

Simultaneous boundary hitting by coupled reflected Brownian motions*

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Abstract

Mirror coupled reflected Brownian motions can simultaneously hit opposite sides of a wedge at different distances from the origin.

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1 Introduction

Our note is concerned with “mirror” couplings of reflected Brownian motions, defined in Section 2. These couplings were used many times to prove theorems in potential theory, see [1, 2, 3, 4, 5, 6]. The main arguments in all of these articles were based on the analysis of the motion of the “mirror,” i.e., the line of symmetry for two coupled reflected Brownian motions. Mirror motion analysis is simple and intuitive as long as a certain simple construction (see Section 2.2) of the mirror coupling can be applied. We will prove that, unfortunately, the simple construction is limited in its scope because two mirror coupled reflected Brownian motions can hit the sides of a wedge at the same time. The mirror-coupling-based proofs in [1, 2, 3, 4, 5, 6] are correct to our best knowledge. The negative result presented in this paper means that the existing proofs cannot be simplified and at least some future applications of mirror couplings will have to be based on less intuitive and less convenient constructions. To our best knowledge, the first rigorous construction of a mirror coupling in any domain with piecewise C^2 -boundary for all positive times was given in [2].

It is known (see [8]) that two reflected Brownian motions in a disc driven by the same Brownian motion (i.e., a “synchronous coupling”) can hit the boundary of the domain at the same time. Many results on path properties of the stochastic flow of reflected Brownian motions can be found in [12] and references therein.

Later in the paper we will need the following representation of a reflected Brownian motion. Let $D \subset \mathbb{R}^2$ be a bounded connected open set with piecewise C^2 -smooth boundary. Let $\mathbf{n}(x)$ denote the unit inward normal vector at $x \in \partial D$. Let B be standard 2-dimensional Brownian motion, $x \in \overline{D}$, and consider the following Skorokhod equation,

$$X_t = x + B_t + \int_0^t \mathbf{n}(X_s) dL_s, \quad \text{for } t \geq 0. \quad (1.1)$$

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Here L is the local time of X on ∂D . In other words, L is a non-decreasing continuous process which does not increase when X is in D , i.e., $\int_0^\infty \mathbf{1}_D(X_t) dL_t = 0$, a.s. Equation (1.1) has a unique pathwise solution (X, L) such that $X_t \in \bar{D}$ for all $t \geq 0$, for all $x \in \bar{D}$ (see [10]). The reflected Brownian motion X is a strong Markov process.

2 Mirror couplings

We will present three different constructions of “mirror couplings” of Brownian motions and reflected Brownian motions in planar domains, starting with couplings in the whole plane and then moving to domains of greater complexity. These constructions were originally developed in [7] and later applied in [4] and other articles. Our review is similar to that in [5].

2.1 Mirror couplings in the plane

Suppose that $x, y \in \mathbb{R}^2$ are symmetric with respect to a line M and $x \neq y$. Let X be a Brownian motion starting from x , let $T_M^X = \inf\{t \geq 0 : X \in M\}$, and let Y_t be the mirror image of X_t with respect to M for $t \leq T_M^X$. We let $Y_t = X_t$ for $t > T_M^X$. By the strong Markov property applied at T_M^X , the process Y is a Brownian motion starting from y . The pair (X, Y) is a “mirror coupling” of Brownian motions in the plane.

2.2 Mirror couplings in half-planes

Informally speaking, a mirror coupling in a half-plane is the unique coupling of reflected Brownian motions in the half-plane that behaves exactly as the mirror coupling in the whole plane when both processes are away from the boundary. Suppose that D_* is a half-plane, $x, y \in D_*$, and let M be the line of symmetry for x and y . The case when M is parallel to ∂D_* is essentially a one-dimensional problem, so we focus on the case when M intersects ∂D_* . By performing rotation and translation, if necessary, we may suppose that D_* is the upper half-plane and M passes through the origin. We will write $x = (r^x, \theta^x)$ and $y = (r^y, \theta^y)$ in polar coordinates. The points x and y are at the same distance from the origin so $r^x = r^y$. Suppose without loss of generality that $\theta^x < \theta^y$. We first generate a 2-dimensional Bessel process R_t starting from r^x . Then we generate two coupled one-dimensional processes on the “half-circle” as follows. Let $\tilde{\Theta}_t^x$ be a 1-dimensional Brownian motion starting from θ^x . Let $\tilde{\Theta}_t^y = -\tilde{\Theta}_t^x + \theta^x + \theta^y$. Let Θ_t^x be reflected Brownian motion on $[0, \pi]$, constructed from $\tilde{\Theta}_t^x$ by the means of the Skorokhod equation. Thus Θ_t^x solves the stochastic differential equation $d\Theta_t^x = d\tilde{\Theta}_t^x + dL_t$, where L_t is a continuous process that changes only when Θ_t^x is equal to 0 or π and Θ_t^x is always in the interval $[0, \pi]$. The process Θ_t^x is constructed in such a way that the difference $\Theta_t^x - \tilde{\Theta}_t^x$ is constant on every interval of time on which Θ_t^x does not hit 0 or π . The analogous reflected process obtained from $\tilde{\Theta}_t^y$ will be denoted $\hat{\Theta}_t^y$. Let τ^Θ be the smallest t with $\Theta_t^x = \hat{\Theta}_t^y$. Then we let $\Theta_t^y = \hat{\Theta}_t^y$ for $t \leq \tau^\Theta$ and $\Theta_t^y = \Theta_t^x$ for $t > \tau^\Theta$. We define a “clock” by $\sigma(t) = \int_0^t R_s^{-2} ds$. Then $X_t = (R_t, \Theta_{\sigma(t)}^x)$ and $Y_t = (R_t, \Theta_{\sigma(t)}^y)$ are reflected Brownian motions in D_* with normal reflection—one can prove this using the same ideas as in the discussion of the skew-product decomposition for 2-dimensional Brownian motion presented in [9]. Moreover, X and Y behave like free Brownian motions coupled by the mirror coupling as long as they are both strictly inside D_* . The processes will stay together after the first time they meet. We call (X, Y) a “mirror coupling” of reflected Brownian motions in half-plane.

