

## ELLIPTIC STOCHASTIC QUANTIZATION

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We prove an explicit formula for the law in zero of the solution of a class of elliptic SPDE in  $\mathbb{R}^2$ . This formula is the simplest instance of *dimensional reduction*, discovered in the physics literature by Parisi and Sourlas (*Phys. Rev. Lett.* **43** (1979) 744–745), which links the law of an elliptic SPDE in  $d + 2$  dimension with a Gibbs measure in  $d$  dimensions. This phenomenon is similar to the relation between a  $\mathbb{R}^{d+1}$  dimensional parabolic SPDE and its  $\mathbb{R}^d$  dimensional invariant measure. As such, dimensional reduction of elliptic SPDEs can be considered a sort of *elliptic stochastic quantisation* procedure in the sense of Nelson (*Phys. Rev.* **150** (1966) 1079–1085) and Parisi and Wu (*Sci. Sin.* **24** (1981) 483–496). Our proof uses in a fundamental way the representation of the law of the SPDE as a supersymmetric quantum field theory. Dimensional reduction for the supersymmetric theory was already established by Klein et al. (*Comm. Math. Phys.* **94** (1984) 459–482). We fix a subtle gap in their proof and also complete the dimensional reduction picture by providing the link between the elliptic SPDE and the supersymmetric model. Even in our  $d = 0$  context the arguments are nontrivial and a non-supersymmetric, elementary proof seems only to be available in the Gaussian case.

**1. Introduction.** *Stochastic quantization* [21, 22, 51] broadly refers to the idea of sampling a given probability distribution by solving a stochastic differential equation (SDE). This idea is both appealing practically and theoretically since simulating or solving a SDE is sometimes simpler than sampling or studying a given distribution. If, in finite dimensions, this boils down mostly to the idea of the Monte Carlo Markov chain method (which was actually invented before stochastic quantization); it is in infinite dimensions that the method starts to have a real theoretical appeal.

It was Nelson [45–47] and subsequently Parisi and Wu [51] who advocated the constructive use of stochastic partial differential equations (SPDEs) to realize a given Gibbs measure for the use of Euclidean quantum field theory (QFT). Indeed the original (parabolic) stochastic quantization procedure of [51] can be understood as the equivalence

$$(1) \quad \mathbb{E}[F(\varphi(t))] \propto \int F(\phi) e^{-S(\phi)} \mathcal{D}\phi.$$

Here,  $F$  belongs to a suitable space of real-valued test functions,  $\mathcal{D}\phi$  is a heuristic “Lebesgue measure” on  $\mathcal{S}'(\mathbb{R}^d)$ , while on the left-hand side the random field  $\varphi$  depends on  $(t, x) \in \mathbb{R} \times \mathbb{R}^d$  and is a stationary solution to the parabolic SPDE

$$(2) \quad \partial_t \varphi(t, x) + (m^2 - \Delta)\varphi(t, x) + V'(\varphi(t, x)) = \xi(t, x),$$

where  $\xi$  is a Gaussian white noise in  $\mathbb{R}^{d+1}$ ,  $V : \mathbb{R} \rightarrow \mathbb{R}$  a generic local potential bounded from below,  $m^2$  a positive parameter and  $\varphi(t)$  is the fixed time marginal of  $\varphi$  which has a law

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independent of  $t$  by stationarity and on the right-hand side we have the formal expression for a measure on functions on  $\mathbb{R}^d$  with weight factor given by

$$(3) \quad S(\phi) := \int_{\mathbb{R}^d} |\nabla\phi(x)|^2 + m^2|\phi(x)|^2 + V(\phi(x)) \, dx.$$

Equation (1) can be made mathematically precise and rigorous by tools from the theory of Markov processes [18, 20, 42], SDE/SPDEs [1, 37, 43, 57] and Dirichlet forms [4], for example, when  $d = 0$ , or when the equation is regularized appropriately and, in certain cases, for suitable renormalized versions of the SPDE [2, 3, 5, 10, 12, 19, 30–32, 35, 36, 44] when  $d = 1, 2, 3$ . Let us note for example that in the full space it is easier to make sense of equation (2) than of the formal Gibbs measure on the right-hand side of (1); see [30].

In a slightly different context, and inspired by previous perturbative computations of Imry and Ma [34] and Young [62], Parisi and Sourlas [49, 50] considered the solutions of the elliptic SPDEs

$$(4) \quad (m^2 - \Delta)\phi + V'(\phi) = \xi$$

in  $\mathbb{R}^{d+2}$  where  $\xi$  is a Gaussian white noise on  $\mathbb{R}^{d+2}$  and they discovered that its stationary solutions are similarly related to the same  $d$ -dimensional Gibbs measure. If we take  $x \in \mathbb{R}^d$ , then they claimed that, for “nice” test functions  $F$  (e.g., correlation functions) we have

$$(5) \quad \mathbb{E}[F(\phi(0, \cdot))] \propto \int F(\varphi) e^{-4\pi S(\varphi)} \mathcal{D}\varphi.$$

More precisely, the law of the random field  $(\phi(0, y))_{y \in \mathbb{R}^d}$ , obtained by looking at the trace of  $\phi$  on the hyperplane  $\{x = (x_1, \dots, x_{d+2}) \in \mathbb{R}^{d+2} : x_1 = x_2 = 0\} \subset \mathbb{R}^{d+2}$ , should be equivalent to that of the Gibbs measure formally appearing on the right-hand side of (5) and corresponding to the action functional (3). Therefore, one can interpret equation (5) as an *elliptic stochastic quantization* prescription in the same spirit of equation (1).

When  $V = 0$ , one can directly check that the formula (5) is correct. Indeed in this case the unique stationary solution  $\phi$  to the elliptic SPDE (4) is given by a Gaussian process with covariance

$$\mathbb{E}[\phi(x)\phi(x')] = \int_{\mathbb{R}^{d+2}} \frac{e^{ik \cdot (x-x')}}{(m^2 + |k|^2)^2} \frac{dk}{(2\pi)^{d+2}}, \quad x, x' \in \mathbb{R}^{d+2}.$$

Therefore, for all  $y, y' \in \mathbb{R}^d$  we have

$$\begin{aligned} \mathbb{E}[\phi(0, y)\phi(0, y')] &= \int_{\mathbb{R}^d} e^{ik \cdot (y-y')} \int_{\mathbb{R}^2} \frac{dq}{(|q|^2 + m^2 + |k|^2)^2} \frac{dk}{(2\pi)^{d+2}} \\ &= \int_{\mathbb{R}^2} \frac{dq}{(|q|^2 + 1)^2} \int_{\mathbb{R}^d} \frac{e^{ik \cdot (y-y')}}{m^2 + |k|^2} \frac{dk}{(2\pi)^{d+2}} \\ &= \frac{1}{4\pi} \int_{\mathbb{R}^d} \frac{e^{ik \cdot (y-y')}}{m^2 + |k|^2} \frac{dk}{(2\pi)^d}, \end{aligned}$$

where we performed a rescaling of the  $q$  integral in order to decouple the two integrations. The reader can easily check that the expression we obtained describes the covariance of the Gaussian random field formally corresponding to the right-hand side of (5) for  $V = 0$ .

While this last argument is almost trivial, a more general justification outside the Gaussian setting is not so obvious. The equivalence (5) was derived in [49, 50] at the theoretical physics level of rigor going through a representation of the left-hand side via a supersymmetric quantum field theory (QFT) involving a pair of scalar fermion fields. This is one of the instances of the *dimensional reduction* phenomenon which is conjectured in certain random

systems where the randomness effectively decreases the dimension of the space where fluctuations take place. A crucial assumption is that the equation (4) has a unique solution, which is already a nontrivial problem for general  $V$ . Parisi and Sourlas [50] observed that nonuniqueness can lead to a breaking of the supersymmetry, in which case the relation (5) could fail. So, part of the task of clarifying the situation is to determine under which conditions *some* relations in the spirit of (5) could anyway be true.

The dimensional reduction (5) of the elliptic SPDEs (4) seems less amenable to standard probabilistic arguments than its parabolic counterpart (1). Let us remark that from the point of view of theoretical physics it is possible [22, 50] to justify also dimensional reduction in the parabolic case (2) using a supersymmetric argument much like in the elliptic setting.

The only attempt we are aware of to a mathematically rigorous understanding of the relation (5) is the work of Klein, Landau and Perez [38–40] (see also the related work on the density of states of electronic systems with random potentials [41]) which however do not fully prove equation (5) but only the equivalence between the intermediate supersymmetric theory in  $d + 2$  dimensions and the Gibbs measure in  $d$ -dimensions. The reason for this limitation is that the problem of uniqueness of the elliptic SPDE seems to unnecessarily restrict the class of potentials for which (5) can be established and Klein et al. decided to bypass a detailed analysis of the situation by starting directly with the supersymmetric formulation. Their rigorous argument requires a cut-off, both on large momenta in  $d$  “orthogonal” dimensions and on the space variable in  $d + 2$  dimensions in order to obtain a well-defined, finite volume problem. This regularization breaks the supersymmetry which has to be recovered by adding a suitable correction term, spoiling the final result (see Theorem 1 and Theorem 3 below). A subtle gap in their published proof is pointed out, and closed, in Section 4.

Let us remark that, in a different context, dimensional reduction has been proven and exploited in the remarkable work of Brydges and Imbrie on branched polymers [13, 14] and more recently by Helmuth [33].

In the present work, we complete the program of *elliptic stochastic quantization*, in the  $d = 0$  case, by proving relation (5) linking the solution to the elliptic SPDE (4) with the Gibbs measure with action (3) and removing the finite volume cut-off in some cases.

Fix  $d = 0$  and consider the two-dimensional elliptic multidimensional SPDE

$$(6) \quad (m^2 - \Delta)\phi(x) + f(x)\partial V(\phi(x)) = \xi(x), \quad x \in \mathbb{R}^2,$$

where  $\phi = (\phi^1, \dots, \phi^n)$  takes values in  $\mathbb{R}^n$ ,  $(\xi^1, \dots, \xi^n)$  are  $n$  independent Gaussian white noises,  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  a smooth potential function,  $f(x) := \tilde{f}(|x|^2)$  with  $\tilde{f} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  a decreasing cut-off function, such that the derivative  $\tilde{f}'$  of the function  $r \mapsto \tilde{f}(r)$  is defined, tending to 0 at infinity, and  $\partial V = (\partial_i V)_{i=1, \dots, n}$  denotes the gradient of  $V$ . We will denote  $f'(x) := \tilde{f}'(|x|^2)$ .

Equation (6) is the elliptic counterpart of the equilibrium Langevin reversible dynamics for finite dimensional Gibbs measures. Let us note that the elliptic dynamics is already described by a SPDE in two dimensions while in the parabolic setting one would consider a much simpler Markovian SDE [2, 35] (no renormalization being necessary). The question of uniqueness of solutions is however quite similar in difficulty; indeed, it is nontrivial to establish uniqueness of stationary solutions to the SDE and much work in the theory of long time behavior of Markov processes is devoted precisely to this. In the elliptic context of (6), there is no (easy) Markov property helping and the question of uniqueness of weak stationary solutions seems more open, even in the presence of the cut-off  $f$ .

What makes this  $d = 0$  problem very interesting, is above all the fact that while the statements we would like to prove are quite easy to describe (see below), to our surprise their rigorous justification is already quite involved and not yet quite complete in full generality.

Define the probability measure  $\kappa$  on  $\mathbb{R}^n$  by

$$(7) \quad \frac{d\kappa}{dy} := Z_\kappa^{-1} \exp\left[-4\pi\left(\frac{m^2}{2}|y|^2 + V(y)\right)\right],$$

where  $y \in \mathbb{R}^n$ ,  $Z_\kappa := \int_{\mathbb{R}^n} \exp[-4\pi(\frac{m^2}{2}|y|^2 + V(y))] dy$  ( $Z_\kappa$  is well defined since  $V$  is bounded from below).

The main result of this paper is the following theorem which states that on very general conditions on  $V$  there is always a weak solution which satisfies (an approximate) elliptic stochastic quantization relation (of the form (5)). By weak solution to the SPDE (6), we mean a probability measure  $\nu$  on the space of fields  $\phi$  under which  $(m^2 - \Delta)\phi + \partial V(\phi)$  is distributed like Gaussian white noise on  $\mathbb{R}^2$ . A strong solution  $\phi$  to equation (6) is a measurable map  $\xi \mapsto \phi = \phi(\xi)$  satisfying the equation for almost all realizations of  $\xi$ . In order to state precisely our results, we need to introduce the following assumptions on  $V$  and on the finite volume cut-off  $f$ .

*Hypothesis C. (convexity)* The potential  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a positive smooth function such that

$$y \in \mathbb{R}^n \mapsto V(y) + m^2|y|^2$$

is strictly convex and  $V$  with its first and second partial derivatives grow at most exponentially at infinity.

*Hypothesis QC. (quasi-convexity)* The potential  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a positive smooth function, such that it and its first and second partial derivatives grow at most exponentially at infinity and, moreover, it is such that there exists a function  $H : \mathbb{R}^n \rightarrow \mathbb{R}$  with exponential growth at infinity such that we have

$$-\langle \hat{n}, \partial V(y + r\hat{n}) \rangle \leq H(y), \quad \hat{n} \in \mathbb{S}^n, y \in \mathbb{R}^n \text{ and } r \in \mathbb{R}_+,$$

with  $\mathbb{S}^n$  is the  $n - 1$  dimensional unit sphere.

*Hypothesis CO. (cut-off)* The function  $f$  is real valued, has at least  $C^2$  smoothness and in addition satisfies  $f' \leq 0$ , it decays exponentially at infinity and fulfills  $\Delta(f) \leq b^2 f$  for  $b^2 < 4m^2$  (some examples of such functions are given in [39] and the motivations for this hypothesis are explained in Remark 38 below).

**THEOREM 1.** *Under the Hypotheses QC and CO, there exists (at least) one weak solution  $\tilde{\nu}$  to equation (6) such that for all measurable bounded functions  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  we have*

$$(8) \quad \int_{\tilde{\mathcal{W}}} h(\phi(0)) \Upsilon_f(\phi) \tilde{\nu}(d\phi) = Z_f \int_{\mathbb{R}^n} h(y) d\kappa(y),$$

where  $\Upsilon_f(\phi) := e^{4 \int_{\mathbb{R}^2} f'(x)V(\phi(x)) dx}$  and  $Z_f := \int_{\mathcal{W}} \Upsilon_f(\phi) \tilde{\nu}(d\phi)$ .  $\tilde{\mathcal{W}}$  is a suitable Banach space of functions from  $\mathbb{R}^2$  to  $\mathbb{R}^n$  where  $\tilde{\nu}$  is defined (see Section 2, equations (13), (14) and (17) for a precise definition of  $\tilde{\mathcal{W}}$  and  $\tilde{\nu}$ ).

**REMARK 2.** The following families of functions satisfy Hypothesis QC:

- Smooth convex functions (since they satisfy the stronger Hypothesis C),
- Smooth bounded functions,
- Smooth functions having the second derivative semidefinite positive outside a compact set,
- Any positive linear combinations of the previous functions.

The drawback of this result is the lack of constructive determination of the weak solution  $\nu$  for which the dimensional reduction described by equation (8) is realized. This is linked with the fact that Hypothesis QC does not guarantee uniqueness of strong solutions to equation (6). The fact that nonuniqueness is related to a possible breaking of the supersymmetry associated with (6) was already noted by Parisi and Sourlas [50]. If we are willing to assume that the potential  $V$  is convex we can be more precise, as the following theorem shows.

**THEOREM 3.** *Under Hypotheses C and CO there exists an unique strong solution  $\phi = \phi(\xi)$  of equation (6) and for all measurable bounded functions  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ , we have*

$$(9) \quad \mathbb{E}[h(\phi(0))\Upsilon_f(\phi)] = Z_f \int_{\mathbb{R}^n} h(y) d\kappa(y),$$

where  $\Upsilon_f$  is defined as in Theorem 1,  $Z_f := \mathbb{E}[\Upsilon_f(\phi)]$  and  $\mathbb{E}$  denotes expectation with respect to the law of  $\xi$ .

Both theorems require the presence of a suitable cut-off  $f \not\equiv 1$  which is responsible for the weighting factor  $\Upsilon_f(\phi)$  on the left-hand side of the dimensional reduction statements (8) and (9). If we would be allowed to take  $f \equiv 1$ , then we would have proven the  $d = 0$  version of equation (5). However, presently we are not able to do this for all QC potentials but only for those satisfying Hypothesis C (see Section 4 for the proof).

**THEOREM 4.** *Suppose that  $V$  satisfies Hypothesis C and let  $\phi$  be the unique strong solution in  $C^0_{\text{exp}\beta}(\mathbb{R}^2; \mathbb{R}^n)$  (see Section 6 for the definition of this space) of the equation*

$$(10) \quad (m^2 - \Delta)\phi + \partial V(\phi) = \xi.$$

Then for any  $x \in \mathbb{R}^2$  and any measurable and bounded function  $h$  defined on  $\mathbb{R}^n$  we have

$$(11) \quad \mathbb{E}[h(\phi(x))] = \int_{\mathbb{R}^n} h(y) d\kappa(y).$$

This result is the first rigorous result on elliptic stochastic quantization without any cut-off. In fact, in this case the results of Klein, Landau and Perez [39] do not hold, since they use only an integral representation of the solution to equation (6) which has meaning only when  $f \not\equiv 1$ .

**REMARK 5.** It is easy to generalize Theorems 1, 3 and 4 to equations of the form

$$(12) \quad (m^2 - \Delta)\phi^i(x) + \sum_{r=1}^n \gamma_r^i \gamma_r^j f(x) \partial_{\phi_j} V(\phi(x)) = \gamma_r^i \xi^r(x),$$

where  $f$  is as before,  $\Gamma = (\gamma_j^i)_{i,j=1,\dots,n}$  is an  $n \times n$  invertible matrix and the Hypothesis C and QC are generalized accordingly.

*Plan.* The paper is organized as follows. In Section 2, we introduce the notions of strong and weak solutions to equation (6), and we prove, in Theorem 10, the existence of strong solutions (and thus also of weak solutions) under Hypothesis QC. We also provide, in Theorem 14, a representation of weak solutions via the theory of transformation of measures on abstract Wiener spaces developed by Üstünel and Zakai in [59] (whose setting and main facts needed here are summarized in Appendix A).

Section 3 is devoted to the proof Theorem 1 and Theorem 3 about elliptic stochastic quantization, under the Hypothesis QC and CO and using Theorem 17 and PDE techniques.

In Section 4, Theorem 17 is proven, that is, dimensional reduction using Hypothesis  $V_\lambda$  (see Section 3). The proof of Theorem 17 is similar to the rigorous version of Parisi and Sourlas argument proposed in [39], starting from different hypotheses. The proof of Theorem 17 in Section 4 is based on Theorem 26 stating a relation between the expectation involving some bosonic and fermionic-free fields.

In Section 5, we prove Theorem 26 exploiting the properties of supersymmetric Gaussian fields. In Section 5, we also propose a brief introduction to supersymmetry and supersymmetric Gaussian fields.

Section 6 discusses the proof of Theorem 4 on the cut-off removal under Hypothesis C.

Appendix A is a brief introduction to the theory of transformations on abstract Wiener spaces used in this paper, and Appendix B consists of a discussion of some properties of fermionic fields.

**2. The elliptic SPDE.** In order to study equation (6), we have to recall some definitions, notation and conventions. Fix an abstract Wiener space  $(\mathcal{W}, \mathcal{H}, \mu)$  where the noise  $\xi$  is defined (for the concept of abstract Wiener space we refer, for example, to [29, 48, 59]). The Cameron–Martin space  $\mathcal{H}$  is the space

$$\mathcal{H} := L^2(\mathbb{R}^2; \mathbb{R}^n),$$

with its natural scalar product and natural norm given by  $\langle h, g \rangle = \sum_{i=1}^n \int_{\mathbb{R}^2} h^i(x) g^i(x) dx$ . Let  $\mathcal{W}$  (in which  $\mathcal{H}$  is densely embedded) be the space

$$(13) \quad \mathcal{W} = \mathcal{W}_{p,\eta} := W_\eta^{p,-1-2\epsilon}(\mathbb{R}^2; \mathbb{R}^n) \cap (1 - \Delta)(C_\eta^0(\mathbb{R}^2; \mathbb{R}^n)),$$

where  $p \geq 1, \eta > 0$  and  $W_\eta^{p,-1-2\epsilon}(\mathbb{R}^2; \mathbb{R}^n)$  is a fractional Sobolev space with norm

$$\|g\|_{W_\eta^{p,-1-2\epsilon}} := \left( \int_{\mathbb{R}^2} (1 + |x|)^{-\eta} |(1 - \Delta)^{-\frac{1}{2}-\epsilon}(g)|^p dx \right)^{\frac{1}{p}},$$

for some  $\epsilon > 0$  small enough and  $(1 - \Delta)(C_\eta^0(\mathbb{R}^2; \mathbb{R}^n))$  is the space of the second order distributional derivatives of continuous functions on  $\mathbb{R}^n$  growing at infinity at most as  $|x|^\eta$  with norm

$$\|g\|_{(-\Delta+1)(C_\eta^0)} := \|(1 + |x|)^{-\eta} ((1 - \Delta)^{-1}g)(x)\|_{L^\infty}.$$

Thus  $\mathcal{W}_{p,\eta}$  is a Banach space with norm given by the sum of the norms of  $W_\eta^{p,-1-2\epsilon}(\mathbb{R}^2; \mathbb{R}^n)$  and of  $(1 - \Delta)^{-1}(C_\eta^0(\mathbb{R}^2; \mathbb{R}^n))$ . In the following, we usually do not specify the indices  $\eta$  and  $p$  in the definition of  $\mathcal{W}_{p,\eta}$  and we write only  $\mathcal{W}$ . We also introduce the notation

$$(14) \quad \tilde{\mathcal{W}} = (1 - \Delta)^{-1}(\mathcal{W}).$$

The Gaussian measure  $\mu$  on  $\mathcal{W}$  is the standard Gaussian measure with Fourier transform given by  $e^{-\frac{1}{2}\|\cdot\|_{\mathcal{H}}^2}$ . The white noise  $\xi$  is then naturally realized on  $(\mathcal{H}, \mathcal{W}, \mu)$ , in the sense that  $\xi$  is the random variable  $\xi : \mathcal{W} \rightarrow S'(\mathbb{R}^2; \mathbb{R}^n)$  (where  $S'(\mathbb{R}^2; \mathbb{R}^n)$  is the space of  $\mathbb{R}^n$ -valued Schwartz distributions on  $\mathbb{R}^2$ ) defined as  $\xi(w) = \text{id}_{\mathcal{W}}(w) = w$ . In this way, the law of  $\xi$  is simply  $\mu$  (or, better, it is equal to the pushforward of  $\mu$  on  $S' := S'(\mathbb{R}^2, \mathbb{R}^n)$  with respect the natural inclusion map of  $\mathcal{W}$  in  $S'$ ).

Sometimes it is also useful to consider the space  $C_\tau^\alpha$  of  $\alpha$ -Hölder continuous functions such that they and their derivatives (or Hölder norms) grow at infinity at most like  $|x|^\tau$  for a real number  $\tau$  (this notation is used also if  $\tau$  is negative in that case the functions decrease at least like  $\frac{1}{|x|^{-\tau}}$ ). It is important to note that  $C_\eta^\alpha$  can be identified with the Besov space  $B_{\infty,\infty,\eta}^\alpha(\mathbb{R}^2)$  (where  $B_{\infty,\infty,\eta}^\alpha(\mathbb{R}^2)$  is the weighted Besov space  $B_{\infty,\infty}^\alpha(\mathbb{R}^2, (1 + |x|)^\eta)$  of [9],



Chapter 2, Section 2.7). It is also important to realize that  $(1 - \Delta)^{-1}(\mathcal{W}) \subset \mathcal{C}_\eta^\alpha$  if we choose  $p$  big enough and  $\alpha > 0$  small enough.

