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Some conditional limiting theorems for symmetric Markov processes with tightness property

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Abstract

Let X be an μ-symmetric irreducible Markov process on I with strong Feller property. In addition, we assume that X possesses a tightness property. In this paper, we prove some conditional limiting theorems for the process X. The emphasis is on conditional ergodic theorem. These results are also discussed in the framework of one-dimensional diffusions.

Keywords: symmetric Markov process; conditional ergodic theorem; quasi-stationary distribution; one-dimensional diffusions; intrinsic ultracontractivity.
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1 Introduction

Let $X = (\Omega, X_t, \mathbb{P}_x, T)$ be an μ-symmetric irreducible Markov process on a locally compact separable metric space I, where μ is a positive Radon measure with full support and T is the lifetime. We denote by $\mathcal{B}(I)$ the Borel σ-field. Adjoining an extra point ∂ to the measurable set $(I, \mathcal{B}(I))$. Set $I_\partial = I \cup \{\partial\}$ and $\mathcal{B}(I_\partial) = \mathcal{B}(I) \cup \{A \cup \{\partial\} : A \in \mathcal{B}(I)\}$. As usual, we denote by $\mathbb{P}_x$ the law of the process starting from x and by $\mathbb{P}_\pi$ the law of the process starting from a distribution π. The corresponding expectations are respectively denoted by $\mathbb{E}_x$ and $\mathbb{E}_\pi$. We assume that the process X is a right Markov process on I with a finite lifetime $T := \inf\{t > 0 : X_t = \partial\}$, i.e., for all $x \in I$,

$$\mathbb{P}_x(T < \infty) = 1.$$

For such a process, one of the fundamental problems is to study its long-term asymptotic behavior conditional on $\{T > t\}$. A closely related topics is *quasi-ergodic distribution* (see, e.g., [1]), i.e., a probability distribution m on I satisfying that for any $x \in I$ and $A \in \mathcal{B}(I)$,

$$\lim_{t\to\infty} \mathbb{E}_x \left(\frac{1}{t} \int_0^t \mathbf{1}_A(X_s) ds | T > t \right) = m(A),$$

where $\mathbf{1}_A$ denotes the indicator function of the set A.

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The quasi-ergodic distribution is substantially different from the following *quasi-limiting distribution* ρ_0 (see, e.g., [1, 6, 9, 10, 19]), i.e., there exists a probability measure π on I such that for all $A \in \mathcal{B}(I)$,

$$\lim_{t\to\infty} \mathbb{P}_\pi(X_t \in A|T > t) = \rho_0(A).$$

Note that, if ρ_0 is a quasi-limiting distribution, then ρ_0 is a *quasi-stationary distribution* (see, e.g., [12]), i.e., a probability distribution ν on I satisfying that for all $t \geq 0$ and $A \in \mathcal{B}(I)$,

$$\mathbb{P}_\nu(X_t \in A|T > t) = \nu(A).$$

When $T = \infty$, under some suitable conditions, it is well-known that m coincides with ρ_0. When $T < \infty$, it's a little surprising that they have this difference. This difference is worth further study. This paper is a continuation of studying quasi-ergodic distributions for symmetric Markov processes. Our aim of this work is to study the existence of a quasi-ergodic distribution and prove some mixing properties for the symmetric Markov processes.

Recently, the study of quasi-ergodic distributions has received more and more attention. When the absorbing boundary is fixed, the study on quasi-ergodic distribution in a very general framework can be found in [1, 4, 5, 6, 7, 10, 19]. In these works, some works need to assume that the process is λ-positive (see, e.g., [1, 6, 7]). Most of these works assume that the reference measure is a finite measure. If the reference measure is an infinite measure, it is more difficult to prove that the λ-invariant measure is a finite measure, which is the basis for the existence of quasi-stationary distributions and quasi-ergodic distributions. In the present paper, we will consider this case. To be exact, our main results are true whether the reference measure is finite or infinite. When the absorbing boundary is moving, the quasi-stationarity and quasi-ergodicity of discrete-time Markov chains were also studied by Oçafrain [14].

In this work, under suitable assumptions, we prove some mixing properties and provide a conditional ergodic theorem for the symmetric Markov processes. These results can give some interpretation of 'quasi-stationarity' of the quasi-ergodic distribution m. These results also exhibit a *phase transition* due to the limiting distribution changes from $0 < p < 1$ to $p = 1$. Finally, we are committed to studying these results in the framework of one-dimensional diffusions absorbed at 0.

The content of this paper is organized as follows. In Section 2, we present some preliminaries that will be needed in the sequel. Our main results and their proofs are presented in Section 3. In Section 4, we study the case of one-dimensional diffusions taking values in $[0, \infty)$, where 0 is an absorbing regular boundary and $+\infty$ is an entrance boundary.

