $n \in \mathbb{N}$  we have

$$\begin{aligned} |\hat{\varphi}(n) - \hat{\varphi}_k(n)| &= \left| \frac{1}{2\pi} \int_{-\pi}^\pi (\varphi(\theta) - \varphi_k(\theta)) \cos(n\theta) d\theta \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^\pi |\varphi(\theta) - \varphi_k(\theta)| d\theta, \end{aligned}$$

and this integral converges to 0 since  $\varphi_k$  converges to  $\varphi$  in  $L^1(-\pi, \pi)$ . Thus, for large enough  $k$ ,  $|\psi(z) - \psi_k(z)|$  can be bounded by an arbitrarily small number independent of  $z$ , and we have uniform convergence.  $\square$

Having established that  $\psi$  is one-to-one, we can invoke the conformal invariance of Brownian motion, which can be found in [9, Theorem 7.20]:

**Theorem 2.4.** *Let  $U$  be a domain in the complex plain,  $x \in U$  and  $f : U \rightarrow \Omega$  be analytic. Let  $\{X_t \mid t \geq 0\}$  be a planar Brownian motion started in  $x$ . Then the process  $\{f(X_t) \mid 0 \leq t \leq T(X_t, U)\}$  is a time-changed Brownian motion, i.e there exists a planar Brownian motion  $\{Y_t \mid t \geq 0\}$  such that for any  $t \in [0, T(X_t, U))$ ,*

$$f(X_t) = Y_{\zeta(t)}$$

where

$$\zeta(t) = \int_0^t |f'(X_s)|^2 ds.$$

If  $f$  is one-to-one then  $\zeta(T(X_t, U))$  is the first exit time from  $\Omega$  by  $\{Y \mid t \geq 0\}$ .

Thus, the process  $\psi(X_t)$  is a time-changed Brownian motion, i.e there exists a planar Brownian motion  $Y_t$  and a monotone function  $\zeta(t)$  such that  $\psi(X_t) = Y_{\zeta(t)}$ . All that remains is to show that  $\Omega$  and  $Y_t$  have the properties we are looking for:

1. Upon hitting the boundary,  $Y_t^{(1)}$  distributes as  $Y_{T(Y_t, \Omega)}^{(1)} \sim \mu$ : The position of  $X_{T(X_t, D)}$  is uniform on the unit circle and the real part of  $\psi(\partial D)$  is made of two (reflected) copies of  $G = F^{-1}$ , so by Theorem 2.1 we get that  $\text{Re}Y_{T(Y_t, \Omega)}$  distributes as  $\mu$ .
2.  $Y_t$  hits the boundary  $\partial\Omega$  in finite time. This is an immediate consequence of the following lemma:

**Lemma 2.5** (Lemma 1.1 in [1]). *Suppose that  $f(w) = \sum_{n=0}^{\infty} a_n w^n$  is a conformal mapping from the unit disc  $D$  onto a domain  $\Omega$  with  $f(0) = z$ . Then*

$$\mathbb{E}_z(T(Y_t, \Omega)) = \frac{1}{2} \sum_{n=1}^{\infty} |a_n|^2. \tag{2.4}$$

Since  $\varphi$  is in  $L^2(-\pi, \pi)$ , the right hand side of equation (2.4) is finite. In fact, we have that  $\mathbb{E}T(Y_t, \Omega) = \text{Var}\mu$ : Since  $(Y_t^{(1)})^2 - t$  is a martingale with expectation 0, we get by the optional stopping theorem that

$$\mathbb{E}(Y_{T(Y_t, \Omega)}^{(1)})^2 - \mathbb{E}T(Y_t, \Omega) = 0.$$

3. The process  $Y_t$  starts at 0, i.e  $Y_0 = \psi(0) = 0$ : The imaginary component is 0 since  $\psi$  is symmetric to complex conjugation, and the real component is 0 since  $Y_t$  is a martingale and  $\text{Re}Y_t$  distributes as  $\mu$ , so that  $\text{Re}Y_0 = \int_{\mathbb{R}} x d\mu = 0$ .
4.  $\Omega$  is simply connected by Brouwer's invariance of domain theorem, which states that:

**Theorem 2.6** (Theorem 1 in Chapter 1 in [12]). *If  $U \subseteq \mathbb{R}^n$  is open and  $f : U \rightarrow \mathbb{R}^n$  is one-to-one and continuous, then  $f$  is a homeomorphism.*

We apply this theorem for the case  $n = 2$ , identifying  $\mathbb{R}^2$  with  $\mathbb{C}$ . □

