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Closed-form formulas for the distribution of the jumps of doubly-stochastic Poisson processes

Arturo Valdivia*

Abstract

We study the obtainment of closed-form formulas for the distribution of the jumps of a doubly-stochastic Poisson process. The problem is approached in two ways. On the one hand, we translate the problem to the computation of multiple derivatives of the Hazard process cumulant generating function; this leads to a closed-form formula written in terms of Bell polynomials. On the other hand, for Hazard processes driven by Lévy processes, we use Malliavin calculus in order to express the aforementioned distributions in an appealing recursive manner. We outline the potential application of these results in credit risk.

Keywords: Doubly-stochastic Poisson process; Bell polynomials; Malliavin calculus; credit risk; Hazard process; integrated non-Gaussian OU process.
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1 Introduction

Consider an ordered series of random times $\tau_1 \leq ... \leq \tau_m$ accounting for the sequenced occurrence of certain events. In the context of credit risk, these random times can be seen as *credit events* such as the firm's value sudden deterioration, credit rate downgrade, the firm's default, etcetera. The valuation of defaultable claims (see [4, 18]) is closely related to the computation of the quantities

$$\mathbb{P}(\tau_n > T|\mathcal{F}_t), \qquad t \geq 0, \quad n = 1, ..., m,$$

where the *reference filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ accounts for the information generated by all state variables.

An interesting possibility to model these random times consists in considering $\tau_1, ..., \tau_m$ as the successive jumps of a *doubly-stochastic Poisson process* (*DSP process*). That is, a time-changed Poisson process $(P_{\Lambda_t})_{t\geq 0}$, where the time change $(\Lambda_t)_{t\geq 0}$ is a non-decreasing càdlàg $\mathbb{F}$-adapted process starting at zero; and the Poisson process $(P_t)_{t\geq 0}$ has intensity rate equal to 1, and it is independent from the σ-algebra $\mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t$. We refer to $(\Lambda_t)_{t\geq 0}$ as the *Hazard process*.

The purpose of this note is to study the obtainment of closed-form formulas for the distributions of the n-th jump of a doubly-stochastic Poisson process. We address the

*Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, E-08007 Barcelona, Spain. E-mail: arturo@valdivia.xyz

problem from two different approaches. First, we relate this problem to the computation of the first n derivatives of the Hazard process cumulant generating function. As shown below, the result is written in closed-form in terms of *Bell polynomials* on the aforementioned derivatives —see [6, 13, 16] for details on these polynomials.

Theorem 1.1. *For $0 \leq t < T$, denote the conditional cumulant generating function of Λ_T by*

$$\Psi(u) := \log \mathbb{E}\left[\exp\{\mathrm{i}u\Lambda_T\}|\, \mathcal{F}_t\right].$$

*If Λ_T has a finite conditional n-th moment (*i.e.*, $\mathbb{E}\left[\Lambda_T^n|\, \mathcal{F}_t\right] < \infty$), then the following equation holds true*

$$\mathbb{P}(\tau_n > T|\mathcal{F}_t) = \mathbf{1}_{\{\tau_n>t\}} \sum_{k=0}^{n-1} \frac{\mathrm{e}^{\Psi(\mathrm{i})}}{k!\mathrm{i}^k} \mathbf{B}_k \left(\frac{\partial \Psi}{\partial u}(\mathrm{i}), ..., \frac{\partial^k \Psi}{\partial u^k}(\mathrm{i})\right), \tag{1.1}$$

where $\mathbf{B}_k$ is the k-th Bell polynomial, i.e.*, $\mathbf{B}_n(x_1, ..., x_n) := \sum_{k=0}^n \mathbf{B}_{n,k}(x_1, ..., x_{n-k+1})$ with*

$$\mathbf{B}_{n,k}(x_1, ..., x_{n-k+1}) := \sum \frac{n!}{j_1!j_2!\cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

the sum running over all sequences of non-negative indices such that $j_1 + j_2 + \cdots + j_{n-k+1} = k$ and $j_1 + 2j_2 + \cdots + (n-k+1)_{n-k+1} = n$, and $\mathbf{B}_{0,0} := 1$.

In light of this result, two considerations are in order. On the one hand, it is desirable to consider a model for $(\Lambda_t)_{t\geq 0}$ having a cumulant generating function Ψ being analytic (around $\mathrm{i} = \sqrt{-1}$), so that an arbitrary number of jumps of the doubly-stochastic Poisson process can be handled. On the other hand, it is straightforward to compute (1.1) in closed-form given a tractable expression for the cumulant generating function Ψ. See examples in Section 2.

As a second approach, we compute the aforementioned distributions directly, by means of the Malliavin calculus. For this approach we consider a strictly positive pure-jump Lévy process $(L_t)_{t\geq 0}$ with Lévy measure ν, and having moments of all orders —see [1, 17] and [7] for a general exposition about Lévy processes and Malliavin calculus. We then assume that the Hazard process is of the form

$$\Lambda_t = \int_0^t \mu(s)\mathrm{d}s + \int_0^t \int_{\mathbb{R}_0} \sigma(s,z)N(\mathrm{d}s, \mathrm{d}z), \qquad t \geq 0, \tag{1.2}$$

where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, N is the Poisson random measure associated to $(L_t)_{t\geq 0}$, μ is a deterministic positive drift, and σ is a positive deterministic function integrable with respect to N. Assume further that $\mathbb{F}$ is given by the natural filtration generated by the driving Lévy process $(L_t)_{t\geq 0}$. In this setting, we have the following result.

