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The frog model on trees with drift

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Abstract

We provide a uniform upper bound on the minimal drift so that the one-per-site frog model on a d-ary tree is recurrent. To do this, we introduce a subprocess that couples across trees with different degrees. Finding couplings for frog models on nested sequences of graphs is known to be difficult. The upper bound comes from combining the coupling with a new, simpler proof that the frog model on a binary tree is recurrent when the drift is sufficiently strong. Additionally, we describe a coupling between frog models on trees for which the degree of the smaller tree divides that of the larger one. This implies that the critical drift has a limit as d tends to infinity along certain subsequences.

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1 Introduction

We study the one-per-site *frog model with drift* on the rooted d-ary tree $\mathbb{T}_d$. Initially there is a single awake frog at the root and one sleeping frog at each non-root vertex. Awake frogs move towards the root with probability p, and otherwise move away from the root to a uniformly sampled child vertex. Frogs at the root always move to a uniformly sampled child vertex. Whenever an awake frog visits a site with a sleeping frog, the sleeping frog wakes up and begins its own independent p-biased random walk. Denote this process by $\mathrm{FM}(d,p)$ and the total number of visits to the root by $V(d,p)$. The process is *recurrent* if $V(d,p)$ is infinite almost surely, and is otherwise *transient*.

There is a history of investigating recurrence for the frog model with drift. It was first studied by Gantert and Schmidt with i.i.d η frogs per site and a drift in the e_1 direction on $\mathbb{Z}$ [GS09]. They showed that the process is recurrent if and only if $E\log_+\eta=\infty$ regardless of the drift. Here $\log_+$ is defined to be 0 at negative values of $\log$. A follow-up work by Ghosh, Noren, and Roitershtein studied the range of the frog model in the transient case [GNR17]. Similar observations were made by Rosenberg when the frog paths are Brownian motions in $\mathbb{R}$ [Ros17a] and for inhomogenous drift on $\mathbb{Z}$ [Ros17b]. The question is more subtle and challenging in higher dimensions.

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Döbler and Pfeifroth showed that the frog model is recurrent on $\mathbb{Z}^d$ for $d \geq 2$ so long as $E \log_+^{(d+1)/2} \eta = \infty$ [DP14]. It was open for some time whether, unlike the $d = 1$ case, there is a phase transition as the drift is varied. This was recently answered by Döbler, Gantert, Höfelsauer, Popov, and Weidner in [DGH$^+$17]. With one sleeping frog at each site of $\mathbb{Z}^d$, they found that recurrence depends on the strength of the drift with notably different behavior in $d = 2$ and $d \geq 3$. We study here how transience and recurrence of $\mathrm{FM}(d, p)$ depend on the drift.

Trees are a natural setting to study the frog model with drift, because the graph structure already induces one. Indeed, $\mathrm{FM}_d = \mathrm{FM}(d, 1/(d+1))$ is the frog model with simple random walk paths. Hoffman, Johnson, and Junge proved that FM_2 is recurrent, but that FM_d is transient for $d \geq 5$ [HJJ17b]. What happens when $d = 3$ and $d = 4$ for the one-per-site frog model is not currently known. However, follow-up work by Hoffman, Johnson, and Junge showed that the frog model with unbiased random walks is recurrent for any d so long as $\Omega(d)$ sleeping frogs are placed at each site [HJJ16, JJ16a, JJ16b]. So, there is a phase transition as we change the degree of the tree, or the initial density of asleep frogs.

Since FM_d is known to be transient for $d \geq 5$, it is natural to ask what is the minimal drift

$$p_d = \inf\{p\colon \mathrm{FM}(d, p) \text{ is recurrent}\}$$

that makes the process recurrent. In general, we know that $p_d \leq 1/2$, because the initially awake frog will return to the root infinitely often when $p \geq 1/2$. A simple argument shows that if $p < 1/(d+1)$ then, even with all frogs initially awake, there are only finitely many expected visits to the root. This immediately gives the bounds $1/(d+1) \leq p_d \leq \frac{1}{2}$. The theorem from [HJJ17b] that FM_2 is recurrent can be stated as $p_2 \leq 1/3$. This result together with the lower bound imply that $p_2 = \frac{1}{3}$.