### 3 Properties and examples

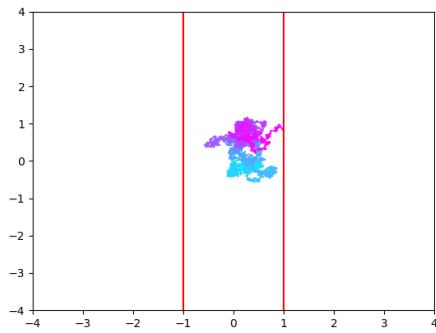
In this section, we will see examples of domains  $\Omega$  for various distributions. Numeric approximations of  $\Omega$  for Bernoulli, Gaussian and Cantor distributions, as well as some distribution on the natural numbers, can be found in Figure 1.

Some easy properties of  $\Omega$  can immediately be gathered from  $\mu$ :

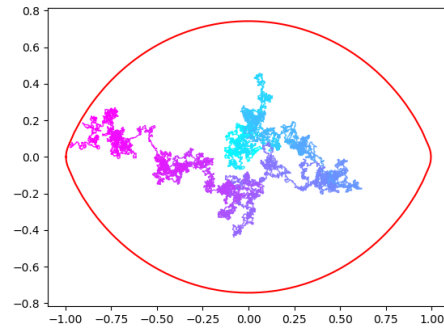
1. If  $x$  is an atom of  $\mu$ , i.e  $\mu(\{x\}) > 0$ , then  $\partial\Omega$  contains a straight line segment at  $x$ , i.e  $\{x\} \times [a, b] \subseteq \partial\Omega$  for some numbers  $a, b$ .
2. If  $(a, b)$  is an interval with  $\mu((a, b)) = 0$  then  $\Omega$  contains the infinite rectangle  $(a, b) \times (-\infty, \infty)$ .
3. If the support of  $\mu$  is infinite, then the projection of  $\Omega$  onto the  $x$  axis is unbounded.

One can describe  $\Omega$  by looking at its boundary  $\partial\Omega$ , and in general it is useful to do this by considering how the curve  $\gamma(\theta) = \psi(e^{i\theta})$  behaves as  $\theta$  traverses from  $-\pi$  to  $\pi$ . Of course, for some distributions both the real part and the imaginary part of  $\psi(e^{i\theta})$  can be infinite, in the latter case even countably many times, so  $\gamma(\theta)$  really sits in the Riemann sphere.

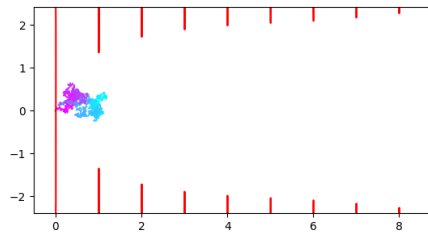
## A conformal Skorokhod embedding



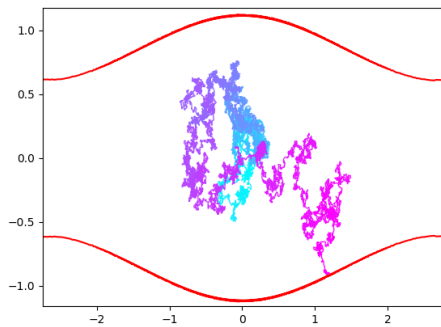
(a)  $\pm 1$  Bernoulli distribution.



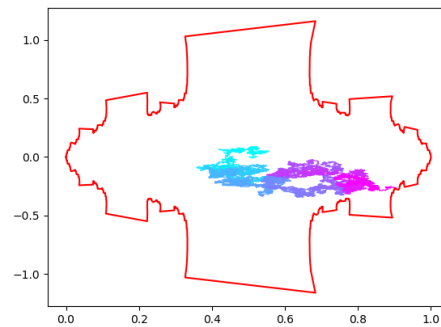
(b) Uniform distribution on  $[-1, 1]$ .



(c)  $\mu(\{k\}) = \frac{1}{2^{k+1}}$  for integer  $k \geq 0$ . Here the Brownian motion does not start at 0, since  $\mu$  is not centered.



(d)  $N(0, 1)$  Gaussian distribution.



(e) Cantor distribution. The fractal boundary is approximated by truncating the very-slowly-converging Fourier series. All vertical line segments should extend to infinity.

Figure 1: Examples of domains  $\Omega$  for various distributions.

