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High minima of non-smooth Gaussian processes*

Zhixin Wu† Arijit Chakrabarty‡ Gennady Samorodnitsky§

Abstract

In this short note we study the asymptotic behaviour of the minima over compact intervals of Gaussian processes, whose paths are not necessarily smooth. We show that, beyond the logarithmic large deviation Gaussian estimates, this problem is closely related to the classical small-ball problem. Under certain conditions we estimate the term describing the correction to the large deviation behaviour. In addition, the asymptotic distribution of the location of the minimum, conditionally on the minimum exceeding a high threshold, is also studied.

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1 Introduction

Let $\mathbf{X} = \big(X(t),\, t \in \mathbb{R}\big)$ be a centered Gaussian process with continuous sample paths. For a compact subinterval $[a,b]$ of the real line we are interested in the right tail of the random variable $\min_{a\le t\le b} X(t)$. This is a complicated object; see e.g. [4] and [1]. On the logarithmic scale, however, this tail can be described as follows:

$$\lim_{u\to\infty} \frac{1}{u^2} \log P\Big(\min_{a\le t\le b} X(t) > u\Big) = -\frac{1}{2\sigma_*^2(a,b)}\,, \tag{1.1}$$

where

$$\sigma_*^2(a,b) = \min_{\nu\in M_1[a,b]} \int_{[a,b]} \int_{[a,b]} R_{\mathbf{X}}(s,t)\,\nu(ds)\,\nu(dt)\,, \tag{1.2}$$

with $R_{\mathbf{X}}$ the covariance function of the process and $M_1[a,b]$ the set of all Borel probability measures ν on $[a,b]$; see Theorem 5.1 in [1]. The quantity in (1.2) is strictly positive whenever the tail probability in (1.1) is strictly positive for $u=0$. In order to obtain more precise results on the right tail of the minimum than (1.1), additional assumptions on the process $\mathbf{X}$, in addition to its continuity, are needed. In [3] such additional assumptions guarantee that the process $\mathbf{X}$ is very smooth. Under these assumptions the optimization

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†Shanghai Jiatong University, China. E-mail: wuzhixin@sjtu.edu.cn
‡Indian Statistical Institute, Kolkata. E-mail: arijit.isi@gmail.com
§Cornell University, United States of America. E-mail: gs18@cornell.edu

problem (1.2) has a unique optimal solution, a probability measure ν_* whose support is a finite set. If k is the cardinality of that set, then (under a non-degeneracy assumption),

$$P\big(\min_{a\le t\le b} X(t) > u\big) \sim cu^{-k}\exp\left\{-\frac{1}{2\sigma_*^2(a,b)}u^2\right\} \tag{1.3}$$

for some $c \in (0,\infty)$.

Our goal in this paper is to obtain results on the asymptotics of the right tail of the Gaussian minimum, more precise than the logarithmic asymptotics (1.1), when the process $\mathbf{X}$ is not so smooth as to satisfy the assumptions of [3] (and, hence, also (1.3)). Such more precise asymptotics are, clearly, related to the support of the optimal measure in (1.2), so the next Section 2 describes certain situations where information on the optimal measure or, at least, on its support, is available. The more precise asymptotic results on the tail of the minima are in Section 3; the results are the most precise in the Markovian case. In Section 4 we show that, in many cases, the law of the location of the minimum of a non-smooth Gaussian process, given that the minimum is high, converges, as the height of the minimum increases, to the minimizer in the optimization problem (1.2). We conclude with examples in Section 5.

2 The optimal measure and its support

When a Gaussian process is very smooth, optimal measures in the optimization problem (1.2) are supported by finite sets; see [3]. On the other hand, processes whose sample paths are sufficiently "rough" may lead to optimal measures with large supports. For example, if $\mathbf{X}$ is the stationary Ornstein-Uhlenbeck process, with covariance function $R_{\mathbf{X}}(s,t) = \exp\{-|s-t|\}$, then the optimal measure in (1.2) is

$$\nu_* = \frac{1}{2+b-a}\delta_a + \frac{1}{2+b-a}\delta_b + \frac{b-a}{2+b-a}\lambda_{a,b}\,,$$

where δ_x is a point mass at x, and $\lambda_{a,b}$ is the uniform probability distribution on the interval (a,b); see Example 6.2 in [1]. In this case the optimal measure has a full support in the interval $[a,b]$. We now demonstrate other situations where this phenomenon holds.

We start with considering certain stationary Gaussian processes, in which case we will use the standard single variable notation for the covariance function $R_{\mathbf{X}}(t) := R_{\mathbf{X}}(s, s+t)$, $s,t \in \mathbb{R}$. By stationarity it is enough to take $a = 0$ and consider intervals of the type $[0,b]$, $b > 0$.