The two processes X and Y in the upper half-plane remain at the same distance from the origin. Suppose now that D_* is an arbitrary half-plane, and x and y belong to D_* . Let M be the line of symmetry for x and y . Then an analogous construction yields a pair of reflected Brownian motions starting from x and y such that the distance from

X_t to $M \cap \partial D_*$ is always the same as for Y_t . Let M_t be the line of symmetry for X_t and Y_t . Note that M_t may move, but only in a continuous way, while the point $M_t \cap \partial D_*$ will never move. We will call M_t the *mirror* and the point $H := M_t \cap \partial D_*$ will be called the *hinge*. The absolute value of the angle between the mirror and the normal vector to ∂D_* at H can only decrease because, assuming that only one of the processes is reflecting from the boundary at some time, the reflecting process will receive an infinitesimal push in the direction of the inner normal vector, due to the local time L_t , and, therefore, the mirror will be pushed away from that process.

2.3 Mirror couplings in polygons

We will present an inductive construction of a mirror coupling (X, Y) of reflected Brownian motions in a planar convex polygonal domain D based on the constructions presented in Sections 2.1 and 2.2. We will construct a coupling only on a (random) time interval $[0, S_\infty]$ such that $X_t \notin \partial D$ or $Y_t \notin \partial D$ for every $t \in [0, S_\infty)$.

Assume that $x, y \in D$, $x \neq y$, and let $\{(X_t^1, Y_t^1), t \geq 0\}$ be the mirror coupling of Brownian motions in the whole plane, starting from $(X_0^1, Y_0^1) = (x, y)$. Let $S_0 = 0$ and $S_1 = \inf\{t \geq 0 : X_t^1 \in \partial D \text{ or } Y_t^1 \in \partial D\}$.

If $X_{S_1}^1 \in \partial D$ and $Y_{S_1}^1 \in \partial D$ then we let $S_\infty = S_1$ and we end the induction.

Suppose that either $X_{S_1}^1 \notin \partial D$ or $Y_{S_1}^1 \notin \partial D$. In the first case let I_1 be the edge of ∂D to which $Y_{S_1}^1$ belongs and let K_1 be the line containing I_1 . In the second case let I_1 be the edge of ∂D to which $X_{S_1}^1$ belongs and let K_1 be the line containing I_1 .

Suppose that $\{(X_t^k, Y_t^k), t \geq S_{k-1}\}$, S_k , I_k and K_k have been defined and either $X_{S_k}^k \notin \partial D$ or $Y_{S_k}^k \notin \partial D$, for some $k \geq 1$. Let $\{(X_t^{k+1}, Y_t^{k+1}), t \geq S_k\}$ be the mirror coupling of Brownian motions starting from $(X_{S_k}^{k+1}, Y_{S_k}^{k+1}) = (X_{S_k}^k, Y_{S_k}^k)$, constructed as in Section 2.2, in the half-plane containing D , with boundary K_k . Let $S_{k+1} = \inf\{t \geq S_k : X_t^{k+1} \in \partial D \text{ or } Y_t^{k+1} \in \partial D\}$.

If $X_{S_{k+1}}^{k+1} \in \partial D$ and $Y_{S_{k+1}}^{k+1} \in \partial D$ then we let $S_\infty = S_{k+1}$ and we end the induction.

Suppose that either $X_{S_{k+1}}^{k+1} \notin \partial D$ or $Y_{S_{k+1}}^{k+1} \notin \partial D$. In the first case let I_{k+1} be the edge of ∂D to which $Y_{S_{k+1}}^{k+1}$ belongs and let K_{k+1} be the line containing I_{k+1} . In the second case let I_{k+1} be the edge of ∂D to which $X_{S_{k+1}}^{k+1}$ belongs and let K_{k+1} be the line containing I_{k+1} .

If there is no k such that $S_\infty = S_k$ then we let $S_\infty = \lim_{k \rightarrow \infty} S_k$.

We define (X_t, Y_t) for $t \in [0, S_\infty)$ by $(X_t, Y_t) = (X_t^k, Y_t^k)$ for $t \in [S_{k-1}, S_k)$ and k such that $S_{k-1} < S_\infty$. If $S_\infty < \infty$ then we extend the definition of (X_t, Y_t) to $t = S_\infty$ by continuity.

The construction of the mirror coupling can be easily continued beyond S_∞ under some circumstances. For example, if $X_{S_\infty} = Y_{S_\infty}$ then X and Y can be continued beyond S_∞ as a single reflected Brownian motion in D .

Let M_t denote the mirror, i.e., the line of symmetry for X_t and Y_t . Since the process which hits I_k does not “feel” the shape of ∂D except for the direction of I_k , it follows that the two processes behave as a mirror coupling in a half-plane and, therefore, they remain at the same distance from the hinge $H_t := M_t \cap K_k$ on the interval $[S_k, S_{k+1})$. The mirror M_t can move but the hinge H_t remains constant on the interval $[S_k, S_{k+1})$. Typically, the hinge H_t jumps at times S_k . The hinge H_t may lie outside \bar{D} at some times.

2.4 Mirror couplings of reflected Brownian motions in a wedge

Remark 2.1. Before we state our result, we will list three events. Each one of these events can occur with strictly positive probability for some polygonal domain and initial conditions. In each case, $X_T \in \partial D$ and $Y_T \in \partial D$ for some stopping time $T \geq 0$, but the construction of the mirror coupling for times greater than T does not pose any technical

difficulties. Hence these three situations are not interesting.

(i) It may happen that $X_T = Y_T \in \partial D$. In this case, one can continue the mirror coupling as a single reflected Brownian motion in D representing both X and Y after time T .

(ii) It may happen that X and Y hit the same edge I at the same time T , at different points. In this case the mirror is orthogonal to I at time T . One can easily continue the mirror coupling after T , on some random time interval, until one of the processes hits a different edge of ∂D .

(iii) If the mirror passes through the intersection point of lines containing two edges I and J then it may happen that X hits I and Y hits J at the same time T . One can easily continue the mirror coupling after T , on some random time interval, until one of the processes hits a different edge of ∂D .

Note that one may take $T \equiv 0$ in cases (i)-(iii), for some domains and starting positions for X and Y . Hence, Theorem 2.2 cannot be strengthened from “some x and y ” to “all x and y .”

We will use complex and vector notation interchangeably.

