Electron. Commun. Probab. **24** (2019), no. 58, 1–15.
https://doi.org/10.1214/19-ECP259
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Discrete harmonic functions in Lipschitz domains

Sami Mustapha[*] Mohamed Sifi[†]

Abstract

We prove the existence and uniqueness of a discrete nonnegative harmonic function for a random walk satisfying finite range, centering and ellipticity conditions, killed when leaving a globally Lipschitz domain in $\mathbb{Z}^d$. Our method is based on a systematic use of comparison arguments and discrete potential-theoretical techniques.

Keywords: random walk in Lipchitz domain; discrete harmonic function; Martin boundary.
AMS MSC 2010: Primary 60G50; 31C35, Secondary 60G40; 30F10.
Submitted to ECP on January 2, 2019, final version accepted on July 25, 2019.

1 Introduction and main results

Random walks conditioned to live in domains $\mathcal{C} \subset \mathbb{Z}^d$ are of growing interest because of the range of their applications in enumerative combinatorics, in probability theory and in harmonic analysis (cf. [7], [9], [11], [17], [18], [30]). Doob h-transforms, where h is harmonic for the random walk, positive within $\mathcal{C}$ and vanishing on its boundary $\partial\mathcal{C}$, are used to perform such conditioning. It is therefore crucial to identify the set of all positive harmonic functions associated with a killed random walk.

General results for homogeneous random walks with non-zero drift killed at the boundary of a half-space or an orthant were obtained in [20], [22], [25]. For random walks with zero drift, only few results are available [6], [11], [19], [30], [31]. The first systematical result was obtained by Raschel, who introduced in [31] a new approach based on the investigation of a functional equation satisfied by the generating function of the values taken by the harmonic function. This approach allows him to establish the existence of positive harmonic functions for random walks with small steps and zero drift killed at the boundary of the quadrant $\mathbb{N}^2$. It should be also mentioned that [31] provides explicit expressions for these harmonic functions.

In a recent work Ignatiouk-Robert [21] investigated the properties of harmonic functions for random walks via ladder heights. Applying her general results to random walks in convex cones she deduced the uniqueness (up to a multiplicative constant) of the harmonic function constructed by Denisov and Wachtel in [11] under some moment condition on the jumps. Alternative constructions of this harmonic function are proposed by Denisov and Wachtel in [12]. These new constructions allow them to remove quite

[*]Institut Mathématiques de Jussieu, Sorbonne Université, Tour 25 5e étage Boite 247. 4, place Jussieu, F-75252 PARIS CEDEX 05, France.
E-mail: sami.mustapha@imj-prg.fr
[†]Université de Tunis El Manar, Faculté des Sciences de Tunis, LR11ES11 Laboratoire d'Analyse Mathématique et Applications LR11ES11. 2092, Tunis, Tunisia.
E-mail: mohamed.sifi@fst.utm.tn

restrictive extendability assumptions imposed in [11]. In [32] Raschel and Tarrago studied the behavior of the Green function for random walks in convex cones which gives the uniqueness of the harmonic function (see also [14]).

Regarding spatially inhomogeneous random walks, the problem is more difficult. Uniqueness of positive harmonic functions for random walks with symmetric spatially inhomogeneous increments, killed at the boundary of a half-space, was established in [28] and more recently in the case of an orthant [8].

The main purpose of the present paper is to extend the results of [8] for the whole class of spatially inhomogeneous centered random walks satisfying finite span and ellipticity conditions and killed when leaving a globally Lipschitz unbounded domain in $\mathbb{Z}^d$.

Consider $\Gamma \subset \mathbb{Z}^d$ a finite subset of $\mathbb{Z}^d$ and let $\pi : \mathbb{Z}^d \times \Gamma \to [0,1]$ such that

$$\sum_{e\in\Gamma} \pi(x,e) = 1, \quad \sum_{e\in\Gamma} \pi(x,e)e = 0; \quad e \in \Gamma,\ x \in \mathbb{Z}^d.$$

Then, we let $\{S(n),\, n \in \mathbb{N}\} = (S_n)_{n\in\mathbb{N}}$ be the Markov chain on $\mathbb{Z}^d$ defined by

$$\mathbb{P}[S_{n+1} = x + e/S_n = x] = \pi(x,e); \quad e \in \Gamma,\ x \in \mathbb{Z}^d,\ n = 0,1,\dots$$

$(S_n)_{n\in\mathbb{N}}$ is a centered random walk with bounded increments which becomes spatially homogeneous if we assume that the probabilities $\pi(x,e)$ are independent of x. We shall assume that the set Γ contains all unit vectors in $\mathbb{Z}^d$, i.e. all the vectors $e_k = (0,\dots,0,1,0,\dots 0) \in \mathbb{Z}^d$, where the 1 is the k-th component. We shall impose to the random walk $(S_n)_{n\in\mathbb{N}}$ to satisfy the following uniform ellipticity condition:

$$\pi(x,e) \geq \alpha, \quad e \in \Gamma,\ x \in \mathbb{Z}^d, \tag{1.1}$$

for some $\alpha > 0$.

We shall denote by:

- $\mathcal{C}$ a globally Lipschitz domain of $\mathbb{Z}^d$ that is, a domain $\mathcal{C} = \mathcal{D} \cap \mathbb{Z}^d$ where

$$\mathcal{D} = \left\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}; x_1 > \varphi(x')\right\}$$

for some Lipschitz function on $\mathbb{R}^{d-1}$ satisfying

$$|\varphi(x') - \varphi(y')| \leq A|x' - y'|, \quad x', y' \in \mathbb{R}^{d-1},$$

for some $A > 0$, where $|.|$ denote the Euclidean norm. We shall assume that $\varphi(0) = 0$.