Theorem 2.1. *Let $\mathbf{X} = \big(X(t),\, t \in \mathbb{R}\big)$ be a centered stationary Gaussian process with continuous sample paths and covariance function $R_{\mathbf{X}}$. Suppose that $R_{\mathbf{X}}$ is strictly convex on $[0,b]$. Then the optimization problem (1.2) has a unique optimal probability measure, which has a full support in the interval $[0,b]$.*

Proof. By Polya's theorem, the spectral measure of the process $\mathbf{X}$ has an absolutely continuous component which is of full support on $\mathbb{R}$; see e.g. [7]. Then there is a unique optimal probability measure ν_* in the optimization problem (1.2); see [1]. Furthermore, the strict convexity of the covariance function implies that it is strictly decreasing on $[0,b]$.

Note that the support of the optimal probability measure ν_* cannot consist of a single point, for in that case the value of the double integral in (1.2) is $R_{\mathbf{X}}(0)$, while any two-point probability measure ν would give a strictly smaller integral. We show now that endpoints 0 and b of the interval belong to the support. By symmetry it is enough to prove that b is in the support of ν_*.

Suppose that, to the contrary, for some $0 < \varepsilon < b$ we have $\nu_*\big([b-\varepsilon, b]\big) = 0$, and let c be the right-most point of the support of ν_*. Then $0 < c \le b - \varepsilon$. Choosing, if necessary, a smaller ε we can assure that $c > \varepsilon$ and that $\nu_*\big([0, c-\varepsilon)\big) > 0$. Construct now a new probability measure, $\hat{\nu}_*$ by translating the positive mass of ν_* in the interval $[c-\varepsilon, c]$ to the interval $[b-\varepsilon, b]$. By the strict monotonicity of the covariance function,

$$
\begin{aligned}
&\int_{[0,b]}\int_{[0,b]} R_{\mathbf{X}}(t-s)\,\hat{\nu}_*(ds)\,\hat{\nu}_*(dt) \\
&= \int_{[0,c-\varepsilon)}\int_{[0,c-\varepsilon)} R_{\mathbf{X}}(t-s)\,\nu_*(ds)\,\nu_*(dt) \\
&+ \int_{[c-\varepsilon,c]}\int_{[c-\varepsilon,c]} R_{\mathbf{X}}(t-s)\,\nu_*(ds)\,\nu_*(dt) \\
&+2 \int_{[c-\varepsilon,c]}\int_{[0,c-\varepsilon)} R_{\mathbf{X}}(b-c+t-s)\,\nu_*(ds)\,\nu_*(dt) \\
&< \int_{[0,c-\varepsilon)}\int_{[0,c-\varepsilon)} R_{\mathbf{X}}(t-s)\,\nu_*(ds)\,\nu_*(dt) \\
&+ \int_{[c-\varepsilon,c]}\int_{[c-\varepsilon,c]} R_{\mathbf{X}}(t-s)\,\nu_*(ds)\,\nu_*(dt) \\
&+2 \int_{[c-\varepsilon,c]}\int_{[0,c-\varepsilon)} R_{\mathbf{X}}(t-s)\,\nu_*(ds)\,\nu_*(dt) \\
&= \int_{[0,b]}\int_{[0,b]} R_{\mathbf{X}}(t-s)\,\nu_*(ds)\,\nu_*(dt)\,,
\end{aligned}
$$

contradicting the optimality of the measure ν_*.

Hence, the endpoints of the interval are in the support of ν_*, and we proceed to prove that the support of ν_* is the entire interval $[0, b]$. Suppose that, to the contrary, there are points $0 \le c_1 < c_2 \le b$, both in the support of ν_*, such that $\nu_*\big((c_1, c_2)\big) = 0$. Denote

$$
m(t) = \int_{[0,b]} R_{\mathbf{X}}(t-s)\,\nu_*(ds),\ 0 \le t \le b\,.
$$

The optimality of the measure ν_* implies that $m(t) \ge \sigma_*^2(0, b)$ (the optimal value of the double integral in (1.2)) for all $0 \le t \le b$, with equality on the support of ν_*; see Theorem 4.3 in [1]. Note that on the interval $[c_1, c_2]$ this function,

$$
m(t) = \int_{[0,c_1]} R_{\mathbf{X}}(t-s)\,\nu_*(ds) + \int_{[c_2,b]} R_{\mathbf{X}}(s-t)\,\nu_*(ds)\,,
$$

is strictly convex by the assumptions. Since $m(c_1) = m(c_2) = \sigma_*^2(0, b)$, this rules out the possibility that $m(t) \ge \sigma_*^2(0, b)$ for $c_1 < t < c_2$. The resulting contradiction completes the proof of the theorem. $\square$

For certain nonstationary Gaussian processes the optimization problem (1.2) can be explicitly solved. Here is one such situation. Let $\big(B(t),\, t \ge 0\big)$ be the standard Brownian motion, and $0 < a < b < \infty$. Consider a centered Gaussian process of the form

$$
X(t) = \frac{1}{g(t)} B(t),\ a \le t \le b\,, \tag{2.1}
$$

where $g:\ [a, b] \to (0, \infty)$ is a continuous function.

