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Convergence of complex martingale for a branching random walk in a time random environment *

Xiaoqiang Wang† Chunmao Huang‡

Abstract

We consider a discrete-time branching random walk in a stationary and ergodic environment $\xi = (\xi_n)$ indexed by time $n \in \mathbb{N}$. Let $W_n(z)$ ($z \in \mathbb{C}^d$) be the natural complex martingale of the process. We show sufficient conditions for its almost sure and quenched L^α convergence, as well as the existence of quenched moments and weighted moments of its limit, and also describe the exponential convergence rate.

Keywords: branching random walk; random environment; complex martingale; moments; weighted moments; convergence rate.
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1 Introduction and main results

We consider a *branching random walk in a time random environment (BRWRE)*, where the distributions of the point processes indexed by particles vary from generation to generation according to a time random environment. First introduced by Biggins and Kyprianou [4], this model was further studied in [9, 11, 21, 23]. For the classical branching walk, Biggins [3] showed a sufficient condition for the almost sure and L^α convergence of the complex martingale of the model for $\alpha \in (1, 2]$, and recently, necessary and sufficient conditions for $\alpha > 1$ were shown by Ikzanove *et al.* [13], while Kolesko and Meiners [15] especially discussed the convergence on the boundary of the uniform convergence region. Aiming to extend the result of [3], this paper focuses on investigating the convergence (in the sense almost sure and in L^α for $\alpha > 1$) of the complex martingale in BRWRE. The main results presented in the paper cannot be derived directly by techniques suitable for classical branching walks. The main reason is that the environment makes it difficult to find useful upper bounds of martingales. Similar problems may appear in other models in random environments, such as branching processes, multiplicative cascades and random fractals in random environments, etc. The techniques used in the paper, especially in the side of dealing with the stationary and ergodic random environments, should provide reference for related topics.

Let us describe the model in detail. The *time random environment*, denoted by $\xi = (\xi_n)$, is a stationary and ergodic sequence of random variables, indexed by the time $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$, taking values in some measurable space $(\Theta, \mathcal{E})$. Without

*Supported by the National Natural Science Foundation of China (Nos. 11601286 and 11501146).
†Shandong University (Weihai), China.
‡Harbin Institute of Technology (Weihai), China. E-mail: cmhuang@hitwh.edu.cn

loss of generality we can suppose that ξ is defined on the product space $(\Theta^{\mathbb{N}}, \mathcal{E}^{\otimes\mathbb{N}}, \nu)$, with ν the law of ξ. Stationarity means that the two random vectors $(\xi_k, \xi_{k+1}, \cdots, \xi_{k+n})$ and $(\xi_{k+h}, \xi_{k+1+h}, \cdots, \xi_{k+n+h})$ have the same joint distribution for any k, n and $h \in \mathbb{N}$; ergodicity can be comprehended as that the following Birkhoff ergodic theorem holds: for any measure-preserving transformation τ and integrable function f on $(\Theta^{\mathbb{N}}, \mathcal{E}^{\otimes\mathbb{N}}, \nu)$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k\xi) = \int_{\Theta^{\mathbb{N}}} f(\xi)d\nu(\xi) \qquad \text{for almost all } \xi.$$

For each realization of ξ_n, there exists a distribution on $\mathbb{N}\times(\mathbb{R}^d)^{\otimes\mathbb{N}^*}$ corresponds it, where $d \geq 1$ is the dimension of the real space and $\mathbb{N}^* = \{1, 2, \cdots\}$. We denote the distribution corresponding to ξ_n by $\eta_n = \eta(\xi_n)$. The notation $\eta(\xi_n)$ can be regarded as a mapping from the space $(\Theta, \mathcal{E})$ to the set of all distributions on $\mathbb{N}\times(\mathbb{R}^d)^{\otimes\mathbb{N}^*}$. Given the environment ξ, the process can be described as follows: at time 0, one initial particle $\emptyset$ of generation 0 is located at $S_\emptyset = 0 \in \mathbb{R}^d$; in general, each particle u of generation n located at $S_u \in \mathbb{R}^d$ is replaced at time $n+1$ by $N(u)$ new particles ui of generation $n+1$, located at

$$S_{ui} = S_u + L_i(u) \qquad (1 \leq i \leq N(u)),$$

where the random vector $X(u) = (N(u), L_1(u), L_2(u), \cdots)$ is of distribution $\eta_n = \eta(\xi_n)$; all particles behave independently conditioned on the environment ξ.

For each realization ξ of the environment sequence, let $(\Gamma, \mathcal{G}, \mathbb{P}_\xi)$ be the probability space on which the process is defined. The probability $\mathbb{P}_\xi$ is usually called *quenched law*, while the total probability $\mathbb{P}$ is usually called *annealed law*. The quenched law $\mathbb{P}_\xi$ may be considered to be the conditional probability of $\mathbb{P}$ given ξ. The expectation with respect to $\mathbb{P}$ (resp. $\mathbb{P}_\xi$) will be denoted by $\mathbb{E}$ (resp. $\mathbb{E}_\xi$).

Let $\mathbb{U} = \{\emptyset\} \cup \bigcup_{n\geq 1}(\mathbb{N}^*)^n$ be the set of all finite sequence $u = u_1 \cdots u_n$. For $u \in \mathbb{U}$, we write $|u|$ for the length of u. Let $\mathbb{T}$ be the Galton-Watson tree with defining elements $\{N(u)\}$ and $\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$ be the set of particles of generation n. For $n \in \mathbb{N}$ and $z = x + \mathbf{i}y \in \mathbb{C}^d$, put

$$m_n(z) = \mathbb{E}_\xi \sum_{i=1}^{N(u)} e^{zL_i(u)} \quad (|u| = n), \tag{1.1}$$

where the product zL should be understood as the inner product that $\sum\limits_{i=1}^{d} z^i L^i$ if $z = (z^1, \cdots, z^d) \in \mathbb{C}^d$ and $L = (L^1, \cdots, L^d) \in \mathbb{R}^d$. We consider the non trivial case that

$$\mathbb{P}_\xi(N = 0) < 1 \qquad a.s., \tag{1.2}$$

so that $m_n(z) \neq 0$ a.s. Set

$$P_0(z) = 1 \qquad \text{and} \qquad P_n(z) = \mathbb{E}_\xi \sum_{u\in\mathbb{T}_n} e^{zS_u} = \prod_{i=0}^{n-1} m_i(z) \quad (n \geq 1). \tag{1.3}$$

For $z \in \mathbb{C}^d$ and $u \in \mathbb{T}$, denote $X_u(z) = \frac{e^{zS_u}}{P_{|u|}(z)}$,

$$W_0(z) = 1, \qquad W_n(z) = \sum_{u\in\mathbb{T}_n} X_u(z) \quad (n \geq 1) \qquad \text{and} \qquad W^* = \sup_{n\geq 0} |W_n(z)|. \tag{1.4}$$

Let $\mathcal{F}_0 = \sigma(\xi)$ and $\mathcal{F}_n = \sigma(\xi, X(u);\ |u| < n)$ for $n \geq 1$. It is well known that for each $z \in \mathbb{C}^d$ fixed, $W_n(z)$ forms a complex martingale with respect to the filtration $\mathcal{F}_n$ under both laws $\mathbb{P}_\xi$ and $\mathbb{P}$. Particularly, for $x \in \mathbb{R}^d$, the martingale $W_n(x)$ is non-negative, hence it converges almost surely (a.s.). In the deterministic environment case, this martingale (with real or complex parameters) has been studied by Kahane and Peyrière [14], Biggins [2, 3], Uchiyama[22], Durrett and Liggett [7], Guivarc'h [10], Lyons [20] and Liu [18, 19], etc. in different contexts. In this paper, we are interested in the convergence of the complex martingale $W_n(z)$ for $z \in \mathbb{C}^d$ fixed. For simplicity, later we write $X_u = X_u(z)$ and $W_n = W_n(z)$ for short.

In deterministic environment, Biggins ([3], Theorem 1) showed a sufficient condition for the almost sure and L^α convergence of W_n for $\alpha \in (1, 2]$, but there was no information for the case $\alpha > 2$. When the environment is independent and identically distributed (i.i.d.), we can deduce the following result from ([11], Theorem 2.4) without effort, which completes and generalizes the results of [3, 11].

For $z = x + \mathbf{i}y \in \mathbb{C}^d$ fixed, write $\rho_z(s) = \mathbb{E}\frac{m_0(sx)}{|m_0(z)|^s}$ $(s \in \mathbb{R})$ if the expectation exists as real number.