Theorem 2.2. Consider a wedge $D = \{re^{i\theta} \in \mathbb{C} : r > 0, 0 < \theta < \alpha\}$ with angle $\alpha \in (0, \pi/2)$. We will denote the edges of D by $E_X = (0, \infty)$ and $E_Y = \{re^{i\alpha} : r > 0\}$. There exist $x, y \in D$ such that if $\{(X_t, Y_t), t \in [0, S_\infty)\}$ is the mirror coupling of reflected Brownian motions in D constructed as in Section 2.3 and $(X_0, Y_0) = (x, y)$ then

$$\mathbb{P}(S_\infty < \infty, X_{S_\infty} \in E_X, Y_{S_\infty} \in E_Y, |X_{S_\infty}| \neq |Y_{S_\infty}|) > 0. \tag{2.1}$$

Remark 2.3. (i) Recall that if $S_\infty < \infty$ then X_{S_∞} is defined as $\lim_{t \uparrow S_\infty} X_t$. A similar remark applies to Y_{S_∞} .

(ii) It is easy to see that if the event in (2.1) holds then none of the situations listed in Remark 2.1 (i)-(iii) could have occurred at time S_∞ .

(iii) It is clear from the construction given in Section 2.3 that Theorem 2.2 applies also to polygonal domains. The reason is that the mirror coupling “can only see” the boundary of the domain locally and, therefore, it evolves in the same way in a polygonal domain as in a wedge formed by the two sides of the polygon that are the closest to the two Brownian motions.

Proof of Theorem 2.2. Step 1. This step is devoted to a purely deterministic lemma. We will study the effect of increasing β on the position of A and the logarithmic transformation of the blue wedge in Fig. 1 (rigorous definitions are given below).

Recall that \angle denotes an angle; we will adopt the convention that all angles are in $[0, \pi]$. For any points F and G in the plane, let $|FG|$ denote the distance between them. We will identify points in the plane with complex numbers and points on the real axis with real numbers. Hence, if F is a point in the positive part of the real axis then $F = |F| = |0F|$. Let $U := \{z \in \mathbb{C} : 0 \leq \text{Im } z \leq \pi\}$.

Fix any $\alpha \in (0, \pi/2)$, consider $H > 0$, $\beta \in (\alpha, \pi/2)$, and let $M = \{H + re^{i\beta} : r \in \mathbb{R}\}$. Define H' by $\{H'\} = M \cap E_Y$. Let \mathcal{S} be the symmetry with respect to M , and define A and A' by $\{A\} = E_X \cap \mathcal{S}(E_Y)$ and $\{A'\} = \{\mathcal{S}(A)\} = E_Y \cap \mathcal{S}(E_X)$. See Fig. 1.

We will consider α and H to be constants and we will treat β as a variable. Note that H', A and A' are uniquely determined given H, α and β .

Elementary geometry shows that $\angle(0AH') = \pi + \alpha - 2\beta$ and $\angle(0A'H) = 2\beta - \alpha$. By the law of sines,

$$\frac{H}{\sin \angle(0A'H)} = \frac{|HA'|}{\sin \angle(H0A')},$$

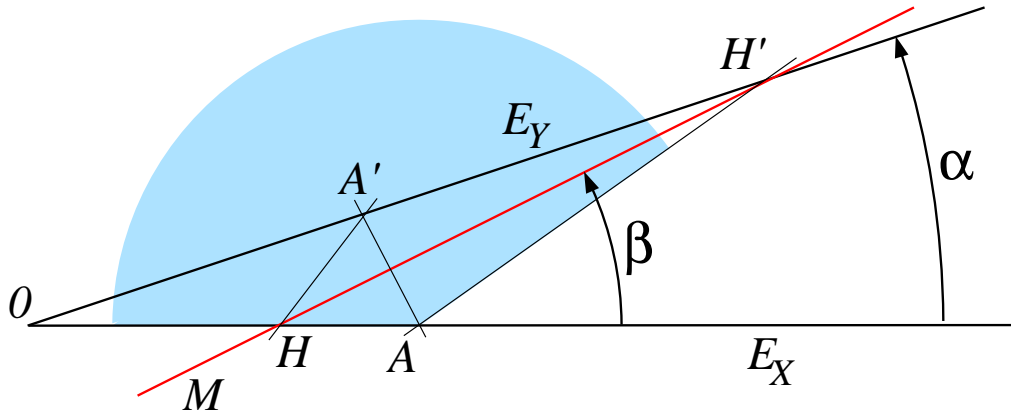


Figure 1: The first step of the proof is devoted to studying the effect of increasing β on the position of A and the logarithmic transformation of the blue wedge.

$$\begin{aligned} \frac{H}{\sin(2\beta - \alpha)} &= \frac{|HA'|}{\sin \alpha}, \\ |HA| = |HA'| &= \frac{H \sin \alpha}{\sin(2\beta - \alpha)}, \\ A = |H| + |HA| &= H(1 + \sin \alpha \csc(2\beta - \alpha)). \end{aligned} \tag{2.2}$$

Let W be the closed wedge with vertex A , such that its sides contain H and H' , and A' lies in its interior. Note that $\pi/4 + \alpha/2 < \pi/2$. For $z \in W$ and $\beta \in (\alpha, \pi/4 + \alpha/2)$ let

$$\begin{aligned} f(\beta, z) &= (\log(z - A) + i(\alpha - 2\beta)) \frac{\pi}{\pi + \alpha - 2\beta} \\ &= \left(\log \left(z - H \left(1 + \frac{\sin \alpha}{\sin(2\beta - \alpha)} \right) \right) + i(\alpha - 2\beta) \right) \frac{\pi}{\pi + \alpha - 2\beta}. \end{aligned} \tag{2.3}$$

The function $f(\beta, z)$ takes values in U and, informally speaking, sends (β, A) to $-\infty$. Consider $r \in (H, A)$. We use (2.2) to see that

$$\begin{aligned} f(\beta, r) &= (\log(r - A) + i(\alpha - 2\beta)) \frac{\pi}{\pi + \alpha - 2\beta} \\ &= (\log(-r + A) + i(\pi + \alpha - 2\beta)) \frac{\pi}{\pi + \alpha - 2\beta} \\ &= \left(\log \left(-r + H \left(1 + \frac{\sin \alpha}{\sin(2\beta - \alpha)} \right) \right) \right) \frac{\pi}{\pi + \alpha - 2\beta} + i\pi. \end{aligned}$$