2 Preliminaries

Before we state the main results of this paper, let us present some preliminaries. We define the semigroup and the resolvent by

$$P_t f(x) = \mathbb{E}_x(f(X_t), t < T), \qquad R_\beta f(x) = \int_0^\infty e^{-\beta t} P_t f(x) dt$$

for all $f \in \mathcal{B}_b(I)$, where $\mathcal{B}_b(I)$ denotes the space of bounded Borel functions on I. Denote by $r(t, x, y)$ the transition density function of the process X. We assume that the process X is symmetric with respect to the reference measure μ, i.e., for $f, g \in \mathcal{B}_b(I)$, $\langle P_t f, g\rangle_\mu = \langle f, P_t g\rangle_\mu$, where the inner product

$$\langle f, g\rangle_\mu := \int_I f(u)g(u)\mu(du).$$

In this paper, we will use the following hypothesis (H).

Definition 2.1. *We say that hypothesis* (H) *holds, if the following three conditions are all satisfied*:

(H1) (**Irreducibility**) *If a Borel set A is P_t-invariant, i.e., for any $f \in \mathbb{L}^2(I,\mu) \cap \mathcal{B}_b(I)$ and $t > 0$, $P_t(\mathbf{1}_A f)(x) = \mathbf{1}_A P_t f(x)$ μ-a.e. then A satisfies either $\mu(A) = 0$ or $\mu(I \setminus A) = 0$.*

(H2) (**Strong Feller property**) *For each t, $P_t(\mathcal{B}_b(I)) \subset C_b(I)$, where $C_b(I)$ is the space of bounded continuous functions on I.*

(H3) (**Tightness**) *For any $\varepsilon > 0$, there exists a compact set K such that*

$$\sup_{x\in I} R_1 \mathbf{1}_{K^c}(x) \le \varepsilon,$$

where K^c denotes the complement of the compact set K.

The hypothesis (H) implies that for any $a > 0$, there exists a compact set K such that

$$\sup_{x\in I} \mathbb{E}_x(e^{a\tau_{K^c}}) < \infty,$$

where τ_{K^c} denotes the first exit time from K^c (see [17]).

Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the Dirichlet form on $\mathbb{L}^2(I,\mu)$ generated by X:

$$\begin{cases} \mathcal{D}(\mathcal{E}) = \left\{ u \in \mathbb{L}^2(I,\mu) : \lim\limits_{t\to 0} \dfrac{1}{t}\langle u - P_t u, u\rangle_\mu < \infty \right\}, \\ \mathcal{E}(u,v) = \lim\limits_{t\to 0} \dfrac{1}{t}\langle u - P_t u, v\rangle_\mu. \end{cases}$$

Let λ be the bottom of spectrum of the infinitesimal operator of the process X, defined by

$$\lambda = \inf\{\mathcal{E}(u,u) : u \in \mathcal{D}(\mathcal{E}), \|u\|_2 = 1\},$$

where $\|\cdot\|_2$ denotes the $\mathbb{L}^2$-norm. Studying the existence of quasi-stationary distributions and quasi-ergodic distributions, it needs the condition that $\lambda > 0$. Under hypothesis (H), from [15, Corollary 3.8], we know that $\lambda > 0$ and for $0 < a < \lambda$, $\sup_{x\in I} \mathbb{E}_x(e^{aT}) < \infty$.

A function $\phi_0(x)$ on I is called a *ground state* of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, if $\phi_0(x) \in \mathcal{D}(\mathcal{E})$, $\|\phi_0\|_2 = 1$ and

$$\mathcal{E}(\phi_0, \phi_0) = \inf\{\mathcal{E}(u,u) : u \in \mathcal{D}(\mathcal{E}), \|u\|_2 = 1\}.$$

The results of this paper are based on the existence of ground states. On the existence of the ground state of the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, from [8, Lemma 6.4.5], we can get the following important result.

Proposition 2.2. ([8, Lemma 6.4.5]) *Assume that hypothesis* (H) *holds. Then, there exists a ground state ϕ_0 of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ uniquely up to sign and ϕ_0 can be taken to be strictly positive on I.*

Under the conditions that hypothesis (H) holds and $\phi_0 \in \mathbb{L}^1(I,\mu) \cap \mathbb{L}^\infty(I,\mu)$, Miura proved that for all $A \in \mathcal{B}(I)$,

$$\lim_{t\to\infty} \mathbb{P}_x(X_t \in A | T > t) = \nu(A), \ \mu\text{-a.e.},$$

where

$$\nu(dy) = \frac{\phi_0(y)\mu(dy)}{\langle \phi_0, 1\rangle_\mu}. \tag{2.1}$$

He has also proved that ν is the unique quasi-stationary distribution of the process X (see [13, Theorem 2.4]). Recently, however, Takeda obtained the following important results.

Proposition 2.3. ([16, Lemma 3.4]) *Assume that hypothesis* (H) *holds. Then, the ground state* $\phi_0 \in \mathbb{L}^1(I,\mu)$.