Theorem 1.2. *The conditional distribution of the n-th jump of doubly-stochastic Poisson process with Hazard process satisfying (1.2) is given by*

$$\mathbb{P}(\tau_n > T|\mathcal{F}_t) = \mathbf{1}_{\{\tau_n>t\}} \mathrm{e}^{\Lambda_t + \ell(t)} \left(\sum_{k=0}^{n-1} \sum_{j=0}^{k} \frac{(\Lambda_t + \ell(t))^j}{j!(k-j)!} m_{k-j}(t)\right),$$

where $\ell(t) := \int_t^T \mu(s)\mathrm{d}s + \int_t^T \int_{\mathbb{R}_0} \sigma(s,z)\nu(\mathrm{d}z)\mathrm{d}s$, and the quantities $m_0, m_1, ..., m_n$ are given recursively according to

$$m_0(t) := \exp\left\{\int_t^T \int_{\mathbb{R}_0} \left[\mathrm{e}^{-\sigma(s,z)} - 1 + \sigma(s,z)\right] \mathrm{d}s\nu(\mathrm{d}z)\right\}, \tag{1.3}$$

and for $r \geq 1$

$$m_{r+1}(t) = m_r(t) \int_t^T \int_{\mathbb{R}_0} (\mathrm{e}^{-\sigma(s,z)} - 1)\sigma(s,z)\mathrm{d}s\nu(\mathrm{d}z) \qquad (1.4)$$
$$+ \sum_{k=1}^{r} \binom{r}{k} m_{r-k}(t) \int_t^T \int_{\mathbb{R}_0} \mathrm{e}^{-\sigma(s,z)}\sigma^{k+1}(s,z)\mathrm{d}s\nu(\mathrm{d}z).$$

The rest of the paper is organized as follows. In Section 2 we present relevant examples appearing the literature. Finally in Section 3 we provide the proofs of our results.

Let us remark that even though our study is motivated by the valuation of defaultable claims, our results can potentially be also used in other areas; see for instance [2, 14, 21] and references therein.

2 Examples

In many traditional models (*e.g.*, [8]) the Hazard processes $(\Lambda_t)_{t\geq 0}$ is assumed to be absolutely continuous with respect to the Lebesgue measure, that is,

$$\Lambda_t := \int_0^{\mathrm{t}} \lambda_{\mathrm{s}} \mathrm{ds}, \qquad t \geq 0, \qquad (2.1)$$

where the process $(\lambda_t)_{t\geq 0}$ is usually refer to as the *hazard rate*, and it is seen as the instantaneous rate of default in the credit risk context. The following two examples show how to use Theorem 1.1 using two prominent particular cases for the hazard rate —and consequently for the Hazard process.

Example 2.1. The *integrated square-root process* $(\Lambda_t^{intSR})_{t\geq 0}$ (see [9]) defined by means of (2.1) where the hazard rate is given by the solution of

$$\mathrm{d}\lambda_t^{SR} = \vartheta(\kappa - \lambda_t^{SR})\mathrm{d}t + \sigma\sqrt{\lambda_t^{SR}}\mathrm{d}W_t,$$

where $(W_t)_{t\geq 0}$ is a Brownian motion, and we assume $\sigma > 0$ and $\vartheta\kappa \geq \sigma^2$ in order to ensure that $(\lambda_t^{SR})_{t\geq 0}$ remains positive. Take now $\mathbb{F}$ as the natural filtration generated by $(W_t)_{t\geq 0}$. It is well-known that the correspondent Hazard process has an analytic cumulant generating function given by

$$\Psi^{intSR}(u) := A(u, T-t) + \lambda_t^{SR} B(u, T-t), \qquad T \geq t \geq 0,$$

where the functions A and B are given by

$$A(u, T-t) = \frac{2\vartheta\kappa}{\sigma^2} \log\left(\frac{2\gamma \mathrm{e}^{\frac{1}{2}(\gamma+\vartheta)(T-t)}}{(\gamma+\vartheta)\mathrm{e}^{-\gamma(T-t)} - 2\gamma}\right), \quad \text{and} \quad B(u, T-t) = \frac{2\gamma(\mathrm{e}^{-\gamma(T-t)} - 1)}{(\gamma+\vartheta)\mathrm{e}^{-\gamma(T-t)} - 2\gamma}$$

with $\gamma := \gamma(u) := \sqrt{\vartheta^2 - 2\mathrm{i}u\sigma^2}$. The simplicity of Ψ^{intSR} allows to compute its partial derivatives involved in (1.1). And finally we can use the n-th Bell polynomial $\mathbf{B}_n$ characterization given by

$$\mathbf{B}_n(x_1, ..., x_n) := \det \begin{bmatrix} \binom{n-1}{0}x_1 & \binom{n-1}{1}x_2 & \binom{n-1}{2}x_3 & \cdots & \binom{n-1}{n-2}x_{n-1} & \binom{n-1}{n-2}x_n \\ -1 & \binom{n-2}{1}x_1 & \binom{n-2}{1}x_2 & \cdots & \binom{n-2}{n-3}x_{n-2} & \binom{n-2}{n-2}x_{n-1} \\ 0 & -1 & \binom{n-3}{1}x_1 & \cdots & \binom{n-3}{n-4}x_{n-3} & \binom{n-3}{n-3}x_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{1}{0}x_1 & \binom{1}{1}x_2 \\ 0 & 0 & 0 & \cdots & -1 & \binom{0}{0}x_1 \end{bmatrix},$$

where in each column the remaining entries below the -1 are equal to zero. For instance, one can easily see that the first three Bell polynomials are $\mathbf{B}_1(x_1) = x_1$, $\mathbf{B}_2(x_1, x_2) = x_1^2 + x_2$ and $\mathbf{B}_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3$.

