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Expectation of the largest bet size in the Labouchere system

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Abstract

For the Labouchere system with winning probability p at each coup, we prove that the expectation of the largest bet size under any initial list is finite if $p > \frac{1}{2}$, and is infinite if $p \le \frac{1}{2}$, solving the open conjecture in [6]. The same result holds for a general family of betting systems, and the proof builds upon a recursive representation of the optimal betting system in the larger family.

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1 Introduction

The Labouchere system, also known as the cancellation system, is one of the most well-known betting systems used in roulette. It was popularized by Henry Du Pré Labouchere, an English politician, writer and journalist. Before the betting, the bettor chooses an initial list L_0 of positive real numbers (e.g., $L_0 = (1, 2, 3, 4)$). During each bet, the bet size equals the sum of the first and last numbers on the list (if only one number remains on the list, then the bet size equals that number). After a win, the first and last terms are canceled from the list; after a loss, the amount just lost is appended to the last term of the list. This system is continued until the list is empty. Table 1 illustrates an example of the Labouchere system.

We introduce the following notations. Let L_n be the list after the n-th coup, l_n be the corresponding list length, B_n be the bet size at the n-th coup, T_n be the remaining target profit (i.e., the sum of the numbers in the list) after the n-th coup, and N be the stopping time that the list first becomes empty, i.e., $L_N = \varnothing$. In this paper, we investigate the behavior of the largest bet size $B^\star \triangleq \max_{1\le n\le N} B_n$ (or $\sup_{n\ge 1} B_n$ if $N = \infty$) in the Labouchere system, and in particular, whether or not $B^\star$ has a finite expectation.

There is very limited literature on analyzing the Labouchere system. Let $p \in [0, 1]$ be the winning probability at each coup, where we assume that the outcomes at different coups are independent. By the standard theory of asymmetric random walks, it is straightforward to show that $N < \infty$ almost surely if and only if $p \ge \frac{1}{3}$ and $\mathbb{E}[N] < \infty$ if and only if $p > \frac{1}{3}$. Downton [3] found a recursion for the distribution of the stopping time N in the case that the initial list L_0 is $(1, 2, 3, 4)$, and Ethier [4] generalized this result to arbitrary initial lists and gave an explicit formula using a generalized version of the

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Table 1: An illustration of the Labouchere system with initial list $L_0 = (1, 2, 3, 4)$.

Coup n	Bet Size B_n	Result	List L_n	Target Profit T_n
			1, 2, 3, 4	10
1	5	Win	2, 3	5
2	5	Loss	2, 3, 5	10
3	7	Loss	2, 3, 5, 7	17
4	9	Loss	2, 3, 5, 7, 9	26
5	11	Win	3, 5, 7	15
6	10	Loss	3, 5, 7, 10	25
7	13	Win	5, 7	12
8	12	Win	$\varnothing$	0

ballot theorem [2, 1]. Specifically, the stopping time N has a finite k-th moment for any k if and only if $p > \frac{1}{3}$. However, Grimmett and Stirzaker [6, Problem 12.9.15] showed that both $\max_{1\le n\le N} T_n$ and $\sum_{n=1}^{N} B_n$ have infinite expectations if $p = \frac{1}{2}$. It was also stated in [6] that $\mathbb{E}[B^\star] = \infty$, but we were informed by Ethier that the proof was incomplete (via an email exchange between him and Grimmett in February 2006). A recent work [7] shows that a sufficient condition of $\mathbb{E}[B^\star] < \infty$ is that $p > p_0 \approx 0.613763$, while matching necessary conditions are still missing. Hence, it remains an open conjecture for more than a decade if the largest bet size $B^\star$ also has an infinite expectation when $\frac{1}{3} < p \le \frac{1}{2}$, which is the main focus of this paper.

There is also another betting system which is similar to the Labouchere system, i.e., the Fibonacci system. Instead of considering the first and last numbers in the list at each coup, the *last* two numbers are added or canceled in the Fibonacci system. Ethier [5] showed that $\mathbb{E}[B^\star] = +\infty$ in Fibonacci system if and only if $p \le \frac{1}{2}$. However, the proof heavily relies on the fact that any list in a Fibonacci system is uniquely determined by its length, which does not hold for the Labouchere system where the list evolves in a more complicated "history dependent" manner.