Proposition 2.4. ([17, Theorem 5.4]) *Assume that hypothesis* (H) *holds. Then, the ground state* ϕ_0 *has a bounded continuous version.*

From the above results, we can see that, under hypothesis (H), $\phi_0 \in \mathbb{L}^1(I,\mu)\cap\mathbb{L}^\infty(I,\mu)$. Only under hypothesis (H), Takeda also showed that ν is the unique quasi-stationary distribution of the process X (see [16, Theorem 3.1]). Inspired by [13] and [16], in this paper, we study quasi-ergodic distributions for the process X, which is substantially different from the above limit distribution.

3 Main results

We will present our main results and their proofs in this section. Note that, our main results are true whether the reference measure is finite or infinite. The following theorem is one of our main results. Under the assumptions that the reference measure is a finite measure and the semigroup $\{P_t\}_{t\geq 0}$ is *ultracontractive*, that is, for any $t>0$, there exists a constant $c_t > 0$ such that

$$r(t,x,y) \leq c_t < \infty \quad \text{for } x,\ y \in I,$$

such a form of theorem has been given in [19, Theorem 3.2]. Compared with [19], we don't need harsh constraints and our approach only uses elementary probability tools and can be easily applied to many other settings.

Theorem 3.1. *Assume that hypothesis* (H) *holds. Then, for any bounded and measurable functions* f,g *on* I *and* $0<p<q<1$, *we have*

(i) $\lim_{t\to\infty} \mathbb{E}_x[f(X_{pt})g(X_t)|T>t] = \int_I f(y)m(dy)\int_I g(y)\nu(dy),\ \mu\text{-a.e.}$,

(ii) $\lim_{t\to\infty} \mathbb{E}_x[f(X_{pt})g(X_{qt})|T>t] = \int_I f(y)m(dy)\int_I g(y)m(dy),\ \mu\text{-a.e.}$,

(iii) $\lim_{t\to\infty} \mathbb{E}_x\left(\frac{1}{t}\int_0^t f(X_s)ds|T>t\right) = \int_I f(y)m(dy),\ \mu\text{-a.e.}$,

where

$$m(dy) = \phi_0^2(y)\mu(dy)$$

and ν *is as in* (2.1).

Proof. (i) Our results depend on the existence of the ground state ϕ_0. Thanks to Proposition 2.2, we know that the ground state ϕ_0 of the Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ exists. Thus, we have $\int_I \phi_0^2(y)\mu(dy) = 1$. Then, m is a probability measure on I. Besides, from Proposition 2.3 and Proposition 2.4, we know that the ground state $\phi_0 \in \mathbb{L}^1(I,\mu)\cap\mathbb{L}^\infty(I,\mu)$.

Next, we first assume that f and g are nonnegative and bounded. According to [13] and [16], we have

$$\lim_{t\to\infty} \frac{e^{\lambda t}P_t g(x)}{\phi_0(x)} = \int_I g(y)\phi_0(y)\mu(dy),\ \mu\text{-a.e.} \tag{3.1}$$

For fixed u, we set

$$h_u(x) = \inf\{e^{\lambda r}P_r g(x)/\phi_0(x) : r \geq u\}.$$

For each u, when $(1-p)t \geq u$, by the Markov property, we obtain

$$\begin{aligned}
\mathbb{E}_x[f(X_{pt})g(X_t)|T>t] &= \frac{\mathbb{E}_x\left[f(X_{pt})g(X_t), T>t\right]}{\mathbb{P}_x(T>t)} \\
&= \frac{e^{\lambda pt}\mathbb{E}_x\left[f(X_{pt})\mathbf{1}_{\{T>pt\}}\cdot e^{\lambda(1-p)t}\mathbb{E}_{X_{pt}}\left(g(X_{(1-p)t})\mathbf{1}_{\{T>(1-p)t\}}\right)\right]}{e^{\lambda t}\mathbb{P}_x(T>t)} \\
&= \frac{e^{\lambda pt}\mathbb{E}_x\left[f(X_{pt})\mathbf{1}_{\{T>pt\}}\cdot e^{\lambda(1-p)t}P_{(1-p)t}\left(g(X_{pt})\right)\right]}{e^{\lambda t}\mathbb{P}_x(T>t)} \\
&\geq \frac{e^{\lambda pt}\mathbb{E}_x\left[f(X_{pt})h_u(X_{pt})\phi_0(X_{pt})\mathbf{1}_{\{T>pt\}}\right]}{e^{\lambda t}\mathbb{P}_x(T>t)} \\
&= \frac{e^{\lambda pt}P_{pt}(fh_u\phi_0)(x)}{e^{\lambda t}P_t\mathbf{1}(x)}.
\end{aligned}$$

For all $r \geq u$, by the definition of $h_u(x)$, we get

$$|f(x)h_u(x)\phi_0(x)| \leq |f(x)e^{\lambda r}P_r g(x)| \leq e^{\lambda r}\|f\|_\infty\|g\|_\infty.$$

