

GROWTH-FRAGMENTATION PROCESSES IN BROWNIAN MOTION INDEXED BY THE BROWNIAN TREE

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We consider the model of Brownian motion indexed by the Brownian tree. For every $r \geq 0$ and every connected component of the set of points where Brownian motion is greater than r , we define the boundary size of this component, and we then show that the collection of these boundary sizes evolves when r varies like a well-identified growth-fragmentation process. We then prove that the same growth-fragmentation process appears when slicing a Brownian disk at height r and considering the perimeters of the resulting connected components.

1. Introduction. The main goal of the present work is to prove that the collection of boundary sizes of excursions of Brownian motion indexed by the Brownian tree above a fixed level evolves according to a well-identified growth-fragmentation process when the level increases. Because of the close connections between Brownian motion indexed by the Brownian tree and the Brownian map or the Brownian disk, this result also implies that the collection of boundary sizes of the connected components of the set of points of a Brownian disk whose distance from the boundary is greater than r evolves according to the same growth-fragmentation process. The latter fact may be viewed as a continuous analog of a recent result of Bertoin, Curien and Kortchemski [6] identifying the growth-fragmentation process arising as the scaling limit for the collection of lengths of cycles obtained by slicing random Boltzmann triangulations with a boundary at a given height, when the size of the boundary grows to infinity. In fact, the growth-fragmentation process of [6] is the same as in our main results, and this strongly suggests that the results of [6] could be extended to more general planar maps with a boundary (see also [5] for related results).

In order to give a more precise description of our main results, we first need to recall the notion of Brownian motion indexed by the Brownian tree. The Brownian tree of interest here is a variant of Aldous' continuum random tree, which is also called the CRT. This tree is conveniently defined as the tree \mathcal{T}_ζ coded by a Brownian excursion $(\zeta_s)_{0 \leq s \leq \sigma}$ under the σ -finite Itô measure of positive excursions (see, e.g., [29], or Section 2.1 below for the definition of this coding). We write ρ for the root of \mathcal{T}_ζ , and we note that \mathcal{T}_ζ is canonically equipped with a volume measure $\text{vol}(\cdot)$. We then consider Brownian motion indexed by \mathcal{T}_ζ , which we denote by $(V_a)_{a \in \mathcal{T}_\zeta}$ —we sometimes also say that V_a is a Brownian label assigned to a . Informally, conditionally on \mathcal{T}_ζ , $(V_a)_{a \in \mathcal{T}_\zeta}$ is just the centered Gaussian process such that $V_\rho = 0$, and $\mathbb{E}[(V_a - V_b)^2] = d_\zeta(a, b)$ for every $a, b \in \mathcal{T}_\zeta$, where d_ζ is the distance on \mathcal{T}_ζ . A formal definition leads to certain technical difficulties because the indexing set is random, but these difficulties can be overcome easily using the formalism of snake trajectories as recalled in Section 2.1. Within this formalism, the Brownian tree \mathcal{T}_ζ , and the Brownian motion $(V_a)_{a \in \mathcal{T}_\zeta}$ are defined under a σ -finite measure \mathbb{N}_0 ; see Section 2.3 for more details. We note that both the CRT and Brownian motion indexed by the Brownian tree are important probabilistic objects that appear as scaling limits for several combinatorial models, interacting

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particle systems and statistical physics models (see the introduction of [1] for a few related references). Furthermore, Brownian motion indexed by the Brownian tree is very closely related to the measure-valued process called super-Brownian motion (see, in particular, [23]).

Let us now discuss growth-fragmentation processes, referring to [4] and [5] for additional details. The basic ingredient in the construction of a (self-similar) growth-fragmentation process is a self-similar Markov process $(X_t)_{t \geq 0}$ with values in $[0, \infty)$ and only negative jumps, which is stopped upon hitting 0. Suppose that $X_0 = z > 0$, and view $(X_t)_{t \geq 0}$ as the evolution in time of the mass of an initial particle called the Eve particle. At each time t where the process X has a jump, we consider that a new particle with mass $-\Delta X_t$ (a child of the Eve particle) is born, and the mass of this new particle evolves (from time t) again according to the law of the process X , independently of the evolution of the mass of the Eve particle. Then each child of the Eve particle has children at discontinuity times of its mass process, and so on. We consider that a particle dies when its mass vanishes. Under suitable assumptions (see [4]), we can make sense of the process $(\mathbf{X}(t))_{t \geq 0}$ giving for every time t the sequence (in nonincreasing order) of masses of all particles alive at that time (if there are only finitely many such particles, the sequence is completed by adding terms all equal to 0). The process \mathbf{X} is Markovian and is called the growth-fragmentation process with Eve particle process X . In the preceding description, the process starts from $(z, 0, 0, \dots)$, but we can get a more general initial value by considering several or countably many Eve particles that evolve independently—some assumption is needed on the initial values of these Eve particles so that at every time t the masses of the particles alive can be ranked in a nonincreasing sequence.

THEOREM 1. *Almost everywhere under the measure \mathbb{N}_0 , for every $r \geq 0$ and for every connected component \mathcal{C} of the open set $\{a \in \mathcal{T}_\zeta : V_a > r\}$, the limit*

$$|\partial\mathcal{C}| := \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \text{vol}(\{a \in \mathcal{C} : V_a < r + \varepsilon\})$$

exists in $(0, \infty)$ and is called the boundary size of \mathcal{C} . For every $r \geq 0$, let $\mathbf{X}(r)$ denote the sequence of boundary sizes of all connected components of $\{a \in \mathcal{T}_\zeta : V_a > r\}$ ranked in non-increasing order. Then, under \mathbb{N}_0 , the process $(\mathbf{X}(r))_{r \geq 0}$ is a growth-fragmentation process whose Eve particle process $(X_t)_{t \geq 0}$ can be described as follows. The process $(X_t)_{t \geq 0}$ is the self-similar Markov process with index $\frac{1}{2}$ which in the case $X_0 = 1$ can be represented as

$$X_t = \exp(\xi(\chi(t))),$$

where $(\xi(s))_{s \geq 0}$ is the Lévy process with only negative jumps and Laplace exponent

$$(1) \quad \psi(\lambda) = \sqrt{\frac{3}{2\pi}} \left(-\frac{8}{3}\lambda + \int_{-\log 2}^0 (e^{\lambda y} - 1 - \lambda(e^y - 1)) e^{-3y/2} (1 - e^y)^{-5/2} dy \right),$$

and $(\chi(t))_{t \geq 0}$ is the time change

$$(2) \quad \chi(t) = \inf \left\{ s \geq 0 : \int_0^s e^{\xi(v)/2} dv > t \right\}.$$

In the setting of Theorem 1, we consider the infinite measure \mathbb{N}_0 , but the statement still makes sense by conditioning on the initial value $\mathbf{X}(0)$. The representation of the self-similar Markov process X in terms of the Lévy process ξ is the classical Lamperti representation of self-similar Markov processes [21]. We note that the process ξ drifts to $-\infty$ and $\chi(t) = \infty$ for $t \geq H_0 := \int_0^\infty e^{\xi(v)/2} dv$, which simply means that X_t is absorbed at 0 at time H_0 .

It is interesting to relate the growth-fragmentation process of Theorem 1 to the local times of the process $(V_a)_{a \in \mathcal{T}_\zeta}$. It is known [9] (see also [33] for closely related results concerning

super-Brownian motion) that there exists, $\mathbb{N}_0(d\omega)$ a.e., a continuous function $(\mathcal{L}_x, x \in \mathbb{R})$ such that, for every nonnegative measurable function f on \mathbb{R} ,

$$\int_{\mathcal{T}_\zeta} \text{vol}(da) f(V_a) = \int_{\mathbb{R}} dx f(x) \mathcal{L}_x,$$

and we call \mathcal{L}_x the local time at level x . Then, for every $r > 0$, if $\mathcal{N}_\varepsilon^r$ denotes the number of connected components of $\{a \in \mathcal{T}_\zeta : V_a > r\}$ with boundary size greater than ε , Proposition 26 below gives

$$\varepsilon^{3/2} \mathcal{N}_\varepsilon^r \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{\sqrt{6\pi}} \mathcal{L}_r, \quad \mathbb{N}_0 \text{ a.e.}$$

In other words, the suitably rescaled number of fragments of $\mathbf{X}(r)$ with size greater than ε converges to \mathcal{L}_r . As a side remark, one might expect the process $(\mathcal{L}_r)_{r \geq 0}$ to be Markovian under \mathbb{N}_0 , by analogy with the classical Ray–Knight theorems for local times of linear Brownian motion. This is not the case, but the previous display shows that \mathcal{L}_r is a function of the Markov process $\mathbf{X}(r)$ which obviously contains more information than the local time.

Thanks to the excursion theory developed in [1], we can in fact deduce Theorem 1 from a simpler statement valid under the “positive Brownian snake excursion measure” \mathbb{N}_0^* introduced and studied in [1]. We refer to Section 2.3 for more details, but note that we can still make sense of the “genealogical tree” \mathcal{T}_ζ and the “labels” $V_a, a \in \mathcal{T}_\zeta$ under \mathbb{N}_0^* . However, we now have $V_a \geq 0$ for every $a \in \mathcal{T}_\zeta$, and more precisely the labels V_b are positive along the ancestral line of a , except at the root and possibly at a . Informally, the measure \mathbb{N}_0^* describes the subtree and the labels corresponding under \mathbb{N}_0 to a connected component of the set of points with positive labels. One can make sense under \mathbb{N}_0^* of the boundary size

$$\mathcal{Z}_0^* := \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \text{vol}(\{a \in \mathcal{T}_\zeta : V_a < \varepsilon\}), \quad \mathbb{N}_0^* \text{ a.e.}$$

and define the conditional probability measures $\mathbb{N}_0^{*,z}(\cdot) = \mathbb{N}_0^*(\cdot \mid \mathcal{Z}_0^* = z)$ for every $z > 0$.

THEOREM 2. *Let $z > 0$. Almost surely under the measure $\mathbb{N}_0^{*,z}$, for every $r \geq 0$ and for every connected component \mathcal{C} of the open set $\{a \in \mathcal{T}_\zeta : V_a > r\}$, the limit*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \text{vol}(\{a \in \mathcal{C} : V_a < r + \varepsilon\})$$

exists in $(0, \infty)$ and is called the boundary size of \mathcal{C} . For every $r \geq 0$, let $\mathbf{Y}(r)$ denote the sequence of boundary sizes of all connected components of $\{a \in \mathcal{T}_\zeta : V_a > r\}$ ranked in nonincreasing order. Then, under $\mathbb{N}_0^{,z}$, the process $(\mathbf{Y}(r))_{r \geq 0}$ is distributed as the growth-fragmentation process of Theorem 1 with initial value $\mathbf{Y}(0) = (z, 0, 0, \dots)$.*

Theorem 1 will be derived as a straightforward consequence of Theorem 2 and the excursion theory of [1]. Both Theorems 1 and 2 have direct applications to the models of random geometry known as the Brownian map and the Brownian disk. Recall that the Brownian map is a random compact metric space homeomorphic to the sphere \mathbb{S}^2 , which is the scaling limit of various classes of random planar maps equipped with the graph distance (see, in particular, [25, 30]). Similarly, the Brownian disk is a random compact metric space homeomorphic to the closed unit disk of the plane, which appears as the scaling limit of rescaled Boltzmann quadrangulations with a boundary, when the size of the boundary grows to infinity (see [7, 8, 19]). We note that the papers [7, 8] consider Brownian disks with fixed boundary size and volume, but in the present work we will be interested in the free Brownian disk [8], Section 1.5, which has a fixed boundary size but a random volume. Let us write \mathbb{D}_z for the free Brownian disk with boundary size $z > 0$. The space \mathbb{D}_z is equipped with a volume measure

denoted by $\mathbf{V}(dx)$. The boundary $\partial\mathbb{D}_z$ may be defined as the set of all points of \mathbb{D}_z that have no neighborhood homeomorphic to the open unit disk, and for every $x \in \mathbb{D}_z$, we write $H(x)$ for the “height” of x , meaning the distance from x to the boundary $\partial\mathbb{D}_z$.

THEOREM 3. *Almost surely, for every $r \geq 0$, for every connected component \mathcal{C} of $\{x \in \mathbb{D}_z : H(x) > r\}$, the limit*

$$|\partial\mathcal{C}| := \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \mathbf{V}(\{x \in \mathcal{C} : H(x) < r + \varepsilon\})$$

exists and is called the boundary size of \mathcal{C} . For every $r \geq 0$, let $\mathbf{Z}(r)$ denote the sequence of boundary sizes of all connected components of $\{x \in \mathbb{D}_z : H(x) > r\}$ ranked in nonincreasing order. Then the process $(\mathbf{Z}(r))_{r \geq 0}$ is distributed as the growth-fragmentation process of Theorem 1 with initial value $\mathbf{Z}(0) = (z, 0, 0, \dots)$.

As the similarity between the two statements suggests, Theorem 3 is closely related to Theorem 2, and in fact can be derived from the latter result thanks to the construction of the free Brownian disk (with boundary size z) from a snake trajectory distributed according to $\mathbb{N}_0^{*,z}$, which is developed in [28]. Similarly, we could use Theorem 1 to derive a result analogous to Theorem 3 for the free Brownian map, thanks to the construction of the latter metric space from a snake trajectory distributed according to \mathbb{N}_0 (see, e.g., [28], Section 3). Rather than writing down this statement about the free Brownian map, we give in Section 11 an analog of Theorem 3 for the Brownian plane, which is an infinite-volume version of the Brownian map that has been shown [10, 14] to be the universal scaling limit of infinite random lattices such as the UIPT or the UIPQ. Theorem 24 below shows that the collection of boundary sizes of the connected components of the complement of the ball of radius r centered at the root of the Brownian plane evolves like the same growth-fragmentation process with indefinite growth starting from 0 (see [5], Section 4.2, for a thorough discussion of this process).

We next state another result for the Brownian disk, which is closely related to Theorem 3.

THEOREM 4. *Let $r > 0$. On the event $\{\sup\{H(x) : x \in \mathbb{D}_z\} > r\}$, let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \dots$ be the connected components of $\{x \in \mathbb{D}_z : H(x) > r\}$ ranked in nonincreasing order of their boundary sizes, and for every $j = 1, 2, \dots$, let d_j denote the intrinsic metric induced by the Brownian disk metric on the open set \mathcal{C}_j . Then, a.s. on the event $\{\sup\{H(x) : x \in \mathbb{D}_z\} > r\}$, for every $j = 1, 2, \dots$ the metric d_j has a continuous extension to the closure $\bar{\mathcal{C}}_j$ of \mathcal{C}_j in \mathbb{D}_z , and this extension is a metric on $\bar{\mathcal{C}}_j$. Furthermore, conditionally on the sequence of boundary sizes $(|\partial\mathcal{C}_1|, |\partial\mathcal{C}_2|, \dots)$, the metric spaces $(\bar{\mathcal{C}}_1, d_1), (\bar{\mathcal{C}}_2, d_2), \dots$ are independent free Brownian disks with respective boundary sizes $|\partial\mathcal{C}_1|, |\partial\mathcal{C}_2|, \dots$*

In the same way as Theorem 3 follows from Theorem 2, Theorem 4 is a consequence of a statement (Theorem 23 below) that describes the conditional distribution of the snake trajectories corresponding to the “excursions above level h ” under $\mathbb{N}_0^{*,z}$, conditionally on the boundary sizes of these excursions. One might expect that Theorem 23, which essentially corresponds to the branching property of growth-fragmentation processes, would be a basic tool for the proof of Theorem 2, but in fact our proof of Theorem 2 does not use this branching property. We also note that Theorem 4 is a Brownian disk analog of a result of [28] showing that the connected components of the complement of a ball in the Brownian map are independent Brownian disks conditionally on their volumes and boundary sizes.

Let us finally mention an interesting corollary to our results.

COROLLARY 5. *There exist positive constants \mathbf{c}_1 and \mathbf{c}_2 such that, for every $r \geq 1$,*

$$\mathbf{c}_1 r^{-6} \leq \mathbb{N}_0^{*,1} \left(\sup_{a \in \mathcal{T}_\zeta} V_a > r \right) = \mathbb{P} \left(\sup_{x \in \mathbb{D}_1} H(x) > r \right) \leq \mathbf{c}_2 r^{-6}.$$

Corollary 5 immediately follows from Theorem 2 and Theorem 3, by using the asymptotics for the extinction time of growth-fragmentation processes found in [5], Corollary 4.5.

The proof of Theorem 2 occupies much of the remaining part of the paper. Let us briefly outline the main steps of this proof. For every $a \in \mathcal{T}_\zeta$ such that $V_a > 0$, one can define a function $(Z_r^{(a)})_{0 \leq r < V_a}$ such that, for every $r \in [0, V_a)$, $Z_r^{(a)}$ is the boundary size of the connected component of $\{b \in \mathcal{T}_\zeta : V_b > r\}$ that contains a (see Proposition 15 below). The function $r \mapsto Z_r^{(a)}$ is càdlàg (right continuous with left limits) with only negative jumps, and every discontinuity time r_0 of this function corresponds to a “splitting” of the connected component containing a into two components, namely the one containing a , which has boundary size $Z_{r_0}^{(a)}$, and another one with boundary size $|\Delta Z_{r_0}^{(a)}|$. It is not a priori obvious that a splitting cannot yield more than two components, but this follows from the fact that local minima of the process V are distinct; see Section 3.1 below. It turns out (Proposition 16) that there exists a unique $a^\bullet \in \mathcal{T}_\zeta$, called the terminal point of the locally largest evolution, such that, for every discontinuity time r_0 of $r \mapsto Z_r^{(a^\bullet)}$, we have $Z_{r_0}^{(a^\bullet)} > |\Delta Z_{r_0}^{(a^\bullet)}|$ (meaning that a^\bullet “stays” in the component with the larger boundary size) and V_{a^\bullet} is maximal among the labels of points satisfying the latter property. Furthermore, the distribution of $(Z_r^{(a^\bullet)})_{0 \leq r < V_{a^\bullet}}$ is the law of the process X of Theorem 1 up to its hitting time of 0 (Proposition 17). The process $(Z_r^{(a^\bullet)})_{0 \leq r < V_{a^\bullet}}$ thus plays the role of the evolution of the mass of the Eve particle. Furthermore, one verifies that, conditionally on $(Z_r^{(a^\bullet)})_{0 \leq r < V_{a^\bullet}}$, for every discontinuity time r_0 , the connected component that splits off the one containing a^\bullet at time r_0 is represented by a snake trajectory distributed according to $\mathbb{N}_0^{*, |\Delta Z_{r_0}^{(a^\bullet)}|}$ (Proposition 19). This provides the recursive structure needed to identify the process $\mathbf{Y}(r)$ of Theorem 2 as a growth-fragmentation process.

We finally mention a few recent papers that are related to the present work. We refer to [4, 5, 32] for the theory of growth-fragmentation processes. As we already mentioned, Theorem 3 can be viewed as a continuous version of the main result of [6]. In addition to [7, 8, 19], free Brownian disks are discussed in the paper [31], which develops an axiomatic characterization of the Brownian map as part of a program aiming to equip the Brownian map with a canonical conformal structure. Brownian disks also play an important role in the recent papers [17, 18] of Gwynne and Miller motivated by the study of statistical physics models on random planar maps. Finally, we observe that there is an interesting analogy between Theorem 1 and the fragmentation process occurring when cutting the CRT at a fixed height. According to [3], the sequence of volumes of the connected components of the complement of the ball of radius r centered at the root in the CRT is a self-similar fragmentation process whose dislocation measure has the form $(2\pi)^{-1/2}(x(1-x))^{-3/2} dx$. Notice that the Lévy measure of the process ξ of Theorem 1 is the pushforward of the measure $\mathbf{1}_{[1/2, 1]}(x)\sqrt{3/2\pi}(x(1-x))^{-5/2} dx$ under the mapping $x \mapsto \log x$.

The paper is organized as follows. Section 2 gives a number of preliminaries. In particular, we recall the formalism of snake trajectories, which provides a convenient set-up for the study of Brownian motion indexed by the Brownian tree. We also give a “rerooting” representation of the measure $\mathbb{N}_0^{*, z}$, which is a key tool in several subsequent proofs. Section 3 discusses the connected components of the tree \mathcal{T}_ζ above a fixed level and also the components “above the minimum”: the independence and distributional properties of the latter have been studied already in the paper [1] and play a basic role in the proof of Theorem 2. Section 4 is devoted to the existence and properties of the boundary size processes $(Z_r^{(a)})_{0 \leq r < V_a}$. In this section, we rely on the theory of exit measures for the Brownian snake [23]. Section 5 introduces the locally largest evolution, and Section 6 identifies the law of the associated boundary size process (Proposition 17). A key tool for this identification is Proposition 11, which gives the

distribution under \mathbb{N}_0 of the exit measure process time-reversed at its last visit to $z > 0$. Section 7 studies the excursions from the locally largest evolution. Roughly speaking, this study provides the recursive structure that shows that the “children” of the Eve particle evolve according to the same Markov process. Theorems 1 and 2 are then proved in Section 8, and Theorem 3 is derived from Theorem 2 in Section 9. Section 10 gives the proof of Theorem 4. Finally, Section 11 contains some complements. In particular, we provide a direct derivation of the cumulant function associated with our growth-fragmentation processes, which is independent of the proof of the main results. We also discuss the analog of Theorem 3 for the Brownian plane, and we investigate the relations between local times of $(V_a)_{a \in \mathcal{T}_\zeta}$ and the growth-fragmentation process of Theorem 1. The Appendix gives the proof of two technical results.

2. Preliminaries.

2.1. *Snake trajectories.* Most of this work is devoted to the study of random processes indexed by continuous random trees. The formalism of snake trajectories, which has been introduced in [1], provides a convenient framework for this study, and we recall the main definitions that will be needed below.

A (one-dimensional) finite path w is a continuous mapping $w : [0, \zeta] \rightarrow \mathbb{R}$, where the number $\zeta = \zeta_{(w)}$ is called the lifetime of w . We let \mathcal{W} denote the space of all finite paths, which is a Polish space when equipped with the distance

$$d_{\mathcal{W}}(w, w') = |\zeta_{(w)} - \zeta_{(w')}| + \sup_{t \geq 0} |w(t \wedge \zeta_{(w)}) - w'(t \wedge \zeta_{(w')})|.$$

The endpoint or tip of the path w is denoted by $\widehat{w} = w(\zeta_{(w)})$. We set $\mathcal{W}_0 = \{w \in \mathcal{W} : w(0) = 0\}$. The trivial element of \mathcal{W}_0 with zero lifetime is identified with the point 0 of \mathbb{R} . Occasionally, we will use the notation $\underline{w} = \min\{w(t) : 0 \leq t \leq \zeta_{(w)}\}$.

DEFINITION 6. A snake trajectory (with initial point 0) is a continuous mapping $s \mapsto \omega_s$ from \mathbb{R}_+ into \mathcal{W}_0 which satisfies the following two properties:

- (i) We have $\omega_0 = 0$ and the number $\sigma(\omega) := \sup\{s \geq 0 : \omega_s \neq 0\}$, called the duration of the snake trajectory ω , is finite (by convention $\sigma(\omega) = 0$ if $\omega_s = 0$ for every $s \geq 0$).
- (ii) For every $0 \leq s \leq s'$, we have

$$\omega_s(t) = \omega_{s'}(t) \quad \text{for every } t \in \left[0, \min_{s \leq r \leq s'} \zeta_{(\omega_r)}\right].$$

We will write \mathcal{S}_0 for the set of all snake trajectories. If $\omega \in \mathcal{S}_0$, we often write $W_s(\omega) = \omega_s$ and $\zeta_s(\omega) = \zeta_{(\omega_s)}$ for every $s \geq 0$. The set \mathcal{S}_0 is equipped with the distance

$$d_{\mathcal{S}_0}(\omega, \omega') = |\sigma(\omega) - \sigma(\omega')| + \sup_{s \geq 0} d_{\mathcal{W}}(W_s(\omega), W_s(\omega')).$$

A snake trajectory ω is completely determined by the knowledge of the lifetime function $s \mapsto \zeta_s(\omega)$ and of the tip function $s \mapsto \widehat{W}_s(\omega)$; see [1], Proposition 8.