- τ the first exit time from $\mathcal{C}$, i.e.,

$$\tau = \inf\{n = 0,1,\dots;\ S_n \notin \mathcal{C}\}.$$

- G_x^y, $x, y \in \mathcal{C}$, the Green function defined by

$$G_x^y = \sum_{n\in\mathbb{N}} \mathbb{P}_x(S_n = y, \tau > n).$$

We are interested in positive functions h which are discrete harmonic for the random walk $(S_n)_{n\in\mathbb{N}}$ killed at the boundary of $\mathcal{C}$, i.e. in functions $h : \overline{\mathcal{C}} \to \mathbb{R}_+$ such that:

i) For any $x \in \mathcal{C}$, $h(x) = \sum_{e\in\Gamma} \pi(x,e)h(x+e)$;

ii) If $x \in \partial\mathcal{C}$, then $h(x) = 0$;

iii) If $x \in \mathcal{C}$, then $h(x) > 0$;

where $\overline{\mathcal{C}} = \partial\mathcal{C} \cup \mathcal{C}$. The boundary of a set $A \subset \mathbb{Z}^d$ is defined by

$$\partial A = \{x \in A^c, x = z + e \text{ for some } z \in A \text{ and } e \in \Gamma\}.$$

In terms of the first exit time of the random walk from $\mathcal{C}$, we have that

$$h(x) = \mathbb{E}^x\left(h(S_1), \tau > 1\right), \quad x \in \mathcal{C}.$$

Theorem 1.1. *Let $(S_n)_{n\in\mathbb{N}}$ be a centered random walk satisfying the above finite support and ellipticity conditions. Assume that $\mathcal{C}$ is a globally Lipschitz domain of $\mathbb{Z}^d$. Then, up to a multiplicative constant, there exists a unique positive function, harmonic for the random walk killed at the boundary.*

The previous result has an important consequence on the Martin boundary theory attached to the random walk $(S_n)_{n\in\mathbb{N}}$ killed on the boundary of $\mathcal{C}$. Recall that for a (transient) Markov chain on a countable state space E, the Martin compactification of E is the unique smallest compactification E_M of the discrete set E for which the Martin kernels $y \to k_y^x = G_y^x/G_y^{x_0}$ (where x_0 is a given reference state in E) extend continuously for all $x \in E$. The minimal Martin boundary $\partial_m E_M$ is the set of all those $\gamma \in \partial E_M$ for which the function $x \to k_\gamma^x$ is minimal harmonic. Recall that a harmonic function h is minimal if $0 \le g \le h$ with g harmonic implies $g = ch$ with some $c > 0$. By the Poisson-Martin boundary representation theorem, every nonnegative harmonic function h can be written as

$$h(x) = \int_{\partial_m E_M} k_\gamma^x \mu(d\gamma),$$

for some positive Borel measure μ on $\partial_m E_M$ (cf. [13], [27], [29]).

An immediate consequence of Theorem 1.1 is the following.

Theorem 1.2. *Let $(S_n)_{n\in\mathbb{N}}$ be a centered random walk satisfying the above finite support and ellipticity conditions. Assume that $\mathcal{C}$ is a globally Lipschitz domain of $\mathbb{Z}^d$. Then the minimal Martin boundary of $(S_n)_{n\in\mathbb{N}}$ killed at the boundary of $\mathcal{C}$ is reduced to one point.*

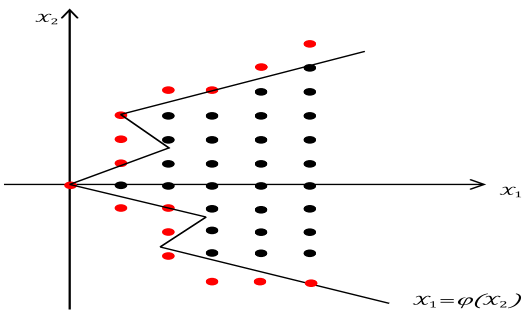

Figure 1: The domain above the graph of a Lipschitz function φ and its boundary with the respect to simple random walk.

A basic example of globally Lipschitz domain is a convex cone $\mathcal{C}$ of $\mathbb{Z}^d$ with vertex 0, contained in the half-space $\{x_1 \ge 0\}$ and containing e_1 as an interior vector. The defining function φ is given in this case by $\varphi(x') = \inf\{x_1 \ge 0, \quad (x_1, x') \in \mathcal{C}\}$.

Lipschitz domains naturally appear in the study of boundary behavior of harmonic functions even in the case of smooth domains. This is well illustrated by Fatou theorem [10] which reduces to studying existence of nontangential boundary values for a bounded (or positive) harmonic function in a union of nontangential circular cones. Such union constitutes a Lipschitz domain.

Beyond cones and union of cones other Lipschitz domains can be naturally considered in probability theory or combinatorics. Consider for example a nondecreasing step function and its corresponding epigraph in $\mathbb{R}^2$. A simple change of axis allows us to transform such domain in a Lipschitz one.

We can also consider a harmonic function globally defined in $\mathbb{Z}^2$, for example

$$u(x_1, x_2) = x_1^4 - 6x_1^2x_2^2 + x_2^4 - x_1^2 - x_2^2,$$

which is harmonic with respect to the simple random walk (i.e., the value of u at any point of $\mathbb{Z}^2$ is equal to the average of the four values at the adjacent points), and look at a connected component of the set of points (x_1, x_2) such that $u(x_1, x_2) > 0$, for example

$$\mathcal{C} = \left\{ (x_1, x_2) \in \mathbb{Z}^2, \quad x_1 > \sqrt{\frac{6x_2^2 + 1 + \sqrt{32x_2^4 + 16x_2^2 + 1}}{2}} \right\}.$$

It is easy to see that $\mathcal{C}$ defines a discrete Lipschitz domain in $\mathbb{Z}^2$. We notice that adding constants allows to produce a whole family of Lipschitz domains whose associated harmonic functions (by Theorem 1.1) have a growth controlled by that of the function u, a fact which is easily verified by a direct application of Theorem 2.3.

A fundamental property of globally Lipschitz domains is that they satisfy both uniform interior and exterior cone conditions. More precisely each point $x \in \partial\mathcal{D}$ is the common vertex of two closed circular cones $\mathcal{C}_+(x)$ and $\mathcal{C}_-(x)$ such that $\mathcal{C}_+(x)\setminus\{x\}$ lies inside $\mathcal{D}$ and $\mathcal{C}_-(x)\setminus\{x\}$ lies inside the complement of $\overline{\mathcal{D}}$. Moreover all these cones are congruent to a fixed circular cone $\mathcal{C}_0 = \{x = (x_1, x'); \quad x_1 \geq \mu|x'|\}$, where $\mu > 0$ is sufficiently small and depends on the Lipschitz constant A.

It should be mentioned that satisfying both interior and exterior cone conditions is not sufficient to characterize globally Lipschitz domains.