It is not much more difficult to establish a non-vanishing lower bound on p_d. The frog model is dominated by the branching random walk (BRW) on $\mathbb{T}_d$ in which particles do not branch when moving towards the root (with probability p), but split in two when moving away. This corresponds to $\mathrm{FM}(\infty, p)$. This BRW is a common tool for analyzing the frog model. By replacing $1/(d+1)$ with p in the calculation at [HJJ16, Proposition 15] it follows that the BRW, and thus $\mathrm{FM}(d, p)$, is transient for $p < q^* = (2 - \sqrt{2})/4 \approx .1464$. Thus, $p_d \geq q^*$. Our main contribution is an upper bound.

Theorem 1.1. *$p_d \leq .4155$ for all $d \geq 2$.*

Proof. This follows from Lemma 2.1 combined with Proposition 1.3 and Proposition 1.4. □

It is interesting to ask how the frog model relates to the dominating BRW. The extra drift the frog model needs to be recurrent, $p_d - q^*$, is one way to measure the difference. By using a BRW that approximates two steps of the frog model, it is not overly taxing to show that $p_d - q^* > 0$ for all d (see [HJJ17b, Proposition 19] for an example of a more refined BRW). Since the dominating BRW corresponds in a sense to $d = \infty$, it is natural to ask if $p_d \to q^*$, and, if so, at what rate? The answer is not obvious, because, as time elapses, the frog model branches less. The region with less branching may grow quickly. Hoffman, Johnson, and Junge proved in [HJJ17a] that, when the density of frogs is $\Omega(d^2)$, the set of activated sites on the d-ary tree contains a linearly expanding ball. Awake frogs in this region cause no branching. We are not sure if this prevents p_d from converging to q^*. In fact, we are not sure whether p_d converges at all. This question of convergence is the second reason we are interested in $\mathrm{FM}(d, p)$.

Coupling frog models on different graphs is known to be difficult. Past work by Fontes, Machado, and Sarkar established that the critical probability for the frog model

with death is not monotonic in the graph [FMS04]. However, Lebensztayn, Machado, and Popove in [LMP05] conjectured that monotonicity holds on regular trees. Before our modest advancement in Proposition 1.2, no coupling had ever been exhibited between the process on different degree trees. A more robust coupling would be nice, because it might help understand how the frog model behaves on random trees and help investigate the convergence of p_d. Note that a coupling that projects the location of a frog in $\mathbb{Z}^{d+1}$ into $\mathbb{Z}^d$ is used by Ramírez and Sidoravicius in [RS04] in their argument that establishes the set of visited sites satisfies a shape theorem.

It ought to hold that $V(d+1,p) \succeq V(d,p)$; here '$\succeq$' denotes stochastic domination. This is because the drift is the same, but there are significantly more frogs in the higher degree tree. Despite considerable effort, we were unable to construct a coupling to this effect. It remains an open problem to prove that $p_{d+1} \leq p_d$, and thus that p_d has a limit. Additionally, there is no obvious coupling so that $\mathrm{FM}(d,p)$ visits the root less than $\mathrm{FM}(d,p')$ does when $p < p'$. Although it would be preposterous, we cannot rule out the possibility $\mathrm{FM}(d,p)$ switches between being transient and recurrent multiple times as we increase p.

The obvious coupling to try between $\mathrm{FM}(d,p)$ and $\mathrm{FM}(d+1,p)$ is to have paired frogs mimic one anothers' displacement from the root, but to move to uniformly chosen vertices when moving away. One can readily find realizations where the frog on $\mathbb{T}_d$ wakes a new frog, while the coupled frog on $\mathbb{T}_{d+1}$ does not. This breaks the coupling. We tried several more sophisticated couplings with no luck. However, a special case in which a coupling works is for trees in which the degree of the smaller tree divides that of the larger tree.