We introduce now two notions of solutions for equation (6). For later convenience, it is better to discuss the equation in term of the variable  $\eta := (m^2 - \Delta)\phi$  for which it reads

$$(15) \quad \eta + f \partial V(\mathcal{I}\eta) = \eta + U(\eta) = \xi,$$

where

$$\mathcal{I} := (m^2 - \Delta)^{-1}$$

and where we introduced the map  $U : \mathcal{W} \rightarrow \mathcal{H}$  given by

$$(16) \quad U(w) := f \partial V(\mathcal{I}w), \quad w \in \mathcal{W}.$$

Under the condition of (at most) exponential growth at infinity of  $V$ , required by Hypothesis QC and Hypothesis C, it is possible to prove, that for  $\eta < 1$  in the definition of  $\mathcal{W}$ , for each  $w \in \mathcal{W}$  we have  $U(w) \in \mathcal{W}$ . Indeed we have

$$\|U(w)\|_{(L^2(\mathbb{R}^2))^n} \leq \|\sqrt{f(x)}\|_{L^2(\mathbb{R}^2)} \cdot \|\sqrt{f(x)}|\partial V(\mathcal{I}w(x))\|_\infty$$

and  $\|\sqrt{f(x)}|\partial V(\mathcal{I}w(x))\|_\infty$  is finite since  $f$  decreases exponentially at infinity and  $V$  grow at most exponentially at infinity.

Furthermore, we introduce the map  $T : \mathcal{W} \rightarrow \mathcal{W}$  as

$$T(w) := w + U(w).$$

It is clear that a map  $S : \mathcal{W} \rightarrow \mathcal{W}$  satisfies equation (15), that is,  $T(S(w)) = \xi(w) = w$ , for  $(\mu)$ -almost all  $w \in \mathcal{W}$ , if and only if  $\mathcal{I}S(w)$  satisfies equation (6). The law  $\nu$  on  $\mathcal{W}$  associated to a solution of equation (15) must satisfy the relation  $T_*(\nu) = \mu$ . For these reasons, we introduce the following definition.

**DEFINITION 6.** A measurable map  $S : \mathcal{W} \rightarrow \mathcal{W}$  is a *strong solution* to equation (15) if  $T \circ S = \text{Id}_{\mathcal{W}}$   $\mu$ -almost surely. A probability measure  $\nu \in \mathcal{P}(\mathcal{W})$  (where  $\mathcal{P}(\mathcal{W})$  is the space of probability measures on  $\mathcal{W}$ ) on the space  $\mathcal{W}$  is a *weak solution* to equations (15) if  $T_*(\nu) = \mu$ , where  $T_*$  is the pushforward related with the map  $T$ .

If  $\nu$  is a probability measure on the space  $\mathcal{W}$ , we write  $\tilde{\nu}$  the unique probability measure on  $\tilde{\mathcal{W}}$  such that

$$(17) \quad (-\Delta + m^2)_*^{-1}(\nu) = \tilde{\nu}.$$

**2.1. Strong solutions.** In order to study the existence of strong solutions to equation (6), we introduce an equivalent version of the same equation that is simpler to study. Indeed if we write

$$\bar{\phi} = \phi - \mathcal{I}\xi,$$

and we suppose that  $\phi$  satisfies equation (6), then the function  $\bar{\phi}$  satisfies the following (random) PDE:

$$(18) \quad (m^2 - \Delta)\bar{\phi} + f \partial V(\bar{\phi} - \mathcal{I}\xi) = 0.$$

Equation (18) can be studied pathwise for any realization of the random field  $\mathcal{I}\xi$ . Hereafter, the symbol  $\lesssim$  stands for inequality with some positive constant standing on the right-hand side.

LEMMA 7. *Suppose that  $V$  satisfies Hypothesis QC, and let  $\bar{\phi}$  be a classical  $C^2$  solution to the equation (18), such that  $\lim_{x \rightarrow \infty} \bar{\phi}(x) = 0$ , then for any  $0 < \tau < 2$  and  $\eta > 0$  we have*

$$(19) \quad \|\bar{\phi}\|_\infty \lesssim 1 + \|f e^{\alpha_1 |\mathcal{I}\xi|}\|_\infty,$$

$$(20) \quad \|\bar{\phi}\|_{C_{-\eta}^{2-\tau}} \lesssim 1 + e^{\alpha_1 \|\bar{\phi}\|_\infty} \|( |x| + 1 )^\eta f e^{\alpha_1 |\mathcal{I}\xi|}\|_\infty,$$

for some positive constant  $\alpha_1$  and where it and the constants involved in the symbol  $\lesssim$  depend only on the function  $H$  in Hypothesis QC.

PROOF. Putting  $r_{\bar{\phi}}(x) = \sqrt{\sum_i (\bar{\phi}^i(x))^2} = |\bar{\phi}(x)|$ ,  $x \in \mathbb{R}^2$ , since the  $C^2$  function  $\bar{\phi}$  converges to zero at infinity, the function  $r_{\bar{\phi}}$  must have a global maximum at some point  $\bar{x} \in \mathbb{R}^2$ . This means that  $-\Delta(r_{\bar{\phi}}^2)(\bar{x}) \geq 0$ . On the other hand, since  $\bar{\phi}$  solves equation (18) we have

$$\begin{aligned} m^2 r_{\bar{\phi}}^2(\bar{x}) &\leq -\frac{1}{2} \Delta(r_{\bar{\phi}}^2)(\bar{x}) + m^2 r_{\bar{\phi}}^2(\bar{x}) \\ &\leq (-\bar{\phi} \cdot \Delta \bar{\phi} - |\nabla \bar{\phi}|^2 + m^2 |\bar{\phi}|^2) \\ &\leq -f(\bar{x}) r_{\bar{\phi}}(\bar{x}) (\hat{n}_{\bar{\phi}}(\bar{x}) \cdot \partial V(\mathcal{I}\xi(\bar{x}) + \hat{n}_{\bar{\phi}}(\bar{x}) r_{\bar{\phi}}(\bar{x}))), \end{aligned}$$

where  $\hat{n}_{\bar{\phi}} = \frac{\bar{\phi}}{|\bar{\phi}|} \in S^n$  when  $r_{\bar{\phi}} \neq 0$ , and 0 elsewhere. Using Hypothesis QC, we obtain

$$\|r_{\bar{\phi}}\|_\infty \leq \frac{f(\bar{x}) H(\mathcal{I}\xi(\bar{x}))}{m^2} \lesssim 1 + \|f e^{\alpha_1 |\mathcal{I}\xi|}\|_\infty,$$

since  $H$  grows at most exponentially at infinity. This result implies inequality (19).

The bound (20) can be obtained directly using the fact  $\|\phi\|_{C^{2-\tau}} \lesssim \|(-\Delta + m^2)(\phi)\|_\infty$ , where we use the properties of the Besov spaces  $C^\alpha(\mathbb{R}^2) = B_{\infty, \infty}^\alpha(\mathbb{R}^2)$  with respect to derivatives (see [58], Chapter 2, Section 2.3.8).  $\square$

REMARK 8. It is simple to prove that the inequalities (19) and (20) can be chosen to be uniform with respect to some rescaling of the potential of the form  $\lambda V$ , or satisfying Hypothesis  $V_\lambda$  below, where  $\lambda \in [0, 1]$ .

In the following, we denote by  $\mathcal{F} : \mathcal{W} \rightarrow \mathcal{P}(C^{2-\tau}(\mathbb{R}^2; \mathbb{R}^n))$  the set valued function which associates to a given  $w \in \mathcal{W}$  the (possible empty) set of solutions to equation (18) in  $C^{2-\tau}(\mathbb{R}^2; \mathbb{R}^n)$ , where  $\tau > 0$ , when  $\mathcal{I}\xi$  is evaluated in  $w$ .

THEOREM 9. *Let  $V$  be a smooth positive function satisfying Hypothesis QC, then for any  $w \in \mathcal{W}$  the set  $\mathcal{F}(w)$  is nonempty and closed. Furthermore,  $\mathcal{F}(w) \subset C^2(\mathbb{R}^2; \mathbb{R}^n)$  and if  $B \subset \mathcal{W}$  is a bounded set then  $\mathcal{F}(B) = \bigcup_{w \in B} \mathcal{F}(w)$  is compact in  $C_{-\eta}^{2-\tau}(\mathbb{R}^2; \mathbb{R}^n)$  for any  $\tau > 0$  and  $\eta \geq 0$ .*

PROOF. We introduce the map  $C_{-\eta}^{2-\tau}(\mathbb{R}^2; \mathbb{R}^n) \times \mathcal{W} \ni (\bar{\phi}, w) \mapsto \mathcal{K}(\bar{\phi}, w) \in C^{2-\tau'}(\mathbb{R}^2; \mathbb{R}^n)$ , where  $\tau' < \tau$ , given by

$$\mathcal{K}^i(\bar{\phi}, w) := -\mathcal{I}(f \partial V(\bar{\phi} + \mathcal{I}\xi(w))).$$

The map  $\mathcal{K}$  is continuous with respect to its first argument; indeed if  $\bar{\phi}, \bar{\phi}_1 \in C^{2-\tau}(\mathbb{R}^2; \mathbb{R}^n)$ ,

$$\begin{aligned} &\|\mathcal{K}^i(\bar{\phi}, w) - \mathcal{K}^i(\bar{\phi}_1, w)\|_{C_{-\eta}^{2-\tau'}} \\ &\lesssim \|( |x| + 1 )^\eta f(\partial V(\bar{\phi}, \mathcal{I}\xi(w)) - \partial V(\bar{\phi}_1, \mathcal{I}\xi(w)))\|_\infty \end{aligned}$$



$$\begin{aligned} &\lesssim \left\| \int_0^1 (|x| + 1)^\eta f \partial^2 V(\bar{\phi} - t(\bar{\phi} - \bar{\phi}_1) + \mathcal{I}\xi(w)) \cdot (\bar{\phi} - \bar{\phi}_1) dt \right\|_\infty \\ &\lesssim \|\bar{\phi} - \bar{\phi}_1\|_\infty \left( (|x| + 1)^\eta \sqrt{f} \right)_\infty \left( \|\partial^2 V_B\|_\infty + e^{\alpha \|\bar{\phi} - \bar{\phi}_1\|_\infty} \|\sqrt{f} e^{\alpha |\mathcal{I}\xi|}\|_\infty \right), \end{aligned}$$

where the positive constant  $\alpha$  depends on the exponential growth of  $\partial^2 V$  at infinity. By a similar reasoning, we can prove that  $\mathcal{K}$  sends bounded sets of  $\mathcal{C}_{-\eta}^{2-\tau}$  into bounded sets of  $\mathcal{C}_{-\eta'}^{2-\tau'}$ , where  $\tau' < \tau$  and  $\eta' > \eta$ . Since the immersion  $\mathcal{C}_{-\eta'}^{2-\tau'} \hookrightarrow \mathcal{C}_{-\eta}^{2-\tau}$  is compact, we have that  $\mathcal{K}$  is a compact map.

Since  $\mathcal{I}\xi \in \mathcal{C}_\alpha^{1-}$  and  $\bar{\phi} \in \mathcal{C}_{-\eta}^{2-\tau}(\mathbb{R}^2; \mathbb{R}^n)$ , we have  $(-\Delta + m^2)\mathcal{K}^i(\bar{\phi}, w) \in \mathcal{C}^{1-}(\mathbb{R}^2; \mathbb{R}^n)$ . This implies, using the regularity results for Poisson equations (see Theorem 4.3 in [27]) and a bootstrap argument, that if  $\bar{\phi} = \mathcal{K}(\bar{\phi}, w)$  then  $\bar{\phi} \in \mathcal{C}^2(\mathbb{R}^2)$ . From this fact, it follows that, using inequalities (19) and (20) of Lemma 7 and Remark 8, the solutions to the equation  $\bar{\phi} = \lambda \mathcal{K}(\bar{\phi}, w)$  are uniformly bounded for  $\lambda \in [0, 1]$ . Thanks to these properties of the map  $\mathcal{K}$  we can use Schaefer’s fixed-point theorem (see [25] Theorem 4, Section 9.2, Chapter 9) to prove the existence of at least one solution to equation (18). Finally, using again Lemma 7 we have that  $\mathcal{F}(B)$  is compact for any bounded set  $B \subset \mathcal{W}$ .  $\square$

**THEOREM 10.** *Under Hypothesis QC on  $V$ , there exists a strong solution to equation (6) (or equivalently to equation (15)).*

**PROOF.** For proving the existence of a strong solution to the equation (15) (in the sense of Definition 6), it is sufficient to prove that we can choose the solutions to equation (18), whose existence for any  $w \in \mathcal{W}$  is guaranteed by Theorem 9, in a measurable way with respect  $w \in \mathcal{W}$ . More precisely, we have to prove that there exists a measurable selection for the function set map  $\mathcal{F}$ , that is, there exists a map  $\bar{S} : \mathcal{W} \rightarrow \mathcal{C}_{-\eta}^{2-\tau}$  such that  $\bar{S}(w) \in \mathcal{F}(w)$ .

Fix a sequence of balls  $B_1, \dots, B_n, \dots \subset \mathcal{W}$  of increasing radius and such that  $\lim_{n \rightarrow +\infty} B_n = \mathcal{W}$ , then by Theorem 9, the map  $\mathcal{F}|_{B_n \setminus B_{n-1}}$  takes values in a compact set. As proven in Theorem 9, the map  $\mathcal{K}$  is continuous in  $\bar{\phi}$  and measurable in  $w$  and, therefore, a Carathéodory map. As a consequence, by Filippov’s implicit function theorem (see Theorem 18.17 in [6]), there exists a (Borel) measurable function  $\bar{S}_n$  defined on  $B_n \setminus B_{n-1}$  such that  $\bar{S}_n(w) \in \mathcal{F}(w)$ . The map  $\bar{S}$  defined on  $B_n \setminus B_{n-1}$  by  $\bar{S}|_{B_n \setminus B_{n-1}} = \bar{S}_n$  is the measurable selection that we need (since  $B_n \setminus B_{n-1}$  is measurable).

A strong solution  $S$  to equation (15) is then given by  $S(w) := w + (m^2 - \Delta)\bar{S}(w)$ ,  $w \in \mathcal{W}$ .  $\square$

**COROLLARY 11.** *Under Hypothesis C, there exists only one strong solution to equation (15).*

**PROOF.** Suppose that  $S_1, S_2$  are two strong solutions to equation (15), then letting  $\phi_j(x, w) = \mathcal{I}(S_j(w(x)))$ ,  $j = 1, 2$ , writing  $\delta\phi(x, w) = \phi_1(x, w) - \phi_2(x, w)$  and  $\delta r(x, w) = \sqrt{\sum_{i=1}^n (\delta\phi^i(x, w))^2}$ , we obtain

$$(m^2 - \Delta)(\delta r^2) + 2 \sum_i (|\nabla \delta\phi^i|^2) + f \delta r [\hat{n}_{\delta\phi} \cdot (\partial V(\phi_1) - \partial V(\phi_2))] = 0.$$

By Lagrange’s theorem, there exists a function  $g(x)$ ,  $x \in \mathbb{R}^2$ , taking values in the segment  $[\phi_1(x), \phi_2(x)] \subset \mathbb{R}^n$  such that  $\hat{n}_{\delta\phi} \cdot (\partial V(\phi_1) - \partial V(\phi_2)) = \delta r \partial^2 V(g)(\hat{n}_{\delta\phi}, \hat{n}_{\delta\phi})$ . From this fact, we obtain

$$(m^2 - \Delta)(\delta r^2) + f(\partial^2 V(g)(\hat{n}_{\delta\phi}, \hat{n}_{\delta\phi}))\delta r^2 \leq 0.$$

Since  $m^2 + \partial^2 V(g)(\hat{n}_{\delta\phi}, \hat{n}_{\delta\phi}) \geq \varepsilon > 0$ ,  $y \mapsto V(y) + m^2|y|^2$  being strictly convex by our Hypothesis C, and  $\delta r^2(x)$  is positive and goes to zero as  $x \rightarrow +\infty$ , we have  $\phi_1 = \phi_2$  and, therefore,  $S_1(w) = S_2(w)$ .  $\square$

2.2. *Weak solutions.* First of all, we prove that the map  $U$ , given by (16), is a  $H - C^1$  function (in the sense of [59], see Appendix A) for the abstract Wiener space  $(\mathcal{W}, \mathcal{H}, \mu)$ .

PROPOSITION 12. *If  $V$  and its derivatives grow at most exponentially at infinity, then the map  $U$  is a  $H - C^1$  function, on the abstract Wiener space  $(\mathcal{W}, \mathcal{H}, \mu)$  and we have*

$$\nabla U^i(w)[h] = f(x) \partial_{\phi^i}^2 V(\mathcal{I}w) \cdot \mathcal{I}(h^j).$$

Furthermore,  $U$  is  $C^2$  Fréchet differentiable as a map from  $\mathcal{W}$  into  $H$ .

PROOF. The proof is essentially based on the fundamental theorem of calculus and the use of the Fourier transform. In order to give an idea of the proof, we only prove the most difficult part, namely that  $\nabla U$  is continuous with respect to translations by elements of  $\mathcal{H}$ , where continuity is understood with respect to the Hilbert–Schmidt norm for operators acting on  $\mathcal{H}$ .

For fixed  $w \in \mathcal{W}$ ,  $h, h' \in \mathcal{H}$ , we have for  $i = 1, \dots, n$ :

$$\begin{aligned} & \nabla U^i(w + h')[h] - \nabla U^i(w)[h] \\ (21) \quad &= f(x) \int_0^1 \partial_{\phi^i \phi^j \phi^r}^3 V((m^2 - \Delta)^{-1}(w + th')) \cdot \mathcal{I}(h^j) \cdot \mathcal{I}(h'^r) dt, \end{aligned}$$

where the sum over  $j, r = 1, \dots, n$  is implied. We recall that the Hilbert–Schmidt norm of an integral kernel is the integral of the square of the absolute value of the kernel. In our case, the Fourier transform of the integral kernel representing the difference (21) is given by

$$\hat{K}_j^i(k, k') = \sum_{r=1}^n \int_{\mathbb{R}^4} \int_0^1 \frac{\hat{V}_{t,jr,f}^i(k - k_1)}{(|k_1 - k_2|^2 + m^2)} \cdot \frac{\hat{h}^r(k_1 - k_2)}{(|k_2 - k'|^2 + m^2)} \frac{dk_1 dk_2}{(2\pi)^4},$$

where  $\hat{V}_{t,jk,f}^i(k, l)$  is the Fourier transform of  $f \partial_{\phi^i \phi^j \phi^k}^3 V(\mathcal{I}(w + th'))$ ,  $t \in [0, 1]$ . It is simple to prove that

$$\begin{aligned} \|\nabla U(w + h')[\cdot] - \nabla U(w)[\cdot]\|_2^2 &\lesssim \int_{\mathbb{R}^4} |\hat{K}_r^i(k, k') \hat{K}_i^r(k', k)| dk dk' \\ &\lesssim \|\sqrt{f} e^{\alpha|\mathcal{I}w| + \alpha|\mathcal{I}h'}\|_\infty^2 \|\sqrt{f}\|_{L^2}^2 \|h'\|_{\mathcal{H}}^2, \end{aligned}$$

where  $\alpha$  depends on the exponential growth of  $\partial^3 V$ . Since  $\|\sqrt{f} e^{\alpha|\mathcal{I}w| + \alpha|\mathcal{I}h'}\|_\infty$  is always finite in  $\mathcal{W}$  (for  $\eta$  positive and small enough), we have proved the continuity of the map  $h' \mapsto \nabla U(w + h')$  with respect to the Hilbert–Schmidt norm.  $\square$

By the notation  $\text{deg}_2(\mathcal{I}\mathcal{H} + K)$ , we denote the regularized Fredholm determinant (see Appendix A and also [56], Chapter 9) which is well defined when  $K$  is a Hilbert–Schmidt operator. The function  $\text{det}_2(\mathcal{I}\mathcal{H} + \cdot)$  is a smooth functional from the space of Hilbert–Schmidt operators (with its natural norm) to  $\mathbb{R}$  (see [56] Theorem 9.2 for the proof of this fact).

We define the measurable map  $N : \mathcal{W} \rightarrow \mathbb{N} \cup \{+\infty\}$ :

$$N(w) := (\text{number of solutions } y \in \mathcal{W} \text{ to the equation } T(y) = w);$$

moreover, let  $M \subset \mathcal{W}$  be the set of zeros of the continuous function  $w \in \mathcal{W} \mapsto \text{det}_2(\mathcal{I}\mathcal{H} + \nabla U(w))$ .

THEOREM 13. *The function  $N$  is greater or equal to 1 and it is  $\mu$ -almost surely finite. Furthermore, the map  $T$  is proper.*

PROOF. We define  $\mathcal{T}(\hat{\phi}, w) = \hat{\phi} + U(\hat{\phi} + w)$ . Obviously, we have that  $z$  is a solution to the equation  $T(z) = w$  if and only if  $\hat{\phi} = z - w$  is a solution to the equation  $\mathcal{T}(\hat{\phi}, w) = 0$ . On the other hand,  $\hat{\phi}$  is solution to the equation  $\mathcal{T}(\hat{\phi}, w) = 0$  if and only if  $\bar{\phi} = \mathcal{I}(\hat{\phi})$  is a solution to equation (18). By Theorem 9, equation (18) has at least one solution for any  $w \in \mathcal{W}$  and so  $N(w) \geq 1$  for any  $w \in \mathcal{W}$ .

Let  $K$  be a compact set in  $\mathcal{W}$  we have that  $T^{-1}(K) \subset K + (m^2 - \Delta)(\mathcal{F}(K))$  (where  $\mathcal{F}$  is the set valued map introduced in Theorem 9). Since  $K$  is compact, by Theorem 9,  $\mathcal{F}(K)$  is compact in  $\mathcal{C}_{-\eta}^{2-}$  which implies that  $(m^2 - \Delta)(\mathcal{F}(K))$  is compact in  $\mathcal{C}_{-\eta}^{0-}$ . Since the immersion  $\mathcal{C}_{-\eta}^{0-} \hookrightarrow \mathcal{W}$  is compact and the sum of two compact sets is compact, we obtain that  $T$  is a proper map.

Since by Proposition 52,  $\mu(T(M)) = 0$ , for proving the theorem it is sufficient to prove that  $N(w) < +\infty$  for  $w \notin T(M)$ . If  $w \notin T(M)$ , then  $\text{id}_H + \nabla U(w)|_{\mathcal{H}}$  is a linear invertible operator on  $\mathcal{H}$  and so  $\text{id}_{\mathcal{W}} + \nabla U(w)$  is a linear invertible operator on  $\mathcal{W}$ . By the implicit function theorem, we have that  $T$  is a  $C^1$  diffeomorphism between a neighborhood  $B_w$  of  $w$  onto  $T(B_w)$ . This implies that the set  $T^{-1}(w)$  consists of isolated points. Since the map  $T$  is proper, this means that  $T^{-1}(w)$  is a compact set made only by isolated points which implies that  $T^{-1}(w)$  is a finite set.  $\square$

If  $K : \mathcal{W} \rightarrow \mathcal{H}$  is an  $H - C^1$  function, we denote by  $\delta(K) : \mathcal{W} \rightarrow \mathbb{R}$  the well-defined Skorokhod integral of the map  $K$  (see Appendix A for an informal introduction of the concept, Appendix B of [59] for a more detailed treatment and Proposition 3.4.1 of [59] for the proof of the fact that the Skorokhod integral of a  $H - C^1$  function is well defined).

THEOREM 14. *A probability measure  $\nu$  is a weak solution to equation (15) if and only if it is absolutely continuous with respect to  $\mu$  and there exists a nonnegative function  $A \in L^\infty(\mu)$  such that  $\sum_{y \in T^{-1}(w)} A(y) = 1$  for  $\mu$ -almost all  $w \in \mathcal{W}$  and  $\frac{d\nu}{d\mu} = A|\Lambda_U|$  with*

$$\Lambda_U(w) := \det_2(I + \nabla U(w)) \exp\left(-\delta(U)(w) - \frac{1}{2}\|U(w)\|_{\mathcal{H}}^2\right).$$

PROOF. Recall that, by Proposition 52,  $\mu(T(M)) = 0$ . This implies that for any weak solution  $\nu$  we have  $\nu(T^{-1}(T(M))) = 0$ . Letting  $\mathbb{W}^n := T^{-1}(N = n) \cap T^{-1}(T(M))$  we deduce that  $\nu(\bigcup_n \mathbb{W}^n) = \sum_n \nu(\mathbb{W}^n) = 1$  and if we prove that  $\nu$  is absolutely continuous with respect to  $\mu$  on each  $\mathbb{W}^n$  we have proved that  $\nu$  is absolutely continuous with respect to  $\mu$ .