Theorem 1.1 (Annealed L^α convergence)**.** *Assume that the environment (ξ_n) is i.i.d. Let $\alpha > 1$. If $\mathbb{E}(\sum_{u\in\mathbb{T}_1} |X_u|)^\alpha < \infty$ and $\max\{\rho_z(\alpha), \rho_z(\beta)\} < 1$ for some $1 < \beta \le \min\{2, \alpha\}$, then $\mathbb{E}(W^*)^\alpha < \infty$, so that W_n converges a.s., in $\mathbb{P}_\xi$-L^α for almost all ξ and in L^α.*

However, when the environment is stationary and ergodic rather than i.i.d, there were no corresponding results in the literature. Many times the methods available for i.i.d environments could not be applied directly to stationary and ergodic environments. For our problem, the main trouble is that it is difficult to estimate the upper bounds for $\mathbb{E}_\xi(\sum_{u\in\mathbb{T}_n} |X_u|^\beta)^{\frac{\alpha}{\beta}}$. Similar trouble was also encountered during our study on the L^α convergence rate of the real martingale in [23], where we obtained satisfactory result for the i.i.d environment case, but failed to acquire the corresponding result for all $\alpha > 1$ in the stationary and ergodic environment case. Such difficulty has been overcome in this paper. Instead of finding the direct upper bounds, we have discovered the asymptotic upper bounds (see Theorem 2.3), with which we successfully obtain the corresponding results of Theorem 1.1 for the stationary and ergodic environment case.

For $z = x + \mathbf{i}y \in \mathbb{C}^d$ fixed, write $f_z(s) = \mathbb{E}\log m_0(sx) - s\mathbb{E}\log|m_0(z)|$ $(s \in \mathbb{R})$ if the expectations exist as real numbers.

Theorem 1.2 (Quenched L^α convergence)**.** *Let $\alpha > 1$. If $\mathbb{E}\log^+ \mathbb{E}_\xi(\sum_{u\in\mathbb{T}_1} |X_u|)^\alpha < \infty$ and $\max\{f_z(\alpha), f_z(\beta)\} < 0$ for some $1 < \beta \le \min\{2, \alpha\}$, then $\mathbb{E}_\xi(W^*)^\alpha < \infty$ a.s., so that W_n converges a.s. and in $\mathbb{P}_\xi$-L^α for almost all ξ.*

Remark 1.3. (a) Apparently, the long-term behaviors of branching random walks can be investigated with the help of the additive martingale W_n. For example, we can use Theorem 1.2 to give a sharp upper bound for the deviation $Z_{n+1}(z) - m_n(z)Z_n(z)$, where $Z_n(z) := \sum_{u\in\mathbb{T}_n} e^{zS_u}$. Besides, in the study of the asymptotic behaviors of BRWRE, it is often necessary to check the convergence of the series in the form of $\sum_n a_0 \cdots a_{n-1}\mathbb{E}_{T^n\xi}(W^*)^\alpha$, where a_n is a random variable depending on ξ_n and T is the shift operator satisfying $T^n\xi = (\xi_n, \xi_{n+1}, \cdots)$ if $\xi = (\xi_0, \xi_1, \cdots)$. In this case, we need to first ensure the finiteness of the moment $\mathbb{E}_\xi(W^*)^\alpha$ before going a step further. It is also worth mentioning that the method presented in this paper may provide an available approach for the study of the convergence of the series mentioned above.

(b) From Theorem 1.2, we can see that W_n converges a.s. and in $\mathbb{P}_\xi$-L^α for almost all ξ to a non-trivial limit (pointwisely) on the set

$$\Lambda = \bigcup_{1<\alpha\le 2} \{z \in \mathbb{C}^d : \mathbb{E}\log^+ \mathbb{E}_\xi(\sum_{u\in\mathbb{T}_1} |X_u|)^\alpha < \infty \text{ and } f_z(\alpha) < 0\}. \tag{1.5}$$

In deterministic environment, Kolesko and Meiners [15] studied the convergence of W_n on the boundary of Λ. Their method can be extended to work on the analogous boundary problem for BRWRE with i.i.d. environment. However, for the stationary and ergodic environment case, as the boundary condition cannot ensure the trueness of the so-called many-to-one formula, the convergence of W_n on the boundary of Λ is still an open question.

In order to help readers better understand the set Λ in BRWRE and distinguish it from the one in classical branching random walk, we present below a simple example corresponding to Example 3.1 of [15].

Example 1.4 (Binary splitting with Gaussian increments in a random environment)**.** Given the environment $\xi = (\xi_n)$, we consider a branching random walk on $\mathbb{R}$ with independent Gaussian increments and binary splitting, i.e., $X(u) = (2, L_1(u), L_2(u))$, where $L_1(u)$, $L_2(u)$ are i.i.d. with one-dimensional Gaussian distribution $\mathcal{N}(\mu_{|u|}, \sigma^2_{|u|})$ conditioned on ξ. The parameter $(\mu_n, \sigma_n^2) = (\mu(\xi_n), \sigma^2(\xi_n))$ depends on the random variable ξ_n. Assume that $\mathbb{E}|\mu_0| < \infty$ and $\mathbb{E}(1/\sigma_0^2) \in (0,\infty)$. Similarly to ([15], Example 3.1), we calculate that $\mathbb{E}\log|m_0(z)| = \log 2 + x\mathbb{E}\mu_0 + \frac{1}{2}(x^2 - y^2)\mathbb{E}(1/\sigma_0^2)$ for $z = x + \mathbf{i}y \in \mathbb{C}$ and the set Λ defined in (1.5) now becomes

$$\Lambda = \{z = x + \mathbf{i}y \in \mathbb{C} : \alpha(\alpha-1)x^2 + \alpha y^2 + (1-\alpha)\frac{2\log 2}{\mathbb{E}(1/\sigma_0^2)} < 0 \text{ for some } \alpha \in (1,2]\}.$$

It is not hard to detect that the shape of Λ is similar to ([15], Figure 1) but with some minor changes in coordinates. Particularly, in the case where $\mathbb{E}(1/\sigma_0^2) = 1$, the figure of Λ coincides with ([15], Figure 1).

Under stronger conditions, we can further obtain the existence of the quenched weighted moments of W^*, of the forms $\mathbb{E}_\xi(W^*)^\alpha \ell(W^*)$, where $\alpha > 1$ and the measurable function $\ell : [0,\infty) \mapsto [0,\infty)$ is slowly varying at ∞, which means that $\lim_{s\to\infty} \frac{\ell(\lambda s)}{\ell(s)} = 1$ for all $\lambda > 0$.

Theorem 1.5 (Quenched weighted moments)**.** *Let $\alpha > 1$ and $\ell : [0,\infty) \mapsto [0,\infty)$ be a function slowly varying at ∞. If $\mathbb{E}\log^+ \mathbb{E}_\xi \left[(\sum_{u\in\mathbb{T}_1} |X_u|)^\alpha \ell(\sum_{u\in\mathbb{T}_1} |X_u|) \right] < \infty$ and $\alpha, \beta \in \operatorname{int}\{s \in \mathbb{R} : f_z(s) < 0\}$ for some $1 < \beta < \min\{2, \alpha\}$, then $\mathbb{E}_\xi(W^*)^\alpha \ell(W^*) < \infty$ a.s.*

In i.i.d. environment, corresponding annealed weighted moments can be deduced from Liang and Liu ([17], Theorem 1.1).

Moreover, thanks to Theorem 2.3, we can further investigate the exponential rate of the quenched L^α convergence of W_n to its limit, denoted by W if it exists.

Theorem 1.6 (Quenched L^α convergence rate)**.** *Let $\alpha > 1$ and $\rho > 1$.*

(a) If $1 < \alpha < 2$, $\mathbb{E}\log^+ \mathbb{E}_\xi(\sum_{u\in\mathbb{T}_1} |X_u|)^r < \infty$ and $\rho < \exp\{-\frac{1}{r} f_z(r)\}$ for some $r \in [\alpha, 2]$, then $W_n - W = o(\rho^{-n})$ a.s. and in $\mathbb{P}_\xi$-L^α for almost all ξ.

(b) Assume that $\alpha \geq 2$ and $\mathbb{E}\log^+ \mathbb{E}_\xi(\sum_{u\in\mathbb{T}_1} |X_u|)^\alpha < \infty$. Then for almost all ξ, the statement $W_n - W = o(\rho^{-n})$ in $\mathbb{P}_\xi$-L^α holds if $\rho < \rho_c$, and does not hold if $\rho > \rho_c$ and $\mathbb{E}\log^- \mathbb{E}_\xi|W_1 - 1|^2 < \infty$, where $\rho_c = \exp\{-\max\{\frac{1}{2} f_z(2), \frac{1}{\alpha} f_z(\alpha)\}\}$.