Let $\omega \in \mathcal{S}_0$ be a snake trajectory and $\sigma = \sigma(\omega)$. The lifetime function $s \mapsto \zeta_s(\omega)$ codes a compact \mathbb{R} -tree, which will be denoted by \mathcal{T}_ζ and called the *genealogical tree* of the snake trajectory. This \mathbb{R} -tree is the quotient space $\mathcal{T}_\zeta := [0, \sigma] / \sim$ of the interval $[0, \sigma]$ for the equivalence relation

$$s \sim s' \quad \text{if and only if} \quad \zeta_s = \zeta_{s'} = \min_{s \wedge s' \leq r \leq s \vee s'} \zeta_r,$$

and \mathcal{T}_ζ is equipped with the distance induced by

$$d_\zeta(s, s') = \zeta_s + \zeta_{s'} - 2 \min_{s \wedge s' \leq r \leq s \vee s'} \zeta_r$$

(notice that $d_\zeta(s, s') = 0$ if and only if $s \sim s'$ and see, for example, [29], Section 3, for more information about the coding of \mathbb{R} -trees by continuous functions). Let $p_\zeta : [0, \sigma] \rightarrow \mathcal{T}_\zeta$ stand for the canonical projection. By convention, \mathcal{T}_ζ is rooted at the point $\rho := p_\zeta(0) = p_\zeta(\sigma)$, and the volume measure $\text{vol}(\cdot)$ on \mathcal{T}_ζ is defined as the pushforward of Lebesgue measure on $[0, \sigma]$ under p_ζ . For every $a, b \in \mathcal{T}_\zeta$, $[[a, b]]$ denotes the line segment from a to b , and the ancestral line of a is the segment $[[\rho, a]]$ (a point b of $[[\rho, a]]$ is called an ancestor of a , and we also say that a is a descendant of b). We use the notation $]]a, b[[$ or $]]a, b[[$ with an obvious meaning. Branching points of \mathcal{T}_ζ are points c such that $\mathcal{T}_\zeta \setminus \{c\}$ has at least 3 connected components.

Let us now make a crucial observation: By property (ii) in the definition of a snake trajectory, the condition $p_\zeta(s) = p_\zeta(s')$ implies that $W_s(\omega) = W_{s'}(\omega)$. So the mapping $s \mapsto W_s(\omega)$ can be viewed as defined on the quotient space \mathcal{T}_ζ (this is indeed the main motivation for introducing snake trajectories: replacing mappings defined on trees, which later will be random trees, by mappings defined on intervals of the real line). For $a \in \mathcal{T}_\zeta$, we set $V_a(\omega) := \widehat{W}_s(\omega)$ whenever $s \in [0, \sigma]$ is such that $a = p_\zeta(s)$ — by the previous observation this does not depend on the choice of s . We interpret V_a as a “label” assigned to the “vertex” a of \mathcal{T}_ζ . Notice that the mapping $a \mapsto V_a$ is continuous on \mathcal{T}_ζ .

We will use the notation

$$W_* := \min\{W_s(t) : s \geq 0, t \in [0, \zeta_s]\} = \min\{V_a : a \in \mathcal{T}_\zeta\},$$

$$W^* := \max\{W_s(t) : s \geq 0, t \in [0, \zeta_s]\} = \max\{V_a : a \in \mathcal{T}_\zeta\}.$$

Finally, we will use the notion of a subtrajectory. Let $\omega \in \mathcal{S}_0$ and assume that the mapping $s \mapsto \zeta_s(\omega)$ is not constant on any nontrivial subinterval of $[0, \sigma]$ (this will always hold in our applications). Let $a \in \mathcal{T}_\zeta \setminus \{\rho\}$ such that a has strict descendants and a is not a branching point. Then there exist two times $s_1 < s_2$ in $(0, \sigma)$ such that $p_\zeta(s_1) = p_\zeta(s_2) = a$, and the set $p_\zeta([s_1, s_2])$ consists of all descendants of a in \mathcal{T}_ζ . We define a new snake trajectory ω' with duration $s_2 - s_1$ by setting, for every $s \geq 0$,

$$\omega'_s(t) := \omega_{(s_1+s) \wedge s_2}(\zeta_{s_1} + t) - \widehat{\omega}_{s_1}, \quad \text{for } 0 \leq t \leq \zeta'_s := \zeta_{(s_1+s) \wedge s_2} - \zeta_{s_1}.$$

We call ω' the subtrajectory of ω rooted at a . Informally, ω' represents the subtree of descendants of a and the associated labels.

2.2. Rerooting and truncation of snake trajectories. We now introduce two important operations on snake trajectories in \mathcal{S}_0 . The first one is the re-rooting operation on \mathcal{S}_0 (see [1], Section 2.2). Let $\omega \in \mathcal{S}_0$ and $r \in [0, \sigma(\omega)]$. Then $\omega^{[r]}$ is the snake trajectory in \mathcal{S}_0 such that $\sigma(\omega^{[r]}) = \sigma(\omega)$ and for every $s \in [0, \sigma(\omega)]$,

$$\zeta_s(\omega^{[r]}) = d_\zeta(r, r \oplus s),$$

$$\widehat{W}_s(\omega^{[r]}) = \widehat{W}_{r \oplus s} - \widehat{W}_r,$$

where we use the notation $r \oplus s = r + s$ if $r + s \leq \sigma$, and $r \oplus s = r + s - \sigma$ otherwise. By a remark following the definition of snake trajectories, these prescriptions completely determine $\omega^{[r]}$.

We will write $\zeta_s^{[r]}(\omega) = \zeta_s(\omega^{[r]})$ and $W_s^{[r]}(\omega) = W_s(\omega^{[r]})$. The tree $\mathcal{T}_{\zeta^{[r]}}$ is then interpreted as the tree \mathcal{T}_ζ rerooted at the vertex $p_\zeta(r)$: More precisely, the mapping $s \mapsto r \oplus s$ induces an isometry from $\mathcal{T}_{\zeta^{[r]}}$ onto \mathcal{T}_ζ , which maps the root of $\mathcal{T}_{\zeta^{[r]}}$ to $p_\zeta(r)$. Furthermore, the vertices of $\mathcal{T}_{\zeta^{[r]}}$ receive the “same” labels as in \mathcal{T}_ζ , shifted so that the label of the root is still 0.

The second operation is the truncation of snake trajectories. For any $w \in \mathcal{W}_0$ and $y \in \mathbb{R}$, we set

$$\tau_y(w) := \inf\{t \in [0, \zeta_{(w)}] : w(t) = y\}, \quad \tau_y^*(w) := \inf\{t \in (0, \zeta_{(w)}] : w(t) = y\}$$

with the usual convention $\inf \emptyset = \infty$ (this convention will be in force throughout this work unless otherwise indicated). Notice that $\tau_y(w) = \tau_y^*(w)$ except possibly if $y = 0$.

Let $\omega \in \mathcal{S}_0$ and $y \in \mathbb{R}$. We set, for every $s \geq 0$,

$$\eta_s(\omega) = \inf\left\{t \geq 0 : \int_0^t du \mathbf{1}_{\{\zeta_{(\omega_u)} \leq \tau_y^*(\omega_u)\}} > s\right\}$$

(note that the condition $\zeta_{(\omega_u)} \leq \tau_y^*(\omega_u)$ holds if and only if $\tau_y^*(\omega_u) = \infty$ or $\tau_y^*(\omega_u) = \zeta_{(\omega_u)}$). Then, setting $\omega'_s = \omega_{\eta_s(\omega)}$ for every $s \geq 0$ defines an element ω' of \mathcal{S}_0 , which will be denoted by $\text{tr}_y(\omega)$ and called the truncation of ω at y (see [1], Proposition 10). The effect of the time change $\eta_s(\omega)$ is to “eliminate” those paths ω_s that hit y (at a positive time when $y = 0$) and then survive for a positive amount of time. The genealogical tree of $\text{tr}_y(\omega)$ is canonically and isometrically identified with the closed subset of \mathcal{T}_ζ consisting of all a such that $V_b(\omega) \neq y$ for every strict ancestor b of a (excluding the root when $y = 0$). By abuse of notation, we often write $\text{tr}_y(W)$ instead of $\text{tr}_y(\omega)$.

2.3. *Measures on snake trajectories.* We will be interested in two important measures on \mathcal{S}_0 . First, the Brownian snake excursion measure \mathbb{N}_0 is the σ -finite measure on \mathcal{S}_0 that satisfies the following two properties: Under \mathbb{N}_0 ,

(i) the distribution of the lifetime function $(\zeta_s)_{s \geq 0}$ is the Itô measure of positive excursions of linear Brownian motion, normalized so that, for every $\varepsilon > 0$,

$$\mathbb{N}_0\left(\sup_{s \geq 0} \zeta_s > \varepsilon\right) = \frac{1}{2\varepsilon};$$

(ii) conditionally on $(\zeta_s)_{s \geq 0}$, the tip function $(\widehat{W}_s)_{s \geq 0}$ is a centered Gaussian process with covariance function

$$K(s, s') = \min_{s \wedge s' \leq r \leq s \vee s'} \zeta_r.$$

Informally, the lifetime process $(\zeta_s)_{s \geq 0}$ evolves under \mathbb{N}_0 like a Brownian excursion, and conditionally on $(\zeta_s)_{s \geq 0}$, each path W_s is a linear Brownian path started from 0, which is “erased” from its tip when ζ_s decreases and is “extended” when ζ_s increases. The measure \mathbb{N}_0 can be interpreted as the excursion measure away from 0 for the Markov process in \mathcal{W}_0 called the Brownian snake. We refer to [23] for a detailed study of the Brownian snake. For every $r > 0$, we have

$$\mathbb{N}_0(W^* > r) = \mathbb{N}_0(W_* < -r) = \frac{3}{2r^2}$$

(see, e.g., [23], Section VI.1).

The following scaling property is often useful. For $\lambda > 0$, for every $\omega \in \mathcal{S}_0$, we define $\theta_\lambda(\omega) \in \mathcal{S}_0$ by $\theta_\lambda(\omega) = \omega'$, with

$$\omega'_s(t) := \sqrt{\lambda} \omega_{s/\lambda^2}(t/\lambda), \quad \text{for } s \geq 0, 0 \leq t \leq \zeta'_s := \lambda \zeta_{s/\lambda^2}.$$

Then $\theta_\lambda(\mathbb{N}_0) = \lambda \mathbb{N}_0$.

Under \mathbb{N}_0 , the paths W_s , $0 < s < \sigma$, take both positive and negative values, simply because they behave like one-dimensional Brownian paths started from 0. We will now introduce another important measure on \mathcal{S}_0 , which is supported on snake trajectories taking only nonnegative values. For $\delta \geq 0$, let $\mathcal{S}_0^{(\delta)}$ be the set of all $\omega \in \mathcal{S}_0$ such that $\sup_{s \geq 0} (\sup_{t \in [0, \zeta_s(\omega)]} |\omega_s(t)|) > \delta$. Also set

$$\mathcal{S}_0^+ = \{\omega \in \mathcal{S}_0 : \omega_s(t) \geq 0 \text{ for every } s \geq 0, t \in [0, \zeta_s(\omega)]\} \cap \mathcal{S}_0^{(0)}.$$

There exists a σ -finite measure \mathbb{N}_0^* on \mathcal{S}_0 , which is supported on \mathcal{S}_0^+ , and gives finite mass to the sets $\mathcal{S}_0^{(\delta)}$ for every $\delta > 0$, such that

$$\mathbb{N}_0^*(G) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{N}_0(G(\text{tr}_{-\varepsilon}(W))),$$

for every bounded continuous function G on \mathcal{S}_0 that vanishes on $\mathcal{S}_0 \setminus \mathcal{S}_0^{(\delta)}$ for some $\delta > 0$ (see [1], Theorem 23). Under \mathbb{N}_0^* , each of the paths W_s , $0 < s < \sigma$, starts from 0, then stays positive during some time interval $(0, \alpha)$, and is stopped immediately when it returns to 0, if it does return to 0.

One can in fact make sense of the ‘‘quantity’’ of paths W_s that return to 0 under \mathbb{N}_0^* . To this end, one proves that the limit

$$(3) \quad \mathcal{Z}_0^* := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_0^\sigma ds \mathbf{1}_{\{\widehat{W}_s < \varepsilon\}}$$

exists \mathbb{N}_0^* a.e. See [1], Proposition 30, for a slightly weaker result—the stronger form stated above follows from the results of [28], Section 10. Notice that replacing the limit by a \liminf in (3) allows us to make sense of $\mathcal{Z}_0^*(\omega)$ for every $\omega \in \mathcal{S}_0^+$. The following conditional versions of the measure \mathbb{N}_0^* play a fundamental role in the present work. According to [1], Proposition 33, there exists a unique collection $(\mathbb{N}_0^{*,z})_{z>0}$ of probability measures on \mathcal{S}_0^+ such that:

(i) We have

$$\mathbb{N}_0^* = \sqrt{\frac{3}{2\pi}} \int_0^\infty dz z^{-5/2} \mathbb{N}_0^{*,z}.$$

(ii) For every $z > 0$, $\mathbb{N}_0^{*,z}$ is supported on $\{\mathcal{Z}_0^* = z\}$.

(iii) For every $z, z' > 0$, $\mathbb{N}_0^{*,z'} = \theta_{z'/z}(\mathbb{N}_0^{*,z})$.

Informally, $\mathbb{N}_0^{*,z} = \mathbb{N}_0^*(\cdot \mid \mathcal{Z}_0^* = z)$.

2.4. *Exit measures.* Let $r \in \mathbb{R}$, $r \neq 0$. In a way similar to the definition of \mathcal{Z}_0^* above, one can make sense of a quantity that measures the number of paths W_s that hit level r under \mathbb{N}_0 . Precisely, the limit

$$(4) \quad \mathcal{Z}_r := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\sigma ds \mathbf{1}_{\{\tau_r(W_s) \leq \zeta_s < \tau_r(W_s) + \varepsilon\}}$$

exists \mathbb{N}_0 a.e. Furthermore, $\mathcal{Z}_r > 0$ if and only if $r \in [W_*, W^*]$, \mathbb{N}_0 a.e. This definition of \mathcal{Z}_r is a particular case of the theory of exit measures; see [23], Chapter V. We note that \mathcal{Z}_r is \mathbb{N}_0 a.e. equal to a measurable function of the truncated snake $\text{tr}_r(W)$: When $r < 0$, this can be seen by observing that \mathcal{Z}_r is the a.e. limit of the quantities $\widetilde{\mathcal{Z}}_r^\varepsilon$ introduced in Remark (ii) after Proposition 8 below.

We now recall the special Markov property of the Brownian snake under \mathbb{N}_0 (see, in particular, the Appendix of [27]).

PROPOSITION 7 (Special Markov property). *Let $(s_i, s'_i), i \in I$ be the connected components of the open set $\{s \in [0, \sigma] : \tau_r(W_s) < \zeta_s\}$. For every $i \in I$, set $a_i := p_\zeta(s_i) = p_\zeta(s'_i)$ and let ω_i be the subtrajectory of ω rooted at a_i . Then, under the probability measure $\mathbb{N}_0(\cdot \mid r \in [W_*, W^*])$, conditionally on $\text{tr}_r(W)$, the point measure $\sum_{i \in I} \delta_{\omega_i}$ is Poisson with intensity $\mathcal{Z}_r \mathbb{N}_0(\cdot)$.*

Let us now explain the relations between exit measures and a certain continuous-state branching process. For $\lambda > 0$, we set

$$\phi(\lambda) := \sqrt{\frac{8}{3}} \lambda^{3/2}.$$

This notation will be used throughout this work. The continuous-state branching process with branching mechanism ϕ , or in short the ϕ -CSBP, is the Feller Markov process X in \mathbb{R}_+ whose transition kernels are given by the following Laplace transform:

$$(5) \quad \mathbb{E}[\exp(-\lambda X_t) \mid X_0 = x] = \exp(-x(\lambda^{-1/2} + t\sqrt{2/3})^{-2}),$$

for every $x, t \geq 0$ and $\lambda > 0$. See, for example, [23], Chapter II, for basic facts about continuous-state branching processes.

For reasons that will appear later, we now concentrate on the variables \mathcal{Z}_r with $r < 0$. According to [15], formula (6), we have, for every $t > 0$,

$$(6) \quad \mathbb{N}_0(1 - e^{-\lambda \mathcal{Z}_{-t}}) = (\lambda^{-1/2} + t\sqrt{2/3})^{-2}.$$

Using both the latter formula and the special Markov property, we get that the process $(\mathcal{Z}_{-r})_{r>0}$ is Markovian under \mathbb{N}_0 with the transition kernels of the ϕ -CSBP, with respect to the filtration $(\mathcal{G}_r)_{r>0}$, where \mathcal{G}_r denotes the σ -field generated by $\text{tr}_{-r}(W)$ and the \mathbb{N}_0 -negligible sets (see [1], Section 2.5, for more details). Although \mathbb{N}_0 is an infinite measure, the preceding statement makes sense by considering the process $(\mathcal{Z}_{-\delta-r})_{r \geq 0}$ under the probability measure $\mathbb{N}_0(\cdot \mid W_* \leq -\delta)$, for every $\delta > 0$. As a consequence, the process $(\mathcal{Z}_{-r})_{r>0}$ has a càdlàg modification under \mathbb{N}_0 , which we consider from now on.

The distribution of $(\mathcal{Z}_{-r})_{r>0}$ under \mathbb{N}_0 can be interpreted as an excursion measure for the ϕ -CSBP, in the following sense. Let $\alpha > 0$, and let

$$\sum_{i \in I} \delta_{\omega_i}$$

be a Poisson measure with intensity $\alpha \mathbb{N}_0$. Set $Y_0 = \alpha$ and for every $t > 0$,

$$Y_t = \sum_{i \in I} \mathcal{Z}_{-t}(\omega_i)$$

(note that this is a finite sum since $\mathbb{N}_0(W_* \leq r) < \infty$ if $r < 0$). Then the process $(Y_t)_{t \geq 0}$ is a ϕ -CSBP started from α . It is enough to verify that Y has the desired one-dimensional marginals and to this end we write, for every $t > 0$, $\mathbb{E}[\exp(-\lambda Y_t)] = \exp(-\alpha \mathbb{N}_0(1 - e^{-\lambda \mathcal{Z}_{-t}}))$ and we use (6).

We note that, for every $z > 0$,

$$(7) \quad \lim_{\varepsilon \rightarrow 0} \downarrow \mathbb{N}_0\left(\sup_{0 < t \leq \varepsilon} \mathcal{Z}_{-t} \geq z\right) = 0.$$

Indeed, assuming that this convergence does not hold, the preceding Poisson representation with $\alpha = z/2$ would imply that the probability of the event $\{\sup\{Y_t : 0 < t \leq \varepsilon\} \geq z\}$ is bounded below by a positive constant independent of ε , which contradicts the right-continuity of paths of Y at time 0. It follows from (7) that $\mathcal{Z}_{-t} \rightarrow 0$ as $t \downarrow 0$, \mathbb{N}_0 a.e.

Exit measures allow us to state the following formula, which relates the measures \mathbb{N}_0^* and \mathbb{N}_0 via a rerooting procedure. Let G be a nonnegative measurable function on \mathcal{S}_0 . Then

$$(8) \quad \mathbb{N}_0^* \left(\int_0^\sigma dr G(W^{[r]}) \right) = 2 \int_{-\infty}^0 db \mathbb{N}_0(\mathcal{Z}_b G(\text{tr}_b(W))).$$

See [1], Theorem 28.

In view of the subsequent developments, it will be important to have a uniform approximation of the exit measure process $(\mathcal{Z}_r)_{r < 0}$ under \mathbb{N}_0 . This is the goal of the next proposition. For $w \in \mathcal{W}_0$ and $r \in \mathbb{R}$, we use the notation

$$T_r(w) := \inf\{t \in [0, \zeta_{(w)}] : w(t) < r\}.$$

PROPOSITION 8. For $r < 0$ and $\varepsilon > 0$, set

$$\mathcal{Z}_r^\varepsilon := \varepsilon^{-2} \int_0^\sigma ds \mathbf{1}_{\{T_r(W_s) = \infty, \widehat{W}_s < r + \varepsilon\}}.$$

Then, for every $\beta > 0$,

$$\sup_{r \in (-\infty, -\beta]} |\mathcal{Z}_r^\varepsilon - \mathcal{Z}_r| \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \mathbb{N}_0 \text{ a.e.}$$

REMARKS. (i) The process $(\mathcal{Z}_r)_{r < 0}$ is càglàd (left continuous with right limits), and the same is true for the process $(\mathcal{Z}_r^\varepsilon)_{r < 0}$ for every $\varepsilon > 0$: If $r_n \uparrow r < 0$, we have $\mathbf{1}_{\{T_{r_n}(W_s) = \infty\}} \downarrow \mathbf{1}_{\{T_r(W_s) = \infty\}}$ and $\mathbf{1}_{\{\widehat{W}_s < r_n + \varepsilon\}} \uparrow \mathbf{1}_{\{\widehat{W}_s < r + \varepsilon\}}$. The right limits of the process $(\mathcal{Z}_r^\varepsilon)_{r < 0}$ are given by

$$(9) \quad \mathcal{Z}_{r+}^\varepsilon = \varepsilon^{-2} \int_0^\sigma ds \mathbf{1}_{\{\underline{W}_s > r, \widehat{W}_s \leq r + \varepsilon\}}.$$

(ii) The reader may notice that a slightly different approximation is used in [1], Lemma 14, or in [28], Proposition 34, where the quantities

$$\widetilde{\mathcal{Z}}_r^\varepsilon := \varepsilon^{-2} \int_0^\sigma ds \mathbf{1}_{\{\zeta_s \leq \tau_r(W_s), \widehat{W}_s < r + \varepsilon\}}$$

are considered. If r is fixed, this makes no difference since $\widetilde{\mathcal{Z}}_r^\varepsilon = \mathcal{Z}_r^\varepsilon$ for every $\varepsilon > 0$, \mathbb{N}_0 a.e. (we may have $\widetilde{\mathcal{Z}}_r^\varepsilon \neq \mathcal{Z}_r^\varepsilon$ only if r is a local minimum of one of the paths W_s , and this occurs with zero \mathbb{N}_0 -measure). The point in using $\mathcal{Z}_r^\varepsilon$ rather than $\widetilde{\mathcal{Z}}_r^\varepsilon$ is the fact that we want a uniform approximation of $(\mathcal{Z}_r)_{r < 0}$ and to this end we are looking for càglàd approximating processes, which is the case for $r \mapsto \mathcal{Z}_r^\varepsilon$ but not for $r \mapsto \widetilde{\mathcal{Z}}_r^\varepsilon$.

We postpone the proof of Proposition 8 to the Appendix below. We note that, for every fixed value of $r < 0$, the convergence $\mathcal{Z}_r^\varepsilon \rightarrow \mathcal{Z}_r$, \mathbb{N}_0 a.e., follows from [28], Proposition 34. Unfortunately, the uniform convergence stated in the proposition requires more work.

2.5. A representation for the measure $\mathbb{N}_0^{*,z}$. For every $z > 0$, set

$$L_z := \inf\{r < 0 : \mathcal{Z}_r = z\}$$

if $\{r < 0 : \mathcal{Z}_r = z\}$ is not empty, and $L_z = 0$ otherwise.

LEMMA 9. We have

$$(10) \quad \mathbb{N}_0(L_z < 0) = \frac{1}{2z}.$$

PROOF. The fact that $\mathbb{N}_0(L_z < 0) = C/z$ for some positive constant C is easy by a scaling argument, but we need another argument to get the value of C . Let $\varepsilon > 0$. We have

$$\begin{aligned} \mathbb{N}_0(L_z < 0) &= \mathbb{N}_0\left(\sup_{t>0} \mathcal{Z}_{-t} \geq z\right) \\ &= \mathbb{N}_0\left(\sup_{0<t\leq\varepsilon} \mathcal{Z}_{-t} \geq z\right) + \mathbb{N}_0\left(\mathbf{1}_{\{\sup_{0<t\leq\varepsilon} \mathcal{Z}_{-t} < z\}} \mathbf{P}_{\mathcal{Z}_{-\varepsilon}}\left(\sup_{t\geq 0} \mathcal{Y}_t \geq z\right)\right), \end{aligned}$$

where we use the notation $(\mathcal{Y}_t)_{t\geq 0}$ for a ϕ -CSBP that starts from x under the probability measure \mathbf{P}_x , for every $x \geq 0$. By the classical Lamperti representation for CSBPs [12, 20], $(\mathcal{Y}_t)_{t\geq 0}$ can be written as a time change of a stable Lévy process with index $3/2$ and no negative jumps. The explicit solution of the two-sided exit problem for such Lévy processes (see [2], Theorem VII.8) now gives

$$\mathbf{P}_{\mathcal{Z}_{-\varepsilon}}\left(\sup_{t\geq 0} \mathcal{Y}_t \geq z\right) = 1 - \sqrt{\left(1 - \frac{\mathcal{Z}_{-\varepsilon}}{z}\right)^+}.$$

Using also (7), we get that

$$\mathbb{N}_0(L_z < 0) = \mathbb{N}_0\left(1 - \sqrt{\left(1 - \frac{\mathcal{Z}_{-\varepsilon}}{z}\right)^+}\right) + o(1), \quad \text{as } \varepsilon \rightarrow 0.$$

For every $\delta > 0$, $\mathbb{N}_0(\mathbf{1}_{\{\mathcal{Z}_{-\varepsilon} > \delta\}} \mathcal{Z}_{-\varepsilon}) = o(1)$ as $\varepsilon \rightarrow 0$. Indeed, this follows from the formula $\mathbb{N}_0(\mathcal{Z}_\varepsilon(1 - e^{-\lambda \mathcal{Z}_{-\varepsilon}})) = 1 - (1 + \sqrt{2/3} \varepsilon \lambda^{1/2})^{-3}$, which is a consequence of (6). Thanks to this observation and to the fact that $\sqrt{1-x} = 1 - \frac{x}{2} + o(x)$ when $x \rightarrow 0$, we find that

$$\mathbb{N}_0(L_z < 0) = \frac{1}{2} \mathbb{N}_0\left(\frac{\mathcal{Z}_{-\varepsilon}}{z}\right) + o(1)$$

as $\varepsilon \rightarrow 0$. The lemma follows since $\mathbb{N}_0(\mathcal{Z}_{-\varepsilon}) = 1$ for every $\varepsilon > 0$. \square

The following proposition, which will play an important role, provides an analog of formula (8) where \mathbb{N}_0^* is replaced by the conditional measure $\mathbb{N}_0^{*,z}$.