Proposition 1.2. *$V(d,p) \preceq V(kd,p)$ for all $k \geq 1$.*

The argument relies on a natural way to map rays from the root in $\mathbb{T}_k$ to embedded k-ary trees in $\mathbb{T}_{kd}$. It does not appear to generalize to any other degrees. The monotonicity proved in Proposition 1.2 implies that any subsequence $(p_{d_i})_{i=1}^{\infty}$ in which d_i divides d_{i+1} for each i converges to some limit. However, it is unclear if the limits of these subsequences are the same, or equal to q^*. Furthermore, since $p_2 = 1/3$, Proposition 1.2 implies that $p_d \leq 1/3$ for all even d.

We make more substantial progress coupling across different graphs with a subprocess of $\mathrm{FM}(d,p)$ that we call the *recursive frog model* $\mathrm{RFM}(d,p)$. It is obtained by trimming and halting the random walk paths of awake frogs. This ensures that $\mathrm{RFM}(d,p)$ visits the root less than $\mathrm{FM}(d,p)$. See Section 2 for the formal definition. A related, but slightly different process known as the *self-similar frog model* has been a useful tool for studying recurrence [HJJ16, JJ16a, HJJ18, HJJ17a, Ros17c].

Let $p'_d = \inf\{p\colon \mathrm{RFM}(d,p) \text{ is recurrent}\}$ be the critical drift for the recursive frog model on $\mathbb{T}_d$. It follows from the dominance relation in Lemma 2.1 that $p_d \leq p'_d$. As mentioned above, the usual frog model is difficult to couple on two trees of different degrees. Finding a coupling for the self-similar frog model also appears challenging. This is because, even with p fixed, the non-backtracking paths used in the self-similar frog model have different laws on different degree trees. It is both useful for our main theorem and of independent interest that there is a coupling where $\mathrm{RFM}(d,p)$ is dominated by $\mathrm{RFM}(d+1,p)$. We use this to show that p'_d is decreasing.

Proposition 1.3. *$p'_{d+1} \leq p'_d$ for all $d \geq 2$.*

The recursive frog model is useful because a coupling is possible across trees of different degrees. However, the coupling comes at the cost of removing a lot of awake frogs. Because so many frogs are removed, it is not obvious whether $\mathrm{RFM}(d,p)$ is ever recurrent. Old techniques do not apply easily here. We provide a new, simpler argument for recurrence for large enough p.

Proposition 1.4. $p_2' \leq .4155$.

All previous results that establish recurrence for the frog model on trees rely on bootstrapping a recursive distributional equation involving $V(d, 1/(d+1))$. See (3.3) for the equation. The recursive frog model is intuitively less recurrent than the self-similar frog model because more frogs are being removed. So, it is not clear that the bootstrapping approach will work. Fortunately, we find a simpler way to proceed. It starts with the usual recursive distributional equation, but uses the second moment method to finish. This is similar to an argument used to prove that the *parking process* visits the origin infinitely often [DGJ$^+$17]. To finish we prove a 0-1 law for the recursive frog model. This is necessary because the recursive frog model is not covered by the frog model 0-1 laws in [KZ17] and [HJJ17b].

2 The recursive frog model and proof of Proposition 1.3

In the *recursive frog model* $\mathrm{RFM}(d,p)$, awake frogs move towards the root at each step with probability $\rho = p/(1-p)$ when $p < 1/2$, and with probability 1 for $p \geq 1/2$. If a frog reaches the root, it is removed. Once a frog moves away from the root, it moves to a uniformly sampled child vertex (possibly the vertex from which it just came) and will thereafter continue to move away from the root to a uniformly sampled child. Frogs are removed if they move away from the root to an already visited site. The process starts with the frog at the root taking a uniform step away from its starting location. At each subsequent step all awake frogs move. If, at one step, several frogs at a vertex are moving away from the root, we arbitrarily select an order for them to move one at a time. This way we can decide which ones, if any, to remove. We say that $\mathrm{RFM}(d,p)$ is *recurrent* if the root is visited infinitely often almost surely.

Lemma 2.1. *If* $\mathrm{RFM}(d,p)$ *is recurrent, then so is* $\mathrm{FM}(d,p)$. *Hence,* $p_d \leq p_d'$.