Since the real part of  $\psi(e^{i\theta})$  is already known to be  $\varphi(\theta)$ , in order to study the boundary  $\partial\Omega$  we need to obtain the imaginary part of  $\psi(e^{i\theta})$ . As mentioned in the previous section, the imaginary part is equal to

$$\text{Im}\psi(e^{i\theta}) = \sum_{n=1}^{\infty} \hat{\varphi}(n) \sin \theta n,$$

which in turn is equal to the Hilbert transform of the real part. There are therefore

two straightforward ways to go about studying  $\text{Im}\psi$ . The first is to inspect the series  $\sum_{n=1}^{\infty} \hat{\varphi}(n) \sin \theta n$  directly, and the second is to consider the Hilbert transform of  $\varphi$ . In the next two subsections, we will see both approaches.

**3.1 Discrete distributions**

Let  $\mu$  be an atomic distribution, supported on a (possibly infinite) discrete set of values  $\{x_i\}_i$ , which obtains  $x_i$  with probability  $p_i$ :

$$\mu(x) = \sum_i p_i \mathbf{1}_{\{x=x_i\}}(x).$$

The cumulative distribution function is then just a sum of step functions,

$$F(x) = \sum_i p_i \mathbf{1}_{\{x \geq x_i\}},$$

and so is the inverse  $G(y)$  and the symmetric  $\varphi_\mu(\theta)$ . We write

$$\varphi(\theta) = \sum_i \alpha_i \mathbf{1}_{\{|\theta| \geq \theta_i\}}$$

for some weights  $\alpha_i$  and thresholds  $\theta_i$  (these can be calculated explicitly from  $p_i$  and  $x_i$ , but we omit the calculations for brevity).

As noted in items (1) and (2) above, the boundary  $\partial\Omega$  consists of infinite rays of the form  $\{x_i\} \times (y_i, \infty]$  and  $\{x_i\} \times (-y_i, \infty]$  for some values  $y_i \geq 0$ . If there is an extremal value  $x = \min_i \{x_i\}$  or  $x = \max_i \{x_i\}$  among the  $x_i$ 's, then clearly  $\partial\Omega$  contains the infinite line  $\{x\} \times (-\infty, \infty)$ .

As the  $y_i$ 's give a complete characterization of  $\Omega$ , calculating them may be of interest.

**Theorem 3.1.** *Each  $y_i$  is given by*

$$y_i = \frac{1}{\pi} \sum_j \alpha_j \log \left| \frac{\sin\left(\frac{\theta_j}{2} - \frac{x}{2}\right)}{\sin\left(\frac{\theta_j}{2} + \frac{x}{2}\right)} \right|,$$

where  $x$  is a solution to the equation

$$\sum_i \alpha_i \left( \cot\left(\frac{\theta_i}{2} - \frac{x}{2}\right) + \cot\left(\frac{\theta_i}{2} + \frac{x}{2}\right) \right) = 0.$$

*Proof.* Although  $\text{Im}\psi(e^{i\theta})$  may be discontinuous and even infinite, it is differentiable almost everywhere; in fact, by section III.2.10 in [6], if  $\text{Re}\psi(e^{i\theta})$  is constant in an interval, then  $\text{Im}\psi(e^{i\theta})$  is analytic there. This means that  $\text{Im}\psi(e^{i\theta})$  is piecewise analytic, since  $\text{Re}\psi(e^{i\theta})$  is piecewise constant except for a discrete set of jumps at  $\{x_i\}_i$ .

The  $y_i$ 's can therefore be calculated by finding the local minima and maxima of  $\text{Im}\psi(e^{i\theta})$ , and these in turn are obtained by differentiation.

Our first step is to calculate  $\text{Im}\psi(e^{i\theta})$ . As noted in the previous section, the relation between  $\text{Re}\psi(e^{i\theta})$  and  $\text{Im}\psi(e^{i\theta})$  is given by the Hilbert transform  $H$ . Since  $H$  is a linear operator, we first compute the Hilbert transform of a single step function.

**Lemma 3.2.** *Let  $u(x)$  be a periodic step function, i.e*

$$u(x) = \begin{cases} 0 & 0 < |x| < \theta_0 \\ 1 & \theta_0 \leq |x| \leq \pi \end{cases}$$

for some  $0 \leq \theta_0 \leq \pi$ , with  $u(x) = u(x + 2\pi)$ . Then

$$(Hu)(x) = \frac{1}{\pi} \log \left| \frac{\sin\left(\frac{\theta_0}{2} - \frac{x}{2}\right)}{\sin\left(\frac{\theta_0}{2} + \frac{x}{2}\right)} \right|. \tag{3.1}$$