Theorem 2.2. *(a) Suppose that g is a nondecreasing concave and twice continuously differentiable function on $[a, b]$. Define*

$$
\begin{aligned}
f(x) &= -g(x)g''(x) \ge 0,\ a < x < b\,, \\
p_a &= \frac{g(a)}{a}\big(g(a) - ag'(a)\big) \ge 0\,, \\
p_b &= g(b)g'(b) \ge 0\,.
\end{aligned}
$$

Then the finite measure μ on $[a,b]$ defined by

$$\mu(dx) = p_a\delta_a(dx) + p_b\delta_b(dx) + f(x)\,dx,\ a \le x \le b\,, \tag{2.2}$$

is equal, up to a multiplicative constant, to an optimal solution to the optimization problem (1.2)*.*

(b) Suppose that g is concave on $[a,b]$, and nondecreasing and twice continuously differentiable on $[a_0,b]$, for some $a < a_0 < b$ such that $g(a_0) = a_0 g'(a_0)$. If p_b is as in part (a), and

$$f(x) = -g(x)g''(x) \ge 0,\ a_0 < x < b\,,$$

then the finite measure μ on $[a,b]$ defined by

$$\mu(dx) = p_b\delta_b(dx) + f(x)\,dx,\ a_0 \le x \le b\,, \tag{2.3}$$

is equal, up to a multiplicative constant, to an optimal solution to the optimization problem (1.2)*.*

Proof. Observe that the covariance function of the process $\mathbf{X}$ is given by

$$R_{\mathbf{X}}(s,t) = \frac{s}{g(s)g(t)},\ a \le s \le t \le b\,.$$

With the measure μ defined by (2.2),

$$\begin{aligned}
&\int_{[a,b]} R_{\mathbf{X}}(s,t)\,\mu(ds) = p_a R_{\mathbf{X}}(a,t) + p_b R_{\mathbf{X}}(b,t) + \int_a^b R_{\mathbf{X}}(x,t)f(x)\,dx \\
=&\frac{g(a)}{a}\big(g(a) - ag'(a)\big)\frac{a}{g(a)g(t)} + g(b)g'(b)\frac{t}{g(b)g(t)} \\
&- \int_a^t \frac{x}{g(t)g(x)} g(x)g''(x)\,dx - \int_t^b \frac{t}{g(t)g(x)} g(x)g''(x)\,dx \\
=&\frac{1}{g(t)}\left[g(a) - ag'(a) + tg'(b) - \int_a^t xg''(x)\,dx - t\int_t^b g''(x)\,dx\right] \\
=&\frac{1}{g(t)}\left[g(a) + \int_a^t g'(x)\,dx\right] = 1
\end{aligned}$$

for each $a \le t \le b$. By Theorem 4.3 in [1] this implies the claim of part (a).

For part (b) note that by the above argument we already know that

$$\int_{[a,b]} R_{\mathbf{X}}(s,t)\,\mu_1(ds) = 1 \tag{2.4}$$

for all $a_0 \le t \le b$. Appealing, once again, to Theorem 4.3 in [1] we see that the claim of part (b) will follow once we check that the value of the integral in (2.4) is at least 1 for $a \le t < a_0$. For such t,

$$\begin{aligned}
&\int_{[a,b]} R_{\mathbf{X}}(s,t)\,\mu_1(ds) = p_b R_{\mathbf{X}}(b,t) + \int_{a_0}^b R_{\mathbf{X}}(x,t)f(x)\,dx \\
=&\frac{1}{g(t)}\left[tg'(b) - t\int_{a_0}^b g''(x)\,dx\right] \\
=&\frac{tg'(a_0)}{g(t)}\,.
\end{aligned}$$

Since by concavity of g,

$$g(a_0) - g(t) = \int_t^{a_0} g'(x)\,dx \geq g'(a_0)(a_0 - t)\,,$$

we conclude that

$$g(t) \leq g(a_0) - a_0 g'(a_0) + t g'(a_0) = t g'(a_0)\,,$$

which gives the required lower bound on the integral of the covariance function. □

Remark 2.3. It is clear that the assumption of continuous second derivative of the function g in Theorem 2.2 can be replaced by the assumption of absolutely continuous first derivative, in which case the function g'' in the statement of the theorem is simply a nonpositive derivative of g' in the sense of absolute continuity.

3 Tails of the minima

In this section we describe certain situations in which we can give more precise asymptotics of the tail of the minimum of a Gaussian process $\mathbf{X}$ beyond the logarithmic asymptotics in (1.1). In these situations the smoothness assumptions of [3] are not satisfied, and asymptotics of the type (1.3) are no longer applicable. Our most precise results apply to Gaussian Markov processes, of which the processes of the type defined in (2.1) are a special case.