Using  $n$  times iteratively the Kuratowski–Ryll–Nardzewski selection theorem (see Theorem 18.13 in [6]) due to the fact that  $T^{-1}(x) \cap \mathbb{W}^n$  is composed by zero or  $n$  elements, we can decompose the set  $\mathbb{W}^n$  into  $n$  measurable subsets  $\mathbb{W}_1^n, \dots, \mathbb{W}_n^n$  where the map  $T|_{\mathbb{W}_i^n}$  is invertible. This means that if  $\Omega \subset \mathbb{W}^n$  we have  $\nu(\Omega \cap \mathbb{W}_i^n) \leq \mu(T(\Omega))$ . On the other hand, we have that  $\mu(T(\Omega)) = \int_{\Omega \cap \mathbb{W}_i^n} |\Lambda_U| d\mu$ . This implies that if  $\mu(\Omega) = 0$  then  $\nu(\Omega \cap \mathbb{W}_i^n) \leq \mu(T(\Omega)) = \int_{\Omega \cap \mathbb{W}_i^n} |\Lambda_U| d\mu = 0$ . As a consequence,  $\nu(\Omega) = \sum_i \nu(\Omega \cap \mathbb{W}_i^n) = 0$  and  $\nu$  is absolutely continuous with respect to  $\mu$ .

Theorem 53 below implies that for any measurable positive functions  $f, A$  we have

$$(22) \quad \int f \circ T(w) A(w) |\Lambda_U(w)| d\mu = \int f(w) \left( \sum_{y \in T^{-1}(w)} A(y) \right) d\mu.$$

Taking  $f = \mathbb{I}_{T(M)}$  and  $A = 1$ , we deduce that  $\int_{T^{-1}(T(M))} |\Lambda_U| d\mu = \mu(T(M)) = 0$ . Therefore, we can suppose that there exists a specific nonnegative function  $A$  such that  $d\nu = A|\Lambda_U|d\mu$  and since  $T_*(\nu) = \mu$  we must have

$$\int f(w) d\mu = \int f \circ T(w) d\nu = \int f \circ T(w) A(w) |\Lambda_U(w)| d\mu,$$

for any bounded measurable function  $f$ . Comparing this with (22), we deduce that  $\sum_{y \in T^{-1}(w)} A(y) = 1$  for  $(\mu_-)$  almost all  $w \in \mathcal{W}$ .

On the other hand, using again Theorem 53 it is simple to prove that if  $d\nu = A|\Lambda_U| d\mu$  and  $\sum_{y \in T^{-1}(w)} A(y) = 1$  then  $\nu$  is a weak solution to equation (15).  $\square$

REMARK 15. If  $S$  is any strong solution to equation (15), then  $\nu = S_*\mu$  is a weak solution. Furthermore, it is simple to prove that the weak solutions of the form  $S_*\mu$ , where  $S$  is some strong solution to (6), are the extremes of the convex set  $\mathfrak{M} := \{\nu \text{ satisfying } T_*\nu = \mu\}$ . Using a lemma (precisely Lemma 21) that we shall prove below, it follows from this that  $\mathfrak{M}$  is weakly compact, and thus, by Krein–Milman theorem (see Theorem 3.21 in [53]), any measure  $\nu \in \mathfrak{M}$  can be written as convex combination of measures induced by strong solutions.

COROLLARY 16. *If  $V$  satisfies Hypothesis C, there exists only one weak solution  $\nu$  to equation (15) and we have that  $\frac{d\nu}{d\mu} = |\Lambda_U|$  and  $\nu = S_*\mu$  (where  $S$  is the only strong solution to equation (15) and  $\Lambda_U$  is as in Theorem 14).*

PROOF. If  $V$  satisfies Hypothesis C, by Corollary 11,  $T$  is invertible and by Theorem 14 we have that  $\nu$  is unique and  $\frac{d\nu}{d\mu} = |\Lambda_U|$ . By Remark 15, we have that  $S_*\mu$ , where  $S$  is the unique strong solution of (15), is the unique weak solution to the same equation.  $\square$

**3. Elliptic stochastic quantization.** In this section, we want to prove the dimensional reduction of equation (6), namely that the law in 0 of at least a (weak) solution to equation (15), has an explicit expression in terms of the potential  $V$ .

The original idea of Parisi and Sourlas [49] for proving this relations was to transform expectations involving the solution  $\phi$  to equation (6) (taken at the origin) into an integral of the form

$$(23) \quad \mathbb{E}[h(\phi(0))] = \int h(\mathcal{I}w(0)) \det(I + \nabla U(\mathcal{I}w)) e^{-\langle U(\mathcal{I}w), \mathcal{I}w \rangle - \frac{1}{2} \|\mathcal{I}w\|_{\mathcal{H}}^2} d\mu(w),$$

where  $U$  is defined in equation (16). Then one can express the weight on the right-hand side of (23) as the exponential  $e^{\int V(\Phi) dx d\theta d\bar{\theta}}$  involving the superfield

$$\Phi(x, \theta, \bar{\theta}) = \varphi(x) + \psi(x)\theta + \bar{\psi}(x)\bar{\theta} + \omega(x)\theta\bar{\theta}$$

(see Section 4 and Section 5 for a more precise description) constructed from the real Gaussian free field  $\varphi$  over  $\mathbb{R}^2$ , two additional fermionic (i.e., anticommuting) fields  $\psi, \bar{\psi}$  and the complex Gaussian field  $\omega$ . Introducing these new anticommuting fields, it can be argued that the integral (23) admits an invariance property with respect to supersymmetric transformations. This implies the dimensional reduction, that is,

$$(24) \quad (23) = \int_{\mathbb{R}^n} h(\varphi(0)) e^{-\int V(\Phi) dx d\theta d\bar{\theta}} \mathcal{D}\Phi = \int_{\mathbb{R}^n} h(y) d\kappa(y).$$

Unfortunately, this reasoning is only heuristic since the integral on the right-hand side of (23) is not well defined without a spatial cut-off, given that both the determinant and the exponential are infinite.

For polynomial potentials  $V$ , a rigorous version of this reasoning was proposed by Klein et al. [39]. More precisely Klein et al. give a rigorous proof of the relationship (24) introducing a suitable modification due to the presence of the spatial cut-off  $f$ , but they do not discuss the relationship between equation (6) and the reduction (23).

In this section, we do not want to propose a rigorous version of the previous reasoning which will be given in Section 4. Here, we only assume that the conclusion of Parisi and Sourlas' formal argument holds for a general enough class of potentials. More precisely, we assume Theorem 17 below.

For technical reasons, which will become clear in the following (see Remark 38 below), in order to state Theorem 17, we need first to introduce an additional class of potentials.

*Hypothesis  $V_\lambda$ .* We have the decomposition

$$V = V_B + \lambda V_U, \quad V_U(y) = \sum_{i=1}^n (y^i)^4, \quad y = (y^1, \dots, y^n) \in \mathbb{R}^n,$$

with  $\lambda > 0$  and  $V_B$  a bounded function with all bounded derivatives on  $\mathbb{R}^n$ .

In Section 4 below, we will exploit a supersymmetric argument, described briefly at the beginning of this section, for the family of potentials  $V$  satisfying the more restrictive Hypothesis  $V_\lambda$  to prove that in this case a cut-off version of equation (24).

**THEOREM 17.** *Under the Hypotheses CO and  $V_\lambda$  if  $h$  is any real measurable bounded function defined on  $\mathbb{R}^n$ , then we have*

$$\int_{\mathcal{W}} h(\mathcal{I}w(0)) \Lambda_U(w) \Upsilon_f(\mathcal{I}w) \, d\mu(w) = Z_f \int_{\mathbb{R}^n} h(y) \, d\kappa(y),$$

where  $Z_f = \int_{\mathcal{W}} \Lambda_U(w) \Upsilon_f(\mathcal{I}w) \, d\mu(w) > 0$ .

**PROOF.** The proof is given in Section 4 below.  $\square$

In the rest of this section, we want to show how to derive from Theorem 17 the dimensional reduction result for the solution to the elliptic SPDE. More precisely, the goal of the rest of this section is to prove the following theorem.

**THEOREM 18.** *Under the Hypotheses CO and QC, there exists (at least) one weak solution  $\nu$  to equation (6) such that for any measurable bounded function  $h$  defined on  $\mathbb{R}^n$  we have*

$$\begin{aligned} \int_{\mathcal{W}} h(\mathcal{I}w(0)) \Upsilon_f(\mathcal{I}w) \, d\nu(w) &= \int_{\mathcal{W}} h(\mathcal{I}w(0)) \Upsilon_f(\mathcal{I}w) \Lambda_U(w) \, d\mu(w) \\ (25) \qquad \qquad \qquad &= Z_f \int_{\mathbb{R}^n} h(y) \, d\kappa(y), \end{aligned}$$

where  $Z_f = \int_{\mathcal{W}} \Upsilon_f(\mathcal{I}w) \, d\nu(w) > 0$ .

This result is very important since it implies Theorem 1 and Theorem 3.

**PROOF OF THEOREM 1 AND THEOREM 3.** The relation (25) can be expressed in the following more probabilistic way. Suppose that on a given probability space  $(\Omega^\nu, \mathbb{P}^\nu)$ , the map  $\phi : \mathbb{R}^2 \times \Omega^\nu \rightarrow \mathbb{R}^n$  gives the weak solution  $\nu$  of Theorem 18, namely that the law of the  $\mathcal{W}$ -random variable  $(m^2 - \Delta)\phi(\cdot, \omega)$  is the measure  $\nu$ . Then we have that, for any real measurable bounded function defined on  $\mathbb{R}^n$ ,

$$\mathbb{E}_{\mathbb{P}^\nu} \left[ h(\phi(0)) \frac{\Upsilon_f(\phi)}{Z_f} \right] = \int_{\mathcal{W}} h(y) \, d\kappa(y),$$

namely we have proven Theorem 1. If we assume Hypothesis C, then by Corollary 11, Corollary 16 and Theorem 18 there exists a unique strong solution satisfying (25) and we have proven as a consequence Theorem 3.  $\square$

The proof of Theorem 18 will be given in several steps of wider degree of generality with respect to the hypothesis on the potential  $V$ , before we prove an auxiliary result.

LEMMA 19. *Under the Hypothesis  $V_\lambda$ , we have that*

$$(26) \quad \int_{\mathcal{W}} g \circ T(w) \Lambda_U(w) \, d\mu(w) = \int_{\mathcal{W}} g(w) \, d\mu(w),$$

where  $g$  is any bounded measurable function defined on  $\mathcal{W}$ .

PROOF. Using the methods of Section 2, we can prove that the map  $T$  satisfies Hypotheses DEG1, DEG2, DEG3 of Appendix A. The claim then follows from Theorem 54 and Theorem 55 below, where we can choose the function  $g$  to be any bounded continuous function since  $\Lambda_U \in L^1(\mu)$  under Hypothesis  $V_\lambda$ .  $\square$

PROPOSITION 20. *Under the Hypotheses CO and  $V_\lambda$ , there exists at least one weak solution  $v$  to equation (15) satisfying (25).*

PROOF. Let  $\mathcal{V} \subset L^1(|\Lambda_U| \, d\mu)$  be the span of the two linear spaces  $\mathcal{V}_1, \mathcal{V}_2 \subset L^1(|\Lambda_U| \, d\mu)$  where  $\mathcal{V}_1$  is composed by the functions of the form  $g \circ T$ , where  $g$  is a measurable function defined on  $\mathcal{W}$  such that  $g \circ T \in L^1(|\Lambda_U| \, d\mu)$ , and  $\mathcal{V}_2$  is formed by the functions of the form  $h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)$ , where  $h$  is a measurable function defined on  $\mathbb{R}^n$  such that  $h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w) \in L^1(|\Lambda_U| \, d\mu)$ . Note that  $\mathcal{V}_1$  and  $\mathcal{V}_2$ , and so  $\mathcal{V} = \text{span}\{\mathcal{V}_1, \mathcal{V}_2\}$ , are non-void since, under the Hypotheses  $V_\lambda$  and CO (see Lemma 40 below),  $\Lambda_U \in L^p(\mu)$  and so  $g \circ T, h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w) \in L^1(\mu)$  whenever  $g, h$  are bounded. Define a positive functional  $\hat{L} : \mathcal{V} \rightarrow \mathbb{R}$  by extending via linearity the relations

$$(27) \quad \hat{L}(h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)) := \int h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w) \Lambda_U(w) \, d\mu(w),$$

$$(28) \quad \hat{L}(g \circ T) := \int g(w) \, d\mu(w)$$

to the whole  $\mathcal{V}$ . We have to verify that  $\hat{L}$  is well defined and positive on  $\mathcal{V}$ . Suppose that there exist functions  $g$  and  $h$  such that  $g \circ T = h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)$ . Then, by Lemma 19, we have

$$(29) \quad \int_{\mathcal{W}} g \, d\mu = \int_{\mathcal{W}} g \circ T \Lambda_U \, d\mu = \int_{\mathcal{W}} h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w) \Lambda_U \, d\mu.$$

This implies that  $\hat{L}$  is well defined on  $\mathcal{V}_1 \cap \mathcal{V}_2$  and so on  $\mathcal{V}$ . Obviously,  $\hat{L}$  is positive on  $\mathcal{V}_2$ , and, by Theorem 17 we have

$$(30) \quad \begin{aligned} \hat{L}(h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)) &= \int_{\mathcal{W}} h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w) \Lambda_U \, d\mu \\ &= Z_f \int_{\mathbb{R}^n} h(y) \, d\kappa(y) \geq 0 \end{aligned}$$

whenever  $h$ , and so  $h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)$ , is positive. This means that  $\hat{L}$  is positive.

For any  $f = g \circ T \in \mathcal{V}_1$ , by Theorem 53 and Theorem 13, we have

$$\begin{aligned} |\hat{L}(f)| &= \left| \int_{\mathcal{W}} g(w) \, d\mu(w) \right| \\ &\leq \int_{\mathcal{W}} |g(w)| N(w) \, d\mu(w) \\ &= \int_{\mathcal{W}} |g \circ T(w)| \Lambda_U(w) \, d\mu(w) = \|f \Lambda_U\|_1. \end{aligned}$$

On the other hand, if  $f \in \mathcal{V}_2$ , by relation (27),  $\hat{L}(f) \leq \|f \Lambda_U\|_1$ . These two inequalities and the positivity of  $\hat{L}$  imply, by Theorem 8.31 of [6] on the extension of positive functionals on Riesz spaces, that there exists at least one positive continuous linear functional  $L$  on  $L^1(|\Lambda_U| \, d\mu)$ , such that  $L(f) = \hat{L}(f)$  for any  $f \in \mathcal{V}$ . The functional  $L$  defines the weak solution to equation (15) we are looking for. Indeed, since  $L$  is a continuous positive functional on  $L^1(|\Lambda_U| \, d\mu)$  there exists a measurable positive function  $B \in L^\infty(|\Lambda_U| \, d\mu) \subset L^\infty(d\mu)$  such that  $L(f) = \int_{\mathcal{W}} f(w) B(w) |\Lambda_U(w)| \, d\mu(w)$ . Since  $\Lambda_U \in L^p$  by Lemma 40 below, we have  $1 \in \mathcal{V}_1$  and so  $L(1) = \int_{\mathcal{W}} 1 \, d\mu(w) = 1$ . This implies, since the function  $B$  is positive, that the  $\sigma$ -finite measure  $d\nu = B|\Lambda_U| \, d\mu$  is a probability measure. Furthermore, since  $\mathcal{V}_1$  contains all the functions  $g \circ T$ , where  $g$  is measurable and bounded, equality (28) implies that  $T_*(\nu) = \mu$ . This means that  $\nu$  is a weak solution to equation (15). Finally, since  $\mathcal{V}_2$  contains all the functions of the form  $h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)$  where  $h$  is measurable and bounded on  $\mathbb{R}^n$  the measure  $\nu$  satisfies the thesis of the theorem.  $\square$

Unfortunately, we cannot repeat this reasoning for general potentials satisfying the weaker Hypothesis QC since both Theorem 17 and Proposition 20 exploit a  $L^p$  bound on  $\Lambda_U$  (see Lemma 40 below) that cannot be obtained for more general potentials. Thus the idea is to generalize equation (25) without passing from equation (24). Indeed it is possible to approximate any potential  $V$  satisfying Hypothesis QC by a sequence of potentials  $(V_i)_i$  satisfying Hypothesis  $V_\lambda$  in such a way that the sequence of weak solutions  $(\nu_i)_i$  associated with  $(V_i)_i$  converges (weakly) to a weak solution associated with the potential  $V$  (see Lemma 21, Lemma 24 and Lemma 25 below). Since equation (25) involves only integrals with respect to a weak solution to equation (6), we are able to prove that equation (25) holds for any potential  $V$  approximating its weak solution  $\nu$  by the sequence  $(\nu_i)_i$  satisfying equation (25).

Let us now set up the approximation argument, starting with a series of lemmas about convergence of weak solutions.

LEMMA 21. *Let  $\{T_i\}_{i \in \mathbb{N}}$  be a sequence of continuous maps on  $\mathcal{W}$  such that for any compact  $K \subset \mathcal{W}$  we have that  $\bigcup_{i \in \mathbb{N}} T_i^{-1}(K)$  is precompact and there exists a continuous map  $T$  such that  $T_i \rightarrow T$  uniformly on the compact subsets of  $\mathcal{W}$ . Let  $\mathbb{M}_i$  be a set of probability measures on  $\mathcal{W}$  defined as follows:*

$$\mathbb{M}_i := \{ \nu \text{ probability measure on } \mathcal{W} \text{ such that } T_{j,*}(\nu) = \mu \text{ for some } j \geq i \}.$$

*Then  $\mathbb{M} := \bigcap_{i \in \mathbb{N}} \bar{\mathbb{M}}_i$ , where the closure is taken with respect to the weak topology on the set of probability measures on  $\mathcal{W}$ , is nonvoid and*

$$\mathbb{M} \subset \{ \nu \text{ probability measure on } \mathcal{W} \text{ such that } T_*(\nu) = \mu \}.$$

PROOF. First of all, we prove that  $\mathbb{M}_i$  is pre-compact for any  $i \in \mathbb{N}$ . This is equivalent to proving that the measures in  $\mathbb{M}_i$  are tight. Let  $\bar{K}$  be a compact set such that  $\mu(\bar{K}) \geq 1 - \epsilon$



for a fixed  $0 < \epsilon < 1$ , then  $K := \overline{\bigcup_{i \in \mathbb{N}} T_i^{-1}(\tilde{K})}$  is a compact set in  $\mathcal{W}$ . Consider  $\nu \in \mathbb{M}_j$  then there exists  $T_k$  such that  $T_{k,*}\nu = \mu$ . This implies

$$\nu(K) \geq \nu\left(\bigcup_i T_i^{-1}(\tilde{K})\right) \geq \nu(T_k^{-1}(\tilde{K})) \geq \mu(\tilde{K}) \geq 1 - \epsilon,$$

for any  $k \in \mathbb{N}$ . Since  $\mathbb{M}_i$  are precompact,  $\bar{\mathbb{M}}_i$  are compact and  $\bar{\mathbb{M}}_i \subset \bar{\mathbb{M}}_j$  if  $i \geq j$ . This implies that  $\mathbb{M}$  is nonvoid. If we consider a  $\nu \in \mathbb{M}$ , there exists a sequence  $\nu_k$  weakly converging to  $\nu$ , for  $k \rightarrow +\infty$ , such that  $T_{i_k,*}(\nu_k) = \mu$  and  $i_k \rightarrow +\infty$ . Proving that  $T_*(\nu) = \mu$  is equivalent to prove that for any  $C^1$  bounded function  $g$  with bounded derivatives defined on  $\mathcal{W}$  taking values in  $\mathbb{R}$  we have  $\int g \circ T \, d\nu = \int g \, d\mu$ . Let  $K$  the compact set defined before, then there exists a  $k \in \mathbb{N}$  such that  $\sup_{w \in K} \|T_{i_k}(w) - T(w)\| \leq \epsilon$  and that  $|\int_{\mathcal{W}} g \circ T \, d\nu - \int_{\mathcal{W}} g \circ T \, d\nu_k| \leq \epsilon$ , for the arbitrary  $0 < \epsilon < 1$ . This implies that

$$\begin{aligned} \left| \int_{\mathcal{W}} g \circ T \, d\nu - \int_{\mathcal{W}} g \, d\mu \right| &\leq \left| \int_{\mathcal{W}} g \circ T \, d\nu - \int_{\mathcal{W}} g \circ T \, d\nu_i \right| \\ &\quad + \left| \int_K (g \circ T - g \circ T_{i_k}) \, d\nu_k \right| \\ &\quad + \|g\|_{\infty} \epsilon + \left| \int_{\mathcal{W}} g \circ T_{i_k} \, d\nu_k - \int_{\mathcal{W}} g \, d\mu \right| \\ &\leq \epsilon + \|\nabla g\|_{\infty} \epsilon + \|g\|_{\infty} \epsilon. \end{aligned}$$

Since  $\epsilon$  is arbitrary, from this it follows that  $\int_{\mathcal{W}} g \circ T \, d\nu = \int_{\mathcal{W}} g \, d\mu$ .  $\square$

REMARK 22. The proof of Lemma 21 proves also that given any sequence of  $\nu_i \in \mathbb{M}_i$  there exists a subsequence converging weakly to  $\nu \in \mathbb{M}$ .

REMARK 23. In the following, we consider a sequence of functions  $V_i$  satisfying Hypothesis QC. To each function  $V_i$  of the sequence, it is possible to associate a map  $U_i : \mathcal{W} \rightarrow \mathcal{H}$  defined by  $U_i(w) := f \partial V_i(\mathcal{I}w)$  and the corresponding map  $T_i : \mathcal{W} \rightarrow \mathcal{W}$  defined by  $T_i(w) = w + U_i(w)$ .

LEMMA 24. *Let  $\{V_i\}_{i \in \mathbb{N}}$  be a sequence of potentials satisfying the Hypothesis QC and converging to the potential  $V$ , and such that  $\partial V_i$  converges uniformly to  $\partial V$  on compact subsets of  $\mathbb{R}^n$ ; moreover, we assume that  $V_i, V, \partial V_i$  and  $\partial V$  are uniformly exponentially bounded and there exists a common function  $H$  entering Hypothesis QC for  $\{V_i\}_{i \in \mathbb{N}}$  and  $V$ . Let  $T_i, T$  be the maps on  $\mathcal{W}$  associated with  $V_i$  and  $V$ , respectively, as in Remark 23. Then the sequence  $\{T_i\}_{i \in \mathbb{N}}$  satisfies the hypothesis of Lemma 21.*

PROOF. Note that the a priori estimates (19) and (20) in Lemma 7 are uniform in  $i \in \mathbb{N}$  since they depend only on the function  $H$  and the exponential growth of  $V_i, V, \partial V_i, \partial V$ . From this, we can deduce the precompactness of the set  $K = \bigcup_{i \in \mathbb{N}} T_i^{-1}(\tilde{K})$  for any compact set  $\tilde{K} \subset \mathcal{W}$  using a reasoning similar to the one proposed in Theorem 9 and Theorem 13.

