A forward-backward random process for the spectrum of 1D Anderson operators

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Abstract

We give a new expression for the law of the eigenvalues of the discrete Anderson model on the finite interval $[1, N]$, in terms of two random processes starting at both ends of the interval. Using this formula, we deduce that the tail of the eigenvectors behaves approximately like $\exp(\sigma B_{|n-k|} - \gamma \frac{|n-k|}{4})$ where $\gamma, \sigma > 0$, B_s is the Brownian motion and k is uniformly chosen in $[1, N]$ independently of B_s. A similar result has recently been shown by B. Rifkind and B. Virag in the critical case, that is, when the random potential is multiplied by a factor $\frac{1}{\sqrt{N}}$.

1 Introduction

We are interested in the one dimensional discrete Anderson model on a finite domain $[1, N]$. This model is very classical and has been studied extensively since the 70s. See for example the monograph of Carmona Lacroix [3]. Compared to higher dimensions, it can be considered as a solved problem. However new approaches can always shed new light on this famous system.

The usual approach to tackle this system is the transfer matrix framework. The eigenvectors of the random Schrodinger operator satisfy a recursive relation of order 2, $u_{n+2} = (V_{n+1} - \lambda)u_{n+1} - u_n$, which can be written in a matrix form. Using this relation, one can construct an eigenvector on $[1, N]$ from the product of the transfer matrices applied to the boundary values. The advantage of such a formulation is that one can then use the very powerful results for random matrices product and from ergodic theory such as the Oseledets theorem.

In the historical approach of Kunz and Souillard [8] or in the proof from the book [4] (section 9.5, page 191) a change of variables is used to deal with the conditional probability of the potential V with a fixed eigenvalue λ. In this short note, we propose another calculation of this conditional probability. We define a random variable k whose random law is close to the uniform law on $[1, N]$. This variable splits the interval $[1, N]$ into two parts $[1, k]$ and $[k, N]$. On the left part, the matrices product is made from left to right. On the right part, the matrices product is made from right to left. And far from the cut, the laws of the matrices are very close to be independent.

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The main interest of our approach is that the connection with the theorems for products of random matrices is more transparent in this setup. From this formula we can recover several known results. Relying on the positivity of Lyapunov exponent, the formula can be used as a new proof of exponential Anderson localization of eigenvectors where the center of localization is uniformly distributed on $[1, N]$. Moreover, because it gives a explicit random law we can go beyond the exponential decay of the eigenvectors far from the center of localization and give an explicit law for their tail.

In the first section, we detail the model and we state our result. Then we give some applications of our theorem in the second section. In particular, we write an asymptotic result similar to the result of Rifkind and Virag in [10]. In section 3, we finally give the proof of the theorem.

2 Model and main result

We consider the one dimensional Anderson model [1] defined on $L^2([1, N] \cap \mathbb{Z}, \mathbb{R})$ through the operator.

$$H^{(N)} = -\Delta^{(N)} + V_\omega^{(N)}.$$

Here $V_\omega^{(N)}$ is a random potential

$$V_\omega^{(N)}(x, y) = \begin{cases} v_k & \text{if } x = y = k \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in [1, N] \cap \mathbb{Z}$ where $(v_k)_{k \in [1, N] \cap \mathbb{Z}}$ are iid random variables and

$$\Delta^{(N)}(x, y) = \begin{cases} 1 & \text{if } |x - y| = 1 \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in [1, N] \cap \mathbb{Z}$ is the discrete Laplacian. Hence $H^{(N)}$ is just the $N \times N$ symmetric matrix

$$H^{(N)} = \begin{pmatrix} v_1 & -1 & & & \\ -1 & & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & & -1 \\ & & & -1 & v_n \end{pmatrix}.$$

We make the following assumption:

(H1) The random law of the (v_k) is absolutely continuous with respect to the Lebesgue measure.

2.1 Transfer matrices

The transfer matrices have been one of the main tools to study the 1D Anderson model. One is interested in the eigenvectors, identified by $(H^{(N)}u)_n = \lambda u_n$, which satisfy the recurrence relation

$$\forall n \in [1, N], \quad -u_{n+1} + (v_n - \lambda)u_n - u_{n-1} = 0 \tag{2.1}$$

with convention that $u_0 = u_{N+1} = 0$. This can be written with transfer matrices

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = T(v_n - \lambda) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}$$

where

$$T(x) = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}.$$

We can then write the matrix product (from right to left)

$$M_n(\lambda) = \prod_{k=1}^{n} T(v_k - \lambda)$$

(by convention $M_0(\lambda) = I_2$ we will also use the shorter notation $T_\lambda(v_k) := T(v_k - \lambda)$) and we have

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = M_n(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The parameter λ is an eigenvalue if and only if there exists $c \in \mathbb{R}$ such that

$$M_N(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix},$$

the condition $u_{N+1} = 0$ is then satisfied. It will be convenient to denote the vector $\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}$ as a complex number in the fashion

$$u_{n+1} + iu_n = z_n = r_n e^{i\phi_n}$$

where $r_n \in \mathbb{R}_+$ and $\phi_n \in \mathbb{R}/2\pi\mathbb{Z}$. We also introduce the lifting of ϕ_k, which we denote by θ_k. This is just a discrete version of the continuous lifting from $\mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}$. It is defined recursively by

$$\theta_k = \begin{cases} \theta_0 = 0 \\ \phi_k[2\pi] \quad \forall k \in [1, N] \end{cases}$$

and

$$\theta_k - \frac{\pi}{2} \leq \theta_{k+1} < \theta_k + \frac{3\pi}{2}.$$

It can be seen that ϕ_{k+1} does not depend on r_k but only on ϕ_k and $T_\lambda(v_k)$. Therefore, for simplicity of notation, we use the same notation T_λ for the (non linear) function $T_\lambda(v_k) : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$:

$$\phi_{k+1} = T_\lambda(v_k)\phi_k.$$

Note that it is possible to recover r_k from $\phi_0, \phi_1, ..., \phi_N$ with the formula

$$\frac{r_{k+1}}{r_k} = \frac{r_{k+1}}{u_{k+1}} \frac{u_{k+1}}{r_k} = \frac{\cos \phi_k}{\sin \phi_{k+1}}.$$