For $\mathbb{R}$-valued BRWRE (i.e. the space dimension $d = 1$), Wang and Huang ([23], Theorem 1.1) showed the exponential rate of the quenched L^α convergence of the non-negative martingale $W_n(x)$ for $1 < \alpha \leq \alpha_x$, where $\alpha_x \in (0,\infty]$ depending on x is a general constant that can be calculated accurately. The evident pity in that result is the lack of the description for the case $\alpha > \alpha_x$. Theorem 1.6 remedies this lack, and meanwhile generalizes the result to the complex martingale $W_n(z)$ in $\mathbb{R}^d$-valued BRWRE.

2 Mandelbrot martingale and auxiliary results

The proofs of theorems rely on the asymptotic properties of the Mandelbrot martingale in the random environment $\xi = (\xi_n)$. For each realization of ξ_n, there exists a distribution (denoted by $\tilde{\eta}_n = \tilde{\eta}(\xi_n)$) on $\mathbb{N} \times (0,\infty)^{\otimes \mathbb{N}^*}$ corresponds to it. Suppose that when the environment ξ is given, $\{(N(u), A_1(u), A_2(u), \cdots), u \in \mathbb{U}\}$ is a sequence of independent random variables taking values in $\mathbb{N} \times (0,\infty)^{\otimes \mathbb{N}^*}$; each $(N(u), A_1(u), A_2(u), \cdots)$ has distribution $\tilde{\eta}(\xi_n)$ if $|u| = n$. For simplicity, we write $(N, A_1, A_2, \cdots)$ for $(N(\emptyset), A_1(\emptyset), A_2(\emptyset), \cdots)$. For $u = u_1 \cdots u_n$ of length n, set $\tilde{X}_\emptyset = 1$ and $\tilde{X}_u = A_{u_1} A_{u_2}(u_1) \cdots A_{u_n}(u_1 \cdots u_{n-1})$. For $n \in \mathbb{N}$ and $s \in \mathbb{R}$, define

$$Y_0^{(s)} = 1 \qquad \text{and} \qquad Y_n^{(s)} = \sum_{u \in \mathbb{T}_n} \tilde{X}_u^s \quad (n \geq 1), \tag{2.1}$$

$$\tilde{m}_n(s) = \mathbb{E}_\xi \sum_{i=1}^{N(u)} A_i(u)^s \quad (|u| = n), \tag{2.2}$$

$$\tilde{P}_0(s) = 1 \qquad \text{and} \qquad \tilde{P}_n(s) = \prod_{i=0}^{n-1} \tilde{m}_i(s) \quad (n \geq 1). \tag{2.3}$$

Then $\tilde{P}_n(s) = \mathbb{E}_\xi Y_n^{(s)}$. We still assume that (1.2) holds, so that $\tilde{P}_n(s) > 0$ a.s. Let $\tilde{\Lambda}(s) = \mathbb{E} \log \tilde{m}_0(s)$ if the expectation exists as real number. Denote $\bar{Y}_n^{(s)} = \frac{Y_n^{(s)}}{\tilde{P}_n(s)} = \frac{Y_n^{(s)}}{\mathbb{E}_\xi Y_n^{(s)}}$. In particular, we write $Y_n = Y_n^{(1)}$ and $\bar{Y}_n = \bar{Y}_n^{(1)}$ for short. Let $\mathcal{E}_0 = \sigma(\xi)$ and $\mathcal{E}_n = \sigma(\xi, (N(u), A_1(u), A_2(u), \cdots);\ |u| < n)$ for $n \geq 1$. Then $\{\bar{Y}_n^{(s)}, \mathcal{E}_n\}$ forms a non-negative martingale under both laws $\mathbb{P}_\xi$ and $\mathbb{P}$. It is called the *Mandelbrot martingale in random environment*. For example, in the model of BRWRE introduced in Section 1, we can construct Mandelbrot martingales $\{\bar{Y}_n^{(s)}, \mathcal{F}_n\}$ in random environment by setting $\tilde{X}_u = |X_u|$. To complete the proofs of Theorems 1.2-1.6, an important step is to investigate the quenched moments of $Y_n^{(s)}$. Before that work, we present below a lemma about the random environment.

Lemma 2.1. *Let $(\alpha_n, \beta_n, \gamma_n)_{n \geq 0}$ be a stationary and ergodic sequence of non-negative random variables. If $\mathbb{E}|\log \alpha_0| < \infty$, $\mathbb{E}|\log \beta_0| < \infty$ and $\mathbb{E}\log^+ \gamma_0 < \infty$, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \left(\sum_{k=0}^{n-1} \alpha_0 \cdots \alpha_{k-1} \gamma_k \beta_{k+1} \cdots \beta_{n-1} \right) \leq \max\{\mathbb{E}\log\alpha_0, \mathbb{E}\log\beta_0\} \quad a.s. \tag{2.4}$$

Proof. Let $c_k = \frac{\gamma_k}{\beta_k}$ and $d_k = \frac{\alpha_k}{\beta_k}$. Since $\mathbb{E}\log^+ c_0 \leq \mathbb{E}\log^+ \gamma_0 + \mathbb{E}\log^- \beta_0 < \infty$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log^+ c_n = 0 \qquad a.s. \tag{2.5}$$

Besides, the ergodic theorem yields

$$\lim_{n\to\infty} \frac{1}{n} \log(\beta_0 \cdots \beta_{n-1}) = \mathbb{E}\log\beta_0 \qquad a.s., \tag{2.6}$$

$$\lim_{n\to\infty} \frac{1}{n} \log(d_0 \cdots d_{n-1}) = \mathbb{E}\log d_0 \qquad a.s. \tag{2.7}$$

By (2.5) and (2.7), a.s., for every $\varepsilon > 0$, there exists a random integer n_ε such that $d_0 \cdots d_{n-1} c_n \leq e^{(\mathbb{E}\log d_0 + \varepsilon)n}$ for all $n \geq n_\varepsilon$, so that

$$\sum_{k=0}^{n-1} d_0 \cdots d_{k-1} c_k \leq A_\varepsilon + \sum_{k=n_\varepsilon}^{n-1} e^{(\mathbb{E}\log d_0 + \varepsilon)k} \leq \begin{cases} A_\varepsilon + n, & \text{if } \mathbb{E}\log d_0 + \varepsilon \leq 0, \\ A_\varepsilon + n e^{(\mathbb{E}\log d_0 + \varepsilon)n}, & \text{if } \mathbb{E}\log d_0 + \varepsilon > 0, \end{cases}$$

where $A_\varepsilon = \sum_{k=0}^{n_\varepsilon - 1} d_0 \cdots d_{k-1} c_k < \infty$. Thus

$$\limsup_{n\to\infty} \frac{1}{n} \log \left(\sum_{k=0}^{n-1} d_0 \cdots d_{k-1} c_k \right) \quad \leq \quad \max\{0,\ \mathbb{E} \log d_0 + \varepsilon\} \qquad a.s.$$

Letting $\varepsilon \to 0$ and noticing (2.6), we get

$$\begin{aligned} & \limsup_{n\to\infty} \frac{1}{n} \log \left(\sum_{k=0}^{n-1} \alpha_0 \cdots \alpha_{k-1} \gamma_k \beta_{k+1} \cdots \beta_{n-1} \right) \\ = \quad & \limsup_{n\to\infty} \left(\frac{1}{n} \log \beta_0 \cdots \beta_{n-1} + \frac{1}{n} \log \left(\sum_{k=0}^{n-1} d_0 \cdots d_{k-1} c_k \right) \right) \\ \leq \quad & \max\{\mathbb{E} \log \alpha_0, \mathbb{E} \log \beta_0\} \quad a.s. \end{aligned}$$

The proof is complete. □

Now let us consider the quenched moments of $Y_n^{(s)}$.

Lemma 2.2. *Let $\alpha > 1$. If $\mathbb{E}_\xi Y_1^\alpha < \infty$ a.s., then for every n, we have $\mathbb{E}_\xi Y_n^\alpha < \infty$ a.s.*

Proof. It can be seen that for $s \in [1, \alpha]$, $\tilde{m}_0(s) \leq \mathbb{E}_\xi Y_1^s \leq (\mathbb{E}_\xi Y_1^\alpha)^{s/\alpha} < \infty$ a.s., hence $\tilde{P}_n(s) \in (0, \infty)$ a.s. for each n. Assume that $\alpha \in (2^m, 2^{m+1}]$ for some integer $m \geq 0$. We will prove the assertion by induction on m.