Proving that  $T_i$  converges to  $T$  uniformly on the compact sets is equivalent to prove that the map  $U_i(w)(x) = f(x) \partial V_i(\mathcal{I}w(x))$  converges to  $U(w)(x) = f(x) \partial V(\mathcal{I}w(x))$  in  $L^2$  uniformly on the compact subsets of  $\mathcal{W}$ . Let  $K$  be a compact set of  $\mathcal{W}$ , then there exists an  $M > 0$  such that  $|\mathcal{I}w(x)| \leq M(1 + |x|^\eta)$  (where we suppose without loss of generality that  $\eta < 1$ ). By hypotheses, we have that there exist two constants  $\alpha, \beta > 0$  such that  $|\partial V_i(y)|, |\partial V(y)| \leq e^{\alpha|y| + \beta}$ , thus there exists a compact subset  $\mathfrak{K}$  of  $\mathbb{R}^2$  such that

$\int_{\mathbb{R}^c} (f(x))^2 \exp(2\alpha M(1 + |x|^\eta) + 2\beta) dx \leq \epsilon$ , for some  $\epsilon \in (0, 1)$ . Denote by  $B_\epsilon$  the ball of radius  $\sup_{x \in \mathbb{R}} M(1 + |x|^\eta)$  then we have

$$\begin{aligned} \sup_{w \in K} \|U_i(w) - U(w)\|_{\mathcal{H}}^2 &\leq 2 \left| \int_{\mathbb{R}^c} (f(x))^2 e^{2\alpha M(1+|x|^\eta)+2\beta} dx \right| \\ &\quad + \sup_{w \in K} \left| \int_{\mathbb{R}} (f(x))^2 |\partial V(\mathcal{I}w) - \partial V_i(\mathcal{I}w)|^2 dx \right| \\ &\leq 2\epsilon + \left( \sup_{y \in B_\epsilon} |\partial V(y) - \partial V_i(y)| \right)^2 \int_{\mathbb{R}} (f(x))^2 dx \\ &\rightarrow 2\epsilon, \end{aligned}$$

as  $i \rightarrow +\infty$ . This means that  $\lim_{i \rightarrow +\infty} (\sup_{w \in K} \|U_i(w) - U(w)\|_{\mathcal{H}}^2) \leq 2\epsilon$ , and since  $\epsilon$  is arbitrary in  $(0, 1)$  the theorem is proved.  $\square$

LEMMA 25. *Let  $V$  be a potential satisfying Hypothesis QC, then there exists a sequence  $\{V_i\}_{i \in \mathbb{N}}$  of bounded smooth potentials converging to  $V$  and satisfying the hypothesis of Lemma 24.*

PROOF. Let  $V$  be a potential satisfying the Hypothesis QC and let  $\tilde{H}$  the function whose existence is guaranteed by Hypothesis QC. Let, for any  $N \in \mathbb{N}$ ,  $v_N := \sup_{y \in B(0, N)} |V(y)|$  and let  $\tilde{V}^N := G_{v_N} \circ V$  where

$$G_k(z) := \begin{cases} z & \text{if } |z| \leq k, \\ k & \text{if } |z| > k. \end{cases}$$

Let  $\rho$  be a smooth compactly supported mollifier and denote by  $\rho_\epsilon$  the function  $\rho_\epsilon(y) := \epsilon^{-n} \rho(\frac{y}{\epsilon})$ . We want to prove that  $V^N = \tilde{V}^N * \rho_{\epsilon_N}$ , for a suitable sequence  $\epsilon_N \in \mathbb{R}_+$ , is the approximation requested by the lemma. Without loss of generality, we can suppose that  $\tilde{H}$  is a positive function depending only on the radius  $|y|$  and increasing as  $|y| \rightarrow +\infty$ . Under these conditions, Hypothesis QC is equivalent to say that for any unit vector  $\hat{n} \in \mathbb{S}^n$  we have that for any  $y \in \mathbb{R}^n$

$$\max(-\hat{n} \cdot \partial V(y + r\hat{n}), 0) \leq \tilde{H}(y).$$

We want to prove that  $H(|y|) = \tilde{H}(|y| + \sup_N(\epsilon_N))$  is the function requested by the lemma.

Since for any unit vector  $\hat{n} \in \mathbb{S}^n$  we have  $|\hat{n} \cdot \partial \tilde{V}^N| \leq |\hat{n} \cdot \partial V|$  and since  $\tilde{V}^N$  is absolutely continuous, we obtain

$$\begin{aligned} -\hat{n} \cdot \partial V^N(y + r\hat{n}) &= ((-\hat{n} \cdot \partial \tilde{V}^N) * \rho_{\epsilon_N})(y + r\hat{n}) \\ &\leq (\max(-\hat{n} \cdot \partial V(\cdot + r\hat{n}), 0) * \rho_{\epsilon_N})(y) \leq \tilde{H} * \rho_{\epsilon_N}(y). \end{aligned}$$

Furthermore, we have that  $\tilde{V}^N = V$  on  $B(0, N - 1)$  and so there exists a sequence  $\{\epsilon_N\}_N$  such that  $\epsilon_N \rightarrow 0$  and  $\sup_{x \in B(0, N-1)} |\partial V^N(x) - \partial V(x)| \leq \frac{1}{N}$ . Since  $V^N$  is smooth and bounded and

$$\tilde{H} * \rho_{\epsilon_N}(y) \leq \tilde{H}\left(|y| + \sup_N(\epsilon_N)\right) = H(y),$$

we conclude the claim.  $\square$

Finally, we are able to prove (25) for all QC potentials, which will conclude this section.

**PROOF OF THEOREM 18.** By Proposition 20, the equality (25) holds when  $V$  satisfies the Hypothesis  $V_\lambda$  for some  $\lambda > 0$ , that is, if  $V(y) = V_{\lambda, V_B}(y) = V_B(y) + \lambda \sum_{k=1}^n (y^k)^4$  for some bounded potential  $V_B$ . It is clear that if  $\lambda_i \rightarrow 0$  the potentials  $V_{\lambda_i, V_B}$  converge to the potential  $V_B$  and the hypothesis of Lemma 24 hold. This means that if  $\hat{\nu}_i$  is a sequence of probability measures such that  $\hat{\nu}_i$  is a weak solution to the equation associated with  $V_{\lambda_i, V_B}$  satisfying the thesis of Proposition 20, by Remark 22 and Lemma 21, there exists a probability measure  $\hat{\nu}$ , that is a weak solution to the equation associated with  $V_B$ , such that  $\hat{\nu}_i \rightarrow \hat{\nu}$  in the weak sense, as  $i \rightarrow \infty$  and  $\lambda_i \rightarrow 0$ .

We want to prove that  $\hat{\nu}$  is a weak solution to the equation associated with  $V_B$  satisfying equation (25). The previous claim is equivalent to proving that

$$(31) \quad \begin{aligned} & \int_{\mathcal{W}} g(\mathcal{I}w(0)) e^{4 \int f'(x) V_{\lambda_i, B}(\mathcal{I}w(x)) dx} d\hat{\nu}^i(w) \\ & \rightarrow \int_{\mathcal{W}} g(\mathcal{I}w(0)) e^{4 \int f'(x) V_B(\mathcal{I}w(x)) dx} d\hat{\nu}(w), \end{aligned}$$

as  $\lambda \rightarrow 0$ , for any continuous bounded function  $g$ , and that  $\kappa_{\lambda_i} \rightarrow \kappa_B$  weakly, where  $d\kappa_{\lambda_i} = \exp(-4\pi V_{\lambda_i, B}) dx / Z_{\lambda_i}$  and  $d\kappa_B = \exp(-4\pi V_B) dx / Z_B$ . Proving relation (31) is equivalent to prove that

$$\int f'(x) V_{\lambda_i, B}(\mathcal{I}w(x)) dx \rightarrow \int f'(x) V_B(\mathcal{I}w(x)) dx$$

uniformly on compact sets of  $\mathcal{W}$ . This assertion can be easily proved using the methods of Lemma 24. Indeed for any  $w$  in the compact set  $K \subset \mathcal{W}$ , using the same notation of the proof of Lemma 24, we have

$$\begin{aligned} & \left| \int f' V_{\lambda_i, B}(\mathcal{I}w) dx - \int f' V_B(\mathcal{I}w) dx \right| \\ & \lesssim \lambda_i \int |f'(x)| (M(1 + |x|^\eta))^4 dx = C_K \lambda_i \rightarrow 0. \end{aligned}$$

The weak convergence of  $\kappa_{\lambda_i}$  to  $\kappa_B$  easily follows from Lebesgue’s dominated convergence theorem.

The previous reasoning proves the theorem for any bounded potential  $V_B$ . Using Lemma 25, we can approximate any potential  $V$  satisfying Hypothesis QC by a sequence of bounded potentials  $V_{B,i}$ . Using Lemma 24, Remark 22, Lemma 21 and a reasoning similar to the one exploited in the first part of the proof we obtain the thesis of the theorem for a general potential satisfying Hypothesis QC.  $\square$

**4. Dimensional reduction.** Define

$$(32) \quad \Xi(h) := \int_{\mathcal{W}} h(\mathcal{I}w(0)) \Lambda_U(w) \frac{\Upsilon_f(\mathcal{I}w)}{Z_f} d\mu(w),$$

with the notation as in Section 2 (Theorem 14) and Section 3 (Theorem 18). In this section, we prove Theorem 17, that is, the identity

$$(33) \quad \Xi(h) = \int_{\mathbb{R}^n} h(y) d\kappa(y).$$

It is important to note that  $\Lambda_U$  appears without the modulus in (32).

Let us start by unfolding the definition of  $\Lambda_U$  and  $\Upsilon_f(\mathcal{I}w)$  in (32) to get the expression

$$Z_f \Xi(h) = \int h(\mathcal{I}w(0)) \det_2(I_{\mathcal{H}} + \nabla U(w)) \times \exp\left(-\delta(U) - \frac{1}{2} \|U\|_{\mathcal{H}}^2 + 4 \int_{\mathbb{R}^2} V(\mathcal{I}w(x)) f'(x) dx\right) d\mu(w).$$

In order to manipulate the regularized Fredholm determinant, we approximate the right-hand side by

$$Z_f^\chi \Xi_\chi(h) := \int h(\mathcal{J}_\chi w(0)) \det_2(I_{\mathcal{H}} + \nabla U^\chi) \times \exp\left(-\delta(U^\chi) - \frac{1}{2} \|U^\chi\|_{\mathcal{H}}^2 + 4 \int_{\mathbb{R}^2} V(\mathcal{J}_\chi w(x)) f'(x) dx\right) d\mu(w),$$

where  $\chi > 0$  is a regularization parameter,  $\mathcal{J}_\chi := \mathcal{I}^{1+\chi} = (m^2 - \Delta)^{-1-\chi}$ ,  $Z_f^\chi$  is the normalization constant such that  $\Xi_\chi(h) = 1$  and

$$(34) \quad U^\chi(w) := \frac{1}{1+2\chi} \mathcal{I}^\chi \partial V(\mathcal{J}_\chi w).$$

We will prove below that  $\lim_{\chi \rightarrow 0} \Xi_\chi(h) = \Xi(h)$ . When  $\chi > 0$ ,  $\nabla U^\chi(w) = \frac{1}{1+2\chi} \times \mathcal{I}^\chi \partial V(\mathcal{J}_\chi w) \mathcal{J}_\chi$  is almost surely a trace class operator and  $U^\chi \in \mathcal{W}^*$ . This means that we can rewrite the regularized Fredholm determinant  $\det_2$  in term of the unregularized one (denoted by  $\det$ ) (see equation (69) and the discussion in Appendix A) obtaining

$$(35) \quad Z_f^\chi \Xi_\chi(h) = \int h(\mathcal{J}_\chi w(0)) \det(I_{\mathcal{H}} + \nabla U^\chi) \times \exp\left(-\langle U^\chi, w \rangle - \frac{1}{2} \|U^\chi\|_{\mathcal{H}}^2 + 4 \int_{\mathbb{R}^2} V(\mathcal{J}_\chi w(x)) f'(x) dx\right) d\mu(w).$$

The determinant is invariant with respect to conjugation and so we can multiply  $\nabla U^\chi$  by  $(-\Delta + m^2)^\chi$  at the left-hand side and by  $(-\Delta + m^2)^{-\chi}$  at the right-hand side (this last multiplication can be done since  $\mathcal{I}^\chi = (-\Delta + m^2)^{-\chi}$  is a bounded operator from  $L^2(\mathbb{R}^2)$  into the Sobolev space  $W^{2\chi,2}(\mathbb{R}^2)$  and  $(-\Delta + m^2)^\chi$  is a bounded operator from  $W^{2\chi,2}(\mathbb{R}^2)$  into  $L^2(\mathbb{R}^2)$ ). In this way, we obtain

$$\det(I_{\mathcal{H}} + \nabla U^\chi) = \det(I_{\mathcal{H}} + \varpi \mathcal{I}^\chi f \partial^2 V(\mathcal{J}_\chi w) \mathcal{J}_\chi) = \det(I_{\mathcal{H}} + \varpi f \partial^2 V(\mathcal{J}_\chi w) \mathcal{I}^{1+2\chi}),$$

where  $\varpi = \frac{1}{1+2\chi}$ , and featuring the operator  $\varpi f \partial^2 V(\mathcal{J}_\chi w) \mathcal{I}^{1+2\chi}$ . Let  $\gamma$  be the Gaussian measure given by the law of  $\varphi = \mathcal{J}_\chi w \in \widetilde{\mathcal{W}}$  under  $\mu$ . In other words, the Gaussian measure  $\gamma$  is the one whose Fourier transform is

$$\int_{\widetilde{\mathcal{W}}} \exp\left(i \int_{\mathbb{R}^2} k(x) \varphi(x) dx\right) d\gamma(\varphi) = \exp\left(-\frac{1}{2} \|\mathcal{J}_\chi(k)\|_{\mathcal{H}}^2\right).$$

The expression (35) is then equivalent to

$$\int h(\varphi(0)) \det(I_{\mathcal{H}} + \varpi f \partial^2 V(\mathcal{J}_\chi w) \mathcal{I}^{1+2\chi}) \exp(-\langle \varpi f \partial V(\varphi), (m^2 - \Delta)\varphi \rangle) \times \exp\left(-\frac{\varpi^2}{2} \|\mathcal{I}^\chi f \partial V(\varphi)\|_{\mathcal{H}}^2 + 4 \int_{\mathbb{R}^2} V(\varphi(x)) f'(x) dx\right) \gamma(d\varphi).$$

At this point, we introduce an auxiliary Gaussian field  $\eta$  distributed as the Gaussian white noise  $\mu$  to write

$$\exp\left(-\frac{\varpi^2}{2}\|\mathcal{I}^\chi f \partial V(\phi)\|_{\mathcal{H}}^2\right) = \int \exp(-i\varpi\langle f \partial V(\phi), \mathcal{I}^\chi \eta \rangle)\mu(d\eta).$$

We also introduce two fermionic fields  $\psi, \bar{\psi}$  realized as bounded operators on a suitable Hilbert space  $\mathfrak{H}_{\psi, \bar{\psi}}$  with a state  $\text{Tr}(\rho \cdot) = \langle \cdot \rangle_{\psi, \bar{\psi}}$  for which

$$\begin{aligned} \{\psi(x), \psi(x')\} &= \{\bar{\psi}(x), \bar{\psi}(x')\} = \{\psi(x), \bar{\psi}(x')\} = 0, \\ \langle \bar{\psi}(x)\bar{\psi}(x') \rangle_{\psi, \bar{\psi}} &= \langle \psi(x)\psi(x') \rangle_{\psi, \bar{\psi}} = 0, \\ \langle \psi(x)\bar{\psi}(x') \rangle_{\psi, \bar{\psi}} &= \varpi \mathcal{G}_{1+2\chi}(x - x'), \end{aligned}$$

where  $\{\cdot, \cdot\}$  is the anticommutator between bounded operators, that is,  $\{K_1, K_2\} = K_1K_2 + K_2K_1$  for any bounded operators defined on  $\mathfrak{H}_{\psi, \bar{\psi}}$ , and  $\mathcal{G}_\alpha$  is the kernel of the operator  $\mathcal{I}^\alpha$  (see Appendix B for the definition of fermionic fields and Theorem 57 for the existence of such fields). By Theorem 58 and Remark 59, these additional fields allow to represent the determinant as

$$\begin{aligned} \det(I_{\mathcal{H}} + \varpi f \partial^2 V(\mathcal{J}_\chi w) \mathcal{I}^{1+2\chi}) \\ = \left\langle \exp\left(\int \psi^i(x) f(x) \partial_{\phi^i \phi^j}^2 V(\varphi(x)) \bar{\psi}^j(x) dx\right) \right\rangle_{\psi, \bar{\psi}}. \end{aligned}$$

By tensorizing the fermionic Hilbert space  $\mathfrak{H}_{\psi, \bar{\psi}}$  with the  $L^2$  space of the product measure  $\gamma \otimes \mu$ , one can realize the fermionic fields  $\psi, \bar{\psi}$  and the Gaussian fields  $\varphi, \eta$  as operators on the same Hilbert space  $\mathfrak{H}$  with a state which we denote by  $\langle \cdot \rangle_\chi$  when this does not cause any ambiguity. As a consequence, we have

$$(36) \quad Z_f^\chi \Xi_\chi(h) = \langle h(\varphi(0)) \exp(Q_\chi(V, f)) \rangle_\chi,$$

with an operator  $Q_\chi(V, f)$  given by

$$\begin{aligned} Q_\chi(V, f) := \int \psi(x) f(x) \partial^2 V(\varphi(x)) \bar{\psi}(x) dx \\ - \varpi \langle f \partial V(\varphi), (m^2 - \Delta)\varphi + i\mathcal{I}^\chi \eta \rangle + 4 \int V(\varphi(x)) f'(x) dx. \end{aligned}$$

The operator  $Q$  satisfies the following important theorem.

**THEOREM 26.** *For all polynomials  $p, P : \mathbb{R}^n \rightarrow \mathbb{R}$  and all  $n \geq 0$  and all  $\chi > 0$ , we have*

$$(37) \quad \langle p(\varphi(0)) (Q_\chi(P, f))^n \rangle_\chi = \langle p(\varphi(0)) (-4\pi P(\varphi(0)))^n \rangle_\chi.$$

This theorem is the key to our results and will be proved with the aid of supersymmetry in Section 5. Going back to equation (36), a possible strategy would be to expand the exponential getting

$$(38) \quad \langle h(\varphi(0)) \exp(Q_\chi(V, f)) \rangle_\chi = \sum_{n \geq 0} \frac{1}{n!} \langle h(\varphi(0)) (Q_\chi(V, f))^n \rangle_\chi$$

and then to use Theorem 26 to prove that each average on the right-hand side is equal to

$$\langle h(\varphi(0)) (-4\pi V(\varphi(0)))^n \rangle_\chi.$$

Since

$$\langle h(\varphi(0))(-4\pi V(\varphi(0)))^n \rangle_\chi = Z_f^\chi \int_{\mathbb{R}^n} h(y) d\kappa(y),$$

the equality (33) would be proved by taking the limit  $\chi \rightarrow 0$ . Unfortunately, equation (38) is not easy to prove since the series on the right-hand side of (38) does not converge absolutely for a general  $V$ . For this reason, we present below an indirect proof of (33). Given Theorem 26, we will deduce Theorem 17 from it via a sequence of successive generalizations:

1. first, we consider potentials  $V$  bounded and such that  $\|\partial^2 V\|_\infty < m^2/2$ ;
2. then the class of  $V$  satisfying Hypothesis  $V_\lambda$  and  $C$ ;
3. finally those  $V$  satisfying only  $V_\lambda$ .

4.1. *Bounded potentials.*

PROPOSITION 27. *For all  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  bounded such that  $\|\partial^2 V\|_\infty < m^2/2$  and  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  bounded and measurable, we have*

$$(39) \quad \langle h(\varphi(0)) \exp(Q_\chi(V, f)) \rangle_\chi = \langle h(\varphi(0)) \exp(-4\pi V(\varphi(0))) \rangle_\chi$$

and for  $\chi > 0$  small enough.

Let us introduce

$$G_\chi(t) := \langle h(\varphi(0)) \exp(t Q_\chi(V, f)) \rangle_\chi,$$

$$H_\chi(t) := \langle h(\varphi(0)) \exp(-t 4\pi V(\varphi(0))) \rangle_\chi$$

for  $t \in [0, 1]$ .

PROOF OF PROPOSITION 27. It is clear that  $H_\chi$  is real analytic in  $t \in [0, 1]$ . By Lemma 29 below, the function  $G_\chi(t)$  is real analytic in  $[-1, 1]$ . It is enough then to prove  $\partial_t^n G_\chi(0) = \partial_t^n H_\chi(0)$  for any  $n \in \mathbb{N}$ . Now

$$\partial_t^n G_\chi(0) = \langle h(\varphi(0))(Q_\chi(V, f))^n \rangle_\chi,$$

$$\partial_t^n H_\chi(0) = \langle h(\varphi(0))(-4\pi V(\varphi(0)))^n \rangle_\chi.$$

By the density of polynomials in the space of two-times differentiable functions with respect to the Malliavin derivative (see [48] Corollary 1.5.1), we can approximate both  $\partial_t^n G_\chi(0)$  and  $\partial_t^n H_\chi(0)$  with expressions of the form  $\langle p(\varphi(0))(Q_\chi(P, f))^n \rangle_\chi$  and  $\langle p(\varphi(0))(-4\pi \times P(\varphi(0)))^n \rangle_\chi$  where  $p, P$  are polynomials and, therefore, conclude from (37) that  $\partial_t^n G_\chi(0) = \partial_t^n H_\chi(0)$  for all  $n \geq 0$ .  $\square$

The following two lemmas prove the claimed analyticity of  $G_\chi$ .

LEMMA 28. *If  $V$  is a bounded potential satisfying the Hypothesis C, then  $\exp(-t\delta(U^\chi)) \in L^1(\mu)$  for any  $|t| \leq 1$  and  $\chi = 0$  and for  $\chi > 0$  small enough. Furthermore, the integral  $\int \exp(-t\delta(U^\chi)) d\mu$  is uniformly bounded for  $\chi = 0$  and for  $\chi > 0$  small enough, and  $t$  in the compact subsets of  $[-1, 1]$ .*

PROOF. Under the hypothesis of the lemma, we have that

$$\|\nabla U^\chi\| \leq \frac{\|\partial^2 V\|_\infty}{m^{2(1+\chi)}},$$

where  $\|\cdot\|$  is the usual operator norm on  $\mathcal{L}(H)$ . Proposition B.8.1 of [59] states that

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\delta(K)\right)\right] \leq (\mathbb{E}[\exp(\|K\|_{\mathcal{H}}^2)])^{\frac{1}{4}} \cdot \left(\mathbb{E}\left[\exp\left(\frac{\|\nabla K\|_2^2}{(1 - \|\nabla K\|_{\mathcal{H}})}\right)\right]\right)^{\frac{1}{4}}$$

whenever  $K$  is a  $H - C^1$  map such that  $\|\nabla K\| < 1$ . Taking  $K = 2tU^\chi$  in the previous inequality, we obtain the thesis.  $\square$

LEMMA 29. *The function  $G_\chi(t)$  is real analytic in  $[-1, 1]$  for  $\chi = 0$  and for  $\chi > 0$  small enough.*

PROOF. First of all, we have that for any  $t \in \mathbb{R}$  the map  $r \rightarrow \det_2(I + (t+r)\nabla U^\chi) =: D_t(r)$  is holomorphic in  $r$  (see [56] Theorem 9.3). By the Cauchy theorem, this means that

$$|\partial_t^n (\det_2(I + t\nabla U^\chi))| \leq \frac{n! \sup_{\theta \in \mathbb{S}^1} |D_t(Re^{i\theta})|}{R^n}.$$

On the other hand, we have for any  $\chi \in [0, 1]$ ,

$$|D_t(r)| \leq \exp\left(\frac{1}{2}\|(t+r)\nabla U^\chi\|_2^2\right) \leq \exp(C(t^2 + |r|^2)\|\partial^2 V\|_\infty^2),$$

where  $C \in \mathbb{R}_+$  is some positive constant depending on  $f$  but not on  $V$ . Thus we obtain

$$|\partial_t^n (\det_2(I + t\nabla U^\chi))| \leq \frac{n! \exp(C(t^2 + |R|^2)\|\partial^2 V\|_\infty^2)}{R^n}.$$

With a similar reasoning, we obtain a uniform bound of this kind for  $\partial_t^n \exp(-\frac{1}{2}|tU^\chi|^2)$ . Finally, we note that

$$\mathbb{E}[\exp(-\delta((t+r)U^\chi))] = \sum \frac{(-1)^n r^n}{n!} \mathbb{E}[\exp(-\delta(tU^\chi))(\delta(U^\chi))^n].$$

By Lemma 28, we note that

$$\begin{aligned} |\mathbb{E}[\partial_t^n e^{-\delta((t+r)U^\chi)}]| &= |\mathbb{E}[e^{-\delta((t+r)U^\chi)}(\delta(U^\chi))^n]| \\ &\leq \frac{1}{\epsilon^n} \mathbb{E}[e^{-\delta((t+\epsilon)U^\chi)} e^{-\delta((t-\epsilon)U^\chi)}] < +\infty \end{aligned}$$

for any  $|t| \leq 1$  and  $0 < \epsilon < \frac{m^{2+2\chi}}{2\|\partial^2 V\|_\infty} - |t|$ . Using the previous inequality, it follows that  $G_\chi(t)$  is real analytic in the required interval.  $\square$

PROPOSITION 30. *We have that  $G_0(t) = H_0(t)$  for  $t \in [-1, 1]$ .*

PROOF. By Proposition 27, we need only to prove that  $G_\chi(t) \rightarrow G_0(t)$  as  $\chi \rightarrow 0$ . Since  $\det_2, \delta, |\cdot|_{\mathcal{H}}$  are continuous with respect to the natural norm of  $\mathcal{H}$  and the Hilbert-Schmidt norm on  $\mathcal{H} \otimes \mathcal{H}$  (see [56] Theorem 9.2 for the continuity of  $\det_2$  and [48] Proposition 1.5.4 for the continuity of  $\delta$ ), and since  $\exp(-\delta(tU^\chi))$  is bounded uniformly in  $L^p$  (for  $p$  small enough), we only have to prove that, for  $\chi \rightarrow 0$ ,  $U^\chi(w) \rightarrow U(w)$  in  $\mathcal{H}$  and  $\nabla U^\chi(w) \rightarrow \nabla U(w)$  in  $\mathcal{H} \otimes \mathcal{H}$  for almost every  $w \in \mathcal{W}$ . We present only the proof of the second convergence, the proof of the first one being simpler and similar.