To see this, consider in $\mathbb{R}^3$ the open cone obtained by bringing together the four orthants $[x_1 > 0, x_2 < 0, x_3 < 0]$, $[x_1 < 0, x_2 < 0, x_3 < 0]$, $[x_1 < 0, x_2 > 0, x_3 > 0]$ and $[x_1 > 0, x_2 < 0, x_3 < 0]$. The boundary of this cone includes the second and fourth quadrants in the (x_1, x_2)-plan. These quadrants intersect at the origin and the normals for the corresponding two surfaces point in opposite directions. As a consequence, the boundary of this cone cannot be represented as the graph of a Lipschitz function and such cone is not a globally Lipschitz domain.

We conclude this introduction with some comments which may be helpful in placing the results of this paper in their proper perspective.

(i) The proof of Theorem 1.1 given in [8] uses in a crucial way the parabolic Harnack principle. We noted in [8] that a more satisfactory approach should dispense with parabolic information and restrict to elliptic tools. A way to get round the difficulties encountered in [8] is to use a lower estimate for superharmonic extensions of discrete positive harmonic functions derived by Kuo and Trudinger in [23]. This lower estimate encompasses three powerful ingredients: the Aleksandrov-Bakel'man-Pucci's maximum principle, cut-off techniques and a Calderón-Zygmund type lemma. Going trough the superharmonic extension gives an alternative to the use of [8, Lemma 2.5] and provides a purely elliptic derivation of [8, Proposition 2.6]. An advantage of this approach is that it allows us to relax the assumptions $0 \in \Gamma$ and $\Gamma = -\Gamma$ made in [8].

(ii) In case of homogeneous symmetric random walks on unbounded Lipschitz domains, the main results of this paper follow from [19]. Although the work of Gyrya and Saloff-Coste concerns diffusion on Dirichlet spaces in the context of infinite graphs G, to derive the desired results for symmetric random walks, it suffices to consider the corresponding cable process (see [3, §2]). More precisely, each edge can be thought of

as an interval of the real line: a "cable". A metric space consisting of G together with cables, one for each edge can be defined. The distance between two points x and y is defined as follows: if x and y are on the same cable, one uses the Euclidean distance. For points on different cables one uses the distance given by the length of a shortest path between the two points. A diffusion X on the resulting metric space is defined as follows. On the interior of each cable, X performs a linear Brownian motion until it reaches an end point. When X is at a vertex, it makes excursions along each of the cables joining x to other vertices according to the transition probabilities given on G. Such process generalizes Walsh's Brownian motion [33]. Since the harmonic functions for cable process and the random walk on the graph G are essentially the same [3, Lemma 2.4] one has all the desired results (namely Theorem 1.1, Theorem 1.2 and Theorem 2.4).

(iii) Spatially inhomogeneous random walks can be considered as the discrete analogues of diffusions generated by second-order differential operators in nondivergence form. As in [8], the main tools in this paper are discrete versions of Carleson estimate and boundary Harnack inequality (cf. [4], [5], [15], [16]).

(iv) We restrict ourselves in this paper to random walks in Lipschitz domains. However, the proofs given below should work for a larger class of domains, for instance uniform or inner uniform domains (cf. [1]). A domain $\mathcal{D}$ is called uniform if there is a constant A such that each pair of points $x, y \in \mathcal{D}$ can be jointed by a path $\gamma \subset \mathcal{D}$ for which $\ell(\gamma) \le A|x-y|$ and $\min\{\ell(\gamma(x,z), \gamma(y,z))\} \le A\ \mathrm{dist}(z, \partial\mathcal{D})$ for all $z \in \gamma$; where $\ell(\gamma)$ denotes the length of γ and $\gamma(x,z)$ (resp. $\gamma(y,z)$) denotes the subpath of γ that connects x (resp. y) and z. Inner uniform domains are defined by replacing the Euclidean distance by the intrinsic distance associated with $\mathcal{D}$. The main advantage of inner uniformity is that, being an intrinsic notion, it can be used in rather general metric spaces.

2 Proof of Theorem 1.1

2.1 Harnack principle

We say that a function $u : \overline{A} = A \cup \partial A \to \mathbb{R}$ is harmonic in $A \subset \mathbb{Z}^d$ if $Lu = 0$ in A, where L is the difference operator defined by

$$Lu(x) = \sum_{e\in\Gamma} \pi(x,e)u(x+e) - u(x).$$

In addition to an obvious maximum principle, harmonic functions satisfy, when they are positive, a Harnack principle. For convenience this principle is formulated in balls. The discrete Euclidean ball of center $y \in \mathbb{Z}^d$ and radius $R \ge 1$ is denoted $B_R(y)$ and simply B_R when y is clearly understood. We shall also have to use cubes. The cube of center $y \in \mathbb{Z}^d$ and sides $2R$, parallel to the coordinate axes is denoted $Q_R(y)$ and simply Q_R when y is clear. The following theorem (see [23, Eq. (4.21)] and [24, Eq. (3.27)]) is a centered version of Harnack principle established by Lawler [26] for random walks with symmetric bounded increments (as well homogeneous and inhomogeneous); see Figure 2 for an example.

Theorem 2.1 (Harnack principle)**.** *Assume that u is a nonnegative harmonic function associated to a random walk satisfying centering, finite support and uniform ellipticity conditions in a ball $B_{2R}(y)$. Then*

$$\max_{B_R(y)} u \le C \min_{B_R(y)} u,$$

where $C = C(d, \alpha, \Gamma) > 0$.