We use (2.2) once again to get,

$$\begin{aligned} \frac{\partial}{\partial \beta} f(\beta, r) &= - \frac{2\pi H \sin(\alpha) \cot(2\beta - \alpha) \csc(2\beta - \alpha)}{(\pi + \alpha - 2\beta)(H(1 + \sin(\alpha) \csc(2\beta - \alpha)) - r)} \\ &\quad + \frac{2\pi}{(\pi + \alpha - 2\beta)^2} \log(H(1 + \sin(\alpha) \csc(2\beta - \alpha)) - r) \\ &= - \frac{2\pi H \sin(\alpha) \cot(2\beta - \alpha) \csc(2\beta - \alpha)}{(\pi + \alpha - 2\beta)(A - r)} + \frac{2\pi}{(\pi + \alpha - 2\beta)^2} \log(A - r). \end{aligned} \tag{2.4}$$

Recall that $\alpha \in (0, \pi/2)$ and fix β_1^* and β_2^* such that $\alpha < \beta_1^* < \beta_2^* < \pi/2$ and $2\beta_2^* - \alpha < \pi/2$. If $\beta \in [\beta_1^*, \beta_2^*]$ then $\sin(\alpha) \cot(2\beta - \alpha) \csc(2\beta - \alpha) > 0$. Hence we can find

$c_1^* = c_1^*(\alpha, H, \beta_1^*, \beta_2^*) > 0$ such that if $\beta \in [\beta_1^*, \beta_2^*]$ then

$$\frac{\partial}{\partial \beta} f(\beta, r) < -c_1^*(A - r)^{-1}. \tag{2.5}$$

Next we calculate the normal derivative of f with respect to the second variable. If we write $z = re^{i\theta}$ then

$$\left| \frac{1}{r} \cdot \frac{\partial}{\partial \theta} f(\beta, re^{i\theta}) \right|_{\theta=0} = \frac{\pi}{\pi + \alpha - 2\beta} (A - r)^{-1}. \tag{2.6}$$

We will derive an analogous estimate for a mapping corresponding to the other side of the wedge D . Let $\gamma = \angle(0H'H)$ and note that $\gamma = \beta - \alpha$. We will now consider α and H' to be constants and we will treat γ as a variable. Note that H, A and A' are uniquely determined given H', α and γ .

We have $\angle(0AH') = \pi - \alpha - 2\gamma$. By the law of sines,

$$\begin{aligned} \frac{|H'|}{\sin \angle(0AH')} &= \frac{|H'A|}{\sin \angle(H'0A)}, \\ \frac{|H'|}{\sin(\pi - \alpha - 2\gamma)} &= \frac{|H'A|}{\sin \alpha}, \\ |H'A'| = |H'A| &= \frac{|H'| \sin \alpha}{\sin(2\gamma + \alpha)}, \\ |A'| = |H'| - |H'A'| &= |H'| (1 - \sin \alpha \csc(2\gamma + \alpha)). \end{aligned} \tag{2.7}$$

Let W' be the closed wedge with vertex A' , such that its sides contain H and H' , and A lies in its interior. Let $(v)^-$ denote the complex conjugate of $v \in \mathbb{C}$. For $z \in W'$ and $\gamma \in (0, \pi/4 - \alpha/2)$ let

$$\begin{aligned} g(\gamma, z) &= \left((\log(z - A') - i\alpha) \frac{\pi}{\pi - \alpha - 2\gamma} \right)^- \\ &= \left((\log(z - H'(1 - \sin \alpha \csc(2\gamma + \alpha))) - i\alpha) \frac{\pi}{\pi - \alpha - 2\gamma} \right)^-. \end{aligned} \tag{2.8}$$

The function $g(\gamma, z)$ takes values in U and, informally speaking, sends (γ, A') to $-\infty$. Consider $z = re^{i\alpha}$ with $r \in (|A'|, |H'|)$. Using (2.7), we obtain

$$\begin{aligned} g(\gamma, z) &= \left((\log(re^{i\alpha} - A') - i\alpha) \frac{\pi}{\pi - \alpha - 2\gamma} \right)^- \\ &= \left(\log(r - |A'|) \frac{\pi}{\pi - \alpha - 2\gamma} \right)^- \\ &= \log(r - |A'|) \frac{\pi}{\pi - \alpha - 2\gamma} \\ &= \log(r - |H'| (1 - \sin \alpha \csc(2\gamma + \alpha))) \frac{\pi}{\pi - \alpha - 2\gamma}, \end{aligned}$$

so, using (2.7) once again,

$$\begin{aligned} \frac{\partial}{\partial \gamma} g(\gamma, z) &= - \frac{2\pi |H'| \sin(\alpha) \cot(\alpha + 2\gamma) \csc(\alpha + 2\gamma)}{(\pi - \alpha - 2\gamma)(r - |H'| (1 - \sin(\alpha) \csc(\alpha + 2\gamma)))} \\ &\quad + \frac{2\pi \log(r - |H'| (1 - \sin(\alpha) \csc(\alpha + 2\gamma)))}{(\pi - \alpha - 2\gamma)^2} \\ &= - \frac{2\pi |H'| \sin(\alpha) \cot(\alpha + 2\gamma) \csc(\alpha + 2\gamma)}{(\pi - \alpha - 2\gamma)(r - |A'|)} + \frac{2\pi \log(r - |A'|)}{(\pi - \alpha - 2\gamma)^2}. \end{aligned}$$

Brownian couplings

Recall that $\alpha \in (0, \pi/2)$ and let $\gamma_1^* = \beta_1^* - \alpha$ and $\gamma_2^* = \beta_2^* - \alpha$. Then

$$0 < \gamma_1^* < \gamma_2^* < \pi/2 - \alpha, \quad 2\gamma_2^* + \alpha < \pi/2. \quad (2.9)$$

If $\gamma \in [\gamma_1^*, \gamma_2^*]$ then $\sin(\alpha) \cot(2\gamma + \alpha) \csc(2\gamma + \alpha) > 0$. Hence we can find $c_2^* = c_2^*(\alpha, H', \gamma_1^*, \gamma_2^*) > 0$ such that if $\gamma \in [\gamma_1^*, \gamma_2^*]$ then

$$\frac{\partial}{\partial \gamma} g(\gamma, r) < -c_2^*(r - |A'|)^{-1}. \quad (2.10)$$

We will now calculate the normal derivative of g with respect to the second variable. Write $z = re^{i\theta}$. Then

$$\left| \frac{1}{r} \cdot \frac{\partial}{\partial \theta} g(\gamma, re^{i\theta}) \Big|_{\theta=\alpha} \right| = \frac{\pi}{\pi - \alpha - 2\gamma} (r - |A'|)^{-1}. \quad (2.11)$$

Recall that $\gamma = \beta - \alpha$ and note that for fixed α, β and H , we have for $z \in W'$,

$$g(\gamma, z) = f(\beta, \mathcal{S}(z)). \quad (2.12)$$

Step 2. Let L^X and L^Y denote the local times in the representation (1.1) for X and Y , resp. Recall that M_t denotes the line of symmetry for X_t and Y_t , reflected Brownian motions in D . Assume that $M_0 = \{K + re^{i\beta_0} : r \in \mathbb{R}\}$ for some $K \in E_X$ and $\alpha < \beta_0 < \pi/4 + \alpha/2$.