Therefore, the function $fh_u\phi_0$ is bounded and measurable. Thus, by (3.1), we obtain

$$\begin{aligned}
\liminf_{t\to\infty}\mathbb{E}_x[f(X_{pt})g(X_t)|T>t] &\geq \lim_{t\to\infty}\frac{e^{\lambda pt}P_{pt}(fh_u\phi_0)(x)}{e^{\lambda t}P_t\mathbf{1}(x)} \\
&= \frac{\int_I f(y)h_u(y)m(dy)}{\langle\phi_0, 1\rangle_\mu}.
\end{aligned}$$

From the definition of $h_u(x)$, we can see that

$$h_u(x)\uparrow\langle\phi_0,1\rangle_\mu\int_I g(y)\nu(dy), \quad \text{as } u\to\infty.$$

Then, letting $u\to\infty$, by the monotone convergence theorem, we obtain

$$\liminf_{t\to\infty}\mathbb{E}_x[f(X_{pt})g(X_t)|T>t] \geq \int_I f(y)m(dy)\cdot\int_I g(y)\nu(dy). \tag{3.2}$$

Conversely, since f and g are bounded, we can repeat the argument, replacing $f(X_{pt})g(X_t)$ by

$$(\|f\|_\infty - f(X_{pt}))(\|g\|_\infty + g(X_t)) \quad \text{and} \quad (\|f\|_\infty + f(X_{pt}))(\|g\|_\infty - g(X_t)),$$

which gives

$$\begin{aligned}
& 2\|f\|_\infty\|g\|_\infty - \limsup_{t\to\infty}\mathbb{E}_x[2f(X_{pt})g(X_t)|T>t] \\
\geq\ & \liminf_{t\to\infty}\mathbb{E}_x[(\|f\|_\infty - f(X_{pt}))(\|g\|_\infty + g(X_t))|T>t] \\
+\ & \liminf_{t\to\infty}\mathbb{E}_x[(\|f\|_\infty + f(X_{pt}))(\|g\|_\infty - g(X_t))|T>t] \\
\geq\ & 2\|f\|_\infty\|g\|_\infty - 2\int_I f(y)m(dy)\cdot\int_I g(y)\nu(dy).
\end{aligned}$$

So, we have

$$\limsup_{t\to\infty}\mathbb{E}_x[f(X_{pt})g(X_t)|T>t] \leq \int_I f(y)m(dy)\cdot\int_I g(y)\nu(dy). \tag{3.3}$$

For nonnegative and bounded functions f and g, by (3.2) and (3.3), we have

$$\lim_{t\to\infty}\mathbb{E}_x[f(X_{pt})g(X_t)|T>t] = \int_I f(y)m(dy)\cdot\int_I g(y)\nu(dy). \tag{3.4}$$

We can extend (3.4) to arbitrary bounded f and g by subtraction. So (i) holds.

(ii) For each u, when $(1-q)t \geq u$ and $(q-p)t \geq u$, by the Markov property, we obtain

$$\begin{aligned}
\mathbb{E}_x[f(X_{pt})g(X_{qt})|T>t] &= \frac{\mathbb{E}_x[f(X_{pt})g(X_{qt}), T>t]}{\mathbb{P}_x(T>t)} \\
&= \frac{\mathbb{E}_x\left[f(X_{pt})g(X_{qt})\mathbf{1}_{\{T>qt\}}\mathbb{P}_{X_{qt}}(T>(1-q)t)\right]}{\mathbb{P}_x(T>t)} \\
&\geq \frac{e^{\lambda(q-1)t}\mathbb{E}_x\left[f(X_{pt})g(X_{qt})\mathbf{1}_{\{T>qt\}}h'_u(X_{qt})\phi_0(X_{qt})\right]}{\mathbb{P}_x(T>t)} \\
&= \frac{e^{\lambda(q-1)t}\mathbb{E}_x\left[f(X_{pt})\rho(X_{qt})\mathbf{1}_{\{T>qt\}}\right]}{\mathbb{P}_x(T>t)} \\
&= \frac{e^{\lambda pt}\mathbb{E}_x\left[f(X_{pt})\mathbf{1}_{\{T>pt\}}\cdot e^{\lambda(q-p)t}\mathbb{E}_{X_{pt}}\left(\rho(X_{(q-p)t})\mathbf{1}_{\{T>(q-p)t\}}\right)\right]}{e^{\lambda t}\mathbb{P}_x(T>t)} \\
&= \frac{e^{\lambda pt}\mathbb{E}_x\left[f(X_{pt})\mathbf{1}_{\{T>pt\}}\cdot e^{\lambda(q-p)t}P_{(q-p)t}\left(\rho(X_{pt})\right)\right]}{e^{\lambda t}\mathbb{P}_x(T>t)} \\
&\geq \frac{e^{\lambda pt}\mathbb{E}_x\left[f(X_{pt})h''_u(X_{pt})\phi_0(X_{pt})\mathbf{1}_{\{T>pt\}}\right]}{e^{\lambda t}\mathbb{P}_x(T>t)} \\
&= \frac{e^{\lambda pt}P_{pt}(fh''_u\phi_0)(x)}{e^{\lambda t}P_t\mathbf{1}(x)},
\end{aligned}$$

where $\rho(x) = g(x)h'_u(x)\phi_0(x)$, and

$$h'_u(x) = \inf\{e^{\lambda r}P_r\mathbf{1}(x)/\phi_0(x) : r \geq u\}, \quad h''_u(x) = \inf\{e^{\lambda r}P_r\rho(x)/\phi_0(x) : r \geq u\}.$$

