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On the tails of the limiting QuickSort density*

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Abstract

We give upper and lower asymptotic bounds for the left tail and for the right tail of the continuous limiting QuickSort density f that are nearly matching in each tail. The bounds strengthen results from a paper of Svante Janson (2015) concerning the corresponding distribution function F. Furthermore, we obtain similar bounds on absolute values of derivatives of f of each order.

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1 Introduction

Let X_n denote the (random) number of comparisons when sorting n distinct numbers using the algorithm QuickSort. Clearly $X_0 = 0$, and for $n \geq 1$ we have the recurrence relation

$$X_n \stackrel{\mathcal{L}}{=} X_{U_n - 1} + X^*_{n-U_n} + n - 1,$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in law (i.e., in distribution); $X_k \stackrel{\mathcal{L}}{=} X^*_k$; the random variable U_n is uniformly distributed on $\{1, \ldots, n\}$; and $U_n, X_0, \ldots, X_{n-1}$, $X^*_0, \ldots, X^*_{n-1}$ are all independent. It is well known that

$$\mathbb{E}X_n = 2(n+1)H_n - 4n,$$

where H_n is the nth harmonic number $H_n := \sum_{k=1}^n k^{-1}$ and (from a simple exact expression) that $\operatorname{Var} X_n = (1 + o(1))(7 - \frac{2\pi^2}{3})n^2$. To study distributional asymptotics, we first center and scale X_n as follows:

$$Z_n = \frac{X_n - \mathbb{E}X_n}{n}.$$

Using the Wasserstein d_2-metric, Rösler [8] proved that Z_n converges to Z weakly as $n \to \infty$. Using a martingale argument, Régnier [7] proved that the slightly renormalized $\frac{n}{n+1} Z_n$ converges to Z in L^p for every finite p, and thus in distribution; equivalently, the same conclusions hold for Z_n. The random variable Z has everywhere finite moment

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generating function with $\mathbb{E}Z = 0$ and $\operatorname{Var} Z = 7 - (2\pi^2/3)$. Moreover, Z satisfies the distributional identity

$$Z \stackrel{\mathcal{L}}{=} UZ + (1-U)Z^* + g(U).$$

On the right, $Z^* \stackrel{\mathcal{L}}{=} Z$; U is uniformly distributed on $(0,1)$; U, Z, Z^* are independent; and

$$g(u) := 2u \ln u + 2(1-u)\ln(1-u) + 1.$$

Further, the distributional identity together with the condition that $\mathbb{E}Z$ (exists and) vanishes characterizes the limiting `Quicksort` distribution; this was first shown by Rösler [8] under the additional condition that $\operatorname{Var} Z < \infty$, and later in full by Fill and Janson [1].

Fill and Janson [2] derived basic properties of the limiting `QuickSort` distribution $\mathcal{L}(Z)$. In particular, they proved that $\mathcal{L}(Z)$ has a (unique) continuous density f which is everywhere positive and infinitely differentiable, and for every $k \geq 0$ that $f^{(k)}$ is bounded and enjoys superpolynomial decay in both tails, that is, for each $p \geq 0$ and $k \geq 0$ there exists a finite constant $C_{p,k}$ such that $\left|f^{(k)}(x)\right| \leq C_{p,k}|x|^{-p}$ for all $x \in \mathbb{R}$.

In this paper, we study asymptotics of $f(-x)$ and $f(x)$ as $x \to \infty$. Janson [3] concerned himself with the corresponding asymptotics for the distribution function F and wrote this: "Using non-rigorous methods from applied mathematics (assuming an as yet unverified regularity hypothesis), Knessl and Szpankowski [4] found very precise asymptotics of both the left tail and the right tail." Janson specifies these Knessl–Szpankowski asymptotics for F in his equations (1.6)–(1.7). But Knessl and Szpankowski actually did more, producing asymptotics for f, which were integrated by Janson to get corresponding asymptotics for F. We utilize the same abbreviation $\gamma := (2 - \frac{1}{\ln 2})^{-1}$ as Janson [3]. With the *same* constant c_3 as in (1.6) of [3], the density analogues of (1.6) (omitting the middle expression) and (1.7) of [3] are that, as $x \to \infty$, Knessl and Szpankowski [4] find

$$f(-x) = \exp\left[-e^{\gamma x + c_3 + o(1)}\right] \tag{1.1}$$

for the left tail and

$$f(x) = \exp[-x \ln x - x \ln\ln x + (1 + \ln 2)x + o(x)] \tag{1.2}$$

for the right tail.