Example 2.2. *The integrated non-Gaussian Ornstein-Uhlenbeck (intOU) processes* defined by means of

$$\Lambda_t^{intOU} := \frac{1}{\vartheta}(1-\mathrm{e}^{-\vartheta t})\lambda_0 + \frac{1}{\vartheta}\int_0^t (1-\mathrm{e}^{-\vartheta(t-s)})\mathrm{d}L_{\vartheta s}, \qquad t \geq 0, \tag{2.2}$$

where $\vartheta, \lambda_0 > 0$ are free parameters, λ_0 being random, and $(L_t)_{t\geq 0}$ a non-decreasing pure-jump positive Lévy process. Equivalently, we can consider again the model in (2.1) where this time $(\lambda_t)_{t\geq 0}$ is given by the solution of

$$\mathrm{d}\lambda_t = -\vartheta\lambda_t \mathrm{d}t + \mathrm{d}L_{\vartheta t}, \qquad \lambda_0 > 0.$$

An interesting property of this Hazard process, $(\Lambda_t^{intOU})_{t\geq 0}$, is that it has continuous sample paths –for this and further properties on intOU processes refer to [10].

It can be shown that

$$\Psi^{intOU}(u) := \frac{\mathrm{i}u\lambda_0}{\vartheta}(1-\mathrm{e}^{-\vartheta T}) + \vartheta\int_0^T k_L\left(\frac{u}{\vartheta}(1-\mathrm{e}^{-\vartheta(T-s)})\right)\mathrm{d}s, \qquad T \geq 0, \tag{2.3}$$

where we take $\mathcal{F}_0 = \sigma(\lambda_0)$, that is, the σ-algebra generated by λ_0. Particular cases of interest are the following. On the one hand, we have the so-called *Gamma*(a,b)*-OU process* which is obtained by taking $(L_t)_{t\geq 0}$ as a Compound Poisson process

$$L_t = \sum_{n=1}^{Z_t} x_n, \qquad t \geq 0,$$

where $(Z_t)_{t\geq 0}$ is a Poisson process with intensity $a\vartheta$, and $(x_n)_{n\geq 1}$ is a sequence of independent identically distributed $\mathrm{Exp}(b)$ variables. In this case, the correspondent Hazard process in (2.2) has a finite number of jumps in every compact time interval. Moreover, the equation (2.3) becomes

$$\Psi_{Gamma}^{intOU}(u) := \frac{\mathrm{i}u\lambda_0}{\vartheta}(1-\mathrm{e}^{-\vartheta T}) + \frac{\vartheta a}{\mathrm{i}u - \vartheta b}\left(\left(b\log\left(\frac{b}{b-\frac{\mathrm{i}u}{\vartheta}(1-\mathrm{e}^{-\vartheta T})}\right) - \mathrm{i}uT\right)\right).$$

On the other hand, we have the so-called *Inverse-Gausssian*(a,b)*-OU process* (see [15, 20] which is obtained by taking $(L_t)_{t\geq 0}$ as the sum of two independent processes, $(L_t = L_t^{(1)} + L_2^{(2)})_{t\geq 0}$, where $(L_t^{(1)})_{t\geq 0}$ is an Inverse-Gausssian$(\frac{1}{2}a, b)$ process, and $(L_t^{(2)})_{t\geq 0}$ is a Compound Poisson process

$$L_t^{(2)} = b^{-1}\sum_{n=1}^{Z_t} x_n^2, \qquad t \geq 0,$$

where $(Z_t)_{t\geq 0}$ is a Poisson process with intensity $\frac{1}{2}ab$, and $(x_n)_{n\geq 1}$ is a sequence of independent identically distributed $\mathrm{Normal}(0,1)$ variables. In this case, the correspondent Hazard process in (2.2) jumps infinitely often in every interval. Moreover, the equation (2.3) becomes

$$\Psi_{IG}^{intOU}(u) := \frac{\mathrm{i}u\lambda_0}{\vartheta}(1-\mathrm{e}^{-\vartheta T}) + \frac{2a\mathrm{i}u}{b\vartheta}A(u,T),$$

where, using $c := -2b^{-2}\mathrm{i}u\vartheta^{-1}$, the function A is defined by

$$A(u,T) := \frac{1-\sqrt{1+c(1-\mathrm{e}^{-\vartheta T})}}{c} + \frac{1}{\sqrt{1+c}}\left[\operatorname{arctanh}\left(\frac{1-\sqrt{1+c(1-\mathrm{e}^{-\vartheta T})}}{c}\right) - \operatorname{arctanh}\left(\frac{1}{\sqrt{1+c}}\right)\right].$$

In both of the cases above, we can see that the simplicity of Ψ allows to compute (1.1) in a straightforward way.

This traditional approach reduces the analytical tractability of the model, and hinders the calibration of the model parameters. This hindrance is partially due to the fact that the Laplace transform of a Hazard process as in (2.1) is known in closed-form only for a reduced number of Hazard rates models. That is one of the reasons why in more recent contributions the modelling focus is set on the Hazard process itself, without requiring to make a reference to the Hazard rate —see for instance [3, 15]. In this line, consider a Hazard process $(\Lambda_t)_{t\geq 0}$ as given in (1.2). The following example provides an explicit computation of the quantities involved in Theorem 1.2.

Example 2.3. (*CMY Hazard process*) In the financial literature, the *CMY process —or one-sided CGMY process* [5]— with parameters $C, M > 0$ and $Y < 1$ refers to the positive pure-jump Lévy process $(L_t^{CMY})_{t\geq 0}$ having Lévy measure ν_{CMY} given by

$$\nu_{CMY}(z) := \frac{C\mathrm{e}^{-Mz}}{z^{1+Y}}\mathbf{1}_{\{z>0\}}.$$

The *Gamma process* and the *Inverse Gaussian process* can be seen as particular cases by taking $Y = 0$ and $Y = \frac{1}{2}$, respectively, see [18].