Proof. If $p \geq 1/2$ then both processes are recurrent. Suppose for the remainder of the argument that $p < 1/2$. We couple these two models in the following way. Let f_v denote a frog in $\mathrm{FM}(d,p)$ originally placed at vertex $v \in \mathbb{T}_d$ and f_v' denote the corresponding frog in $\mathrm{RFM}(d,p)$. Write $|v|$ as the graph distance from v to the root. If it gets woken up, the frog f_v follows an independent p-biased random walk path γ_v. We can use γ_v to generate the random walk path γ_v' for f_v' in $\mathrm{RFM}(d,p)$.

Let D_v be the smallest graph distance from the root among all the vertices on γ_v and let v^* be the first vertex on γ_v that is at distance D_v. Make the first $|v| - D_v$ steps of (γ_v') the path that goes directly from v to v^*. For a p-biased random walk started at 0, the probability of hitting -1 eventually is exactly $\rho = \frac{p}{1-p}$, and thus $P(D_v = k) = \rho^{|v|-k}(1-\rho)$ for $k > 0$ and $P(D_v = 0) = \rho^{|v|}$. This matches the law of the frog paths in $\mathrm{RFM}(d,p)$; in fact, these are just the probabilities that the vertex closest to the root on γ_v' is at distance k from the root. This takes care of the upward steps in γ_v'. Since $p \leq \frac{1}{2}$, γ_v contains a unique uniformly random ray from v^* to infinity. Have the subsequent steps in γ_v' follow this ray from v^* to infinity. We have now generated paths for frogs in $\mathrm{RFM}(d,p)$, which comply with the rules of frog movement, and frogs will follow these paths until they are removed according to the removal rule of $\mathrm{RFM}(d,p)$.

Our construction ensures that $\gamma_v' \subseteq \gamma_v$ for every vertex v. It follows that if a frog originally placed at v has ever been woken up in $\mathrm{RFM}(d,p)$, the corresponding frog in $\mathrm{FM}(d,p)$ would also be woken up (see also [KZ17, Equation (4)] and [HJJ17b, Proposition 7]). Thus, $\mathrm{RFM}(d,p)$ visits the root no more frequently than $\mathrm{FM}(d,p)$. □

To deduce Proposition 1.3 we will prove that if $\mathrm{RFM}(d,p)$ is recurrent, then so is $\mathrm{RFM}(d+1,p)$. Hence $p_{d+1}' \leq p_d'$.

Proof of Proposition 1.3. We introduce a modified process $\mathrm{RFM}'(d+1,p)$ which is obtained by running $\mathrm{RFM}(d+1,p)$ with an additional procedure for removing frogs. Suppose that a frog f is at a vertex with $0 \le i \le d$ child vertices already visited. If f moves away from the root to a new vertex, then we remove f with probability

$$z_i = 1 - \frac{(d-i)(d+1)}{d(d+1-i)}.$$

Otherwise, f moves to an unvisited child vertex. This ensures that the probability f visits a new site when moving away from the root at a vertex with i already visited children is equal to

$$(1-z_i)\frac{d+1-i}{d+1} = \frac{d-i}{d}.$$

Notice this probability is 0 when $i = d$. This makes it so at most d children of a given vertex can be woken. Thus, all activity in $\mathrm{RFM}'(d+1,p)$ takes place on an embedded copy of $\mathbb{T}_d$. Additionally, frogs move with the same law as in $\mathrm{RFM}(d,p)$, since jumps towards the root occur with probability ρ, and jumps away from the root are to an unvisited child vertex with probability $(d-i)/d$ (when there are i already visited children). It follows that $\mathrm{RFM}'(d+1,p)$ restricted to the embedded d-ary tree has the same law as $\mathrm{RFM}(d,p)$. So, if $\mathrm{RFM}(d,p)$ is recurrent, then so is $\mathrm{RFM}'(d+1,p)$.

It remains to show that recurrence of $\mathrm{RFM}'(d+1,p)$ implies that of $\mathrm{RFM}(d+1,p)$. We will refer to these as F' and F, respectively and couple them as follows. First, we have the frog at v in each model use the same upward path γ'_v towards the root as defined in the proof of Lemma 2.1. We further couple the two models so that child vertices of each vertex of F are woken in the same order as what occurs in F' – this is possible if at any time and for any vertex v, the number of visits from v to its child vertices in F is at least as many as that in F', and we will prove this claim by induction. If additional children of v are visited in F, then they are chosen uniformly. Recall that F' is defined to occasionally remove frogs that would have visited new child vertices. It follows that

($*$) if more frogs attempt to move from vertex v to its child vertices in F, then there are more visited child vertices of v in F than in F'.