Theorem 3.1. *Let $(X(t),\, a \leq t \leq b)$ be a centered Gaussian Markov process with continuous sample paths, such that an optimal measure ν_* in the optimization problem* (1.2) *has an absolutely continuous component ν_{ac}, whose density with respect to the Lebesgue measure has a version with*

$$\eta := \inf_{x\in[a,b]} \frac{d\nu_{ac}(x)}{dx} > 0\,. \tag{3.1}$$

Then

$$\begin{aligned} -\infty \quad &< \quad \liminf_{u\to\infty} u^{-2/3}\left(\log P\left(\min_{a\leq t\leq b} X(t) > u\right) + \frac{1}{2\sigma_*^2(a,b)}u^2\right) \\ &\leq \quad \limsup_{u\to\infty} u^{-2/3}\left(\log P\left(\min_{a\leq t\leq b} X(t) > u\right) + \frac{1}{2\sigma_*^2(a,b)}u^2\right) < 0\,. \end{aligned}$$

Proof of Theorem 3.1. We will use the following easily checkable fact (which also follows from Theorem 4.12.11 (iii) of [2]): if $f : (0,\infty) \to (0,\infty)$ is a bounded measurable function such that

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{\beta} \log f(\varepsilon) = -c\,, \tag{3.2}$$

for some $\beta, c \in (0,\infty)$, then there exists $C \in (0,\infty)$ such that

$$\lim_{x\to\infty} x^{-\beta/(1+\beta)} \log \int_0^\infty e^{-x\varepsilon} f(\varepsilon) d\varepsilon = -C\,. \tag{3.3}$$

Denote

$$Y = \int_{[a,b]} X(t)\,\nu_*(dt)\,. \tag{3.4}$$

Since ν_* has full support, it follows that

$$\mathrm{E}(X(t)|Y) = Y \text{ a.s. for all } t \in [a,b]\,;$$

see e.g. p.8 in [3]. With

$$Z(t) := X(t) - Y,\ t \in [a,b]\,,$$

wee see that Y and $\bigl(Z(t),\, t \in [a,b]\bigr)$ are independent. Since

$$\int_{[a,b]} Z(t)\,\nu_*(dt) = 0 \ \text{ a.s.}\,,$$

it follows that

$$Z_* := \min_{a\le t\le b} Z(t) \le 0 \ \text{ a.s.}\,.$$

Therefore, for $u > 0$,

$$\begin{aligned}
&P\left(\min_{a\le t\le b} X(t) > u\right) = P(Y + Z_* > u)\\
=&\int_u^\infty P(Z_* > u-y)\,P(Y\in dy)\\
=&\int_u^\infty P(Z_* > u-y)\frac{1}{\sigma_*(a,b)\sqrt{2\pi}}e^{-y^2/2\sigma_*^2(a,b)}\,dy\\
=&\frac{1}{\sigma_*(a,b)\sqrt{2\pi}}e^{-u^2/2\sigma_*^2(a,b)}\int_0^\infty e^{-u\varepsilon/\sigma_*^2(a,b)}P(Z_* > -\varepsilon)e^{-\varepsilon^2/2\sigma_*^2(a,b)}\,d\varepsilon\,.
\end{aligned}$$

We will prove that

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^2 \log P(Z_* > -\varepsilon) > -\infty\,. \tag{3.5}$$

By (3.3) with $\beta = 2$ this will prove the lower bound in the statement of the theorem. However, if X_* and X^* are the smallest and the largest values, respectively, of $\mathbf{X}$ on $[a,b]$, then, as $\varepsilon\downarrow 0$,

$$\log P(Z_* > -\varepsilon) \ge \log P\left(X^* - X_* < \varepsilon\right) \sim -\kappa\varepsilon^{-2}$$

for some $\kappa \in (0,\infty)$. The asymptotic equivalence in the last line has been shown in [6]. Thus, (3.5) follows.

In order to prove the upper bound in the statement of the theorem, we use a change of measure. Let $\mathcal{L}_{\mathbf{X}}$ be the closed in L^2 linear span of the process $\mathbf{X}$. For every $Z \in \mathcal{L}_{\mathbf{X}}$, the function $f_Z(t) = E\bigl(ZX(t),\, a\le t\le b\bigr)$ belongs to the reproducing kernel Hilbert space of $\mathbf{X}$ and, hence, the probability measures $\bigl(X(t),\, a\le t\le b\bigr)$ and $\bigl(X(t)+f_Z(t),\, a\le t\le b\bigr)$ generate on $\mathbb{R}^{[a,b]}$ are equivalent. Furthermore, in the obvious notation,

$$\frac{dP^{\mathbf{X}+\mathbf{f}_Z}}{dP^{\mathbf{X}}} = \exp\left\{Z - EZ^2/2\right\}\,;$$

see [8]. In particular, for every such Z,

$$\begin{aligned}
&P\bigl(\min_{a\le t\le b} X(t) + f_Z(t) > 0\bigr) \qquad (3.6)\\
=&\exp\left\{-EZ^2/2\right\} E\left[e^Z \mathbf{1}\bigl(\min_{a\le t\le b} X(t) > 0\bigr)\right]\,.
\end{aligned}$$