Consider now a Hazard process of the form

$$\Lambda_t^{CMY} := \int_0^t \mu(s)\mathrm{d}s + \int_0^t \sigma(s)\left(\mathrm{d}L_s^{CMY} + CM^{Y-1}\Gamma(1-Y)\mathrm{d}s\right), \qquad t \geq 0.$$

This is equivalent to take, in (1.2), a function σ is of the form $\sigma(s,z) = z\sigma(s)$. Then the quantities in (1.3) and (1.4) are given by

$$m_0^{CMY}(t) = \begin{cases} \exp\left\{C\int_t^T M^{Y-1}\Gamma(1-Y)\sigma(s) + \Gamma(-Y)\left[(M+\sigma(s))^Y - M^Y\right]\mathrm{d}s\right\}, & Y \neq 0 \\ \exp\left\{C\int_t^T \frac{\sigma(s)}{M} - \log\left(1+\frac{\sigma(s)}{M}\right)\mathrm{d}s\right\}, & Y = 0 \end{cases}$$

and

$$\begin{aligned} m_{n+1}^{CMY}(t) &= m_n^{CMY}(t)C\Gamma(1-Y)\left[\int_t^T \sigma(s)\left((M+\sigma(s))^{Y-1} - M^{Y-1}\right)\mathrm{d}s\right] \\ &\quad + \sum_{k=1}^n \binom{n}{k} m_{n-k}^{CMY}(t)C\Gamma(k+1-Y)\int_t^T \sigma^{k+1}(s)(M+\sigma(s))^{Y-(k+1)}\mathrm{d}s. \end{aligned}$$

for $n \geq 1$. Further we shall have $\ell^{CMY}(t) = \int_t^T \mu(s)\mathrm{d}s + CM^{Y-1}\Gamma(1-Y)\int_t^T \sigma(s)\mathrm{d}s$.

Finally, let us remark that we when considering a model like (1.2), the quantities appearing in Theorem 1.1 and Theorem 1.2 can be related according to the following.

Example 2.4. Let the Hazard process $(\Lambda_t)_{t\geq 0}$ be given as in (1.2). It can be seen that in this case (Lemma 3.7 below) the cumulant generating function is given by

$$\Psi(u) = \mathrm{i}u(\Lambda_t + \ell(t)) + \int_t^T \int_{\mathbb{R}_0}\left[\mathrm{e}^{\mathrm{i}u\sigma(s,z)} - 1 - \mathrm{i}u\sigma(s,z)\right]\mathrm{d}s\nu(\mathrm{d}z).$$

Consequently, if the function σ has finite moments

$$\int_0^T \int_{\mathbb{R}_0} \sigma^k(s,z)\mathrm{d}s\nu(\mathrm{d}z) < \infty, \qquad k = 1, ..., n, \tag{2.4}$$

then the n-th derivative of Ψ is given by

$$\frac{1}{\mathrm{i}}\frac{\partial \Psi}{\partial u} = \Lambda_t + \ell(t) + \int_t^T \int_{\mathbb{R}_0} \left[\mathrm{e}^{\mathrm{i}u\sigma(s,z)} - 1\right] \sigma(s,z)\mathrm{d}s\nu(\mathrm{d}z),$$

and

$$\frac{1}{\mathrm{i}^k}\frac{\partial^k \Psi}{\partial u^k} = \int_t^T \int_{\mathbb{R}_0} \mathrm{e}^{\mathrm{i}u\sigma(s,z)}\sigma^k(s,z)\mathrm{d}s\nu(\mathrm{d}z), \qquad k = 2, ..., n.$$

Indeed, these equations can be obtained by successive differentiation under the integral sign due to the assumption (2.4).

3 Proofs

Let us start by the construction of the doubly-stochastic Poisson process that we shall consider in what follows. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ denote our reference filtration; we shall assume that it satisfies the usual conditions of $\mathbb{P}$-completeness and right-continuity. Let the i.i.d. random variables $\eta_1, ..., \eta_m$ be exponentially distributed with parameter 1, all being independent of $\mathcal{F}_\infty$. Then the n-th jump of the doubly-stochastic Poisson process with Hazard rate $(\Lambda_t)_{t\geq 0}$ can be characterized as

$$\tau_n = \inf\left\{t > 0 \;:\; \Lambda_t \geq \eta_1 + ... + \eta_n\right\}. \tag{3.1}$$

This construction leads to the following expression for the conditional distribution of the DSP process n-th jump

$$\mathbb{P}(\tau_n > T |\, \mathcal{F}_T) = \mathrm{e}^{-\Lambda_T} \sum_{j=0}^{n-1} \frac{1}{j!}\Lambda_T^j, \qquad T \geq 0. \tag{3.2}$$

Indeed, by construction,

$$\mathbb{P}(\tau_n > t |\, \mathcal{F}_\infty) = \mathbb{P}\left(\sum_{j=1}^{n} \eta_j > \Lambda_t \,\middle|\, \mathcal{F}_\infty\right) = \mathrm{e}^{-\Lambda_t} \sum_{j=0}^{n-1} \frac{\Lambda_t^j}{j!},$$

since conditioned to $\mathcal{F}_\infty$ the random variable $\eta_1 + ... + \eta_n$ has a Gamma distribution. The result then follows by preconditioning to $\mathcal{F}_t$ —recall that $(\Lambda_t)_{t\geq 0}$ is $\mathbb{F}$-adapted.