We have that

$$\nabla U^\chi(w)[h] = \mathcal{I}^\chi(f\partial^2 V(\mathcal{J}_\chi w) \cdot \mathcal{J}_\chi h),$$

thus proving the convergence of  $\nabla U^\chi(w)$  in  $\mathcal{H} \otimes \mathcal{H}$  is equivalent to proving the convergence of  $(m^2 - \Delta)^{-1-\chi}$  to  $(m^2 - \Delta)^{-1}$  in  $\mathcal{H} \otimes \mathcal{H}$  and the convergence of  $f\partial^2 V(\mathcal{J}_\chi w)$  to



$f\partial^2V(\mathcal{I}w)$  in  $C^0(\mathbb{R}^2)$ . The first convergence follows from a direct computation using the Fourier transform of this operators. The second convergence follows from the fact that  $V$  is smooth with bounded derivatives,  $f$  decays exponentially at infinity and  $\mathcal{J}_\chi w$  converges to  $\mathcal{I}w$  pointwise and uniformly on compact sets since  $(m^2 - \Delta)^{-\chi} \rightarrow \text{id}_{L^2}$ , weakly as bounded operator on  $L^2(\mathbb{R}^2)$  and  $(m^2 - \Delta)^{-1}$  is a compact operator from  $L^2(\mathbb{R}^2)$  into  $C^0_{\text{loc}}(\mathbb{R}^2)$ .  $\square$

4.2. *Potentials satisfying Hypothesis  $V_\lambda$  and  $C$ .* Let  $V_B$  denote a bounded smooth potential with all its derivatives bounded. Introduce the following equation for  $\phi_t = \bar{\phi}_t + \mathcal{I}\xi$ :

$$(40) \quad (m^2 - \Delta)\bar{\phi}_t + tf\partial V_B(\bar{\phi}_t + \mathcal{I}\xi) = 0.$$

Denote by  $\lambda_-$  the infimum on  $y \in \mathbb{R}^n$  over the eigenvalues of the  $y$  dependent matrix  $(\partial^2V_B(y))$ , and with  $\lambda_+$  the supremum on  $y \in \mathbb{R}^n$  over the eigenvalues of the same matrix.

For  $t \in (-\frac{m^2}{|\lambda_+ \wedge 0|}, \frac{m^2}{|\lambda_- \wedge 0|})$ , we have that equation (40) has a unique solution that, by the implicit function theorem, is infinitely differentiable with respect to  $t$  when  $V_B \in C^\infty(\mathbb{R}^n)$ . Define the formal series

$$(41) \quad S_t(r) := \sum_{k \geq 1} \frac{\sup_{x \in \mathbb{R}^2} |\partial_t^k \bar{\phi}_t(x)|}{k!} r^k.$$

LEMMA 31. *Suppose that  $V_B$  is a bounded real valued function with all derivatives bounded such that*

$$\|\partial^k V_B\|_\infty \leq C^k k!,$$

where the norm is the one induced by the identification of  $\partial^n V_B$  as a multilinear operator and for some  $C \in \mathbb{R}_+$ , then the  $r$  power series  $S_t(r)$  is holomorphic for any  $t \in (-\frac{m^2}{|\lambda_+ \wedge 0|}, \frac{m^2}{|\lambda_- \wedge 0|})$ . Furthermore, the radius of convergence of  $S_t(r)$  can be chosen uniformly for  $t$  in compact subsets of  $(-\frac{m^2}{|\lambda_+ \wedge 0|}, \frac{m^2}{|\lambda_- \wedge 0|})$ .

PROOF. We define the following functions:

$$\bar{V}^1(r) := \sum_{k \geq 0} \frac{\|\partial^{k+1} V_B\|_\infty}{k!} r^k, \quad \bar{V}^2(r) := \sum_{k \geq 0} \frac{\|\partial^{k+2} V_B\|_\infty}{k!} r^k.$$

We have that the partial derivative  $\partial_t^k \bar{\phi}_t$  solves the following equation:

$$(m^2 - \Delta)\partial_t^k \bar{\phi}_t + t\partial^2 V_B(\bar{\phi}_t) \cdot \partial_t^k \bar{\phi}_t = -\partial_t^{k-1}(\partial V_B(\bar{\phi}_t) + t\partial^2 V_B(\bar{\phi}_t) \cdot \partial_t \bar{\phi}_t) + t\partial^2 V_B(\bar{\phi}_t) \cdot \partial_t^k \bar{\phi}_t.$$

Using a reasoning similar to the one of Lemma 7, it is easy to prove that

$$\|\partial_t^k \bar{\phi}_t\|_\infty \leq \frac{\|-\partial_t^{k-1}(\partial V_B(\bar{\phi}_t) + t\partial^2 V_B(\bar{\phi}_t) \cdot \partial_t \bar{\phi}_t) + t\partial^2 V_B(\bar{\phi}_t) \cdot \partial_t^k \bar{\phi}_t\|_\infty}{m^2 - |t|(\lambda_{\text{sign}(t)} \wedge 0)},$$

where it is important to note that the right-hand side of the previous inequality depends only on the derivatives of order at most  $k - 1$ . The previous inequality and the method of majorants (see [60]) of holomorphic functions permit to get the following differential inequality for  $S_t(r)$ :

$$(42) \quad (m^2 - |t|(\lambda_{\text{sign}(t)} \wedge 0) - r\bar{V}^2(S_t(r)))\partial_r(S_t)(r) \leq \bar{V}^1(S_t(r)).$$

From the previous inequality, we obtain that  $S_t(r)$  is majorized by the holomorphic function  $F_t(r)$  that is the solution of the differential equation (42) (where the symbol  $\leq$  is replaced by  $=$ ) depending parametrically on  $t$  with initial condition  $F_t(0) = 0$ . Since  $F_t(r)$  is majorized by  $F_k(r)$  or by  $F_{-k}(r)$  if  $|t| \leq k$ , the thesis follows.  $\square$

REMARK 32. An example of potential satisfying the hypotheses of Lemma 31 is given by the family of trigonometric polynomials in  $\mathbb{R}^n$ .

LEMMA 33. Under the hypotheses of Lemma 31 with  $V = V_B$  and assuming that  $h$  is an entire function, we have that  $G_0(t) = H_0(t)$  for any  $t \in (-\frac{m^2}{|\lambda_+ \wedge 0|}, \frac{m^2}{|\lambda_- \wedge 0|})$ . In other words, the thesis of Theorem 17 holds if  $\lambda = 0$ ,  $V_B$  satisfies Hypothesis C as well as the hypotheses of Lemma 31.

PROOF. By Proposition 30, we need only to prove that  $G_0$  is real analytic in the required set. By Corollary 16, we have that

$$G_0(t) = \mathbb{E}[h(\mathcal{I}\xi(0) + \bar{\phi}_t(0))e^{4 \int t V_B(\mathcal{I}\xi(x) + \bar{\phi}_t(x)) f'(x) dx}].$$

Then the thesis follows from Lemma 31 and the analyticity of  $h$  and of the exponential.  $\square$

Let  $V$  be a potential satisfying the Hypothesis  $V_\lambda$ . Then there exist  $V_B$  such that  $V = V_B + \lambda V_U$  and we define

$$V_{t,\lambda} = t V_B + \lambda V_U,$$

for any  $t \in \mathbb{R}$ . Denote by  $U_{t,\lambda}$  the corresponding map from  $\mathcal{W}$  into  $\mathcal{H}$ . Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous bounded function. We write

$$G_{0,\lambda}(t) := \int_{\mathcal{W}} h(\mathcal{I}w(0)) \det_2(I_{\mathcal{H}} + \nabla U_{t,\lambda}) \times \exp\left(-\delta(U_{t,\lambda}) - \frac{1}{2} \|U_{t,\lambda}\|_{\mathcal{H}}^2 + 4 \int_{\mathbb{R}^2} V_{t,\lambda}(\mathcal{I}w(x)) f'(x) dx\right) d\mu$$

and

$$H_{0,\lambda}(t) := Z_f \int_{\mathbb{R}^n} h(y) \exp\left(-4\pi \left(m^2 \frac{|y|^2}{2} + t V_B(y) + \lambda V_U(y)\right)\right) dy.$$

It is evident that the thesis of Theorem 17 is equivalent to prove that

$$G_{0,\lambda}(t) = H_{0,\lambda}(t)$$

for any bounded potential  $V_B$ , any  $h$  continuous and bounded and any  $t \in (-\frac{m^2}{|\lambda_+ \wedge 0|}, \frac{m^2}{|\lambda_- \wedge 0|})$ . This fact is the result of the next proposition.

PROPOSITION 34. Under Hypothesis  $V_\lambda$ , we have that  $G_{0,\lambda}(t) = H_{0,\lambda}(t)$  for any  $t \in (-\frac{m^2}{|\lambda_+ \wedge 0|}, \frac{m^2}{|\lambda_- \wedge 0|})$ . In other words, the thesis of Theorem 17 holds if  $V$  satisfies also Hypothesis C.

PROOF. By Lemma 33, we know that Theorem 17 holds for any  $\lambda = 0$  and for any bounded potential satisfying Hypothesis C and the hypothesis of Lemma 31. Thus if we are able to approximate any potential  $V$  satisfying Hypothesis  $V_\lambda$  and Hypothesis C by potentials of the form requested by Lemma 33 the thesis is proved.

We can use the methods of the proof of Lemma 25 for approximating a potential  $V$  satisfying Hypothesis  $V_\lambda$  by a sequence of potentials  $V_{B,N}$  satisfying the hypothesis of Lemma 31. More in detail, using the notation of Lemma 25, we have that the sequence of functions  $V^N$  is composed by smooth, bounded functions and, if  $V$  satisfies Hypothesis  $V_\lambda$ , they are identically equal to  $N$  outside a growing sequence of squares  $Q_N \subset \mathbb{R}^2$ . This means that  $V^{N,p}$ , which is the periodic extension of  $V^N$  outside the square  $Q_N$ , is a smooth function for any

$N \in \mathbb{N}$ . Since  $V^{N,p}$  is periodic, it can be approximated with any precision we want by a trigonometric polynomial  $P^N$ . Furthermore, since  $V$  satisfies Hypothesis C, also  $V^{N,p}$  satisfies Hypothesis C and we can choose the trigonometric polynomial  $P^N$  satisfying Hypothesis C too. In this way, we construct a sequence of potentials  $V_{B,N} = P^N$  satisfying the hypotheses of Lemma 31 and converging to  $V$  uniformly on compact subsets of  $\mathbb{R}^n$ . Thus the thesis follows from Lemma 21, Lemma 24, Corollary 16 and the fact that the functions of the form  $L(\mathcal{I}\xi(0) + \bar{\phi}_t(0))$ , where  $L$  is an entire function, are dense in the set of measurable functions in  $\mathcal{I}\xi(0) + \bar{\phi}_t(0)$  with respect to the  $L^p(\mu)$  norm.  $\square$

4.3. Potentials satisfying only Hypothesis  $V_\lambda$ .

LEMMA 35. Under the Hypothesis  $V_\lambda$ , we have  $\det_2(I + \nabla U(w)) \in L^\infty(\mu)$ .

PROOF. We follow the same reasoning proposed in [39] for polynomials. First of all, by the invariance property of the determinant with respect to conjugation, we have that

$$\det_2(I + \nabla U(w)) = \det_2(I + O(w)),$$

where  $O(w)$  is the self-adjoint operator given by

$$(43) \quad O_j^i(w)[h] = (m^2 - \Delta)^{-\frac{1}{2}} (f \partial_{\phi^i \phi^j}^2 V(\mathcal{I}w)) \cdot (m^2 - \Delta)^{-\frac{1}{2}} h.$$

Since  $V$  satisfies the Hypothesis QC, the eigenvalues of the symmetric matrix  $\partial^2 V(y)$  (where  $y \in \mathbb{R}^n$ ) are bounded from below. Furthermore, we can write the matrix  $\partial^2 V(y)$  as the difference of two commuting matrices  $\partial^2 V(y) = V_+(y) - V_-(y)$  where  $V_+(y)$ ,  $V_-(y)$  are symmetric. They have only eigenvalues greater or equal to zero and  $\ker V_+(y) \cap \ker V_-(y) = \{0\}$ . We denote by  $O^+$ ,  $O^-$  the two operators defined as  $O$  in equation (43) replacing  $\partial^2 V$  by  $V_+$  and  $V_-$ , respectively. Obviously,  $O^+$  and  $O^-$  are positive definite and  $O = O^+ - O^-$ . By Lemma 3.3 [39], we have that

$$|\det_2(I + O(w))| \leq \exp(2\|O^-(w)\|_2^2).$$

Using a reasoning similar to the one of Proposition 12 and the fact that, under the Hypothesis  $V_\lambda$ , the minimum eigenvalue  $\lambda(y)$  of  $\partial^2 V(y)$  has a finite infimum  $\lambda_-$  that is the same as the one for  $V_-$ . We obtain

$$|\det_2(I + \nabla U(w))| = |\det_2(I + O(w))| \leq \exp(C\lambda_0\|f\|_2^2)$$

for some positive constant  $C$ . In particular, we have  $\det_2(I + \nabla U(w)) \in L^\infty(\mu)$ .  $\square$

In order to prove that  $\exp(-\delta(U)) \in L^p(\mu)$ , we split  $U$  into two pieces. First of all, if  $\lambda(y)$  is the minimum eigenvalue of  $\partial^2 V(y)$  we recall that  $\lambda_- = \inf_{y \in \mathbb{R}^n} \lambda(y)$ . Moreover, we shall set

$$\bar{U} := U - (\lambda_- \wedge 0) f \mathcal{I}(w),$$

and  $\hat{U} := U - \bar{U}$ . We also set  $W := V + \frac{\lambda_-}{2}|y|^2$ . We introduce a useful approximation of  $\bar{U}(w)$  for proving Theorem 40. Let  $P_n$  the projection of a  $L^2(\mathbb{R}^2)$  function on the momenta less than  $n$ , that is,

$$P_n(h) = \int_{|k| < n} e^{ik \cdot x} \hat{h}(k) dk,$$

where  $\hat{h}$  is the Fourier transform of  $h$  defined on  $\mathbb{R}^2$ . We can uniquely extend the operator  $P_n$  to all tempered distributions. In this way, we define  $U_n(w)$  as

$$(44) \quad U_n(w) := P_n[f \partial V(\mathcal{I}P_n w)],$$

where  $w \in \mathcal{W}$ . We shall denote by  $\bar{U}_n$  the expression corresponding to (44) where  $V$  is replaced by  $W$ .

LEMMA 36. *Under the Hypothesis  $V_\lambda$ , there exist two positive constants  $C, \alpha$  independent on  $p \geq 2$  and  $n \in \mathbb{N}$  such that*

$$(45) \quad \mathbb{E}[|\delta(\bar{U}_n - \bar{U})|^p] \leq C(p - 1)^{2p} n^{-\alpha},$$

for some constant  $C > 0$ . Furthermore, a similar bound holds also for  $\mathbb{E}[|\|\nabla U_n\|_2^2 - \|\nabla U\|_2^2|^p]$  and  $\mathbb{E}[|\|\mathcal{I}w\|_{\mathcal{H}}^2 - P_n(\mathcal{I}w)\|_{\mathcal{H}}^2|^p]$ .

PROOF. First of all, we write  $\bar{U} = U_B + \bar{U}_U$  where  $U_B = f\partial V_B(\mathcal{I}w)$ , and we consider the corresponding decomposition for  $\bar{U}_n$ . If we prove that an inequality analogous to (45) holds for  $U_B - U_{B,n}$  and  $\bar{U}_U - \bar{U}_{U,n}$  separately, then the inequality (45) holds.

In order to prove the lemma, we use the following inequality (proven in [59] Proposition B.8.1):

$$(46) \quad \mathbb{E}\left[\cosh\left(\frac{\sqrt{\rho}}{2\sqrt{2}}\delta(K)\right)\right] \leq (\mathbb{E}[\exp(\rho\|K\|_{\mathcal{H}}^2)])^{\frac{1}{4}} \times \left(\mathbb{E}\left[\exp\left(\frac{\rho}{1-\rho c}\|\nabla K\|_2^2\right)\right]\right)^{\frac{1}{4}}$$

that holds when  $\|\nabla K\|_2^2 \in L^\infty$ ,  $\|\nabla K\| \leq c < 1$  and  $0 \leq \rho < \frac{1}{2c^2}$ . Putting  $K = \bar{\epsilon}(U_B - U_{B,n})$  for  $\bar{\epsilon}$  small enough, since  $\|\nabla(U_{B,n} - U_B)\|_2^2, \|\nabla(U_{B,n} - U_B)\| \in L^\infty$  with a bound uniform in  $n$ , we have that

$$(47) \quad \mathbb{E}[\cosh(\epsilon\delta(U_B - U_{B,n}))] \leq (\mathbb{E}[\exp(\epsilon'\|U_B - U_{B,n}\|_{\mathcal{H}}^2)])^{\frac{1}{4}} \times (\mathbb{E}[\exp(\epsilon'\|\nabla(U_B - U_{B,n})\|_2^2)]),$$

for suitable  $\epsilon, \epsilon' > 0$  and for all  $n \in \mathbb{N}$ . First of all, we want to give a bound for the right-hand side of (47) providing a precise convergence rate to the constant 1 of the upper bound for the right-hand side as  $n \rightarrow +\infty$ . We first note that

$$(48) \quad \mathbb{E}[\exp(\epsilon'\|U_B - U_{B,n}\|_{\mathcal{H}}^2)] = \sum_{k=1}^{\infty} \frac{\epsilon'^k}{k!} \mathbb{E}[\|U_B - U_{B,n}\|_{\mathcal{H}}^{2k}].$$

Using a reasoning like the one in the proof of Proposition 12, we have that

$$\|U_B - U_{B,n}\|_{\mathcal{H}}^2 \lesssim \|\partial V_B\|_\infty^2 \|Q_n(f)\|_{\mathcal{H}}^2 + \|\partial^2 V_B\|_\infty^2 \int_{\mathbb{R}^2} (f(x)Q_n(\mathcal{I}w)(x))^2 dx,$$

where  $Q_n = I - P_n$ . From the previous inequality and the hypercontractivity of Gaussian random fields, we obtain that

$$\begin{aligned} \mathbb{E}[\|U_B - U_{B,n}\|_{\mathcal{H}}^{2k}] &\lesssim k \left( \|Q_n(f)\|_{\mathcal{H}}^{2k} + \int_{\mathbb{R}^2} f(x)^k \mathbb{E}[(Q_n(\mathcal{I}w)(x))^2] dx \right) \\ &\lesssim k \|Q_n(f)\|_{\mathcal{H}}^{2k} + k(2k - 1)^k \|f\|_1 \mathbb{E}[(Q_n(\mathcal{I}w)(x))^2]^k, \end{aligned}$$

where the constants implied by the symbol  $\lesssim$  do not depend on  $k$ . The right-hand side converges then for  $n \rightarrow +\infty$  to 1 as we have announced. Using the Fourier transform, the fact that  $f$  is a Schwartz function, and the fact that  $\mathcal{I}w$  is equivalent to a white noise transformed by the operator  $(m^2 - \Delta)$  it is simple to obtain that  $\|Q_n(f)\|_{\mathcal{H}}^2, \mathbb{E}[(Q_n(\mathcal{I}w)(x))^2] \lesssim \frac{1}{n^2}$ . Then using the fact that  $(2k - 1)^k \lesssim C_1^k k!$  and inserting the previous inequality in equation (48), we obtain

$$\mathbb{E}[\exp(\epsilon'\|U_B - U_{B,n}\|_{\mathcal{H}}^2)] \leq 1 + C_3 \frac{\epsilon'}{n^2 (1 - \frac{C_2 \epsilon'}{n^2})^2},$$

that holds when  $\epsilon' > 0$  is small enough and for two positive constants  $C_2, C_3$ . Using similar methods, it is possible to prove a similar estimate for  $\mathbb{E}[\exp(\epsilon' \|\nabla(U_B - U_{B,n})\|_2^2)]$ . Inserting these estimates in the inequality (47), we obtain

$$(49) \quad \mathbb{E}[\cosh(\epsilon \delta(U_B - U_{B,n}))] - 1 \lesssim \frac{\epsilon'}{n^2},$$

where the constants implied by the symbol  $\lesssim$  do not depend on  $n$  and on  $\epsilon'$ , when  $\epsilon'$  is smaller than a suitable  $\epsilon'_0 > 0$ . Using the inequality (49), we obtain that

$$\begin{aligned} & \sum_{k,n=1}^{+\infty} \frac{n^{1/2} \epsilon^{2k}}{(2k)!} \mathbb{E}[(\delta(U_B - U_{B,n}))^{2k}] \\ &= \sum_{n=1}^{+\infty} n^{\frac{1}{2}} (\mathbb{E}[\cosh(\epsilon \delta(U_B - U_{B,n}))] - 1) \\ &\lesssim \sum_{n=1}^{\infty} \frac{\epsilon'}{n^{\frac{3}{2}}} < +\infty. \end{aligned}$$

Since the terms of an absolutely convergent series are bounded, we obtain

$$\mathbb{E}[(\delta(U_B - U_{B,n}))^{2k}] \lesssim \frac{(2k)!}{\epsilon^{2k} n^{\frac{1}{2}}} \lesssim (2k - 1)^{4k} n^{-\frac{1}{2}}.$$

Using Young inequality, we obtain that the inequality (45) holds for any  $p \geq 2$ . The estimate for  $\delta(\bar{U}_U - \bar{U}_{U,n})$  follows from the fact that  $\bar{U}_U$  is a polynomial of at most third degree and from hypercontractivity estimates for polynomial expressions of Gaussian random fields.

The result for  $\|\nabla U\|_2^2 - \|\nabla U_n\|_2^2$  can be proved using the same decomposition of  $U$  and  $U_n$  and following a similar reasoning. The result for  $\mathbb{E}[\|f\mathcal{I}w\|_{\mathcal{H}}^2 - \|fP_n(\mathcal{I}w)\|_{\mathcal{H}}^2]^p$  can be proved using hypercontractivity for polynomial expressions of Gaussian random fields.  $\square$

In the following, we write  $c_n = \text{Tr}(P_n \circ \mathcal{I})$ . It is important to note that

$$c_n = \int_{|x| < n} \frac{1}{|x|^2 + m^2} dx \lesssim \log(n),$$

where the integral is taken on the ball  $|x| < n$  on  $\mathbb{R}^2$ .

LEMMA 37. *There exists a  $\lambda_0 > 0$  depending only on  $f$  and  $m^2$  such that for any  $0 < \lambda < \lambda_0$  and  $V$  satisfying the Hypothesis  $V_\lambda$  there exist some constants  $\alpha, C_1, C_2 > 0$  such that*

$$\delta(\bar{U}_n) - R \int_{\mathbb{R}^2} f(P_n \mathcal{I}w)^2 dx - \|\nabla U_n\|_2^2 \geq -C_1 - C_2 c_n^\alpha$$

for any  $R \in \mathbb{R}_+$ .