For this reason, in the rest of the paper we focus mostly on $(\phi_k)_{k=0,...,N}$. We note $\mathcal{F}(\lambda) = (\phi_k)_{k=0,...,N}$ which has been constructed from the recursive formula $\phi_{k+1} = T_\lambda(v_k)\phi_k$ and $\phi_0 = 0$. And for λ an eigenvalue, we note $\mathcal{P}h(\lambda) = (\phi_k)_{k=0,...,N}$ the phase of the corresponding eigenvector. Note that it is equal to $\mathcal{F}(\lambda)$ with the condition $\phi_N = \frac{\pi}{2}[\pi]$

2.2 Forward and backward process

In this subsection, we define two natural random laws on the chain $X = (\phi_k)_{k=0,..,N}$. The first one is the Markov chain starting from ϕ_0 with an initial law μ_f defined on $\mathbb{S}^1$ and transition law $\phi_k \to \phi_{k+1} = T_\lambda(v_k)\phi_k$ with an random measure ν for v_k. We call it the forward process. The second one is the Markov chain starting from ϕ_N with an initial law μ_b and transition law $\phi_k \to \phi_{k-1} = T_\lambda^{-1}(v_{k-1})\phi_k$ with a random measure ν for v_k and we call it the backward process. Then we introduce a cut in $[1, N]$, and we can define the

random law product between these two processes which we call the Forward-backward process.

For a proper definition we use test functions on $\mathbb{R}^{N+1}$ which are bounded and continuous.

Definition 1 (Forward and backward processes).

- The forward process. Let $\mathcal{P}_f$ be the probability on $\mathbb{R}^{N+1}$ defined by

$$\mathcal{P}_f^{(\lambda)}(F) = \int \cdots \int d\mu_f(\phi_0) d\nu(v_1) \cdots d\nu(v_N) F(X)$$

for any test function F.

- The backward process. Let $\mathcal{P}_b^{(\lambda)}$ be the probability on $\mathbb{R}^{N+1}$ defined by

$$\mathcal{P}_b^{(\lambda)}(F) = \int \cdots \int d\nu(v_1)...d\nu(v_N) d\mu_b(\phi_N) F(X)$$

for any test function F.

Remark 2. If we introduce $\xi_{0,n} : \phi_0 \to \phi_n^f = \prod_{k=0}^{n-1} T_\lambda(v_k)\phi_0$ and if for almost surely any $v_1, v_2, ..., v_n$, μ_b and the push measure $\xi(\mu_f)$ are equivalent measures then remark that for any F:

$$\mathcal{P}_b^{(\lambda)}(F) = \int \cdots \int d\nu(v_1)...d\nu(v_n) d\mu_f(X_0) \frac{d\mu_b(X_n)}{d\xi(\mu_f(X_0))}\big|_{v_1,...,v_n} F(X)$$
$$= \mathcal{P}_f^{(\lambda)}\left(F \frac{d\mu_b(X_N)}{d\xi(\mu_f(X_0))}\big|_{v_1,...,v_n} \right).$$

Definition 3 (Forward-backward process). For $k \in [1, N]$, we define $\mathcal{P}_{f,0..k}^{(\lambda)} \otimes \mathcal{P}_{b,k,...,N}^{(\lambda)}$ a forward process for $X^f = \phi_0^f, \phi_1^f..., \phi_k^f$ with $\phi_0^f = 0$, ($\mu_f = \delta_0$) and a backward process for $X^b = \phi_N^b, \phi_{N-1}^b..., \phi_k^b$, with $\phi_N^b = \frac{\pi}{2}$ ($\mu_b = \delta_{\frac{\pi}{2}}$) which are independent from each other.

2.3 Results

We are now ready to state the main theorem of our paper.

Theorem 4 (Law of the spectrum of the 1D Anderson model)**.** For any test function $G(\lambda, X)$, we have

$$\mathbb{E}\Big[\sum_{\lambda \in \sigma(H^{(N)}), X = \mathcal{P}^{(\lambda)} h(\lambda)} G(\lambda, X) \Big]$$
$$= \int_{\mathbb{R}} d\lambda \sum_{k=1}^{N} \mathbb{E}_{\mathcal{P}_{f,1..k}^{(\lambda)} \otimes \mathcal{P}_{b,k,...,N}^{(\lambda)}} \Big[G(\lambda, X) \delta_{\phi_k^f - \phi_k^b[\pi]} \sin^2(\phi_k^f) \Big] \quad (2.2)$$

that we can rewrite

$$\mathbb{E}\Big[\frac{1}{N} \sum_{\lambda \in \sigma(H^{(N)}), X = \mathcal{P}h(\lambda)} G(\lambda, X) \Big]$$
$$= \int_{\mathbb{R}} \rho(\lambda) 1_{\rho(\lambda)>0} \left(\frac{1}{N} \sum_{k=1}^{N} \mathbb{E}_{\mathcal{P}_{f,1..k}^{(\lambda)} \otimes \mathcal{P}_{b,k,...,N}^{(\lambda)}} \Big[G(\lambda, X) \frac{\delta_{\phi_k^f - \phi_k^b[\pi]} \sin^2(\phi_k^f)}{\rho(\lambda)} \Big] \right) d\lambda \quad (2.3)$$

with $\rho(\lambda)$ the density of state.