Firstly, for $m = 0$, we have $\alpha \in (1, 2]$, so that $\frac{\alpha}{2} \in (0, 1]$. Notice that

$$\bar{Y}_{n+1} - \bar{Y}_n = \sum_{u \in \mathbb{T}_n} \frac{\tilde{X}_u}{\tilde{P}_n(1)} (\bar{Y}_1(u) - 1),$$

with $\bar{Y}_1(u) = \frac{Y_1^{(1)}(u)}{\mathbb{E}_\xi [Y_1^{(1)}(u)]}$, where under the quenched law $\mathbb{P}_\xi$, $\{Y_k^{(s)}(u)\}_{|u|=n}$ are i.i.d. and independent of $\mathcal{E}_n$ with common distribution determined by $\mathbb{P}_\xi(Y_k^{(s)}(u) \in \cdot) = \mathbb{P}_{T^n \xi}(Y_k^{(s)} \in \cdot)$. Recall that the notation T represents the shift operator: $T^n \xi = (\xi_n, \xi_{n+1}, \cdots)$ if $\xi = (\xi_0, \xi_1, \cdots)$. Applying Burkholder's inequality (see e.g. [6], Theorem 11.2.1) twice and noticing the sub-additivity of the function $x \mapsto x^{\alpha/2}$, we get

$$\mathbb{E}_\xi |\bar{Y}_n - 1|^\alpha \leq C \sum_{k=0}^{n-1} \mathbb{E}_\xi |\bar{Y}_{k+1} - \bar{Y}_k|^\alpha \leq C \sum_{k=0}^{n-1} \frac{\tilde{P}_k(\alpha)}{\tilde{P}_k(1)^\alpha} \mathbb{E}_{T^k \xi} |\bar{Y}_1 - 1|^\alpha < \infty \quad a.s., \tag{2.8}$$

where $C > 0$ is a constant, and in general it does not necessarily stand for the same constant throughout. Thus the assertion holds for $m = 0$.

Now suppose that the assertion holds for $\alpha \in (2^m, 2^{m+1}]$. For $\alpha \in (2^{m+1}, 2^{m+2}]$, we have $\frac{\alpha}{2} \in (2^m, 2^{m+1}]$. Observe that $\mathbb{E}_\xi (Y_1^{(2)})^{\frac{\alpha}{2}} = \mathbb{E}_\xi (\sum_{u \in \mathbb{T}_1} \tilde{X}_u^2)^{\frac{\alpha}{2}} \leq \mathbb{E}_\xi Y_1^\alpha < \infty$ a.s. By the induction assumption, we have $\mathbb{E}_\xi (Y_n^{(2)})^{\frac{\alpha}{2}} < \infty$ a.s. for each n. Using Burkholder's inequality, Minkowski's inequality and Jensen's inequality, we deduce that

$$\begin{aligned} \left(\mathbb{E}_\xi |\bar{Y}_n - 1|^\alpha \right)^{\frac{2}{\alpha}} \quad & \leq \quad C \sum_{k=0}^{n-1} \left(\mathbb{E}_\xi |\bar{Y}_{k+1} - \bar{Y}_k|^\alpha \right)^{\frac{2}{\alpha}} \\ & \leq \quad C \sum_{k=0}^{n-1} \left(\mathbb{E}_\xi \left[\sum_{u \in \mathbb{T}_k} \left(\frac{\tilde{X}_u}{\tilde{P}_k(1)} \right)^2 (\bar{Y}_1(u) - 1)^2 \right]^{\frac{\alpha}{2}} \right)^{\frac{2}{\alpha}} \\ & \leq \quad C \sum_{k=0}^{n-1} \left(\frac{\mathbb{E}_\xi (Y_k^{(2)})^{\frac{\alpha}{2}}}{\tilde{P}_k(1)^\alpha} \mathbb{E}_{T^k \xi} |\bar{Y}_1 - 1|^\alpha \right)^{\frac{2}{\alpha}} < \infty \quad a.s., \end{aligned}$$

which means that the assertion also holds for $m+1$. $\square$

For $\alpha > 1$ and $\beta \in [1, \min\{2, \alpha\}]$, notice that $\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}} \le \mathbb{E}_\xi Y_n^\alpha$. It follows from Lemma 2.2 that the quenched moment $\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}} < \infty$ a.s. for every n if $\mathbb{E}_\xi Y_1^\alpha < \infty$ a.s. Furthermore, we can even find the asymptotic upper bounds for $\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}}$.

Theorem 2.3. *Let $\beta \in (1,2]$ and $\alpha \in (\beta^m, \beta^{m+1}]$ for some integer $m \ge 1$. If $\mathbb{E}\log^+ \mathbb{E}_\xi Y_1^\alpha < \infty$, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}} \le \alpha h(\alpha, \beta, \cdots, \beta^m) \qquad a.s., \tag{2.9}$$

where $h(s_1, \cdots, s_k) = \max\{\frac{\tilde\Lambda(s_1)}{s_1}, \cdots, \frac{\tilde\Lambda(s_k)}{s_k}\}$.

Remark 2.4. By the convexity of $\tilde\Lambda(s)$, it can be seen that $h(\alpha, \beta, \cdots, \beta^m) \le \sup_{\beta \le s \le \alpha} \{\frac{1}{s}\tilde\Lambda(s)\} = h(\alpha, \beta)$ if $\alpha \in (\beta^m, \beta^{m+1}]$.

Proof of Theorem 2.3. We will prove the assertion by induction on m. Firstly, we consider $m = 1$, in which case $\alpha \in (\beta, \beta^2]$, so that $\frac{\alpha}{\beta} \in (1, \beta] \subset (1, 2]$. Similarly to (2.8), using Burkholder's inequality twice, we obtain

$$\mathbb{E}_\xi (\bar Y_n^{(\beta)})^{\frac{\alpha}{\beta}} \le C\left(\mathbb{E}_\xi |\bar Y_n^{(\beta)} - 1|^{\frac{\alpha}{\beta}} + 1\right) \le C\left(\sum_{k=0}^{n-1} \frac{\tilde P_k(\alpha)}{\tilde P_k(\beta)^{\frac{\alpha}{\beta}}} \frac{\gamma_k}{\tilde m_k(\beta)^{\frac{\alpha}{\beta}}} + 1\right), \tag{2.10}$$

where $\gamma_n = \mathbb{E}_{T^n\xi} |Y_1^{(\beta)} - \tilde m_n(\beta)|^{\frac{\alpha}{\beta}}$. Notice that

$$\gamma_0 \le C\left(\mathbb{E}_\xi (Y_1^{(\beta)})^{\frac{\alpha}{\beta}} + \tilde m_0(\beta)^{\frac{\alpha}{\beta}}\right) \le C\left(\mathbb{E}_\xi Y_1^\alpha + \tilde m_0(\beta)^{\frac{\alpha}{\beta}}\right).$$

Thus $\mathbb{E}\log^+ \gamma_0 < \infty$, since $\mathbb{E}\log^+ \mathbb{E}_\xi Y_1^\alpha < \infty$ and $\tilde\Lambda(\beta)$ exists. From (2.10), we deduce

$$\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}} = \mathbb{E}_\xi (\bar Y_n^{(\beta)})^{\frac{\alpha}{\beta}} \tilde P_n(\beta)^{\frac{\alpha}{\beta}} \le C\left(A_n + \tilde P_n(\beta)^{\frac{\alpha}{\beta}}\right), \tag{2.11}$$

where $A_n = \sum_{k=0}^{n-1} \tilde P_k(\alpha) \gamma_k \tilde m_{k+1}(\beta)^{\frac{\alpha}{\beta}} \cdots \tilde m_{n-1}(\beta)^{\frac{\alpha}{\beta}}$. Since $\mathbb{E}\log^+ \gamma_0 < \infty$, $\tilde\Lambda(\alpha)$ and $\tilde\Lambda(\beta)$ exist, by Lemma 2.1,

$$\limsup_{n\to\infty} \frac{1}{n} \log A_n \le \max\{\tilde\Lambda(\alpha), \frac{\alpha}{\beta}\tilde\Lambda(\beta)\} = \alpha h(\alpha, \beta) \qquad a.s. \tag{2.12}$$

Therefore, by (2.12) and the ergodic theorem,

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log\left(A_n + \tilde P_n(\beta)^{\frac{\alpha}{\beta}}\right) &\le \max\{\limsup_{n\to\infty} \frac{1}{n} \log A_n, \limsup_{n\to\infty} \frac{1}{n} \log \tilde P_n(\beta)^{\frac{\alpha}{\beta}}\} \\ &\le \alpha h(\alpha, \beta) \quad a.s. \end{aligned} \tag{2.13}$$

Combining (2.13) with (2.11) leads to $\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}} \le \alpha h(\alpha, \beta)$ a.s., which means that the assertion holds for $m = 1$.