PROOF. If  $\text{Tr}(|\nabla K|) < +\infty$  and  $K \in \mathcal{W}$ , we have that  $\delta(K) = \langle K, w \rangle_{\mathcal{H}} - \text{Tr}(\nabla K)$ . Using this relation, we obtain that

$$\begin{aligned} \delta(\bar{U}_k) &= \sum_{i=1}^n \int_{\mathbb{R}^2} P_k(f \partial_{\phi^i} W(P_k \mathcal{I}w))(x) w^i(x) dx \\ &\quad - \sum_{i=1}^n \text{Tr}_{L^2}(P_k(f \partial_{\phi^i}^2 W(P_k \mathcal{I}w) \cdot P_k(m^2 - \Delta))). \end{aligned}$$

From this, we obtain the lower bound

$$\begin{aligned}
 \int_{\mathbb{R}^2} P_k(f \partial_{\phi^i} W(P_k \mathcal{I}w)) w^i \, dx &= \int_{\mathbb{R}^2} f \partial_{\phi^i} W(P_k \mathcal{I}w) (m^2 - \Delta)(P_k \mathcal{I}w^i) \, dx \\
 &= \int_{\mathbb{R}^2} f \partial_{\phi^i} W(\mathcal{I}w_k) (m^2 - \Delta)(\mathcal{I}w_k^i) \, dx \\
 &= \int_{\mathbb{R}^2} f \partial_{\phi^i \phi^r} W(\mathcal{I}w_k) \nabla \mathcal{I}w_k^i \cdot \nabla \mathcal{I}w_k^r \, dx \\
 &\quad + m^2 \int_{\mathbb{R}^2} f \mathcal{I}w_k^i \partial_{\phi^i} W(\mathcal{I}w_k) \, dx \\
 &\quad - \int_{\mathbb{R}^2} (\Delta f) W(\mathcal{I}w_k) \, dx \\
 &\geq \int_{\mathbb{R}^2} f (m^2 \mathcal{I}w_k^i \partial_{\phi^i} W(\mathcal{I}w_k) - b^2 W(\mathcal{I}w_k)) \, dx.
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 &\text{Tr}_{L^2}(P_k(f \partial_{\phi^i \phi^i}^2 W(\mathcal{I}w_k) \cdot P_k(m^2 - \Delta))) \\
 &= c_n \int_{\mathbb{R}^2} \partial_{\phi^i \phi^i}^2 W(\mathcal{I}w_k) f \, dx \\
 &\leq \frac{c_n^p}{p} + \frac{1}{q} \int_{\mathbb{R}^2} (\partial_{\phi^i \phi^i}^2 W(\mathcal{I}w_k) (\mathcal{I}w_k))^q f \, dx,
 \end{aligned}$$

where  $\frac{1}{q} + \frac{1}{p} = 1$  and  $q < 2$ . Furthermore, we have that

$$\begin{aligned}
 \|\nabla U_k\|_2^2 &\leq \int_{\mathbb{R}^2} \frac{1}{(|x|^2 + m^2)^2} \, dx \int_{\mathbb{R}^2} (\partial_{\phi^i \phi^i}^2 V(\mathcal{I}w_k))^2 f \, dx \\
 &= \ell \int_{\mathbb{R}^2} (\partial_{\phi^i \phi^i}^2 V(\mathcal{I}w_k))^2 f \, dx,
 \end{aligned}$$

where  $\ell = \int_{\mathbb{R}^2} \frac{1}{(|x|^2 + m^2)^2} \, dx$ . Using the previous inequality, we obtain that

$$\begin{aligned}
 &\delta(\bar{U}_n) - R \int_{\mathbb{R}^2} f |\mathcal{I}w_k|^2 \, dx - \|\nabla U_n\|_2^2 \\
 &\geq -\frac{c_n^p}{p} + \int_{\mathbb{R}^2} f (m^2 \mathcal{I}w_k^i \partial_{\phi^i} W(\mathcal{I}w_k) - b^2 W(\mathcal{I}w_k)) \, dx \\
 &\quad - \int_{\mathbb{R}^2} f \left( \frac{(\partial_{\phi^i \phi^i}^2 W(\mathcal{I}w_k))^q}{q} + \ell (\partial_{\phi^i \phi^i}^2 V(\mathcal{I}w_k))^2 + R |\mathcal{I}w_k|^2 \right) \, dx.
 \end{aligned}$$

It is simple to see that there exists a  $\lambda_0 > 0$  (depending only on  $b^2$  and  $m^2$ ) such that for any potential  $V$  satisfying the Hypothesis  $V_\lambda$  with  $\lambda < \lambda_0$  the expression

$$(50) \quad m^2 y_k^i \partial_{\phi^i} W(y) - b^2 W(y) - \frac{(\partial_{\phi^i \phi^i}^2 W(y))^q}{q} - \ell (\partial_{\phi^i \phi^i}^2 V(\mathcal{I}w_k))^2 - R |y|^2$$

is bounded from below, and thus the thesis of the lemma holds.  $\square$

REMARK 38. Lemma and 36 Lemma 37 are the only places where Hypothesis CO and Hypothesis  $V_\lambda$  are used in an essential way.

Indeed we are able to obtain the estimate (45), using the technique of the proof of Lemma 36, only if  $V$  is a sum of a bounded function and a polynomial. Furthermore, we

can obtain that the expression (50) is bounded from below; for  $\lambda$  small enough and for any  $R > 0$ , only if the expression  $y_k^i \partial_{\phi^i} W(y)$  is positive at infinity and it is able to compensate the growth of all the other terms in expression (50).

The previous conditions are satisfied only if  $b^2 < 4m^2$  and  $V$  is a sum of a bounded function and a polynomial of fourth degree (not less because of the presence of  $-R|y|^2$ , and no more since in the other cases the growth at infinity of  $\ell(\partial_{\phi^i}^2 \phi^i V(\mathcal{I}w_k))^2$  would have been strictly stronger than the growth at infinity of  $y_k^i \partial_{\phi^i} W(y)$ ). This is the main reason for the restriction on  $b^2$  in Hypothesis CO and for the special form of  $V$  required by Hypothesis  $V_\lambda$ .

LEMMA 39. *Given a  $p \in [1, +\infty)$ , there is a  $R > 0$  big enough such that*

$$\exp\left(-\delta(\hat{U}) - R \int_{\mathbb{R}^2} f(x)|\mathcal{I}w(x)|^2 dx\right) \in L^p(\mu).$$

PROOF. This lemma is proven in [39] Lemma 3.2.  $\square$

LEMMA 40. *Suppose that  $f$  satisfies the Hypotheses CO, then there exists  $\lambda_0 > 0$  depending only on  $f$  and  $m^2$  such that for any  $\lambda < \lambda_0$  and any  $V$  satisfying the Hypothesis  $V_\lambda$  we have that*

$$\exp(-\delta(U) + (1 + \|\nabla U\|_2^2)) \in L^p(\mu)$$

for any  $p \in [1, +\infty)$ .

PROOF. The thesis follows from Lemma 35, Lemma 36, Lemma 37 and Lemma 39 using a standard reasoning due to Nelson (see Lemma V.5 of [55] or [28]) due to the fact that from the previous results it follows that there exist two constants  $\alpha, \beta > 0$  independent on  $N$  such that

$$\mu(\{w \in \mathcal{W} | \delta(U^N)(w) \geq \beta(\log(N))\}) \leq e^{-N^\alpha}. \quad \square$$

PROOF OF THEOREM 17. By Proposition 34 in order to prove the theorem, it remains only to prove that  $G_{0,\lambda}(t)$  is real analytic for any  $t \in \mathbb{R}$ . The proof of this fact easily follows from Lemma 40 exploiting a reasoning similar to the one used in Lemma 29.  $\square$

**5. Supersymmetry.** At this point, our main result is reduced to check the claim of Theorem 26, namely that for all polynomials  $p, P : \mathbb{R}^n \rightarrow \mathbb{R}$  and all  $n \geq 0$  and all  $\chi > 0$  we have the equivalence

$$(51) \quad \langle p(\varphi(0))(Q_\chi(P, f))^n \rangle_\chi = \langle p(\varphi(0))(-4\pi P(\varphi(0)))^n \rangle_\chi.$$

Since the expressions in the expectations are polynomials in the fields  $\varphi, \omega, \psi, \bar{\psi}$  which are “free,” namely satisfy either the bosonic or fermionic version of Wick’s theorem (see, e.g., [26] Chapter 3, Section 8), the claim could be checked by explicit computations. However, this is still not trivial and a better understanding of the structure of the required computations can be obtained introducing a supersymmetric formulation involving the *superspace*  $\mathfrak{S}$  and the *superfield*  $\Phi$ . This new formulation exposes a symmetry of the problem which is not obvious from the expressions we obtained so far.

For an introduction to the mathematical formalism of supersymmetry see, for example, [7, 23, 24, 52].



5.1. *The superspace.* Formally, the superspace  $\mathfrak{S}$  can be thought as the set of points  $(x, \theta, \bar{\theta})$  where  $x \in \mathbb{R}^2$  and  $\theta, \bar{\theta}$  are two additional anticommuting coordinates. A more concrete construction is to understand  $\mathfrak{S}$  via the algebra of smooth functions on it.

Let  $\mathfrak{G}(\theta_1, \dots, \theta_n)$  be the (real) Grassmann algebra generated by the symbols  $\theta_1, \dots, \theta_n$ , that is,  $\mathfrak{G}(\theta_1, \dots, \theta_n) = \text{span}(1, \theta_i, \theta_i\theta_j, \theta_i\theta_j\theta_k, \dots, \theta_1 \cdots \theta_n)$  with the relations  $\theta_i\theta_j = -\theta_j\theta_i$ .

A  $C^\infty$  function  $F : \mathbb{R}^2 \rightarrow \mathfrak{G}(\theta, \bar{\theta})$  is just a quadruplet  $(f_\emptyset, f_\theta, f_{\bar{\theta}}, f_{\theta\bar{\theta}}) \in (C^\infty(\mathbb{R}^2))^4$ , via the identification

$$(52) \quad F(x) = f_\emptyset(x) + f_\theta(x)\theta + f_{\bar{\theta}}(x)\bar{\theta} + f_{\theta\bar{\theta}}(x)\theta\bar{\theta}.$$

The function  $F$  can be considered as a function  $F : \mathfrak{S} \rightarrow \mathbb{R}$  by formally writing

$$F(x, \theta, \bar{\theta}) = F(x).$$

In particular, we identify  $C^\infty(\mathfrak{S})$  with  $C^\infty(\mathbb{R}^2; \mathfrak{G}(\theta, \bar{\theta}))$ .  $C^\infty(\mathfrak{S})$  is a noncommutative algebra on which we can introduce a linear functional defined by

$$F \mapsto \int F(x, \theta, \bar{\theta}) dx d\theta d\bar{\theta} := - \int_{\mathbb{R}^2} f_{\theta\bar{\theta}}(x) dx,$$

where  $f_{\theta\bar{\theta}}(x)$  as in equation (52), induced by the standard Berezin integral on  $\mathfrak{S}$  satisfying

$$\int d\theta d\bar{\theta} = \int \theta d\theta d\bar{\theta} = \int \bar{\theta} d\theta d\bar{\theta} = 0, \quad \int \theta\bar{\theta} d\theta d\bar{\theta} = -1.$$

REMARK 41. A norm on  $C^\infty(\mathfrak{S})$  can be defined by

$$\|F\|_{C(\mathfrak{S})} = \sup_{x \in \mathbb{R}^2} (|f_\emptyset(x)| + |f_\theta(x)| + |f_{\bar{\theta}}(x)| + |f_{\theta\bar{\theta}}(x)|),$$

and an involution by

$$\bar{F}(x, \theta, \bar{\theta}) = \overline{f_\emptyset(x)} + \overline{f_\theta(x)}\theta + \overline{f_{\bar{\theta}}(x)}\bar{\theta} + \overline{f_{\theta\bar{\theta}}(x)}\theta\bar{\theta},$$

where the bar on the right-hand side denotes complex conjugation.

Given  $r \in C^1(\mathbb{R}; \mathbb{R})$ , we define the composition  $r \circ F : \mathfrak{S} \rightarrow \mathbb{R}$  by

$$r(F(x, \theta, \bar{\theta})) := r(f_\emptyset(x)) + r'(f_\emptyset(x))f_\theta(x)\theta + r'(f_\emptyset(x))f_{\bar{\theta}}(x)\bar{\theta} + r'(f_\emptyset(x))f_{\theta\bar{\theta}}(x)\theta\bar{\theta},$$

in accordance with the same expression one would get if  $r$  were a monomial. Moreover, we can define similarly the space of Schwartz superfunctions  $\mathcal{S}(\mathfrak{S})$  and the Schwartz superdistributions  $\mathcal{S}'(\mathfrak{S}) = \mathcal{S}'(\mathbb{R}^2; \mathfrak{G}(\theta, \bar{\theta}))$  where  $T \in \mathcal{S}'(\mathfrak{S})$  can be written  $T = T_\emptyset + T_\theta\theta + T_{\bar{\theta}}\bar{\theta} + T_{\theta\bar{\theta}}\theta\bar{\theta}$  with  $T_\emptyset, T_\theta, T_{\bar{\theta}}, T_{\theta\bar{\theta}} \in \mathcal{S}'(\mathbb{R}^2)$  and duality pairing

$$T(f) = -T_\emptyset(f_{\theta\bar{\theta}}) + T_\theta(f_{\bar{\theta}}) - T_{\bar{\theta}}(f_\theta) - T_{\theta\bar{\theta}}(f_\emptyset), \quad f_\emptyset, f_\theta, f_{\bar{\theta}}, f_{\theta\bar{\theta}} \in \mathcal{S}(\mathbb{R}^2).$$

5.2. *The superfield.* We take the generators  $\theta, \bar{\theta}$  to anticommute with the fermionic fields  $\psi, \bar{\psi}$ , and introduce the complex Gaussian field

$$\omega := -\varpi((m^2 - \Delta)\varphi + i\mathcal{I}^X \eta)$$

and put all our fields together in a single object defining the *superfield*

$$\Phi(x, \theta, \bar{\theta}) := \varphi(x) + \bar{\psi}(x)\theta + \psi(x)\bar{\theta} + \omega(x)\theta\bar{\theta},$$

where  $x \in \mathbb{R}^2$ . We also define

$$V(\Phi(x, \theta, \bar{\theta})) = V(\varphi(x)) + \partial V(\varphi(x))(\bar{\psi}(x)\theta + \psi(x)\bar{\theta}) + [\partial V(\varphi(x))\omega(x) + \partial^2 V(\varphi(x))\psi(x)\bar{\psi}(x)]\theta\bar{\theta}$$

and since

$$\tilde{f}(|x|^2 + 4\theta\bar{\theta}) = \tilde{f}(|x|^2) + 4\tilde{f}'(|x|^2)\theta\bar{\theta},$$

where  $\tilde{f} : \mathbb{R}_+ \rightarrow \mathbb{R}$  is the smooth function such that  $f(x) = \tilde{f}(|x|^2)$  and  $f'(x) = \tilde{f}'(|x|^2)$  (see Section 1), we observe that

$$\begin{aligned} & - \int V(\Phi(x, \theta, \bar{\theta}))\tilde{f}(|x|^2 + 4\theta\bar{\theta}) \, dx \, d\theta \, d\bar{\theta} \\ &= \int f(x)\partial V(\varphi(x))\omega(x) \, dx \\ & \quad + \int [f(x)\partial^2 V(\varphi(x))\psi(x)\bar{\psi}(x) + 4V(\varphi(x))f'(x)] \, dx = Q_\chi(V, f). \end{aligned}$$

By introducing the superspace distribution  $\theta\bar{\theta}\delta_0(dx)$  we have also, by similar computations,

$$p(\varphi(0)) = - \int p(\Phi(x, \theta, \bar{\theta}))\theta\bar{\theta}\delta_0(dx) \, d\theta \, d\bar{\theta}.$$

As a consequence, we can rewrite  $\langle p(\varphi(0))(Q_\chi(P, f))^n \rangle_\chi$  as an average over the superfield  $\Phi$ :

$$\begin{aligned} \Xi_\chi(p) &:= \langle p(\varphi(0))(Q_\chi(P, f))^n \rangle_\chi \\ (53) \quad &= \left\langle \left( - \int p(\Phi(x, \theta, \bar{\theta}))\theta\bar{\theta}\delta_0(dx) \, d\theta \, d\bar{\theta} \right) \right. \\ & \quad \left. \times \left( - \int P(\Phi(x, \theta, \bar{\theta}))\tilde{f}(|x|^2 + 4\theta\bar{\theta}) \, dx \, d\theta \, d\bar{\theta} \right)^n \right\rangle_\chi. \end{aligned}$$

While all these rewritings are essentially algebraic, the supersymmetric formulation (53) makes it appear as a symmetry of the expression for  $\Xi_\chi(p)$  which was not clear from the original formulation. In some sense, the reader can think of the superspace  $(x, \theta, \bar{\theta})$  and of the superfield  $\Phi(x, \theta, \bar{\theta})$  as a convenient bookkeeping procedure for a series of relations between the quantities one is manipulating.

A crucial observation is that the superfield  $\Phi$  is a free field with mean zero, namely all its correlation functions can be expressed in terms of the two-point function  $\langle \Phi(x, \theta, \bar{\theta})\Phi(x', \theta', \bar{\theta}') \rangle_\chi$  via Wick's theorem. A direct computation of this two point function gives

$$\begin{aligned} & \langle \Phi(x, \theta, \bar{\theta})\Phi(x', \theta', \bar{\theta}') \rangle_\chi \\ &= \langle \varphi(x)\varphi(x') \rangle_\chi - \langle \bar{\psi}(x)\psi(x') \rangle_\chi \theta\bar{\theta}' - \langle \psi(x)\bar{\psi}(x') \rangle_\chi \bar{\theta}\theta' \\ & \quad + \langle \varphi(x)\omega(x') \rangle_\chi \theta'\bar{\theta}' + \langle \omega(x)\varphi(x') \rangle_\chi \theta\bar{\theta} \\ & \quad + \langle \omega(x)\omega(x') \rangle_\chi \theta\bar{\theta}\theta'\bar{\theta}' \\ &= \mathcal{G}_{2+2\chi}(x-x') + \varpi \mathcal{G}_{1+2\chi}(x-x')(\theta\bar{\theta}' - \bar{\theta}\theta') \\ & \quad - \varpi(m^2 - \Delta)\mathcal{G}_{2+2\chi}(x-x')(\theta'\bar{\theta}' + \theta\bar{\theta}) \\ & \quad + ((m^2 - \Delta)^2\mathcal{G}_{2+2\chi}(x-x') - \mathcal{G}_{2\chi}(x-x'))\theta\bar{\theta}\theta'\bar{\theta}'. \end{aligned}$$

Upon observing that  $(m^2 - \Delta)\mathcal{G}_{2+2\chi} = \mathcal{G}_{1+2\chi}$ ,  $(m^2 - \Delta)^2\mathcal{G}_{2+2\chi} = \mathcal{G}_{2\chi}$  and that  $-\theta\bar{\theta}' + \bar{\theta}\theta' + \theta'\bar{\theta}' + \theta\bar{\theta} = (\theta - \theta')(\bar{\theta} - \bar{\theta}')$ , we conclude

$$(54) \quad \langle \Phi(x, \theta, \bar{\theta})\Phi(x, \theta', \bar{\theta}') \rangle = C_\Phi(x - x', \theta - \theta', \bar{\theta} - \bar{\theta}'),$$

where

$$C_\Phi(x, \theta, \bar{\theta}) := \mathcal{G}_{2+2\chi}(x) - \varpi \mathcal{G}_{1+2\chi}(x)\theta\bar{\theta}.$$

REMARK 42. Note that when  $\chi = 0$ , the superfield  $\Phi$  corresponds to the formal functional integral

$$e^{-\frac{1}{2} \int [\Phi(m^2 - \Delta_S)\Phi] dx d\theta d\bar{\theta}} \mathcal{D}\Phi,$$

where  $\mathcal{D}\Phi = \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\varphi \mathcal{D}\eta$  and where  $\Delta_S = \Delta + \partial_\theta \partial_{\bar{\theta}}$  is the super-Laplacian, where  $\partial_\theta, \partial_{\bar{\theta}}$  are the Grassmannian derivative such that  $\partial_\theta(\theta) = \partial_{\bar{\theta}}(\bar{\theta}) = -1$ ,  $\partial_\theta(\bar{\theta}) = \partial_{\bar{\theta}}(\theta) = 0$ ,  $\partial_\theta(\bar{\theta}\theta) = -\bar{\theta}$  and  $\partial_{\bar{\theta}}(\theta\bar{\theta}) = \theta$  (see, e.g., [61] Chapter 20 or [63], Section 16.8.4).

Then

$$\begin{aligned} & \frac{1}{2} \int [\Phi(m^2 - \Delta_S)\Phi] dx d\theta d\bar{\theta} \\ &= \frac{1}{2} \int [-2\bar{\psi}(m^2 - \Delta)\psi - \omega\omega + 2\omega(m^2 - \Delta)\varphi] dx \\ &= \frac{1}{2} \int [-2\psi(m^2 - \Delta)\bar{\psi} + ((m^2 - \Delta)\varphi)^2 + \eta^2] dx \end{aligned}$$

and this indeed corresponds to the action functional appearing in the formal functional integral for  $(\psi, \bar{\psi}, \varphi, \eta)$ . This is in agreement with the fact that the two-point function satisfies the equation

$$(m^2 - \Delta_S)C_\Phi(x, \theta, \bar{\theta}) = \delta_0(x)\delta(\theta)\delta(\bar{\theta}),$$

where  $\delta(x)\delta(\theta)\delta(\bar{\theta})$  is the distribution such that

$$\int F(x, \theta, \bar{\theta})\delta_0(x)\delta(\theta)\delta(\bar{\theta}) dx d\theta d\bar{\theta} = f_\varnothing(0),$$

namely  $C_\Phi$  is the Green's function for  $(m^2 - \Delta_S)$ .

5.3. *The supersymmetry.* On  $C^\infty(\mathfrak{S})$  one can introduce the (graded) derivations

$$Q := 2\theta\nabla + x\partial_{\bar{\theta}}, \quad \bar{Q} := 2\bar{\theta}\nabla - x\partial_\theta,$$

where  $x \in \mathbb{R}^2$ ,  $\nabla$  (and in the following also  $\Delta = \text{div}(\nabla \cdot)$ ) acts only on the space variables  $x \in \mathbb{R}^2$ , which are such that

$$Q(|x|^2 + 4\theta\bar{\theta}) = \bar{Q}(|x|^2 + 4\theta\bar{\theta}) = 0,$$

namely they annihilate the quadratic form  $|x|^2 + 4\theta\bar{\theta}$ . Moreover, if  $QF = \bar{Q}F = 0$ , for  $F$  as in equation (52), then we must have

$$0 = QF(x, \theta, \bar{\theta}) = 2\theta\nabla f_\varnothing(x) + xf_{\bar{\theta}}(x) + 2\nabla f_{\bar{\theta}}(x)\theta\bar{\theta} - xf_{\theta\bar{\theta}}(x)\theta,$$

$$0 = \bar{Q}F(x, \theta, \bar{\theta}) = 2\bar{\theta}\nabla f_\varnothing(x) + xf_\theta(x) - 2\nabla f_\theta(x)\theta\bar{\theta} + xf_{\theta\bar{\theta}}(x)\bar{\theta}$$

and, therefore,

$$\nabla f_\varnothing(x) = \frac{x}{2} f_{\theta\bar{\theta}}(x) \quad \text{and} \quad f_\theta(x) = f_{\bar{\theta}}(x) = 0.$$

If we also request that  $F$  is invariant with respect to  $\mathbb{R}^2$  rotations in space, then there exists an  $f$  such that  $f(|x|^2) = f_{\varnothing}(x)$  from which we deduce that  $2xf'(|x|^2) = \nabla f(|x|^2) = \nabla f_{\varnothing}(x) = \frac{x}{2}f_{\theta\bar{\theta}}(x)$  which implies

$$f(|x|^2 + 4\theta\bar{\theta}) = f(|x|^2) + 4f'(|x|^2)\theta\bar{\theta} = f_{\varnothing}(x) + f_{\theta\bar{\theta}}(x)\theta\bar{\theta} = F(x, \theta, \bar{\theta}).$$

Namely any function satisfying these two equations can be written in the form

$$F(x, \theta, \bar{\theta}) = f(|x|^2 + 4\theta\bar{\theta}).$$

Observe that if we introduce the linear transformations

$$\tau(b, \bar{b}) \begin{pmatrix} x \\ \theta \\ \bar{\theta} \end{pmatrix} = \begin{pmatrix} x + 2\bar{b}\theta\rho + 2b\bar{\theta}\rho \\ \theta - (x \cdot b)\rho \\ \bar{\theta} + (x \cdot \bar{b})\rho \end{pmatrix} \in \mathfrak{G}(\theta, \bar{\theta}, \rho)$$

for  $b, \bar{b} \in \mathbb{R}^2$  and where  $\rho$  is a new odd variable anticommuting with  $\theta, \bar{\theta}$  and itself, then we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \tau(tb, t\bar{b})F(x, \theta, \bar{\theta}) &= \frac{d}{dt} \Big|_{t=0} F(\tau(tb, t\bar{b})(x, \theta, \bar{\theta})) \\ &= (b \cdot \bar{Q} + \bar{b} \cdot Q)F(x, \theta, \bar{\theta}) \end{aligned}$$

so  $\tau(b, \bar{b}) = \exp(b \cdot \bar{Q} + \bar{b} \cdot Q)$  and  $\tau(tb, t\bar{b})\tau(sb, s\bar{b}) = \tau((t+s)b, (t+s)\bar{b})$ . In particular,  $F \in C^\infty(\mathfrak{G})$  is supersymmetric if and only if  $F$  is invariant with respect to rotations in space and for any  $b, \bar{b} \in \mathbb{R}^2$  we have  $\tau(b, \bar{b})(F) = F$ .