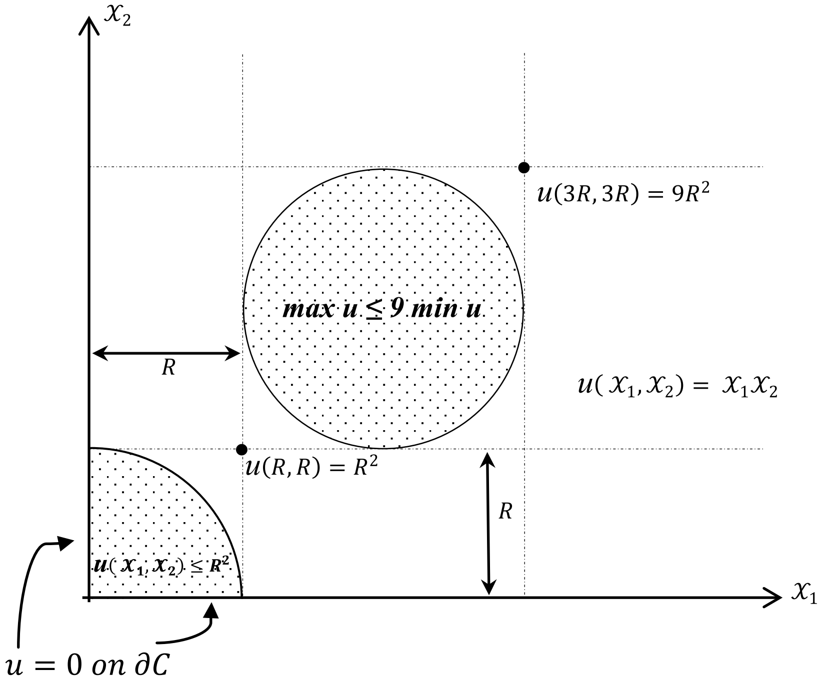

Figure 2: Behavior of the positive harmonic function $u(x_1, x_2) = x_1 x_2$ in the quarter plane.

2.2 Carleson estimate

The classical Carleson estimate asserts that a positive harmonic function vanishing on a portion of the boundary is bounded, up to a smaller portion, by the value at a fixed point in the domain with a multiplicative constant independent of the function. This type of estimate first appears in the paper of Carleson [10, Eq. (4.1)] on Fatou-type theorem for harmonic functions in several variables.

Theorem 2.2. *Assume that u is a nonnegative harmonic function in $\mathcal{C} \cap B_{3R}(y)$. Assume that $u = 0$ on $\partial\mathcal{C} \cap B_{2R}(y)$. Then*

$$\max\{u(x),\ x \in \mathcal{C} \cap B_R(y)\} \leq C\, u(y + Re_1), \quad R \geq C, \tag{2.1}$$

where $C = C(d, \alpha, \Gamma, A) > 0$ is independent of y, R and u and $e_1 = (1, 0, \ldots, 0)$.

The proof of Theorem 2.2 relies on the following Proposition.

Proposition 2.3. *Let $y \in \partial\mathcal{C}$ and R large enough. Let u be a nonnegative harmonic function in $\mathcal{C} \cap B_{3\sqrt{d}R}(y)$ which vanishes on $\partial\mathcal{C} \cap B_{2\sqrt{d}R}(y)$. Then*

$$\max\left\{u(x),\ x \in \overline{\mathcal{C} \cap B_R(y)}\right\} \leq \rho \max\left\{u(x),\ x \in \overline{\mathcal{C} \cap B_{2\sqrt{d}R}(y)}\right\}, \tag{2.2}$$

with a constant $0 < \rho = \rho(d, \alpha, \Gamma, A) < 1$.

The proof of Proposition 2.3 that we give here differs from that of the analogous proposition in [8]. It is interesting to compare with [8, Proposition 2.6] to see that (2.2) is equivalent to a lower estimate

$$\mathbb{P}_x\left[S\left(\tau_{\mathcal{C} \cap B_{2\sqrt{d}R}(y)}\right) \in \partial\mathcal{C} \cap B_{2\sqrt{d}R}(y)\right] \geq 1 - \rho, \quad x \in \mathcal{C} \cap B_R(y),$$

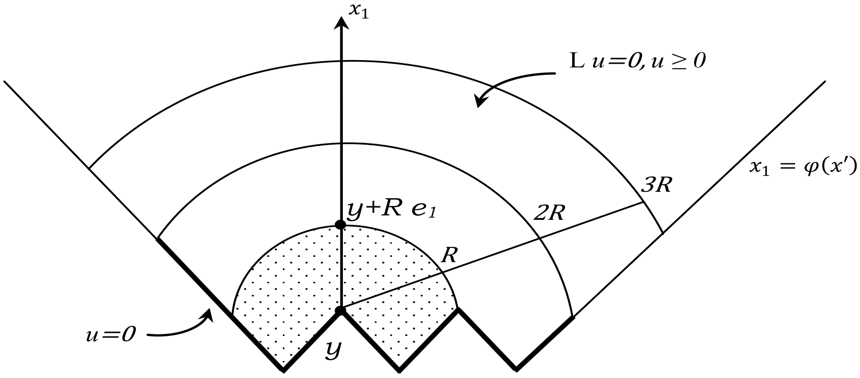

Figure 3: Carleson estimate. The harmonic function u is dominated in the shaded region by its value at $y + R\,e_1$.

where $\tau_{\mathcal{C}\cap B_{2\sqrt{d}R}(y)}$ denotes the exit time from $\mathcal{C} \cap B_{2\sqrt{d}R}(y)$. Another way to reformulate the oscillations estimate of Proposition 2.3 is to say that $\max\left\{u(x),\ x \in \overline{\mathcal{C} \cap B_R(y)}\right\}$ decreases polynomially, i.e., for $r < R$,

$$\max\left\{u(x),\ x \in \overline{\mathcal{C} \cap B_r(y)}\right\} \leq \left(\frac{r}{R}\right)^C \max\left\{u(x),\ x \in \overline{\mathcal{C} \cap B_R(y)}\right\},$$

which constitutes a counterpart to Harnack principle.