Let \bar{E}_X (\bar{E}_Y) be the straight line containing E_X (E_Y). Let $\{H_{X,t}\} = M_t \cap \bar{E}_X$ and $\{H_{Y,t}\} = M_t \cap \bar{E}_Y$. Let \mathcal{S}_t be the symmetry with respect to M_t for $t < S_\infty$. In particular, we have $\mathcal{S}_t(X_t) = Y_t$ for all $t < S_\infty$.

Let β_t be defined by $M_t = \{H_{X,t} + re^{i\beta_t} : r \in \mathbb{R}\}$ and

$$T' = \inf \{t \geq 0 : X_t = Y_t \text{ or } 0 \in M_t \text{ or } \beta_t \notin (\alpha, \pi/4 + \alpha/2)\}, \quad (2.13)$$

$$T'' = \inf \{t \geq 0 : X_t \in \partial D \text{ and } Y_t \in \partial D\}, \quad (2.14)$$

$$T = T' \wedge T''. \quad (2.15)$$

The following definitions apply to $t \in [0, T)$.

Let $A_{X,t}$ and $A_{Y,t}$ be defined by $\{A_{X,t}\} = \bar{E}_X \cap \mathcal{S}_t(\bar{E}_Y)$ and $\{A_{Y,t}\} = \{ \mathcal{S}_t(A_{X,t}) \} = \bar{E}_Y \cap \mathcal{S}_t(\bar{E}_X)$. See Fig. 2.

We will argue that our assumptions on M_0 and β_0 imply that $0 < H_{X,0} < A_{X,0}$ and $0 < |A_{Y,0}| < |H_{Y,0}|$. Since $\alpha \in (0, \pi/2)$, we have $\alpha < \beta_0 < \pi/4 + \alpha/2 < \pi/2$. Hence, $\mathcal{S}_t(\bar{E}_Y)$ must intersect E_X at a point $A_{X,0}$ to the right of $H_{X,0}$ and such that $|0A_{X,0}| < 2|0H_{X,0}|$. The bound $\beta_0 < \pi/2$ also implies that $A_{Y,0}$ must lie between 0 and $H_{Y,0}$.

Let $W_{X,t}$ be the closed wedge with vertex $A_{X,t}$, such that its sides contain $H_{X,t}$ and $H_{Y,t}$, and $A_{Y,t}$ lies in its interior. For $t \in [0, T)$ and $z \in W_{X,t}$ let

$$\mathcal{F}(t, z) = (\log(z - A_{X,t}) + i(\alpha - 2\beta_t)) \frac{\pi}{\pi + \alpha - 2\beta_t}.$$

Let $\gamma_t = \beta_t - \alpha$. Let $W_{Y,t}$ be the closed wedge with vertex $A_{Y,t}$, such that its sides contain $H_{Y,t}$ and $H_{X,t}$, and $A_{X,t}$ lies in its interior. Recall that $(v)^-$ denotes the complex conjugate of $v \in \mathbb{C}$. For $t \in [0, T)$ and $z \in W_{Y,t}$ let

$$\mathcal{G}(t, z) = \left((\log(z - A_{Y,t}) - i\alpha) \frac{\pi}{\pi - \alpha - 2\gamma_t} \right)^-.$$

It follows from (2.3), (2.8) and (2.12) that

$$\mathcal{F}(t, z) = f(\beta_t, z), \quad \text{for } t \in [0, T), z \in W_{X,t},$$

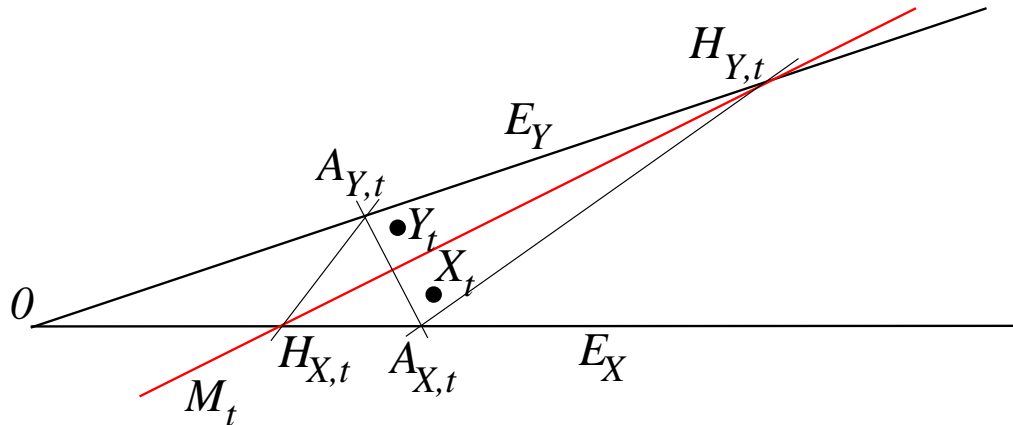


Figure 2: The proof shows that it is possible for X and Y to visit the boundary simultaneously at a time t such that $X_t = A_{X,t}$, $Y_t = A_{Y,t}$ and $|X_t| \neq |Y_t|$.

$$\begin{aligned} \mathcal{G}(t, z) &= g(\gamma_t, z), \quad \text{for } t \in [0, T], z \in W_{Y,t}, \\ \mathcal{G}(t, z) &= \mathcal{F}(t, \mathcal{S}(z)), \quad \text{for } z \in W_{Y,t}, \\ \mathcal{G}(t, Y_t) &= \mathcal{F}(t, \mathcal{S}(Y_t)) = \mathcal{F}(t, X_t), \quad \text{for } t \in [0, T]. \end{aligned}$$

The function $\mathcal{F}(t, z)$ takes values in U and sends $(t, A_{X,t})$ to $-\infty$. The function $\mathcal{G}(t, z)$ also takes values in U and sends $(t, A_{Y,t})$ to $-\infty$.