*Proof.* Denote by  $\square_\alpha(x)$  the square pulse of half-width  $\alpha$  around the origin:

$$\square_\alpha(x) = \begin{cases} 1 & |x| < \alpha \\ 0 & o.w. \end{cases}.$$

Then  $u(x)$  is the sum of infinitely many square pulses, each with half-width  $\pi - \theta_0$ :

$$u(x) = \sum_{k=-\infty}^{\infty} \square_{\pi-\theta_0}(x + \pi + 2k\pi).$$

The Hilbert transform of a single square pulse  $\square_\alpha$  can readily be calculated to be ([8, Equation 9.2 in Appendix 1])

$$(H\square_\alpha)(x) = \frac{1}{\pi} \log \left| \frac{\alpha + x}{\alpha - x} \right|$$

(this basically amounts to computing an integral of the form  $\frac{1}{\pi} \int_{-\alpha}^{\alpha} \frac{1}{x-\tau} d\tau$ ). The Hilbert transform commutes with shifts, and so

$$\begin{aligned} (Hu)(x) &= \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \log \left| \frac{\pi - \theta_0 + x + \pi + 2k\pi}{\pi - \theta_0 - (x + \pi + 2k\pi)} \right| \\ &= \frac{1}{\pi} \log \prod_{k=-\infty}^{\infty} \left| 1 + \frac{\frac{1}{2\pi}(2\pi - 2\theta_0)}{k + \frac{1}{2\pi}(\theta_0 + x)} \right|. \end{aligned}$$

The infinite product is an expression of the form

$$\prod_{k=-\infty}^{\infty} \left| 1 + \frac{b}{k - a} \right|, \tag{3.2}$$

with  $a = -\frac{1}{2\pi}(\theta_0 + x)$  and  $b = \frac{1}{2\pi}(2\pi - 2\theta_0)$ . It can be shown that a product in the form of equation (3.2) is equal to

$$\frac{\sin(\pi(a - b))}{\sin(\pi a)};$$

to see this, recall Euler's infinite product identity for the sine function,

$$\sin(x) = x \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n^2\pi^2} \right).$$

We then have

$$\begin{aligned} (Hu)(x) &= \frac{1}{\pi} \log \left| \frac{\sin\left(\pi\left(-\frac{1}{2\pi}(\theta_0 + x) - \frac{1}{2\pi}(2\pi - 2\theta_0)\right)\right)}{\sin\left(-\pi\frac{1}{2\pi}(\theta_0 + x)\right)} \right| \\ &= \frac{1}{\pi} \log \left| \frac{\sin\left(\frac{\theta_0 - x}{2}\right)}{\sin\left(\frac{\theta_0 + x}{2}\right)} \right|. \quad \square \end{aligned}$$

Differentiating Equation (3.1) in Lemma 3.2, each  $y_i$  is therefore given by  $(H\varphi)(x)$  where  $x$  is a zero of the derivative of  $H\varphi$ :

$$\begin{aligned} 0 &= \frac{d}{dx} \sum_i \frac{\alpha_i}{\pi} \log \left| \frac{\sin\left(\frac{\theta_i - x}{2}\right)}{\sin\left(\frac{\theta_i + x}{2}\right)} \right| \\ &= \frac{1}{2\pi} \sum_i \alpha_i \frac{\sin\left(\frac{\theta_i + x}{2}\right)}{\sin\left(\frac{\theta_i - x}{2}\right)} \cdot \frac{-\frac{1}{2} \cos\left(\frac{\theta_i - x}{2}\right) \sin\left(\frac{\theta_i + x}{2}\right) - \frac{1}{2} \cos\left(\frac{\theta_i + x}{2}\right) \sin\left(\frac{\theta_i - x}{2}\right)}{\sin\left(\frac{\theta_i + x}{2}\right)^2} \\ &= -\frac{1}{4\pi} \sum_i \alpha_i \left( \cot\left(\frac{\theta_i - x}{2}\right) + \cot\left(\frac{\theta_i + x}{2}\right) \right). \quad \square \end{aligned}$$



**3.2 The uniform distribution**

Let  $\mu$  be the uniform distribution on  $[-1, 1]$ , so that

$$F_\mu = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(x + 1) & x \in [-1, 1], \\ 1 & x > 1 \end{cases}$$

the inverse function is  $G(y) = 2y - 1$ , and  $\varphi(\theta) = 2\frac{|\theta|}{\pi} - 1$ . The Fourier series of  $\varphi_\mu$  is given by

$$\varphi(\theta) = -\frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos((2k - 1)\theta)}{(2k - 1)^2},$$

which gives

$$\psi(z) = -\frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{z^{2k-1}}{(2k - 1)^2}.$$