Proof. To prove (2.2) we first observe that it suffices to show that

$$\max\left\{u(x),\ x \in \overline{\mathcal{C} \cap Q_R(y)}\right\} \leq \rho \max\left\{u(x),\ x \in \overline{\mathcal{C} \cap Q_{2R}(y)}\right\}. \tag{2.3}$$

Without loss of generality, we assume $y = 0$ and $\max\left\{u(x),\ x \in \overline{\mathcal{C} \cap Q_{2R}}\right\} = 1$. Then considering the function $v : \overline{Q_{2R}} \to \mathbb{R}$ defined by $v = 1 - u$ in $\overline{\mathcal{C} \cap Q_{2R}}$ and $v = 1$ on $\overline{Q_{2R}} \setminus \mathcal{C}$, we see that (2.3) reduces to the following lower estimate

$$v(x) \geq \lambda = \lambda(d, \alpha, \Gamma, A) > 0, \quad x \in \overline{Q_R}. \tag{2.4}$$

In fact, $Lv = 0$ in $\mathcal{C} \cap Q_{2R}$ and for all $x \in Q_{2R} \setminus \mathcal{C}$

$$v(x) = 1 = \sum_{e \in \Gamma} \pi(x, e) \geq \sum_{e \in \Gamma} \pi(x, e) v(x + e)$$

which shows that $Lv \leq 0$ in Q_{2R}, that is v is superharmonic in Q_R, and we can use [24, Eq. (3.24)] and we deduce that

$$\min_{\overline{Q_R}} v \geq \gamma \left(\frac{\left|\overline{Q_R} \cap \{v \geq 1\}\right|}{|\overline{Q_R}|}\right)^{\frac{\log \gamma}{\log \kappa}} \tag{2.5}$$

where $0 < \gamma, \kappa < 1$ are two positive constants depending on d, α and Γ and where the notation $|S|$ is used to denote the cardinality of a subset $S \subset \mathbb{Z}^d$.

The fact that the same constant γ appears in the prefactor and the exponent of the RHS of (2.5) is a consequence of the cube decomposition argument used by Kuo and Trudinger in the proof of [24, Eq. (3.24)] but has no importance for the proof of (2.4).

The estimate (2.5) says that if we have a superharmonic function that goes above 1 on a non trivial portion of Q_R (in the sense that the cardinality of the set where $v \geq 1$ is larger than a fraction of the cardinality of Q_R) then the minimum of v on Q_R is bounded below by some constant.

On the other hand, the fact that $\mathcal{C}$ is Lipschitz allows us to find a circular cone $\mathcal{C}'$ with vertex at the origin such that $\mathcal{C}' \subset \mathcal{C}^c$. It follows then that there exists a positive constant μ (depending on A) such that for R large enough

$$\left|\overline{Q_R} \cap \{v \geq 1\}\right| \geq \left|\overline{Q_R} \cap \mathcal{C}'\right| \geq \mu|\overline{Q_R}|. \tag{2.6}$$

We conclude from (2.5) and (2.6) that

$$\min_{\overline{Q_R}} v \geq \gamma^{1+\frac{\log \mu}{\log \delta}},$$

which implies (2.4) and completes the proof of (2.3). □

Proof of Theorem 2.2. To prove the Carleson estimate (2.1) we first observe that the uniform ellipticity assumption implies that $u(\xi) \leq Ce^{C\,|\xi-\zeta|}u(\zeta)$, $\xi, \zeta \in \mathcal{C} \cap B_{3R}(y)$; where $C = C(d, \alpha, \Gamma, A) > 0$. This local Harnack principle allows us to assume that the distance of x from $\partial\mathcal{C}$ is sufficiently large. We shall denote by $\delta(x)$ ($x \in \mathcal{C} \cap B_{2R}(y)$) this distance and suppose that $\delta(x) \geq C$. The fact that $\mathcal{C}$ is Lipschitz combined with a Harnack chain argument based on Theorem 2.1 imply that

$$u(x) \leq C \left(\frac{R}{\delta(x)}\right)^{\gamma} u(y + Re_1), \quad x \in \mathcal{C} \cap B_{2R}(y), \tag{2.7}$$

where γ and C are positive constants depending on d, α, Γ and A. More precisely one can link the two points x and $y + Re_1$ by a sequence of balls intersecting successively and transferring the control from the point at the end of the sequence $y + Re_1$ to x. Thanks to the fact that the domain is Lipschitz the total number of balls needed is bounded above by $C\log(\dfrac{R}{\delta(x)})$.

Let $x \in \mathcal{C} \cap B_{2R}(y)$, and let us assume that

$$\delta(x) < \left(1 - \left(\frac{1+\rho}{2}\right)^{\frac{1}{\gamma}}\right) \frac{2R - |x-y|}{8\sqrt{d}} \tag{2.8}$$

where ρ is the constant obtained in (2.2) and γ the exponent that appears in (2.7). Let $x_0 \in \partial\mathcal{C}$ such that $|x - x_0| = \delta(x)$. It follows easily from (2.8) and the fact that $\delta(x)$ is sufficiently large that $\overline{B_{3\sqrt{d}\delta(x)}(x_0)} \subset B_{2R}(y)$. By Proposition 2.3 applied to the harmonic function u in the domain $B_{3\sqrt{d}\delta(x)}(x_0) \cap \mathcal{C}$, we have

$$u(x) \leq \max\left\{u(x'),\ \ x' \in \mathcal{C} \cap \overline{B_{\frac{3}{2}\delta(x)}(x_0)}\right\} \leq \rho \max\left\{u(x'),\ \ x' \in \mathcal{C} \cap \overline{B_{3\sqrt{d}\delta(x)}(x_0)}\right\}. \tag{2.9}$$

Let $z \in \mathcal{C} \cap \overline{B_{3\sqrt{d}\delta(x)}(x_0)}$ satisfying

$$u(z) = \max\left\{u(x'),\ \ x' \in \mathcal{C} \cap \overline{B_{3\sqrt{d}\delta(x)}(x_0)}\right\}.$$

We have

$$(2R - |x - y|) \leq (2R - |z - y|) + 8\sqrt{d}\delta(x).$$

Hence, thanks to (2.8)

$$(2R - |x - y|) \leq \left(\frac{1+\rho}{2}\right)^{-\frac{1}{\gamma}} (2R - |z - y|).$$

It follows that

$$(2R - |x-y|)^\gamma u(x) \le \left(\frac{1+\rho}{2}\right)^{-1} (2R - |z-y|)^\gamma u(x),$$

and therefore, by (2.9)

$$\begin{aligned}(2R - |x-y|)^\gamma u(x) &\le \frac{2\rho}{1+\rho}(2R - |z-y|)^\gamma u(z) \\ &\le \theta_0 \max_{x' \in \mathcal{C}\cap B_{2R}(y)} (2R - |x'-y|)^\gamma u(x')\end{aligned} \tag{2.10}$$

where

$$\theta_0 = \frac{2\rho}{1+\rho} < 1.$$

It remains to consider the case where

$$\delta(x) \ge \left(1 - \left(\frac{1+\rho}{2}\right)^{\frac{1}{\gamma}}\right) \frac{2R - |x-y|}{8\sqrt{d}}. \tag{2.11}$$