Now a little bit of notation. Under this coupling let $\mathcal{A}'_t$ and $\mathcal{A}_t$ be the set of vertices visited in F' and F up to time t. We will show that $\mathcal{A}'_t \subseteq \mathcal{A}_t$ for all $t \ge 0$ and use this to deduce Proposition 1.3. Clearly, $\mathcal{A}'_0 = \mathcal{A}_0$. Now suppose that for $0 \le s \le t$ we have

$$\mathcal{A}'_s \subseteq \mathcal{A}_s. \tag{2.1}$$

Thus at each $v \in \mathcal{A}_t$, the frog at v in F wakes up before or at the same time as its counterpart at v in F' and possibly the frog at v in F' never is woken up. Note that under the coupling, the upward paths γ'_v are the same in both models. It follows that by the next time step every vertex in F has had at least as many visits to it from frogs moving towards the root as what occurs in F'. We can represent this notationally as

$$U'_{t+1}(u) \le U_{t+1}(u) \qquad \forall u \in \mathbb{T}_d, \tag{2.2}$$

where $U'_{t+1}(u)$ are the number of frogs that have moved from a child vertex of u to u up to time $t+1$, and similarly for $U_{t+1}(u)$.

Since the frogs started at v are also coupled to turn away from the root at the same terminal vertex of v^* of γ'_v (see Lemma 2.1), it follows from (2.2) that for all $u \in \mathbb{T}_d$, at least as many frogs in F have turned away from the root at u by time $t+1$ as in F'. That is

$$S'_{t+1}(u) \le S_{t+1}(u) \qquad \forall u \in \mathbb{T}_d. \tag{2.3}$$

Here $S'_{t+1}(u)$ and $S_{t+1}(u)$ are the counts of how many frogs that have reached u by time $t+1$ and u was the vertex at which the frogs turn away from the root ($v^* = u$ in the notation of Lemma 2.1). Note we adopt the convention that S'_t and S_t are both zero at the root, since all frogs are removed after visiting the root in both processes.

Now consider the total number of frogs that have attempted to move from u to one of its child vertices by time $t+1$. Denote the number by $D'_{t+1}(u)$ in F' and by $D_{t+1}(u)$ in F. Note that all visits that have made to u and then attempted to move to one of its child vertices consist of one visit from the parent of vertex u (to wake up the sleeping frog at u) and all others from the frogs using u as the terminal vertex of their upward paths.

If $u \in \mathcal{A}_t$, then necessarily the first visit to u came from a frog that moved from the parent vertex of u. It follows that

$$D'_{t+1}(u) = \mathbf{1}\{u \in \mathcal{A}'_t\}(1 + S'_{t+1}(u))$$

and similarly for $D_{t+1}(u)$. Using (2.1) and (2.3) and the above formula we have

$$D'_{t+1}(u) \le D_{t+1}(u) \qquad \forall u \in \mathbb{T}_d. \tag{2.4}$$

It follows from (2.4) and the observation at $(*)$ that every vertex in F has at least as many children visited as in F' at time $t+1$, which ensures $\mathcal{A}'_{t+1} \subseteq \mathcal{A}_{t+1}$ by choosing child vertices of each vertex to be waken up in the same order. It follows from induction that this containment holds for all t. In particular, (2.2) holds for all t with u taken to be the root. Hence, recurrence of F' implies that of F. □

3 Proof of Proposition 1.4

An advantage of $\mathrm{RFM}(2,p)$ is that the number of visits to the root satisfies a recursive distributional equation. See Figure 1 for a visual representation of the following notation. Let $\varnothing$ be the root of $\mathbb{T}_2$. The frog initially awake at the root will move to one of the two children of the root and then it, or the frog it wakes there, will move down another level. Call these sites $\overline{\varnothing}$ and x, respectively. Let y be the sibling vertex of x.