Recall that $Ph(\lambda)$ is the phase of the eigenvector corresponding of the eigenvalue λ.

This formula is to be understood as follow. One chooses k randomly in $[1, N]$ which splits the segment into two parts $[1, k]$ and $[k, N]$. On the left, we obtain a forward process, on the right, we obtain a backward process. The choice of k is not exactly uniform on $[1, N]$ because of the condition $\delta_{\phi_k^f - \phi_k^b} \sin^2(\phi_k^f)$. However, for large N, and for any $k \leq N$ not too close to 1 or N, the laws of ϕ_k^f and ϕ_k^b are very close to their invariant measure under action of T_λ and then do not depend of k. Therefore the law of k becomes close to the uniform.

There is still a dependence between the two processes at the connection between the forward and backward processes. However, because of the mixing properties of the matrix product, the correlations decay exponentially fast outside of the cut k.

We recall that a stationary process X_k is called $(\alpha_n)_{n \in \mathbb{N}}-$ mixing if

$$\forall k, \max_{A,B} |\mathbb{P}(X_k \in A, X_{k+n} \in B) - \mathbb{P}(X_k \in A)\mathbb{P}(X_{k+n} \in B)| \leq \alpha_n$$

The following is a well known result

Proposition 5. There exists a constant $C > 0$ and $0 < \kappa < 1$ such that the process ϕ_k is $(C\kappa^n)_{n \in \mathbb{N}}$-mixing.

For a proof, see [3] proposition IV.3.12.

3 Applications

We present here three applications of our result. The first one is a formula for the integrated density of states. The second one is about the form of the tails of the eigenvectors. We then finish with a temperature profile from [5].

3.1 A formula for the integrated density of states

The following equality can be found as well in [3] (Proposition VIII.3.10 and Problem VIII.6.8).

Proposition 6. For $\lambda \in \mathbb{R}$, let $\mu_\lambda(d\phi) = \rho_\lambda(\phi)d\phi$ be the T_λ-invariant measure on $\mathbb{R}/\mathbb{Z}$. The density of states

$$dN(\lambda) = \lim_{N \to \infty} \frac{1}{N} \#\{\sigma(H^{(N)}) \cap [\lambda, \lambda + d\lambda]\}$$

is given by

$$\frac{dN(\lambda)}{d\lambda} = \int_{\mathbb{R}/2\pi\mathbb{Z}} \sin^2(\phi)\rho_\lambda(\phi)\rho_\lambda\left(\frac{\pi}{2} - \phi\right)d\phi.$$

Proof. We apply Theorem 4, and we choose $G(s, X) = G(s)$ (that does not depend on X) as an approximation of $1_{s \in [\lambda, \lambda + d\lambda]}$. Then

$$\frac{1}{N}\mathbb{E}\Big[\sum_{\lambda \in \sigma(H^{(N)})} G(\lambda)\Big] = \int G(\lambda)d\lambda \frac{1}{N}\sum_k \mathbb{E}_{\mathcal{P}_{f,1..k}^{(\lambda)} \otimes \mathcal{P}_{b,k,...,N}^{(\lambda)}}[\delta_{\phi_k^f - \phi_k^b[\pi]} \sin^2(\phi_k^f)]$$

$$= \int G(\lambda)d\lambda \frac{1}{N}\sum_k \int \rho_{k,\lambda}(\phi)\tilde{\rho}_{N-k,\lambda}(\phi) \sin^2(\phi)d\phi$$

where $\rho_{k,\lambda}$ is the probability density of the angles of $M_k(\lambda)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\tilde{\rho}_{N-k,\lambda}$ is the probability density of the angles of

$$\tilde{M}_{N-k}\begin{pmatrix} 0 \\ 1 \end{pmatrix} := \prod_{i=k+1}^{N} T_\lambda^{-1}(v_i)(\lambda)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(here the product is made from left to right). Because $T_\lambda^{-1}(v_i) = \begin{pmatrix} 0 & 1 \\ -1 & v_i \end{pmatrix}$ we have that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{M}_{N-k} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T_\lambda^{-1}(v_{k+1}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 T_\lambda^{-1}(v_{k+2}) \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

has the same law as $M_{N-k} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and in particular $\tilde{\rho}_{N-k,\lambda}(\phi) = \rho_{N-k,\lambda}(\frac{\pi}{2} - \phi)$.

We can then conclude using that $\rho_{k,\lambda} \to \rho_\lambda$ and $\rho_{N-k,\lambda} \to \rho_\lambda$ when $k \to \infty$ and $N - k \to \infty$. $\qquad\square$

3.2 Brownian and drift for the eigenvectors

It is well known since the work of Carmona-Klein-Martinelli [2], Goldsheild-Molcha-nov-Pastur [7] and Kunz-Souillard [8] that the eigenvectors are localized and decay exponentially from the center of localization. An exact form of the eigenvectors has been recently proven in the critical case where V is replaced by $\frac{V}{\sqrt{N}}$ in [10]. There Rifkind and Virag proved that the eigenvectors in the bulk have the form $e^{\sigma \frac{B_{|t-u|}}{2} - \gamma|t-u|}$. We claim using our formula of Theorem 4 that a similar result is universal for the tails of the eigenvector in the non critical case.