2 Main results

To study the Labouchere system, we first introduce a larger family of betting systems called (a, b)-*list systems*:

Definition 2.1 ((a, b)-List System)**.** *Let $a < 0 \le b$ be integers. An (a, b)-list system consists of a target sequence $\{T_n\}$, a bet sequence $\{B_n\}$ and a length sequence $\{l_n\}$, which evolve as follows:*

1. *At the beginning, $T_0 > 0$ and $l_0 \in \{1, 2, \cdots\}$;*
2. *At the n-th coup, the system makes a bet size $B_n \in [0, T_{n-1}]$ which may depend on the entire history. Then the target and length sequences evolve as*

$$T_n = \begin{cases} T_{n-1} - B_n & \text{if wins} \\ T_{n-1} + B_n & \text{if loses} \end{cases}, \qquad l_n = \begin{cases} (l_{n-1} + a)_+ & \text{if wins} \\ l_{n-1} + b & \text{if loses} \end{cases}.$$

3. *Termination condition: let $N = \inf\{n : l_n = 0\}$ be the stopping time that the length becomes zero, we must have $T_n = l_n = 0$ for any $n \ge N$ and $B_n = 0$ for any $n > N$.*

In such a list system, the target T_n represents the remaining amount of money one would like to earn at the end of the n-th coup; consequently, T_n shrinks after a win, and increases after a loss. The length l_n represents the length of the "list" at the n-th coup, where it may be some real/virtual list which governs the betting process. For example, the well-known martingale system (where the bet is doubled after each loss) belongs to the $(-1,0)$-list system with $l_0 = 1$ and $B_n = T_{n-1}$, and both the Labouchere and Fibonacci systems fall into the category of $(-2,1)$-list systems. The termination condition ensures that, as long as the list length l_n hits zero, the target must be fulfilled as well (i.e., $T_n = 0$), and the betting process terminates.

In this paper, we only consider $(-2,1)$-list systems where the Labouchere system is included, but our results and proof techniques are generalizable to general (a,b)-list systems. Our first result characterizes the behavior of the largest bet size $B^\star$ under general list systems:

Theorem 2.2. *For any $(-2,1)$-list system, the following holds:*

1. *If $p > \frac{1}{2}$, we have $\mathbb{E}[B^\star] < \infty$;*
2. *If $\frac{1}{3} < p < \frac{1}{2}$, we have $\mathbb{E}[B^\star] = \infty$;*
3. *If $p \le \frac{1}{3}$ and $B_n \ge c_1 l_{n-1} + c_2$ for some constants $c_1 > 0, c_2 \in \mathbb{R}$ almost surely, we have $\mathbb{E}[B^\star] = \infty$.*

Theorem 2.2 shows that for any $(-2,1)$-list systems, the expectation $\mathbb{E}[B^\star]$ of the largest bet size $B^\star$ has a phase transition at $p = \frac{1}{2}$: the expectation is finite if the player is favored, and is infinite if the house takes the advantage. Consequently, we have the following corollary:

Corollary 2.3. *For the Labouchere system with any initial list, we have $\mathbb{E}[B^\star] < \infty$ if $p > \frac{1}{2}$ and $\mathbb{E}[B^\star] = \infty$ if $p < \frac{1}{2}$.*

The fair-game case $p = \frac{1}{2}$ requires more delicate analysis, and is summarized in the following theorem:

Theorem 2.4. *Let $(\bar{b}_l)_{l=1}^\infty, (\underline{b}_l)_{l=1}^\infty$ be two sequences taking value in $[0,1]$. Suppose that some $(-2,1)$-list system satisfies that $T_{n-1}\underline{b}_{l_{n-1}} \le B_n \le T_{n-1}\bar{b}_{l_{n-1}}$ for any n, and one of the following conditions holds:*

1. *$\lim_{l\to\infty} \bar{b}_l = 0$;*
2. *$\inf_l \underline{b}_l > 0$,*

we have $\mathbb{E}[B^\star] = \infty$ under $p = \frac{1}{2}$.