Notice first that by conditioning (3.2) to $\mathcal{F}_t$ we get

$$\mathbb{P}(\tau_n > T |\, \mathcal{F}_t) = \sum_{j=0}^{n-1} \frac{1}{j!}\mathbb{E}\left[\Lambda_T^j \mathrm{e}^{-\Lambda_T} \,\middle|\, \mathcal{F}_t\right], \qquad T \geq t \geq 0. \tag{3.3}$$

Then the purpose of Theorem 1.1 and Theorem 1.2 is to provide a way to compute the conditional expectations in the equation above.

3.1 Proof of Theorem 1.1

Let η_t stand for the conditional (to $\mathcal{F}_t$) law of Λ_T, so that the assumption on the n-th conditional moment reads

$$\int_{\mathbb{R}} x^n \eta_t(\mathrm{d}x) < \infty.$$

As in the unconditional case (*cf.* [12, Theorem 13.2]), the condition above ensures that the conditional characteristic function

$$\varphi(u; t, T) := \mathbb{E}\left[\exp\{\mathrm{i}u\Lambda_T\} |\, \mathcal{F}_t\right]$$

has continuous partial derivatives up to order n, and furthermore the following equation holds true

$$\frac{1}{\mathrm{i}^k}\frac{\partial^k \varphi(u;t,T)}{\partial u^k} = \mathbb{E}\left[\Lambda_T^k \mathrm{e}^{\mathrm{i}u\Lambda_T}\middle| \mathcal{F}_t\right], \qquad k = 0, 1, ..., n.$$

Now, using the Bell polynomials (as defined in Theorem 1.1) we have an expression for the chain rule for higher derivatives:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f \circ g = \sum_{k=0}^{n} (f^{(k)} \circ g)\mathbf{B}_{n,k}(g^{(1)}, ..., g^{(n+k-1)}),$$

where the superscript denotes the correspondent derivative, *i.e.*, $f^{(k)} := \frac{\mathrm{d}^k}{\mathrm{d}x^k} f$ and $g^{(k)} := \frac{\mathrm{d}^k}{\mathrm{d}x^k} g$, which are assumed to exist. This expression is known as the *Riordan's formula* —for these results on Bell polynomials we refer to [6, 13, 16].

It remains to apply Riordan's formula to $f = \exp$ and $g = \Psi$ in order to get

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\varphi = \sum_{k=0}^{n} \varphi\mathbf{B}_{n,k}(\Psi^{(1)}, ..., \Psi^{(n+k-1)}) = \varphi\mathbf{B}_n(\Psi^{(1)}, ..., \Psi^{(n)}).$$

3.2 Proof of Theorem 1.2

From this moment on, we shall work with a strictly positive pure-jump Lévy process $(L_t)_{t\geq 0}$ having a Lévy measure ν satisfying

$$\int_{(-\varepsilon,\varepsilon)} \mathrm{e}^{pz}\nu(\mathrm{d}z) < \infty$$

for every $\varepsilon > 0$ and certain $p > 0$. This condition implies in particular that $(L_t)_{t\geq 0}$ have moments of all orders, and the polynomials are dense in $L^2(\mathrm{d}t \times \nu)$. Notice that this condition is always satisfied if the Lévy measure has compact support.

In other to prove the theorem we need the following.

3.2.1 Preliminaries on Malliavin calculus via chaos expansions

Let us now introduce basic notions of Malliavin calculus for Lévy processes which we shall use as a framework. Here we mainly follow [7].

For every $T > 0$, let $\mathcal{L}_T^2((\mathrm{d}t \times \nu)^n) := \mathcal{L}^2(([0,T] \times \mathbb{R}_0)^n)$ be the space of deterministic functions such that

$$\|f\|_{\mathcal{L}_T^2((\mathrm{d}t\times\nu)^n)} := \left(\int_{([0,T]\times\mathbb{R}_0)^n} f^2(t_1, z_1, ..., t_n, z_n)\mathrm{d}t_1\nu(\mathrm{d}z_1)\cdots\mathrm{d}t_n\nu(\mathrm{d}z_n)\right)^{\frac{1}{2}} < \infty,$$

and such that they are zero over k-*diagonal sets*, see [19, Remark 2.1]. The *symmetrization* $\tilde{f}$ of f is defined by

$$\tilde{f}(t_1, z_1, ..., t_n, z_n) := \frac{1}{n!}\sum_{\sigma} f(t_{\sigma(1)}, z_{\sigma(1)}, ..., t_{\sigma(n)}, z_{\sigma(n)}),$$

where the sum runs over all the permutations σ of $\{1, ..., n\}$. For every f in the subspace of symmetric functions, $\widetilde{\mathcal{L}}_T^2((\mathrm{d}t \times \nu)^n) := \{f \in \mathcal{L}^2((\mathrm{d}t \times \nu)^n) \ : \ f = \tilde{f}\}$, we define the *$n$-fold iterated integral of f by*

$$I_n(f) := n!\int_0^T \int_{\mathbb{R}_0} \cdots \int_0^{t_2-} \int_{\mathbb{R}_0} f(t_1, z_1, ..., t_n, z_n)\widetilde{N}(\mathrm{d}t_1, \mathrm{d}z_1)\cdots\widetilde{N}(\mathrm{d}t_n, \mathrm{d}z_n).$$

For constant values $f_0 \in \mathbb{R}$ we set $I_0(f_0) := f_0$. In these terms, the *Wiener-Itô chaos expansion for Poisson random measures*, due to [11], states that every $\mathcal{F}_T$-measurable random variable $F \in \mathcal{L}^2(\mathbb{P})$ admits a representation

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

via a unique sequence of elements $f_n \in \widetilde{\mathcal{L}}_T^2((\mathrm{d}t \times \nu)^n)$. In virtue of this result, each random field $(X_{t,z})_{(t,z)\in[0,T]\times\mathbb{R}_0}$ has an expassion

$$X_{t,z} = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t, z)), \qquad f_n(\cdot, t, z) \in \widetilde{\mathcal{L}}_T^2((\mathrm{d}t \times \nu)^n),$$

provided, of course, that $X_{t,z}$ is $\mathcal{F}_T$-measurble with $\mathbb{E}[X_{t,z}^2] < \infty$ for all (t, z) in $[0, T] \times \mathbb{R}_0$. Now we are in position to define the *Skorohod integral* and the *Malliavin derivative*.