Let V_t be the number of visits to the root in $\mathrm{RFM}(2,p)$ with frogs placed at all sites up to distance t from the root and the rest of the sites empty. Similar reasoning as the inductive argument with $\mathcal{A}_t$ in Proposition 1.3, gives that $V_t \preceq V_{t+1}$ (in the usual sense of stochastic dominance), and thus there is a distributional limit $V := V_\infty$. Let V_t^x and V_t^y be the number of visits to $\overline{\varnothing}$ from frogs initially in the subtrees rooted at x and y, respectively. We will further be interested in the event A_t that a frog ever enters the subtree rooted at y.

Although the law for paths in $\mathrm{RFM}(2,p)$ is different than in the self-similar frog model from [HJJ17b], it enjoys the same recursive properties. This is because both processes ($\mathrm{RFM}(d,p)$ and the self-similar frog model) have frogs follow non-backtracking paths and get removed when moving away from the root to already visited sites. The first observation is that

$$V_t^x \overset{d}{=} V_{t-1}. \tag{3.1}$$

This is because, by definition, a frog enters the subtree rooted at x. The resulting subprocess in the subtree rooted at $\varnothing$ can then be coupled to a independent version of $\mathrm{RFM}(2,p)$ on a tree with sleeping frogs up to distance $t-1$ from the root. Additionally, since at most one frog will enter the subtree rooted at y, we can use the same reasoning to write

$$V_t^y \overset{d}{=} \mathbf{1}\{A_t\}V'_{t-1} \tag{3.2}$$

with V'_{t-1} an independent copy of V_{t-1} that is also independent of $\mathbf{1}\{A_t\}$.

Let $\mathrm{Bin}(X,p)$ denote a sum of X independent Bernoulli(p) random variables. We claim that V_{t+1} satisfies the following analogue of [HJJ17b, (2)]:

$$V_{t+1} = \mathrm{Bin}(V^x_{t+1}+1,\rho) + \mathrm{Bin}(V^y_{t+1},\rho).$$

The first binomial term in (3.3) counts how many frogs move from x to $\overline{\varnothing}$. This is distributed like $\mathrm{Bin}(V^x_{t+1}+1,\rho)$ since each frog that visits $\overline{\varnothing}$ from x will move to $\varnothing$ with probability ρ, and subsequently be removed. The '+1' term comes from the frog initially sleeping at $\overline{\varnothing}$. The second binomial term counts how many frogs will move from y to $\overline{\varnothing}$, and then to $\varnothing$. The identities at (3.1) and (3.2) then give

$$V_{t+1} = \mathrm{Bin}(V_t+1,\rho) + \mathbf{1}\{A_t\}\,\mathrm{Bin}(V'_t,\rho). \tag{3.3}$$

Analyzing the first and second moments of this recursive distributional equation is sufficient to deduce V is infinite for ρ large enough.

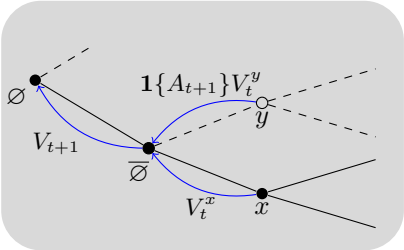

Figure 1: V_{t+1} is the total number of visits to $\varnothing$ in $\mathrm{RFM}(d,p)$ with sleeping frogs placed up to distance $t+1$ from the root. It can be expressed as a binomial thinning of the number of visits to $\overline{\varnothing}$. These quantities are i.i.d. and distributed like V_t.

Proof of Proposition 1.4. Consider $\mathrm{RFM}(2,p)$ and let $x_t = EV_t^2/(EV_t)^2$. We will prove that $\sup_t x_t = C < \infty$. It follows from the Paley-Zygmund inequality that

$$P(V_t > EV_t/2) \geq (4x_t)^{-1} \geq (4C)^{-1} \qquad \text{for all } t \geq 1. \tag{3.4}$$

We will also show that $EV_t \to \infty$ as $t \to \infty$, and the above line implies $P(V=\infty) > 0$.