Note that B_n/T_{n-1} is the bet proportion at the n-th coup, and general $(-2,1)$-list systems correspond to the case where $\bar{b}_l = 1, \underline{b}_l = 0$ for any l. Theorem 2.4 shows that, if the bet proportion either vanishes or is lower bounded from below as the list length l grows, the largest bet size still has an infinite expectation in a fair game. The following corollary follows from Theorem 2.4:

Corollary 2.5. *For the Labouchere system with any initial list, $\mathbb{E}[B^\star] = \infty$ if $p = \frac{1}{2}$.*

Combining Corollaries 2.3 and 2.5, we conclude that for the Labouchere system, $\mathbb{E}[B^\star] = \infty$ if and only if $p \le \frac{1}{2}$, solving the open conjecture in [6]. It also follows directly from Theorems 2.2 and 2.4 that for the Fibonacci system, $\mathbb{E}[B^\star] = \infty$ if and only if $p \le \frac{1}{2}$, recovering the result in [5]. Generalizing the arguments to $(-1,0)$-list systems, this also recovers the famous St. Petersburg paradox that $\mathbb{E}[B^\star] = \infty$ in the martingale system under $p = \frac{1}{2}$.

Based on Theorem 2.4, a natural question would be that whether $\mathbb{E}[B^\star] = \infty$ holds in any $(-2,1)$-list systems. We have the following partial result:

Theorem 2.6. *For any $(-2,1)$-list system and $\varepsilon > 0$, the following holds under $p = \frac{1}{2}$:*

$$\mathbb{E}\left[B^\star (1 \vee \log B^\star)^{-(1+\varepsilon)}\right] < \infty, \qquad \mathbb{E}[B^\star (1 \vee \log B^\star)] = \infty.$$

Theorem 2.6 shows that, the moment $\mathbb{E}[(B^\star)^\alpha]$ always has a phase transition at $\alpha = 1$ in a fair game. However, the exact answer for $\alpha = 1$ is still unknown, and we leave it as a conjecture:

Conjecture 2.7. *For any $(-2,1)$-list systems, $\mathbb{E}[B^\star] = \infty$ under $p = \frac{1}{2}$.*

3 Proof of Theorems 2.2 and 2.6

In this section, we first prove Theorem 2.6, and then apply Theorem 2.6 to proving Theorem 2.2.

3.1 Proof of Theorem 2.6

We make use of the asymptotic tail behavior of the stopping time N in the $(-2,1)$-list system.

Lemma 3.1. *[4] For $p > \frac{1}{3}$, we have*

$$\mathbb{P}_{l_0}(N \geq n+1) \sim D_{l_0}(n) n^{-\frac{3}{2}} \kappa^{\frac{n}{3}},$$

where l_0 is the length of the initial list, $D_{l_0}(n)$ is a constant only depending on l_0 and $n \pmod 3$, and $\kappa \triangleq \frac{27}{4} p(1-p)^2 < 1$.

Based on Lemma 3.1, we are about to prove Theorem 2.6. We first show that $\mathbb{E}\left[B^\star(1 \vee \log B^\star)^{-(1+\varepsilon)}\right] < \infty$. Under $p = \frac{1}{2}$, the target sequence $\{T_n\}$ is a martingale, with $\mathbb{E}[T_n] = T_0$. By Doob's maximal inequality, for any $\lambda > 0$,

$$\mathbb{P}\left(\max_{0 \leq m \leq n} T_m \geq \lambda\right) \leq \frac{\mathbb{E}[T_n]}{\lambda} = \frac{T_0}{\lambda}.$$