In follows from (2.11) that

$$(2R - |x-y|)^\gamma u(x) \le \varepsilon_0^{-\gamma} \delta(x)^\gamma u(x) \le \varepsilon_0^{-\gamma} \max_{x' \in \mathcal{C}\cap B_{2R}(y)} \delta(x')^\gamma u(x')$$

where $8\sqrt{d}\varepsilon_0 = 1 - \left(\frac{1+\rho}{2}\right)^{\frac{1}{\gamma}}$ and, thanks to (2.7),

$$(2R - |x-y|)^\gamma u(x) \le C\,\varepsilon_0^{-\gamma} R^\gamma u(y + Re_1). \tag{2.12}$$

Putting together (2.10) and (2.12) and taking the supremum over $\mathcal{C} \cap B_{2R}(y)$, we deduce that

$$\max_{x \in \mathcal{C}\cap B_{2R}(y)} (2R - |x-y|)^\gamma u(x) \le \theta_0 \max_{x \in \mathcal{C}\cap B_{2R}(y)} (2R - |x-y|)^\gamma u(x) + C\,\varepsilon_0^{-\gamma} R^\gamma u(y + Re_1).$$

Using the fact that $(2R - |x-y|) \ge R$ for $x \in \mathcal{C} \cap B_R(y)$ we deduce the estimate (2.1). □

2.3 Boundary Harnack principle

Carleson estimate can be extended to the ratio u/v of positive harmonic functions.

Theorem 2.4 (Boundary Harnack principle)**.** *Let $y \in \partial\mathcal{C}$ and $K > 0$ large enough. Assume that u and v are two nonnegative harmonic functions in $\mathcal{C} \cap B_{3KR}(y)$. Assume that $u,\, v = 0$ on $\partial\mathcal{C} \cap B_{2KR}(y)$. Then*

$$\max_{x \in \mathcal{C}\cap B_R(y)} \frac{u(x)}{v(x)} \le C\,\frac{u(y+Re_1)}{v(y+Re_1)}, \quad R \ge C, \tag{2.13}$$

where $C = C(d, \alpha, \Gamma, A, K) > 0$.

The above formulation of the boundary Harnack principle follows the classical formulation but the proof of (2.13) which will be given below shows that the assumption $v = 0$ on $\partial\mathcal{C} \cap B_{2KR}(y)$ is not needed so that (2.1) constitutes a special case of (2.13).

The estimate (2.13) is an immediate consequence of the lower estimate contained in the following lemma.

For $y \in \partial\mathcal{C}$ and $R \ge r \ge 1$, we shall denote by

$$\begin{aligned}\mathcal{D}_{R,r}(y) &= B_R(y) \cap \{x \in \mathcal{C}, \delta(x) > r\}; \\ \mathcal{C}_{R,r}(y) &= (B_R(y) \cap \mathcal{C}) \setminus \mathcal{D}_{R,r}(y).\end{aligned}$$

For $R \ge r \ge 1$, the boundary of $\mathcal{C}_{R,r}$ is the union of three sets: the "bottom" $\partial\mathcal{C}_{R,r} \cap \mathcal{C}^c$, the "lateral side" $\partial\mathcal{C}_{R,r} \cap \{x \in \mathcal{C},\ 0 \le \delta(x) \le r\}$ and the "top" $\partial\mathcal{C}_{R,r} \cap \mathcal{D}_{R,r}$.

Lemma 2.5. *There exists a constant $K_0 > 0$ such that for all $K \geq K_0$ and for all $y \in \partial\mathcal{C}$, $r \geq 1$,*

$$\min_{x \in \mathcal{C} \cap B_r(y)} \frac{\mathbb{P}_x\left[S(\tau_{\mathcal{C}_{Kr,r}(y)}) \in \mathcal{D}_{Kr,r}(y)\right]}{\mathbb{P}_x\left[S(\tau_{\mathcal{C}_{Kr,r}(y)}) \in (\partial\mathcal{C}_{Kr,r}(y) \cap \mathcal{C}) \setminus \mathcal{D}_{Kr,r}(y)\right]} \geq 1 \tag{2.14}$$

where $\tau_{\mathcal{C}_{Kr,r}(y)}$ denotes the exit time from $\mathcal{C}_{Kr,r}(y)$.

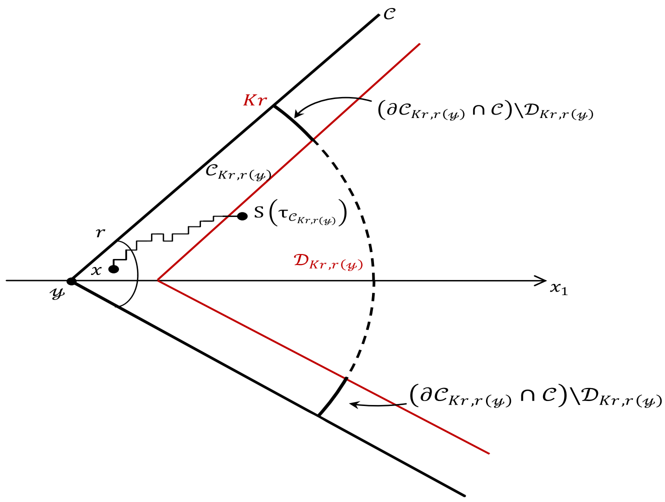

Figure 4: For K large enough and $x \in B_r(y)$ the probability $\mathbb{P}_x\left[S\left(\tau_{\mathcal{C}_{Kr,r}(y)}\right) \in \mathcal{D}_{Kr,r}(y)\right]$ is larger than $\mathbb{P}\left[S\left(\tau_{\mathcal{C}_{Kr,r}(y)}\right) \in (\partial\mathcal{C}_{Kr,r}(y) \cap \mathcal{C}) \setminus \mathcal{D}_{Kr,r}(y)\right]$.