PROPOSITION 10. For any nonnegative measurable function G on \mathcal{S}_0 , for every $z > 0$,

$$z^{-2} \mathbb{N}_0^{*,z}\left(\int_0^\sigma ds G(W^{[s]})\right) = \mathbb{N}_0(G(\text{tr}_{L_z}(W)) | L_z < 0).$$

REMARK. When $G = 1$, one recovers the known formula $\mathbb{N}_0^{*,z}(\sigma) = z^2$, see the remark following Proposition 15 in [28].

PROOF OF PROPOSITION 10. We may and will assume that G is bounded and continuous. We use the same notation $(\mathcal{Y}_t, \mathbf{P}_x)$ as in the previous proof and we also set $\Lambda_z := \sup\{t \geq 0 : \mathcal{Y}_t = z\}$ with the convention $\sup \emptyset = 0$.

As a consequence of (8), the formula

$$(11) \quad \mathbb{N}_0^*\left(\int_0^\sigma dr \varphi(\mathcal{Z}_0^*) G(W^{[r]})\right) = 2 \int_{-\infty}^0 db \mathbb{N}_0(\mathcal{Z}_b \varphi(\mathcal{Z}_b) G(\text{tr}_b(W))),$$

holds for any nonnegative measurable function φ on $[0, \infty)$. To derive (11) from (8), notice that (3) and Proposition 8 allow us to write $\mathcal{Z}_0^* = \Gamma(W^{[r]})$, \mathbb{N}_0^* a.e., and $\mathcal{Z}_b = \Gamma(\text{tr}_b(W))$, \mathbb{N}_0 a.e., with the same measurable function Γ on \mathcal{S}_0 .

Let us fix $z_0 > 0$ and a continuous function φ on \mathbb{R}_+ which is supported on a compact subset of $(0, \infty)$ and such that $\varphi(z_0) > 0$. We observe that, for any $b < 0$, we have

$$(12) \quad \begin{aligned} & \mathbb{N}_0(\mathbf{1}_{\{b-\varepsilon \leq L_{z_0} < b\}} \mathcal{Z}_b \varphi(\mathcal{Z}_b) G(\text{tr}_b(W))) \\ &= \mathbb{N}_0(h_\varepsilon(\mathcal{Z}_b, z_0) \mathcal{Z}_b \varphi(\mathcal{Z}_b) G(\text{tr}_b(W))), \end{aligned}$$

where the function h_ε is defined for every $z > 0$ by

$$h_\varepsilon(z, z_0) = \mathbf{P}_z(0 < \Lambda_{z_0} \leq \varepsilon).$$

To get (12), we use the Markov property of the process $(\mathcal{Z}_{-r})_{r>0}$ (with respect to the filtration $(\mathcal{G}_r)_{r>0}$ introduced in Section 2.4) at time $-b$.

By combining (11) (with $\varphi(z)$ replaced by $h_\varepsilon(z, z_0)\varphi(z)$) and (12), we get

$$(13) \quad \begin{aligned} & \mathbb{N}_0^* \left(\int_0^\sigma dr h_\varepsilon(\mathcal{Z}_0^*, z_0) \varphi(\mathcal{Z}_0^*) G(W^{[r]}) \right) \\ &= 2 \int_{-\infty}^0 db \mathbb{N}_0(\mathbf{1}_{\{b-\varepsilon \leq L_{z_0} < b\}} \mathcal{Z}_b \varphi(\mathcal{Z}_b) G(\text{tr}_b(W))) \\ &= 2 \mathbb{N}_0 \left(\int_{L_{z_0}}^{(L_{z_0} + \varepsilon) \wedge 0} db \mathcal{Z}_b \varphi(\mathcal{Z}_b) G(\text{tr}_b(W)) \right). \end{aligned}$$

Let us multiply the right-hand side of (13) by ε^{-1} and study its limit as $\varepsilon \rightarrow 0$. By Lemma 11 in [1], we know that $\text{tr}_b(W) \rightarrow \text{tr}_{L_{z_0}}(W)$ as $b \downarrow L_{z_0}$, \mathbb{N}_0 a.e. on $\{L_{z_0} < 0\}$. It follows that

$$(14) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon} \mathbb{N}_0 \left(\int_{L_{z_0}}^{(L_{z_0} + \varepsilon) \wedge 0} db \mathcal{Z}_b \varphi(\mathcal{Z}_b) G(\text{tr}_b(W)) \right) \\ &= 2z_0 \varphi(z_0) \mathbb{N}_0(G(\text{tr}_{L_{z_0}}(W)) \mathbf{1}_{\{L_{z_0} < 0\}}), \end{aligned}$$

where dominated convergence is easily justified thanks to our assumptions on φ and the property $\mathbb{N}_0(L_{z_0} < 0) < \infty$. On the other hand, properties (i) and (ii) stated at the end of Section 2.3 allow us to rewrite the left-hand side of (13) as

$$(15) \quad \begin{aligned} & \mathbb{N}_0^* \left(\int_0^\sigma dr h_\varepsilon(\mathcal{Z}_0^*, z_0) \varphi(\mathcal{Z}_0^*) G(W^{[r]}) \right) \\ &= \sqrt{\frac{3}{2\pi}} \int_0^\infty \frac{dz}{z^{5/2}} h_\varepsilon(z, z_0) \varphi(z) \mathbb{N}_0^{*,z} \left(\int_0^\sigma dr G(W^{[r]}) \right). \end{aligned}$$

Consider the special case $G = 1$. We deduce from the convergence (14), using also the formula $\mathbb{N}_0^{*,z}(\sigma) = z^2$ and the identities (13) and (15), that

$$(16) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sqrt{\frac{3}{2\pi}} \int_0^\infty \frac{dz}{z^{1/2}} h_\varepsilon(z, z_0) \varphi(z) = 2z_0 \varphi(z_0) \mathbb{N}_0(L_{z_0} < 0).$$

For a general (bounded and continuous) function G , a simple scaling argument shows that the function $z \mapsto z^{-2} \mathbb{N}_0^{*,z}(\int_0^\sigma dr G(W^{[r]}))$ is also bounded and continuous on $(0, \infty)$. We may thus apply (16) with $\varphi(z)$ replaced by the function

$$z \mapsto \varphi(z) z^{-2} \mathbb{N}_0^{*,z} \left(\int_0^\sigma dr G(W^{[r]}) \right)$$

and we get

$$(17) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sqrt{\frac{3}{2\pi}} \int_0^\infty \frac{dz}{z^{5/2}} h_\varepsilon(z, z_0) \varphi(z) \mathbb{N}_0^{*,z} \left(\int_0^\sigma dr G(W^{[r]}) \right) \\ &= \frac{2\varphi(z_0)}{z_0} \mathbb{N}_0^{*,z_0} \left(\int_0^\sigma dr G(W^{[r]}) \right) \mathbb{N}_0(L_{z_0} < \infty). \end{aligned}$$

From the identities (13) and (15), the right-hand sides of (14) and (17) are equal, which gives the desired result. \square

2.6. *The exit measure process time-reversed at L_z .* The goal of this section is to prove the following proposition. Recall that for a Lévy process ξ with only negative jumps we define its Laplace exponent $\psi(\lambda)$ by

$$\mathbb{E}[\exp(\lambda\xi(t))] = \exp(t\psi(\lambda)), \quad \lambda \geq 0.$$

We use the notation \mathcal{Z}_{r+} for the right limit of $u \mapsto \mathcal{Z}_u$ at r .

PROPOSITION 11. *Set $\mathcal{Z}_r = 0$ for $r \geq 0$. Under $\mathbb{N}_0(\cdot \mid L_z < 0)$, the process $(\mathcal{Z}_{(L_z+r)+})_{r \geq 0}$ is distributed as a self-similar Markov process $(X_r^\circ)_{r \geq 0}$ with index $\frac{1}{2}$ starting from z , which can be represented as*

$$X_t^\circ = z \exp(\xi^\circ(\chi^\circ(z^{-1/2}t))),$$

where $(\xi^\circ(s))_{s \geq 0}$ is the Lévy process with only negative jumps and Laplace exponent

$$\psi^\circ(\lambda) = \sqrt{\frac{3}{2\pi}} \int_{-\infty}^0 (e^{\lambda y} - 1 - \lambda(e^y - 1))e^{y/2}(1 - e^y)^{-5/2} dy,$$

and $(\chi^\circ(t))_{t \geq 0}$ is the time change

$$\chi^\circ(t) = \inf \left\{ s \geq 0 : \int_0^s e^{\xi^\circ(v)/2} dv > t \right\}.$$

We note that the Lévy process ξ° drifts to $-\infty$, and the quantity

$$H_0^\circ := z^{1/2} \int_0^\infty e^{\xi^\circ(v)/2} dv$$

is finite a.s. For $t \geq H_0^\circ$, we have $\chi^\circ(z^{-1/2}t) = \infty$ and $\xi^\circ(\chi^\circ(z^{-1/2}t)) = -\infty$. Thus H_0° is the hitting time of 0 by X° , and X° is absorbed at 0.

PROOF. Let $(U_t)_{t \geq 0}$ denote a stable Lévy process with index $3/2$ and no negative jumps, whose distribution is characterized by the formula

$$\mathbb{E}[\exp(-\lambda U_t)] = \exp(t\phi(\lambda)), \quad \lambda > 0, t \geq 0,$$

where $\phi(\lambda) = \sqrt{8/3}\lambda^{3/2}$ as previously. If $\underline{U}_t := \min\{U_s : 0 \leq s \leq t\}$, the process $U_t - \underline{U}_t$ is a strong Markov process for which 0 is a regular point. Furthermore, $-\underline{U}_t$ serves as a local time at 0 for $U - \underline{U}$. We refer to [2], especially Chapters VII and VIII, for these standard facts about Lévy processes. We denote the excursion measure of $U - \underline{U}$ away from 0, corresponding to the local time $-\underline{U}$, by \mathbf{n} . Then \mathbf{n} is a σ -finite measure on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$.

For notational convenience, we write $\tilde{\mathcal{Z}}_x = \mathcal{Z}_{-x}$ for $x > 0$ and $\tilde{\mathcal{Z}}_0 = 0$. Notice that $\tilde{\mathcal{Z}}$ has càdlàg sample paths. We also set

$$\bar{L}_z = -L_z = \sup\{x > 0 : \tilde{\mathcal{Z}}_x = z\}$$

with the convention $\sup \emptyset = 0$.

LEMMA 12. *For every $x \geq 0$, set*

$$\eta(x) := \inf \left\{ y > 0 : \int_0^y \tilde{\mathcal{Z}}_u du > x \right\}.$$

Let $\mathcal{Y}_x = \tilde{\mathcal{Z}}_{\eta(x)}$ if $\eta(x) < \infty$ and $\mathcal{Y}_x = 0$ otherwise. Then the distribution of $(\mathcal{Y}_x)_{x \geq 0}$ under \mathbb{N}_0 is \mathbf{n} .

This lemma is basically a version for excursion measures of the Lamperti representation [12, 20] connecting continuous-state branching processes with Lévy processes. As we were unable to find a precise reference, we provide a proof in the [Appendix](#) below.

On the event $\{\bar{L}_z > 0\}$, set

$$\Lambda_z := \sup\{x \geq 0 : \mathcal{Y}_x = z\} = \int_0^{\bar{L}_z} \bar{Z}_s \, ds.$$

Still on the event $\{\bar{L}_z > 0\}$, we then introduce the time-reversed processes

$$\check{Z}_u = \begin{cases} \bar{Z}_{(\bar{L}_z - u)-} & \text{if } 0 \leq u < \bar{L}_z, \\ 0 & \text{if } u \geq \bar{L}_z, \end{cases}$$

and

$$\check{Y}_u = \begin{cases} \mathcal{Y}_{(\Lambda_z - u)-} & \text{if } 0 \leq u < \Lambda_z, \\ 0 & \text{if } u \geq \Lambda_z. \end{cases}$$

We note that we have again the Lamperti representation

$$(18) \quad \check{Z}_t = \check{Y}_{\gamma(t)}, \quad \text{with } \gamma(t) = \inf\left\{u \geq 0 : \int_0^u \frac{dv}{\check{Y}_v} > t\right\}.$$

Next, as a consequence of Lemma 12 and Theorem 4 in [13], we know that the process $(\check{Y}_u)_{u \geq 0}$ is distributed under $\mathbb{N}_0(\cdot \mid L_z < 0)$ as the Lévy process $-U$ started from z and conditioned to hit zero continuously before hitting $(-\infty, 0)$, and stopped at that hitting time. We refer to Section 4 of [13] for a discussion of the latter process. Furthermore, we can then use Corollary 3 of Caballero and Chaumont [11] to obtain that the process $(\check{Y}_u)_{u \geq 0}$ under $\mathbb{N}_0(\cdot \mid L_z < 0)$ has the distribution of a self-similar Markov process $(X'_u)_{u \geq 0}$ which can be represented in the form

$$X'_u = z \exp(\xi^\circ(\chi'(z^{-3/2}u))),$$

where ξ° is the Lévy process in the statement of the proposition,¹ and $(\chi'(t))_{t \geq 0}$ is the time change

$$\chi'(t) = \inf\left\{s \geq 0 : \int_0^s e^{3\xi^\circ(v)/2} \, dv > t\right\}.$$

(Note that the self-similarity index of X' is $3/2$ as the one for U .)

Recalling (18), we see that $(\check{Z}_t)_{t \geq 0}$ has the same distribution as $(X'_{\gamma'(t)})_{t \geq 0}$ where

$$\gamma'(t) = \inf\left\{u \geq 0 : \int_0^u \frac{dv}{X'_v} > t\right\}$$

and $X'_\infty = 0$ by convention. Let

$$H'_0 := \inf\{t \geq 0 : X'_t = 0\} \quad \text{and} \quad K'_0 := \int_0^{H'_0} (X'_v)^{-1} \, dv,$$

so that $\gamma'(t) < H'_0$ if $t < K'_0$ and $\gamma'(t) = \infty$ if $t \geq K'_0$. Simple manipulations show that

$$\begin{aligned} \chi'(z^{-3/2}\gamma'(t)) &= \int_0^t \frac{ds}{\sqrt{X'_{\gamma'(s)}}} = \inf\left\{u \geq 0 : z^{1/2} \int_0^u \exp\left(\frac{1}{2}\xi^\circ(v)\right) \, dv > t\right\} \\ &= \chi^\circ(z^{-1/2}t) \end{aligned}$$

¹In order for the reader to recover the exact form of the Laplace exponent ψ° in the proposition, we mention the following minor inaccuracy in [11]: In formula (23) of the latter paper, $+c_-$ should be replaced by $-c_-$.

if $t < K'_0 = z^{1/2} \int_0^\infty e^{\xi^\circ(u)/2} du$, whereas $\chi'(z^{-3/2}\gamma'(t)) = \infty$ if $t \geq K'_0$. In both cases, we get $X'_{\gamma'(t)} = z \exp(\xi^\circ(\chi^\circ(z^{-1/2}t))) = X_t^\circ$, with the notation of the proposition. We conclude that $(\check{Z}_t)_{t \geq 0}$ has the same distribution as $(X_t^\circ)_{t \geq 0}$. This is the desired result since by construction $\check{Z}_t = \check{Z}_{L_z+t}$. \square

3. Special connected components of the genealogical tree.

3.1. *Components above a level.* In this section and the next one, we formulate certain definitions and facts that make sense for a deterministic snake trajectory satisfying some regularity properties. We fix $\omega \in \mathcal{S}_0$ and consider the associated genealogical tree \mathcal{T}_ζ .

DEFINITION 13. We say that $x \in \mathbb{R}$ is a local minimum of ω if there exist two distinct points $a_1, a_2 \in \mathcal{T}_\zeta$ and a point $b \in]]a_1, a_2[[$ such that

$$V_b = \min_{c \in]]a_1, a_2[[} V_c = x.$$

We then also say that b is a point of local minimum.

Clearly, the set of all local minima is countable.

We will assume the following regularity properties:

- (i) local minima are distinct: if b, b' are two distinct points of local minimum, $V_b \neq V_{b'}$;
- (ii) no branching point is a point of local minimum;
- (iii) for every $x \in \mathbb{R}$, $\text{vol}(\{c \in \mathcal{T}_\zeta : V_c = x\}) = 0$.

All these properties hold \mathbb{N}_0 a.e. and $\mathbb{N}_0^{*,z}$ a.e. For (i) under \mathbb{N}_0 , one just uses the fact that the increments of V along disjoint segments of the tree \mathcal{T}_ζ are independent. As for (iii), the case of \mathbb{N}_0 follows from the fact that the pushforward of $\text{vol}(da)$ under the mapping $a \mapsto V_a$ has a continuous density [9], and one can then use Proposition 10 to deal with $\mathbb{N}_0^{*,z}$. Notice that (i) implies that the mapping $c \mapsto V_c$ cannot be constant on a nontrivial line segment of \mathcal{T}_ζ .

In the remaining part of this section, we assume in addition that $\omega \in \mathcal{S}_0^+$. We set

$$\mathcal{T}_\zeta^\circ := \{a \in \mathcal{T}_\zeta : V_a > 0\}.$$

Let us fix $a \in \mathcal{T}_\zeta^\circ$. For every $r \in [0, V_a)$, let $\mathcal{C}_r^{(a)}$ denote the connected component of $\{b \in \mathcal{T}_\zeta : V_b > r\}$ that contains a . We note that $\mathcal{C}_{r'}^{(a)} \subset \mathcal{C}_r^{(a)}$ if $r < r' < V_a$. Let $\overline{\mathcal{C}}_r^{(a)}$ stand for the closure of $\mathcal{C}_r^{(a)}$ and, if $r \in (0, V_a)$, set

$$\mathcal{C}_{r-}^{(a)} := \bigcap_{r' \in [0, r)} \mathcal{C}_{r'}^{(a)}.$$

We always have $\overline{\mathcal{C}}_r^{(a)} \subset \mathcal{C}_{r-}^{(a)}$ and equality holds if and only if $r \notin D^{(a)}$, where the set $D^{(a)}$ is defined by

$$D^{(a)} := \left\{ r \in (0, V_a) : \exists b \in \mathcal{T}_\zeta \setminus \{a\}, V_b > r \text{ and } \min_{c \in]]a, b[[} V_c = r \right\}.$$

Note that $D^{(a)}$ is a subset of the set of all local minima.

If $r \in D^{(a)}$ and $b \neq a$ is such that $V_b > r$ and $\min_{c \in]]a, b[[} V_c = r$, then there exists a unique $c_0 \in]]a, b[[$ such that $V_{c_0} = r$, and c_0 does not depend on the choice of b (because local minima are distinct by (i) above). Note that c_0 cannot be a branching point of the tree \mathcal{T}_ζ , by (ii). We can then set

$$\check{\mathcal{C}}_r^{(a)} = \{b \in \mathcal{T}_\zeta : c_0 \in]]a, b[[\text{ and } V_c > r \text{ for every } c \in]]c_0, b[[\},$$

and $\mathcal{C}_{r-}^{(a)}$ is the closure of the union $\mathcal{C}_r^{(a)} \cup \check{\mathcal{C}}_r^{(a)}$. Notice that $\check{\mathcal{C}}_r^{(a)} = \mathcal{C}_r^{(b)}$ for any $b \in \check{\mathcal{C}}_r^{(a)}$. For future use, we note that the boundary of $\mathcal{C}_r^{(a)}$, or of $\check{\mathcal{C}}_r^{(a)}$, has zero volume (by (iii)).

3.2. *Excursions above the minimum.* Let us consider $\omega \in \mathcal{S}_0$, and assume that the conditions (i)–(iii) of the previous section hold. Recall our notation ρ for the root of \mathcal{T}_ζ and note that $V_\rho = 0$. In a way very similar to the definition of $D^{(a)}$ above we now set

$$D(\omega) = \left\{ r < 0 : \exists a \in \mathcal{T}_\zeta, V_a > r \text{ and } \min_{c \in \llbracket \rho, a \llbracket} V_c = r \right\}.$$

Let us fix $r \in D$. Then r is a local minimum and we let c_0 be the uniquely determined point of local minimum such that $V_{c_0} = r$. The same arguments as in the previous section allow us to single out a particular component of $\{c \in \mathcal{T}_\zeta : V_c > r\}$ by setting

$$\check{C}_r = \{a \in \mathcal{T}_\zeta : c_0 \in \llbracket \rho, a \llbracket \text{ and } V_c > r \text{ for every } c \in \llbracket c_0, a \llbracket \}.$$

Note that this definition of \check{C}_r would correspond to $\check{C}_r^{(\rho)}$ in the notation of the preceding section, but we are now considering negative values of r , instead of $r > 0$ in Section 3.1. It is convenient to represent \check{C}_r and the labels on this component by a snake trajectory ω^r , which may be defined as follows. Since the point c_0 has strict descendants in the tree \mathcal{T}_ζ and is not a branching point, we can make sense of the subtrajectory rooted at c_0 , which we denote by $\tilde{\omega}^r$ (see Section 2.1). We are in fact only interested in those descendants of c_0 that lie in \check{C}_r , and for this reason, we consider the truncation $\omega^r = \text{tr}_0(\tilde{\omega}^r)$.

Write $\mathcal{T}_\zeta(\omega^r)$ for the genealogical tree of ω^r and, as previously, let $\mathcal{T}_\zeta^\circ(\omega^r)$ denote the subset of $\mathcal{T}_\zeta(\omega^r)$ consisting of points with positive labels. Then \check{C}_r is identified with $\mathcal{T}_\zeta^\circ(\omega^r)$ via a volume preserving isometry, in such a way that, for every $a \in \check{C}_r$, we have $V_a(\omega) = r + V_{\tilde{a}}(\omega^r)$ if \tilde{a} is the point of $\mathcal{T}_\zeta^\circ(\omega^r)$ corresponding to a . Consequently, for every $\delta \geq 0$, connected components of $\{a \in \mathcal{T}_\zeta : V_a(\omega) > r + \delta\}$ contained in \check{C}_r are in one-to-one correspondence with connected components of $\{a \in \mathcal{T}_\zeta(\omega^r) : V_a(\omega) > \delta\}$. The latter fact will be important for our applications in Section 8 below.

We call $\omega^r, r \in D$, the excursions of ω above the minimum. We refer to [1], Section 3, for a (slightly different) more detailed presentation.

The following theorem, which is one of the main results of [1], identifies the conditional distribution of the excursions $\omega^r, r \in D$, under \mathbb{N}_0 and conditionally on the exit measure process $(\mathcal{Z}_r)_{r < 0}$.

THEOREM 14 ([1], Proposition 36, Theorem 40). *$\mathbb{N}_0(d\omega)$ a.e., D coincides with the set of all discontinuity times of the process $(\mathcal{Z}_r)_{r < 0}$. We can thus write $D = \{r_1, r_2, \dots\}$ where r_1, r_2, \dots is the sequence of these discontinuity times ordered so that $|\Delta \mathcal{Z}_{r_1}| > |\Delta \mathcal{Z}_{r_2}| > \dots$. Then, under \mathbb{N}_0 and conditionally on $(\mathcal{Z}_r)_{r < 0}$, the random snake trajectories $\omega^{r_1}, \omega^{r_2}, \dots$ are independent, and for every $i \geq 1$ the conditional distribution of ω^{r_i} is $\mathbb{N}_0^{*, |\Delta \mathcal{Z}_{r_i}|}$.*

4. Measuring the boundary size of components above a level. We will now argue under $\mathbb{N}_0^{*,z}$ for a fixed $z > 0$. The measure $\mathbb{N}_0^{*,z}$ is supported on \mathcal{S}_0^+ , and so we may use the notation introduced in Section 3.1. Recall in particular that $\mathcal{T}_\zeta^\circ := \{a \in \mathcal{T}_\zeta : V_a > 0\}$.