Note that $B_n \leq T_{n-1}$, for $\lambda \geq 2$ we therefore have

$$\begin{aligned}
\mathbb{P}\left(\max_{1 \leq m \leq n} B_m(1 \vee \log B_m)^{-(1+\varepsilon)} \geq \lambda\right) &= \mathbb{P}\left(\max_{1 \leq m \leq n} B_m \geq C\lambda(\log \lambda)^{1+\varepsilon}\right) \\
&\leq \mathbb{P}\left(\max_{0 \leq m \leq n-1} T_m \geq C\lambda(\log \lambda)^{1+\varepsilon}\right) \\
&\leq \frac{T_0}{C\lambda(\log \lambda)^{1+\varepsilon}}
\end{aligned}$$

where $C > 0$ is some universal constant. As a result,

$$\begin{aligned}
\mathbb{E}\left[\max_{1 \leq m \leq n} B_m(1 \vee \log B_m)^{-(1+\varepsilon)}\right] &= \int_0^\infty \mathbb{P}\left(\max_{1 \leq m \leq n} B_m(1 \vee \log B_m)^{-(1+\varepsilon)} \geq \lambda\right) d\lambda \\
&\leq 2 + \int_2^\infty \frac{T_0}{C\lambda(\log \lambda)^{1+\varepsilon}} d\lambda < \infty
\end{aligned}$$

where in the last step we have used that

$$\int_2^\infty \frac{dx}{x(\log x)^{1+\varepsilon}} < \infty.$$

Choosing $n \to \infty$, by monotone convergence we arrive at $\mathbb{E}\left[B^\star(1 \vee \log B^\star)^{-(1+\varepsilon)}\right] < \infty$.

Now we show that $\mathbb{E}[B^\star(1 \vee \log B^\star)] = \infty$. We recall the following Fenchel–Young inequality:

$$xy \leq \psi(x) + \psi^\star(y)$$

where $\psi(\cdot)$ is convex, and $\psi^\star(y) = \sup_x(xy - \psi(x))$ is the Fenchel–Legendre dual of ψ. For $\psi(x) = e^{cx}$ with $c > 0$, we have

$$\psi^\star(y) = \sup_{x\in\mathbb{R}} (xy - e^{cx}) = \frac{y}{c}\left(\log\frac{y}{c} - 1\right),$$

and therefore

$$\mathbb{E}[NB^\star] \le \mathbb{E}[\psi(N)] + \mathbb{E}[\psi^\star(B^\star)] = \mathbb{E}[e^{cN}] + \frac{1}{c}\mathbb{E}\left[B^\star\left(\log\frac{B^\star}{c} - 1\right)\right].$$

By Lemma 3.1, for $c > 0$ sufficiently small we have $\mathbb{E}[e^{cN}] < \infty$. Moreover, [6] shows that

$$\mathbb{E}[NB^\star] \ge \mathbb{E}\left[\sum_{n=1}^N B_n\right] = \infty.$$

A combination of the previous two inequalities yields $\mathbb{E}[B^\star(1 \vee \log B^\star)] = \infty$.

3.2 Proof of Theorem 2.2 and Corollary 2.3

Now we prove Theorem 2.2 using Theorem 2.6 and a change of measure.

Fix any $p > \frac{1}{2}$, let P be the probability measure over the betting process under winning probability p, and Q be the counterpart under winning probability $\frac{1}{2}$. Note that for any sample path ω with stopping time $N = n$, there must be $\frac{n}{3} + c$ wins and $\frac{2n}{3} - c$ losses, where c is a constant depending only on the initial length l_0 and $n \pmod 3$. As a result, the likelihood ratio is

$$\frac{dP}{dQ}(\omega) = \frac{p^{\frac{n}{3}+c}(1-p)^{\frac{2n}{3}-c}}{2^{-n}} = \left(\frac{p}{1-p}\right)^c \cdot \left(\frac{p(1-p)^2}{\frac{1}{2}(1-\frac{1}{2})^2}\right)^{\frac{n}{3}} \le C\rho^n$$

where $C > 0, \rho \in (0,1)$ are numerical constants independent of n, and we have used that the function $p \mapsto p(1-p)^2$ is strictly decreasing in $p \in [\frac{1}{3}, 1]$. As a result,

$$\mathbb{E}_P[B^\star] = \mathbb{E}_Q\left[B^\star \cdot \frac{dP}{dQ}\right] \le C \cdot \mathbb{E}_Q[\rho^N B^\star].$$

Since $T_n \le T_{n-1} + B_n \le 2T_{n-1}$ in any list system, we have $B^\star \le \max_{0\le n\le N} T_n \le T_0 \cdot 2^N$, and therefore

$$\mathbb{E}_Q[\rho^N B^\star] \le T_0^\varepsilon \cdot \mathbb{E}_Q[(\rho 2^\varepsilon)^N (B^\star)^{1-\varepsilon}]$$

for any $\varepsilon > 0$. Choosing $\varepsilon > 0$ small enough such that $\rho 2^\varepsilon < 1$, Theorem 2.6 implies that $\mathbb{E}_P[B^\star] < \infty$.