We will come as close to these non-rigorous results for the density as Janson [3] does for the distribution function, and we also obtain similar asymptotic bounds for tail suprema of absolute values of derivatives of the density. Although our asymptotics for f imply the asymptotics for F in Janson's main Theorem 1.1, it is important to note that in the case of upper bounds (but not lower bounds) on f we use his results in the proofs of ours.

The next two theorems are our main results.

Theorem 1.1. *Let* $\gamma := (2 - \frac{1}{\ln 2})^{-1}$. *As* $x \to \infty$, *the limiting* `QuickSort` *density function* f *satisfies*

$$\exp\left[-e^{\gamma x + \ln\ln x + O(1)}\right] \leq f(-x) \leq \exp\left[-e^{\gamma x + O(1)}\right], \tag{1.3}$$

$$\exp[-x \ln x - x \ln\ln x + O(x)] \leq f(x) \leq \exp[-x \ln x + O(x)]. \tag{1.4}$$

To state our second main theorem we let $\underline{F}(x) := F(-x)$ and $\overline{F}(x) := 1 - F(x)$, and for a function $h : \mathbb{R} \to \mathbb{R}$ we write

$$\|h\|_x := \sup_{t \geq x} |h(t)|. \tag{1.5}$$

Theorem 1.2. *Given an integer $k \ge 0$, as $x \to \infty$ the k^{th} derivative of the limiting* `QuickSort` *distribution function F satisfies*

$$\exp\left[-e^{\gamma x+\ln\ln x+O(1)}\right] \le \|\underline{F}^{(k)}\|_x \le \exp\left[-e^{\gamma x+O(1)}\right], \tag{1.6}$$

$$\exp[-x\ln x-(k\vee 1)x\ln\ln x+O(x)] \le \|\overline{F}^{(k)}\|_x \le \exp[-x\ln x+O(x)]. \tag{1.7}$$

Remark 1.3. (a) Using the monotonicity of F, it is easy to see that the assertions of Theorem 1.2 for $k=0$ are equivalent to the main Theorem 1.1 of Janson [3], which agrees with the formulation of our Theorem 1.2 in that case except that the four bounds are on $|\underline{F}(x)|$ and $|\overline{F}(x)|$ instead of the tail suprema $\|\underline{F}\|_x$ and $\|\overline{F}\|_x$. Further, our Theorem 1.1 implies the assertions of Theorem 1.2 for $k=1$. So we need only prove Theorem 1.1 and Theorem 1.2 for $k \ge 2$.

(b) The non-rigorous arguments of Knessl and Szpankowski [4] suggest that the following asymptotics as $x \to \infty$ obtained by repeated formal differentiation of (1.1)–(1.2) are correct for every $k \ge 0$:

$$f^{(k)}(-x) = \exp\left[-e^{\gamma x+c_3+o(1)}\right], \tag{1.8}$$

$$f^{(k)}(x) = (-1)^k \exp[-x\ln x - x\ln\ln x + (1+\ln 2)x + o(x)]. \tag{1.9}$$

But these remain conjectures for now. Unfortunately, for $k \ge 1$ we don't even know how to identify rigorously the asymptotic signs of $f^{(k)}(\mp x)$! Concerning $k=1$, it has long been conjectured that f is unimodal. This would of course imply that $f'(-x) > 0$ and $f'(x) < 0$ for sufficiently large x.

As already mentioned, Fill and Janson [2] proved that or each $p \ge 0$ and $k \ge 0$ there exists a finite constant $C_{p,k}$ such that $\left|f^{(k)}(x)\right| \le C_{p,k}|x|^{-p}$ for all $x \in \mathbb{R}$. Our technique for proving the upper bounds in Theorems 1.1 and 1.2 is to use explicit bounds on the constants $C_k := C_{0,k}$ together with the Landau–Kolmogorov inequality (see, for example, [9]).

Our paper is organized as follows. In Section 2 we deal with preliminaries: We recall an integral equation for f that is the starting point for our lower-bound results in Theorem 1.1, review the Landau–Kolmogorov inequality, and bound C_k explicitly in terms of k. Sections 3 and 4 derive the stated lower bounds on the left and right tails, respectively, of f using an iterative approach similar to that of Janson [3] for the distribution function. In Section 5 we establish the left-tail results claimed in (1.3) and (1.6). In Section 6, we establish the right-tail results claimed in (1.4) and (1.7).