If $X_t \in E_X$ for some t then we will call X active at time t , and similarly for Y . Suppose that X_t is active at time t . Then, over a short time interval $[t, t + \delta]$, the mirror M_t will move from the position M_t to $M_{t+\delta}$, the angle β_t will increase to $\beta_{t+\delta}$, the wedge $W_{X,t}$ will be transformed into the wedge $W_{X,t+\delta}$, and the angle of $W_{X,t}$ will change from $\pi + \alpha - 2\beta_t$ to $\pi + \alpha - 2\beta_{t+\delta}$. As a result, $A_{X,t}$ will move to $A_{X,t+\delta}$ in the direction of X_t . Analogous remarks apply to the situation when Y is active at time t . In that case $A_{Y,t}$ will move to $A_{Y,t+\delta}$ in the direction of Y_t . For a point z between 0 and $A_{X,t+\delta}$, its image under \mathcal{F} will change from $\mathcal{F}(t, z)$ to $\mathcal{F}(t + \delta, z)$.

Let

$$Z_t^* = \mathcal{G}(t, Y_t) = \mathcal{F}(t, X_t), \quad t \in [0, T), \tag{2.16}$$

$$\begin{aligned} \tilde{\rho}(t) &= \int_0^t \left| \left(\frac{d}{dz} \mathcal{F}(s, z) \Big|_{z=X_s} \right) \right|^2 ds, \quad t \in [0, T), \\ s_* &= \lim_{t \uparrow T} \tilde{\rho}(t), \end{aligned} \tag{2.17}$$

$$\begin{aligned} \rho(t) &= \inf\{s \geq 0 : \tilde{\rho}(s) \geq t\}, \\ Z_s &= Z^*(\rho(s)), \quad s \in [0, s_*). \end{aligned}$$

The process $\{Z_t, t \in [0, s_*)\}$ is reflected Brownian motion in U with (random) oblique reflection, by an argument very similar to the proof of [11, Thm. 2.3]. We will not reproduce that proof here but we will point similarities. In [11, Thm. 2.3], a reflected Brownian motion is transformed by a continuous mapping depending on space and time. In that paper, there is a non-decreasing process, the norm of the original reflected Brownian motion, that is constant on time intervals whose union has full Lebesgue measure. On each of these intervals, the mapping does not depend on time and is analytic in the space variable. In our case, the local times L^X and L^Y are constant on

time intervals whose union has full Lebesgue measure. On each of these intervals, the mapping $\mathcal{F}(t, z)$ does not depend on time and is analytic in the space variable.

The obliquely reflected Brownian motion Z_t in U has the following representation. For some two-dimensional Brownian motion B' and $z_0 = \mathcal{F}(0, X_0) \in U$,

$$Z_t = z_0 + B'_t + \int_0^t \mathbf{v}(s, Z_s) dL_s^Z, \quad \text{for } t \in [0, s_*]. \tag{2.18}$$

Here L^Z is the local time of Z on ∂U . In other words, L^Z is a non-decreasing continuous process which does not increase when Z is in the interior U° of U , i.e., $\int_0^{s_*} \mathbf{1}_{U^\circ}(Z_t) dL_t^Z = 0$, a.s. The vector of oblique reflection \mathbf{v} is normalized in (2.18) so that the absolute value of its normal component is equal to 1. The vector \mathbf{v} is random, i.e., $\mathbf{v}(s, \cdot)$ depends on $\{Z_t, 0 \leq t \leq s\}$. We will write $\mathbf{v} = (v_1, v_2) = v_1 + iv_2$, for $v_1, v_2 \in \mathbb{R}$. Hence, $|v_2| = 1$. More precisely, $v_2(z) = 1$ if $z \in \mathbb{R}$ and $v_2(z) = -1$ if $\text{Im}(z) = \pi$.

We will now determine the first component v_1 of the vector of oblique reflection \mathbf{v} . We will use the informal differential notation: $\Delta\beta_t = a(t)\Delta L_t^X$ should be interpreted as $\int_s^u d\beta_t = \int_s^u a(t)dL_t^X$ for all $s < u$. It follows from the construction of the mirror coupling in a half-plane outlined in Section 2.2 that at the time when X_t is active, this process gets an infinitesimal push along the inner normal vector of the size ΔL_t^X so the angular motion of the mirror, in radians, is equal to one half of the angle swept by X , i.e.,

$$\Delta\beta_t = \frac{\Delta L_t^X}{2|X_t - H_{X,t}|}. \tag{2.19}$$

Therefore, by the chain rule,

$$\begin{aligned} \frac{v_1(\tilde{\rho}(t), Z_{\tilde{\rho}(t)})}{|v_2(\tilde{\rho}(t), Z_{\tilde{\rho}(t)})|} &= \frac{\Delta\beta_t}{\Delta L_t^X} \cdot \frac{\frac{\partial}{\partial\beta} f(\beta, r) \Big|_{r=|X_t|, \beta=\beta_t}}{\left| \frac{1}{r} \cdot \frac{\partial}{\partial\theta} f(\beta, re^{i\theta}) \Big|_{\theta=0, r=|X_t|, \beta=\beta_t} \right|} \\ &= \frac{1}{2|X_t - H_{X,t}|} \cdot \frac{\frac{\partial}{\partial\beta} f(\beta, r) \Big|_{r=|X_t|, \beta=\beta_t}}{\left| \frac{1}{r} \cdot \frac{\partial}{\partial\theta} f(\beta, re^{i\theta}) \Big|_{\theta=0, r=|X_t|, \beta=\beta_t} \right|}. \end{aligned} \tag{2.20}$$

Fix some β_1^* and β_2^* such that $\alpha < \beta_1^* < \beta_0 < \beta_2^* < \pi/2$ and $2\beta_2^* - \alpha < \pi/2$. Let $T_1 = T \wedge \inf\{t \geq 0 : \beta_t \notin [\beta_1^*, \beta_2^*]\}$. We combine (2.5), (2.6) and (2.20) to derive the following estimate for times $t \in [0, T_1)$ such that X_t is active,

$$\frac{v_1(\tilde{\rho}(t), Z_{\tilde{\rho}(t)})}{|v_2(\tilde{\rho}(t), Z_{\tilde{\rho}(t)})|} \leq -\frac{c_1^*}{2|X_t - H_{X,t}|} \cdot \frac{\pi + \alpha - 2\beta_2^*}{\pi} =: -\frac{\hat{c}_1}{|X_t - H_{X,t}|}. \tag{2.21}$$

Note that $\hat{c}_1 > 0$.