Definition 3.1. *The random field* $(X_{t,z})_{(t,z)\in[0,T]\times\mathbb{R}_0}$ belongs to $Dom(\delta)$ *if*

$$\sum_{n=0}^{\infty} (n+1)! \left\| \tilde{f}_n \right\|^2_{\mathcal{L}_T^2((\mathrm{d}t\times\nu)^n)} < \infty$$

and has Skorohod integral with respect to $\widetilde{N}$

$$\delta(X) = \int_0^T \int_{\mathbb{R}_0} X_{t,z} \widetilde{N}(\delta t, \mathrm{d}z) := \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n).$$

Definition 3.2. *Let* $\mathbb{D}_{1,2}$ *be the* stochastic Sobolev space *consisting of all* $\mathcal{F}_T$*-measureble random variables* $F \in \mathcal{L}^2(\mathbb{P})$ *with chaos expansion* $F = \sum_{n=0}^{\infty} I_n(f_n)$ *satisfying*

$$\|F\|_{\mathbb{D}_{1,2}} := \sum_{n=1}^{\infty} n\, n! \left\| \tilde{f}_n \right\|^2_{\mathcal{L}_T^2((\mathrm{d}t\times\nu)^n)} < \infty.$$

For every $F \in \mathbb{D}_{1,2}$ *its* Malliavin derivative *is defined as*

$$D_{t,z}F := \sum_{n=0}^{\infty} n I_{n-1}(f_n(\cdot, t, z)).$$

Let us mention here that $Dom(\delta) \subseteq \mathcal{L}^2(\mathbb{P} \times \mathrm{d}t \times \nu)$, $\delta(X) \in \mathcal{L}^2(\mathbb{P})$, $\mathbb{D}_{1,2} \subset \mathcal{L}^2(\mathbb{P})$ and $DF \in \mathcal{L}^2(\mathbb{P} \times \mathrm{d}t \times \nu)$.

The following theorems are central for the results below. We refer to [7] for more details and the proof of these theorems.

Theorem 3.3. *(*Duality formula*) Let* X *be Skorohod integrable and let* $F \in \mathbb{D}_{1,2}$*. Then*

$$\mathbb{E}\left[F \int_0^T \int_{\mathbb{R}_0} X_{t,z} \widetilde{N}(\delta t, \mathrm{d}z)\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} X_{t,z} D_{t,z} F \mathrm{d}t \nu(\mathrm{d}z)\right].$$

Theorem 3.4. *(Product rule) Let* $F, G \in \mathbb{D}_{1,2}$ *with* G *bounded. Then* $FG \in \mathbb{D}_{1,2}$ *and*

$$D_{s,z}(FG) = F D_{s,z} G + G D_{s,z} F + D_{s,z} F D_{s,z} G, \qquad \mathrm{d}t \times \nu - a.e.$$

Theorem 3.5. *(Chain rule) Let* $F \in \mathbb{D}_{1,2}$*, and let* g *be a continuous function such that* $g(F) \in \mathcal{L}^2(\mathbb{P})$ *and* $g(F + D_{s,z}F) \in \mathcal{L}^2(\mathbb{P} \times \mathrm{d}t \times \nu)$*. Then* $g(F) \in \mathbb{D}_{1,2}$ *and*

$$D_{s,z} g(F) = g(F + D_{s,z} F) - g(F).$$

3.2.2 A recursive formula

Lemma 3.6. *For every deterministic Skorohod integrable function f and non-negative integer n, define*

$$F := \int_0^T \int_{\mathbb{R}_0} f(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z), \qquad \textit{and} \qquad X_n := F^n \mathrm{e}^{-F},$$

where F^n is the n-th power of F. If $Y \in \mathbb{D}_{1,2}$ is bounded, then the Malliavin derivative of YX_n is given ($\mathrm{d}t \times \nu-$a.e.) by

$$D_{s,z}(YX_n) = \mathrm{e}^{-f(s,z)}\left(Y + D_{s,z}Y\right)\left(\sum_{k=0}^{n}\binom{n}{k}X_{n-k}f^k(s,z)\right) - YX_n.$$

Proof. By the product rule we have

$$\begin{aligned} D_{s,z}(YX_n) &= D_{s,z}\left((Y\mathrm{e}^{-F})(F^n)\right) \\ &= \left(D_{s,z}(Y\mathrm{e}^{-F})\right)(F^n + D_{s,z}F^n) + Y\mathrm{e}^{-F}D_{s,z}F^n, \qquad \mathrm{d}t \times \nu - a.e, \end{aligned}$$

and

$$D_{s,z}(Y\mathrm{e}^{-F}) = YD_{s,z}\mathrm{e}^{-F} + (D_{s,z}Y)\left(\mathrm{e}^{-F} + D_{s,z}\mathrm{e}^{-F}\right), \qquad \mathrm{d}t \times \nu - a.e.$$