2 Preliminaries

2.1 An integral equation for f

Fill and Janson [2, Theorem 4.1 and (4.2)] produced an integral equation satisfied by f, namely,

$$f(x) = \int_{u=0}^{1}\int_{z\in\mathbb{R}} f(z)\, f\left(\frac{x-g(u)-(1-u)z}{u}\right)\frac{1}{u}\,dz\,du. \tag{2.1}$$

This integral equation will be used in the proofs of our lower-bound results for f.

2.2 Landau–Kolmogorov inequality

For an overview of the Landau–Kolmogorov inequality, see [6, Chapter 1]. Here we state a version of the inequality well-suited to our purposes; see [5] and [9, display (21) and the display following (17)].

Lemma 2.1. *Let $n \geq 2$, and suppose $h : \mathbb{R} \to \mathbb{R}$ has n derivatives. If h and $h^{(n)}$ are both bounded, then for $1 \leq k < n$ so is $h^{(k)}$. Moreover, there exist constants $c_{n,k}$ (not depending on h) such that, for every $x \in \mathbb{R}$, the supremum norm $\| \cdot \|_x$ defined at* (1.5) *satisfies*

$$\|h^{(k)}\|_x \leq c_{n,k} \, \|h\|_x^{1-(k/n)} \, \|h^{(n)}\|_x^{k/n}, \quad 1 \leq k < n.$$

Further, for $1 \leq k \leq n/2$ the best constants $c_{n,k}$ satisfy

$$c_{n,k} \leq n^{(1/2)[1-(k/n)]}(n-k)^{-1/2} \left(\frac{e^2 n}{4k}\right)^k \leq \left(\frac{e^2 n}{4k}\right)^k.$$

2.3 Explicit constant upper bounds for absolute derivatives

We also make use of the following two results extracted from [2, Theorem 2.1 and (3.3)].

Lemma 2.2. *Let ϕ denote the characteristic function corresponding to f. Then for every real $p \geq 0$ we have*

$$|\phi(t)| \leq 2^{p^2+6p}|t|^{-p} \quad \text{for all } t \in \mathbb{R}.$$

Lemma 2.3. *For every integer $k \geq 0$ we have*

$$\sup_{x \in \mathbb{R}} |f^{(k)}(x)| \leq \frac{1}{2\pi} \int_{t=-\infty}^{\infty} |t|^k \, |\phi(t)| \, dt.$$

Using these two results, it is now easy to bound $f^{(k)}$.

Proposition 2.4. *For every integer $k \geq 0$ we have*

$$\sup_{x \in \mathbb{R}} |f^{(k)}(x)| \leq 2^{k^2+10k+17}.$$

Proof. For every integer $k \geq 0$ we have

$$\begin{aligned}
\sup_{x \in \mathbb{R}} |f^{(k)}(x)| &\leq \frac{1}{2\pi} \int_{t=-\infty}^{\infty} |t|^k \, |\phi(t)| \, dt \\
&\leq \frac{1}{2\pi} \left[\int_{|t|>1} |t|^k \, |\phi(t)| \, dt + \int_{|t|\leq 1} |t|^k \, |\phi(t)| \, dt \right] \\
&\leq \frac{1}{2\pi} \left[\int_{|t|>1} 2^{(k+2)^2+6(k+2)} t^{-2} \, dt + \int_{|t|\leq 1} |t|^k \, dt \right] \\
&\leq \frac{1}{\pi} \left[2^{k^2+10k+16} + \frac{1}{k+1} \right] \leq 2^{k^2+10k+17},
\end{aligned}$$

as desired. □

3 Left tail lower bound on f

Our iterative approach to finding the left tail lower bound on f in Theorem 1.1 is similar to the method used by Janson [3] for F. The following lemma gives us an inequality that is essential in this section; as we shall see, it is established from a recurrence inequality. For $z \geq 0$ define

$$m_z := \left(\min_{x \in [-z,0]} f(x) \right) \wedge 1.$$

Lemma 3.1. *Given $\epsilon \in (0, 1/10)$, let $a \equiv a(\epsilon) := -g\left(\frac{1}{2} - \epsilon\right) > 0$. Then for any integer $k \geq 2$ we have*

$$m_{ka} \geq \left(2\epsilon^3 m_{2a}\right)^{2^{k-2}}.$$

We delay the proof of Lemma 3.1 in order to show next how the lemma leads us to the desired lower bound in (1.3) on the left tail of f by using the same technique as in [3] for F.