Let $\gamma_1^* = \beta_1^* - \alpha$ and $\gamma_2^* = \beta_2^* - \alpha$. Since $\gamma_t = \beta_t - \alpha$, we have $\gamma_1^* < \gamma_0 < \gamma_2^*$. Note that $T_1 = T \wedge \inf\{t \geq 0 : \gamma_t \notin [\gamma_1^*, \gamma_2^*]\}$. The conditions imposed on β_1^* and β_2^* imply that $0 < \gamma_1^* < \gamma_0 < \gamma_2^* < \pi/2 - \alpha$ and $2\gamma_2^* + \alpha < \pi/2$. Hence, the assumptions (2.9) are satisfied. It follows that we can use (2.10) and (2.11) to derive the following estimate, analogous to (2.21), for times $t \in [0, T_1)$ such that Y_t is active,

$$\frac{v_1(\tilde{\rho}(t), Z_{\tilde{\rho}(t)})}{|v_2(\tilde{\rho}(t), Z_{\tilde{\rho}(t)})|} \leq -\frac{c_2^*}{2|Y_t - H_{Y,t}|} \cdot \frac{\pi - \alpha - 2\gamma_2^*}{\pi} =: -\frac{\hat{c}_2}{|Y_t - H_{Y,t}|}, \tag{2.22}$$

where $\hat{c}_2 > 0$.

It is easy to see that we can choose $r_1 > 0$ so small that

$$r_1 < \frac{1}{8} (|A_{X,0} - A_{Y,0}| \wedge |A_{X,0} - H_{X,0}| \wedge |A_{Y,0} - H_{Y,0}|), \tag{2.23}$$

and if $T' < \infty$ then $|X_{T'} - A_{X,0}| \geq 3r_1$ and $|Y_{T'} - A_{Y,0}| \geq 3r_1$.

It follows from the construction of the mirror coupling given in Sections 2.2-2.3 that on the interval $[0, T_1]$, the distance $|0H_{X,t}|$ is non-decreasing, the distance $|0H_{Y,t}|$ is non-increasing, and the functions β_t and γ_t are non-decreasing. This implies that there exist $\widehat{\beta}_2 \in (\beta_0, \beta_2^*)$ and $\widehat{\gamma}_2 \in (\gamma_0, \gamma_2^*)$ so small that if

$$T_2 = T \wedge \inf\{t \geq 0 : \beta_t \notin [\beta_0, \widehat{\beta}_2]\} \wedge \inf\{t \geq 0 : \gamma_t \notin [\gamma_0, \widehat{\gamma}_2]\} \tag{2.24}$$

then for $t \in [0, T_2]$,

$$|H_{X,t} - H_{X,0}| \leq r_1, \quad |H_{Y,t} - H_{Y,0}| \leq r_1, \tag{2.25}$$

$$|A_{X,t} - A_{X,0}| \leq r_1, \quad |A_{Y,t} - A_{Y,0}| \leq r_1. \tag{2.26}$$

We will now argue that

$$\{T = T'' = T_2\} \subset \{s_* = \infty\}. \tag{2.27}$$

It follows from (2.4), (2.6) and (2.20) that the vector of oblique reflection $\mathbf{v}(\widehat{\rho}(t), Z_t^*)$ is locally bounded on the upper boundary of U for $t < T_2 \leq T$. The analogous remark applies to the lower boundary of U .

Suppose that $s_* < \infty$ and $T = T'' = T_2$. We will show that this leads to a contradiction. The limit $\lim_{s \uparrow s_*} Z_s$ exists and is finite because s_* is finite and Z is a reflected Brownian motion with a locally bounded vector of oblique reflection, a continuous process. If $X_{T-} \in \partial D$ and $Y_{T-} \in \partial D$ then $X_{T-} = A_{X,T-}$ and $Y_{T-} = A_{Y,T-}$. Hence, (2.16) implies that $\lim_{s \uparrow s_*} \text{Re } Z_s = -\infty$, a contradiction. This implies that either $X_T \notin \partial D$ or $Y_T \notin \partial D$, hence $T \neq T''$, according to the definition (2.14) of T'' . This is a contradiction so we conclude that (2.27) is true.

Let $c_1 = (\widehat{c}_1 \wedge \widehat{c}_2)/r_1$. It follows from (2.21), (2.22) and (2.25)-(2.26) that we have the bound $v_1(\widehat{\rho}(t), Z_{\widehat{\rho}(t)}) \leq -c_1$ if $t \leq T_2$ and either $X_t \in \mathcal{B}(A_{X,0}, r_1) \cap \partial D$ or $Y_t \in \mathcal{B}(A_{Y,0}, r_1) \cap \partial D$.

It follows from (2.19), a formula analogous to (2.19) for γ_t (not stated explicitly), and (2.25)-(2.26) that there exists $c_2 > 0$ such that

$$\left\{ L_t^X \leq c_2, L_t^Y \leq c_2, \sup_{0 \leq s \leq t} |A_{X,0} - X_s| \leq r_1, \sup_{0 \leq s \leq t} |A_{Y,0} - Y_s| \leq r_1 \right\} \tag{2.28}$$

$$\subset \left\{ \beta_t \leq \frac{\beta_0 + \widehat{\beta}_2}{2}, \gamma_t \leq \frac{\gamma_0 + \widehat{\gamma}_2}{2} \right\}.$$