Moreover, since $D_{s,z}F = f(s,z)$, then an application of the chain rule tells us that $D_{s,z}\mathrm{e}^{-F} = \mathrm{e}^{-F}(\mathrm{e}^{-f(s,z)} - 1)$ and

$$D_{s,z}F^n = (F + D_{s,z}F)^n - F^n = (F + f(s,z))^n - F^n = \sum_{k=0}^{n-1}\binom{n}{k}F^k f^{n-k}(s,z).$$

Combining these expressions we get

$$\begin{aligned} D_{s,z}YX_n =& \left(Y\mathrm{e}^{-F}(\mathrm{e}^{-f(s,z)}-1)+(D_{s,z}Y)\left(\mathrm{e}^{-F}+\mathrm{e}^{-F}(\mathrm{e}^{-f(s,z)}-1)\right)\right)\sum_{k=0}^{n}\binom{n}{k}F^k f^{n-k}(s,z) \\ &+Y\mathrm{e}^{-F}\sum_{k=0}^{n-1}\binom{n}{k}F^k f^{n-k}(s,z) \\ =& Y\mathrm{e}^{-F}\left((\mathrm{e}^{-f(s,z)}-1)\sum_{k=0}^{n}\binom{n}{k}F^k f^{n-k}(s,z)+\sum_{k=0}^{n-1}\binom{n}{k}F^k f^{n-k}(s,z)\right) \\ &+(D_{s,z}Y)\mathrm{e}^{-(F+f(s,z))}\sum_{k=0}^{n}\binom{n}{k}F^k f^{n-k}(s,z) \\ =& Y\mathrm{e}^{-F}\left(\mathrm{e}^{-f(s,z)}\sum_{k=0}^{n}\binom{n}{k}F^k f^{n-k}(s,z)^{n-k}-F^n\right) \\ &+(D_{s,z}Y)\mathrm{e}^{-(F+f(s,z))}\sum_{k=0}^{n}\binom{n}{k}F^k f^{n-k}(s,z). \end{aligned}$$

Thus, rewriting the last equivalence in terms of $X_1, ..., X_n$, we get the result. ☐

In what follows let us write, for every $t \geq 0$, $\Lambda_t = \mu(t) + M_t + A(t)$ with

$$M_t = \int_0^t \int_{\mathbb{R}_0} \sigma(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \qquad \text{and} \qquad A(t) = \int_0^t \int_{\mathbb{R}_0} \sigma(s,z)\nu(\mathrm{d}z)\mathrm{d}s.$$

Lemma 3.7. *The conditional characteristic function of such Hazard processes in (1.2),*

$$\varphi(u;t,T) := \mathbb{E}\left[\mathrm{e}^{\mathrm{i}u\Lambda_T}\,\middle|\,\mathcal{F}_t\right],$$

is given by

$$\varphi(u;t,T) = \exp\left\{\mathrm{i}u(\Lambda_t + \ell(t)) + \int_t^T \int_{\mathbb{R}_0} \left[\mathrm{e}^{\mathrm{i}u\sigma(s,z)} - 1 - \mathrm{i}u\sigma(s,z)\right] \mathrm{d}s\nu(\mathrm{d}z)\right\}. \tag{3.4}$$

Proof. Since the integrands μ and σ are deterministic, then the increment $M_T - M_t$ is independent of $\mathcal{F}_t$ and

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}u\Lambda_T}\,\middle|\,\mathcal{F}_t\right] &= \exp\{\mathrm{i}u\Lambda_t\}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}u(\Lambda_T-\Lambda_t)}\,\middle|\,\mathcal{F}_t\right]\\ &= \exp\{\mathrm{i}u\Lambda_t\}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}u([\int_0^T \mu(s)\mathrm{d}s + M_T + A(T)] - [\int_0^t \mu(s)\mathrm{d}s + M_t + A(t)])}\,\middle|\,\mathcal{F}_t\right]\\ &= \exp\left\{\mathrm{i}u(\Lambda_t + \ell(t))\right\}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}u(M_T - M_t)}\right].\end{aligned}$$

Notice that for every deterministic function f the process $(\mathcal{E}_t(f))_{t\geq 0}$ defined by

$$\mathcal{E}_t(f) := \exp\left\{\int_0^t \int_{\mathbb{R}_0} f(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z) - \int_0^t \int_{\mathbb{R}_0} \left[\mathrm{e}^{f(s,z)} - 1 - f(s,z)\right] \mathrm{d}s\nu(\mathrm{d}z)\right\}$$

is a Doléans-Dade exponential martingale. Thus $\mathbb{E}[\mathcal{E}_T(f)] = 1$, and so

$$\mathbb{E}\left[\mathcal{E}_T(f)\mathrm{e}^{\int_0^T \int_{\mathbb{R}_0}\left[\mathrm{e}^{f(s,z)} - 1 - f(s,z)\right]\mathrm{d}s\nu(\mathrm{d}z)}\right] = \mathrm{e}^{\int_0^T \int_{\mathbb{R}_0}\left[\mathrm{e}^{f(s,z)} - 1 - f(s,z)\right]\mathrm{d}s\nu(\mathrm{d}z)}.$$

In our case this reads as

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}u(M_T - M_t)}\right] &= \mathbb{E}\left[\exp\left\{\int_0^T \int_{\mathbb{R}_0} \mathrm{i}u\mathbf{1}_{[t,T]}(s)\sigma(s,z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right\}\right]\\ &= \exp\left\{\int_0^T \int_{\mathbb{R}_0} \left[\mathrm{e}^{\mathrm{i}u\mathbf{1}_{[t,T]}(s)\sigma(s,z)} - 1 - \mathrm{i}u\mathbf{1}_{[t,T]}(s)\sigma(s,z)\right] \mathrm{d}s\nu(\mathrm{d}z)\right\}.\end{aligned}$$