For $a \in \mathcal{T}_\zeta^\circ, r \in [0, V_a)$ and $\varepsilon > 0$, we set

$$\mathcal{Z}_r^{(a),\varepsilon} := \varepsilon^{-2} \text{vol}(\{b \in \mathcal{C}_r^{(a)} : V_b \leq r + \varepsilon\}).$$

PROPOSITION 15. *The following properties hold $\mathbb{N}_0^{*,z}$ a.e.:*

(i) *For every $a \in \mathcal{T}_\zeta^\circ, (\mathcal{Z}_r^{(a),\varepsilon})_{r \in [0, V_a)}$ converges as $\varepsilon \rightarrow 0$, uniformly on $[0, V_a - \beta]$ for every $\beta > 0$, to a limiting càdlàg function $(\mathcal{Z}_r^{(a)})_{r \in [0, V_a)}$ with only negative jumps, which takes positive values on $[0, V_a)$ and is such that $\mathcal{Z}_0^{(a)} = z$.*

- (ii) If $a, a' \in \mathcal{T}_\zeta^\circ$, we have $Z_r^{(a)} = Z_r^{(a')}$ for every $r \in [0, \min_{c \in \llbracket a, a' \rrbracket} V_c)$.
- (iii) Let $a \in \mathcal{T}_\zeta^\circ$. The set of discontinuities of $r \mapsto Z_r^{(a)}$ is $D^{(a)}$. If $r \in D^{(a)}$ we have

$$Z_{r-}^{(a)} = Z_r^{(a)} + Z_r^{(b)},$$

where b is an arbitrary point of $\check{\mathcal{C}}_r^{(a)}$. Moreover, $Z_r^{(a)} \neq Z_r^{(b)}$.

PROOF. Recall from Proposition 8 the notation $Z_r^\varepsilon(\omega)$ for $\omega \in \mathcal{S}_0$ and $r < 0$, and the fact that $r \mapsto Z_r^\varepsilon(\omega)$ is càglàd. We let Θ_z be the set of all snake trajectories $\omega \in \mathcal{S}_0$ such that $W_*(\omega) < 0$ and:

- (a) $(Z_r^\varepsilon(\omega))_{r \in [W_*(\omega), 0)}$ converges as $\varepsilon \rightarrow 0$ to a limiting càdlàg function $(Z_r(\omega))_{r \in [W_*(\omega), 0)}$, uniformly on $[W_*(\omega), -\beta]$ for every $\beta \in (0, -W_*(\omega))$;
- (b) the set of discontinuity times of this limiting function is $D(\omega) \cap (W_*(\omega), 0)$;
- (c) the function $(Z_r(\omega))_{r \in [W_*(\omega), 0)}$ takes positive values on $[W_*(\omega), 0)$, and takes the value z for $r = W_*(\omega)$;
- (d) $Z_r(\omega) \neq |\Delta Z_r(\omega)|$ for every $r \in D(\omega) \cap (W_*(\omega), 0)$.

It follows from Proposition 8 and the first assertion of Theorem 14 that $\text{tr}_{L_z}(W)$ belongs to Θ_z, \mathbb{N}_0 a.e. on the event $\{L_z < 0\}$. We also use the fact that the process $(Z_{-r}(\omega))_{r > 0}$ evolves under \mathbb{N}_0 as a ϕ -CSBP (and, therefore, as the time change of a stable Lévy process) to obtain the property $Z_r(\omega) \neq |\Delta Z_r(\omega)|$ when $r \in D(\omega)$.

Taking $G = \mathbf{1}_{\Theta_z}$ in Proposition 10, we also get that, $\mathbb{N}_0^{*,z}(\text{d}\omega)$ a.s., for $\text{d}s$ a.e. $s \in [0, \sigma]$, the rerooted snake trajectory $W^{[s]}$ belongs to Θ_z . So let us fix $\omega \in \mathcal{S}_0$ such that the preceding assertion holds. We can then take a sequence s_1, s_2, \dots dense in $[0, \sigma]$ such that $\omega^{[s_i]}$ belongs to Θ_z for every $i = 1, 2, \dots$. Setting $a_i = p_\zeta(s_i)$, we also know that a_1, a_2, \dots all belong to \mathcal{T}_ζ° (otherwise $W_*(\omega^{[s_i]}) = 0$). We now observe that $W_*(\omega^{[s_i]}) = -V_{a_i}(\omega)$, and, for every $r \in [0, V_{a_i}(\omega))$,

$$Z_r^{(a_i),\varepsilon}(\omega) = Z_{(r-V_{a_i}(\omega))^+}^\varepsilon(\omega^{[s_i]}).$$

This is a simple consequence of our definitions and formula (9) for the right limits Z_{r+}^ε .

Since $\omega^{[s_i]}$ belongs to Θ_z , we deduce from the last display and assertion (a) above that the convergence stated in part (i) of the proposition holds when $a = a_i$, and that, for every $r \in [0, V_{a_i}(\omega))$,

$$Z_r^{(a_i)}(\omega) = Z_{(r-V_{a_i}(\omega))^+}(\omega^{[s_i]}).$$

The function $r \mapsto Z_r^{(a_i)}(\omega)$ then satisfies the properties stated in (i). Moreover, it is immediate that the set of discontinuity times of this function is

$$\{V_{a_i} + r : r \in D(\omega^{[s_i]})\} = D^{(a_i)}(\omega),$$

where the last equality is again a consequence of our definitions. Furthermore, if $r \in D^{(a_i)}(\omega)$ and if j is an index such that $a_j \in \check{\mathcal{C}}_r^{(a_i)}$, the fact that $\mathcal{C}_r^{(a_i)}$ is the closure of the union $\mathcal{C}_r^{(a_i)} \cup \check{\mathcal{C}}_r^{(a_i)}$ implies that

$$Z_{r-}^{(a_i),\varepsilon}(\omega) = Z_r^{(a_i),\varepsilon}(\omega) + Z_r^{(a_j),\varepsilon}(\omega)$$

and by passing to the limit $\varepsilon \rightarrow 0$,

$$Z_{r-}^{(a_i)}(\omega) = Z_r^{(a_i)}(\omega) + Z_r^{(a_j)}(\omega).$$

Finally, property (d) gives $Z_r^{(a_i)}(\omega) \neq Z_r^{(a_j)}(\omega)$.

The preceding discussion shows that properties (i) and (iii) of the proposition hold if we restrict our attention to points in the dense sequence a_1, a_2, \dots . However, it readily follows from our definitions that we have $C_r^{(a)} = C_r^{(a_i)}$, and thus also $Z_r^{(a), \varepsilon} = Z_r^{(a_i), \varepsilon}$ as soon as $r < \min_{c \in \llbracket a, a_i \rrbracket} V_c$. We infer that we can define $(Z_r^{(a)})_{r \in [0, V_a]}$ in a unique way so that

$$Z_r^{(a)} = Z_r^{(a_i)}, \quad \text{for every } r < \min_{c \in \llbracket a, a_i \rrbracket} V_c, \text{ for every } i \geq 1.$$

It is then a simple matter to verify that assertions (i) and (iii) hold in the stated form, and assertion (ii) is also immediate. \square

5. The locally largest evolution. We say that a point $a \in \mathcal{T}_\zeta^\circ$ is regular if

$$\bigcap_{r \in [0, V_a)} C_r^{(a)} = \{a\}.$$

PROPOSITION 16. *There exists $\mathbb{N}_0^{*,z}$ a.e. a unique point a^\bullet of \mathcal{T}_ζ° such that the following two properties hold:*

- (i) *We have $Z_r^{(a^\bullet)} > |\Delta Z_r^{(a^\bullet)}|$, for every $r \in [0, V_a)$.*
- (ii) *The point a^\bullet is regular.*

We will call a^\bullet the terminal point of the locally largest evolution. Note that condition (i) is relevant only for $r \in D^{(a^\bullet)}$ since $Z^{(a^\bullet)}$ takes positive values.

PROOF. We first establish uniqueness. Suppose that a_1 and a_2 are two distinct points of \mathcal{T}_ζ° that satisfy the properties stated in (i) and (ii). We notice that we must have

$$\min_{c \in \llbracket a_1, a_2 \rrbracket} V_c < V_{a_1} \wedge V_{a_2}$$

because if the latter minimum is equal say to V_{a_1} the whole segment $\llbracket a_1, a_2 \rrbracket$ is contained in

$$\bigcap_{r \in [0, V_{a_1})} C_r^{(a_1)}$$

contradicting the regularity of a_1 . Set $u = \min_{c \in \llbracket a_1, a_2 \rrbracket} V_c$. By definition, we have then $u \in D^{(a_1)} \cap D^{(a_2)}$ and, by property (iii) in Proposition 15, we get

$$Z_{u-}^{(a_1)} = Z_{u-}^{(a_2)} = Z_u^{(a_1)} + Z_u^{(a_2)},$$

and thus $|\Delta Z_u^{(a_1)}| = Z_u^{(a_2)}$, $|\Delta Z_u^{(a_2)}| = Z_u^{(a_1)}$. This shows that property (i) in Proposition 16 cannot hold simultaneously for a_1 and for a_2 .

Let us turn to existence. We let r_∞ be the supremum of the set of all reals $r \geq 0$ such that there exists $a \in \mathcal{T}_\zeta^\circ$ with $V_a \geq r$ and $Z_s^{(a)} > |\Delta Z_s^{(a)}|$ for every $s < r$. By the definition of r_∞ , we can then find a nondecreasing sequence $(r_n)_{n \geq 1}$ in \mathbb{R}_+ and a corresponding sequence $(a_n)_{n \geq 1}$ in \mathcal{T}_ζ° such that $V_{a_n} \geq r_n$, $Z_s^{(a_n)} > |\Delta Z_s^{(a_n)}|$ for every $s < r_n$, and $r_n \uparrow r_\infty$ as $n \uparrow \infty$. By compactness, we may assume that $a_n \rightarrow a_\infty \in \mathcal{T}_\zeta$ as $n \uparrow \infty$, and it is clear that $a_\infty \in \mathcal{T}_\zeta^\circ$ (the case $V_{a_\infty} = 0$ is excluded since it would imply that $r_\infty = 0$).

Using property (ii) in Proposition 15, we have, for every n ,

$$Z_s^{(a_\infty)} > |\Delta Z_s^{(a_\infty)}|, \quad \text{for every } s < r_n \wedge \min_{c \in \llbracket a_n, a_\infty \rrbracket} V_c.$$

Since

$$\min_{c \in \llbracket a_n, a_\infty \rrbracket} V_c \xrightarrow{n \rightarrow \infty} V_{a_\infty} \geq r_\infty$$

(the last inequality because $V_{a_n} \geq r_n$ for every n), it follows that

$$Z_s^{(a_\infty)} > |\Delta Z_s^{(a_\infty)}|, \quad \text{for every } s < r_\infty.$$

We next claim that $r_\infty = V_{a_\infty}$. Otherwise, we would have $r_\infty < V_{a_\infty}$ and then either r_∞ would be a continuity time of $Z^{(a_\infty)}$, which immediately gives a contradiction with the definition of r_∞ , or r_∞ would be a discontinuity time of $Z^{(a_\infty)}$, and, by taking $a = a_\infty$ or $a \in \check{C}_{r_\infty}^{(a_\infty)}$, we would again get a contradiction with the definition of r_∞ .

It remains to verify that a_∞ is regular. If a_∞ is not regular, then we can choose

$$b \in \bigcap_{r \in [0, V_{a_\infty})} \mathcal{C}_r^{(a_\infty)}$$

with $V_b > V_{a_\infty} = r_\infty$. If r_∞ is a continuity point of $r \mapsto Z_r^{(b)}$, we get a contradiction with the definition of r_∞ . If r_∞ is a discontinuity point of $r \mapsto Z_r^{(b)}$, then by taking $a = b$ or $a \in \check{C}_{r_\infty}^{(b)}$, we again get a contradiction. This completes the proof. \square

Let $u \in (0, V_{a^\bullet})$. For future use, we note that, for every $a \in \mathcal{T}_\zeta^\circ$, we have $a \in \mathcal{C}_u^{(a^\bullet)}$ if and only if $V_a > u$ and $Z_r^{(a)} > |\Delta Z_r^{(a)}|$ for every $r \leq u$. The ‘‘only if’’ part is trivial. Conversely, assuming that $V_a > u$ and $Z_r^{(a)} > |\Delta Z_r^{(a)}|$ for every $r \leq u$, the property $a \notin \mathcal{C}_u^{(a^\bullet)}$ would lead to a contradiction by using Proposition 15 (iii) with $r = \min_{c \in \llbracket a^\bullet, a \rrbracket} V_c \leq u$.

6. The law of the locally largest evolution. Our next goal is to compute the distribution of $(Z_t^{(a^\bullet)})_{0 \leq t < V_{a^\bullet}}$ under $\mathbb{N}_0^{*,z}$.

PROPOSITION 17. *The process $(Z_t^{(a^\bullet)})_{0 \leq t < V_{a^\bullet}}$ is distributed under $\mathbb{N}_0^{*,z}$ as $(X_t)_{0 \leq t < H_0}$, where $(X_t)_{t \geq 0}$ is the self-similar Markov process with index $\frac{1}{2}$ starting from z , which can be represented as*

$$X_t = z \exp(\xi(\chi(z^{-1/2}t))),$$

where $(\xi(s))_{s \geq 0}$ is the Lévy process with only negative jumps whose Laplace exponent ψ is given by formula (1) and $(\chi(t))_{t \geq 0}$ is the time change defined in (2), and $H_0 = \inf\{t \geq 0 : X_t = 0\}$.

PROOF. We fix $u > 0$ and consider a bounded measurable function F on the Skorokhod space $\mathbb{D}([0, u], \mathbb{R})$. We observe that

$$F((Z_t^{(a^\bullet)})_{0 \leq t \leq u}) \mathbf{1}_{\{V_{a^\bullet} > u\}} = \int \text{vol}(da) F((Z_t^{(a)})_{0 \leq t \leq u}) \mathbf{1}_{\{a \in \mathcal{C}_u^{(a^\bullet)}\}} \frac{1}{\text{vol}(\mathcal{C}_u^{(a)})},$$

simply because if $a \in \mathcal{C}_u^{(a^\bullet)}$ we have $Z_t^{(a)} = Z_t^{(a^\bullet)}$ for $0 \leq t \leq u$ and $\mathcal{C}_u^{(a)} = \mathcal{C}_u^{(a^\bullet)}$. From the definition of $\text{vol}(\cdot)$ and a previous observation, the right-hand side can also be written as

$$\int_0^u ds F((Z_t^{(s)})_{0 \leq t \leq u}) \mathbf{1}_{\{\widehat{W}_s > u; Z_t^{(s)} > |\Delta Z_t^{(s)}|, \forall t \leq u\}} \frac{1}{\text{vol}(\mathcal{C}_u^{(p_\zeta(s))})},$$

where we have written $Z_t^{(s)} = Z_t^{(a)}$ if $a = p_\zeta(s)$ to simplify notation.

The preceding considerations show that

$$\begin{aligned} & \mathbb{N}_0^{*,z}(F((Z_t^{(a^\bullet)})_{0 \leq t \leq u}) \mathbf{1}_{\{V_{a^\bullet} > u\}}) \\ &= \mathbb{N}_0^{*,z}\left(\int_0^u ds F((Z_t^{(s)})_{0 \leq t \leq u}) \mathbf{1}_{\{\widehat{W}_s > u; Z_t^{(s)} > |\Delta Z_t^{(s)}|, \forall t \leq u\}} \frac{1}{\text{vol}(\mathcal{C}_u^{(p_\zeta(s))})}\right) \end{aligned}$$

$$= z^2 \mathbb{N}_0 \left(F((\mathcal{Z}_{(L_z+t)+})_{0 \leq t \leq u}) \right. \\ \left. \times \mathbf{1}_{\{L_z < -u; \mathcal{Z}_{(L_z+t)+} > |\Delta \mathcal{Z}_{L_z+t}|, \forall t \leq u\}} \frac{1}{\text{vol}(\mathcal{C}_{L_z+u})} \Big| L_z < 0 \right),$$

where for $r < 0$, we use (under \mathbb{N}_0) the notation \mathcal{C}_r for the connected component of $\{a \in \mathcal{T}_\zeta : V_a > r\}$ containing the ‘‘root’’ $p_\zeta(0)$. The second equality of the last display is a consequence of Proposition 10 and the way the functions $Z_t^{(a)}$ were constructed in Section 4.

In the terminology of [1], \mathcal{C}_{L_z+u} is (up to a set of zero volume) the union of the subsets of \mathcal{T}_ζ corresponding to the excursions above the minimum that start at a level greater than $L_z + u$. Using Theorem 14 and [1], Proposition 31, we obtain that the conditional distribution of $\text{vol}(\mathcal{C}_{L_z+u})$ under $\mathbb{N}_0(\cdot | L_z < 0)$ and knowing $(\mathcal{Z}_r)_{r < 0}$ is the law of

$$\sum_{i=1}^\infty |\Delta \mathcal{Z}_{r_i}|^2 \nu_i,$$

where r_1, r_2, \dots is an enumeration of the jumps of \mathcal{Z} on $(L_z + u, 0)$, and the random variables ν_1, ν_2, \dots are independent and distributed according to the density

$$\frac{1}{\sqrt{2\pi}} x^{-5/2} \exp\left(-\frac{1}{2x}\right) \mathbf{1}_{\{x > 0\}}.$$

Writing $\mathbb{E}^{(v)}[\cdot]$ for the expectation with respect to the variables ν_1, ν_2, \dots , we can thus also write

$$(19) \quad \mathbb{N}_0^{*,z} (F((Z_t^{(a^*)})_{0 \leq t \leq u}) \mathbf{1}_{\{V_{a^*} > u\}}) \\ = z^2 \mathbb{N}_0 \left(F((\mathcal{Z}_{L_z+t})_{0 \leq t \leq u}) \right. \\ \left. \times \mathbf{1}_{\{L_z < -u; \mathcal{Z}_{(L_z+t)+} > |\Delta \mathcal{Z}_{L_z+t}|, \forall t \leq u\}} \mathbb{E}^{(v)} \left[\frac{1}{\sum |\Delta \mathcal{Z}_{r_i}|^2 \nu_i} \right] \Big| L_z < 0 \right),$$

and the right-hand side is an integral under \mathbb{N}_0 of a quantity depending only on the exit measure process $(\mathcal{Z}_r)_{r < 0}$.

Thanks to Proposition 11, we can replace the right-hand side of (19) by

$$(20) \quad z^2 \mathbb{E} \left[F((X_t^\circ)_{0 \leq t \leq u}) \mathbf{1}_{\{H_0^\circ > u; X_t^\circ > |\Delta X_t^\circ|, \forall t \leq u\}} \mathbb{E}^{(v)} \left[\frac{1}{\sum |\Delta X_{s_i}^\circ|^2 \nu_i} \right] \right],$$

where s_1, s_2, \dots is an enumeration of the jump times of X° over $[u, H_0^\circ)$. Then, by the Markov property and the self-similarity of X° , the conditional expectation of the quantity

$$\mathbb{E}^{(v)} \left[\frac{1}{\sum |\Delta X_{s_i}^\circ|^2 \nu_i} \right]$$

given $(X_t^\circ)_{0 \leq t \leq u}$ is $C/(X_u^\circ)^2$, for some constant $C \in (0, \infty)$ (the cases $C = 0$ and $C = \infty$ are excluded since the preceding equalities would give an absurd statement). So the quantity (20) is also equal to C times

$$(21) \quad z^2 \mathbb{E} [F((X_t^\circ)_{0 \leq t \leq u}) \mathbf{1}_{\{H_0^\circ > u; X_t^\circ > |\Delta X_t^\circ|, \forall t \leq u\}} (X_u^\circ)^{-2}].$$

We will rewrite this quantity in a different form. In the remaining part of the proof, we take $z = 1$ for the sake of simplicity (of course the self-similarity of X° will then allow us to

get a similar result for an arbitrary value of z). Using the representation in Proposition 11, we obtain that the quantity (21) is equal for $z = 1$ to

$$(22) \quad \mathbb{E}[F((\exp(\xi^\circ(\chi^\circ(t))))_{0 \leq t \leq u}) \mathbf{1}_{\{\chi^\circ(u) < \infty\}} \times \mathbf{1}_{\{\Delta \xi^\circ(s) > -\log 2, \forall s \in [0, \chi^\circ(u)]\}} \exp(-2\xi^\circ(\chi^\circ(u)))]$$

LEMMA 18. For every $v \geq 0$, set

$$M_v = \mathbf{1}_{\{\Delta \xi^\circ(s) > -\log 2, \forall s \in [0, v]\}} \exp(-2\xi^\circ(v)).$$

Then $(M_v)_{v \geq 0}$ is a martingale with respect to the canonical filtration of the process ξ° . Let ξ be as in Proposition 17. Then, for every fixed $v > 0$ the process $(\xi^\circ(t))_{0 \leq t \leq v}$ is distributed under the probability measure $M_v \cdot \mathbb{P}$ as $(\xi(t))_{0 \leq t \leq v}$ under \mathbb{P} .

PROOF. To simplify notation, we write $\alpha = \sqrt{3/2\pi}$. Thanks to the properties of Lévy processes, in order to verify that $(M_v)_{v \geq 0}$ is a martingale, it suffices to prove that $\mathbb{E}[M_v] = 1$ for every $v > 0$. It is convenient to set

$$\xi''(t) = \sum_{0 \leq s \leq t} \Delta \xi^\circ(s) \mathbf{1}_{\{\Delta \xi^\circ(s) \leq -\log 2\}},$$

so that we can write $\xi^\circ(t) = \xi'(t) + \xi''(t)$, where ξ' and ξ'' are two independent Lévy processes. The Laplace exponent of ξ'' is

$$\psi''(\lambda) = \alpha \int_{-\infty}^{-\log 2} (e^{\lambda y} - 1) e^{y/2} (1 - e^y)^{-5/2} dy,$$

and the Laplace exponent of ξ' is

$$(23) \quad \begin{aligned} \psi'(\lambda) &= \psi^\circ(\lambda) - \psi''(\lambda) \\ &= \alpha \left(\int_{-\log 2}^0 (e^{\lambda y} - 1 - \lambda(e^y - 1)) e^{y/2} (1 - e^y)^{-5/2} dy \right. \\ &\quad \left. - \lambda \int_{-\infty}^{-\log 2} (e^y - 1) e^{y/2} (1 - e^y)^{-5/2} dy \right) \\ &= \alpha \left(\int_{-\log 2}^0 (e^{\lambda y} - 1 - \lambda(e^y - 1)) e^{y/2} (1 - e^y)^{-5/2} dy + 2\lambda \right) \end{aligned}$$

using the simple calculation

$$\int_{-\infty}^{-\log 2} e^{y/2} (1 - e^y)^{-3/2} dy = \int_0^{1/2} (1 - x)^{-3/2} \frac{dx}{\sqrt{x}} = 2.$$

Note that ξ' has bounded jumps and therefore exponential moments of any order, so that $\psi'(\lambda)$ makes sense for every $\lambda \in \mathbb{R}$ and not only $\lambda \geq 0$.

We have then

$$\mathbb{E}[M_v] = \mathbb{P}(\xi''(v) = 0) \mathbb{E}[e^{-2\xi'(v)}].$$

On one hand, $\mathbb{E}[e^{-2\xi'(v)}] = \exp(\alpha K v)$, where

$$\begin{aligned} K &= -4 + \int_{-\log 2}^0 (e^{-2y} - 1 + 2(e^y - 1)) e^{y/2} (1 - e^y)^{-5/2} dy \\ &= -4 + \int_{1/2}^1 (x^{-2} - 1 + 2(x - 1)) (1 - x)^{-5/2} \frac{dx}{\sqrt{x}} \end{aligned}$$

$$\begin{aligned}
 &= -4 + 2 \int_{1/2}^1 x^{-3/2}(1-x)^{-1/2} dx + \int_{1/2}^1 x^{-5/2}(1-x)^{-1/2} dx \\
 &= \frac{8}{3},
 \end{aligned}$$

where the last equality follows from classical formulas for incomplete Beta functions. On the other hand,

$$\mathbb{P}(\xi''(v) = 0) = \exp\left(-\alpha v \int_{-\infty}^{-\log 2} e^{y/2}(1-e^y)^{-5/2} dy\right) = \exp\left(-\frac{8}{3}\alpha v\right).$$

By combining the last two displays, we get the desired result $\mathbb{E}[M_v] = 1$.