Let $c_3 > 0$ be so small that if

$$D_X = \{w \in D : \text{dist}(w, \partial D) \leq c_3, |w - A_{X,0}| \leq 2r_1\}, \tag{2.29}$$

$$D_Y = \{w \in D : \text{dist}(w, \partial D) \leq c_3, |w - A_{Y,0}| \leq 2r_1\}, \tag{2.30}$$

$$\widehat{T}_X = \inf\{t \geq 0 : X_t \notin D_X\}, \quad \widehat{T}_Y = \inf\{t \geq 0 : Y_t \notin D_Y\}, \tag{2.31}$$

$$F_1 = \{L^X(\widehat{T}_X) \leq c_2 \text{ and } L^Y(\widehat{T}_Y) \leq c_2\}, \tag{2.32}$$

then

$$\mathbb{P}(F_1 \mid x \in D_X, y \in D_Y) > 3/4. \tag{2.33}$$

In view of the definition (2.24)-(2.26), we can fix a deterministic $c_4 \in \mathbb{R}$ so small that

$$\text{if } t \in [0, T_2] \text{ and } w \notin D_X \text{ then } \text{Re } \mathcal{F}(t, w) > c_4, \text{ and} \tag{2.34}$$

$$\text{if } t \in [0, T_2] \text{ and } w \notin D_Y \text{ then } \text{Re } \mathcal{G}(t, w) > c_4. \tag{2.35}$$

Assume for a moment that s_* defined in (2.17) is infinite, a.s. We will argue that no matter what oblique vector of reflection is, the local time on the boundary increases at a linear rate in the sense that for some $c_5 > 0$, not depending on the reflection vector,

$$\lim_{t \rightarrow \infty} L_t^Z / t = c_5, \text{ a.s.} \tag{2.36}$$

The process $\{L_t^Z, t \geq 0\}$ is the same as the local time of one dimensional reflected Brownian motion $\text{Im } Z_t$ in $(0, \pi)$. The process L_t^Z increases when $\text{Im } Z_t$ is reflecting from 0 or π . Let ξ_k be the consecutive hitting times of the boundary, i.e., $\xi_0 = \inf\{t \geq 0 : \text{Im } Z_t = 0\}$, $\xi_k = \inf\{t \geq \xi_{k-1} : |\text{Im } Z_{\xi_{k-1}} - \text{Im } Z_{\xi_t}| = \pi\}$, $k \geq 1$. The vectors $(\xi_k - \xi_{k-1}, L_{\xi_k}^Z - L_{\xi_{k-1}}^Z)$ are i.i.d., by the strong Markov property. Standard methods show that both components of these vectors have finite expectations so the strong law of large numbers implies (2.36).

We use (2.36) to see that there exists $c_6 < c_4$ such that if the vector of oblique reflection (v_1, v_2) satisfies $v_1(t, z) \leq -c_1$ for $t \geq 0$ and $z \in \partial U$ with $\text{Re } z \leq c_4$, and

$$F_2 = \left\{ \lim_{t \rightarrow \infty} \text{Re } Z_t = -\infty, \sup_{t \geq 0} \text{Re } Z_t < c_4 \right\} \tag{2.37}$$

then

$$\mathbb{P}(F_2 \mid \text{Re } Z_0 \leq c_6) > 3/4. \tag{2.38}$$

It follows that if

$$F_3 = \left\{ \sup_{0 \leq t < \tilde{\rho}(T_2) \wedge s_*} \text{Re } Z_t < c_4 \right\} \tag{2.39}$$

then

$$\mathbb{P}(F_3 \mid \text{Re } Z_0 \leq c_6) > 3/4. \tag{2.40}$$

Recall that $(X_0, Y_0) = (x, y)$. We choose $x, y \in D$ such that $x \in D_X$, $y \in D_Y$ and $\text{Re } \mathcal{F}(0, x) = \text{Re } \mathcal{G}(0, y) \leq c_6$.

The event $F_1 \cap F_2 \cap F_3$ has probability greater than 1/4 because of (2.33), (2.38) and (2.40). Suppose that $F_1 \cap F_2 \cap F_3$ occurred.

Assume that $\tilde{\rho}(T_2) < s_*$. We will show that this assumption leads to a contradiction. Since F_3 occurred and $\tilde{\rho}(T_2) < s_*$,

$$\sup_{0 \leq t < \tilde{\rho}(T_2) \wedge s_*} \text{Re } Z_t = \sup_{0 \leq t < \tilde{\rho}(T_2)} \text{Re } Z_t < c_4. \tag{2.41}$$

This, (2.24), (2.31) and (2.34)-(2.35) imply that

$$T_2 < \hat{T}_X \wedge \hat{T}_Y. \tag{2.42}$$

This, the assumption that F_1 occurred and (2.32) imply that $L^X(T_2) \leq c_2$ and $L^Y(T_2) \leq c_2$. Since $T_2 < \hat{T}_X \wedge \hat{T}_Y$ holds, it follows from (2.29)-(2.30) that $\sup_{0 \leq s \leq T_2} |A_{X,0} - X_s| \leq r_1$ and $\sup_{0 \leq s \leq T_2} |A_{Y,0} - Y_s| \leq r_1$ hold. This, coupled with the earlier observations, shows that the event on the left hand side of (2.28) holds with t replaced with T_2 . Hence, the event on the right hand side of (2.28) holds with t replaced with T_2 . But this contradicts the definitions of T_2 and s_* and the assumption that $\tilde{\rho}(T_2) < s_*$. The proof that $\tilde{\rho}(T_2) \geq s_*$ is complete.

The fact that $\tilde{\rho}(T_2) \geq s_*$ and the definitions of s_* and T_2 given in (2.17) and (2.24) imply that $T = T_2$ and, therefore, $\tilde{\rho}(T_2) = s_*$. This, in turn implies that (2.41) and (2.42)

remain valid. The definition of r_1 (see the paragraph containing (2.23)) and (2.29)-(2.31) imply that $\widehat{T}_X \wedge \widehat{T}_Y < T'$. Hence, the fact that $T = T_2 < \widehat{T}_X \wedge \widehat{T}_Y$ implies that $T < T'$. Now it follows from (2.15) that $T = T''$. According to (2.27), $s_* = \infty$. This, the fact that $\tilde{\rho}(T_2) = s_*$ and the definition (2.39) of F_3 imply that $\sup_{0 \leq t < \infty} \operatorname{Re} Z_t < c_4$. Comparing (2.37) and (2.39), and recalling that we are assuming that F_2 holds, we conclude that $T < \infty$, $X_T = A_{X,T} \in E_X$ and $Y_T = A_{Y,T} \in E_Y$. We have $|X_T| \neq |Y_T|$ because $T < T'$. The theorem holds with $S_\infty = T$. \square

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