For $p \in (\frac{1}{3}, \frac{1}{2})$, we use the same argument to obtain $\frac{dP}{dQ} \ge C\rho^N$ for some $\rho > 1$. Then

$$\mathbb{E}_P[B^\star] \ge C \cdot \mathbb{E}_Q[\rho^N B^\star] \ge CT_0^{-\varepsilon} \cdot \mathbb{E}_Q[(\rho 2^{-\varepsilon})^N (B^\star)^{1+\varepsilon}],$$

and by choosing $\varepsilon > 0$ small enough, Theorem 2.6 yields $\mathbb{E}_P[B^\star] = \infty$.

Finally, for $p \le \frac{1}{3}$, we have $\mathbb{E}[\sup_{0\le n<N} l_n] = \infty$ by the theory of asymmetric random walks. Hence, by assumption we have

$$\mathbb{E}[B^\star] \ge c_1 \mathbb{E}[\sup_{0\le n<N} l_n] + c_2 = \infty$$

as desired. The proof of Theorem 2.2 is completed.

As for Corollary 2.3, it suffices to verify that the condition $B_n \ge c_1 l_{n-1} + c_2$ holds for the Labouchere system. Let $a > 0$ be the minimum number in the initial list L_0, a simple induction on n yields that $B_n \ge a(l_{n-1} - l_0)_+$, which shows that the condition is fulfilled with $c_1 = a > 0, c_2 = -al_0$.

4 Proof of Theorem 2.4 and Corollary 2.5

In this section, we first use a recursive representation of the optimal list system to prove Theorem 2.4. Then we investigate the specific properties of the Labouchere system and show that the condition in Theorem 2.4 holds, thereby proving Corollary 2.5.

4.1 Proof of Theorem 2.4

If $\inf_l \underline{b}_l \geq c > 0$, we have $B^\star \geq c \max_{0\leq n\leq N} T_n$, which has an infinite expectation [6]. Now we assume that $\lim_{l\to\infty} \bar{b}_l = 0$ and prove Theorem 2.4 by contradiction. We first introduce the following definition:

Definition 4.1. *For any $x > 0$ and $l \in \{1, 2, \cdots\}$, we define $f(x, l)$ to be the infimum of $\mathbb{E}[B^\star]$ over all possible $(-2, 1)$-list systems with initial target x and initial length l, such that $B_n \leq \bar{b}_{l_{n-1}} T_{n-1}$ for any n.*

Definition 4.1 considers an *optimal* $(-2, 1)$-list system with initial target x and initial length l, where optimality is measured in terms of a smallest expectation of the largest bet size $B^\star$. The quantity $f(x, l) \in \mathbb{R}_+ \cup \{+\infty\}$ is the corresponding expectation, and it is well-defined even if the optimal list system does not exist. The next lemma presents recursive relations between $f(x, l)$ with different l.

Lemma 4.2. *There exists some sequence $(a_l)_{l=1}^\infty$ taking value in $\mathbb{R}_+ \cup \{+\infty\}$ such that $f(x, l) = xa_l$ for any $x > 0$. Moreover, the sequence $(a_l)_{l=1}^\infty$ satisfies the following inequalities:*

$$a_l \geq \min_{b\in[0,\bar{b}_l]} \frac{\max\{b, (1-b)a_{l-2}\} + \max\{b, (1+b)a_{l+1}\}}{2}, \qquad l \geq 3$$

$$a_1 \geq a_2 + \frac{1}{2} \geq a_3 + 1.$$

Proof. When the initial target x is scaled by $\lambda > 0$, we may always scale all bet sizes by λ to arrive at a new list system with the initial target λx, and vice versa. Hence, $f(x, l)$ is proportional to x, and $f(x, l) = xa_l$.