Proposition 3.2. *As $x \to \infty$ we have*

$$\ln f(-x) \geq -e^{\gamma x + \ln \ln x + O(1)}.$$

Proof. By Lemma 3.1, for $x > a$ we have

$$f(-x) \geq m_x \geq m\left(\left\lceil \frac{x}{a} \right\rceil a\right) \geq \left(2\epsilon^3 m_{2a}\right)^{2^{\lceil x/a \rceil - 2}} \geq \left(2\epsilon^3 m_{2a}\right)^{2^{x/a}},$$

provided ϵ is sufficiently small that $2\epsilon^3 m_{2a} < 1$. The same as Janson [3], we pick $\epsilon = x^{-1/2}$ and, setting $\gamma = (2 - \frac{1}{\ln 2})^{-1}$, get $\frac{1}{a} = \frac{\gamma}{\ln 2} + O(x^{-1})$ and

$$\begin{aligned} \ln f(-x) &\geq 2^{\frac{\gamma}{\ln 2} x + O(1)} \cdot \ln\left(2\epsilon^3 m_{2a}\right) \\ &= e^{\gamma x + O(1)} \cdot \left(-\tfrac{3}{2} \ln x + \ln m_{2a} + \ln 2\right) \\ &\geq -e^{\gamma x + \ln \ln x + O(1)}. \end{aligned}$$

□

Now we go back to prove Lemma 3.1:

Proof of Lemma 3.1. By the integral equation (2.1) satisfied by f (and symmetry in u about $u = 1/2$), for arbitrary z and a we have

$$f(-z-a) = 2\int_{u=0}^{1/2} \int_{y \in \mathbb{R}} f(y) f\left(\frac{-z-a-g(u)-(1-u)y}{u}\right) \frac{1}{u}\, dy\, du. \tag{3.1}$$

Since f is everywhere positive, we can get a lower bound on $f(-z-a)$ by restricting the range of integration in (3.1). Therefore,

$$f(-z-a) \geq 2\int_{u=\frac{1}{2}-\frac{\epsilon}{2}}^{1/2} \int_{y=-z}^{-z+\epsilon^2} f(y) f\left(\frac{-z-a-g(u)-(1-u)y}{u}\right) \frac{1}{u}\, dy\, du. \tag{3.2}$$

We claim that in this integral region, we have $\frac{-z-a-g(u)-(1-u)y}{u} \geq -z$, which is equivalent to $y + z \leq \frac{-a-g(u)}{1-u}$. Here is a proof. Observe that when ϵ is small enough and $u \in [\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2}]$, we have

$$\begin{aligned} \frac{-a-g(u)}{1-u} &\geq \frac{g\left(\frac{1}{2}-\epsilon\right) - g\left(\frac{1}{2}-\frac{\epsilon}{2}\right)}{\frac{1}{2}+\frac{\epsilon}{2}} \\ &\geq \frac{\frac{\epsilon}{2}\left|g'\left(\frac{1}{2}-\frac{\epsilon}{2}\right)\right|}{\frac{1}{2}+\frac{\epsilon}{2}} = \frac{\epsilon}{1+\epsilon}\left|2\ln\left(1-\frac{2\epsilon}{1+\epsilon}\right)\right| \\ &\geq \frac{4\epsilon^2}{(1+\epsilon)^2} \geq \epsilon^2. \end{aligned}$$

Also, in this integral region we have $y + z \leq \epsilon^2$. So we conclude that $y + z \leq \frac{-a-g(u)}{1-u}$.

Next, we claim that $\frac{-z-a-g(u)-(1-u)y}{u} \leq 0$ in this integral region if z is large enough. Here is a proof. Let $\frac{-z-a-g(u)-(1-u)y}{u} = -z + \delta$ with $\delta \geq 0$. Then in the integral region we have $0 \leq y + z = \frac{-a-g(u)-u\delta}{1-u}$. Therefore

$$\begin{aligned} \delta \leq \frac{-a-g(u)}{u} \leq \frac{-a-g\left(\frac{1}{2}\right)}{\frac{1}{2}-\frac{\epsilon}{2}} &= \frac{2}{1-\epsilon}\left[g\left(\frac{1}{2}-\epsilon\right) - g\left(\frac{1}{2}\right)\right] \\ &\leq \frac{2\epsilon}{1-\epsilon}\left|2\ln\left(1-\frac{4\epsilon}{1+2\epsilon}\right)\right| \\ &\leq 19\epsilon^2, \end{aligned}$$

where the last inequality can be verified to hold for $\epsilon < 1/10$. That means if we pick z large enough, for example, $z \geq 20\epsilon^2$, then $\frac{-z-a-g(u)-(1-u)y}{u} = -z + \delta$ will be negative. It can also be verified that $a \geq 30\epsilon^2$ for $\epsilon < 1/10$.