Proof of Theorem 2.4. In order to derive estimate (2.13) from (2.14), we first observe that it is always possible to assume that $u(y + Re_1) = v(y + Re_1) = 1$. For a large R, Carleson estimate (2.1) combined with Harnack principle imply that the function u is dominated by a positive constant c_0 in the region $B_{KR}(y) \cap \mathcal{C}$. This constant c_0 can be chosen so that by Harnack principle the lower estimate $v \geq \frac{1}{c_0}$ holds on $\mathcal{D}_{KR,R}(y)$. Let $v_0 = c_0 v$ and $u_0 = \dfrac{u}{c_0} - v_0$. Let $x \in B_R(y) \cap \mathcal{C}$. We have:

$$\begin{aligned} u_0(x) &\leq \mathbb{P}_x\left[S(\tau_{\mathcal{C}_{KR,R}(y)}) \in (\partial\mathcal{C}_{KR,R}(y) \cap \mathcal{C}) \setminus \mathcal{D}_{KR,R}(y)\right] \\ &\leq \mathbb{P}_x\left[S(\tau_{\mathcal{C}_{KR,R}(y)}) \in \mathcal{D}_{KR,R}(y)\right] \\ &\leq v_0(x), \end{aligned}$$

where the second inequality follows from (2.14). We deduce then that

$$\frac{u(x)}{c_0} - c_0 v(x) \leq c_0 v(x), \quad x \in B_R(y) \cap \mathcal{C}.$$

So that

$$\frac{u(x)}{v(x)} \leq 2c_0^2, \quad x \in B_R(y) \cap \mathcal{C},$$

which completes the proof of (2.13). □

Proof of Lemma 2.5. To prove estimate (2.14) it suffices to show that if u, $v : \overline{\mathcal{C}_{Kr,r}(y)} \to$

$\mathbb{R}$ (where $y \in \partial\mathcal{C}$, $r \geq 1$ are fixed) satisfy

$$\begin{cases} u(x) = \sum_{e\in\Gamma} \pi(x,e)u(x+e), & x \in \mathcal{C}_{Kr,r}(y) \\ u(x) \geq 0 & \text{in} \quad \overline{\mathcal{C}_{Kr,r}(y)} \\ u(x) \geq 1 & \text{on} \quad \partial\mathcal{C}_{Kr,r}(y) \cap \mathcal{D}_{Kr,r}(y) \end{cases} \tag{2.15}$$

$$\begin{cases} v(x) = \sum_{e\in\Gamma} \pi(x,e)v(x+e), & x \in \mathcal{C}_{Kr,r}(y) \\ v(x) \leq 1 & \text{in} \quad \overline{\mathcal{C}_{Kr,r}(y)} \\ v(x) \leq 0 & \text{on} \quad \partial\mathcal{C}_{Kr,r}(y) \cap \mathcal{D}_{Kr,r}(y) \end{cases} \tag{2.16}$$

then we have

$$v(x) \leq u(x), \quad x \in B_r(y) \cap \mathcal{C} \tag{2.17}$$

provided that $K \geq K_0$ is large enough.

First we prove that under (2.15) the function u satisfies

$$u(x) \geq 2\alpha \left(\frac{\delta(x)}{r}\right)^{\beta}, \quad x \in B_r(y) \cap \mathcal{C}, \tag{2.18}$$

for appropriate constants α, $\beta > 0$.

The proof of (2.18) relies on the following construction.

We assume K large enough and we define $\tilde{u} : \overline{B_{Mr}(y) \cap \mathcal{C}} \longrightarrow \mathbb{R}$ by

$$\begin{cases} \tilde{u}(x) = \sum_{e\in\Gamma} \pi(x,e)\tilde{u}(x+e), & x \in B_{Mr}(y) \cap \mathcal{C} \\ \tilde{u} = \min(u,1) & \text{on} \quad \partial\left(B_{Mr}(y) \cap \mathcal{C}\right) \setminus \mathcal{D}_{Kr,r}(y) \\ \tilde{u} = 1 & \text{on} \quad \partial\left(B_{Mr}(y) \cap \mathcal{C}\right) \cap \mathcal{D}_{Kr,r}(y) \end{cases}$$

where $0 < M \leq K$ is chosen so that $\tilde{y} = y + (M-1)re_1$ satisfies

$$\delta(\tilde{y}) \geq 10r.$$

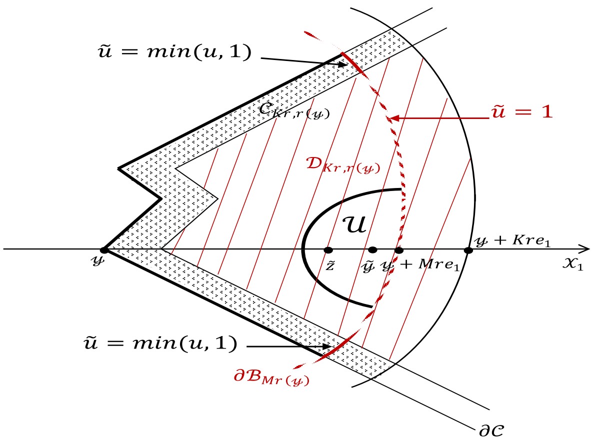

Figure 5: The boundary values of $\tilde{u}$ on $\partial\big(B_{Mr}(y) \cap \mathcal{C}\big)$.

Let $\mathcal{U} = B_{Mr}(y) \cap B_{2r}(\tilde{y})$ and $w : \overline{B_{2r}(\tilde{y})} \longrightarrow \mathbb{R}$ be defined by

$$w(x) = \tilde{u}(x), \quad x \in \overline{\mathcal{U}}; \; w(x) = 1, \quad x \in \mathcal{U}^c \cap \overline{B_{2r}(\tilde{y})}.$$

It is easy to see that w is superharmonic. Let $\tilde{z} = y + (M-2)re_1$. By the same argument used in the proof of the lower estimate (2.4) combined with Harnack principle, we see that $w(\tilde{z})$ satisfies a lower estimate $w(\tilde{z}) \geq c$. It follows then that $\tilde{u}(\tilde{z}) = \tilde{w}(\tilde{z}) \geq c$.

Since $u \geq 1$ on $\partial\mathcal{C}_{Kr,r}(y) \cap \mathcal{D}_{Kr,r}(y)$, we deduce by the maximum principle that $u \geq \tilde{u}$ on $B_r(y) \cap \mathcal{C}$. It follows that $u(\tilde{z}) \geq c$. Using a chain of appropriate balls connecting $\tilde{z}$ to $x \in B_r(y) \cap \mathcal{C}$ and applying Harnack inequality successively in each ball we deduce (2.18).