Taking expectation in (3.3) and using independence between $\mathbf{1}\{A_{t+1}\}$ and V^y_{t+1} gives

$$EV_{t+1} = \rho(1+P(A_{t+1}))EV_t + \rho. \tag{3.5}$$

It is easy to show that $P(A_{t+1}) \geq 1 - \prod_{i=0}^{t}(1-\rho^i(1-\rho)/2)$. This is because there always exists a line segment L_{t+1} from $\overline{\varnothing}$ to a vertex at distance $t+1$ from the root along which all the frogs have been woken up. For the frog on this line segment at distance $i+1$ from the root to visit y, it must take i steps toward the root, then move to y from $\overline{\varnothing}$. This occurs with probability $\rho^i(1-\rho)/2$. Since there are t frogs along this line segment, we obtain the claimed bound on $P(A_{t+1})$ by only considering these frogs guaranteed to be awake.

A computer can easily verify that $\rho(1+P(A_{51})) > 1$ for $\rho > .7107$. Converting from $\rho = p/(1-p)$ back to p implies that this holds for $p > .4155$. Using (3.5), for such p we have $\epsilon > 0$ so that $EV_{t+1} \geq (1+\epsilon)EV_t + \rho$ for $t \geq 50$. It follows that EV_t diverges as $t \to \infty$. This alone is not enough to conclude that the root is visited infinitely often almost surely. To establish this, we need to control the second moment.

Let $X_t = \mathrm{Bin}(V^x_{t+1} + 1, \rho)$ and $Y_t = \mathrm{Bin}(V^y_{t+1}, \rho)$ so that $V_{t+1} = X_t + \mathbf{1}\{A_{t+1}\}Y_t$ and thus

$$V^2_{t+1} = X^2_t + \mathbf{1}\{A_{t+1}\}Y^2_t + 2\mathbf{1}\{A_{t+1}\}X_tY_t. \tag{3.6}$$

Taking expectations and using independence as well as the bound $\mathbf{1}\{A_t\} \le 1$ we have

$$EV^2_{t+1} \le EX^2_t + P(A_{t+1})EY^2_t + 2EX_tEY_t. \tag{3.7}$$

Using the formula for the second moment of a random sum of i.i.d. $Z_i = \mathrm{Ber}(\rho)$ random variables

$$E\left(\sum_{i=1}^{N} Z_i\right)^2 = \rho(1-\rho)EN + \rho^2 EN^2,$$

we have $EX^2_t = \rho(1-\rho)E(V_t+1) + \rho^2 E(V_t+1)^2$, and similarly $EY^2_t = \rho(1-\rho)EV_t + \rho^2 EV^2_t$. Plugging these expressions into (3.7) and gathering smaller order terms yields

$$EV^2_{t+1} = \rho^2(1 + P(A_{t+1}))EV^2_t + 2\rho^2(EV_t)^2 + O(EV_t). \tag{3.8}$$

Squaring (3.5) gives

$$(EV_{t+1})^2 = (\rho(1 + P(A_{t+1}))EV_t)^2 + O(EV_t). \tag{3.9}$$

Recall that $x_t = EV^2_t/(EV_t)^2$. Dividing (3.8) by (3.9) gives

$$x_{t+1} = \frac{1}{1 + P(A_{t+1})}x_t + O(1).$$

Since $1 + P(A_{t+1}) \ge 1 + \epsilon > 1$ for all t, the leading coefficient is less than 1. This ensures that $\sup_t x_t = C < \infty$ which gives (3.4).

There is a quick way to go from $P(V = \infty) > 0$ to $P(V = \infty) = 1$. Recall the definition of L_{t+1} from just below (3.5). We can extend this to obtain a ray L from the root to ∞ with an awake frog at each site. Let $\varnothing_t$ be the site at distance t on this ray. The awake frog at $\varnothing_t$ moves to the child $y_t \notin L$ beneath it with probability $(1-\rho)/2$. When this occurs, an independent V-distributed number of frogs will visit $\varnothing_t$. If this quantity is infinite, then $\varnothing$ is visited infinitely often. Since $P(V = \infty) > 0$ and there are infinitely many independent trials along L, we must have V is infinite almost surely. □