With Y as in (3.4) we choose $Z = -uY/EY^2$. Since ν_* has a full support, we have $f_Z(t) = -u$ for all $a\le t\le b$. By (3.6),

$$\begin{aligned}
P\bigl(\min_{a\le t\le b} X(t) > u\bigr) &= \exp\left\{-\frac{1}{2\sigma_*^2(a,b)}u^2\right\} \qquad (3.7)\\
E&\left[\exp\left\{-u\frac{1}{\sigma_*^2(a,b)}\int_{[a,b]} X(t)\,\nu_*(dt)\right\}\mathbf{1}\bigl(\min_{a\le t\le b} X(t) > 0\bigr)\right]\,.
\end{aligned}$$

Next,

$$\begin{aligned}
&E\left[\exp\left\{-u\frac{1}{\sigma_*^2(a,b)}\int_{[a,b]}X(t)\,\nu_*(dt)\right\}\mathbf{1}\big(\min_{a\le t\le b}X(t)>0\big)\right]\\
\le&E\left[\exp\left\{-u\frac{1}{\sigma_*^2(a,b)}\int_{[a,b]}|X(t)|\,\nu_*(dt)\right\}\right]\\
\le&\exp\left\{-u^{2/3}\frac{1}{\sigma_*^2(a,b)}\right\}+P\left(\int_{[a,b]}|X(t)|\,\nu_*(dt)\le u^{-1/3}\right).
\end{aligned}$$

Appealing, once again, to [6], we have, by (3.1),

$$\begin{aligned}
&\log P\left(\int_{[a,b]}|X(t)|\,\nu_*(dt)\le u^{-1/3}\right)\\
\le&\log P\left(\eta\int_{[a,b]}|X(t)|\,dt\le u^{-1/3}\right)\sim-\kappa_1u^{2/3}
\end{aligned}$$

for some $\kappa_1\in(0,\infty)$. In conjunction with (3.7) this establishes the upper bound in the theorem. □

It is clear from the proof of Theorem 3.1 that there is a close connection between the improvements on the logarithmic asymptotics (1.1) of the minima of Gaussian processes and small ball problems for these processes. Availability of bounds on small ball probabilities is often helpful in obtaining bounds on the tail of the Gaussian minimum. The following theorem is another example of this.

Theorem 3.2. *Let $(X(t),\,a\le t\le b)$ be a centered Gaussian process with continuous sample paths, such that an optimal measure ν_* in the optimization problem* (1.2) *has a full support in $[a,b]$. Suppose that there exists a function $\sigma:[0,\infty)\to[0,\infty)$ satisfying*

$$\lim_{h\downarrow0}h^{-\beta}\sigma(h)=c\in(0,\infty)\tag{3.8}$$

for some $\beta>0$, such that such that

$$\mathrm{E}\left[(X(t)-X(s))^2\right]\le\sigma(|t-s|)^2,\ s,t\in[a,b].$$

Then,

$$\liminf_{u\to\infty}u^{-1/(\beta+1)}\left(\log P\left(\min_{t\in[a,b]}X(t)>u\right)+\frac{1}{2\sigma_*^2(a,b)}u^2\right)>-\infty,$$

where $\sigma_*^2(a,b)$ is as in (1.2)*, and should not be confused with the σ of* (3.8).

Proof. An argument identical to the proof of the lower bound in Theorem 3.1 gives us

$$\begin{aligned}
&P\left(\min_{a\le t\le b}X(t)>u\right)\\
\ge\ &\frac{1}{\sigma_*(a,b)\sqrt{2\pi}}e^{-u^2/2\sigma_*^2(a,b)}\\
&\int_0^\infty e^{-u\varepsilon/\sigma_*^2(a,b)}P\left(\max_{a\le t\le b}|X(t)-X(a)|<\varepsilon/2\right)e^{-\varepsilon^2/2\sigma_*^2(a,b)}\,d\varepsilon.
\end{aligned}$$

Since by the assumption (3.8) we have, for some $K\in(0,\infty)$,

$$P\left(\max_{a\le t\le b}|X(t)-X(a)|\le\varepsilon\right)\ge\exp\left(-K\varepsilon^{-1/\beta}\right),\ \varepsilon>0,$$

by Theorem 4.1 in [5], the claim of the theorem follows from (3.3). □

4 The location of the minimum

For a continuous centered Gaussian process $\mathbf{X} = \big(X(t),\, t \in \mathbb{R}\big)$ consider the location of the minimum of the process on an interval $[a,b]$:

$$T_* := \arg \min_{a\le t\le b} X(t)\,,$$

where we choose the leftmost location of the minimum in case there are ties. For very smooth Gaussian processes considered in [3] it was proved that, as $u \to \infty$,

$$¶\left(T_* \in \cdot \,\Big|\, \min_{a\le t\le b} X(t) > u\right) \Rightarrow \nu_*\,, \tag{4.1}$$

with ν_* the unique minimizer in the optimization problem (1.2). In that case the latter optimal measure is always supported by a finite set. Our goal in this section is to show that (4.1) continues to hold for Gaussian processes whose sample paths are not smooth, and for which the optimal measure may have full support.