For all $r \geq u$, by the definition of $h'_u(x)$ and $h''_u(x)$, we get

$$|f(x)h''_u(x)\phi_0(x)| \leq |f(x)e^{\lambda r}P_r\rho(x)| \leq e^{2\lambda r}\|f\|_\infty\|g\|_\infty.$$

Therefore, the function $fh''_u\phi_0$ is bounded and measurable. Moreover, as seen from the proof of (i), we have

$$h''_u(x) \uparrow \langle\phi_0, 1\rangle_\mu \int_I g(y)m(dy), \quad \text{as } u \to \infty.$$

Thus, the proof of (ii) then follows from the same arguments as in the proof of (i).

(iii) By (i) or (ii) and the dominated convergence theorem, we get

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}_x\left(\frac{1}{t}\int_0^t f(X_s)ds|T>t\right) &= \lim_{t\to\infty}\mathbb{E}_x\left(\int_0^1 f(X_{pt})dp|T>t\right) \\
&= \lim_{t\to\infty}\int_0^1 \mathbb{E}_x(f(X_{pt})|T>t)dp \\
&= \int_I f(y)m(dy).
\end{aligned}$$

Hence, there exists a quasi-ergodic distribution for the process X. □

Remark 3.2. Suppose the assumptions of Theorem 3.1 hold and the semigroup $\{P_t\}_{t\geq 0}$ is ultracontractive. Then,"μ-a.e. x" in Theorem 3.1 can be strengthened to "all x". In fact, based on [16, Corollary 2.1], using the same method as in [13, Theorem 2.4], it can prove that for all $x \in I$, the equality (3.1) holds. And then, the conclusion can be established.

As an interesting application of Theorem 3.1, we obtain the following conditional functional weak laws of large numbers. Under an admittedly more restricted setting, such a form of result has appeared in [19, Theorem 3.6]. However, compared with [19], we don't need harsh constraints.

Proposition 3.3. *Assume that hypothesis* (H) *holds. Then, for any bounded and measurable function* f *on* I *and any* $\varepsilon > 0$*, we have*

$$\lim_{t\to\infty} \mathbb{P}_x\left(\left|\frac{1}{t}\int_0^t f(X_s)ds - \int_I f(y)m(dy)\right| \geq \varepsilon | T > t\right) = 0,\ \mu\text{-a.e.}$$

Proof. First note that for any bounded and measurable function f on I, we have

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}_x\left[\left(\frac{1}{t}\int_0^t f(X_s)ds\right)^2 |T > t\right] \\
= &\lim_{t\to\infty} \mathbb{E}_x\left[\left(\int_0^1 f(X_{pt})dp\right)^2 |T > t\right] \\
= &\lim_{t\to\infty} \mathbb{E}_x\left[\int_0^1 f(X_{pt})dp \int_0^1 f(X_{qt})dq|T > t\right] \\
= &\lim_{t\to\infty} \int_0^1\int_0^1 \mathbb{E}_x[f(X_{pt})f(X_{qt})|T > t]dpdq \\
= &\lim_{t\to\infty} \left(\int_0^1\int_0^q \mathbb{E}_x[f(X_{pt})f(X_{qt})|T > t]dpdq + \int_0^1\int_0^p \mathbb{E}_x[f(X_{pt})f(X_{qt})|T > t]dqdp\right) \\
= &\left(\int_I f(y)m(dy)\right)^2.
\end{aligned}$$

Let $h(x) = f(x) - \int_I f(y)m(dy)$. So, by Markov inequality, we have

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{P}_x\left(\left|\frac{1}{t}\int_0^t f(X_s)ds - \int_I f(y)m(dy)\right| \geq \varepsilon | T > t\right) \\
\leq &\lim_{t\to\infty} \frac{\mathbb{E}_x\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \int_I f(y)m(dy)\right|^2 |T > t\right]}{\varepsilon^2} \\
= &\lim_{t\to\infty} \frac{\mathbb{E}_x\left[\left(\frac{1}{t}\int_0^t h(X_s)ds\right)^2 |T > t\right]}{\varepsilon^2} \\
= &\frac{\left(\int_I h(y)m(dy)\right)^2}{\varepsilon^2} \\
= &0.
\end{aligned}$$

This completes the proof of the proposition. □

Next, we remark that if the semigroup $\{P_t\}_{t\geq 0}$ is *intrinsically ultracontractive*, then for any probability measure π on I, Theorem 3.1 is still holds. The semigroup $\{P_t\}_{t\geq 0}$ is said to be intrinsically ultracontractive, if for any $t > 0$, there exist two constants $\alpha_t, c_t > 0$ such that

$$\alpha_t\phi_0(x)\phi_0(y) \leq r(t,x,y) \leq c_t\phi_0(x)\phi_0(y) \quad \text{for } x,\ y \in I. \tag{3.5}$$

Before we give the result, let's first prove the following result.