4 Proof of Proposition 1.2

Proof. We impose coordinates on $\mathbb{T}_d$ by writing a vertex v at distance n from the root as $v = x_1 \cdots x_n$ with $x_i \in \{1, 2, \ldots, d\}$. For $1 \le i \le d$ define the set-valued function $G(i) = \{k(i-1)+1, \ldots, ki\}$. Let L_v be the line segment that consists of vertices $\varnothing = v_0, v_1, \ldots, v_n = v$ on the shortest path from the root to v. For each L_v we define a subgraph of $\mathbb{T}_{kd}$:

$$\mathbb{T}_k(L_v) = \bigcup_{x_1 \cdots x_i \in L_v} G(x_1) \times \cdots \times G(x_i).$$

See Figure 2 for a depiction. If v has distance n from the root, then $\mathbb{T}_k(L_v)$ is a k-ary tree of height n. Call vertices $\{v' \in \mathbb{T}_k(L_v) \colon |v'| = n\}$ the *leaves*. The embedding is such that for $v, v' \in \mathbb{T}_d$ we have

$$\mathbb{T}_k(L_v) \cap \mathbb{T}_k(L_{v'}) = \mathbb{T}_k(L_v \cap L_{v'}). \tag{4.1}$$

We will define a modified version of $\mathrm{FM}(kd, p)$ that sometimes removes frogs early. Awake frogs f' in the modified version will be coupled to a unique frog f in $\mathrm{FM}(d, p)$. Frogs move according to the random paths in $\mathrm{FM}(kd, p)$. The rules for the coupling are that:

(i) If f moves towards the root, then f' moves towards the root.

(ii) Suppose that f is at $x_1 \cdots x_n$ and f' is at $x'_1 \cdots x'_n$. If f moves away from the root to $x_1 \cdots x_n x_{n+1}$, then f' moves to a uniformly random vertex in $x'_1 \cdots x'_n \times G(x_{n+1})$.

(iii) f' only wakes a frog when f does. Upon doing so these newly awakened frogs (one from each process) are also coupled. If f' moves to a site with a sleeping frog, but f does not, then that sleeping frog is removed from the process

We conclude by explaining why the coupling is well-defined and gives the claimed dominance. These rules ensure that f and f' have the same displacement from the root so Rule (ii) always holds. Moreover, Rule (ii) combined with (4.1) ensure that the first visit to $v \in \mathbb{T}_d$ corresponds to the first visit to a leaf of $\mathbb{T}_k(L_v)$. So, when the frog at v is woken by f, then there will be a sleeping frog at whatever leaf vertex of $\mathbb{T}_k(L_v)$ that f' moves to. It follows that Rule (iii) holds for all steps in the coupling. The resulting process is a restricted version of $\mathrm{FM}(kd, p)$ that visits each distance from the root the same number of times as in $\mathrm{FM}(d, p)$. This gives the claimed result. □

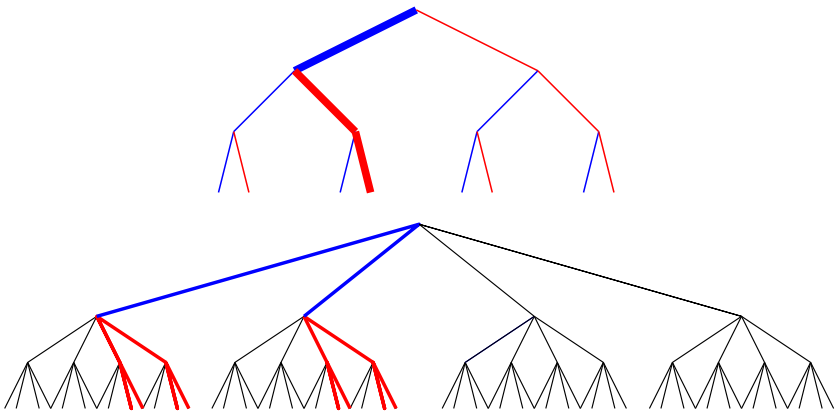

Figure 2: Each segment L_v is associated to a k-ary tree in $\mathbb{T}_{kd}$. The bolded line $L_{122} \subseteq \mathbb{T}_2$ above corresponds to the binary tree in $\mathbb{T}_4$ shown beneath. The color coding represents the coordinate entries of v.

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