Theorem 4.1. *Let $\mathbf{X} = \big(X(t),\, t \in \mathbb{R}\big)$ be a centered stationary Gaussian process with continuous sample paths and covariance function $R_{\mathbf{X}}$. Suppose that $R_{\mathbf{X}}$ is strictly convex on $[0,b]$. Then* (4.1) *holds with $a = 0$ and any $b > 0$, where ν_* is the unique optimal probability measure for the optimization problem* (1.2)*.*

Proof. The fact that the optimization problem (1.2) has a unique optimal solution ν_* was established in Theorem 2.1. We use (3.7) (with $a = 0$). Let $A \subseteq [a,b]$ be a Borel set that is a continuity set for ν_*. Recalling the notation (3.4) we obtain

$$\begin{aligned}&P\big(T_* \in A |\min_{a\le t\le b} X(t) > u\big)\\ =&\frac{E\left[\exp\left\{-uY/\sigma_*^2(a,b)\right\}\mathbf{1}\big(\min_{a\le t\le b} X(t) > 0,\, T_* \in A\big)\right]}{E\left[\exp\left\{-uY/\sigma_*^2(a,b)\right\}\mathbf{1}\big(\min_{a\le t\le b} X(t) > 0\big)\right]}\,.\end{aligned}$$

By Fubini's theorem this can be rewritten in the form

$$\begin{aligned}&P\big(T_* \in A |\min_{a\le t\le b} X(t) > u\big)\\ =&\frac{\int_0^\infty \exp\left\{-ux/\sigma_*^2(a,b)\right\} P\big(Y \le x,\, \min_{a\le t\le b} X(t) > 0,\, T_* \in A\big)\,dx}{\int_0^\infty \exp\left\{-ux/\sigma_*^2(a,b)\right\} P\big(Y \le x,\, \min_{a\le t\le b} X(t) > 0\big)\,dx}\,,\end{aligned}$$

and so it is enough to prove that

$$\begin{aligned}\nu_*(A) &= \lim_{x\to 0} \frac{P\big(Y \le x,\, \min_{a\le t\le b} X(t) > 0,\, T_* \in A\big)}{P\big(Y \le x,\, \min_{a\le t\le b} X(t) > 0\big)}\\ &= \lim_{x\to 0} P\Big(T_* \in A \Big| Y \le x,\, \min_{a\le t\le b} X(t) > 0\Big)\,.\end{aligned}$$

If we denote by m_x the probability measure described by the right hand side of this statement, then we need to prove that

$$m_x \Rightarrow \nu_* \text{ as } x \to 0. \tag{4.2}$$

To this end, we use a discrete approximation. Let $\mathcal{P}_k = \left\{bi2^{-k},\, i = 0, 1, \ldots, 2^k\right\}$ be the kth binary partition of the interval $[0,b]$, $k = 1, 2, \ldots$. For each k we consider the following restricted version of the optimization problem (1.2):

$$\min_{\nu \in M_1(\mathcal{P}_k)} \int_{[0,b]}\int_{[0,b]} R_{\mathbf{X}}(s,t)\,\nu(ds)\,\nu(dt)\,, \tag{4.3}$$

where the probability measures are required to be supported by the finite set $\mathcal{P}_k$. As in the case of the full optimization problem (1.2), the fact that the spectral measure of the process $\mathbf{X}$ is of full support guarantees that the problem (4.3) has a unique optimal solution, which we will denote by $\nu_{*,k}$. We also denote by $\sigma^2_{*,k}$ the corresponding value of the double integral. The same argument as in the case of the restricted optimization problem shows that, because of strict convexity of $R_{\mathbf{X}}$, $\nu_{*,k}$ assigns a positive mass to each point in $\mathcal{P}_k$.

Clearly, $\sigma^2_{*,1} \geq \sigma^2_{*,2} \geq \ldots \geq \sigma^2_*[0,b]$. On the other hand, the obvious discretizations of the measure ν_* produce a sequence of probability measures $\nu'_k \in \mathcal{P}_k$, $k = 1, 2, \ldots$ such that $\nu'_k \Rightarrow \nu_*$ as $k \to \infty$. By continuity,

$$\int_{[0,b]} \int_{[0,b]} R_{\mathbf{X}}(s,t)\, \nu'_k(ds)\, \nu'_k(dt) \to \int_{[0,b]} \int_{[0,b]} R_{\mathbf{X}}(s,t)\, \nu_*(ds)\, \nu_*(dt)$$

as $k \to \infty$, so by the optimality of the measures $(\nu_{*,k})$ we conclude that $\sigma^2_{*,k} \to \sigma^2_*[0,b]$. We claim that $\nu_{*,k} \Rightarrow \nu_*$. Since the space $M_1[0,b]$ is weakly compact, it is enough to prove that every subsequential limit of the sequence $(\nu_{*,k})$ is equal to ν_*. However, for every subsequence of the sequence $(\nu_{*,k})$ the value of the double integral in the optimization problem (1.2) converges to $\sigma^2_*[0,b]$ and, by weak continuity of the double integral, it also converges to the double integral with respect to the subsequential limit. Since under the assumptions of the theorem the optimization problem (1.2) has a unique optimal solution, we conclude that every subsequential limit of the sequence $(\nu_{*,k})$ is equal to ν_*.