Proposition 3.4. *Assume that hypothesis* (H) *holds. If the semigroup* $\{P_t\}_{t\geq 0}$ *is intrinsically ultracontractive, then there exist a probability measure* ν *on* I *and two constants* $C, \gamma > 0$ *such that, for all initial distribution* π *on* I,

$$\|\mathbb{P}_\pi(X_t \in \cdot | T > t) - \nu(\cdot)\|_{TV} \leq C e^{-\gamma t}, \quad \forall t \geq 0, \tag{3.6}$$

where $\|\cdot\|_{TV}$ *is the total variation norm.*

Proof. The main method of the proof is similar to that in [2, Theorem 5.1]. The approach is to use the following condition, which is actually equivalent to the exponential convergence (3.6) (see [3, Theorem 2.1])

Condition (A) There exists a probability measure ν_1 on I such that

(A1) there exist $t_0, c_1 > 0$ such that, for all $x \in I$,

$$\mathbb{P}_x(X_{t_0} \in \cdot | T > t_0) \geq c_1 \nu_1(\cdot);$$

(A2) there exists $c_2 > 0$ such that, for all $x \in I$ and $t \geq 0$,

$$\mathbb{P}_{\nu_1}(T > t) \geq c_2 \mathbb{P}_x(T > t).$$

If the semigroup $\{P_t\}_{t\geq 0}$ is intrinsically ultracontractive, then for all $x \in I$, by (3.5), we have

$$\frac{\alpha_t}{c_t}\nu(\cdot) \leq \mathbb{P}_x(X_{t_0} \in \cdot | T > t_0) \leq \frac{c_t}{\alpha_t}\nu(\cdot), \tag{3.7}$$

where ν is as in (2.1). Hence, (A1) holds with $c_1 = \frac{\alpha_t}{c_t}$ and $\nu_1 = \nu$.

If $t \leq t_0$, from (3.5), we have $\mathbb{P}_\pi(T > t) \geq \mathbb{P}_\pi(T > t_0) \geq \alpha_t \pi(\phi_0)\langle \phi_0, 1\rangle_\mu$, where $\pi(\phi_0) := \int_I \phi_0(y)\pi(dy)$. We can adjust the value of $c_t\|\phi_0\|_\infty$ such that for any $x \in I$,

$$\begin{aligned} \mathbb{P}_\pi(T > t) &\geq \alpha_t \pi(\phi_0)\langle \phi_0, 1\rangle_\mu \\ &\geq \frac{\alpha_t \pi(\phi_0)}{c_t\|\phi_0\|_\infty} \\ &\geq \frac{\alpha_t \pi(\phi_0)}{c_t\|\phi_0\|_\infty}\mathbb{P}_x(T > t). \end{aligned}$$

If $t \geq t_0$, from (3.5), we get

$$\begin{aligned} \mathbb{P}_\pi(X_{t_0} \in \cdot) &\geq \alpha_t \pi(\phi_0)\nu(\cdot)\langle \phi_0, 1\rangle_\mu \\ &\geq \frac{\phi_0(x)}{\|\phi_0\|_\infty}\alpha_t \pi(\phi_0)\nu(\cdot)\langle \phi_0, 1\rangle_\mu \\ &\geq \frac{\alpha_t \pi(\phi_0)}{c_t\|\phi_0\|_\infty}\mathbb{P}_x(X_{t_0} \in \cdot). \end{aligned}$$

Thus, by the Markov property, we obtain

$$\begin{aligned} \mathbb{P}_\pi(T > t) &= \mathbb{E}_\pi(\mathbb{P}_{X_{t_0}}(T > t - t_0)) \\ &\geq \frac{\alpha_t \pi(\phi_0)}{c_t\|\phi_0\|_\infty}\mathbb{E}_x(\mathbb{P}_{X_{t_0}}(T > t - t_0)) \\ &= \frac{\alpha_t \pi(\phi_0)}{c_t\|\phi_0\|_\infty}\mathbb{P}_x(T > t). \end{aligned}$$

Therefore, for all $x \in I$ and $t \geq 0$, $\mathbb{P}_\pi(T > t) \geq \frac{\alpha_t \pi(\phi_0)}{c_t\|\phi_0\|_\infty}\mathbb{P}_x(T > t)$. Thus, taking $\pi = \nu_1 = \nu$, this entails (A2) for $c_2 = \frac{\alpha_t \pi(\phi_0)}{c_t\|\phi_0\|_\infty}$. This completes the proof of the proposition. □

As seen from the proof of Theorem 3.1, the proof only uses the fact that the equality (3.1) holds and $\phi_0 \in \mathbb{L}^1(I,\mu)\cap\mathbb{L}^\infty(I,\mu)$. If the semigroup $\{P_t\}_{t\geq 0}$ is intrinsically ultracontractive, we know from Proposition 3.4 or [16, Lemma 3.7] that for any initial distribution π on I,

$$\lim_{t\to\infty}\frac{\int_I e^{\lambda t}P_t g(x)\pi(dx)}{\langle\phi_0,1\rangle_\pi}=\int_I g(y)\phi_0(y)\mu(dy). \tag{3.8}$$

That is to say, for any initial distribution π on I, the equality (3.1) holds. Thus, due to the reason mentioned above, by using a similar argument as in the proof of Theorem 3.1, we have the following result. We point out that Theorem 3.5 complements results of [19] for the reference measure being a finite measure and the semigroup $\{P_t\}_{t\geq 0}$ being ultracontractive.