For the reader's convenience we recall the heuristic of the following classical results. One can write any product of random matrices $M_N = \prod_{i=1}^N T_i$ as

$$\log(\|M_N\|) = \log\Big(\prod_{i=1}^N \frac{\|M_i\|}{\|M_{i-1}\|}\Big) = \sum_{i=1}^N \log\Big(\|T_i\big(\frac{M_{i-1}}{\|M_{i-1}\|}\big)\|\Big)$$

In the case when T_i are iid and there are some strong mixing property on $\frac{M_{i-1}}{\|M_{i-1}\|}$, the terms $Y_i = \log\Big(\|T_i\big(\frac{M_{i-1}}{\|M_{i-1}\|}\big)\|\Big)$ should behave like iid random variables. One can then prove the strong law of large number, the central limit theorem, and Donsker theorem. See the paper of Le Page [9] for this matter. One therefore defines a "mean", a "variance" and a "random walk" as follows.

- The Lyapunov exponent is

$$\gamma(\lambda) := \lim_{N \to \infty} \frac{1}{N} \mathbb{E}\big[\log\|M_N(\lambda)\|\big].$$

- The limit variance is

$$\sigma^2(\lambda) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}\big[(\log\|M_N(\lambda)\| - \gamma(\lambda)N)^2\big].$$

- The "Random Walk" is

$$S_n = \frac{1}{\sigma(\lambda)}\big(\log\|M_n(\lambda)\| - \gamma(\lambda)n\big),$$

and

$$W_N(t) = \frac{1}{\sqrt{N}} S_{\lfloor Nt \rfloor}.$$

Finally, we denote by W the Wiener measure.

Theorem 7 (Limit theorem for products of random matrices)**.** We have the following:

- There exists $\gamma(\lambda) > 0$ such that $\lim_{N \to \infty} \frac{1}{N} \log\|M_N(\lambda)\| = \gamma(\lambda)$, almost surely,

- $\sigma^2(\lambda) > 0$,
- $W_N \to W$ in law.

We refer to [9, Theorems 2 and 3] for the proof of Theorem 7.

We recover then the form of a Brownian with drift, both on the right hand side and the left hand side of the cut. For λ an eigenvalue, and $r_k e^{i\phi_k}$ constructed from the corresponding eigenvector, we note $q_k^\lambda = \log(r_k)$. For scaling, we set $q^\lambda(s) = \frac{q_{\lfloor Ns \rfloor}^\lambda}{N}$

Proposition 8 (Tail of eigenvectors)**.**

1) Choosing $\lambda^{(N)}$ uniformly in $\sigma(H^{(N)})$, we have the following convergence in law

$$(\lambda^{(N)}, \frac{q_{\lfloor Ns \rfloor}^{\lambda^{(N)}}}{N}) \to (\tilde{\lambda}, -|\gamma(\tilde{\lambda})(s - x)|)$$

   when $N \to \infty$, where $\tilde{\lambda}$ is a random variable with law the limiting density of state ρ and x an independent variable on $[0, 1]$ with uniform law.

2) There exists a sequence of random variables $\{x^{(N)}\}$ with uniform law on $[0, 1]$ such that

$$(\lambda^{(N)}, \frac{q_{\lfloor Ns \rfloor}^{\lambda^{(N)}} + N\gamma(\lambda^N)|s - x^{(N)}|}{\sqrt{N}}) \to (\tilde{\lambda}, \sigma(\tilde{\lambda})W_{s-x})$$

   when $N \to \infty$, where W is the Wiener measure.

The first statement is the very classical result of Anderson localization for the one dimensional model. The eigenvectors decay exponentially from their center of localization and this center is chosen uniformly on the domain. The second statement says that the typical deviation from the decay is the exponential of a Brownian (see Figure 1 for an ilustration).

Rifkind and Virag [10] studied the eigenvectors in the bulk of the one dimensional Anderson model in the continuous case where the potential is a white noise. It is the limit of the discrete model in the called critical regime where the potential is scaled like $V_\omega^{(N)} = \frac{1}{\sqrt{N}} V_\omega$. In this regime, we are not able to speak of localization because the length of the decay is as large as the size of the domain. However they prove the exact law of the form of the eigenvectors

$$q^\lambda(s) = -|\gamma(\lambda)(s - x)| + \sigma(\lambda)W_{s-x}$$

To make the connection with our previous proposition, one can actually show that for $V_\omega = \epsilon v_\omega$, with $\mathbb{E}(v_\omega^2) = \sigma^2$, in the limit $\epsilon \to 0$ and $|\lambda| < 2$, we have

$$\gamma(\lambda) = \frac{\sigma^2}{4 - \lambda^2}\epsilon^2 + o(\epsilon^2)$$

and

$$\frac{\sigma(\lambda)^2}{2} = \frac{\sigma^2}{4 - \lambda^2}\epsilon^2 + o(\epsilon^2).$$

Proof of Proposition 8. If in our formula (2.3) the term $\delta_{\phi_k^f - \phi_k^b}$ were not there, then the forward and the backward process would have been completely independent. Our proposition would have then followed from Theorem 7, under the conditions that r_k obtained by the forward process and the r_k obtained by the backward process are the same. And that the normalization $\sum_{n=1}^{N} |u_n|^2 = 1$ which is replaced at the limit by $\sup q^\lambda(s) = 0$.