Now suppose that the assertion holds for $\alpha \in (\beta^m, \beta^{m+1}]$ with $m \ge 1$. For $\alpha \in (\beta^{m+1}, \beta^{m+2}]$, we have $\frac{\alpha}{\beta} \in (\beta^m, \beta^{m+1}]$ and $\frac{\alpha}{\beta^2} > \beta^{m-1} \ge 1$. Let

$$D_n^{(\beta)} = Y_{n+1}^{(\beta)} - Y_n^{(\beta)} \tilde m_n(\beta) = \sum_{u \in \mathbb{T}_n} \tilde X_u^\beta \left(Y_1^{(\beta)}(u) - \tilde m_n(\beta)\right).$$

Then we can write $Y_n^{(\beta)} = \sum_{k=0}^{n-1} D_k^{(\beta)} \tilde{m}_{k+1}(\beta) \cdots \tilde{m}_{n-1}(\beta) + \tilde{P}_n(\beta)$. Therefore,

$$\begin{aligned}
\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}} &= \mathbb{E}_\xi \left(\sum_{k=0}^{n-1} D_k^{(\beta)} \tilde{m}_{k+1}(\beta) \cdots \tilde{m}_{n-1}(\beta) + \tilde{P}_n(\beta) \right)^{\frac{\alpha}{\beta}} \\
&\leq (n+1)^{\frac{\alpha}{\beta}-1} \left(\sum_{k=0}^{n-1} \mathbb{E}_\xi |D_k^{(\beta)}|^{\frac{\alpha}{\beta}} \tilde{m}_{k+1}(\beta)^{\frac{\alpha}{\beta}} \cdots \tilde{m}_{n-1}(\beta)^{\frac{\alpha}{\beta}} + \tilde{P}_n(\beta)^{\frac{\alpha}{\beta}} \right). \quad (2.14)
\end{aligned}$$

Applying the BDG-inequality (see e.g. [6], Theorem 11.3.2) to $\{D_n^{(\beta)}\}$, we get

$$\begin{aligned}
&\mathbb{E}_\xi \left[|D_n^{(\beta)}|^{\frac{\alpha}{\beta}} | \mathcal{E}_n \right] \\
&\leq C \left(\left(\sum_{u \in \mathbb{T}_n} \mathbb{E}_\xi \left[\tilde{X}_u^{\beta^2} |Y_1^{(\beta)}(u) - \tilde{m}_n(\beta)|^\beta \Big| \mathcal{E}_n \right] \right)^{\frac{\alpha}{\beta^2}} + \sum_{u \in \mathbb{T}_n} \mathbb{E}_\xi \left[\tilde{X}_u^{\alpha} |Y_1^{(\beta)}(u) - \tilde{m}_n(\beta)|^{\frac{\alpha}{\beta}} \Big| \mathcal{E}_n \right] \right) \\
&\leq C \left((Y_n^{(\beta^2)})^{\frac{\alpha}{\beta^2}} \gamma_n + Y_n^{(\alpha)} \gamma_n \right).
\end{aligned}$$

Taking the expectation $\mathbb{E}_\xi$ yields

$$\mathbb{E}_\xi |D_n^{(\beta)}|^{\frac{\alpha}{\beta}} \leq C \left(\mathbb{E}_\xi (Y_n^{(\beta^2)})^{\frac{\alpha}{\beta^2}} + \tilde{P}_n(\alpha) \right) \gamma_n. \tag{2.15}$$

Combining (2.15) with (2.14), we get

$$\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}} \leq C(n+1)^{\frac{\alpha}{\beta}-1} \left(B_n + A_n + \tilde{P}_n(\beta)^{\frac{\alpha}{\beta}} \right), \tag{2.16}$$

where $B_n = \sum_{k=0}^{n-1} C_{n,k}$ and $C_{n,k} = \mathbb{E}_\xi (Y_k^{(\beta^2)})^{\frac{\alpha}{\beta^2}} \gamma_k \tilde{m}_{k+1}(\beta)^{\frac{\alpha}{\beta}} \cdots \tilde{m}_{n-1}(\beta)^{\frac{\alpha}{\beta}}$. Since $\frac{\alpha}{\beta} \in (\beta^m, \beta^{m+1}]$ and $\mathbb{E} \log^+ \mathbb{E}_\xi (Y_1^{(\beta)})^{\frac{\alpha}{\beta}} \leq \mathbb{E} \log^+ \mathbb{E}_\xi Y_1^\alpha < \infty$, by the induction assumption,

$$\begin{aligned}
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}_\xi (Y_n^{(\beta^2)})^{\frac{\alpha}{\beta^2}} &\leq \frac{\alpha}{\beta} \max\{ \frac{\beta}{\alpha} \tilde{\Lambda}(\alpha), \frac{1}{\beta} \tilde{\Lambda}(\beta^2), \cdots, \frac{1}{\beta^m} \tilde{\Lambda}(\beta^{m+1}) \} \\
&= \alpha h(\alpha, \beta^2, \cdots, \beta^{m+1}) \qquad a.s.
\end{aligned}$$

It follows that for every $\varepsilon > 0$, there exists a random integer $n_\varepsilon > 0$ such that for $n \geq n_\varepsilon$,

$$\mathbb{E}_\xi (Y_n^{(\beta^2)})^{\frac{\alpha}{\beta^2}} \leq \exp\{ \left(\alpha h(\alpha, \beta^2, \cdots, \beta^{m+1}) + \varepsilon \right) n \} \qquad a.s. \tag{2.17}$$

For n large enough, decompose

$$B_n = \sum_{k=0}^{n_\varepsilon - 1} C_{n,k} + \sum_{k=n_\varepsilon}^{n-1} C_{n,k} =: B_{1,n}(\varepsilon) + B_{2,n}(\varepsilon).$$

For $B_{2,n}(\varepsilon)$, it follows from (2.17) that

$$B_{2,n}(\varepsilon) \leq \sum_{k=0}^{n-1} \exp\{ \left(\alpha h(\alpha, \beta^2, \cdots, \beta^{m+1}) + \varepsilon \right) k \} \gamma_k \tilde{m}_{k+1}(\beta)^{\frac{\alpha}{\beta}} \cdots \tilde{m}_{n-1}(\beta)^{\frac{\alpha}{\beta}} \qquad a.s.$$

Using Lemma 2.1, we obtain

$$\limsup_{n \to \infty} \frac{1}{n} \log B_{2,n}(\varepsilon) \leq \max\{ \alpha h(\alpha, \beta^2, \cdots, \beta^{m+1}) + \varepsilon, \frac{\alpha}{\beta} \tilde{\Lambda}(\beta) \} \leq \alpha h(\alpha, \beta, \cdots, \beta^{m+1}) + \varepsilon \; a.s. \tag{2.18}$$

For $B_{1,n}(\varepsilon)$, notice that $\mathbb{E}_\xi (Y_k^{(\beta^2)})^{\frac{\alpha}{\beta^2}} \le \mathbb{E}_\xi (\bar{Y}_k^{(\beta)})^{\frac{\alpha}{\beta}} \tilde{P}_k(\beta)^{\frac{\alpha}{\beta}}$. Thus

$$\begin{aligned} B_{1,n}(\varepsilon) &\le \sum_{k=0}^{n_\varepsilon - 1} \mathbb{E}_\xi (\bar{Y}_k^{(\beta)})^{\frac{\alpha}{\beta}} \tilde{P}_k(\beta)^{\frac{\alpha}{\beta}} \gamma_k \tilde{m}_{k+1}(\beta)^{\frac{\alpha}{\beta}} \cdots \tilde{m}_{n-1}(\beta)^{\frac{\alpha}{\beta}} \\ &= \tilde{P}_n(\beta)^{\frac{\alpha}{\beta}} \sum_{k=0}^{n_\varepsilon - 1} \mathbb{E}_\xi (\bar{Y}_k^{(\beta)})^{\frac{\alpha}{\beta}} \frac{\gamma_k}{\tilde{m}_k(\beta)^{\frac{\alpha}{\beta}}}. \end{aligned}$$

By Lemma 2.2, we have $\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}} < \infty$ a.s. for each n, so that

$$C_\varepsilon(\xi) := \sum_{k=0}^{n_\varepsilon - 1} \mathbb{E}_\xi (\bar{Y}_k^{(\beta)})^{\frac{\alpha}{\beta}} \frac{\gamma_k}{\tilde{m}_k(\beta)^{\frac{\alpha}{\beta}}} < \infty \qquad a.s.$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \log B_{1,n}(\varepsilon) \le \limsup_{n\to\infty} \left(\frac{1}{n} \log \tilde{P}_n(\beta)^{\frac{\alpha}{\beta}} + \frac{1}{n} \log C_\varepsilon(\xi) \right) = \frac{\alpha}{\beta} \tilde{\Lambda}(\beta) \qquad a.s. \tag{2.19}$$