Now consider $\epsilon < 1/10$, an integer $k \geq 3$, $z \in [(k-2)a, (k-1)a]$, and $x = z + a \in [(k-1)a, ka]$. Noting $z \geq a \geq 30\epsilon^2 > 20\epsilon^2$, by (3.2) we have

$$f(-x) \geq 2 \cdot \frac{\epsilon}{2} \cdot m_z^2 \cdot \epsilon^2 \cdot 2 \geq 2\epsilon^3 m_{(k-1)a}^2.$$

Further, for $x \in [0, (k-1)a]$ we have

$$f(-x) \geq m_{(k-1)a} > 2\epsilon^3 m_{(k-1)a}^2$$

since $2\epsilon^3 < 1$ and $m_{(k-1)a} \leq 1$ by definition. Combine these two facts, we can conclude that for $x \in [0, ka]$ we have $f(-x) \geq 2\epsilon^3 m_{(k-1)a}^2$. This implies the recurrence inequality

$$m_{ka} \geq 2\epsilon^3 m_{(k-1)a}^2.$$

The desired inequality follows by iterating:

$$m_{ka} \geq \left(2\epsilon^3\right)^{2^{k-2}-1} m_{2a}^{2^{k-2}} \geq \left(2\epsilon^3 \cdot m_{2a}\right)^{2^{k-2}}. \qquad \square$$

4 Right tail lower bound on f

Once again we use an iterative approach to derive our right-tail lower bound on f in Theorem 1.1. The following key lemma is established from a recurrence inequality. Define

$$c := 2[F(1) - F(0)] \in (0, 2)$$

and

$$m_z := \min_{x \in [0,z]} f(x), \quad z \geq 0.$$

Lemma 4.1. *Suppose $b \in [0, 1)$ and that $\delta \in (0, 1/2)$ is sufficiently small that $g(\delta) \geq b$. Then for any integer $k \geq 1$ satisfying*

$$2 + (k-1)b \leq [g(\delta) - b]/\delta$$

we have

$$m_{2+kb} \geq (c\delta)^{k-1} m_3.$$

We delay the proof of Lemma 4.1 in order to show next how the lemma leads us to the desired lower bound in (1.4) on the right tail of f.

Proposition 4.2. *As $x \to \infty$ we have*

$$f(x) \geq \exp[-x \ln x - x \ln \ln x + O(x)].$$

Proof. Given $x \geq 3$ suitably large, we will show next that we can apply Lemma 4.1 for suitably chosen $b > 0$ and δ and $k = \lceil (x-2)/b \rceil \geq 2$. Then, by the lemma,

$$f(x) \geq m_{2+kb} \geq (c\delta)^{k-1} m_3 \geq (c\delta)^{(x-2)/b} m_3, \tag{4.1}$$

and we will use (4.1) to establish the proposition.

We make the same choices of δ and b as in [3, Sec. 4], namely, $\delta = 1/(x \ln x)$ and $b = 1 - (2/\ln x)$. To apply Lemma 4.1, we need to check that $g(\delta) \geq b$ and $2 + (k-1)b \leq [g(\delta) - b]/\delta$, for the latter of which it is sufficient that $x \leq [g(\delta) - b]/\delta$. Indeed, if x is sufficiently large, then

$$g(\delta) \geq 1 + 3\delta \ln \delta = 1 - \tfrac{3}{x \ln x}(\ln x + \ln \ln x) \geq 1 - \tfrac{4}{x},$$

where the elementary first inequality is (4.1) in [3], and so

$$g(\delta) - b \geq \tfrac{2}{\ln x} - \tfrac{4}{x} \geq \tfrac{1}{\ln x} > 0$$

and

$$\frac{g(\delta) - b}{\delta} \geq \frac{1/\ln x}{1/(x \ln x)} = x.$$

Finally, we use (4.1) to establish the proposition. Indeed,

$$\begin{aligned} -\ln f(x) &\leq \tfrac{x-2}{b} \ln(\tfrac{1}{c\delta}) - \ln m_3 \\ &\leq \tfrac{x}{1-(2/\ln x)} [\ln(x \ln x) + \ln(\tfrac{1}{c})] - \ln m_3 \\ &= \tfrac{x}{1-(2/\ln x)} \ln(x \ln x) + O(x). \end{aligned}$$