The boundary  $\partial\Omega$  is given in Figure 1b. The  $y$  component of  $\gamma(\theta) = \psi(e^{i\theta}) = (x(\theta), y(\theta))$  is then given by

$$y(\theta) = -\frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{\sin((2k - 1)\theta)}{(2k - 1)^2}.$$

Differentiating by  $\theta$ , we get

$$\frac{d}{d\theta}y(\theta) = -\frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos((2k - 1)\theta)}{2k - 1}.$$

If we denote by  $\text{gd}^{-1}(x) = \tanh^{-1}(\tan(\frac{x}{2}))$  the inverse Gudermannian function, then a short calculation reveals that

$$\begin{aligned} \frac{d}{d\theta}y(\theta) &= -\frac{8}{\pi^2} \text{gd}^{-1}\left(\frac{\pi}{2} - |\theta|\right) \\ &= -\frac{8}{\pi^2} \tanh^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{|\theta|}{2}\right)\right). \end{aligned}$$

Thus the domain  $\Omega$  is bounded by the parametric curve

$$\gamma(\theta) = \left(\frac{2|\theta|}{\pi} - 1, -\int_{-\pi}^{\theta} \frac{8}{\pi^2} \tanh^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{|s|}{2}\right)\right) ds\right).$$

This integral may be solved with the assistance of computational software such as Wolfram Mathematica; the antiderivative of  $\tanh^{-1}(\tan(\frac{\pi}{4} - \frac{x}{2}))$  is given by

$$x \tanh^{-1}\left(\cot\left(\frac{\pi}{4} - \frac{x}{2}\right)\right) + \frac{1}{2}(i(\text{Li}_2(-e^{ix}) - \text{Li}_2(e^{ix}))) + x \log(1 - e^{ix}) - \log(1 + e^{ix})),$$

where  $\text{Li}_n(x)$  is the polylogarithm function,

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

## A conformal Skorokhod embedding

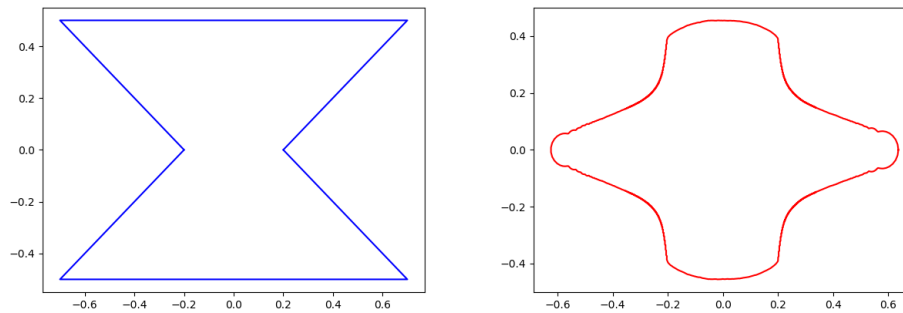


Figure 2: Two domains giving rise to the same continuous distribution. The domain on the right hand side is an approximation obtained by using the empirical distribution given by the domain on the left hand side.

## 4 Other directions and open questions

For a given measure  $\mu$ , the domain  $\Omega$  in Theorem 1.1 is not unique, even if we require  $\Omega$  to be simply connected and symmetric to conjugation (see Figure 2 for two different domains giving the same distribution). Is there any distinguishing trait to the construction given above? Can we deduce properties of  $\Omega$  from properties of  $\mu$ ? For example,

**Question 4.1.** For what measures  $\mu$  is the domain  $\Omega$  in Theorem 1.1 convex?

Theorem 1.1 utilizes the conformal invariance of Brownian motion and uses complex-analytic tools, and so relies on the fact that the Brownian motion is planar. However, we may ask similar hitting-time questions concerning the marginal distribution of the first  $n$  coordinates of a stopped Brownian motion in higher dimensions. It is already well known that not every distribution  $\mu$  on  $\mathbb{R}^n$  can be generated this way: For example, high dimensional Brownian motion does not hit points, so any  $\mu$  containing atoms cannot be obtained. What can we say about measures  $\mu$  for which we already know that there is a Skorokhod embedding?

**Question 4.2.** Let  $\mu$  be a distribution on  $\mathbb{R}^n$  for which there exists a Skorokhod embedding. Is there a domain  $\Omega \subseteq \mathbb{R}^{n+m}$  for some  $m$  so that  $(Y_{T(X_t, \Omega)}^{(1)}, \dots, Y_{T(X_t, \Omega)}^{(n)})$  is a Skorokhod embedding for  $\mu$ ?

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