For $l \geq 3$ and any $(-2, 1)$-list system, let $b \in [0, \bar{b}_l]$ be any bet size at the first coup with initial target $T_0 = 1$ and initial length l. Let $B_1^\star, B_2^\star$ be the largest bet sizes (excluding the first bet) after winning/losing the first coup, respectively. Then by definition of $f(x, l)$, we have

$$\begin{aligned}
\mathbb{E}B_1^\star &\geq f(1-b, l-2) = (1-b)a_{l-2}, \\
\mathbb{E}B_2^\star &\geq f(1+b, l+1) = (1+b)a_{l+1}.
\end{aligned}$$

Note that $B^\star$ is either $\max\{b, B_1^\star\}$ or $\max\{b, B_2^\star\}$, we have

$$\begin{aligned}
\mathbb{E}[B^\star] &= \frac{\mathbb{E}\max\{b, B_1^\star\} + \mathbb{E}\max\{b, B_2^\star\}}{2} \\
&\geq \frac{\max\{b, \mathbb{E}B_1^\star\} + \max\{b, \mathbb{E}B_2^\star\}}{2} \\
&\geq \frac{\max\{b, (1-b)a_{l-2}\} + \max\{b, (1+b)a_{l+1}\}}{2}
\end{aligned}$$

where the first inequality is due to the convexity of $x \mapsto \max\{b, x\}$. Note that this inequality holds for any list systems, taking infimum over the LHS gives the desired inequality for $l \geq 3$. The other inequalities for $l \leq 2$ can be established analogously. □

Based on Lemma 4.2, we may investigate more properties of a_l. If $a_1 = \infty$, it is obvious that $a_l = \infty$ for any $l \in \mathbb{N}$ (since any initial list may evolve into length one with a non-zero probability), and Theorem 2.4 holds. Next we show that $a_1 < \infty$ is impossible. Assume by contradiction that $a_1 < \infty$, we will have the following lemma.

Lemma 4.3. *If $a_1 < \infty$, the sequence $\{a_l\}$ will be strictly decreasing, i.e., $a_1 > a_2 > a_3 > \cdots$.*

Proof. For $l \geq 3$, by Lemma 4.2 we have

$$\begin{aligned} a_l &\geq \min_{b\in[0,\bar{b}_l]} \frac{(1-b)a_{l-2} + (1+b)a_{l+1}}{2} \\ &\geq \min_{b\in[0,1]} \frac{(1-b)a_{l-2} + (1+b)a_{l+1}}{2} \\ &= \min\left\{\frac{a_{l-2}+a_{l+1}}{2}, a_{l+1}\right\}, \end{aligned}$$

where in the last step we have used the fact that an affine function attains its minimum at the boundary. Consequently, if we already know that $a_1 \geq a_2 \geq \cdots \geq a_l$, we must also have $a_l \geq a_{l+1}$. Hence, by induction on l, the sequence $\{a_l\}$ is decreasing.

To show the strict decreasing property, by Lemma 4.2 again we have

$$\begin{aligned} a_l &\geq \min_{b\in[0,\bar{b}_l]} \frac{\max\{b,(1-b)a_{l-2}\} + (1+b)a_{l+1}}{2} \\ &\geq \min_{b\in[0,1]} \frac{\max\{b,(1-b)a_{l-2}\} + (1+b)a_{l+1}}{2} \\ &= \frac{1}{2}\min_{b\in[0,1]} \max\{b + (1+b)a_{l+1}, (1-b)a_{l-2} + (1+b)a_{l+1}\}. \end{aligned}$$

For real numbers r_1, r_2, s_1, s_2 with $r_1 > 0 \geq r_2, s_1 \leq s_2, r_1 + s_1 \geq r_2 + s_2$, straightforward computation yields

$$\min_{x\in[0,1]} \max\{r_1x + s_1, r_2x + s_2\} = \frac{r_1s_2 - r_2s_1}{r_1 - r_2}.$$