But

$$\begin{aligned} &\frac{x}{1-(2/\ln x)} \ln(x \ln x) \\ &\qquad = x \left[1 + \frac{2}{\ln x} + O\left(\frac{1}{(\log x)^2}\right)\right] (\ln x + \ln \ln x) \\ &\qquad = (x \ln x) \left[1 + \frac{2}{\ln x} + O\left(\frac{1}{(\log x)^2}\right)\right] \left(1 + \frac{\ln \ln x}{\ln x}\right) \\ &\qquad = (x \ln x) \left[1 + \frac{\ln \ln x}{\ln x} + \frac{2}{\ln x} + \frac{2 \ln \ln x}{(\ln x)^2} + O\left(\frac{1}{(\log x)^2}\right)\right] \\ &\qquad = x \ln x + x \ln \ln x + 2x + \frac{2x \ln \ln x}{\ln x} + O\left(\frac{x}{\log x}\right) \\ &\qquad = x \ln x + x \ln \ln x + O(x). \end{aligned}$$

So

$$-\ln f(x) \leq x \ln x + x \ln \ln x + O(x),$$

as claimed. □

Now we go back to prove Lemma 4.1, but first we need two preparatory results.

Lemma 4.3. *Suppose* $z \geq 2$, $b \geq 0$, *and* $\delta \in (0, 1/2)$ *satisfy* $g(\delta) \geq b$ *and* $z \leq [g(\delta) - b]/\delta$. *Then*

$$f(z + b) \geq c\,\delta\, m_z.$$

Proof. By the integral equation (2.1) satisfied by f (and symmetry in u about $u = 1/2$), for arbitrary z and b we have

$$f(z + b) = 2 \int_{u=0}^{1/2} \int_{y \in \mathbb{R}} f(y) f\left(\frac{z + b - g(u) - (1-u)y}{u}\right) \frac{1}{u}\, dy\, du.$$

Since f is positive everywhere, a lower bound on $f(z + b)$ can be achieved by shrinking the region of integration:

$$\begin{aligned} f(z + b) &\geq 2 \int_{u=0}^{\delta} \int_{y=0}^{z} f(y) f\left(\frac{z + b - g(u) - (1-u)y}{u}\right) \frac{1}{u}\, dy\, du \\ &\geq 2 m_z \int_{u=0}^{\delta} \int_{y=0}^{z} f\left(\frac{z + b - g(u) - (1-u)y}{u}\right) \frac{1}{u}\, dy\, du \\ &= 2 m_z \int_{u=0}^{\delta} \int_{\xi = z + \frac{b - g(u)}{u}}^{\frac{z + b - g(u)}{u}} f(\xi) \frac{1}{1-u}\, d\xi\, du. \end{aligned} \tag{4.2}$$

The equality comes from a change of variables. We next claim that the integral of integration for ξ contains $(0, z-1)$, and then the desired result follows. Indeed, if $u \in (0, \delta)$ and $\xi \in (0, z-1)$ then

$$\xi < z - 1 < \tfrac{z-1}{u} \leq \tfrac{z+b-g(u)}{u},$$

where the last inequality holds because $b \geq 0$ and $g(u) \leq 1$; and, because $g(u) \geq g(\delta)$ and $g(\delta) \geq b$ and $z \leq [g(\delta) - b]/\delta$, we have

$$\begin{aligned}\xi > 0 &= z + \tfrac{b-g(u)}{u} - [z + \tfrac{b-g(u)}{u}] \geq z + \tfrac{b-g(u)}{u} - [z + \tfrac{b-g(\delta)}{u}] \\ &\geq z + \tfrac{b-g(u)}{u} - [z + \tfrac{b-g(\delta)}{\delta}] \geq z + \tfrac{b-g(u)}{u}.\end{aligned}$$

□

Lemma 4.4. *Suppose $b \geq 0$ and that $\delta \in (0, 1/2)$ is sufficiently small that $g(\delta) \geq b$. Then for any integer $k \geq 2$ satisfying*

$$2 + (k-1)b \leq [g(\delta) - b]/\delta$$

we have

$$m_{2+kb} \geq c\,\delta\, m_{2+(k-1)b}.$$

Proof. For $y \in [2 + (k-1)b, 2 + kb]$, application of Lemma 4.3 with $z = y - b$ yields

$$f(y) \geq c\,\delta\, m_{y-b} \geq c\,\delta\, m_{2+(k-1)b}.$$

Also, for $y \in [0, 2 + (k-1)b]$ we certainly have

$$f(y) \geq m_{2+(k-1)b} > c\,\delta\, m_{2+(k-1)b}.$$

The result follows. □

We are now ready to complete this section by proving Lemma 4.1.