Define, analogously to (3.4),

$$Y_k = \int_{[a,b]} X(t)\, \nu_{*,k}(dt)\,,$$

and let

$$T_{*,k} := \arg\min_{t \in \mathcal{P}_k} X(t)\; k = 1, 2, \ldots\,,$$

once again choosing the leftmost location in the case of a tie. For each k we define a probability measure on $[a,b]$ by

$$m_{x,k}(A) = P\big(T_{*,k} \in A \big| Y_k \leq x,\, \min_{t \in \mathcal{P}_k} X(t) > 0\big),\; A \text{ Borel.}$$

It is clear that $T_{*,k} \to T_*$ and $\min_{t \in \mathcal{P}_k} X(t) \to \min_{a \leq t \leq b} X(t)$ a.s. Furthermore, $Y_k \to Y$ in L^2. Furthermore, the distribution of $\min_{a \leq t \leq b} X(t)$ is atomless (see Lemma 1 in [9]). We conclude that, for each fixed $x > 0$, $m_{x,k} \Rightarrow m_x$ as $k \to \infty$. It follows that the claim (4.2) will follow if we prove that

$$m_{x,k} \Rightarrow \nu_{*,k} \text{ uniformly in } k \text{ as } x \to 0. \tag{4.4}$$

Consider the zero mean Gaussian random vector $\mathbf{X}^{(k)} = \big(X(bi2^{-k}),\, i = 0, 1, \ldots, 2^k\big)$. Let Σ_k denote its covariance matrix. The uniqueness of the minimizing measure $\nu_{*,k}$ implies that the vector $\mathbf{X}^{(k)}$ has full support, so Σ_k is invertible. For any $j = 0, 1, \ldots, 2^k$ we can write

$$\begin{aligned} m_{x,k}\big(\{bj2^{-k}\}\big) =& \frac{P\big(\mathbf{X}^{(k)} \in E_j(x)\big)}{\sum_{i=0}^{2^k} P\big(\mathbf{X}^{(k)} \in E_i(x)\big)} \\ =& \frac{\int_{E_j(x)} \exp\big\{-\mathbf{z}^T \Sigma_k^{-1} \mathbf{z}/2\big\}\, d\mathbf{z}}{\sum_{i=0}^{2^k} \int_{E_i(x)} \exp\big\{-\mathbf{z}^T \Sigma_k^{-1} \mathbf{z}/2\big\}\, d\mathbf{z}}\,, \end{aligned} \tag{4.5}$$

where

$$E_j(x) = \left\{ \mathbf{z} \in (0,\infty)^{2^k+1},\, z_j < z_i,\, i \neq j,\, \sum_{i=0}^{2^k+1} \nu_{*,k}\big(\{bi2^{-k}\}\big) z_i \leq x \right\},$$

$j = 0, 1, \ldots, 2^k + 1$. It is straightforward to compute that

$$\int_{E_j(x)} d\mathbf{z} = \frac{x^{2^k+1}}{(2^k+1)!} \frac{\nu_{*,k}\big(\{bj2^{-k}\}\big)}{\prod_{i=0}^{2^k+1} \nu_{*,k}\big(\{bi2^{-k}\}\big)}$$

Therefore, if we prove that

$$\mathbf{z}^T \Sigma_k^{-1} \mathbf{z} \to 0 \text{ as } x \to 0 \tag{4.6}$$

uniformly on $\cup_{i=0}^{2^k+1} E_i(x)$, then we obtain uniform convergence in (4.4) (even in total variation).

To this end, let $\mathbf{w} = \Sigma_k^{-1}\mathbf{z}$, so that

$$\mathbf{z}^T \Sigma_k^{-1} \mathbf{z} = \mathbf{w}^T \Sigma_k \mathbf{w}\,.$$

Let $\boldsymbol{\theta} = \Sigma^{-1}\mathbf{1}$. The vector $\boldsymbol{\theta}$ is equal, up to a multiplicative scale, to the probability vector of the measure $\nu_{*,k}$; see [3]. Therefore,

$$\|\boldsymbol{\theta}\|_1 = \mathbf{1}^T\boldsymbol{\theta} = \mathbf{1}^T \Sigma_k^{-1} \mathbf{1} = \boldsymbol{\theta}^T \Sigma_k \boldsymbol{\theta} = (\|\boldsymbol{\theta}\|_1)^2 \sigma_{*,k}^2\,,$$

so that

$$\|\boldsymbol{\theta}\|_1 = \frac{1}{\sigma_{*,k}^2}\,.$$

In particular,

$$\mathbf{w}^T\mathbf{1} = \mathbf{z}^T\boldsymbol{\theta} \leq \|\boldsymbol{\theta}\|_1 x = \frac{x}{\sigma_{*,k}^2}$$

on $\cup_{i=0}^{2^k+1} E_i(x)$. We conclude that

$$\mathbf{w}^T \Sigma_k \mathbf{w} \leq R_X(0)\big(\mathbf{w}^T\mathbf{1}\big)^2 = R(0)\frac{x^2}{\sigma_{*,k}^4}\,.$$