Therefore we only have to check that the little perturbation around the cut k has no influence. We fix ϕ. Conditionally on $\phi_k^b = \phi$ and $\phi_k^f = \phi$ the F-B processes are independent. The results of Theorem 7 are true asymptotically with probability 1. Therefore for any ϕ in a set of full Lebesgue measure on $\mathbb{S}^1$ the results of Theorem 7 are true conditionally of $\phi_k^b = \phi$ and $\phi_k^f = \phi$. $\qquad\square$
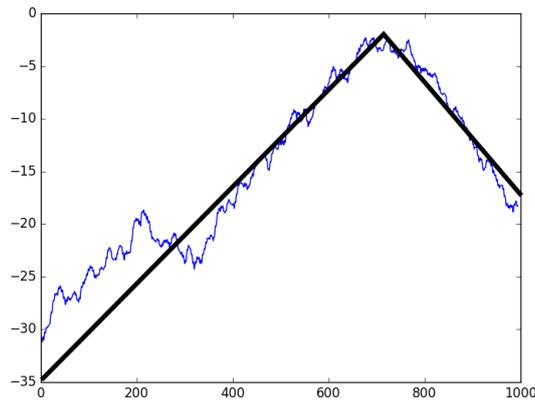
Figure 1: A realization of $\log\|M_n(\lambda)\|$ for $N = 1000$, v_ω uniform on $[0,1]$ with Dirichlet boundary conditions. We add a fit of the form $|\gamma(\lambda)(s - x)|$.

3.3 A temperature profile

We will use our result to explain some numerical observations which have been made in [5]. In this article, the authors are interested in the temperature profile of a disordered chain connected to two thermal baths of temperatures T_L at the left hand side (connected to the site 1) and T_R at the right hand side (connected to the site N). According to [5], the temperature $T(x)$ at site x is expected to be given (under certain limiting assumptions for the thermalisation process) by

$$T(x) = \sum_{\lambda \in \sigma(H^{(N)})} |\psi_\lambda(x)|^2 \Big(T_L \frac{|\psi_\lambda(1)|^2}{|\psi_\lambda(1)|^2 + |\psi_\lambda(N)|^2} + T_R \frac{|\psi_\lambda(N)|^2}{|\psi_\lambda(1)|^2 + |\psi_\lambda(N)|^2}\Big) \tag{3.1}$$

where $H_N^{(N)}$ is our one dimensional random Schrödinger operator and ψ_λ are its eigenvectors.

We prove that T converge to a step function where the transition from T_L and T_R happens in a neibourghood of $x = \frac{N}{2}$ at a scale $\sqrt{N}$. We denote $x = \lfloor \frac{N}{2} + \sqrt{N}y \rfloor$ and we expect variation of T with y of order 1. This has been observed numerically in [5].

Proposition 9. We have the following convergence

$$\mathbb{E}\big[T(\lfloor\sqrt{N}y + \frac{N}{2}\rfloor)\big] = T_L + (T_R - T_L)\int_{\mathbb{R}} \mathbb{P}\Big(\mathcal{N}(0,1) \leq \frac{2\gamma(\lambda)}{\sigma(\lambda)}y\Big)dN(\lambda) + o_{N\to\infty}(1)$$

where $dN(\lambda)$ is the integrated density of states as defined in Proposition 6, $\gamma(\lambda)$ the Lyapunov exponent and $\sigma(\lambda)$ the limit variance.

The Lyapunov exponent is positive, continuous and so is bounded from below on the support of $\sigma(H^{(N)})$. The variance $\sigma(\lambda)$ is bounded, therefore uniformly on λ,

$$\mathbb{P}(\mathcal{N}(0,1) \leq \frac{\sigma(\lambda)}{2\gamma(\lambda)}y) \to 0$$

for $y \to -\infty$ and

$$\mathbb{P}(\mathcal{N}(0,1) \leq \frac{\sigma(\lambda)}{2\gamma(\lambda)}y) \to 1$$

for $y \to \infty$. We have then $T(x) \approx T_L$ for $\frac{N}{2} - x \gg \sqrt{N}$ and $T(x) \approx T_R$ for $x - \frac{N}{2} \gg \sqrt{N}$, the step function numerically observed.

Proof. We use our formula and write:

$$\mathbb{E}(T(x)) = \sum_{k\in[1,N]} \int_{\mathbb{R}} d\lambda \mathbb{E}_{\mathcal{P}_{f,1..k}^{(\lambda)}\otimes\mathcal{P}_{b,k,...,N}^{(\lambda)}} \Big[|\psi_\lambda(x)|^2$$

$$\Big(T_L \frac{|\psi_\lambda(1)|^2}{|\psi_\lambda(1)|^2 + |\psi_\lambda(N)|^2} + T_R \frac{|\psi_\lambda(N)|^2}{|\psi_\lambda(1)|^2 + |\psi_\lambda(N)|^2}\Big) \times$$

$$\times \delta_{\phi_k^f - \phi_k^b} \sin^2(\phi_k^f) \Big].$$

With the notation of Proposition 8, we write

$$T_L \frac{|\psi_\lambda(1)|^2}{|\psi_\lambda(1)|^2 + |\psi_\lambda(N)|^2} + T_R \frac{|\psi_\lambda(N)|^2}{|\psi_\lambda(1)|^2 + |\psi_\lambda(N)|^2}$$

$$= T_L \frac{e^{Nq_1^\lambda}}{e^{Nq_1^\lambda} + e^{Nq_N^\lambda}} + T_R \frac{e^{Nq_N^\lambda}}{e^{Nq_1^\lambda} + e^{Nq_N^\lambda}}$$

Therefore for N large, this converges to T_L for $q_1^\lambda > q_N^\lambda$ and T_1 for $q_1^\lambda < q_N^\lambda$. We have then at the limit a Bernoulli T_{int} with parameter given by Proposition 8:

$$T_{int} = \begin{cases} T_R & \text{with probability } \mathbb{P}(\mathcal{N}(0,1) \leq \frac{(2k-N)\gamma(\lambda)}{\sqrt{N}\sigma(\lambda)}) \\ T_L & \text{with probability } \mathbb{P}(\mathcal{N}(0,1) \geq \frac{(2k-N)\gamma(\lambda)}{\sqrt{N}\sigma(\lambda)}) \end{cases}.$$