Hence,

$$a_l \geq \frac{2a_{l-2}a_{l+1} + a_{l-2} + a_{l+1}}{2(a_{l-2}+1)} = a_{l+1} + \frac{a_{l-2} - a_{l+1}}{2(a_{l-2}+1)}.$$

If we have $a_l = a_{l+1}$, we will also have $a_{l-2} = a_{l+1}$ based on the previous inequality. Due to the decreasing property of $\{a_l\}$, $a_{l-1} = a_l$ also holds, and repeating this process yields $a_2 = a_3$, a contradiction to Lemma 4.2. Hence $a_l > a_{l+1}$ for any l. $\square$

Based on Lemmas 4.2 and 4.3, we are about to arrive at the desired contradiction. Fix any $\varepsilon > 0$ such that $\rho \triangleq \frac{1-\varepsilon}{1+\varepsilon} + (\frac{1-\varepsilon}{1+\varepsilon})^2 > 1$. Since $\lim_{l\to\infty} \bar{b}_l = 0$, we take $l_0 > 0$ large enough such that $\bar{b}_l < \varepsilon$ for any $l > l_0$. Then for $l > l_0$, Lemma 4.2 yields

$$\begin{aligned} a_l &\geq \min_{b\in[0,\bar{b}_l]} \frac{(1-b)a_{l-2} + (1+b)a_{l+1}}{2} \\ &\geq \min_{b\in[0,\varepsilon]} \frac{(1-b)a_{l-2} + (1+b)a_{l+1}}{2} \\ &= \min_{b\in[0,\varepsilon]} \frac{(a_{l+1} - a_{l-2})b + a_{l+1} + a_{l-2}}{2} \\ &= \frac{(a_{l+1} - a_{l-2})\varepsilon + a_{l+1} + a_{l-2}}{2}, \end{aligned}$$

where in the last step we have used $a_{l+1} \leq a_{l-2}$ by Lemma 4.3. A rearrangement of the previous inequality gives

$$a_l - a_{l+1} \geq \frac{1-\varepsilon}{1+\varepsilon} \cdot (a_{l-2} - a_l)$$

for any $l > l_0$. Similarly,

$$\begin{aligned} a_{l+1} - a_{l+2} &\geq \frac{1-\varepsilon}{1+\varepsilon} \cdot (a_{l-1} - a_{l+1}) \\ &\geq \frac{1-\varepsilon}{1+\varepsilon} \cdot (a_l - a_{l+1}) \\ &\geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 \cdot (a_{l-2} - a_l). \end{aligned}$$

Adding them together yields

$$a_l - a_{l+2} \geq \left[\frac{1-\varepsilon}{1+\varepsilon} + \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2\right] \cdot (a_{l-2} - a_l) = \rho(a_{l-2} - a_l).$$

Our choice of ε implies $\rho > 1$, and therefore $a_{l+2k-2} - a_{l+2k} \geq \rho^k (a_{l-2} - a_l)$ for any $k \in \mathbb{N}$ and $l > l_0$. Since $a_{l+2k-2} - a_{l+2k} \leq a_1$, and $a_{l-2} > a_l$ by Lemma 4.3, this inequality implies that

$$a_1 \geq \rho^k (a_{l-2} - a_l)$$

for any $k = 1, 2, \cdots$, a contradiction to the assumption $a_1 < \infty$. The proof of Theorem 2.4 is complete.

4.2 Proof of Corollary 2.5

First we observe that it suffices to prove the case where the initial list consists of a single positive number. This observation is due to that there is a positive probability to reduce the list length to $l_n = 1$ after finitely many coups for any initial list L_0.

To study the combinatorial properties of the Labouchere system, we introduce the following definition:

Definition 4.4. *A list of positive real numbers $(a_1, a_2, a_3, \cdots, a_n)$ is called* good *if it satisfies the following conditions:*

- *Every element in the list is positive, i.e., $a_i > 0$ for any i;*
- *The list is non-decreasing, i.e., $a_1 \leq a_2 \leq \cdots \leq a_n$;*
- *The difference of the list is non-decreasing with difference at most a_1, i.e., $a_2 - a_1 \leq a_3 - a_2 \leq \cdots \leq a_n - a_{n-1} \leq a_1$.*

The key properties of a good list are summarized in the following lemmas.