By duality, the operators  $Q, \bar{Q}$  and  $\tau(b, \bar{b})$  also act on the space  $S'(\mathfrak{G})$  and we say that the distribution  $T \in S'(\mathfrak{G})$  is supersymmetric if it is invariant with respect to rotations in space and  $Q(T) = \bar{Q}(T) = 0$ . For supersymmetric functions and distribution, the following fundamental theorem holds.

**THEOREM 43.** *Let  $F \in \mathcal{S}(\mathfrak{G})$  and  $T \in S'(\mathfrak{G})$  such that  $T_0$  is a continuous function. If both  $F$  and  $T$  are supersymmetric, then we have the reduction formula*

$$(55) \quad \int T(x, \theta, \bar{\theta}) \cdot F(x, \theta, \bar{\theta}) \, dx \, d\theta \, d\bar{\theta} = 4\pi T_{\varnothing}(0)F_{\varnothing}(0).$$

**PROOF.** The proof can be found in [39], Lemma 4.5 (see also [54] for a general proof on supermanifolds).  $\square$

Let us note that

$$QC_{\Phi}(x, \theta, \bar{\theta}) = \bar{Q}C_{\Phi}(x, \theta, \bar{\theta}) = 0,$$

indeed we can check that

$$\begin{aligned} \nabla \mathcal{G}_{2+2\chi}(x) &= \int_{\mathbb{R}^2} \frac{dk}{(2\pi)^2} \frac{(ik)e^{ik \cdot x}}{(m^2 + |k|^2)^{2+2\chi}} \\ &= -\frac{i}{2(1+2\chi)} \int_{\mathbb{R}^2} \frac{dk}{(2\pi)^2} e^{ik \cdot x} \nabla_k \frac{1}{(m^2 + |k|^2)^{1+2\chi}} \\ &= \frac{i}{2(1+2\chi)} \int_{\mathbb{R}^2} \frac{dk}{(2\pi)^2} \frac{(ix)e^{ik \cdot x}}{(m^2 + |k|^2)^{1+\delta}} = -\frac{x}{2(1+2\chi)} \mathcal{G}_{1+2\chi}(x) \\ &= -\frac{x \varpi}{2} \mathcal{G}_{1+2\chi}(x). \end{aligned}$$

As a consequence, expectation values of polynomials over the superfield  $\Phi$  are invariant under the supersymmetry generated by any linear combinations of  $Q, \bar{Q}$ .

REMARK 44. The previous discussion implies that

$$(56) \quad \tau(b, \bar{b})C_\Phi(x, \theta, \bar{\theta}) = C_\Phi(x, \theta, \bar{\theta}).$$

As a consequence, the superfield  $\Phi' := \tau(b, \bar{b})\Phi$  is a Gaussian free field and has the same correlation function  $C_{\Phi'}$  as  $\Phi$  given by equation (54). However, it is important to stress that this does not imply that  $\Phi'$  has the same “law” as  $\Phi$ , namely that  $\langle F(\Phi') \rangle = \langle F(\Phi) \rangle$  for nice arbitrary functions. Indeed the correlation function given in equations (54) involves only the product  $\langle \Phi(x, \theta, \bar{\theta})\Phi(x, \theta', \bar{\theta}') \rangle$  of the complex superfield  $\Phi$  and not also the product  $\langle \Phi(x, \theta, \bar{\theta})\bar{\Phi}(x, \theta', \bar{\theta}') \rangle$  of  $\Phi$  with its complex conjugate  $\bar{\Phi}$ . The law of  $\Phi$  would have been invariant with respect super transformations if and if only  $\langle \Phi(x, \theta, \bar{\theta})\Phi(x, \theta', \bar{\theta}') \rangle$  and  $\langle \Phi(x, \theta, \bar{\theta})\bar{\Phi}(x, \theta', \bar{\theta}') \rangle$  had been both supersymmetric. Unfortunately, the function  $\langle \Phi(x, \theta, \bar{\theta})\bar{\Phi}(x, \theta', \bar{\theta}') \rangle$  is not invariant with respect to super transformations.

5.4. *Expectation of supersymmetric polynomials.* As explained in Remark 44, the law of  $\Phi$  is not supersymmetric. Nevertheless, we can deduce important consequences from the supersymmetry of the correlation function  $C_\Phi$ . More precisely, since  $\Phi$  is a free field Wick’s theorem (see, e.g., [26] Chapter 3, Section 8) hold and

$$(57) \quad \left\langle \prod_{i=1}^{2n} \Phi(x_i, \theta_i, \bar{\theta}_i) \right\rangle_\chi = \sum_{\{(i_k, j_k)\}_k} \prod_{k=1}^n C_\Phi(x_{i_k} - x_{j_k}, \theta_{i_k} - \theta_{j_k}, \bar{\theta}_{i_k} - \bar{\theta}_{j_k}),$$

$$(58) \quad \left\langle \prod_{i=1}^{2n+1} \Phi(x_i, \theta_i, \bar{\theta}_i) \right\rangle_\chi = 0.$$

By the supersymmetry of  $C_\Phi(x - x', \theta - \bar{\theta}, \theta - \bar{\theta}')$  and of its products, we obtain that

$$\left\langle \prod_{i=1}^{2n} \tau(b, \bar{b})(\Phi)(x_i, \theta_i, \bar{\theta}_i) \right\rangle_\chi = \left\langle \prod_{i=1}^{2n} \Phi(x_i, \theta_i, \bar{\theta}_i) \right\rangle_\chi.$$

The previous equality implies that

$$(59) \quad \begin{aligned} & \left\langle \prod_{i=1}^n \int P_i(\Phi) \cdot \tau(b, \bar{b})(F^i) dx d\theta d\bar{\theta} \right\rangle_\chi \\ &= \left\langle \prod_{i=1}^n \int \tau(b, \bar{b})(P_i(\Phi)) \cdot F^i dx d\theta d\bar{\theta} \right\rangle_\chi \\ &= \left\langle \prod_{i=1}^n \int P_i(\tau(b, \bar{b})(\Phi)) \cdot F^i dx d\theta d\bar{\theta} \right\rangle_\chi \\ &= \left\langle \prod_{i=1}^n \int P_i(\Phi) \cdot F^i dx d\theta d\bar{\theta} \right\rangle_\chi, \end{aligned}$$

where  $P_1, \dots, P_n$  are arbitrary polynomials and  $F^1, \dots, F^n$  arbitrary functions on super-space.

LEMMA 45. Let  $F^1, \dots, F^n \in \mathcal{S}(\mathfrak{S})$  be supersymmetric smooth functions and  $P_1, \dots, P_n$  be  $n$  polynomials then

$$\left\langle \prod_{i=1}^n \int P_i(\Phi)(x, \theta, \bar{\theta}) \cdot F^i(x, \theta, \bar{\theta}) dx d\theta d\bar{\theta} \right\rangle_\chi = (4\pi)^n \left\langle \prod_{i=1}^n f_{\mathcal{O}}^i(0) P_i(\phi(0)) \right\rangle_\chi.$$

PROOF. We define the distribution  $\mathcal{H}^1 \in \mathcal{S}'(\mathfrak{G})$  in the following way:

$$\mathcal{H}^1(G) := \left\langle \int P_1(\Phi)(x, \theta, \bar{\theta}) \cdot G(x, \theta, \bar{\theta}) \, dx \, d\theta \, d\bar{\theta} \right. \\ \left. \times \prod_{i=2}^n \int P_i(\Phi)(x, \theta, \bar{\theta}) \cdot F^i(x, \theta, \bar{\theta}) \, dx \, d\theta \, d\bar{\theta} \right\rangle_{\mathcal{X}}$$

for any  $G \in \mathcal{S}(\mathfrak{G})$ . Using the fact that  $F^2, \dots, F^n$  are supersymmetric and relation (59), we have that

$$\mathcal{H}^1(\tau(b, \bar{b})(G)) \\ = \left\langle \int P_1(\Phi) \cdot \tau(b, \bar{b})(G) \, dx \, d\theta \, d\bar{\theta} \prod_{i=2}^n \int P_i(\Phi) \cdot F^i \, dx \, d\theta \, d\bar{\theta} \right\rangle_{\mathcal{X}} \\ = \left\langle \int P_1(\Phi) \cdot \tau(b, \bar{b})(G) \, dx \, d\theta \, d\bar{\theta} \prod_{i=2}^n \int P_i(\Phi) \cdot \tau(b, \bar{b})(F^i) \, dx \, d\theta \, d\bar{\theta} \right\rangle_{\mathcal{X}} \\ = \mathcal{H}^1(G).$$

This means that  $\mathcal{H}^1$  is supersymmetric and since  $F^1$  is also supersymmetric, by Theorem 43 we conclude

$$\mathcal{H}^1(F^1) = f_{\emptyset}^1(0) \mathcal{H}_0^1(0) \\ = (4\pi) \left\langle f_{\emptyset}^1(0) P_i(\phi(0)) \prod_{i=2}^n \int F^i \cdot P_i(\Phi) \, dx \, d\theta \, d\bar{\theta} \right\rangle_{\mathcal{X}} = \mathcal{H}^1(K),$$

where  $K := (4\pi)\delta_0(dx)\theta\bar{\theta}$ . Setting

$$\mathcal{H}^2(G) := \left\langle \left( \int P_i(\Phi) K \, dx \, d\theta \, d\bar{\theta} \right) \right. \\ \left. \times \left( \int P_i(\Phi) G \, dx \, d\theta \, d\bar{\theta} \right) \prod_{i=3}^n \int P_i(\Phi) F^i \, dx \, d\theta \, d\bar{\theta} \right\rangle_{\mathcal{X}}$$

and reasoning similarly we also conclude that  $\mathcal{H}^2(F^2) = \mathcal{H}^2(V)$ . Proceeding by transforming each subsequent factor, we can deduce that

$$\left\langle \prod_{i=1}^n \int P_i(\Phi) F^i \, dx \, d\theta \, d\bar{\theta} \right\rangle_{\mathcal{X}} = \left\langle \prod_{i=1}^n \int P_i(\Phi) K \, dx \, d\theta \, d\bar{\theta} \right\rangle_{\mathcal{X}} \\ = (4\pi)^n \left\langle \prod_{i=1}^n f_{\emptyset}^i(0) P_i(\phi(0)) \right\rangle_{\mathcal{X}}. \quad \square$$

PROOF OF THEOREM 26. It is enough to use Lemma 45 with  $P_1 = p, P_2 = \dots = P_{n+1} = P, F_1 = -\theta\bar{\theta}\delta_0(x)$  and  $F_2 = \dots = F_{n+1} = \tilde{f}(|x|^2 + 4\theta\bar{\theta})$  to conclude.  $\square$

REMARK 46. The dimensional reduction proof via supersymmetry is already present in [39] and indeed our result is analogous, under different hypotheses, to Theorem II in [39]. The proofs of Lemma 35, Lemma 37 and Lemma 39 above follows the same ideas of Lemma 3.1, Lemma 3.2 and Lemma 3.3 in [39]. We decided to propose a detailed proof of Theorem 17 mainly for two reasons:

1. The first reason is that the hypotheses on the potential  $V$  of Theorem 17 and of Theorem II in [39] are quite different. Indeed in [39], only polynomial potentials are considered while Hypothesis  $V_\lambda$  permits to consider polynomial of at most fourth degree perturbed by any bounded function. In order to prove the boundedness of  $\Lambda_U$  in  $L^p(\mu)$  under these different hypotheses, we need to prove Lemma 36 which is a trivial consequence of hypercontractivity when the potential  $V$  is polynomial but is based on the nontrivial inequality (46) (proven in [59]) for general potentials  $V$ .

2. The second main reason is the difference in the use of supersymmetry and of the supersymmetric representation of the integral (32). Indeed, in our opinion, there is a little gap in the proof of Theorem III of [39] that cannot be fixed without developing a longer proof. More precisely, in the proof of Theorem III of [39] it is tacitly assumed that the expression

$$\Psi(F) := \left\langle g(\varphi(0)) \exp\left(-\int V(\Phi)F \, d\theta \, d\bar{\theta} \, dx\right) \right\rangle_\chi,$$

is supersymmetric with respect to the function  $F$ , that is, if  $F$  is a smooth function in  $\mathcal{S}(\mathfrak{S})$  and  $\tau(b, \bar{b})$  is a supersymmetric transformation, then we have that  $\Psi(\tau(b, \bar{b})(F)) = G(F)$ . In our opinion, this fact is nontrivial since the law of  $\Phi$  is not supersymmetric (see Remark 44). What can be easily proven is only that the expressions

$$\Psi^n(F) := \left\langle g(\varphi(0)) \left(\int V(\Phi)F \, d\theta \, d\bar{\theta} \, dx\right)^n \right\rangle_\chi$$

are supersymmetric in  $F$  (see Theorem 26 above). This fact alone does not easily imply that  $\Psi(F)$  is supersymmetric. Indeed for the discussion in Section 4, we cannot guarantee that the series (38), which is equivalent to  $\Psi(F) = \sum_{n \geq 0} \frac{1}{n!} \Psi^n(F)$ , converges absolutely when  $V$  growth at infinity at least as a polynomial of fourth degree (and we do not know under which conditions on  $V$  and  $F$  it converges relatively). In order to overcome this problem, we propose a proof of Theorem 17 which exploits only indirectly the supersymmetric representation of the integral (32) in a way which permits to use only the supersymmetry of the expressions  $\Psi^n(F)$  and avoiding the proof of the supersymmetry of the expression  $\Psi(F)$ .

**6. Removal of the spatial cut-off.** In this section, we prove Theorem 4 on the removal of the spatial cut-off in the setting of Hypothesis C. It is important to note that, differently from Theorem 18, we explicitly require that the potential  $V$  satisfies Hypothesis C and not only Hypothesis QC. This is not due to problems in proving the existence of solutions to equation (10) or in proving the convergence of the cut-offed solution to the noncut-offed one without the Hypothesis C (see Lemma 48). The main difficulty is instead to prove the convergence of  $\Upsilon_f(\phi)/Z_f$  to 1. Indeed the previous factor does not actually converge and what we can reliably expect is that

$$(60) \quad \lim_{f \rightarrow 1} Z_f^{-1} \mathbb{E}[\Upsilon_f(\phi_f) | \sigma(\phi_f(0))] \rightarrow 1,$$

where hereafter  $\phi_f$  denotes the solution to the equation (6) with cut-off  $f$ , that is,  $\Upsilon_f(\phi_f)/Z_f$  becomes independent with respect to the  $\sigma$ -algebra generated by  $\phi_f(0)$ .

To prove (60) directly is quite difficult due to the nonlinearity of the equation or equivalently to the presence of the regularized Fredholm determinant in the expressions (26) and (25) (which is a strongly nonlocal operator). For this reason, we want to exploit a reasoning similar to the one used in Section 4. With this aim, we introduce the equation

$$(61) \quad (m^2 - \Delta)\phi_{f,t} + t f \partial V(\phi_{f,t}) = \xi$$

and the functions

$$F_f^L(t) := Z_f^{-1} \mathbb{E}[L(\phi_{f,t}(0)) e^{4t \int_{\mathbb{R}^2} f'(x) V(\phi_{f,t}(x)) \, dx}],$$



where  $t$  is taken such that  $t\partial^2V(y) + m^2$  is positive definite, and  $F^L(t) = \mathbb{E}[L(\phi_t(0))]$  (where  $\phi_t$  is the solution to (61) with  $f \equiv 1$ ). By Lemma 31, (whose proof does not use in any point the cut-off  $f$ )  $F^L(t)$  is real analytic whenever  $V$  is a trigonometric polynomial,  $t\partial^2V(y) + m^2$  is definite positive for any  $y \in \mathbb{R}^n$  and  $L$  is an entire bounded function. Furthermore, by Theorem 18,  $F_f^L(t) = H^L(t)$  (where  $H^L(t) = \int L(y) d\kappa_t(y)$ , see Section 4) which is real analytic. Thus if we are able to prove that  $\lim_{f \rightarrow 1} \partial_t^n F_f^L(0) = \partial_t^n F^L(0)$  we have that  $H^L(t) = F^L(t)$  whenever  $t\partial^2V + m^2$  is definite positive proving that Theorem 4 when  $V$  is a trigonometric polynomial satisfying Hypothesis C. The idea then is to apply a generalization of Lemma 21, Lemma 24, Lemma 25 and the reasoning in the proof of Proposition 34 and in the proof of Theorem 18 in order to obtain Theorem 4.

REMARK 47. Hypothesis C is required in an essential way in the proof of the holomorphy of  $F^L(t)$ , in particular in Lemma 31. The fact that the cut-off is removed does not allow to reason by approximation as we did in Theorem 17.

Since the proof is composed by many steps which is a straightforward generalization of the results of the previous sections of the paper, we write here only some details of the parts of the proof of Theorem 4 which largely differ from what has been obtained before.

Hereafter, we denote by  $\omega_\beta(x)$  the function

$$\omega_\beta(x) := \exp(-\beta\sqrt{(1 + |x|^2)})$$

and introduce the space  $\mathcal{W}_\beta$  where  $\beta > 0$  in the following way:

$$\mathcal{W}_\beta := (-\Delta + 1)C_{\exp\beta}^0(\mathbb{R}^2; \mathbb{R}^n),$$

where  $C_{\exp\beta}^0$  is the space of continuous function with respect to the weighted  $L^\infty$  norm

$$\|g\|_{\infty, \exp\beta} := \sup_{x \in \mathbb{R}^2} |\omega_\beta(x)g(x)|.$$

The triple  $(\mathcal{W}_\beta, \mathcal{H}, \mu)$  is an abstract Wiener space. We introduce the obvious generalization of equation (18)

$$(62) \quad (m^2 - \Delta)\bar{\phi} + \partial V(\bar{\phi} + \mathcal{I}\xi) = 0,$$

where  $\bar{\phi} = \phi - \mathcal{I}\xi$ .

Now we want to prove a result that can replace Lemma 7. Indeed Lemma 7 plays a central role in the previous sections of the paper, allowing to prove the existence of strong solutions to equation (6), the characterization of weak solutions in Theorem 13 and Theorem 14 and finally allowing to show the convergence of weak solutions using the convergence of potentials in Lemma 21, Lemma 24.

LEMMA 48. *Suppose that  $V$  satisfies the Hypothesis QC and suppose that  $\bar{\phi}$  is a classical solution to equation (62). Then there exists a  $\beta_0$  depending only on  $m^2$  such that, for any  $\beta < \beta_0$ ,*

$$(63) \quad \|\bar{\phi}\|_{\infty, \exp\beta} \lesssim \|\exp(\alpha|\mathcal{I}\xi|\)|_{\infty, \exp\beta},$$

where  $\|\exp(\alpha|\mathcal{I}\xi|\)|_{\infty, \exp\beta}$  is almost surely finite and the constants implied by the symbol  $\lesssim$  depend only on  $H$  and  $m^2$ . Furthermore, for any  $U$  open and bounded we have

$$(64) \quad \|\bar{\phi}\|_{C^{2-\tau}(U)} \lesssim \|\exp(\alpha p|\mathcal{I}\xi|\)|_{U_\epsilon, \infty} \exp(\alpha p)\|\bar{\phi}\|_{\infty, \exp\beta} \|\omega_\beta^{-1}\|_{U_\epsilon, \infty},$$

where  $U_\epsilon := \{x \in \mathbb{R}^2 | \exists y \in U, |y - x| \leq \epsilon\}$  and  $\epsilon > 0$ .

PROOF. The proof is very similar to the proof of Lemma 7. We report here only the passages having the main differences. For any  $\epsilon > 0$ , there is a  $\beta_\epsilon > 0$  and for any  $\beta < \beta_\epsilon$  we have

$$\left| \frac{\Delta(\omega_{2\beta}(x))}{\omega_{2\beta}(x)} - \frac{|\nabla\omega_{2\beta}(x)|^2}{\omega_{4\beta}(x)} \right| < \epsilon, \quad x \in \mathbb{R}^2.$$

Without loss of generality (using the result of Lemma 7), we have that  $\lim_{x \rightarrow \infty} |\bar{\phi}(x)|^2 \times \omega_{2\beta}(x) = 0$  and so  $x \mapsto |\bar{\phi}(x)|^2 \omega_{2\beta}(x)$  has a positive maximum at  $\bar{x} \in \mathbb{R}^2$ . This means that  $-\Delta(|\bar{\phi}|^2 \omega_{2\beta})(\bar{x}) \geq 0$  and  $\nabla \bar{\phi} = -\frac{\bar{\phi}}{2\omega_{2\beta}} \nabla \omega_{2\beta}$  we have that

$$\begin{aligned} (m^2 - \epsilon) |\bar{\phi}(\bar{x})|^2 \omega_{2\beta}(\bar{x}) &\leq \frac{-\Delta(|\bar{\phi}|^2 \omega_{2\beta})(\bar{x})}{2} + m^2 |\bar{\phi}(\bar{x})|^2 \omega_{2\beta}(\bar{x}) \\ &\leq -\omega_{2\beta}(\bar{x}) (\bar{\phi}(\bar{x}) \cdot \partial V(\mathcal{I}\xi(\bar{x}) + \bar{\phi}(\bar{x}))). \end{aligned}$$

Using a reasoning similar to the one of Lemma 7, the thesis follows.  $\square$

Since the bounds (63) and (64) in  $C_{\text{exp}\beta}^0$  and  $C_{\text{loc}}^{2-\tau}$  imply the compactness in  $C_{\text{exp}\beta'}^0$  when  $\beta' < \beta$ , Lemma 48 permits to prove the existence of strong solutions to equation (10), their uniqueness when  $V$  satisfies Hypothesis C and the generalization of Lemma 21, Lemma 24, Lemma 25, Proposition 34 and Theorem 18 needed in order to prove Theorem 4.

At this point, the proof of Theorem 4 requires only the following additional statement.

THEOREM 49. *Let  $V$  be a trigonometric polynomial, let  $L$  be a polynomial and let  $f_r$  be a sequence of cut-offs satisfying Hypothesis CO, such that  $f_r \equiv 1$  on the ball of radius  $r \in \mathbb{N}$  and such that  $f_r'(x) \leq C_1 \exp(-C_2(|x| - r))$  for some positive constants  $C_1, C_2 \in \mathbb{R}_+$  independent on  $r$ . Then*

$$\partial_t^k H^L(0) = \lim_{r \rightarrow +\infty} \partial_t^k F_{f_r}^L(0) = \partial_t^k F^L(0).$$

To make the proof easy to follow, we restrict ourselves to the scalar case, that is, the case where  $n = 1$ . The general case is a straightforward generalization. We will also need certain results about iterated Gaussian integrals. So let us introduce first some notation.