Theorem 3.5. *Assume that hypothesis* (H) *holds. If the semigroup* $\{P_t\}_{t\geq 0}$ *is intrinsically ultracontractive, then for any bounded and measurable functions* f,g *on* I*, any probability measure* π *on* I *and* $0<p<q<1$*, we have*

(i) $\lim_{t\to\infty}\mathbb{E}_\pi[f(X_{pt})g(X_t)|T>t]=\int_I f(y)m(dy)\int_I g(y)\nu(dy)$,

(ii) $\lim_{t\to\infty}\mathbb{E}_\pi[f(X_{pt})g(X_{qt})|T>t]=\int_I f(y)m(dy)\int_I g(y)m(dy)$,

(iii) $\lim_{t\to\infty}\mathbb{E}_\pi\left(\frac{1}{t}\int_0^t f(X_s)ds|T>t\right)=\int_I f(y)m(dy)$,

where m *and* ν *are as in Theorem* 3.1*.*

4 One-dimensional diffusions

This section is devoted to study the case of one-dimensional diffusions taking values in $[0,\infty)$, where 0 is an absorbing regular boundary and $+\infty$ is an entrance boundary.

Let $Y=(Y_t,t\geq 0)$ be a one-dimensional drifted Brownian motion in $[0,\infty)$ such that 0 is an absorbing boundary. More formally, Y is defined as the solution of the stochastic differential equation

$$dY_t=dB_t-q(Y_t)dt,\qquad Y_0=y>0, \tag{4.1}$$

where $(B_t,t\geq 0)$ is a standard one-dimensional Brownian motion and $q(x)\in C^1[0,\infty)$.

Define $Q(y):=\int_0^y 2q(x)dx$. In this section, we will use the following hypothesis.

Hypothesis (B). $\int_0^\infty e^{Q(y)}\left(\int_y^\infty e^{-Q(z)}dz\right)dy<\infty.$

According to [18], we know that if hypothesis (B) holds, then 0 is an absorbing regular boundary and $+\infty$ is an entrance boundary. And, for all $x\in(0,\infty)$, we have

$$\mathbb{P}_x(T<\infty)=1,$$

where $T=\inf\{t>0:Y_t=0\}$. Define

$$\mu(dy):=e^{-Q(y)}dy. \tag{4.2}$$

Notice that μ is the speed measure of the process Y. From [18], we know that if hypothesis (B) holds, then $\mu(0,\infty)<\infty$.

It is well known (see, e.g., [8]) that the one-dimensional diffusion process Y is symmetric with respect to μ and satisfies (H1) and (H2). When studying quasi-stationary distributions and quasi-ergodic distributions of one-dimensional diffusion processes, it is often necessary to classify the boundary: *regular boundary, exit boundary, entrance boundary and natural boundary.* Following [11, Chapter 5], we know that:

(a) If $+\infty$ is a regular or exit boundary, then $\lim_{x\to+\infty} R_1\mathbf{1}(x) = 0$.

(b) If $+\infty$ is an entrance boundary, then $\lim_{r\to+\infty} \sup_{x\in(0,\infty)} R_1\mathbf{1}_{(r,\infty)}(x) = 0$.

(c) If $+\infty$ is a natural boundary, then $\lim_{x\to+\infty} R_1\mathbf{1}_{(r,\infty)}(x) = 1$ and thus $\sup_{x\in(0,\infty)} R_1\mathbf{1}_{(r,\infty)}(x) = 1$.

Therefore, (H3) is satisfied if and only if no natural boundaries are present. Hence, under hypothesis (B), the process Y satisfies hypothesis (H) and then Theorem 3.1 holds.

We remark that if hypothesis (B) holds, then for all $x \in (0,\infty)$, the equality (3.1) holds (see [18]). Hence, if hypothesis (B) holds, then for the process Y, "μ-$a.e.$ x" in Theorem 3.1 can be strengthened to "all x".

Note that, if hypothesis (B) is satisfied, then we know from the proof of [18, Theorem 4.3] that for any bounded and measurable function g on $(0,\infty)$, and any initial distribution π on $(0,\infty)$,

$$\lim_{t\to\infty} \frac{\int_0^\infty e^{\lambda t} P_t g(x)\pi(dx)}{\int_0^\infty \phi_0(x)\pi(dx)} = \int_0^\infty g(y)\phi_0(y)\mu(dy).$$

This means that for any initial distribution π on $(0,\infty)$, the equality (3.1) holds and then

$$\nu(dy) = \frac{\phi_0(y)\mu(dy)}{\int_0^\infty \phi_0(z)\mu(dz)} \tag{4.3}$$

is the unique quasi-stationary distribution of the process Y. Due to the same reason mentioned in Section 3, by using a similar argument as in the proof of Theorem 3.1, we have the following result.