Lemma 4.5. *If the initial list L_0 is good, the list L_n after n-th coup is also good for any n.*

Proof. It suffices to prove that, if $L_{n-1} = (a_1, \cdots, a_l)$ is a good list, so is L_n. Based on the outcome at n-th coup, there are only two possibilities:

- $L_n = (a_1, a_2, \cdots, a_l, a_1 + a_l)$, or
- $L_n = (a_2, a_3, \cdots, a_{l-1})$.

In either case, one can check from Definition 4.4 directly that L_n is a good list, as desired. □

Lemma 4.6. *If the list L_{n-1} is good and has length $l \geq 2$, in the Labouchere system we have*

$$\frac{B_n}{T_{n-1}} \leq \sqrt{\frac{2}{l}} + \frac{2}{l}.$$

Proof. Let $L_{n-1} = (a_1, \cdots, a_l)$. By definition, for any $k \leq l$ we have

$$a_l \leq a_{l-1} + a_1 \leq a_{l-2} + 2a_1 \leq \cdots \leq a_{l-k} + ka_1.$$

As a result,

$$a_l \leq \frac{1}{k} \sum_{j=1}^{k} (a_{l-j} + ja_1) \leq \frac{1}{k} \sum_{j=1}^{l} a_j + \frac{k+1}{2} \cdot a_1.$$

Note that the current bet size is $B_n = a_1 + a_l$, and the current target is $T_{n-1} = \sum_{j=1}^{l} a_j$. Hence, for any $k \leq l$ we have

$$\frac{B_n}{T_{n-1}} = \frac{a_1 + a_l}{\sum_{j=1}^{l} a_j} \leq \frac{(k+3)a_1}{2\sum_{j=1}^{l} a_j} + \frac{1}{k} \leq \frac{k+3}{2l} + \frac{1}{k}.$$

Setting $k = \lceil \sqrt{2l} \rceil \leq l$ arrives at

$$\frac{B_n}{T_{n-1}} \leq \frac{\lceil \sqrt{2l} \rceil + 3}{2l} + \frac{1}{\lceil \sqrt{2l} \rceil} \leq \frac{\sqrt{2l} + 4}{2l} + \frac{1}{\sqrt{2l}} = \sqrt{\frac{2}{l}} + \frac{2}{l},$$

as claimed. □

Note that the initial list L_0 consisting of a single positive number is good, by Lemma 4.5 we know that all future lists L_n are also good. Moreover, by setting

$$\bar{b}_l = \min\left\{\sqrt{\frac{2}{l}} + \frac{2}{l}, 1\right\},$$

by Lemma 4.6 we know that $B_n \leq \bar{b}_{l_{n-1}} T_{n-1}$ always holds. Note that $\lim_{l \to \infty} \bar{b}_l = 0$, Theorem 2.4 yields $\mathbb{E}[B^\star] = \infty$ in the Labouchere system, as desired.

References

[1] Émile Barbier. Généralisation du problème résolu par M. J. Bertrand. *Comptes Rendus des Séances de l'Académie des Sciences*, 105:407, 1887.

[2] Joseph Bertrand. Solution d'un problème. *Comptes Rendus des Séances de l'Académie des Sciences*, 105:369, 1887.

[3] F Downton. A note on Labouchere sequences. *Journal of the Royal Statistical Society, Series A*, 143:363–366, 1980. MR-0590362

[4] Stewart N Ethier. Absorption time distribution for an asymmetric random walk. In SN Ethier, J Feng, and RH Stockbridge, editors, *Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz*, pages 31–40. IMS Collections 4, Institute of Mathematical Statistics, Beachwood, OH, 2008. MR-2574222

[5] Stewart N Ethier. *The Doctrine of Chances: Probabilistic Aspects of Gambling*. Springer Science & Business Media, 2010. MR-2656351

[6] Geoffrey Grimmett and David Stirzaker. *One Thousand Exercises in Probability*. Oxford University Press, Oxford, 2001.

[7] Nina Zubrilina. On the expected value of the maximal bet in the Labouchere system. *arXiv preprint arXiv:1808.06642*, 2018.

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