By (2.18) and (2.19),

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log B_n &\le \max\{\limsup_{n\to\infty} \frac{1}{n} \log B_{1,n}(\varepsilon), \limsup_{n\to\infty} \frac{1}{n} \log B_{2,n}(\varepsilon)\} \\ &\le \alpha h(\alpha, \beta, \cdots, \beta^{m+1}) + \varepsilon \qquad a.s. \end{aligned}$$

Since ε is arbitrary, we get

$$\limsup_{n\to\infty} \frac{1}{n} \log B_n \le \alpha h(\alpha, \beta, \cdots, \beta^{m+1}) \qquad a.s. \tag{2.20}$$

Finally, combining (2.20) and (2.13) with (2.16), the assertion also holds for $m+1$. □

Lemma 2.5. *Let $(\zeta_n)_{n\ge 0}$ be a stationary and ergodic sequence of non-negative random variables satisfying $\mathbb{E} \log^+ \zeta_0 < \infty$. Assume that $\mathbb{E} \log^+ \mathbb{E}_\xi Y_1^\alpha < \infty$ for some $\alpha > 1$.*

- *(a) If $\max\{\tilde{\Lambda}(\alpha), \tilde{\Lambda}(\beta)\} < 0$ for some $\beta \in (1, 2]$ with $\beta \le \alpha$, then for every constant $\gamma > 0$, the series $\sum_n \left(\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}} \right)^\gamma \zeta_n < \infty$ a.s.*
- *(b) If $\alpha, \beta \in \operatorname{int}\{s \in \mathbb{R} : \tilde{\Lambda}(s) < 0\}$ for some $\beta \in (1, 2]$ with $\beta < \alpha$, then there exists a constant $\varepsilon_0 > 0$ such that for every constant $0 < \varepsilon < \varepsilon_0$ and every constant $\gamma > 0$, the series $\sum_n \left(\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha \pm \varepsilon}{\beta}} \right)^\gamma \zeta_n < \infty$ a.s.*

Proof. Proof of the assertion (a). If $\alpha = \beta \in (1, 2]$, it follows from ([12], Lemma 3.1) that the series $\sum_n \left(\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}} \right)^\gamma \zeta_n = \sum_n \tilde{P}_n(\alpha)^\gamma \zeta_n < \infty$ a.s., since $\tilde{\Lambda}(\alpha) < 0$ and $\mathbb{E} \log^+ \zeta_0 < \infty$. Now we consider the case where $\alpha > \beta$. Notice that $\mathbb{E} \log^+ \zeta_0 < \infty$ implies that $\lim_{n\to\infty} \frac{1}{n} \log^+ \zeta_n = 0$ a.s. By Theorem 2.3 and Remark 2.4,

$$\limsup_{n\to\infty} \frac{1}{n} \log \left[\left(\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}} \right)^\gamma \zeta_n \right] \le \gamma \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}} \le \gamma \alpha h(\alpha, \beta) < 0 \quad a.s.,$$

which implies that the series $\sum_n \left(\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}} \right)^\gamma \zeta_n < \infty$ a.s.

Proof of the assertion (b). Denote $\alpha_\varepsilon = \alpha \pm \varepsilon$. Since $\alpha, \beta \in \operatorname{int}\{s \in \mathbb{R} : \tilde{\Lambda}(s) < 0\}$, we can take $1 < \beta' < \beta$ and $\alpha' > \alpha$ such that $\tilde{\Lambda}(s) < 0$ on $[\beta', \alpha']$. Let $M_n = \sup_{u \in \mathbb{T}_n} \tilde{X}_u^{\beta - \beta'}$. Then

$$Y_n^{(\beta)} = \sum_{u \in \mathbb{T}_n} \tilde{X}_u^{\beta'} \tilde{X}_u^{\beta - \beta'} \le M_n Y_n^{(\beta')}. \tag{2.21}$$

For $\delta>0$, set $\frac{1}{q}=\frac{\alpha_\varepsilon\beta'}{(\alpha-\delta)\beta}$, $\frac{1}{p}=1-\frac{1}{q}=\frac{(\alpha-\delta)\beta-\alpha_\varepsilon\beta'}{(\alpha-\delta)\beta}$ and $\alpha^*=(\beta-\beta')\frac{\alpha_\varepsilon}{\beta}p=\frac{(\beta-\beta')\alpha_\varepsilon(\alpha-\delta)}{(\alpha-\delta)\beta-\alpha_\varepsilon\beta'}$. Take δ and ε small enough such that $p,q>1$ and $\alpha-\delta,\alpha^*\in(\beta',\alpha')$. By (2.21) and Hölder's inequality,

$$\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha_\varepsilon}{\beta}} \leq \left[\mathbb{E}_\xi M_n^{\frac{\alpha_\varepsilon}{\beta}p}\right]^{1/p}\left[\mathbb{E}_\xi (Y_n^{(\beta')})^{\frac{\alpha_\varepsilon}{\beta}q}\right]^{1/q} \leq \tilde{P}_n(\alpha^*)^{1/p}\left[\mathbb{E}_\xi (Y_n^{(\beta')})^{\frac{\alpha-\delta}{\beta'}}\right]^{1/q}. \quad (2.22)$$

Since $\mathbb{E}\log^+\mathbb{E}_\xi Y_1^{\alpha-\delta}<\infty$, by Theorem 2.3 and Remark 2.4,

$$\limsup_{n\to\infty}\frac{1}{n}\log \mathbb{E}_\xi (Y_n^{(\beta')})^{\frac{\alpha-\delta}{\beta'}} \leq (\alpha-\delta)h(\alpha-\delta,\beta')<0 \qquad a.s. \quad (2.23)$$

Noticing (2.22) and (2.23), we calculate that

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n}\log\left[\left(\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha_\varepsilon}{\beta}}\right)^\gamma \zeta_n\right]\\
\leq\ & \lim_{n\to\infty}\frac{1}{n}\log\tilde{P}_n(\alpha^*)^{\gamma/p}+\limsup_{n\to\infty}\frac{1}{n}\log\left[\mathbb{E}_\xi (Y_n^{(\beta')})^{\frac{\alpha-\delta}{\beta'}}\right]^{\gamma/q}+\lim_{n\to\infty}\frac{1}{n}\log^+\zeta_n\\
\leq\ & \frac{\gamma}{p}\tilde{\Lambda}(\alpha^*)+\frac{\gamma}{q}(\alpha-\delta)h(\alpha-\delta,\beta')<0 \quad a.s.,
\end{aligned}$$

which implies the a.s. convergence of the series $\sum_n \left(\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha_\varepsilon}{\beta}}\right)^\gamma \zeta_n$. □

3 Proof of Theorems 1.2-1.6

Let us come back to BRWRE and give the proofs of Theorems 1.2-1.6. As Theorem 1.1 can be proved by arguments in the proof of ([11], Theorem 2.4) with tiny modifications, we omit its proof.

For $z=x+\mathbf{i}y\in\mathbb{C}^d$ fixed, set $\tilde{X}_u=|X_u|=\frac{e^{xS_u}}{|P_{|u|}(z)|}$. We will use the notations introduced in Section 2 for the Mandelbrot martingale. Note that now $\tilde{\Lambda}(s)=f_z(s)$. Moreover, for $u\in\mathbb{U}$, denote

$$B_u=\sum_{i=1}^{N(u)}\frac{e^{zL_i(u)}}{m_{|u|}(z)}-1 \qquad \text{and} \qquad \zeta_n^{(s)}=\mathbb{E}_\xi|B_u|^s \quad (|u|=n).$$

It can be seen that if $\mathbb{E}\log^+\mathbb{E}_\xi(\sum_{u\in\mathbb{T}_1}\tilde{X}_u)^\alpha<\infty$, then $\mathbb{E}\log^+\zeta_0^{(s)}<\infty$ for $0<s\leq\alpha$. In the following proofs, we will use the generalized BDG-inequality for complex martingales (we still call it the BDG-inequality later). Such inequality can be obtained by applying the classical BDG-inequality for real-valued martingales to the real and imaginary parts respectively of the complex martingales and noticing the convexity and monotonicity of the related functions.