Theorem 4.1. *Assume that hypothesis* (B) *holds. Then, for any bounded and measurable functions* f, g *on* $(0,\infty)$*, any probability measure* π *on* $(0,\infty)$ *and* $0 < p < q < 1$*, we have*

(i) $\lim_{t\to\infty} \mathbb{E}_\pi[f(Y_{pt})g(Y_t)|T>t] = \int_0^\infty f(y)m(dy) \int_0^\infty g(y)\nu(dy)$,

(ii) $\lim_{t\to\infty} \mathbb{E}_\pi[f(Y_{pt})g(Y_{qt})|T>t] = \int_0^\infty f(y)m(dy) \int_0^\infty g(y)m(dy)$,

(iii) $\lim_{t\to\infty} \mathbb{E}_\pi\left(\frac{1}{t}\int_0^t f(Y_s)ds|T>t\right) = \int_0^\infty f(y)m(dy)$,

where

$$m(dy) = \phi_0^2(y)\mu(dy)$$

and ν *is as in* (4.3).

References

[1] L. A. Breyer and G. O. Roberts. A quasi-ergodic theorem for evanescent processes. *Stochastic Process. Appl.* **84** (1999), 177–186. MR-1719286

[2] N. Champagnat, K. Coulibaly-Pasquier, and D. Villemonais. Criteria for exponential convergence to quasi-stationary distributions and applications to multi-dimensional diffusions. *Séminaire de Probabilités XLIX*, Lecture Notes in Math., vol. 2215, Springer, Cham (2018), 165–182. MR-3837104

[3] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution and Q-process. *Probab. Theory Related Fields* **164** (2016), 243–283. MR-3449390

[4] N. Champagnat and D. Villemonais. Uniform convergence to the Q-process. *Electron. Commun. Probab.* **22**(33) (2017), 1–7. MR-3663104

[5] J. Chen and S. Jian. A deviation inequality and quasi-ergodicity for absorbing Markov processes. *Ann. Mat. Pur. Appl.* **197** (2018), 641–650. MR-3772920

[6] J. Chen, H. Li, and S. Jian. Some limit theorems for absorbing Markov processes. *J. Phys. A: Math. Theor.* **45** (2012), 345003. MR-2960634

[7] D. C. Flaspohler. Quasi-stationary distributions for absorbing continuous-time denumerable Markov chains. *Ann. Inst. Statist. Math.* **26** (1974), 351–356. MR-0368155

[8] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet Forms and Symmetric Markov Processes*, 2nd rev. and ext. ed. Walter de Gruyter, Berlin (2010). MR-2778606

[9] G. He. A note on the quasi-ergodic distribution of one-dimensional diffusions. *C. R. Math. Acad. Sci. Paris* **356** (2018), 967–972. MR-3849083

[10] G. He, H. Zhang, and Y. Zhu. On the quasi-ergodic distribution of absorbing Markov processes. *Statist. Probab. Lett.* **149** (2019), 116–123. MR-3915361

[11] K. Itô. *Essentials of Stochastic Processes.* American Mathematical Society, Providence (2006). MR-2239081

[12] S. Méléard and D. Villemonais. Quasi-stationary distributions and population processes. *Probab. Surv.* **9** (2012), 340–410. MR-2994898

[13] Y. Miura. Ultracontractivity for Markov semigroups and quasi-stationary distributions. *Stoch. Anal. Appl.* **32** (2014), 591–601. MR-3219695

[14] W. Oçafrain. Quasi-stationarity and quasi-ergodicity for discrete-time Markov chains with absorbing boundaries moving periodically. *ALEA, Lat. Am. J. Probab. Math. Stat.* **15** (2018), 429–451. MR-3800481

[15] M. Takeda. A tightness property of a symmetric Markov process and the uniform large deviation principle. *Proc. Am. Math. Soc.* **141** (2013), 4371–4383. MR-3105879

[16] M. Takeda. Existence and uniqueness of quasi-stationary distributions for symmetric Markov processes with tightness property. To appear in *J. Theoret. Probab.* (2019). MR-3828191

[17] M. Takeda. Compactness of symmetric Markov semigroups and boundedness of eigenfunctions. To appear in *Trans. Amer. Math. Soc.* (2019).

[18] H. Zhang and G. He. Domain of attraction of quasi-stationary distribution for one-dimensional diffusions. *Front. Math. China* **11** (2016), 411–421. MR-3473743

[19] J. Zhang, S. Li, and R. Song. Quasi-stationarity and quasi-ergodicity of general Markov processes. *Sci. China Math.* **57** (2014), 2013–2024. MR-3247530

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