It follows from (2.18) that if $x \in \overline{\mathcal{C}_{r,r}(y)} \setminus \mathcal{C}_{r,r/K}(y)$ then we have

$$u(x) \geq 2\alpha K^{-\beta}. \tag{2.19}$$

Let us now prove that there exists $N > 0$ such that

$$v(x) \leq e^{-NK}, \quad x \in \overline{\mathcal{C}_{r,r}(y)}. \tag{2.20}$$

Let $j = 1, \ldots, \lfloor \frac{K-1}{2} \rfloor$ and let $x_j \in \partial\mathcal{C}_{(2j-1)r,r}(y)$ be such that

$$v(x_j) = \max\left\{v(x), \quad x \in \overline{\mathcal{C}_{(2j-1)r,r}(y)}\right\}.$$

Let $\mathcal{U}_j = B_{2r}(x_j) \cap \mathcal{C}_{(2j+1)r,r}(y)$ and $\tau_{\mathcal{U}_j}$ be the exit time from $\mathcal{U}_j$. By the same argument used in the proof of (2.4) we see that

$$\mathbb{P}_{x_j}\left[S(\tau_{\mathcal{U}_j}) \in \mathcal{D}_{Kr,r}(y)\right] \geq c > 0.$$

Using (2.16) (in particular, the fact that $v \leq 0$ on $\partial\mathcal{C}_{Kr,r}(y) \cap \mathcal{D}_{Kr,r}(y)$) we deduce then that

$$v(x_j) \leq \theta \max_{\overline{\mathcal{U}_j}} v,$$

where $0 < \theta < 1$. Hence

$$\max\left\{v(x), \quad x \in \overline{\mathcal{C}_{(2j-1)r,r}(y)}\right\} \leq \theta \max\left\{v(x), \quad x \in \overline{\mathcal{C}_{(2j+1)r,r}(y)}\right\}.$$

Iterating this estimate we obtain

$$\max\left\{v(x), \quad x \in \overline{\mathcal{C}_{r,r}(y)}\right\} \leq \theta^{\lfloor \frac{K-1}{2} \rfloor} \max\left\{v(x), \quad x \in \overline{\mathcal{C}_{Kr,r}(y)}\right\} \leq e^{-NK},$$

which proves (2.20). It follows from (2.20) that

$$v \leq \alpha K^{-\beta} \quad \text{in} \quad \overline{\mathcal{C}_{r,r}(y)} \tag{2.21}$$

provided that K is large enough.

From the previous considerations it follows that

$$u_1 = \frac{K^\beta}{2\alpha} u \geq 0 \quad \text{in} \quad \overline{\mathcal{C}_{r,r/K}(y)}$$

with

$$u_1 \geq 1 \quad \text{in} \quad \partial\mathcal{C}_{r,r/K}(y) \cap \mathcal{D}_{r,r/K}(y)$$

thanks to (2.19) and, thanks to (2.21),

$$v_1 = \frac{K^\beta}{2\alpha}(2v - u) \leq \frac{K^\beta}{\alpha} v \leq 1 \quad \text{in} \quad \overline{\mathcal{C}_{r,r/K}(y)},$$

with

$$v_1 \leq 0 \quad \text{on} \quad \partial\mathcal{C}_{r,r/K}(y) \cap \mathcal{D}_{r,r/K}(y).$$

In particular, we have

$$u_1 - v_1 = \frac{K^\beta}{\alpha}(u - v) \geq 0 \quad \text{on} \quad \overline{\mathcal{C}_{r,r}(y) \setminus \mathcal{C}_{r,r/K}(y)}.$$

It follows that u_1, v_1 satisfy the same assumptions as u, v with r replaced by r/K. We can then iterate the construction at different scales and define u_i, v_i such that

$$u_i - v_i = \left(\frac{K^\beta}{\alpha}\right)^i (u-v) \geq 0 \quad \text{on} \quad \overline{\mathcal{C}_{r/K^i,r/K^i}(y) \setminus \mathcal{C}_{r/K^i,r/K^{i+1}}(y)}$$

$i = 1, 2 \ldots$. We deduce then that

$$u - v \geq 0 \quad \text{on} \quad S(y) = \bigcup_{i \geq 0} \overline{\mathcal{C}_{r/K^i,r/K^i}(y) \setminus \mathcal{C}_{r/K^i,r/K^{i+1}}(y)}.$$

Let $x \in B_r(y)$ and $\tilde{x} \in \partial\mathcal{C}$ satisfying $\delta(x) = |x - \tilde{x}|$. Then

$$\mathcal{C}_{Kr,r}(\tilde{x}) \subset \mathcal{C}_{(K+2)r,r}(y).$$

Replacing K by $K+2$ in the previous considerations and repeating the argument we deduce that $u \geq v$ on $S(\tilde{x})$ that contains x. This shows that $u(x) \geq v(x)$ and completes the proof of (2.17). □

Proof of Theorem 1.1. Set

$$v_l(y) = \frac{G^y_{e_1}(l+1) - G^y_{e_1}(l)}{G^{e_1}_{e_1}(l+1) - G^{e_1}_{e_1}(l)}, \quad y \in B_{2^l}(0) \cap \mathcal{C}, \quad l = 1, 2, \ldots,$$

where $G^y_x(l)$ is the Green function of $(S_n)_{n\in\mathbb{N}}$ in $B_{2^l}(0) \cap \mathcal{C}$, $l = 1, 2, \ldots$. It is easy to see that the estimate

$$u(\xi) \leq Ce^{C|\xi-\zeta|}u(\zeta), \quad \xi, \zeta \in \mathcal{C},$$

implies that v_l satisfies

$$v_l(y) \leq C, \quad y \in B_{2^k}(0) \cap \mathcal{C}, \quad l \geq k+1,$$

with a constant C depending only on k. The diagonal process then allows us to deduce the existence of a positive harmonic function defined globally on $\mathcal{C}$ and vanishing on $\partial\mathcal{C}$.

As in [8] the uniqueness can be deduced by essentially the same method as in [1, Proof of Theorem 3] and [2, Lemma 6.2]. □

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Acknowledgments. The authors would like to thank the referee for the exceptional reading effort that he has agreed to make twice and for all the time he has spent with our manuscript.

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