Proof of Theorem 1.2. We shall prove that $\mathbb{E}_\xi\sup_{n\geq0}|W_n-1|^\alpha<\infty$ a.s., which is equivalent to $\mathbb{E}_\xi(W^*)^\alpha<\infty$ a.s. By the BDG-inequality, we see that for $1\leq\beta\leq\min\{2,\alpha\}$,

$$\begin{aligned}
\mathbb{E}_\xi\sup_{n\geq0}|W_n-1|^\alpha &\leq C\left(\mathbb{E}_\xi\left(\sum_{n=0}^\infty \mathbb{E}_\xi\left[|W_{n+1}-W_n|^\beta\middle|\mathcal{F}_n\right]\right)^{\frac{\alpha}{\beta}}+\sum_{n=0}^\infty\mathbb{E}_\xi|W_{n+1}-W_n|^\alpha\right)\\
&=: C(A_1(\xi)+B_1(\xi)).
\end{aligned}$$

We need to show that $A_1(\xi)$ and $B_1(\xi)$ are finite a.s. Applying the BDG-inequality to the martingale difference $\{W_{n+1} - W_n\}$, we have for $1 \le \beta \le \min\{2, \gamma\}$,

$$\begin{aligned}\mathbb{E}_\xi\left[|W_{n+1} - W_n|^\gamma | \mathcal{F}_n\right] &\le C\left(\mathbb{E}_\xi\left[\left(\sum_{u\in\mathbb{T}_n} \tilde{X}_u^\beta \mathbb{E}_\xi|B_u|^\beta\right)^{\frac{\gamma}{\beta}}\middle|\mathcal{F}_n\right] + \sum_{u\in\mathbb{T}_n} \tilde{X}_u^\gamma \mathbb{E}_\xi|B_u|^\gamma\right)\\ &= C\left((Y_n^{(\beta)})^{\frac{\gamma}{\beta}}(\zeta_n^{(\beta)})^{\frac{\gamma}{\beta}} + Y_n^{(\gamma)}\zeta_n^{(\gamma)}\right). \end{aligned} \tag{3.1}$$

For $A_1(\xi)$, using (3.1) with $\gamma = \beta$ and Minkowski's inequality, we have

$$A_1(\xi)^{\frac{\beta}{\alpha}} \le C\left[\mathbb{E}_\xi\left(\sum_{n=0}^\infty Y_n^{(\beta)}\zeta_n^{(\beta)}\right)^{\frac{\alpha}{\beta}}\right]^{\frac{\beta}{\alpha}} \le C\sum_{n=0}^\infty \left[\mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}}\right]^{\frac{\beta}{\alpha}} \zeta_n^{(\beta)}.$$

Notice that here $\mathbb{E}\log^+ \zeta_0^{(s)} < \infty$ ($s = \alpha, \beta$), $\mathbb{E}\log^+ \mathbb{E}_\xi Y_1^\alpha < \infty$ and $\max\{\tilde{\Lambda}(\alpha),\ \tilde{\Lambda}(\beta)\} < 0$. Hence $A_1(\xi) < \infty$ a.s. by Lemma 2.5(a). For $B_1(\xi)$, using (3.1) with $\gamma = \alpha$, we have

$$B_1(\xi) \le C\left(\sum_{n=0}^\infty \mathbb{E}_\xi (Y_n^{(\beta)})^{\frac{\alpha}{\beta}}(\zeta_n^{(\beta)})^{\frac{\alpha}{\beta}} + \sum_{n=0}^\infty \tilde{P}_n(\alpha)\zeta_n^{(\alpha)}\right). \tag{3.2}$$

The first series in the right hand side of (3.2) converges a.s. by Lemma 2.5(a), and the second series $\sum_n \tilde{P}_n(\alpha)\zeta_n^{(\alpha)} < \infty$ a.s. by ([12], Lemma 3.1). Thus $B_1(\xi) < \infty$ a.s. □

Proof of Theorem 1.5. Take $\beta < \beta_1 < \min\{2, \alpha\}$. Clearly, $\beta_1 \in \operatorname{int}\{s \in \mathbb{R} : \tilde{\Lambda}(s) < 0\}$. Put $\phi(x) = x^\alpha \ell(x)$. Without loss of generality, we assume that $\phi(x)$ and $\phi(x^{\frac{1}{\beta_1}})$ are increasing, convex on $[0, \infty)$ and $\ell(x) > 0$ for all $x \ge 0$ (see [16], Lemma 3.1). By the BDG-inequality and using (3.1) (with $\alpha = \beta = \beta_1$), we obtain

$$\begin{aligned}\mathbb{E}_\xi \phi(\sup_{n\ge 0}|W_n - 1|) &\le C\left(\mathbb{E}_\xi\phi(\left(\sum_{n=0}^\infty Y_n^{(\beta_1)}\zeta_n^{(\beta_1)}\right)^{\frac{1}{\beta_1}}) + \sum_{n=0}^\infty \mathbb{E}_\xi\phi(|W_{n+1} - W_n|)\right)\\ &=: C(A_2(\xi) + B_2(\xi)).\end{aligned}$$

We will prove that $A_2(\xi)$ and $B_2(\xi)$ are finite a.s. For $A_2(\xi)$, by Potter's theorem (see [5], Theorem 1.5.6), for $0 < \varepsilon < \alpha - \beta_1$, we have

$$A_2(\xi) \le C\left(\mathbb{E}_\xi\left(\sum_{n=0}^\infty Y_n^{(\beta_1)}\zeta_n^{(\beta_1)}\right)^{\frac{\alpha+\varepsilon}{\beta_1}} + \mathbb{E}_\xi\left(\sum_{n=0}^\infty Y_n^{(\beta_1)}\zeta_n^{(\beta_1)}\right)^{\frac{\alpha-\varepsilon}{\beta_1}}\right) =: C\left(A_2^+ + A_2^-\right).$$

By Minkowski's inequality and Lemma 2.5(b), we have $A_2^\pm < \infty$ a.s. for ε small enough, so that $A_2(\xi) < \infty$ a.s. Now we consider $B_2(\xi)$. Denote $\Delta_n = \mathbb{E}_\xi\phi(|W_{n+1} - W_n|)$. By the BDG-inequality,

$$\Delta_n \le C\left(\mathbb{E}_\xi\phi(\left(Y_n^{(\beta_1)}\zeta_n^{(\beta_1)}\right)^{\frac{1}{\beta_1}}) + \mathbb{E}_\xi\phi(\sup_{u\in\mathbb{T}_n}\tilde{X}_u|B_u|)\right) =: C\left(\Delta_{1,n} + \Delta_{2,n}\right).$$

For $\Delta_{1,n}$, again by Potter's theorem, we have for $\varepsilon > 0$,

$$\Delta_{1,n} \le C\left(\mathbb{E}_\xi\left(Y_n^{(\beta_1)}\zeta_n^{(\beta_1)}\right)^{\frac{\alpha+\varepsilon}{\beta_1}} + \mathbb{E}_\xi\left(Y_n^{(\beta_1)}\zeta_n^{(\beta_1)}\right)^{\frac{\alpha-\varepsilon}{\beta_1}}\right) =: C\left(\Delta_{1,n}^+ + \Delta_{1,n}^-\right).$$

It can be deduced from Lemma 2.5(b) that the series $\sum_n \Delta^{\pm}_{1,n} < \infty$ a.s. for ε small enough.

For $\Delta_{2,n}$, by the increasing and convex property of the function $\phi(x^{\frac{1}{\beta_1}})$ and using Potter's theorem,

$$\begin{aligned}
\Delta_{2,n} &\le \mathbb{E}_\xi \phi\Big(\Big(\sum_{u\in\mathbb{T}_n} \tilde{X}_u^{\beta_1} |B_u|^{\beta_1}\Big)^{\frac{1}{\beta_1}}\Big) \\
&\le \mathbb{E}_\xi \left(\sum_{u\in\mathbb{T}_n} \frac{\tilde{X}_u^{\beta_1}}{Y_n^{(\beta_1)}} \phi(|B_u|(Y_n^{(\beta_1)})^{\frac{1}{\beta_1}}) \right) \\
&\le C\left(\Delta^+_{2,n} + \Delta^-_{2,n}\right),
\end{aligned}$$

where $\Delta^{\pm}_{2,n} = \mathbb{E}_\xi (Y_n^{(\beta_1)})^{\frac{\alpha\pm\varepsilon}{\beta_1}} \mathbb{E}_{T^n\xi}\phi(|B_\emptyset|)$. Since $\mathbb{E}\log^+ \mathbb{E}_\xi \phi(|B_\emptyset|) < \infty$, it follows from Lemma 2.5(b) that $\sum_n \Delta^{\pm}_{2,n} < \infty$ a.s. if ε small enough. Thus $B_2(\xi) = \sum_n \Delta_n < \infty$ a.s. $\square$

Proof of Theorem 1.6. The proof is very similar to the proof of ([23], Theorem 1.1). For $\rho > 1$, set $A(\rho) = \sum_{n=0}^{\infty} \rho^n (W - W_n)$ and $\hat{A}_n(\rho) = \sum_{k=0}^{n} \rho^k (W_{k+1} - W_k)$. By ([1], Lemma 3.1 and Remark 3.1), $A(\rho)$ converges in $\mathbb{P}_\xi$-L^α if and only if $\sup_n \mathbb{E}_\xi |\hat{A}_n(\rho)|^\alpha < \infty$, and the L^α convergence of $A(\rho)$ implies its a.s. convergence.