Then, let us fix $v > 0$. It is straightforward to verify that the properties of stationarity and independence of the increments of ξ° are preserved under the probability measure $M_v \cdot \mathbb{P}$, so that $(\xi^\circ(t))_{0 \leq t \leq v}$ remains a Lévy process under this probability measure. To evaluate the Laplace exponent of this Lévy process, we write

$$\begin{aligned}
 \mathbb{E}[M_v e^{\lambda \xi^\circ(v)}] &= \exp\left(-\frac{8}{3}\alpha v\right) \mathbb{E}[e^{(\lambda-2)\xi'(v)}] = \exp\left(\left(\psi'(\lambda-2) - \frac{8}{3}\alpha\right)v\right) \\
 &= \exp(v\psi(\lambda)),
 \end{aligned}$$

where ψ is as in (1). The last equality follows from formula (23) for ψ' and simple calculations left to the reader. This completes the proof of the lemma. \square

Let us come back to (22). We first observe that, for every $v > 0$,

$$\begin{aligned}
 (24) \quad &\mathbb{E}[F((\exp(\xi^\circ(\chi^\circ(t))))_{0 \leq t \leq u}) \mathbf{1}_{\{\chi^\circ(u) \leq v\}} M_{\chi^\circ(u)}] \\
 &= \mathbb{E}[F((\exp(\xi^\circ(\chi^\circ(t))))_{0 \leq t \leq u}) \mathbf{1}_{\{\chi^\circ(u) \leq v\}} M_v] \\
 &= \mathbb{E}[F((\exp(\xi(\chi(t))))_{0 \leq t \leq u}) \mathbf{1}_{\{\chi(u) \leq v\}}],
 \end{aligned}$$

where $\chi(t)$ is as in formula (2). As $v \rightarrow \infty$, the right-hand side of (24) converges to $\mathbb{E}[F(\exp(\xi(\chi(t))))_{0 \leq t \leq u} \mathbf{1}_{\{\chi(u) < \infty\}}]$. Similarly, the left-hand side of (24) converges to the quantity (22): Dominated convergence is easily justified by noting that

$$\mathbb{E}[M_{\chi^\circ(u)} \mathbf{1}_{\{\chi^\circ(u) < \infty\}}] \leq \liminf_{t \rightarrow \infty} \mathbb{E}[M_{\chi^\circ(u) \wedge t} \mathbf{1}_{\{\chi^\circ(u) \leq t\}}]$$

and $\mathbb{E}[M_{\chi^\circ(u) \wedge t}] = \mathbb{E}[M_t] = 1$ by the optional stopping theorem.

Hence the quantity (22) is equal to $\mathbb{E}[F((\exp(\xi(\chi(t))))_{0 \leq t \leq u}) \mathbf{1}_{\{\chi(u) < \infty\}}]$. We conclude that

$$\mathbb{N}_0^{*,1}(F((Z_t^{(a^*)})_{0 \leq t \leq u}) \mathbf{1}_{\{V_{a^*} > u\}}) = C \mathbb{E}[F((\exp(\xi(\chi(t))))_{0 \leq t \leq u}) \mathbf{1}_{\{\chi(u) < \infty\}}].$$

At this stage, we can take $F = 1$ and let u tend to 0, and we find that $C = 1$. This gives the statement of Proposition 17 for $z = 1$, and it is easily extended by self-similarity.

7. Excursions from the locally largest evolution. If $\omega \in \mathcal{S}_0$ satisfies the regularity properties stated in Section 3.1, we can define the excursions above the minimum ω' , $r \in D$ in the way described in Section 3.2.

Let us now argue under $\mathbb{N}_0^{*,z}(d\omega)$ and for $u \in [0, \sigma]$, consider the re-rooted snake trajectory $\omega^{[u]}$. Let $a = p_\zeta(u)$ and recall that $D^{(a)}$ is the set of the discontinuity times of $(Z_r^{(a)})_{r \in [0, V_a]}$. As we already noticed in the proof of Proposition 15, we have $D^{(a)}(\omega) = \{V_a + r : r \in D(\omega^{[u]})\}$. If $r \in D^{(a)}(\omega)$ we can thus associate with $r - V_a \in D(\omega^{[u]})$ an excursion of $\omega^{[u]}$ above the minimum, which we denote by $\omega^{a,r}$ (it is easy to see that this excursion only

depends on a and not on u such that $a = p_\zeta(u)$. We already noticed that, if $a, a' \in \mathcal{T}_\zeta^\circ$ are such that $V_a \wedge V_{a'} > u$ and $\mathcal{C}_u^{(a)} = \mathcal{C}_u^{(a')}$, we have $D^{(a)} \cap [0, u] = D^{(a')} \cap [0, u]$, and it is also true that $\omega^{a,r} = \omega^{a',r}$ for $r \in D^{(a)} \cap [0, u]$.

We will now apply the preceding considerations to $a = a^\bullet$. We write $D^{(a^\bullet)} = \{r_1, r_2, \dots\}$, where $|\Delta Z_{r_1}^{(a^\bullet)}| > |\Delta Z_{r_2}^{(a^\bullet)}| > \dots$.

PROPOSITION 19. *Under $\mathbb{N}_0^{*,z}$, conditionally on $(Z_r^{(a^\bullet)})_{0 \leq r < V_{a^\bullet}}$, the excursions ω^{a^\bullet, r_i} , $i = 1, 2, \dots$ are independent, and for every fixed $i \geq 1$ the conditional distribution of ω^{a^\bullet, r_i} is $\mathbb{N}_0^{*, |\Delta Z_{r_i}^{(a^\bullet)}|}$.*

PROOF. We proceed in a way similar to the one used above to determine the law of $(Z_r^{(a^\bullet)}, 0 \leq r \leq V_{a^\bullet})$. We fix $u > 0$ and consider a bounded measurable function F on the Skorokhod space $\mathbb{D}([0, u], \mathbb{R}_+)$, and a bounded measurable function H on $\mathbb{R}_+ \times \mathcal{S}_0$ such that $H = 1$ on $\mathbb{R}_+ \times (\mathcal{S}_0 \setminus \mathcal{S}_0^{(\delta)})$ for some $\delta > 0$. The latter condition ensures that $H(r, \omega^{a^\bullet, r}) = 1$ except for finitely many values of $r \in D$. Then

$$\begin{aligned}
 & \mathbb{N}_0^{*,z} \left(F((Z_t^{(a^\bullet)})_{0 \leq t \leq u}) \mathbf{1}_{\{V_{a^\bullet} > u\}} \prod_{r \in D^{(a^\bullet)} \cap [0, u]} H(r, \omega^{a^\bullet, r}) \right) \\
 &= \mathbb{N}_0^{*,z} \left(\int_0^\sigma ds F((Z_t^{(s)})_{0 \leq t \leq u}) \mathbf{1}_{\{\widehat{W}_s > u; Z_t^{(s)} > |\Delta Z_t^{(s)}|, \forall t \leq u\}} \right) \\
 (25) \quad & \times \frac{1}{\text{vol}(\mathcal{C}_u^{(p_\zeta(s))})} \prod_{r \in D^{(p_\zeta(s))} \cap [0, u]} H(r, \omega^{p_\zeta(s), r}) \\
 &= z^2 \mathbb{N}_0 \left(F((\mathcal{Z}_{(L_z+t)+})_{0 \leq t \leq u}) \mathbf{1}_{\{L_z < -u; \mathcal{Z}_{(L_z+t)+} > |\Delta \mathcal{Z}_{L_z+t}|, \forall t \leq u\}} \right) \\
 & \times \frac{1}{\text{vol}(\mathcal{C}_{L_z+u})} \prod_{r \in D \cap [L_z, L_z+u]} H(r - L_z, \omega^r) \Big).
 \end{aligned}$$

We used the remarks preceding the statement of the proposition in the first equality, and Proposition 10 in the second one. At this stage, we use Theorem 14, which shows that conditionally on the exit measure process $(\mathcal{Z}_r)_{r < 0}$ (whose set of discontinuities is D) the excursions $\omega_r, r \in D$, are independent and the conditional distribution of ω^r is $\mathbb{N}_0^{*, |\Delta \mathcal{Z}_r|}$. Since the quantity $\text{vol}(\mathcal{C}_{L_z+u})$ only depends on the excursions ω^r with $r > L_z + u$, we can rewrite the last line of the preceding display as

$$\begin{aligned}
 & z^2 \mathbb{N}_0 \left(F((\mathcal{Z}_{(L_z+t)+})_{0 \leq t \leq u}) \mathbf{1}_{\{L_z < -u; \mathcal{Z}_{(L_z+t)+} > |\Delta \mathcal{Z}_{L_z+t}|, \forall t \leq u\}} \right) \\
 & \times \frac{1}{\text{vol}(\mathcal{C}_{L_z+u})} \prod_{r \in D \cap [L_z, L_z+u]} \mathbb{N}_0^{*, |\Delta \mathcal{Z}_r|} (H(r - L_z, \cdot))
 \end{aligned}$$

but then, we can reuse the same arguments “backwards” to see that the latter quantity is also equal to

$$(26) \quad \mathbb{N}_0^{*,z} \left(F((Z_t^{(a^\bullet)})_{0 \leq t \leq u}) \mathbf{1}_{\{V_{a^\bullet} > u\}} \prod_{r \in D^{(a^\bullet)} \cap [0, u]} \mathbb{N}_0^{*, |\Delta Z_r^{(a^\bullet)}|} (H(r, \cdot)) \right).$$

The quantity (26) is thus equal to the left-hand-side of (25). Simple arguments show that this also implies that, for any bounded measurable function G on the appropriate space of càdlàg

paths,

$$\begin{aligned} & \mathbb{N}_0^{*,z} \left(G((Z_t^{(a^\bullet)})_{0 \leq t < V_{a^\bullet}}) \prod_{r \in D^{(a^\bullet)}} H(r, \omega^{a^\bullet, r}) \right) \\ &= \mathbb{N}_0^{*,z} \left(G((Z_t^{(a^\bullet)})_{0 \leq t < V_{a^\bullet}}) \prod_{r \in D^{(a^\bullet)}} \mathbb{N}_0^{*, |\Delta Z_r^{(a^\bullet)}|} (H(r, \cdot)) \right). \end{aligned}$$

This gives the statement of the proposition. \square

We will call ω^{a^\bullet, r_i} , $i = 1, 2, \dots$ the excursions of ω from the locally largest evolution. The number r_i is called the starting level of ω^{a^\bullet, r_i} . As previously, these excursions will always be listed in decreasing order of their “boundary sizes” $|\Delta Z_{r_i}^{(a^\bullet)}|$. To simplify notation, we set $\omega^{(i)} = \omega^{a^\bullet, r_i}$.

8. The growth-fragmentation process. In this section, we argue again under $\mathbb{N}_0^{*,z}(d\omega)$. Recall that $\omega^{(1)}, \omega^{(2)}, \dots$ are the excursions of ω from the locally largest evolution and r_1, r_2, \dots stand for the respective starting levels of these excursions.

Next, for every $i \geq 1$, since the conditional distribution of $\omega^{(i)}$ knowing $(Z_r^{(a^\bullet)})_{0 \leq r < V_{a^\bullet}}$ is $\mathbb{N}_0^{*, |\Delta Z_{r_i}^{(a^\bullet)}|}$, we can also define a point a_i^\bullet as the terminal point of the locally largest evolution in $\omega^{(i)}$, and the excursions $\omega^{(i,1)}, \omega^{(i,2)}, \dots$ from the locally largest evolution in $\omega^{(i)}$ (ranked as explained at the end of the previous section). We write $r_{i,j}$ for the starting level of $\omega^{(i,j)}$.

Obviously, we can continue the construction by induction. Assuming that we have defined $\omega^{(i_1, \dots, i_k)}$, we let $a_{i_1, \dots, i_k}^\bullet$ be the terminal point of the locally largest evolution in $\omega^{(i_1, \dots, i_k)}$, and we denote the excursions from the locally largest evolution in $\omega^{(i_1, \dots, i_k)}$ by $\omega^{(i_1, \dots, i_k, 1)}, \omega^{(i_1, \dots, i_k, 2)}, \dots$. For every $j \geq 1$, we let $r_{i_1, \dots, i_k, j}$ be the starting level of $\omega^{(i_1, \dots, i_k, j)}$.

We also set, for every (i_1, \dots, i_k) ,

$$h_{i_1, \dots, i_k} = r_{i_1} + r_{i_1, i_2} + \dots + r_{i_1, \dots, i_k},$$

and we let β_{i_1, \dots, i_k} be the label of $a_{i_1, \dots, i_k}^\bullet$ (in $\omega^{(i_1, \dots, i_k)}$). Note that $h_{i_1, \dots, i_k} + \beta_{i_1, \dots, i_k}$ is the label of the point corresponding to $a_{i_1, \dots, i_k}^\bullet$ in ω .

Let $r \in [h_{i_1, \dots, i_k}, h_{i_1, \dots, i_k} + \beta_{i_1, \dots, i_k})$, and consider the connected component $\mathcal{C}_{r-h_{i_1, \dots, i_k}}^{(a_{i_1, \dots, i_k}^\bullet)}(\omega^{(i_1, \dots, i_k)})$: as in Section 3.1, this is the connected component of

$$\{a \in \mathcal{T}_\zeta(\omega^{(i_1, \dots, i_k)}) : V_a(\omega^{(i_1, \dots, i_k)}) > r - h_{i_1, \dots, i_k}\}$$

that contains $a_{i_1, \dots, i_k}^\bullet$. As explained in Section 3.2, this connected component corresponds (via a volume-preserving isometry) to a connected component of $\{a \in \mathcal{T}_\zeta(\omega^{(i_1, \dots, i_{k-1})}) : V_a(\omega^{(i_1, \dots, i_{k-1})}) > r - h_{i_1, \dots, i_{k-1}}\}$ and inductively to a connected component of $\{a \in \mathcal{T}_\zeta(\omega) : V_a(\omega) > r\}$. The latter component is denoted by $\mathcal{D}_r^{(i_1, \dots, i_k)}$. Recall that this definition makes sense only if $r \in [h_{i_1, \dots, i_k}, h_{i_1, \dots, i_k} + \beta_{i_1, \dots, i_k})$ (otherwise we may take $\mathcal{D}_r^{(i_1, \dots, i_k)} = \emptyset$).

We set

$$\mathcal{U} = \bigcup_{k=0}^\infty \mathbb{N}^k$$

with the convention $\mathbb{N}^0 = \{\emptyset\}$. We define $h_\emptyset = 0$, $\beta_\emptyset = V_{a^\bullet}(\omega)$ and $\mathcal{D}_r^\emptyset = \mathcal{C}_r^{(a^\bullet)}$ if $0 \leq r < V_{a^\bullet}(\omega)$.

LEMMA 20. *Let $r \geq 0$. The sets $\mathcal{D}_r^{(i_1, \dots, i_k)}$, for all $(i_1, \dots, i_k) \in \mathcal{U}$ such that $h_{i_1, \dots, i_k} \leq r < h_{i_1, \dots, i_k} + \beta_{i_1, \dots, i_k}$, are all the connected components of $\{a \in \mathcal{T}_\zeta : V_a > r\}$.*

PROOF. We already know that any of the sets $\mathcal{D}_r^{(i_1, \dots, i_k)}$, $(i_1, \dots, i_k) \in \mathcal{U}$, is a connected component of $\{a \in \mathcal{T}_\zeta : V_a > r\}$, and we need to show that any connected component is of this type. Let \mathcal{C} be a connected component of $\{a \in \mathcal{T}_\zeta : V_a > r\}$, and choose any $a \in \mathcal{C}$. The process $(Z_t^{(a)})_{0 \leq t \leq r}$ has only finitely many jump times $s \in [0, r]$ such that $|\Delta Z_s^{(a)}| > Z_s^{(a)}$ (note that $Z^{(a)}$ is bounded below by a positive constant on $[0, r]$). We denote these jump times by $0 < t_1 < t_2 < \dots < t_k \leq r$, where $k \geq 0$.

- If $k = 0$, this means that $|\Delta Z_s^{(a)}| < Z_s^{(a)}$ for every $s \in [0, r]$, and we have already seen that this implies that $a \in \mathcal{C}_r^{(a^\bullet)}$. We have thus $\mathcal{C} = \mathcal{C}_r^{(a^\bullet)} = \mathcal{D}_r^\emptyset$ in that case.

- Suppose that $k \geq 1$. We have $Z_s^{(a)} = Z_s^{(a^\bullet)}$ if and only if $0 \leq s < t_1$. In particular $a \in \mathcal{C}_s^{(a^\bullet)}$ if and only if $0 \leq s < t_1$, so that a belongs to $a \in \mathcal{C}_{t_1-}^{(a^\bullet)}$, which is the closure of $\mathcal{C}_{t_1}^{(a^\bullet)} \cup \check{\mathcal{C}}_{t_1}^{(a^\bullet)}$. Since points of the boundary of $\mathcal{C}_{t_1}^{(a^\bullet)} \cup \check{\mathcal{C}}_{t_1}^{(a^\bullet)}$ have label t_1 whereas $V_a > t \geq t_1$, it follows that $a \in \check{\mathcal{C}}_{t_1}^{(a^\bullet)}$, and $\mathcal{C} \subset \check{\mathcal{C}}_{t_1}^{(a^\bullet)}$. Furthermore, t_1 is a jump time of $Z^{(a^\bullet)}$, so that we have $t_1 = r_{i_1}$ for some $i_1 \geq 1$. As explained in Section 3.2, $\check{\mathcal{C}}_{t_1}^{(a^\bullet)}$ is identified with $\mathcal{T}_\zeta^\circ(\omega^{(i_1)})$, and through this identification \mathcal{C} is identified with a connected component \mathcal{C}' of $\{b \in \mathcal{T}_\zeta^\circ(\omega^{(i_1)}) : V_b(\omega^{(i_1)}) > r - r_{i_1}\}$ and a is identified with a point a' of \mathcal{C}' . We have then

$$(Z_t^{(a')}(\omega^{(i_1)}), 0 \leq t \leq r - t_1) = (Z_{r_1+t}^{(a)}, 0 \leq t \leq r - t_1).$$

In particular, if $k = 1$, there are no jump times $s \in [0, r - t_1]$ such that $|\Delta Z_s^{(a')}(\omega^{(i_1)})| > Z_s^{(a')}(\omega^{(i_1)})$, and we conclude that $\mathcal{C}' = \mathcal{C}_{r-r_{i_1}}^{(a_1^\bullet)}(\omega^{(i_1)})$, which means that $\mathcal{C} = \mathcal{D}_r^{(i_1)}$.

- Suppose $k \geq 2$. Then $t_2 - t_1$ is the first jump time of $(Z_t^{(a')}(\omega^{(i_1)}))_{0 \leq t \leq r-t_1}$ such that $|\Delta Z_s^{(a')}(\omega^{(i_1)})| > Z_s^{(a')}(\omega^{(i_1)})$, we have $Z_s^{(a_1^\bullet)}(\omega^{(i_1)}) = Z_s^{(a')}(\omega^{(i_1)})$ for $0 \leq s < t_2 - t_1$, and there exists $i_2 \geq 1$ such that $t_2 - t_1 = r_{i_1, i_2}$. We have then $\mathcal{C}' \subset \check{\mathcal{C}}_{t_2-t_1}^{(a_1^\bullet)}(\omega^{(i_1)})$. It follows that \mathcal{C}' is identified with a connected component \mathcal{C}'' of $\{b \in \mathcal{T}_\zeta^\circ(\omega^{(i_1, i_2)}) : V_b(\omega^{(i_1, i_2)}) > r - r_{i_1} - r_{i_1, i_2}\}$. If $k = 2$, we conclude as in the preceding step that $\mathcal{C}'' = \mathcal{C}_{r-h_{i_1, i_2}}^{(a_{i_1, i_2}^\bullet)}(\omega^{(i_1, i_2)})$, which means that $\mathcal{C} = \mathcal{D}_r^{(i_1, i_2)}$.

The proof is easily completed by induction, and we omit the details. \square

PROOF OF THEOREM 2. The first part of Theorem 2 (concerning the definition and approximation of boundary sizes) is a consequence of Proposition 15, which in fact gives a stronger result. So we only need to prove the second part of the statement. If \mathcal{C} is a connected component of $\{a \in \mathcal{T}_\zeta : V_a > r\}$, we write $Z_{(\mathcal{C})} = Z_r^{(a)}$ where a is any point of \mathcal{C} (this does not depend on the choice of a). To simplify notation, for every $(i_1, \dots, i_k) \in \mathcal{U}$ and every $j \geq 1$, we write $\Delta_{i_1, \dots, i_k, j}$ for the jump at time $h_{i_1, \dots, i_k, j}$ of the function $r \mapsto Z_{(\mathcal{D}_r^{(i_1, \dots, i_k)})}$ —from our construction this is also the jump at time $r_{i_1, \dots, i_k, j}$ of the function $r \mapsto Z_r^{(a_{i_1, \dots, i_k}^\bullet)}(\omega^{(i_1, \dots, i_k)})$.

From the preceding lemma, we get that $\mathbf{Y}(r)$ is obtained as the collection of the quantities $Z_{(\mathcal{D}_r^{(i_1, \dots, i_k)})}$ (ranked in nondecreasing order) for all $(i_1, \dots, i_k) \in \mathcal{U}$ such that $h_{i_1, \dots, i_k} \leq r < h_{i_1, \dots, i_k} + \beta_{i_1, \dots, i_k}$.

We know that the process $(Z_{(\mathcal{D}_r^\emptyset)})_{0 \leq r < \beta_\emptyset} = (Z_r^{(a^\bullet)})_{0 \leq r < V_a^\bullet}$ is distributed as the self-similar Markov process X of Proposition 17 started from z and killed when it hits 0. Thanks to Proposition 19, we then get that, conditionally on $(Z_{(\mathcal{D}_r^\emptyset)})_{0 \leq r < \beta_\emptyset}$, the excursions $\omega^{(i)}$, $i = 1, 2, \dots$, are independent and for every fixed j the conditional distribution of

$\omega^{(j)}$ is $\mathbb{N}_0^{*,|\Delta_j|}$. Consequently, under the same conditioning, the processes $(Z_{(\mathcal{D}_{h_i+r}^{(i)})})_{0 \leq r \leq \beta_i}$, $i = 1, 2, \dots$, are independent copies of X started respectively at $|\Delta_i|$, $i = 1, 2, \dots$.

We can continue by induction, using Proposition 19 at every step. We obtain that, conditionally on the processes

$$(Z_{(\mathcal{D}_r^{(i_1, \dots, i_\ell)})})_{h_{i_1, \dots, i_\ell} \leq r < h_{i_1, \dots, i_\ell} + \beta_{i_1, \dots, i_\ell}}, \quad 0 \leq \ell \leq k, (i_1, \dots, i_\ell) \in \mathbb{N}^\ell,$$

the excursions $\omega^{(i_1, \dots, i_k, j)}$, $(i_1, \dots, i_k, j) \in \mathbb{N}^{k+1}$, are independent and for every fixed (i_1, \dots, i_k, j) the conditional distribution of $\omega^{(i_1, \dots, i_k, j)}$ is $\mathbb{N}_0^{*,|\Delta_{i_1, \dots, i_k, j}|}$. Hence, under the same conditioning, the processes

$$(Z_{(\mathcal{D}_{h_{i_1, \dots, i_k, j+r}}^{(i_1, \dots, i_k, j)})})_{0 \leq r \leq \beta_{i_1, \dots, i_k, j}}, \quad (i_1, \dots, i_k, j) \in \mathbb{N}^{k+1}$$

are independent copies of X started respectively at $|\Delta_{i_1, \dots, i_k, j}|$.