In order to conclude, we recall that most of the mass of $|\psi_\lambda|^2$ is around a few number of sites around k so

$$\mathbb{E}(T(x)) = \int_{\mathbb{R}} d\lambda \sum_{k\in[x-\alpha(N),x+\alpha(N)]} \mathbb{E}_{\mathcal{P}_{f,1..k}^{(\lambda)}\otimes\mathcal{P}_{b,k,...,N}^{(\lambda)}} \Big[|\psi_\lambda(x)|^2$$

$$\Big(T_L \frac{|\psi_\lambda(1)|^2}{|\psi_\lambda(1)|^2 + |\psi_\lambda(N)|^2} + T_R \frac{|\psi_\lambda(N)|^2}{|\psi_\lambda(1)|^2 + |\psi_\lambda(N)|^2}\Big)\delta_{\phi_k^f - \phi_k^b[\pi]}$$

$$\sin^2(\phi_k) \Big] + O(e^{-\gamma(\lambda)\alpha(N)})$$

where we chose $\alpha(N)$ such that $\sqrt{N} \gg \alpha(N) \gg 1$. Moreover for large N,

$$\mathbb{P}(\mathcal{N}(0,1) \geq \frac{(2x-N)\gamma(\lambda)}{\sqrt{N}\sigma(\lambda)}) = \mathbb{P}(\mathcal{N}(0,1) \geq \frac{(2k-N)\gamma(\lambda)}{\sqrt{N}\sigma(\lambda)}) + o(1),$$

we have then

$$\mathbb{E}(T(x)) = \int_{\mathbb{R}} d\lambda \sum_{k\in[x-\alpha(N),x+\alpha(N)]} \mathbb{E}_{\mathcal{P}_{f,1..k}^{(\lambda)}\otimes\mathcal{P}_{b,k,...,N}^{(\lambda)}} [|\psi_\lambda(x)|^2$$

$$(T_L + (T_R - T_L)\mathbb{P}(\mathcal{N}(0,1) \geq \frac{(2x-N)\gamma(\lambda)}{\sqrt{N}\sigma(\lambda)}))\delta_{\phi_k^f - \phi_k^b}$$

$$\sin^2(\phi_k^f)) + o(1).$$

Finally we use the following formula, for x such that $\min(x, N-x) \gg 1$

$$\sum_{k\in[1,N]} \mathbb{E}_{\mathcal{P}_{f,1..k}^{(\lambda)}\otimes\mathcal{P}_{b,k,...,N}^{(\lambda)}} [|\psi_\lambda(x)|^2 \delta_{\phi_k^f - \phi_k^b} \sin^2(\phi_k)] = \frac{dN(\lambda)}{d\lambda} + o(1).$$

Indeed, for any A Borel set of $\mathbb{R}$:

$$\frac{1}{N}\mathbb{E}(\#\{\sigma(H^{(N)}) \cap A\})$$

$$= \frac{1}{N}\mathbb{E}(Tr(1_A(H^{(N)})))$$

$$= \frac{1}{N}\sum_{x \in [1,N]}\mathbb{E}\Big[\sum_{\lambda \in A \cap \sigma(H^{(N)})}|\psi_\lambda(x)|^2\Big]$$

$$= \int 1_A(\lambda)d\lambda\frac{1}{N}\sum_{x \in [1,N]}\sum_{k \in [1,N]}\mathbb{E}_{\mathcal{P}_{f,1..k}^{(\lambda)} \otimes \mathcal{P}_{b,k,...,N}^{(\lambda)}}\Big[|\psi_\lambda(x)|^2\delta_{\phi_k^f - \phi_k^b}\sin^2(\phi_k^f)\Big]$$

We then note that the sums are asymptotically independent of x for x not close to the edges. Indeed,

$$\mathbb{E}_{\mathcal{P}_{f,1..k}^{(\lambda)} \otimes \mathcal{P}_{b,k,...,N}^{(\lambda)}}\Big[|\psi_\lambda(x)|^2\delta_{\phi_k^f - \phi_k^b}\sin^2(\phi_k^f)\Big]$$

$$= \iint \rho_{x-2\alpha(N)}(\phi_{x-2\alpha(N)})\bar{\rho}_{x-2\alpha(N)}(\phi_{x+2\alpha(N)})d\phi_{x-2\alpha(N)}d\phi_{x+2\alpha(N)}$$

$$\mathbb{E}_{\mathcal{P}_{f,x-2\alpha(N)..k}^{(\lambda)} \otimes \mathcal{P}_{b,k,...,x-2\alpha(N)}^{(\lambda)}}\Big[|\psi_\lambda(x)|^2\delta_{\phi_k^f - \phi_k^b}\sin^2(\phi_k^f)\Big] + o(1)$$

where we have the forward-backward process starting from $\phi_{x-2\alpha(N)}$ at the left and from $d\phi_{x+2\alpha(N),\lambda}$ at the right. But $\rho_{x-2\alpha(N),\lambda}$ and $\bar{\rho}_{x+2\alpha(N),\lambda}$ converge to the invariant measure ρ_λ and $\bar{\rho}_\lambda$ of the forward and the backward process. Finally we have

$$\int_{\mathbb{R}} 1_A(\lambda)\frac{dN(\lambda)}{d\lambda}d\lambda$$

$$= \int 1_A(\lambda)d\lambda\sum_{k \in [1,N]}\mathbb{E}_{\mathcal{P}_{f,1..k}^{(\lambda)} \otimes \mathcal{P}_{b,k,...,N}^{(\lambda)}}[|\psi_\lambda(x)|^2\delta_{\phi_k^f - \phi_k^b}\sin^2(\phi_k)] + o(1)$$