For the assertion (a), using Burkholder's inequality twice gives

$$\sup_n \mathbb{E}_\xi |\hat{A}_n(\rho)|^\alpha \le C \sum_{n=0}^{\infty} \rho^{\alpha n} \tilde{P}_n(r)^{\alpha/r} (\zeta_n^{(r)})^{\alpha/r} < \infty \qquad a.s.,$$

since $\mathbb{E}\log^+ \zeta_0^{(r)} < \infty$ and $\log\rho + \frac{1}{r}\tilde{\Lambda}(r) < 0$. Thus we have $\lim_{n\to\infty} \rho^n (W - W_n) = 0$ a.s. and in $\mathbb{P}_\xi$-L^α for almost all ξ.

For the assertion (b), by Burkholder's inequality, Minkowski's inequality and the formula (2.8) in [23],

$$\sup_n \mathbb{E}_\xi |\hat{A}_n(\rho)|^\alpha \le C \left(\sum_{n=0}^{\infty} \rho^{2n} \left(\mathbb{E}_\xi (Y_n^{(2)})^{\frac{\alpha}{2}} \right)^{\frac{2}{\alpha}} (\zeta_n^{(\alpha)})^{\frac{2}{\alpha}} \right)^{\frac{\alpha}{2}}.$$

Since $\mathbb{E}\log^+ \mathbb{E}_\xi Y_1^\alpha < \infty$, by Theorem 2.3 and Remark 2.4,

$$\limsup_{n\to\infty} \frac{1}{n} \log \left[\rho^{2n} \left(\mathbb{E}_\xi (Y_n^{(2)})^{\frac{\alpha}{2}} \right)^{\frac{2}{\alpha}} (\zeta_n^{(\alpha)})^{\frac{\alpha}{2}} \right] \le 2\log\rho + 2h(\alpha, 2) < 0 \qquad a.s.$$

if $\rho < \rho_c = \exp\{-h(\alpha,2)\}$. It follows that $\sup_n \mathbb{E}_\xi |\hat{A}_n(\rho)|^\alpha < \infty$ a.s., which implies that $\lim_{n\to\infty} \rho^n \left(\mathbb{E}_\xi |W - W_n|^\alpha\right)^{\frac{1}{\alpha}} = 0$ a.s. Conversely, if $\rho > \rho_c$ and $\mathbb{E}\log^- \zeta_0^{(2)} < \infty$, we suppose that $\lim_{n\to\infty} \rho^n \left(\mathbb{E}_\xi |W - W_n|^\alpha\right)^{\frac{1}{\alpha}} = 0$ a.s. Now we can deduce that for any $\rho_1 \in (1, \rho)$, $\sup_n \mathbb{E}_\xi |\hat{A}_n(\rho_1)|^\alpha < \infty$ a.s. Thus, by Burkholder's inequality,

$$\sum_{n=0}^{\infty} \rho_1^{\alpha n} \tilde{P}_n(r)^{\alpha/r} (\zeta_n^{(r)})^{\alpha/r} \le C \sup_n \mathbb{E}_\xi |\hat{A}_n(\rho_1)|^\alpha < \infty \qquad a.s.$$

for all $r \in [2, \alpha]$. Since $\mathbb{E}|\log \zeta_0^{(r)}| \le \frac{r}{\alpha}\mathbb{E}\log^+ \zeta_0^{(\alpha)} + \frac{r}{2}\mathbb{E}\log^- \zeta_0^{(2)} < \infty$, Lemma 3.1 of [12] yields $\log\rho_1 \le -\frac{1}{r}\tilde{\Lambda}(r)$ for all $r \in [2, \alpha]$, namely, $\log\rho_1 \le \inf_{2\le r\le\alpha} \{-\frac{1}{r}\tilde{\Lambda}(r)\} = \log\rho_c$. Letting $\rho_1 \uparrow \rho$ yields $\rho \le \rho_c$, which contradicts the fact that $\rho > \rho_c$. $\square$

References

[1] Alsmeyer, G., Iksanov, A., Polotskiy, S. and Rösler, U.: Exponential rate of L_p-convergence of instrinsic martingales in supercritical branching random walks. *Theory Stoch. Process* **15**, (2009), 1–18. MR-2598524

[2] Biggins, J.D.: Martingale convergence in the branching random walk. *J. Appl. prob.* **14**, (1977), 25–37. MR-0433619

[3] Biggins, J.D.: Uniform convergence of martingales in the branching random walk. *Ann. Prob.* **20**, (1992), 137–151. MR-1143415

[4] Biggins, J.D. and Kyprianou, A.E.: Measure change in multitype branching. *Adv. Appl. Probab.* **36**, (2004), 544–581. MR-2058149

[5] Bingham, N.H., Goldie, C.M. and Teugels, J.L.: *Regular Variation,* Cambridge Univ. Press, Cambridge, 1987.

[6] Chow, Y.S. and Teicher, H.: *Probability theory: Independence, Interchangeability and Martingales,* Springer-Verlag, New York, 1988.

[7] Durrett, R. and Liggett, T.: Fixed points of the smoothing transformation. *Z. Wahrsch. verw. Geb.* **64**, (1983), 275–301. MR-0716487

[8] Gao, Z., Liu, Q. and Wang, H.: Central limit theorems for a branching random walk with a random environment in time. *Acta Math. Sci.* **34**, B (2) (2014), 501–512. MR-3174096

[9] Gao, Z. and Liu, Q.: Exact convergence rates in central limit theorems for a branching random walk with a random environment in time. *Stoch. Proc. Appl.* **126**, (2016), 2634–2664. MR-3522296

[10] Guivarc'h, Y.: Sur une extension de la notion de loi semi-stable. *Ann. Inst. H. Poincaré. Probab. Statist.* **26**, (1990), 261–285. MR-1063751

[11] Huang, C., Liang, X. and Liu, Q.: Branching random walks with random environments in time, *Front. Math. China* **9**, (2014), 835–842. MR-3240345

[12] Huang, C., and Liu, Q.: Convergence in L^p and its exponential rate for a branching process in a random environment. *Electro. J. Probab.* **19**, (2014), no. 104, 1–22. MR-3275856

[13] Iksanov, A., Liang, X. and Liu, Q.: On L^p-convergence of the Biggins martingale with complex parameter, arXiv:1903.00524

[14] Kahane, J.P. and Peyrière, J.: Sur certaines martingales de Benoit Mandelbrot. *Adv. Math.* **22**, (1976), 131–145. MR-0431355

[15] Kolesko, K. and Meiners, M.: Convergence of complex martingales in the branching random walk: the boundary. *Electron. Commun. Probab.* **22**, (2017), no. 18, 1–14. MR-3615669

[16] Liang, X. and Liu, Q.: Weighted moments of the limit of a branching process in a random environment. *Proceedings of the Steklov Institute of Mathematics* **282**, (2013), 127–145. MR-3308588

[17] Liang, X. and Liu, Q.: Weighted moments for Mandelbrot's martingales. *Electron. Commun. Probab.* **20**, (2015), no. 85, 1–12. MR-3434202

[18] Liu, Q.: On generalized multiplicative cascades. *Stoch. Proc. Appl.* **86**, (2000), 61–87. MR-1741808

[19] Liu, Q.: Asymptotic properties absolute continuity of laws stable by random weighted mean. *Stoch. Proc. Appl.* **95**, (2001), 83–107. MR-1847093

[20] Lyons, R.: A simple path to Biggins's martingale convergence for branching random walk. In K.B. Athreya, P. Jagers, (eds.), *Classical and Modern Branching Processes*, IMA Vol. Math. Appl. **84**, 217–221, Springer-Verlag, New York, 1997. MR-1601749

[21] Mallein, B. and Miloś, P.: Maximal displacement of a supercritical branching random walk in a time-inhomogeneous random environment. *Stoch. Proc. Appl.*, doi: 10.1016/j.spa.2018.09.008.

[22] Uchiyama, K.: Spatial growth of a branching process of particles living in $\mathbb{R}^d$, *Ann. Probab.* **10**, (1982), no.4, 896–918. MR-0672291

[23] Wang, X. and Huang, C.: Convergence of martingale and moderate deviations for a branching random walk with a random environment in time. *J. Theor. Probab.* **30**, (2017), 961–995. MR-3687246

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