From these observations, we conclude that $(\mathbf{Y}(r))_{r \geq 0}$ is a growth fragmentation process whose Eve particle process is the self-similar Markov process X and with initial value $\mathbf{Y}(0) = (z, 0, 0, \dots)$. \square

PROOF OF THEOREM 1. Under \mathbb{N}_0 , the connected components $\mathcal{C}_1, \mathcal{C}_2, \dots$ of $\{a \in \mathcal{T}_\zeta : V_a > 0\}$, and the labels on these components can be represented by snake trajectories $\omega_1, \omega_2, \dots$ defined in a way very similar to the excursions above the minimum in Section 3.2 (see also the introduction of [1]). By [1], Theorem 4, conditionally on the boundary sizes $(|\partial\mathcal{C}_1|, |\partial\mathcal{C}_2|, \dots)$, $\omega_1, \omega_2, \dots$ are independent and the conditional distribution of ω_i is $\mathbb{N}_0^{*,|\partial\mathcal{C}_i|}$. In the notation of Theorem 1, $\mathbf{X}(0)$ is just the (ranked) sequence $(|\partial\mathcal{C}_1|, |\partial\mathcal{C}_2|, \dots)$, and we get that, conditionally on $\mathbf{X}(0)$, the process $(\mathbf{X}(r))_{r \geq 0}$ is obtained by superimposing independent processes $(\mathbf{Y}_i(r))_{r \geq 0}$ such that, for every $i \geq 1$, \mathbf{Y}_i is a growth-fragmentation process started from $(|\partial\mathcal{C}_i|, 0, 0, \dots)$ (by Theorem 2). The desired result follows. \square

9. Slicing the Brownian disk at heights. In this section, we prove Theorem 3. We rely on the construction of the free Brownian disk \mathbb{D}_z from a random snake trajectory distributed according to $\mathbb{N}_0^{*,z}$. This construction is given in [28], to which we refer for additional details. Throughout this section, we argue under $\mathbb{N}_0^{*,z}$, and the following statements hold $\mathbb{N}_0^{*,z}$ a.s.

The free Brownian disk \mathbb{D}_z is a random geodesic compact metric space, which is constructed (under $\mathbb{N}_0^{*,z}$) as a quotient space of \mathcal{T}_ζ . The canonical projection, which is a continuous mapping from \mathcal{T}_ζ onto \mathbb{D}_z , is denoted by Π . We note that the pushforward of the volume measure vol on \mathcal{T}_ζ is the volume measure \mathbf{V} on \mathbb{D}_z .

Recall the notation $H(x)$, for the ‘‘height’’ of $x \in \mathbb{D}_z$ (the distance from x to the boundary $\partial\mathbb{D}_z$). We will not need the details of the construction of \mathbb{D}_z , but we record the following two facts:

- (a) If $a \in \mathcal{T}_\zeta$ and $x = \Pi(a)$, we have $H(x) = V_a$.
- (b) For every $a, b \in \mathcal{T}_\zeta$ such that $\Pi(a) = \Pi(b)$, we have $V_a = V_b = \min_{c \in \llbracket a, b \rrbracket} V_c$.

The following lemma is an analog for the Brownian disk of Proposition 3.1 of [24] for the Brownian map. The proof is similar, but we provide details because this result is the key to the derivation of Theorem 3.

LEMMA 21. *Let $a, b \in \mathcal{T}_\zeta$ and let $(\gamma(t))_{0 \leq t \leq T}$ be a continuous path in \mathbb{D}_z such that $\gamma(0) = \Pi(a)$ and $\gamma(T) = \Pi(b)$. Then*

$$\min_{0 \leq t \leq T} H(\gamma(t)) \leq \min_{c \in \llbracket a, b \rrbracket} V_c.$$

PROOF. We may assume that

$$V_a \wedge V_b > \min_{c \in \llbracket a, b \rrbracket} V_c$$

since the result is trivial otherwise. Then we can find $c_0 \in \llbracket a, b \rrbracket$ such that

$$V_{c_0} = \min_{c \in \llbracket a, b \rrbracket} V_c.$$

The points a and b are in different connected components of $\mathcal{T}_\zeta \setminus \{c_0\}$. Let \mathcal{C}_1 be the connected component of $\mathcal{T}_\zeta \setminus \{c_0\}$ that contains a , and let $\mathcal{C}_2 = \mathcal{T}_\zeta \setminus \overline{\mathcal{C}}_1$, so that $b \in \mathcal{C}_2$. Set

$$t_0 := \inf\{t \in [0, T] : \gamma(t) \in \Pi(\overline{\mathcal{C}}_2)\}.$$

Since $\Pi(\overline{\mathcal{C}}_2)$ is closed, we have $\gamma(t_0) \in \Pi(\overline{\mathcal{C}}_2)$. Furthermore, $t_0 > 0$ because otherwise this would mean that $\Pi(a) \in \Pi(\overline{\mathcal{C}}_2)$, and thus that there would exist $a' \in \overline{\mathcal{C}}_2$ such that $\Pi(a) = \Pi(a')$: Noting that $c_0 \in \llbracket a, a' \rrbracket$, property (b) above would imply that $V_a \leq V_{c_0}$, which is a contradiction.

We can then choose a sequence $(s_n)_{n \geq 1}$ in $[0, t_0]$ such that $s_n \uparrow t_0$ as $n \uparrow \infty$. Since $\gamma(s_n) \in \Pi(\mathcal{C}_1)$, there exists $a_n \in \mathcal{C}_1$ such that $\gamma(s_n) = \Pi(a_n)$. Up to extracting a subsequence, we can assume that $a_n \rightarrow a_\infty \in \overline{\mathcal{C}}_1$. Then necessarily $\Pi(a_\infty) = \gamma(t_0) = \Pi(b')$ for some $b' \in \overline{\mathcal{C}}_2$. By properties (a) and (b), we must have

$$H(\gamma(t_0)) = V_{b'} = V_{a_\infty} = \min_{c \in \llbracket a_\infty, b' \rrbracket} V_c \leq V_{c_0}.$$

This completes the proof. \square

PROPOSITION 22. *Let $r > 0$ and $a, b \in \mathcal{T}_\zeta$. Then $\Pi(a)$ and $\Pi(b)$ belong to the same connected component of $\{x \in \mathbb{D}_z : H(x) > r\}$ if and only if a and b belong to the same connected component of $\{c \in \mathcal{T}_\zeta : V_c > r\}$.*

PROOF. If a and b belong to the same connected component of $\{c \in \mathcal{T}_\zeta : V_c > r\}$, then the line segment $\llbracket a, b \rrbracket$ is contained in $\{c \in \mathcal{T}_\zeta : V_c > r\}$, and $\Pi(\llbracket a, b \rrbracket)$ provides a path going from $\Pi(a)$ to $\Pi(b)$ that stays in $\{x \in \mathbb{D}_z : H(x) > r\}$, by property (a).

Conversely, if a and b belong to different connected components of $\{c \in \mathcal{T}_\zeta : V_c > r\}$, then

$$\min_{c \in \llbracket a, b \rrbracket} V_c \leq r,$$

and, by Lemma 21, any continuous path from $\Pi(a)$ to $\Pi(b)$ must visit a point x with $H(x) \leq r$. It follows that $\Pi(a)$ and $\Pi(b)$ belong to different connected components of $\{x \in \mathbb{D}_z : H(x) > r\}$. \square

PROOF OF THEOREM 3. By Proposition 22, for every $r \geq 0$, the projection Π induces a one-to-one correspondence between connected components of $\{c \in \mathcal{T}_\zeta : V_c > r\}$ and connected components of $\{x \in \mathbb{D}_z : H(x) > r\}$. Furthermore, let \mathcal{D} be a connected component of $\{x \in \mathbb{D}_z : H(x) > r\}$, and let \mathcal{C} be the associated connected component of $\{c \in \mathcal{T}_\zeta : V_c > r\}$ (such that $\Pi(\mathcal{C}) = \mathcal{D}$). Together with property (a) above, the fact that Π maps the volume measure vol to \mathbf{V} immediately shows that the boundary size $|\partial \mathcal{D}|$ can be defined by the approximation in Theorem 3, and that $|\partial \mathcal{D}| = |\partial \mathcal{C}|$. Theorem 3 is now a direct consequence of Theorem 2. \square

10. The law of components above a fixed level. Our goal in this section is to prove Theorem 4. To this end, we will first state and prove a theorem about excursions “above a fixed height” for a snake trajectory distributed according to $\mathbb{N}_0^{*,z}$.

Let us fix $r \geq 0$. Let $\omega \in \mathcal{S}_0$ be chosen according to \mathbb{N}_0 , or to $\mathbb{N}_0^{*,z}$, and consider all connected components of $\{a \in \mathcal{T}_\zeta : V_a > r\}$. If \mathcal{C} is one of these connected components, we can represent \mathcal{C} and the labels on \mathcal{C} by a snake trajectory $\tilde{\omega}$, which is defined as follows. First, we observe that there is a unique $a_0 \in \mathcal{T}_\zeta$ such that $a_0 \in \partial\mathcal{C}$ and every point of \mathcal{C} is a descendant of a_0 . Note that $V_{a_0} = r$, and that the point a_0 cannot be a branching point (no branching point can have label r , \mathbb{N}_0 a.e. or $\mathbb{N}_0^{*,z}$ a.s.). Hence we can make sense of the subtrajectory rooted at a_0 , which we denote by $\tilde{\omega}$. Finally, we let $\tilde{\omega} = \text{tr}_0(\tilde{\omega})$.

We can define the boundary size $\mathcal{Z}_0^*(\tilde{\omega})$ of $\tilde{\omega}$, using Proposition 15 (setting $\mathcal{Z}_0^*(\tilde{\omega}) = \mathcal{Z}_r^{(a)}$, where a is an arbitrary point of \mathcal{C}) if ω is chosen according to $\mathbb{N}_0^{*,z}$, or the excursion theory of [1] if ω is chosen according to \mathbb{N}_0 . We call the snake trajectories $\tilde{\omega}$ obtained when varying \mathcal{C} among the connected components of $\{a \in \mathcal{T}_\zeta : V_a > r\}$ the excursions of ω above level r .

THEOREM 23. *Let $r > 0$. On the event $\{W^*(\omega) > r\}$, let $\tilde{\omega}^1, \tilde{\omega}^2, \dots$ be the excursions of ω above level r , ranked in decreasing order of their boundary sizes. Write $\tilde{z}_1 > \tilde{z}_2 > \dots$ for these boundary sizes. Then, under $\mathbb{N}_0^{*,z}(\cdot \mid W^* > r)$, conditionally on the collection $(\tilde{z}_i)_{i \geq 1}$, the snake trajectories $\tilde{\omega}^1, \tilde{\omega}^2, \dots$ are independent with respective distributions $\mathbb{N}_0^{*,\tilde{z}_1}, \mathbb{N}_0^{*,\tilde{z}_2}, \dots$*

REMARK. It is not immediately obvious that the boundary sizes $\tilde{z}^1, \tilde{z}^2, \dots$ are distinct a.s. This can however be deduced from the arguments of the proof below.

PROOF OF THEOREM 23. We will derive the theorem from the excursion theory of [1], and to this end we first need to argue under $\mathbb{N}_0(d\omega)$. We write $\omega^1, \omega^2, \dots$ for the excursions above 0 ranked in decreasing order of their boundary sizes $\mathcal{Z}_0^*(\omega^1), \mathcal{Z}_0^*(\omega^2), \dots$. Theorem 4 in [1] then implies that, under \mathbb{N}_0 and conditionally on $\mathcal{Z}_0^*(\omega^1) = z_1, \mathcal{Z}_0^*(\omega^2) = z_2, \dots$, the excursions $\omega^1, \omega^2, \dots$ are independent and the conditional distribution of ω^i is \mathbb{N}_0^{*,z_i} .

Let A be the event where exactly one excursion above 0 hits r , and let ω^{i_0} be this excursion. It follows from the preceding observations that, under $\mathbb{N}_0(\cdot \mid A)$, the conditional distribution of ω^{i_0} knowing $\mathcal{Z}_0^*(\omega^{i_0}) = z$ is $\mathbb{N}_0^{*,z}(\cdot \mid W^* > r)$. Hence, if φ is a bounded nonnegative measurable function on \mathbb{R}_+ , and h is a nonnegative measurable function on \mathcal{S}_0 , we have

$$\begin{aligned}
 (27) \quad & \mathbb{N}_0 \left(\mathbf{1}_A \varphi(\mathcal{Z}_0^*(\omega^{i_0})) \exp \left(- \sum_{i=1}^{\infty} h(\tilde{\omega}^i) \right) \right) \\
 & = \mathbb{N}_0 \left(\mathbf{1}_A \varphi(\mathcal{Z}_0^*(\omega^{i_0})) \mathbb{N}_0^{*,\mathcal{Z}_0^*(\omega^{i_0})} \left(\exp \left(- \sum_{i=1}^{\infty} h(\tilde{\omega}^i) \right) \right) \right),
 \end{aligned}$$

where we use the notation $\tilde{\omega}^1, \tilde{\omega}^2, \dots$ introduced in the theorem for the excursions above level r (notice that this makes sense both under \mathbb{N}_0 and under $\mathbb{N}_0^{*,z}$ and that, on the event A , the excursions of ω and of ω^{i_0} above level r are the same).

We will now rewrite the left-hand side of (27) in a different form. To this end (arguing under $\mathbb{N}_0(d\omega \mid W^* > r)$), it is convenient to introduce all excursions of ω away from r : each such excursion $\bar{\omega}^i, i = 1, 2, \dots$, corresponds to one connected component of $\{a \in \mathcal{T}_\zeta : V_a \neq r\}$, but we exclude the connected component containing the root ρ (which may be represented by $\text{tr}_r(\omega)$) and, apart from this fact, the definition of these excursions is exactly the same as that of excursions above level r . As previously, the excursions $\bar{\omega}^i, i = 1, 2, \dots$ are listed in decreasing order of their boundary sizes $\bar{z}^i := \mathcal{Z}_0^*(\bar{\omega}^i), i = 1, 2, \dots$. For every $i \geq 1$, we also

let $\bar{\varepsilon}^i$ be the sign of $\bar{\omega}^i$: $\bar{\varepsilon}^i = 1$ if the values of V on the corresponding connected component are greater than r and $\bar{\varepsilon}^i = -1$ otherwise. The sequence $\tilde{\omega}^1, \tilde{\omega}^2, \dots$ is obtained by keeping only the excursions $\bar{\omega}^i$ with $\bar{\varepsilon}^i = 1$ in the sequence $\bar{\omega}^i, i = 1, 2, \dots$. Let \mathcal{B} be the σ -field generated by $\text{tr}_r(\omega)$, the sequence $(\bar{\varepsilon}^i, \bar{z}^i)_{i=1,2,\dots}$ and the excursions $\bar{\omega}^i$ for all i such that $\bar{\varepsilon}^i = -1$ (in other words the excursions below level r). By combining the special Markov property with [1], Theorem 4, we get that conditionally on \mathcal{B} the excursions $\tilde{\omega}^i, i = 1, 2, \dots$ are independent and the conditional distribution of $\tilde{\omega}^i$ is $\mathbb{N}_0^{*,\tilde{z}_i}$, where we write $\tilde{z}_i = \mathcal{Z}_0^*(\tilde{\omega}^i)$ as in the statement of the theorem—notice that the quantities \tilde{z}_i are \mathcal{B} -measurable.

The point is now that the event A (and the variable $\mathbf{1}_A \varphi(\mathcal{Z}_0^*(\omega^{i_0}))$) is \mathcal{B} -measurable. In fact, it is not hard to check that A is determined by the knowledge of $\text{tr}_r(\omega)$ and of the excursions below level r (for A to hold, no such excursion is allowed to contain a path that comes back to 0 and then visits r again). Thanks to this observation, we can rewrite the left-hand side of (27) as

$$\mathbb{N}_0 \left(\mathbf{1}_A \varphi(\mathcal{Z}_0^*(\omega^{i_0})) \prod_{i=1}^{\infty} \mathbb{N}_0^{*,\tilde{z}_i}(e^{-h}) \right).$$

By the same argument that led us to (27), this is also equal to

$$(28) \quad \mathbb{N}_0 \left(\mathbf{1}_A \varphi(\mathcal{Z}_0^*(\omega^{i_0})) \mathbb{N}_0^{*,\mathcal{Z}_0^*(\omega^{i_0})} \left(\prod_{i=1}^{\infty} \mathbb{N}_0^{*,\tilde{z}_i}(e^{-h}) \right) \right).$$

Notice that the law of $\mathcal{Z}_0^*(\omega^{i_0})$ under $\mathbb{N}_0(\cdot | A)$ has a positive density with respect to Lebesgue measure (under \mathbb{N}_0 , the boundary sizes of excursions away from 0 are the jumps of a ϕ -CSBP under its excursion measure; see [1], Theorem 4). The equality between the quantity (28) and the right-hand side of (27) for every function φ implies that we have

$$(29) \quad \mathbb{N}_0^{*,z} \left(\exp \left(- \sum_{i=1}^{\infty} h(\tilde{\omega}^i) \right) \right) = \mathbb{N}_0^{*,z} \left(\prod_{i=1}^{\infty} \mathbb{N}_0^{*,\tilde{z}_i}(e^{-h}) \right)$$

for Lebesgue a.a. $z > 0$. We claim that (29) in fact holds for every $z > 0$. To see this, we need a continuity argument. We restrict our attention to functions h of the type $h(\omega) = h_1(\mathcal{Z}_0^*(\omega))h_2(\omega)$, where h_1 and h_2 are both (nonnegative and) bounded and continuous on \mathbb{R}_+ and \mathcal{S}_0 , respectively, and there exists $\delta > 0$ such that $h_1(x) = 0$ if $x \leq \delta$ and $h_2(\omega) = 0$ if $W^*(\omega) \leq \delta$. Under these assumptions on h , one can verify that both sides of (29) are left-continuous functions of z , which will yield our claim. Let us briefly explain this. We write $g_1(z)$ and $g_2(z)$ for the left-hand side and the right-hand side of (29), respectively. We use the scaling transformation $\theta_{z/z'}$ that maps $\mathbb{N}_0^{*,z'}$ to $\mathbb{N}_0^{*,z}$ (see Section 2.3) to check that $g_i(z') \rightarrow g_i(z)$ as $z' \uparrow z$, for $i = 1$ or 2 . We note that this scaling transformation maps excursions above level r to excursions above level $r\sqrt{z/z'}$, and, for the function g_2 , we observe that the collection $(\tilde{z}_i)_{i \geq 1}$ is the value at time r of the growth-fragmentation process \mathbf{X} of Theorem 2, and we use the continuity properties of this growth-fragmentation process (see, in particular, Corollary 4 in [4]). We omit a few details that are left to the reader.

Once we know that (29) holds for a fixed $z > 0$ and for a sufficiently large class of functions h , we obtain that the conditional distribution of the random point measure

$$\sum_{i=1}^{\infty} \delta_{\tilde{\omega}^i}$$

given $(\tilde{z}_i)_{i \geq 1}$ is as prescribed in the statement of the theorem. This completes the proof. \square

PROOF OF THEOREM 4. We can derive Theorem 4 from Theorem 23 by arguments very similar to those of the proof of Theorem 38 in [28] and, for this reason we only sketch the

main steps of the proof. As we already noticed in the proof of Theorem 3 in the previous section, the connected components $\mathcal{C}_1, \mathcal{C}_2, \dots$ (in the notation of Theorem 4) are in one-to-one correspondence with the excursions $\tilde{\omega}^1, \tilde{\omega}^2, \dots$, in such a way that the boundary size of \mathcal{C}_i is equal to the boundary size \tilde{z}_i of $\tilde{\omega}_i$. Following [28], Section 8, we can associate a random compact metric space $\Theta(\tilde{\omega}^i)$ with each excursion $\tilde{\omega}^i$, and we know, by Theorem 23 and the main result of [28], that conditionally on $(\tilde{z}_i)_{i \geq 1}$, the random metric spaces $\Theta(\tilde{\omega}^i)$, $i \geq 1$, are independent free Brownian disks with respective perimeters \tilde{z}_i , $i \geq 1$. So, all that remains is to show that, for every $i \geq 1$, the random metric space $(\bar{\mathcal{C}}_i, d_i)$ can be constructed in the way explained in the statement of Theorem 4 and is isometric to $\Theta(\tilde{\omega}^i)$. This is exactly similar to the proof of the identity (60) in [28], to which we refer for additional details. \square

11. Complements.

11.1. *The cumulant function.* It is known [32] that a (self-similar) growth-fragmentation process is characterized by a pair consisting of the self-similarity index (here $\alpha = -1/2$) and a cumulant function κ , which is a convex function defined on $(0, \infty)$ possibly taking the values $+\infty$. The cumulant function κ is given explicitly in terms of the Laplace exponent ψ and the Lévy measure $\pi(dy)$ of the Lévy process ξ appearing in the Lamperti representation of the self-similar process describing the evolution of the Eve particle, via the formula

$$\kappa(p) = \psi(p) + \int_{(-\infty, 0)} (1 - e^y)^p \pi(dy), \quad p > 0.$$

This identity is used in [6] to give an explicit formula for $\kappa(p)$ (see formula (33) in [6]). We will now describe a different approach to the formula for κ , which is independent of the derivation of the Laplace exponent. This suggests that another proof of Theorem 2 should be possible without the identification of the law of the locally largest evolution, provided one knows a priori that the process $(\mathbf{Y}(r))_{r \geq 0}$ is a growth-fragmentation process—note that Theorem 23 does not provide enough information for this.

In view of recovering the expression of κ , we observe that the negative values of the cumulant function are given by the following formula [5], Section 3. We consider the growth-fragmentation process $(\mathbf{Y}(r))_{r \geq 0}$ of Theorem 2 started at $(1, 0, 0, \dots)$. For every $r \geq 0$, write $\mathbf{Y}(r) = (Y_r^1, Y_r^2, \dots)$, and, for every $p \in \mathbb{R}$,

$$\|\mathbf{Y}(r)\|_p = \sum_{i=1}^{\infty} |Y_r^i|^p.$$

Then, for every $p > 1/2$, the quantity

$$(30) \quad \mathbb{N}_0^{*,1} \left(\int_0^\infty dr \|\mathbf{Y}(r)\|_{p-1/2} \right)$$

is finite if and only if $\kappa(p) < 0$, and is then equal to $-1/\kappa(p)$ [5], Formula (16).