The proposition then follows, namely we have:

$$\mathbb{E}(T(x)) = T_L + \int_{\mathbb{R}}(T_R - T_L)\mathbb{P}\Big(\mathcal{N}(0,1) \geq \frac{(2x-N)\gamma(\lambda)}{\sqrt{N}\sigma(\lambda)}\Big)\frac{dN(\lambda)}{d\lambda}d\lambda + o(1)$$

as we wanted. □

3.4 Periodic boundary conditions

We tried to obtain a similar result for periodic boundary conditions. With the multi-scale analysis tools [6], one has the exponential decay from the center of localization. But because the periodic boundary conditions is among the most popular model it would be also interesting to have an interpretation with forward backward process in this case.

In the critical regime, one would expect the form of the eigenvectors to be like $e^{F(s)}$, on $\mathbb{R}/2\pi\mathbb{Z}$ with $F(s) = -\gamma d(s,u) + \sigma\tilde{B}_{s-u}$ with u uniformly chosen on $[0,2\pi]$, $d(s,u) = \min(|s-u|,|2\pi+s-u|,|s-2\pi-u|)$ the distance on $\mathbb{R}/2\pi\mathbb{Z}$ and $\tilde{B}$ a Brownian bridge (see Figure 2). So far we have not been able to prove this statement. However we think the following is interesting for further development:

Proposition 10. The condition $u_{-1} = u_{N+1} = 0$ in the Dirichlet case has to be replaced by $\text{Tr}(M_N(\lambda)) = 2$.

Proof. Let $(u_n)_{n \in [1,N]}$ be an eigenvector of eigenvalue λ and $z = \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$ where periodic boundary conditions means that by convention $u_N = u_0$ and $u_{N+1} = u_1$). Therefore

λ is a eigenvalue if and only if $M_N(\lambda)z = z$ ie 1 is an eigenvalue of $M_N(\lambda)$. Because $\det(M_N) = 1$ we have that 1 is a solution of $x^2 - Tr(M_N(\lambda))x + 1 = 0$ and so $\mathrm{Tr}(M_N(\lambda)) = 2$. Conversely if 1 is a eigenvalue of $M_N(\lambda)$ then one can chose (u_0, u_1) such that $\begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$ is the corresponding eigenvector of $M_N(\lambda)$ and construct iterativelly (u_n) with $\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = M_n(\lambda) \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$ to obtain a eigenvector of $H^{(N)}$ of eigvalue λ. □
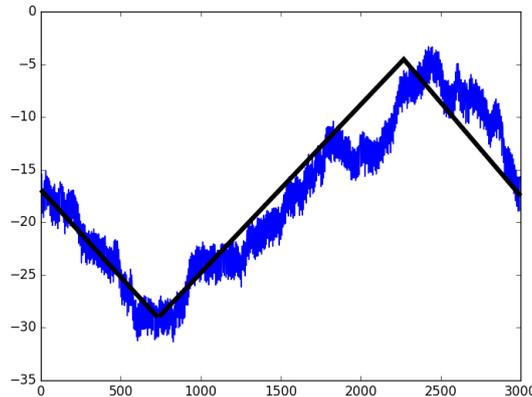

Figure 2: A realization of $\log \|M_n(\lambda)\|$ with periodic boundary conditions for $N = 3000$, v_ω uniform on $[0, 0.3]$. We add a fit of the form $-\gamma \min(|s - u|, |s - u + \pi|)$.

4   Proof of Theorem 4

Proof. We recall that ϕ_N is the angle for the complex $u_{N+1} + iu_N$, $\theta_N = \phi_N[2\pi]$ and the Dirichlet boundary condition states that $u_{N+1} = 0$ for an eigenvalue. We have that λ is an eigenvalue if and only if $\phi_N = \frac{\pi}{2}[\pi]$, therefore

$$\mathbb{E}\Big[\sum_{\lambda \in \sigma(H^{(N)}), X = \mathcal{P}h(\lambda)} G(\lambda, X) \Big] = \mathbb{E}\Big[\sum_{\lambda : \theta_N(\lambda) \in \frac{\pi}{2} + \pi\mathbb{Z}, \ X = \mathcal{F}(\lambda)} G(\lambda, X) \Big].$$

Remark 11. $\theta_N : \lambda \to \theta_N(\lambda)$ is continuous and strictly increasing (see calculation below).

For finite N, the inverse function θ_N^{-1} is continuous, so are G (continuous function of λ and θ_i), X. We can therefore write

$$\mathbb{E}\Big[\sum_{\lambda : \theta_N(\lambda) \in \pi\mathbb{Z} + \frac{\pi}{2}, X = \mathcal{F}(\lambda)} G(\lambda, X) \Big]$$
$$= \lim_{\epsilon \to 0} \mathbb{E}\Big[\frac{1}{2\epsilon} \sum_{n \in \mathbb{Z}} \int_{\pi n + \pi/2 - \epsilon}^{\pi n + \pi/2 + \epsilon} \sum_{\lambda : \theta_N(\lambda) = s, \ X = \mathcal{F}(\lambda)} G(\lambda, X) ds \Big].$$