It remains to notice that $\mathrm{e}^{\mathrm{i}u\mathbf{1}_{[t,T]}\sigma} - 1 - \mathrm{i}u\mathbf{1}_{[t,T]}\sigma = \left[\mathrm{e}^{\mathrm{i}u\sigma} - 1 - \mathrm{i}u\sigma\right]\mathbf{1}_{[t,T]}$. □

Lemma 3.8. *Under the notation of Lemma 3.6 we have*

$$\mathbb{E}\left[X_0\right] = \exp\left\{\int_0^T \int_{\mathbb{R}_0} \left[\mathrm{e}^{-f(s,z)} - 1 + f(s,z)\right] \mathrm{d}s\nu(\mathrm{d}z)\right\},$$

and for $n \geq 1$ the following recursive formula holds true

$$\begin{aligned}\mathbb{E}\left[X_{n+1}\right] = \mathbb{E}[X_n] &\int_0^T \int_{\mathbb{R}_0} (\mathrm{e}^{-f(s,z)} - 1)f(s,z)\mathrm{d}s\nu(\mathrm{d}z)\\ &+ \sum_{k=1}^n \binom{n}{k}\mathbb{E}[X_{n-k}]\int_0^T \int_{\mathbb{R}_0} \mathrm{e}^{-f(s,z)}f^{k+1}(s,z)\mathrm{d}s\nu(\mathrm{d}z).\end{aligned}$$

Proof. The Lemma 2.4 provides the base case ($n = 0$). For $n \geq 1$, notice that

$$\begin{aligned}\mathbb{E}[X_{n+1}] &= \mathbb{E}[FX_n]\\ &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} f(s,z)D_{s,z}X_n\mathrm{d}s\nu(\mathrm{d}z)\right]\\ &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} f(s,z)\left(\mathrm{e}^{-f(s,z)}\sum_{k=0}^n \binom{n}{k}X_{n-k}f^k(s,z) - X_n\right)\mathrm{d}s\nu(\mathrm{d}z)\right],\end{aligned}$$

where the second line follows from the duality formula, and the last one from Lemma 3.6 by setting $Y = 1$. The result then follows by the linearity of the expectation. □

3.2.3 Proof of Theorem 1.2

Notice that (3.3) can be rewritten as

$$\begin{aligned}\mathbb{P}(\tau_n > T|\mathcal{F}_t) &= \sum_{k=0}^{n-1} \frac{1}{k!}\mathbb{E}\left[\Lambda_T^k \mathrm{e}^{-\Lambda_T}\,\middle|\,\mathcal{F}_t\right] \\ &= \mathbf{1}_{\{\tau_n > t\}}\mathrm{e}^{\Lambda_t+\ell(t)}\left(\sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{(\Lambda_t+\ell(t))^j}{j!(k-j)!}\mathbb{E}\left[(M_T-M_t)^{k-j}\mathrm{e}^{-(M_T-M_t)}\,\middle|\,\mathcal{F}_t\right]\right).\end{aligned}$$

Indeed, it suffices to expand the factor

$$\begin{aligned}\Lambda_T^k &= ([\Lambda_T - \Lambda_t] + \Lambda_t)^k \\ &= \left(\left[\int_0^T \mu(s)\mathrm{d}s + M_T + A(T) - \int_0^t \mu(s)\mathrm{d}s - M_t - A(t)\right] + \Lambda_t\right)^k \\ &= \sum_{j=0}^{k}\binom{k}{j}(M_T-M_t)^{k-j}\left(\Lambda_t+\ell(t)\right)^j,\end{aligned}$$

and use that $\Lambda_t + \ell(t)$ is $\mathcal{F}_t$-measurable. Now, since the integrand in (1.2) is deterministic, we have that the increment $M_T - M_t$ is independent of $\mathcal{F}_t$ and thus

$$\mathbb{E}\left[(M_T-M_t)^{k-j}\,\mathrm{e}^{-(M_T-M_t)}\,\middle|\,\mathcal{F}_t\right] = \mathbb{E}\left[(M_T-M_t)^{k-j}\,\mathrm{e}^{-(M_T-M_t)}\right] = m_{k-j}.$$

Applying Lemma 3.8 with $f(s,z) := \mathbf{1}_{[t,T]}(s)\sigma(s,z)$ we show that the quantities $m_0, m_1, \ldots,$ m_n satisfy the recursion claimed. In order to remove factor $\mathbf{1}_{[t,T]}(s)$ from the expression, it remains to take into account the basic identities

$$\int_0^T\int_{\mathbb{R}_0}(\mathrm{e}^{-\mathbf{1}_{[t,T]}(s)\sigma(s,z)}-1)\mathbf{1}_{[t,T]}(s)\sigma(s,z)\mathrm{d}s\nu(\mathrm{d}z) = \int_t^T\int_{\mathbb{R}_0}(\mathrm{e}^{-\sigma(s,z)}-1)\sigma(s,z)\mathrm{d}s\nu(\mathrm{d}z),$$

and

$$\int_0^T\int_{\mathbb{R}_0}\mathrm{e}^{-\mathbf{1}_{[t,T]}(s)\sigma(s,z)}\mathbf{1}_{[t,T]}(s)\sigma^{n-k+1}(s,z)\mathrm{d}s\nu(\mathrm{d}z) = \int_t^T\int_{\mathbb{R}_0}\mathrm{e}^{-\sigma(s,z)}\sigma^{n-k+1}(s,z)\mathrm{d}s\nu(\mathrm{d}z).$$

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