For every $i \geq 1$, let σ_r^i be the volume of the i th connected component of $\{a \in \mathcal{T}_\xi : V_a > r\}$ (for our purposes here the way the connected components are ordered is irrelevant). Let $q \in (-1, 1)$. As a consequence of (11), we have, for $r > 0$,

$$(31) \quad \mathbb{N}_0^* \left(\sum_{i=1}^{\infty} \sigma_r^i (Y_r^i)^q e^{-Z_0^*} \right) = 2\mathbb{N}_0 \left(\int_{-\infty}^{-r} db \mathcal{Z}_b (\mathcal{Z}_{b+r})^q e^{-Z_b} \right).$$

Let us consider first the left-hand side of (31). Using Theorem 23 and the identity $\mathbb{N}_0^{*,z}(\sigma) = z^2$ in the second equality, we get

$$\begin{aligned} \mathbb{N}_0^* \left(\sum_{i=1}^{\infty} \sigma_r^i (Y_r^i)^q e^{-Z_0^*} \right) &= \sqrt{\frac{3}{2\pi}} \int_0^\infty dz z^{-5/2} e^{-z} \mathbb{N}_0^{*,z} \left(\sum_{i=1}^{\infty} \sigma_r^i (Y_r^i)^q \right) \\ &= \sqrt{\frac{3}{2\pi}} \int_0^\infty dz z^{-5/2} e^{-z} \mathbb{N}_0^{*,z} \left(\sum_{i=1}^{\infty} (Y_r^i)^{q+2} \right) \\ &= \sqrt{\frac{3}{2\pi}} \int_0^\infty dz z^{q-1/2} e^{-z} \mathbb{N}_0^{*,1} (\|\mathbf{Y}(rz^{-1/2})\|_{q+2}), \end{aligned}$$

by scaling. If we integrate with respect to dr , we arrive at

$$(32) \quad \int_0^\infty dr \mathbb{N}_0^* \left(\sum_{i=1}^{\infty} \sigma_r^i (Y_r^i)^q e^{-Z_0^*} \right) = \sqrt{\frac{3}{2\pi}} \Gamma(q+1) \mathbb{N}_0^{*,1} \left(\int_0^\infty dr \|\mathbf{Y}(r)\|_{q+2} \right).$$

Consider then the right-hand side of (31). Recalling formula (5) and the fact that the process $(Z_{-r})_{r>0}$ is Markovian under \mathbb{N}_0 with the transition kernels of the ϕ -CSBP, we get

$$\begin{aligned} &\mathbb{N}_0 \left(\int_{-\infty}^{-r} db \mathcal{Z}_b (\mathcal{Z}_{b+r})^q e^{-Z_b} \right) \\ &= \int_{-\infty}^{-r} db \mathbb{N}_0 \left((\mathcal{Z}_{b+r})^{q+1} \times \left(1 + r \sqrt{\frac{2}{3}} \right)^{-3} \exp \left(-\mathcal{Z}_{b+r} \left(1 + r \sqrt{\frac{2}{3}} \right)^{-2} \right) \right) \\ &= \mathbb{N}_0 \left(\int_{-\infty}^0 db (\mathcal{Z}_b)^{q+1} \times \left(1 + r \sqrt{\frac{2}{3}} \right)^{-3} \exp \left(-\mathcal{Z}_b \left(1 + r \sqrt{\frac{2}{3}} \right)^{-2} \right) \right). \end{aligned}$$

Integrating with respect to dr , we find

$$(33) \quad \begin{aligned} &\int_0^\infty dr \mathbb{N}_0 \left(\int_{-\infty}^{-r} db \mathcal{Z}_b (\mathcal{Z}_{b+r})^q e^{-Z_b} \right) \\ &= \frac{1}{2\sqrt{2/3}} \mathbb{N}_0 \left(\int_{-\infty}^0 db (\mathcal{Z}_b)^q (1 - e^{-Z_b}) \right). \end{aligned}$$

To compute the right-hand side, write $x^{q-1} = \Gamma(1-q)^{-1} \int_0^\infty d\lambda \lambda^{-q} e^{-\lambda x}$ (for $x > 0$), and recall (6), which gives $\mathbb{N}_0(\mathcal{Z}_b e^{-\lambda \mathcal{Z}_b}) = \lambda^{-3/2} (\lambda^{-1/2} - b\sqrt{2/3})^{-3}$ for $b < 0$. It follows that

$$\begin{aligned} &\mathbb{N}_0 \left(\int_{-\infty}^0 db (\mathcal{Z}_b)^q (1 - e^{-Z_b}) \right) \\ &= \frac{1}{\Gamma(1-q)} \int_0^\infty d\lambda \lambda^{-q} \mathbb{N}_0 \left(\int_{-\infty}^0 db \mathcal{Z}_b (e^{-\lambda \mathcal{Z}_b} - e^{-(\lambda+1)\mathcal{Z}_b}) \right) \\ &= \frac{1}{\Gamma(1-q)} \int_0^\infty d\lambda \lambda^{-q} \int_{-\infty}^0 db \left(\lambda^{-3/2} (\lambda^{-1/2} - b\sqrt{2/3})^{-3} \right. \\ &\quad \left. - (\lambda+1)^{-3/2} ((\lambda+1)^{-1/2} - b\sqrt{2/3})^{-3} \right) \\ &= \frac{1}{2\sqrt{2/3}\Gamma(1-q)} \int_0^\infty d\lambda \lambda^{-q} (\lambda^{-1/2} - (\lambda+1)^{-1/2}). \end{aligned}$$

The right-hand side is finite if and only if $-1/2 < q < 1/2$, and then an elementary calculation gives

$$\int_0^\infty d\lambda \lambda^{-q} (\lambda^{-1/2} - (\lambda + 1)^{-1/2}) = -\frac{1}{\sqrt{\pi}} \Gamma(1 - q) \Gamma\left(q - \frac{1}{2}\right).$$

Coming back to (33), we see that

$$\int_0^\infty dr \mathbb{N}_0 \left(\int_{-\infty}^{-r} db \mathcal{Z}_b (\mathcal{Z}_{b+r})^q e^{-Z_b} \right) = -\frac{3}{8\sqrt{\pi}} \Gamma\left(q - \frac{1}{2}\right).$$

Combining this equality with (31) and (32) leads to

$$\mathbb{N}_0^{*,1} \left(\int_0^\infty dr \|\mathbf{Y}(r)\|_{q+2} \right) = -\sqrt{\frac{3}{8}} \frac{\Gamma(q - \frac{1}{2})}{\Gamma(q + 1)}.$$

Replacing q by $p = q + 5/2$, we finally obtain that, for $2 < p < 3$, the quantity (30) is finite, and

$$\kappa(p) = \sqrt{\frac{8}{3}} \frac{\Gamma(p - \frac{3}{2})}{\Gamma(p - 3)},$$

which is in agreement with formula (33) in [6]—note that the value of κ in the latter formula should be multiplied by the factor $\sqrt{3/2\pi}$ that appears in the formula for ψ in Theorem 1.

Finally, an argument of analytic continuation shows that the preceding formula for $\kappa(p)$ holds for every $p > 3/2$, whereas $\kappa(p) = +\infty$ for $p \in (0, 3/2]$. The function $p \mapsto \kappa(p)$ is (finite and) convex on $(3/2, \infty)$, and vanishes at $p = 2$ and $p = 3$ (with the notation of [5], we have $\omega_- = 2$ and $\omega_+ = 3$).

11.2. *A growth-fragmentation process in the Brownian plane.* In this section, we consider the random pointed metric space $(\mathcal{P}_\infty, d_\infty)$ called the Brownian plane, which has been introduced and studied in [14]. The space \mathcal{P}_∞ has a distinguished point ρ_∞ , and, for every $r > 0$, we may define the boundary sizes of the connected components of $\{x \in \mathcal{P}_\infty : d(\rho_\infty, x) > r\}$, via the same approximation as used above in Section 4: To see that this definition makes sense, one may argue that there exists a coupling of the Brownian plane and the Brownian map such that small balls centered at the distinguished point in the two spaces are isometric [14], then rely on Proposition 8 to treat the case when r is small enough, and finally use the scale invariance of the Brownian plane. Notice that there is exactly one unbounded component, whose boundary is also the boundary of the so-called hull of radius r in \mathcal{P}_∞ (see, in particular, [15]).

We will relate this collection of boundary sizes to the growth-fragmentation process of Theorem 1 subject to a special conditioning. Precisely, we consider this growth-fragmentation process starting from 0 and conditioned to have indefinite growth (see [5], Section 4.2). Let us briefly describe this process, referring to [5] for more details. We start with one Eve particle, whose mass process $(\widehat{X}_t)_{t \geq 0}$ evolves as the process X of Theorem 1 conditioned to start from 0 and to stay positive for all times. To be specific, the process \widehat{X} is a self-similar Markov process with index $1/2$, which can be obtained via the Lamperti representation from a Lévy process $\widehat{\xi}$ with no positive jumps and Laplace exponent

$$(34) \quad \widehat{\psi}(\lambda) := \kappa(3 + \lambda) = \sqrt{\frac{8}{3}} \frac{\Gamma(\lambda + \frac{3}{2})}{\Gamma(\lambda)}, \quad \lambda > 0.$$

See [5], Lemma 2.1, for the fact that the function $\widehat{\psi}(\lambda)$ corresponds to the Laplace exponent of a Lévy process without positive jumps. Then, as previously, each jump time t of \widehat{X}

corresponds to the birth of a new particle (child of the Eve particle) with mass $|\Delta\widehat{X}_t|$, but the masses of these new particles evolve independently according to the distribution of the process X , and similarly for the children of these particles, and so on. We emphasize that only the mass process of the Eve particle evolves according to a different Markov process \widehat{X} , while the masses of its children, grandchildren, etc., evolve according to the law of the process X . As previously, we write $\widehat{\mathbf{X}}(r)$ for the collection of masses of all particles present at time r .

THEOREM 24. *As a process indexed by the variable $r > 0$, the collection of the boundary sizes of all connected components of $\{x \in \mathcal{P}_\infty : d_\infty(\rho_\infty, x) > r\}$ is distributed as the process $(\widehat{\mathbf{X}}(r))_{r>0}$.*

PROOF. We first explain that the role of the Eve particle (for the process $\widehat{\mathbf{X}}$) is played by the evolution of the boundary size of the unbounded component of $\{x \in \mathcal{P}_\infty : d_\infty(\rho_\infty, x) > r\}$. Let \widehat{Z}_r be this boundary size, with the convention that $\widehat{Z}_0 = 0$. The distribution of the process $(\widehat{Z}_r)_{r \geq 0}$ is described in [15], Proposition 1.2. From this description, using also the arguments of [16], Section 4.4, one gets that \widehat{Z}_r can be written as

$$\widehat{Z}_r = U_{\eta_r}^\uparrow,$$

where $(U_t^\uparrow)_{t \geq 0}$ is the Lévy process with no positive jumps and Laplace exponent $\phi(\lambda) = \sqrt{8/3}\lambda^{3/2}$ conditioned to start from 0 and to stay positive for all times $t > 0$ —see [2], Chapter VII, for a rigorous definition of this process—and η_r is the time change

$$\eta_r = \inf\left\{t \geq 0 : \int_0^t \frac{ds}{U_s^\uparrow} > r\right\}.$$

It follows from this representation that $(\widehat{Z}_r)_{r \geq 0}$ is a self-similar Markov process with index $1/2$ with values in $[0, \infty)$. We will now verify that the Laplace exponent of the Lévy process arising in the Lamperti representation of this self-similar Markov process is equal to $\widehat{\psi}(\lambda)$, which will imply that the process $(\widehat{Z}_t)_{t \geq 0}$ has the same distribution as the mass process of the Eve particle in the description of the process $(\widehat{\mathbf{X}}(t))_{t \geq 0}$. We slightly abuse notation by introducing, for every $x \geq 0$, a probability measure \mathbb{P}_x under which the Markov process \widehat{Z} starts from x . By self-similarity, for every $a > 0$, the law of $(a^{-2}\widehat{Z}_{at})_{t \geq 0}$ under \mathbb{P}_x coincides with the law of $(\widehat{Z}_t)_{t \geq 0}$ under $\mathbb{P}_{a^{-2}x}$. We recall from [15], Proposition 1.2, that, for every $r > 0$, \widehat{Z}_r follows (under \mathbb{P}_0) a Gamma distribution with parameter $3/2$ and mean r^2 .

Let $q \in (-\frac{3}{2}, -\frac{1}{2})$. Then

$$\begin{aligned} \mathbb{E}_0\left[\int_1^\infty \widehat{Z}_t^q dt\right] &= \int_1^\infty t^{2q} \mathbb{E}_0[\widehat{Z}_1^q] dt \\ (35) \qquad &= -\frac{\mathbb{E}_0[\widehat{Z}_1^q]}{2q+1} = -\left(\frac{2}{3}\right)^q \frac{1}{2q+1} \frac{\Gamma(q+\frac{3}{2})}{\Gamma(\frac{3}{2})}. \end{aligned}$$

We may compute the left-hand side of (35) in a different manner by applying the Markov property at time 1. We get

$$(36) \quad \mathbb{E}_0\left[\int_1^\infty \widehat{Z}_t^q dt\right] = \mathbb{E}_0\left[\mathbb{E}_{\widehat{Z}_1}\left[\int_0^\infty \widehat{Z}_t^q dt\right]\right] = \mathbb{E}_0[\widehat{Z}_1^{q+1/2}] \times \mathbb{E}_1\left[\int_0^\infty \widehat{Z}_t^q dt\right]$$

using the self-similarity of \widehat{Z} . Then $\mathbb{E}_0[\widehat{Z}_1^{q+1/2}] = (2/3)^{q+1/2} \Gamma(q+2) / \Gamma(3/2)$ and, on the other hand, if $\widehat{\xi}$ denotes the Lévy process (started from 0) arising in the Lamperti representation of the self-similar process \widehat{Z} , we have

$$\mathbb{E}_1\left[\int_0^\infty \widehat{Z}_t^q dt\right] = \mathbb{E}\left[\int_0^\infty e^{(q+\frac{1}{2})\widehat{\xi}(t)} dt\right].$$

The quantity in the right-hand side must be finite, which implies that $\mathbb{E}[e^{(q+\frac{1}{2})\hat{\xi}(t)}] < \infty$ for every $t > 0$, and $\mathbb{E}[e^{(q+\frac{1}{2})\hat{\xi}(t)}] = \exp(t\psi^*(q + \frac{1}{2}))$ with $\psi^*(q + \frac{1}{2}) < 0$. From (35) and (36), we get

$$\frac{1}{\psi^*(q + \frac{1}{2})} = \frac{1}{2} \left(\frac{2}{3}\right)^{-1/2} \frac{\Gamma(q + \frac{1}{2})}{\Gamma(q + 2)}.$$

Finally, we find that, for $-1 < q < 0$, we have

$$\psi^*(q) = \sqrt{\frac{8}{3}} \frac{\Gamma(q + \frac{3}{2})}{\Gamma(q)} = \hat{\psi}(q).$$

An argument of analytic continuation now allows us to obtain that the Laplace exponent of the Lévy process $\hat{\xi}$ is equal to $\hat{\psi}(\lambda)$ as desired.

Once we have identified $(\hat{Z}_t)_{t \geq 0}$ as the mass process of the Eve particle in the description of $(\hat{X}(t))_{t \geq 0}$, the remaining steps of the proof are very similar to those of the proof of Theorem 2, and we will only sketch the main ingredients. We first recall the relevant features of the construction of the Brownian plane $(\mathcal{P}_\infty, d_\infty)$ which is developed in [15], Section 3.2, to which we refer for further details. The random metric space \mathcal{P}_∞ is obtained as a quotient space of a (noncompact) random tree \mathcal{T}_∞ , which itself is constructed by grafting a Poisson collection of (compact) \mathbb{R} -trees to an infinite spine isometric to $[0, \infty)$. The point 0 of the spine corresponds to the distinguished point ρ_∞ of \mathcal{P}_∞ . Furthermore, every point a of \mathcal{T}_∞ is assigned a nonnegative label Λ_a , and this label coincides with $d_\infty(\rho_\infty, x)$, if x is the point of \mathcal{P}_∞ corresponding to a . Then, as in the proof of Theorem 3, it is not hard to check that connected components of $\{x \in \mathcal{P}_\infty : d_\infty(\rho_\infty, x) > r\}$ are in one-to-one correspondence with connected components of $\{a \in \mathcal{T}_\infty : \Lambda_a > r\}$, for every fixed $r > 0$.

For every $a \in \mathcal{T}_\infty$, let $[[a, \infty[[$ stand for the range of the unique geodesic path from a to ∞ in \mathcal{T}_∞ , and set $\underline{\Lambda}_a = \min\{\Lambda_b : b \in [[a, \infty[[\}$. If \mathcal{C} is a (necessarily bounded) connected component of $\{a \in \mathcal{T}_\infty : \Lambda_a - \underline{\Lambda}_a > 0\}$, then both \mathcal{C} and the labels $(\Lambda_a)_{a \in \mathcal{C}}$ can be represented by a snake trajectory $\omega_{\mathcal{C}}$, in a way very similar to what we did for \check{C}_r in Section 3.2.

PROPOSITION 25. *Setting $\inf\{\Lambda_a : a \in \mathcal{C}\} = r$ yields a one-to-one correspondence between connected components \mathcal{C} of $\{a \in \mathcal{T}_\infty : \Lambda_a - \underline{\Lambda}_a > 0\}$ and jump times r of the process $(\hat{Z}_t)_{t \geq 0}$. Let r_1, r_2, \dots be an enumeration of these jump times, which is measurable with respect to the σ -field generated by $(\hat{Z}_r)_{r \geq 0}$, and for every $i = 1, 2, \dots$, let \mathcal{C}_i be the connected component associated with r_i . Then, conditionally on the process $(\hat{Z}_r)_{r \geq 0}$, the snake trajectories $\omega_{\mathcal{C}_i}, i = 1, 2, \dots$, are independent, and the conditional distribution of $\omega_{\mathcal{C}_i}$ is $\mathbb{N}_0^{*, |\Delta \hat{Z}_{r_i}|}$.*

Proposition 25 is obviously an analog of Theorem 14, and can be derived from the latter result using the relations between the labeled tree $(\mathcal{T}_\infty, (\Lambda_a)_{a \in \mathcal{T}_\infty})$ and the pair $(\mathcal{T}_\zeta, (V_a)_{a \in \mathcal{T}_\zeta})$ under \mathbb{N}_0 (compare the decomposition of the Brownian snake at the minimum found in [15], Theorem 2.1, with the construction of $(\mathcal{T}_\infty, (\Lambda_a)_{a \in \mathcal{T}_\infty})$ in the same reference).

Given Proposition 25, the end of the proof of Theorem 24 follows the same general pattern as that of Theorem 2 in Section 8, and we leave the details to the reader. \square

REMARK. We could have used Corollary 2 of [11] to identify the Laplace exponent of the Lévy process $\hat{\xi}$ in the preceding proof. This would still have required some calculations, and for this reason we preferred the more direct approach presented above.

11.3. *Local times.* In this section, we argue under $\mathbb{N}_0(d\omega)$, but similar results hold under $\mathbb{N}_0^*(d\omega)$. Recall from the **Introduction** the definition of the local times $(\mathcal{L}_x, x \in \mathbb{R})$ of the process $(V_a)_{a \in \mathcal{T}_\zeta}$. In this section, we relate the values of \mathcal{L}_x for $x > 0$ to the growth-fragmentation process of Theorem 1 or equivalently to the connected components of $\{a \in \mathcal{T}_\zeta : V_a > x\}$.

We fix $h > 0$. It is convenient to introduce the “local time exit process” $(\mathcal{X}_r^h)_{r \geq 0}$, which roughly speaking measures for every $r \geq 0$ the “quantity” of paths W_s that have hit level h and accumulated a local time equal to r at level h . The precise definition of this process fits in the general framework of exit measures [23], Chapter V, and we refer to the Introduction of [1] for more details (only the case $h = 0$ is considered there, but the extension to the case $h > 0$ is straightforward). Note that $\mathcal{X}_0^h = \mathcal{Z}_h$ is just the usual exit measure from $(-\infty, r)$, which can be defined by formula (4). Furthermore, under $\mathbb{N}_0(\cdot | W^* > h)$, the process $(\mathcal{X}_r^h)_{r \geq 0}$ is a ϕ -CSBP started from \mathcal{Z}_h ; see again the Introduction of [1]. Of course, on the event $\{W^* < h\}$, the process $(\mathcal{X}_r^h)_{r \geq 0}$ is identically equal to zero.

Let $\mathcal{C}_1^h, \mathcal{C}_2^h, \dots$ be the connected components of $\{a \in \mathcal{T}_\zeta : V_a > h\}$, ranked in decreasing order of their boundary sizes $|\partial \mathcal{C}_1^h| > |\partial \mathcal{C}_2^h| > \dots$. For every $i = 1, 2, \dots$, let H_i be the height of \mathcal{C}_i^h , defined by

$$H_i := \sup_{a \in \mathcal{C}_i^h} V_a - h.$$

PROPOSITION 26. We have \mathbb{N}_0 a.e.

$$(37) \quad \mathcal{L}_h = \int_0^\infty dr \mathcal{X}_r^h.$$

Moreover, \mathbb{N}_0 a.e.,

$$(38) \quad \delta^{3/2} \#\{i : |\partial \mathcal{C}_i^h| > \delta\} \xrightarrow{\delta \rightarrow 0} \frac{1}{\sqrt{6\pi}} \mathcal{L}_h$$

and

$$(39) \quad \delta^3 \#\{i : H_i > \delta\} \xrightarrow{\delta \rightarrow 0} \mathbf{c} \mathcal{L}_h,$$

where $\mathbf{c} = \frac{3}{2} \pi^{-3/2} \Gamma(1/3)^3 \Gamma(7/6)^3$.

REMARK. The proposition also holds for $h = 0$, but the proof of (37) requires a different argument in that case, see [22], Proposition 3.

PROOF OF PROPOSITION 26. The convergence (39) is already established in [26], in the slightly weaker form of convergence in measure (note that “upcrossings” from h to $h + \delta$, as defined in [26], exactly correspond to connected components of $\{a \in \mathcal{T}_\zeta : V_a > h\}$ with height greater than δ). We will use this fact to prove the identity (37). To simplify notation, we set

$$\mathcal{L}_h^* = \int_0^\infty dr \mathcal{X}_r^h.$$

As in the proof of Theorem 23, we consider all excursions away from h . It follows from [1], Proposition 3, Theorem 4 (and an application of the special Markov property) that these excursions are in one-to-one correspondence with the jumps of the process $(\mathcal{X}_r^h)_{r \geq 0}$, and that conditionally on the latter process, they are independent and the conditional distribution of the excursion corresponding to the jump $\Delta \mathcal{X}_r^h$ is

$$\frac{1}{2} \mathbb{N}_0^{*, \Delta \mathcal{X}_r^h} + \frac{1}{2} \check{\mathbb{N}}_0^{*, \Delta \mathcal{X}_r^h},$$

where we use the notation $\check{\mathbb{N}}_0^{*,z}$ for the pushforward of $\mathbb{N}_0^{*,z}$ under the symmetry $\omega \mapsto -\omega$. We recall that the connected components of $\{a \in \mathcal{T}_\zeta : V_a > h\}$ are in one-to-one correspondence with the excursions above level h , in such a way that the boundary size of a component is equal to the corresponding jump of $(\mathcal{X}_r^h)_{r \geq 0}$.

Write U for the (stopped) Lévy process obtained from the ϕ -CSBP \mathcal{X}^h by the Lamperti transformation (note that U is stopped upon hitting 0 and that $U_0 = \mathcal{Z}_h$). Notice that the hitting time of 0 by U is \mathcal{L}_h^* . Since the jumps of $(\mathcal{X}_r^h)_{r \geq 0}$ are also the jumps of U , we obtain the identity in distribution

$$\left(\mathcal{L}_h^*, \sum_{i=1}^\infty \delta_{|\partial \mathcal{C}_i^h|} \right) \stackrel{(d)}{=} \left(\mathcal{L}_h^*, \sum_{i=1}^\infty \mathbf{1}_{\{\epsilon_i=1\}} \delta_{\Delta U_{r_i}} \right),$$

where r_1, r_2, \dots are the jump times of U , and $\epsilon_1, \epsilon_2, \dots$ is a sequence of independent Bernoulli variables with parameter $1/2$, which is independent of U . Since the Lévy measure of U is

$$\pi(dz) = \sqrt{\frac{3}{2\pi}} z^{-5/2} dz,$$

so that $\pi((\delta, \infty)) = \sqrt{2/3\pi} \delta^{-3/2}$, it easily follows that, \mathbb{N}_0 a.e.,

$$\delta^{3/2} \#\{i : \epsilon_i = 1 \text{ and } \Delta U_{r_i} > \delta\} \xrightarrow{\delta \rightarrow 0} \frac{1}{\sqrt{6\pi}} \mathcal{L}_h^*.$$

This gives the convergence (38), except that we have not yet verified that $\mathcal{L}_h = \mathcal{L}_h^*$.

To this end, using again the conditional distribution of the excursions away from h given the process $(\mathcal{X}_r^h)_{r \geq 0}$, we observe that we have also

$$\left(\mathcal{L}_h^*, \sum_{i=1}^\infty \delta_{H_i} \right) \stackrel{(d)}{=} \left(\mathcal{L}_h^*, \sum_{i=1}^\infty \mathbf{1}_{\{\epsilon_i=1\}} \delta_{\sqrt{\Delta U_{r_i}} M_i} \right),$$

where M_1, M_2, \dots is a sequence of independent random variables distributed according to the law of W^* under $\mathbb{N}_0^{*,1}$, which is also independent of $(U, (\epsilon_i)_{i \geq 1})$. Now observe that, if z is chosen according to $\pi(dz)$ and M according to the law of W^* under $\mathbb{N}_0^{*,1}$, $\sqrt{z}M$ is distributed according to the “law” of W^* under \mathbb{N}_0^* , which satisfies

$$\mathbb{N}_0^*(W^* > \delta) = 2c\delta^{-3}$$

by [1], Lemma 25. It follows that, \mathbb{N}_0 a.e.,

$$\delta^3 \#\{i : H_i > \delta\} \xrightarrow{\delta \rightarrow 0} c\mathcal{L}_h^*.$$

By comparing this convergence with [26], Theorem 6, we get that $\mathcal{L}_h = \mathcal{L}_h^*$, which completes the proof. \square

APPENDIX

This appendix is devoted to the proofs of Lemma 12 and Proposition 8.

PROOF OF LEMMA 12. It is convenient to write \mathbf{N} for the distribution of $(\mathcal{Z}_{-t})_{t \geq 0}$ under \mathbb{N}_0 (we agree that $\mathcal{Z}_0 = 0$). Then \mathbf{N} is a σ -finite measure on the Skorokhod space $\mathbb{D}([0, \infty), \mathbb{R})$. For $\varepsilon > 0$, let

$$\sum_{i \in I_\varepsilon} \delta_{w_i^\varepsilon}(dw)$$

be a Poisson point measure on $\mathbb{D}([0, \infty), \mathbb{R})$ with intensity $\varepsilon \mathbf{N}$. As already noticed in Section 2.4, we can construct a ϕ -CSBP started from ε by setting, for $t > 0$,

$$Y_t^\varepsilon = \sum_{i \in I_\varepsilon} w_i^\varepsilon(t)$$

and $Y_0^\varepsilon = \varepsilon$. Set $T_0^\varepsilon := \inf\{t \geq 0 : Y_t^\varepsilon = 0\}$. The classical Lamperti transformation [12, 20] allows us to relate Y^ε to another process X^ε distributed as a stable Lévy process with no negative jumps and index $3/2$, started from ε and stopped at the first time when it hits 0, via the formula

$$X_t^\varepsilon = Y_{\gamma_t^\varepsilon}^\varepsilon,$$

where $\gamma_t^\varepsilon := \inf\{s \geq 0 : \int_0^s Y_u^\varepsilon du > t\}$ if $t < \int_0^\infty Y_u^\varepsilon du$ and $\gamma_t^\varepsilon = T_0^\varepsilon$ otherwise.