The rest follows from a change of variables. Let us denote

$$I_\epsilon = \frac{\pi}{2} + \cup_{n \in \mathbb{Z}} [\pi n - \epsilon, \pi n + \epsilon]$$

and

$$P_\epsilon(G) = \mathbb{E}\Big[\frac{1}{2\epsilon} \sum_{n \in \mathbb{Z}} \int_{2\pi n - \epsilon}^{2\pi n + \epsilon} \sum_{\lambda : \theta_N(\lambda) = s} G(\lambda, \mathcal{F}(\lambda)) ds \Big]$$
$$= \mathbb{E}\Big[\int_{\mathbb{R}} G(\lambda, \mathcal{F}(\lambda)) | \frac{d\theta_N(\lambda)}{d\lambda} | \frac{1}{2\epsilon} 1_{\theta_N(\lambda) \in I_\epsilon} d\lambda \Big].$$

Then

$$\frac{d\theta_N(\lambda)}{d\lambda} = \frac{d\phi_N(\lambda)}{d\lambda}$$

$$= \frac{d}{d\lambda}\Big[\prod_{k=1}^{N} T(v_\omega(k) - \lambda)\phi_0 \Big]$$

$$= \sum_{k=1}^{N} \frac{d\phi_N}{d\phi_k}\big|_{v_\omega(N),\dots,v_\omega(k+1)} \cdot \frac{d}{d\lambda}[T(v_\omega(k) - \lambda)](\phi_{k-1}).$$

In this formula appears the term $\frac{d\phi_N}{d\phi_k}\big|_{v_\omega(N),\dots,v_\omega(k+1)}$. It is this term that changes the law from a forward process to a backward process. We then calculate $\frac{d}{d\lambda}[T(v_\omega(k) - \lambda)](\phi_{k-1})]$ with

$$\begin{pmatrix} u_{k+1} \\ u_k \end{pmatrix} = \begin{pmatrix} (v - \lambda)u_k + u_{k-1} \\ u_k \end{pmatrix},$$

$$\frac{d}{d\lambda}\begin{pmatrix} u_{k+1} \\ u_k \end{pmatrix} = \begin{pmatrix} -u_k \\ 0 \end{pmatrix},$$

and thus

$$\frac{d}{d\lambda}[T(V_\omega(k) - \lambda)](\phi_{k-1})] = \frac{\begin{pmatrix} u_{k+1} \\ u_k \end{pmatrix} \wedge \begin{pmatrix} -u_k \\ 0 \end{pmatrix}}{\Big\| \begin{pmatrix} u_{k+1} \\ u_k \end{pmatrix} \Big\|^2} = \frac{u_k^2}{u_k^2 + u_{k+1}^2} = \sin^2\phi_k.$$

We carry on the calculation,

$$P_\epsilon(G) = \mathbb{E}\Big[\int_{\mathbb{R}} G(\lambda, \mathcal{F}(\lambda))\big|\frac{d\theta_N(\lambda)}{d\lambda}\big| \frac{1}{2\epsilon}1_{\theta_N(\lambda)\in I_\epsilon} d\lambda \Big]$$

$$= \sum_{k=1}^{N} \int_{\mathbb{R}} d\lambda\Big[\int \dots \int d\nu(v_1)\dots d\nu(v_n)G(\lambda, \mathcal{F}(\lambda))\frac{d\phi_N}{d\phi_k} \cdot \sin^2(\phi_k) \Big] \frac{1}{2\epsilon}1_{\theta_N(\lambda)\in I_\epsilon}.$$

We artificially add a variable ϕ:

$$P_\epsilon(G) = \sum_{k=1}^{N} \int_{\mathbb{R}} d\lambda\Big[\int \dots \int d\nu(v_1)\dots d\nu(v_k) \int_{\mathbb{S}^1} d\phi\delta_{\phi_k}(\phi)$$

$$\int \dots \int d\nu(v_{k+1})\dots d\nu(v_N)G(\lambda, \mathcal{F}(\lambda))\frac{d\phi_N}{d\phi} \cdot \sin^2(\phi_k^f) \Big]\frac{1}{2\epsilon}1_{\theta_N(\lambda)\in I_\epsilon}.$$

Then we use the remark 2 and by taking the limit

$$\frac{1}{2\epsilon}1_{\phi_N\in I_\epsilon/\pi\mathbb{Z}}d\phi_N \to \delta_{\phi_N=0[\pi]}$$

we get

$$P_\epsilon(G) = \sum_{k=0}^{N} \int_{\mathbb{R}} d\lambda\Big[\int \dots \int d\nu(v_1)\dots d\nu(v_k) \int_{\mathbb{S}^1}$$

$$\int \dots \int d\phi_N d\nu(v_{k+1})\dots d\nu(v_N)\delta_{\phi_k}(\phi)G(\lambda, \mathcal{F}(\lambda))\sin^2(\phi_k^f) \Big]\frac{1}{2\epsilon}1_{\theta_N(\lambda)\in I_\epsilon},$$

$$\text{with } \mathcal{P}_{f,1..k}^{(\lambda)} \otimes \mathcal{P}_{b,k,\dots,N}^{(\lambda)}$$

$$= \sum_{k=0}^{N} \int_{\mathbb{R}} d\lambda\Big[\mathbb{E}_{\mathcal{P}_{f,1..k}^{(\lambda)} \otimes \mathcal{P}_{b,k,\dots,N}^{(\lambda)}}[G(\lambda, X)\delta_{\phi_k^f - \phi_k^b[\pi]}\sin^2(\phi_k^f)\frac{1}{2\epsilon}1_{\phi_N\in I_\epsilon/\pi\mathbb{Z}}]$$

is the forward-backward process with μ_b the uniform law on $\mathbb{S}^1$ and we can then conclude. $\qquad\square$

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