Since $\sigma_{*,k}^2 \to \sigma_*^2[0,b] > 0$, for all k large enough we have $\sigma_{*,k}^2 \geq \sigma_*^2[0,b]/2$, and we have obtained the desired uniform convergence, thus completing the proof. □

5 Examples

In this section, the results in Sections 2–4 are applied to two examples. The first example illustrates applications of Theorems 2.2 and 3.1.

Example 5.1. Let $(B(t) : t \geq 0)$ be a standard Brownian motion, and fix $0 < \alpha < 1$. Define

$$X(t) = t^{-\alpha} B(t),\, t > 0\,.$$

Fix $0 < a < b < \infty$, and set

$$X_* = \min_{t\in[a,b]} X(t)\,.$$

Theorem 2.2 implies that the finite measure μ on $[a,b]$ defined by

$$\mu_\alpha(dx) = \alpha(1-\alpha)x^{2\alpha-2}\,dx + (1-\alpha)a^{2\alpha-1}\delta_a(dx) + \alpha b^{2\alpha-1}\delta_b(dx)\,,$$

is a constant multiple of the optimal measure, that is, the solution to the optimization problem (1.2). Let

$$\sigma_*^2(a,b;\alpha) = \mu_\alpha([a,b])^{-2}\mathrm{Var}\left(\int_a^b X(t)\mu_\alpha(dt)\right)\,.$$

As the Radon-Nykodym derivative of the absolutely continuous component of μ_α with respect to the Lebesgue measure is bounded away from 0 on $[a,b]$, the hypotheses of Theorem 3.1 are clearly satisfied, which implies that

$$\liminf_{u\to\infty} u^{-2/3}\left(\log P(X_* > u) + \frac{1}{2\sigma_*^2(a,b;\alpha)}u^2\right) > -\infty\,,$$

and

$$\limsup_{u\to\infty} u^{-2/3}\left(\log P(X_* > u) + \frac{1}{2\sigma_*^2(a,b;\alpha)}u^2\right) < 0\,.$$

In other words, as $u\to\infty$,

$$P(X_* > u) = \exp\left(-\frac{1}{2\sigma_*^2(a,b;\alpha)}u^2 - u^{\frac{2}{3}+O(1/\log u)}\right)\,. \tag{5.1}$$

When $\alpha = 1/2$, $X(t)$ is a time-changed Ornstein-Uhlenbeck process. That is,

$$\big(X(e^{2t}) : t\in\mathbb{R}\big) \overset{d}{=} (Z_t : t\in\mathbb{R})\ ,$$

the process on the right hand side being an Ornstein-Uhlenbeck process. Therefore, a special case of (5.1) is that for any compact interval $[a,b]\subset\mathbb{R}$,

$$P\left(\min_{t\in[a,b]} Z_t > u\right) = \exp\left(-\frac{1}{2\sigma_*^2(e^{2a},e^{2b};1/2)}u^2 - u^{\frac{2}{3}+O(1/\log u)}\right)\,, \tag{5.2}$$

as $u\to\infty$.

The second example illustrates applications of Theorems 2.1, 3.2 and 4.1.

Example 5.2. Let $(X(t) : t\in\mathbb{R})$ be a stationary Gaussian process with mean zero and covariance function

$$R_{\mathbf{X}}(t) = \exp\{-|t|^\alpha\},\ t\in\mathbb{R}\,,$$

for a fixed $0<\alpha\le 1$. The assumptions of Theorem 2.1 are satisfied for any $b>0$ and, hence, the optimal measure, say ν_*, in the optimization problem (1.2) is of full support. If $\alpha=1$, this follows from the explicit solution of the optimization problem in [1].

The hypotheses of Theorem 3.2 are therefore satisfied with $\beta=\alpha/2$ and $c=\sqrt{2}$, which implies the existence of $C\in(0,\infty)$ satisfying

$$\log P\left(\min_{t\in[a,b]} X(t) > u\right) \ge -\frac{1}{2\sigma_*^2(a,b)}u^2 - Cu^{-2/(\alpha+2)}\,,$$

for large u. When $\alpha=1$ this reduces to the upper bound in (5.2).

Finally, an appeal to Theorem 4.1 shows that the conditional law of the location of the minimum (the leftmost one to be chosen in case of ties) on $[a,b]$ given that the minimum if above u, converges weakly to ν_* as $u\to\infty$.

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