Let us fix $\delta > 0$ and assume from now on that $\varepsilon \in (0, \delta)$. Let B_ε stand for the event $\{\sup_{t \geq 0} Y_t^\varepsilon \geq \delta\} = \{\sup_{t \geq 0} X_t^\varepsilon \geq \delta\}$. By the solution of the two-sided exit problem already used in the proof of Lemma 9, we have

$$\mathbb{P}(B_\varepsilon) = 1 - \sqrt{\frac{\delta - \varepsilon}{\delta}} = \frac{\varepsilon}{2\delta} + O(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0.$$

On the other hand, let $A_\varepsilon \subset B_\varepsilon$ be the event where there is exactly one $i \in I_\varepsilon$ such that $\sup_{t \geq 0} w_i^\varepsilon(t) \geq \delta$. By Lemma 9,

$$\mathbb{P}(A_\varepsilon) = \frac{\varepsilon}{2\delta} \exp\left(-\frac{\varepsilon}{2\delta}\right) = \frac{\varepsilon}{2\delta} + O(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0.$$

If F is a bounded measurable function on the space $\mathbb{D}([0, \infty), \mathbb{R})$, we deduce from the last two displays that

$$(40) \quad \mathbb{E}[F((X_t^\varepsilon)_{t \geq 0}) \mid B_\varepsilon] = \mathbb{E}[F((X_t^\varepsilon)_{t \geq 0}) \mid A_\varepsilon] + O(\varepsilon), \quad \text{as } \varepsilon \rightarrow 0.$$

We can associate with X^ε the process “reflected above the minimum” defined by

$$\tilde{X}_t^\varepsilon := X_t^\varepsilon - \inf\{X_s^\varepsilon : 0 \leq s \leq t\}.$$

We have obviously $0 \leq X_t^\varepsilon - \tilde{X}_t^\varepsilon \leq \varepsilon$ for every $t \geq 0$. If $\tilde{B}_\varepsilon \subset B_\varepsilon$ stands for the event $\{\sup_{t \geq 0} \tilde{X}_t^\varepsilon \geq \delta\}$, it is easily checked that $\mathbb{P}(B_\varepsilon \setminus \tilde{B}_\varepsilon) = O(\varepsilon^2)$, so that we can replace B_ε by \tilde{B}_ε in (40). Furthermore, on the event \tilde{B}_ε we can introduce the first excursion of \tilde{X}^ε away from 0 that hits δ and denote this excursion by $(\mathcal{X}_t^\varepsilon)_{t \geq 0}$. Notice that the distribution of $(\mathcal{X}_t^\varepsilon)_{t \geq 0}$ is $\mathbf{n}_\delta(de) := \mathbf{n}(de \mid \sup\{e(t) : t \geq 0\} \geq \delta)$. Let d_{Sk} be a distance on $\mathbb{D}([0, \infty), \mathbb{R})$ that induces the Skorokhod topology. It is a simple matter to verify that, for every $\alpha > 0$,

$$\mathbb{P}(d_{\text{Sk}}((\tilde{X}_t^\varepsilon)_{t \geq 0}, (\mathcal{X}_t^\varepsilon)_{t \geq 0}) > \alpha \mid \tilde{B}_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Assume from now on that F is (bounded and) Lipschitz with respect to d_{Sk} . We deduce from (40) (with B_ε replaced by \tilde{B}_ε) and the last display that

$$(41) \quad \mathbb{E}[F((X_t^\varepsilon)_{t \geq 0}) \mid A_\varepsilon] - \mathbb{E}[F((\mathcal{X}_t^\varepsilon)_{t \geq 0}) \mid \tilde{B}_\varepsilon] \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Note that $\mathbb{E}[F((\mathcal{X}_t^\varepsilon)_{t \geq 0}) \mid \tilde{B}_\varepsilon]$ does not depend on ε and is equal to $\int \mathbf{n}_\delta(de) F(e)$.

On the other hand, conditionally on the event A_ε there is a unique index $i_0 \in I_\varepsilon$ such that $\sup_{t \geq 0} w_{i_0}^\varepsilon(t) \geq \delta$, and the distribution of $w_{i_0}^\varepsilon$ is $\mathbf{N}_\delta(dw) := \mathbf{N}(dw \mid \sup_{t \geq 0} w(t) \geq \delta)$. We then set $\mathcal{Y}_t^\varepsilon = w_{i_0}^\varepsilon(t)$, and

$$\eta_t^\varepsilon = \inf\left\{s \geq 0 : \int_0^s du \mathcal{Y}_u^\varepsilon > t\right\}$$

if $t < \int_0^\infty du \mathcal{Y}_u^\varepsilon$, and $\eta_t^\varepsilon = \inf\{s \geq 0 : \mathcal{Y}_s^\varepsilon = 0\}$ otherwise. Observing that the conditional distribution of $Y^\varepsilon - \mathcal{Y}^\varepsilon$ given A_ε is dominated by the law of a Φ -continuous state branching process started from ε , one verifies that, for every $\alpha > 0$,

$$\mathbb{P}(\text{dsk}((Y_{\gamma_t^\varepsilon}^\varepsilon)_{t \geq 0}, (\mathcal{Y}_{\eta_t^\varepsilon}^\varepsilon)_{t \geq 0}) > \alpha \mid A_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

(Here, we omit a few details that are left to the reader.) Recalling that $Y_{\gamma_t^\varepsilon}^\varepsilon = X_t^\varepsilon$ and using (41), we get

$$\mathbb{E}[F((\mathcal{Y}_{\eta_t^\varepsilon}^\varepsilon)_{t \geq 0}) \mid A_\varepsilon] - \int \mathbf{n}_\delta(\text{de})F(e) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Since the conditional distribution of $(\mathcal{Y}_{\eta_t^\varepsilon}^\varepsilon)_{t \geq 0}$ given A_ε is \mathbf{N}_δ (independently of ε), using the equalities $\mathbf{N}(\{w : \sup_{t \geq 0} w(t) \geq \delta\}) = \frac{1}{2\delta} = \mathbf{n}(\{e : \sup_{t \geq 0} e(t) \geq \delta\})$ (the first one by Lemma 9 and the second one as an easy consequence of the two-sided exit problem), we arrive at the result of the lemma. \square

PROOF OF PROPOSITION 8. *First step.* Recall that, for every $t > 0$, \mathcal{G}_t denotes the σ -field on \mathcal{S}_0 generated by the mapping $\omega \mapsto \text{tr}_{-t}(\omega)$ and completed by the collection of all \mathbb{N}_0 -negligible sets. We also define \mathcal{G}_0 as the σ -field generated by the \mathbb{N}_0 -negligible sets. For every $\eta > 0$, the process $(\mathcal{Z}_{-t})_{t \geq \eta}$ is Markov with respect to the filtration $(\mathcal{G}_t)_{t \geq \eta}$ under the probability measure $\mathbb{N}_0^{[\eta]} := \mathbb{N}_0(\cdot \mid W_* \leq -\eta)$. By the Feller property of the semigroup, the strong Markov property holds even for stopping times of the filtration $(\mathcal{G}_{t+})_{t \geq \eta}$.

We fix two reals $\eta \in (0, 1)$ and $M > 1$. Let $\varepsilon \in (0, \eta)$. From the proof of Proposition 34 in the Appendix of [28], we have, for every $r \leq -\eta$,

$$\mathbb{N}_0((\mathcal{Z}_r^\varepsilon - \mathcal{Z}_r)^2) \leq 4\varepsilon^2.$$

We note that [28] deals with the quantity $\tilde{\mathcal{Z}}_r^\varepsilon$ defined in Remark (ii) after Proposition 8, rather than with $\mathcal{Z}_r^\varepsilon$, but as explained in this remark, this makes no difference for a fixed value of r . Furthermore, [28] gives the latter bound only for “truncated versions” of $\tilde{\mathcal{Z}}_r^\varepsilon$ and \mathcal{Z}_r , but an application of Fatou’s lemma then yields the preceding display.

Let $\delta \in (0, 1)$. By Markov’s inequality, for $r \leq -\eta$,

$$\mathbb{N}_0(|\mathcal{Z}_r^\varepsilon - \mathcal{Z}_r| > \delta) \leq \delta^{-2} \times 4\varepsilon^2.$$

We apply this to $r = -j\varepsilon^{3/2}$ for all integers j such that $\eta \leq j\varepsilon^{3/2} \leq M + 1$. It follows that

$$(42) \quad \begin{aligned} &\mathbb{N}_0\left(|\mathcal{Z}_{-j\varepsilon^{3/2}}^\varepsilon - \mathcal{Z}_{-j\varepsilon^{3/2}}| > \frac{\delta}{2}, \text{ for some } j \text{ s.t. } \eta \leq j\varepsilon^{3/2} \leq M + 1\right) \\ &\leq 16(M + 1)\delta^{-2}\varepsilon^{1/2}. \end{aligned}$$

Fix a real $K > 0$, and consider the random time

$$S := \inf\{t \geq \eta : \mathcal{Z}_{-t} < K \text{ and } |\mathcal{Z}_{-t}^\varepsilon - \mathcal{Z}_{-t}| > \delta\}.$$

Note that S is a stopping time of the filtration $(\mathcal{G}_{t+})_{t \geq \eta}$ (because both processes $(\mathcal{Z}_{-t}^\varepsilon)_{t \geq \eta}$ and $(\mathcal{Z}_{-t})_{t \geq \eta}$ have càdlàg paths and are adapted to the filtration $(\mathcal{G}_t)_{t \geq \eta}$). On the event $\{S < \infty\}$, we have $|\mathcal{Z}_{-S}^\varepsilon - \mathcal{Z}_{-S}| \geq \delta$ and $\mathcal{Z}_{-S} \leq K$.

Our goal is now to bound $\mathbb{N}_0(S \leq M)$. To this end, we will use (42). On the event $\{S < \infty\}$ write $[-S]_\varepsilon$ for the greatest number of the form $-j\varepsilon^{3/2}$ in the interval $(-\infty, -S)$. Then

$$\begin{aligned} \{S \leq M\} &= \{S \leq M, |\mathcal{Z}_{[-S]_\varepsilon}^\varepsilon - \mathcal{Z}_{[-S]_\varepsilon}| > \delta/2\} \\ &\cup \{S \leq M, |\mathcal{Z}_{[-S]_\varepsilon}^\varepsilon - \mathcal{Z}_{[-S]_\varepsilon}| \leq \delta/2\}. \end{aligned}$$

By (42), the \mathbb{N}_0 -measure of the first set in the right-hand side is bounded above by $c_1 \varepsilon^{1/2}$ for some constant c_1 depending on M and δ . On the other hand, recalling that $|\mathcal{Z}_{-S}^\varepsilon - \mathcal{Z}_{-S}| \geq \delta$ on $\{S < \infty\}$, we obtain that the second set is contained in

$$\{S \leq M, |\mathcal{Z}_{[-S]_\varepsilon} - \mathcal{Z}_{-S}| \geq \delta/4\} \cup \{S \leq M, |\mathcal{Z}_{[-S]_\varepsilon}^\varepsilon - \mathcal{Z}_{-S}^\varepsilon| \geq \delta/4\}.$$

Using the strong Markov property of $(\mathcal{Z}_{-t})_{t \geq \eta}$ at time S , the bound $\mathcal{Z}_{-S} \leq K$ on $\{S < \infty\}$, and the fact that a ϕ -CSBP can be written as a time change of a Lévy process, it is easy to verify that

$$(43) \quad \mathbb{N}_0(S \leq M, |\mathcal{Z}_{[-S]_\varepsilon} - \mathcal{Z}_{-S}| \geq \delta/4) \leq c_2 \varepsilon^{3/2}$$

for some constant c_2 depending on δ and K .

In the second and the third step below, we will get similar estimates for the \mathbb{N}_0 -measure of (appropriate subsets of) the event $\{S \leq M, |\mathcal{Z}_{[-S]_\varepsilon}^\varepsilon - \mathcal{Z}_{-S}^\varepsilon| \geq \delta/4\}$. We will explain in the fourth step how the proof of the proposition is completed by combining all these estimates.

Second step. We first study the quantity

$$\mathbb{N}_0(S \leq M, \mathcal{Z}_{[-S]_\varepsilon}^\varepsilon - \mathcal{Z}_{-S}^\varepsilon \geq \delta/4).$$

From our definitions, on the event $\{S < \infty\}$, the quantity $\mathcal{Z}_{[-S]_\varepsilon}^\varepsilon - \mathcal{Z}_{-S}^\varepsilon$ is bounded above by

$$F_\varepsilon := \varepsilon^{-2} \int_0^\sigma ds \mathbf{1}_{\{T_{-S}(W_s) < \infty, T_{-S-\varepsilon^{3/2}}(W_s) = \infty\}} \mathbf{1}_{\{\widehat{W}_s < [-S]_\varepsilon + \varepsilon\}}.$$

For every integer $n \geq 1$, write $[-S]_{(n)}$ for the greatest number of the form $-j2^{-n}$ in $(-\infty, -S)$, and set, still on the event $\{S < \infty\}$,

$$F_{\varepsilon,n} := \varepsilon^{-2} \int_0^\sigma ds \mathbf{1}_{\{T_{[-S]_{(n)}}(W_s) < \infty, T_{[-S]_{(n)}-\varepsilon^{3/2}}(W_s) = \infty\}} \mathbf{1}_{\{\widehat{W}_s < [-S]_{(n)} + \varepsilon\}}.$$

Observing that $\mathbf{1}_{\{T_{-S}(W_s) < \infty\}} = \lim_{n \rightarrow \infty} \mathbf{1}_{\{T_{[-S]_{(n)}}(W_s) < \infty\}}$ and using Fatou's lemma, we have

$$\mathbb{N}_0(\mathbf{1}_{\{S < \infty\}} F_\varepsilon) \leq \liminf_{n \rightarrow \infty} \mathbb{N}_0(\mathbf{1}_{\{S < \infty\}} F_{\varepsilon,n}).$$

Then

$$\begin{aligned} \mathbb{N}_0(\mathbf{1}_{\{S < \infty\}} F_{\varepsilon,n}) &= \varepsilon^{-2} \sum_{k=1}^\infty \mathbb{N}_0 \left(\mathbf{1}_{\{(k-1)2^{-n} \leq S < k2^{-n}\}} \right. \\ &\quad \left. \times \int_0^\sigma ds \mathbf{1}_{\{T_{-k2^{-n}}(W_s) < \infty, T_{-k2^{-n}-\varepsilon^{3/2}}(W_s) = \infty\}} \mathbf{1}_{\{\widehat{W}_s < -k2^{-n} + \varepsilon\}} \right). \end{aligned}$$

We can apply the special Markov property (Proposition 7) to each term of the sum in the right-hand side. Note that the variable $\mathbf{1}_{\{(k-1)2^{-n} \leq S < k2^{-n}\}}$ is measurable with respect to $\mathcal{G}_{k2^{-n}}$, whereas the subsequent integral is a function of the snake trajectories ω_i introduced in Proposition 7 when $r = -k2^{-n}$. We obtain

$$\begin{aligned} \mathbb{N}_0(\mathbf{1}_{\{S < \infty\}} F_{\varepsilon,n}) &= \varepsilon^{-2} \sum_{k=1}^\infty \mathbb{N}_0(\mathbf{1}_{\{(k-1)2^{-n} \leq S < k2^{-n}\}} \mathcal{Z}_{k2^{-n}}) \\ &\quad \times \mathbb{N}_0 \left(\int_0^\sigma ds \mathbf{1}_{\{T_{-\varepsilon^{3/2}}(W_s) = \infty\}} \mathbf{1}_{\{\widehat{W}_s < \varepsilon\}} \right). \end{aligned}$$

By the first-moment formula for the Brownian snake [23], Proposition 4.2, we have

$$\mathbb{N}_0 \left(\int_0^\sigma ds \mathbf{1}_{\{T_{-\varepsilon^{3/2}}(W_s) = \infty\}} \mathbf{1}_{\{\widehat{W}_s < \varepsilon\}} \right) = \mathbb{E}_0 \left[\int_0^{t_{-\varepsilon^{3/2}}} dt \mathbf{1}_{\{B_t < \varepsilon\}} \right] \leq c_3 \varepsilon^{5/2},$$

where $(B_t)_{t \geq 0}$ is a standard linear Brownian motion starting from x under the probability measure \mathbb{P}_x , $\tau_r = \inf\{t \geq 0 : B_t = r\}$ for every $r \in \mathbb{R}$, and c_3 is a constant. We conclude that $\mathbb{N}_0(\mathbf{1}_{\{S < \infty\}} F_{\varepsilon, n}) \leq c_3 \varepsilon^{1/2} \mathbb{N}_0(\mathbf{1}_{\{S < \infty\}} \mathcal{Z}_{[S]_{(n)}}) \leq c_3 \varepsilon^{1/2}$, and the same bound holds for $\mathbb{N}_0(\mathbf{1}_{\{S < \infty\}} F_\varepsilon)$. Finally, Markov's inequality gives

$$\mathbb{N}_0(S \leq M, \mathcal{Z}_{[-S]_\varepsilon}^\varepsilon - \mathcal{Z}_{-S}^\varepsilon \geq \delta/4) \leq \mathbb{N}_0(S < \infty, F_\varepsilon \geq \delta/4) \leq \frac{4}{\delta} c_3 \varepsilon^{1/2}.$$

Third step. We now consider the event $\{S \leq M, \mathcal{Z}_{-S}^\varepsilon - \mathcal{Z}_{[-S]_\varepsilon}^\varepsilon \geq \delta/4\}$. We observe that, if $S < \infty$,

$$\mathcal{Z}_{-S}^\varepsilon - \mathcal{Z}_{[-S]_\varepsilon}^\varepsilon \leq \varepsilon^{-2} \int_0^\sigma ds \mathbf{1}_{\{T_{-S}(W_s) = \infty, \widehat{W}_s \in [-S]_\varepsilon + \varepsilon, -S + \varepsilon\}}.$$

Notice that $T_{-S}(W_s) = \infty$ implies $T_{[-S]_\varepsilon}(W_s) = \infty$ and that $-S + \varepsilon \leq [-S]_\varepsilon + \varepsilon + \varepsilon^{3/2}$. Hence, on the event where $S \leq M$ and $\mathcal{Z}_{-S}^\varepsilon - \mathcal{Z}_{[-S]_\varepsilon}^\varepsilon \geq \delta/4$, we can find a real $r \in [-M - 1, -\eta]$ of the form $r = j\varepsilon^{3/2}$ with $j \in \mathbb{Z}$, such that

$$\varepsilon^{-2} \int_0^\sigma ds \mathbf{1}_{\{T_r(W_s) = \infty, \widehat{W}_s \in [r + \varepsilon, r + \varepsilon + \varepsilon^{3/2}]\}} \geq \frac{\delta}{4}.$$

Let us fix $r \in [-M - 1, -\eta]$ in the following lines, and bound the probability of the event in the last display. From the first-moment formula for the Brownian snake, we have, with the same notation as in the second step,

$$\begin{aligned} & \mathbb{N}_0\left(\varepsilon^{-2} \int_0^\sigma ds \mathbf{1}_{\{T_r(W_s) = \infty, \widehat{W}_s \in [r + \varepsilon, r + \varepsilon + \varepsilon^{3/2}]\}}\right) \\ &= \varepsilon^{-2} \mathbb{E}_0\left[\int_0^{\tau_r} dt \mathbf{1}_{\{B_t \in [r + \varepsilon, r + \varepsilon + \varepsilon^{3/2}]\}}\right] \leq c_4 \varepsilon^{1/2}, \end{aligned}$$

with some constant c_4 . To get a better estimate, we use higher moments, but to this end we need to perform a suitable truncation. We fix $A > 0$, and we observe that, for every integer $k \geq 1$, for any nonnegative measurable function f on \mathbb{R} , we have

$$\begin{aligned} & \mathbb{N}_0\left(\left(\int_0^\sigma ds \mathbf{1}_{\{T_r(W_s) = \infty, \tau_A(W_s) = \infty\}} f(\widehat{W}_s)\right)^k\right) \\ & \leq C_{k, A, M} \left(\sup_{x \in [r, A]} \mathbb{E}_x\left[\int_0^{\tau_r \wedge \tau_A} dt f(B_t)\right]\right)^k, \end{aligned}$$

where $C_{k, A, M}$ is a constant depending only on k , A and M . The bound in the previous display can be derived in a straightforward way from the k th moment formula for the Brownian snake [23], Proposition IV.2. We omit the details. We apply this bound with $f = \mathbf{1}_{[r + \varepsilon, r + \varepsilon + \varepsilon^{3/2}]}$, and we arrive at the estimate

$$\begin{aligned} & \mathbb{N}_0\left(\left(\varepsilon^{-2} \int_0^\sigma ds \mathbf{1}_{\{T_r(W_s) = \infty, \tau_A(W_s) = \infty\}} \mathbf{1}_{\{\widehat{W}_s \in [r + \varepsilon, r + \varepsilon + \varepsilon^{3/2}]\}}\right)^k\right) \\ & \leq C_{k, A, M} (c_5 \varepsilon^{1/2})^k, \end{aligned}$$

with a constant c_5 depending on A and M . From Markov's inequality, we then get

$$\begin{aligned} & \mathbb{N}_0\left(W^* < A, \varepsilon^{-2} \int_0^\sigma ds \mathbf{1}_{\{T_r(W_s) = \infty, \widehat{W}_s \in [r + \varepsilon, r + \varepsilon + \varepsilon^{3/2}]\}} \geq \frac{\delta}{4}\right) \\ & \leq \left(\frac{\delta}{4}\right)^{-k} \mathbb{N}_0\left(\left(\varepsilon^{-2} \int_0^\sigma ds \mathbf{1}_{\{T_r(W_s) = \infty, \tau_A(W_s) = \infty\}} \mathbf{1}_{\{\widehat{W}_s \in [r + \varepsilon, r + \varepsilon + \varepsilon^{3/2}]\}}\right)^k\right) \\ & \leq C_{k, A, M} \left(\frac{\delta}{4}\right)^{-k} (c_5 \varepsilon^{1/2})^k. \end{aligned}$$

We take $k = 4$ and sum the preceding estimate over possible values of $r = -j\varepsilon^{3/2}$ in $[-M - 1, -\eta]$, and we arrive at the estimate

$$\mathbb{N}_0(W^* < A, S \leq M, \mathcal{Z}_{-S}^\varepsilon - \mathcal{Z}_{[-S]_\varepsilon}^\varepsilon \geq \delta/4) \leq c_6 \varepsilon^{1/2}$$

with a constant c_6 depending on A , M and δ .

Fourth step. We deduce from the second and third steps that we have

$$(44) \quad \mathbb{N}_0(W^* < A, S \leq M, |\mathcal{Z}_{[-S]_\varepsilon}^\varepsilon - \mathcal{Z}_{-S}^\varepsilon| \geq \delta/4) \leq c_7 \varepsilon^{1/2},$$

with a constant c_7 depending on δ , A , M , K . Combining (43) and (44) and recalling the considerations of the end of the first step, we arrive at the bound

$$(45) \quad \mathbb{N}_0(W^* < A, S \leq M) \leq (c_1 + c_2 + c_7) \varepsilon^{1/2}.$$

Let us write $S = S^{(\varepsilon)}$ to recall the dependence on ε . Let n_0 be the first integer such that $(n_0)^{-3} < \eta$. The bound (45) gives

$$\sum_{n=n_0}^{\infty} \mathbb{N}_0(W^* < A, S^{(n^{-3})} \leq M) < \infty.$$

Hence, \mathbb{N}_0 a.e. on the event $W^* < A$, we have $S^{(n^{-3})} > M$ for all large enough n . This means that, \mathbb{N}_0 a.e. on the event where $\sup\{\mathcal{Z}_{-t} : t > 0\} < K$ and $W^* < A$, we have for all large enough n ,

$$\sup_{\eta \leq u \leq M} |\mathcal{Z}_{-u}^{(n^{-3})} - \mathcal{Z}_{-u}| \leq \delta.$$

Since δ , K and A are arbitrary, we obtain that, \mathbb{N}_0 a.e.,

$$\lim_{n \rightarrow \infty} \left(\sup_{\eta \leq u \leq M} |\mathcal{Z}_{-u}^{(n^{-3})} - \mathcal{Z}_{-u}| \right) = 0.$$

We can replace $\eta \leq u \leq M$ by $\eta \leq u < \infty$ since $\mathcal{Z}_{-u}^{(\varepsilon)} - \mathcal{Z}_{-u} = 0$ for $u > -W_* + \varepsilon$. The statement of the proposition then follows by a monotonicity argument. \square

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