Proof of Lemma 4.1. By iterating the recurrence inequality of Lemma 4.4, it follows that

$$m_{2+kb} \geq (c\,\delta)^{k-1} m_{2+b}.$$

Lemma 4.1 then follows since $b < 1$. □

5 Left tail bounds for tail suprema of absolute derivatives

From Section 3 (respectively, Section 4) we know the left-tail lower bound of (1.3) [resp., the right-tail lower bound of (1.4)]. In this section we establish the left-tail bounds of (1.3) and (1.6), and in the next section we do the same for right tails.

5.1 Lower bounds

As discussed in Remark 1.3(a), in light of the main theorem of Janson [3] and our Section 3, to finish our treatment of left-tail lower bounds we need only prove the lower bound in (1.6) for fixed $k \geq 2$. For that, choose any x and apply the Landau–Kolmogorov Lemma 2.1, bounding the function $\underline{F}'(\cdot) = -f(-\cdot)$ in terms of the functions $\underline{F}$ and $\underline{F}^{(k)}$. This gives

$$f(-x) \leq \|\underline{F}'\|_x \leq c_{k,1}\, \|\underline{F}\|_x^{(k-1)/k}\, \|\underline{F}^{(k)}\|_x^{1/k},$$

i.e.,

$$\|\underline{F}^{(k)}\|_x \geq c_{k,1}^{-k}\, \|\underline{F}\|_x^{-(k-1)}\, [f(-x)]^k.$$

But recall

$$c_{k,1} \leq e^2 k/4, \quad \|\underline{F}\|_x \leq \exp\left[-e^{\gamma x + O(1)}\right], \quad f(-x) \geq \exp\left[-e^{\gamma x + \ln\ln x + O(1)}\right].$$

Plugging in these bounds, we obtain the desired result.

5.2 Upper bounds

The left-tail upper bounds in (1.6) of Theorem 1.2 can be written in the equivalent form

$$\lambda_k := \limsup_{x\to\infty} \left[\gamma x - \ln\left(-\ln \|\underline{F}^{(k)}\|_x\right)\right] < \infty; \tag{5.1}$$

note also that the left-tail upper bound in (1.3) of Theorem 1.1 follows from $\lambda_1 < \infty$. As discussed in Remark 1.3(a), (5.1) is known for $k = 0$ from Janson [3]. So to finish our treatment of left-tail upper bounds in Theorems 1.1–1.2 we need only prove (5.1) for $k \geq 1$.

In this subsection we prove the following stronger Proposition 5.1, which implies that λ_k is non-increasing in $k \geq 0$ and therefore that $\lambda_k < \infty$ for every k. In preparation for the proof, see the definition of μ_j in (5.2) and note that if $\mu_j \leq 0$ for $j = 0, \dots, k-1$, then λ_j is non-increasing for $j = 0, \dots, k$; in particular, (5.1) then holds.

Proposition 5.1. *For each fixed $k \geq 0$ we have*

$$\mu_k := \limsup_{x\to\infty} \left[-\ln\left(-\ln \|\underline{F}^{(k+1)}\|_x\right) + \ln\left(-\ln \|\underline{F}^{(k)}\|_x\right)\right] \leq 0. \tag{5.2}$$

Proof. We proceed by induction on k. Choosing any x and applying the Landau–Kolmogorov inequality Lemma 2.1 to the function $h = \underline{F}^{(k)}$, we find for $n \geq 2$ that

$$\|\underline{F}^{(k+1)}\|_x \leq \tfrac{1}{4} e^2 n \, \|\underline{F}^{(k)}\|_x^{1-(1/n)} \, \|\underline{F}^{(k+n)}\|_x^{1/n}.$$

We can bound the norm $\|\underline{F}^{(k+n)}\|_x$ using Proposition 2.4 simply by

$$a_{n,k} := 2^{(k+n-1)^2 + 10(k+n-1) + 17}. \tag{5.3}$$

Thus the argument of the $\limsup$ in (5.2) can be bounded above by

$$-\ln\left[1 - \frac{1}{n} - \frac{2 - \ln 4 + \ln n + n^{-1} \ln a_{n,k}}{-\ln \|\underline{F}^{(k)}\|_x}\right].$$

By Janson's bound giving $\lambda_0 < \infty$ if $k = 0$ and by induction on k if $k \geq 1$, we know that (5.1) holds. Thus, letting $n \equiv n(x) \to \infty$ with $n(x) = o(e^{\gamma x})$, the claimed inequality follows. □

Remark 5.2. According to Remark 1.3, it is natural to conjecture that for every k the $\limsup$ in (5.1) is a limit and equals $-c_3$ and hence the $\limsup$ in (5.2) is a vanishing limit.

6 Right tail bounds for tail suprema of absolute derivatives

In this section we establish the right-tail bounds of (1.4) and (1.7).

6.1 Lower bounds

As discussed in Remark 1.3(a), in light of the main theorem of [3] and our Section 4, to finish our treatment of right-tail lower bounds we need only prove the lower bound in (1.7) for fixed $k \geq 2$. For that, proceed using the Landau–Kolmogorov Lemma 2.1 as in Section 5.1 to obtain

$$\|\overline{F}^{(k)}\|_x \geq c_{k,1}^{-k} \, \|\overline{F}\|_x^{-(k-1)} \, [f(x)]^k.$$

But recall

$$c_{k,1} \leq e^2 k/4, \quad \|\overline{F}\|_x \leq \exp[-x \ln x + O(x)],$$
$$f(x) \geq \exp\left[-x \ln x - x \ln\ln x + O(x)\right].$$

Plugging in these bounds, we obtain the desired result.

6.2 Upper bounds

The right-tail upper bounds in (1.7) of Theorem 1.2 can be written in the equivalent form

$$\rho_k := \limsup_{x\to\infty} \, x^{-1} \left(x \ln x + \ln \|\overline{F}^{(k)}\|_x \right) < \infty; \tag{6.1}$$

note also that the right-tail upper bound in (1.4) of Theorem 1.1 follows from $\rho_1 < \infty$. As discussed in Remark 1.3(a), (6.1) is known for $k = 0$ from Janson [3]. So to finish our treatment of right-tail upper bounds in Theorems 1.1–1.2 we need only prove (6.1) for $k \geq 1$.

In this subsection we prove the next stronger Proposition 6.1, a right-tail analogue of Proposition 5.1, and it then follows by choosing $r(x) \equiv x$ that ρ_k is non-increasing in $k \geq 0$ and therefore that $\rho_k < \infty$ for every k.

Proposition 6.1. *Let r be a function satisfying $r(x) = \omega(\sqrt{x \log x})$ as $x \to \infty$. Then for each fixed $k \geq 0$ we have*

$$\sigma_k := \limsup_{x\to\infty} \, r(x)^{-1} \left(\ln \|\overline{F}^{(k+1)}\|_x - \ln \|\overline{F}^{(k)}\|_x \right) \leq 0. \tag{6.2}$$

Proof. Proceeding as in the proof of Proposition 5.1, for any x and any $n \geq 2$ we have

$$\|\overline{F}^{(k+1)}\|_x \leq \tfrac{1}{4} e^2 n \, \|\overline{F}^{(k)}\|_x^{1-(1/n)} \, \|\overline{F}^{(k+n)}\|_x^{1/n};$$

we again bound the norm $\|\overline{F}^{(k+n)}\|_x$ by (5.3). Thus the argument of the $\limsup$ in (6.2) can be bounded above by

$$r(x)^{-1} \left[\tfrac{1}{n} (-\ln \|\overline{F}^{(k)}\|_x) + 2 - \ln 4 + \ln n + \tfrac{1}{n} \ln a_{n,k} \right].$$

By the right-tail *lower* bound for $\|\overline{F}^{(k)}\|_x$ in (1.7) (established in the preceding subsection), we know that

$$-\ln \|\overline{F}^{(k)}\|_x \leq x \ln x + (k \vee 1) x \ln \ln x + O(x) = (1 + o(1)) x \ln x.$$

Thus, letting $n \equiv n(x)$ satisfy $n(x) = \omega((x \log x)/r(x))$ and $n(x) = o(r(x))$, the claimed inequality follows. □

Remark 6.2. According to Remark 1.3, it is natural to conjecture that for every k we have $\rho_k = -\infty$ and the $\limsup$ in (6.2) with $r(x) \equiv x$ is a vanishing limit.

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