We denote by  $\mathcal{T}$  the set of rooted trees with at least a external vertex which is not the root. We denote by  $\tau_0$  the tree with only one vertex other than the root. In this set, we introduce two operations: if  $\tau \in \mathcal{T}$ , we denote by  $[\tau]$  the tree obtained from  $\tau$  by adding a new vertex at the root which becomes the new root, and if  $\tau' \in \mathcal{T}$  we denote by  $\tau \cdot \tau'$  the tree obtained by identifying the root of  $\tau$  and  $\tau'$ . It is possible to obtain any element of  $\mathcal{T}$  by applying iteratively a finite number of times the previous operations to  $\tau_0$ . Furthermore, we define  $\mathcal{I}_\tau^f(x) \in C^0(\mathbb{R}^2)$  by induction in the following way:

$$\begin{aligned} \mathcal{I}_{\tau_0}^f(x) &:= \mathcal{I}\xi, & \mathcal{I}_{[\tau]}^f(x) &:= \int_{\mathbb{R}^2} \mathcal{G}(x - y) f(y) \mathcal{I}_\tau^f(y) dy, \\ \mathcal{I}_{\tau \cdot \tau'}^f(x) &:= \mathcal{I}_\tau^f(x) \cdot \mathcal{I}_{\tau'}^f(x), \end{aligned}$$

where  $\mathcal{G}(x)$  is the Green function of the operator  $\mathcal{I} = (m^2 - \Delta)^{-1}$ . We need also to introduce the following notation. Suppose that  $\tau, \tau' \in \mathcal{T}$  and let  $\mathcal{P}_{\tau, \tau'}$  be the set of all possible pairing between the external vertices (except their roots) of the forest  $\tau \sqcup \tau'$  and let  $\mathcal{P}_{\tau, \tau'}^{\text{int}} \subset \mathcal{P}_{\tau, \tau'}$  the set of all possible pairing involving separately the vertices of  $\tau$  and  $\tau'$ . If  $\pi \in \mathcal{P}$ , we write

$$\mathfrak{J}_{\tau, \tau'}^{\pi, f}(x, y) = \mathbb{E}[\tilde{\mathcal{I}}_\tau^{\pi, f}(x) \cdot \tilde{\mathcal{I}}_{\tau'}^{\pi, f}(y)],$$

where  $\tilde{\mathcal{I}}_{\tau}^{\pi, f}(x)$ ,  $\tilde{\mathcal{I}}_{\tau'}^{\pi, f}(y)$  are the expression  $\mathcal{I}_{\tau}^f(x)$  where  $\xi$  is replaced by some copies of Gaussian white noises  $\xi_V$  one for each vertex  $V$  of  $\tau$  and  $\tau'$  which have correlation 0 if  $(V, V') \notin \pi$  and are identically correlated otherwise.

LEMMA 50. *With the notation and the hypotheses of Theorem 49, we have that for any  $\tau, \tau' \in \mathcal{T}$ ,*

$$\lim_{r \rightarrow +\infty} \left( \mathbb{E} \left[ \mathcal{I}_{\tau}^{f_r}(0) \cdot \prod_{i=1}^p \int f'_r(x) \mathcal{I}_{\tau_i}^{f_r}(x) dx \right] - \mathbb{E}[\mathcal{I}_{\tau}^{f_r}(0)] \cdot \mathbb{E} \left[ \prod_{i=1}^p \int f'_r(x) \mathcal{I}_{\tau_i}^{f_r}(x) dx \right] \right) = 0.$$

PROOF. We present the proof only for the case  $p = 1$ , since the general case is a straightforward generalization. Since  $\mathcal{I}_{\tau}^{f_r}$  are Gaussian random variables depending polynomially with respect to the white noise  $\xi$ , using the notation previously introduced we have

$$\begin{aligned} & \mathbb{E} \left[ \mathcal{I}_{\tau}^{f_r}(0) \cdot \int f'_r(x) \mathcal{I}_{\tau'}^{f_r}(x) dx \right] - \mathbb{E}[\mathcal{I}_{\tau}^{f_r}(0)] \cdot \mathbb{E} \left[ \int f'_r(x) \mathcal{I}_{\tau'}^{f_r}(x) dx \right] \\ &= \sum_{\pi \in \mathcal{P}_{\tau, \tau'} \setminus \mathcal{P}_{\tau, \tau'}^{\text{int}}} \int_{\mathbb{R}^2} \mathfrak{J}_{\tau, \tau'}^{\pi, f_r}(0, x) f'_r(x) dx. \end{aligned}$$

Let us consider the simplest case when  $\tau = \tau_k := [\dots[\tau_0]\dots]$   $k$  times and  $\tau' = \tau_{k'} = [\dots[\tau_0]\dots]$   $k'$  times. In this case, we have

$$\begin{aligned} \mathfrak{J}_{\tau, \tau'}^{\pi, f_r}(0, x) &= \int \mathcal{G}(0 - y_1) f_r(y_1) \cdots \mathcal{G}(y_k - x_1) \\ &\quad \times \mathcal{G}(x_1 - x_2) f_r(x_2) \cdots f_r(x_{k'}) \mathcal{G}(x_{k'} - x) dy_1 \cdots dy_k dx_1 \cdots dx_{k'}. \end{aligned}$$

In particular, since  $\mathcal{C}(x) = \mathcal{G} * \mathcal{G}$ , which is the Green function of  $\mathcal{I}^2 = (m^2 - \Delta)^{-2}$ , is bounded and positive, and since  $\mathcal{G}$  is positive we obtain that

$$|\mathfrak{J}_{\tau, \tau'}^{\pi, f_r}(0, x)| \leq \underbrace{\mathcal{G} * \dots * \mathcal{G}}_{k+k' \text{ times}}(0 - x) = \int_{\mathbb{R}^2} \frac{e^{-il \cdot x}}{(|l|^2 + m^2)^{k+k'}} dl.$$

Thus we get

$$|\mathfrak{J}_{\tau, \tau'}^{\pi, f_r}(0, x)| \cdot (|x|^2 + 1) \leq \left| \int_{\mathbb{R}^2} (-\Delta_l + 1) \frac{e^{-il \cdot x}}{(|l|^2 + m^2)^{k+k'}} dl \right| \leq C_3,$$

where  $C_3 \in \mathbb{R}_+$ . Thus

$$\begin{aligned} & \int_{\mathbb{R}^2} \mathfrak{J}_{\tau, \tau'}^{\pi, f_r}(0, x) f'_r(x) dx \\ & \leq \int_{B_r^c} \frac{C_3}{(|x|^2 + 1)} C_1 \exp(-C_2(|x| - r)) dx \lesssim_{C_1, C_2, C_3} \frac{1}{r^2 + 1} \rightarrow 0. \end{aligned}$$

For the general case, let us note that  $\mathfrak{J}_{\tau, \tau'}^{\pi, f_r}(0, x)$  is built by taking the product and the convolution with the functions  $\mathcal{G}$ ,  $f_r$  and  $\mathcal{C} = \mathcal{G} * \mathcal{G}$ . We note that  $\mathcal{C}$  appears one time for every pair of vertices  $(V_1, V_2) \in \pi$ . Then, since  $\pi \notin \mathcal{P}_{\tau, \tau'}^{\text{int}}$  there is at least a couple  $(V, V') \in \pi$  such that  $V$  is a vertex of  $\tau$  and  $V'$  is a vertex of  $\tau'$ . Now we can bound the function  $\mathcal{C}$  with a constant  $C_4$  for all pairs of vertices  $(V_1, V_2) \neq (V, V')$  and  $f_r$  by 1 obtaining, for any  $x \in \mathbb{R}^2$ , that

$$\mathfrak{J}_{\tau, \tau'}^{\pi, f_r}(0, x) \lesssim C_4^{k_1} \mathfrak{J}_{\tau_{k_2}, \tau_{k_3}}^{f_r}(0, x)$$

for some  $k_1, k_2, k_3 \in \mathbb{N}$ . The thesis follows from the previous inequality and the bounds obtained on  $\mathfrak{J}_{\tau_{k_2}, \tau_{k_3}}^{f_r}(0, x)$ .  $\square$

PROOF OF THEOREM 49. We write

$$\mathfrak{L}_{f_r}(t) := L(\phi_{f_r,t}(0)) \mathfrak{E}_{f_r}(t) := \exp\left(4t \int_{\mathbb{R}^2} f'_r(x) V(\phi_{f_r,t}(x)) dx\right).$$

We have

$$\begin{aligned} \partial_t^k F_{f_r}^L(t) &= \sum_{0 \leq l \leq k} \binom{k}{l} \mathbb{E}\left[\mathfrak{L}_{f_r}^{(k-l)}(0) \partial_t^l \left(\frac{\mathfrak{E}_{f_r}(t)}{\mathbb{E}[\mathfrak{E}_{f_r}(t)]}\right) \Big|_{t=0}\right] \\ &= \mathbb{E}[\mathfrak{L}_{f_r}^{(k)}(0)] + \sum_{1 \leq l \leq k} \sum_{0 \leq p \leq l-1} \binom{k}{l} \binom{l}{p} (\mathbb{E}[\mathfrak{L}_{f_r}^{(k-l)}(0) \cdot \mathfrak{E}_{f_r}^{(l-p)}(0)] \\ &\quad - \mathbb{E}[\mathfrak{L}_{f_r}^{(k-l)}(0)] \mathbb{E}[\mathfrak{E}_{f_r}^{(l-p)}(0)]) \cdot \partial_t^p \left(\frac{1}{\mathbb{E}[\mathfrak{E}_{f_r}(t)]}\right) \Big|_{t=0}, \end{aligned}$$

where we used the Leibniz rule for the derivative of the product and the relation

$$\partial_t^l \left(\frac{1}{\mathbb{E}[\mathfrak{E}_{f_r}(t)]}\right) \Big|_{t=0} = - \sum_{0 \leq p \leq l-1} \binom{l}{p} \mathbb{E}[\mathfrak{E}_{f_r}^{(l-p)}(0)] \cdot \partial_t^p \left(\frac{1}{\mathbb{E}[\mathfrak{E}_{f_r}(t)]}\right) \Big|_{t=0}.$$

Since  $\partial_t^p \left(\frac{1}{\mathbb{E}[\mathfrak{E}_{f_r}(t)]}\right) \Big|_{t=0}$  is bounded from above and below when  $r \rightarrow +\infty$  if we are able to prove that  $\mathbb{E}[\mathfrak{L}_{f_r}^{(k)}(0)] \rightarrow \partial_t^k F^L(0)$  and  $\mathbb{E}[\mathfrak{L}_{f_r}^{(k-l)}(0) \cdot \mathfrak{E}_{f_r}^{(l-p)}(0)] - \mathbb{E}[\mathfrak{L}_{f_r}^{(k-l)}(0)] \times \mathbb{E}[\mathfrak{E}_{f_r}^{(l-p)}(0)] \rightarrow 0$ , the theorem is proven.

First of all, we note that

$$(65) \quad (m^2 - \Delta) \partial_t^k \phi_{f_r,t} \Big|_{t=0} = k f_r \partial_t^{k-1} (V(\phi_{f_r,t})) \Big|_{t=0}$$

for  $k > 0$  and  $\phi_{f_r,0} = \mathcal{I}\xi$  for  $k = 0$ . This means that  $\mathfrak{L}_{f_r}^{(k-l)}(0)$ ,  $\mathfrak{E}_{f_r}^{(l-p)}(0)$  are given by a finite combination of convolutions and products between the function  $\mathcal{G}$  (i.e., the Green function of  $\mathcal{I}$ ), the functions of the form  $V^{(l)}(\phi_{f_r,0})$  (where  $V^{(l)}$  is the  $l$ th derivative of  $V$ ), the cut-off  $f_r$  and  $f'_r$ . Since  $V$  is a trigonometric polynomial, by developing  $V$  and its derivative by Taylor series, we obtain the following formal expressions:

$$(66) \quad \begin{aligned} \mathfrak{L}_{f_r}^{(k)}(0) &= \sum_{\tau \in \mathcal{T}} A_{\tau}^k \mathcal{I}_{\tau}^{f_r}(0), \\ \mathfrak{E}_{f_r}^{(k)}(0) &= \sum_l \sum_{\tau_1, \dots, \tau_l \in \mathcal{T}} B_{(\tau_1, \dots, \tau_l)}^{k,l} \prod_{i=1}^l \int_{\mathbb{R}^2} f'_r(x) \mathcal{I}_{\tau_i}^{f_r}(x) dx. \end{aligned}$$

The previous series are not only formal but they are actually absolutely convergent series. Furthermore, we can change the integral, the expectation and the limit with the series.

In order to prove this, we now note that there exist two positive constants  $C, \alpha > 0$  such that the function  $V$  is majorized (in the meaning of the majorants method) by  $C \exp(\alpha x)$  and let  $\tilde{L}$  be the polynomial which majorizes the polynomial  $L$ . We now consider  $\tilde{\mathfrak{L}}_{f_r}(t) = \tilde{L}(\phi_{f_r,t}(0))$  and  $\tilde{\mathfrak{E}}_{f_r}(t) = (tC \int_{\mathbb{R}^2} f'_r(x) \exp(\alpha \phi_{f_r,t}(x)) dx)$ . For what we said,  $\tilde{\mathfrak{L}}_{f_r}^{(k)}(0)$  and  $\tilde{\mathfrak{E}}_{f_r}^{(p)}(0)$  are a finite combination of convolutions and products between  $\mathcal{G}$ , the functions of the form  $V^{(l)}(\phi_{f_r,0})$  (where  $V^{(l)}$  is the  $l$ th derivative of  $V$ ), the cut-off  $f_r$  and  $f'_r$ . Let  $\tilde{\mathfrak{L}}_{f_r}^k$  and  $\tilde{\mathfrak{E}}_{f_r}^k$  be some random variables having the same expression of  $\tilde{\mathfrak{L}}_{f_r}^{(k)}(0)$  and  $\tilde{\mathfrak{E}}_{f_r}^{(p)}(0)$  where we replace every appearance of  $V(\phi_{f_r,0}(x))$  by  $C \exp(\alpha |\phi_{f_r,0}(x)|)$ , every appearance of  $V'(\phi_{f_r,0}(x))$  with

$C\alpha \exp(\alpha|\phi_{f_r,0}(x)|)$  and so on. We introduce the following functions dependent on  $\tau \in \mathcal{T}$  and defined recursively as follows:

$$\begin{aligned} \mathcal{J}_{\tau_0}^{f_r}(x) &:= |\mathcal{I}_{\tau_0}^{f_r}(x)| \mathcal{J}_{[\tau]}^{f_r}(x) := \int_{\mathbb{R}^2} \mathcal{G}(x-y) f_r(y) \mathcal{J}_{\tau}^{f_r}(y) dy, \\ \mathcal{J}_{\tau,\tau'}^{f_r}(x) &:= \mathcal{J}_{\tau}^{f_r}(x) \cdot \mathcal{J}_{\tau'}^{f_r}(x). \end{aligned}$$

We then obtain that

$$\hat{\mathcal{L}}_{f_r}^{(k)} = \sum_{\tau \in \mathcal{T}} \hat{A}_{\tau}^k \mathcal{J}_{\tau}^{f_r}(0) \hat{\mathcal{E}}_{f_r}^{(k)} = \sum_l \sum_{\tau_1, \dots, \tau_l \in \mathcal{T}} \hat{B}_{(\tau_1, \dots, \tau_l)}^{k,l} \prod_{i=1}^l \int_{\mathbb{R}^2} f'_r(x) \mathcal{J}_{\tau_i}^{f_r}(x) dx.$$

By our construction, we have that  $\hat{A}_{\tau}^k, \hat{B}_{\tau,i}^{k,l}$  are all greater or equal than zero and also the following inequalities hold  $|A_{\tau}^k| \leq \hat{A}_{\tau}^k, |B_{(\tau_1, \dots, \tau_l)}^{k,l}| \leq \hat{B}_{(\tau_1, \dots, \tau_l)}^{k,l}$ . Furthermore, we have  $|\mathcal{I}_{\tau}^{f_r}(x)| \leq \mathcal{J}_{\tau}^{f_r}(x)$ . Finally,  $\mathbb{E}[|\hat{\mathcal{L}}_{f_r}^{(k)}|^p], \mathbb{E}[|\hat{\mathcal{E}}_{f_r}^{(k)}|^p]$  are finite for any  $p$ , since the  $x_1, \dots, x_l$  function,

$$\mathbb{E} \left[ \exp \left( \beta \alpha \sum_{i=1}^l |\phi_{f_r,0}(x_i)| \right) \right] \leq +\infty,$$

for any  $\beta > 0$ . Since  $\mathcal{G}$  is positive, the bounds on  $\mathbb{E}[|\hat{\mathcal{L}}_{f_r}^{(k)}|^p], \mathbb{E}[|\hat{\mathcal{E}}_{f_r}^{(k)}|^p]$  can be chosen uniformly on  $r$ . This implies that the series (66) are absolutely convergent and by Lebesgue's dominated convergence theorem we can exchange the series with the summation and the limit. This means that

$$\begin{aligned} &\lim_{r \rightarrow +\infty} \mathbb{E}[\mathcal{L}_{f_r}^{(k)}(0) \cdot \mathcal{E}_{f_r}^{(l)}(0)] - \mathbb{E}[\mathcal{L}_{f_r}^{(k)}(0)] \mathbb{E}[\mathcal{E}_{f_r}^{(l)}(0)] \\ &= \lim_{r \rightarrow +\infty} \sum_{l \in \mathbb{N}, \tau, \tau_1, \dots, \tau_l \in \mathcal{T}} A_{\tau}^k B_{(\tau_1, \dots, \tau_l)}^{k,l} \left( \mathbb{E} \left[ \mathcal{I}_{\tau}^{f_r}(0) \cdot \prod_{i=1}^l \int f'_r(x) \mathcal{I}_{\tau_i}^{f_r}(x) dx \right] \right. \\ &\quad \left. - \mathbb{E}[\mathcal{I}_{\tau}^{f_r}(0)] \cdot \mathbb{E} \left[ \prod_{i=1}^l \int f'_r(x) \mathcal{I}_{\tau_i}^{f_r}(x) dx \right] \right) \\ &= \sum_{l \in \mathbb{N}, \tau, \tau_1, \dots, \tau_l \in \mathcal{T}} A_{\tau}^k B_{(\tau_1, \dots, \tau_l)}^{k,l} \lim_{r \rightarrow +\infty} \left( \mathbb{E} \left[ \mathcal{I}_{\tau}^{f_r}(0) \cdot \prod_{i=1}^l \int f'_r(x) \mathcal{I}_{\tau_i}^{f_r}(x) dx \right] \right. \\ &\quad \left. - \mathbb{E}[\mathcal{I}_{\tau}^{f_r}(0)] \cdot \mathbb{E} \left[ \prod_{i=1}^l \int f'_r(x) \mathcal{I}_{\tau_i}^{f_r}(x) dx \right] \right) = 0, \end{aligned}$$

where in the last line we used Lemma 50. In a similar way, it is simple to prove that

$$\mathbb{E}[\mathcal{L}_{f_r}^{(k)}(0)] \rightarrow \partial_t^k F^L(0),$$

and this concludes the proof.  $\square$

**PROOF OF THEOREM 4.** Using the bounds (63) and (64), we can prove the existence of strong solutions to equation (10), and their uniqueness when  $V$  satisfies Hypothesis C.

Furthermore, using again the bounds (63) and (64) and a suitable generalization of Lemma 21, Lemma 24, Lemma 25, we can prove that Theorem 4 holds for any potential satisfying Hypothesis C if and only if Theorem 4 holds for trigonometric potentials satisfying Hypothesis C.

The fact that Theorem 4 holds for trigonometric potentials, satisfying Hypothesis C, is a consequence of Theorem 49.  $\square$

APPENDIX A: TRANSFORMATIONS IN ABSTRACT WIENER SPACES

This Appendix summarizes the results of [59] which are used in the paper and establish the related notation. Hereafter, we consider an abstract Wiener space  $(W, H, \mu)$  where  $W$  is a separable Banach space,  $H$  is an Hilbert space densely and continuously embedded in  $W$  (with inclusion map denoted by  $i : H \rightarrow W$ ) called the Cameron–Martin space and  $\mu$  is the Gaussian measure on  $W$  associated with the Cameron–Martin space, that is,  $\mu$  is the centered Gaussian measure on  $W$  such that for any  $w^* \in W^*$  we have  $\hat{\mu}(w^*) = \int_W \exp(i \langle w^*, w \rangle) d\mu(w) = \exp(-\frac{\|i^*(w^*)\|_H^2}{2})$  where  $i^* : W^* \rightarrow H$  is the dual operator of  $i$ .

If  $u : W \rightarrow \mathbb{R}$  is a measurable nonlinear functional, we denote by  $\nabla u : W \rightarrow H$  the following linear operator:

$$\nabla u(w)[h] = \langle \nabla u(w), h \rangle_H := \lim_{\epsilon \rightarrow 0} \frac{u(w + \epsilon h) - u(w)}{\epsilon}.$$

The operator  $\nabla$  is called a Malliavin derivative and it is possible to prove that it is closable (with unique closure) on the set of measurable  $L^p(\mu)$  functions. We denote the domain of  $\nabla$  in  $L^p(\mu)$  by  $\mathbb{D}_{p,1}$ . The previous operation can be extended for functional  $u : W \rightarrow H^{\otimes k}$  where  $\nabla u : W \rightarrow H^{\otimes k+1}$  with its natural topology. Also, this extension of the operator  $\nabla$  is closable.

If the measurable nonlinear operator  $F : W \rightarrow H$ , where  $|F|_H \in L^p(\mu)$ , is such that  $\mathbb{E}[\langle F, \nabla u \rangle_H] = \mathbb{E}[\tilde{F}u]$  for some  $\tilde{F} \in L^p(\mu)$ , we say that  $F$  is in the domain of the operator  $\delta$  and we denote by  $\delta(F) = \tilde{F} \in L^p(\mu)$  the Skorokhod integral of the measurable operator  $F$ . The following expression for  $\delta(F)$  used in the following holds: suppose that  $F(w) \in i^*(W^*)$  and that  $\nabla F(w)$  is a trace class operator on  $H$  for  $\mu$  almost every  $w \in W$  then

$$(67) \quad \delta(F)(w) = \langle i^{*, -1}(F(w)), w \rangle - \text{Tr}(\nabla F(w)).$$

We introduce a definition for studying the random transformations defined on abstract Wiener spaces.

DEFINITION 51. Let  $U : W \rightarrow H$  be a measurable map. We say that  $U$  is a  $H - C^1$  map if for  $\mu$  almost every  $w \in W$  the map  $U_w : H \rightarrow H$ , defined as  $h \mapsto U_w(h) := U(w + h)$ , is a Fréchet differentiable function in  $H$  and if  $\nabla U_w : H \rightarrow H^{\otimes 2}$ , defined as  $h \mapsto \nabla U_w(h) := \nabla U(w + h)$  where  $\nabla$  is the Malliavin derivative, is continuous for almost every  $w \in W$  and with respect to the natural (Hilbert–Schmidt) topology of  $H^{\otimes 2}$ .

We introduce the shift  $T : W \rightarrow W$  associated with  $U$ , that is, the map defined as  $T(w) = w + U(w)$ , and the nonlinear functional  $\Lambda_U : W \rightarrow \mathbb{R}$  as follows:

$$(68) \quad \Lambda_U(w) = \det_2(I_H + \nabla U(w)) \exp\left(-\delta(U)(w) - \frac{1}{2}|U(w)|_H^2\right),$$

where  $\det_2(I_H + K)$  is the regularized Fredholm determinant (see [56] Chapter 9) that it is well defined for any Hilbert–Schmidt operator  $K$ . In particular, if  $K$  is self-adjoint, we have

$$\det_2(I + K) = \prod_{i \in \mathbb{N}} (1 + \lambda_i) e^{-\lambda_i},$$

where  $\lambda_i$  are the eigenvalues of the operator  $K$ .

Suppose that  $U(w) \in i^*(W)$  and that  $\nabla U(w)$  is a trace class operator for almost any  $w \in W$ , then using the expression (67) and the properties of  $\det_2$  we obtain

$$(69) \quad \Lambda_U(w) = \det(I_H + \nabla U(w)) \exp\left(-\langle i^{*, -1}(U(w)), w \rangle_W - \frac{1}{2}|U(w)|_H^2\right),$$

where  $\det(I_H + K)$  is the standard Fredholm determinant. The functional  $\Lambda_U$  is closely related to the transformation of the measure  $\mu$  with respect to the transformation  $T$ . Indeed suppose that  $W$  is finite dimensional, then we have

$$d\mu = \exp\left(-\frac{1}{2}\langle w, w \rangle_H\right) \frac{dx}{Z} = \exp\left(-\frac{1}{2}\langle i^{*-1}(w), w \rangle_W\right) \frac{dx}{Z},$$

where  $Z \in \mathbb{R}_+$  is a suitable normalization constant and  $dx$  is the Lebesgue measure on  $W$ . Thus, if  $T$  is a diffeomorphism on  $W$ , we evidently have thanks to equation (69),

$$\frac{dT_*(\mu)}{d\mu} = \left| \det(I + \nabla U(w)) \exp\left(-\langle i^{*-1}(U(w)), w \rangle_W - \frac{1}{2}\langle i^{*-1}(U(w)), U(w) \rangle_W\right) \right| = |\Lambda_U(w)|.$$

The previous relation can be extended to the case where  $W$  and  $H$  are infinite dimensional and the transformation  $T$  is not a diffeomorphism but it is only a  $H - C^1$  map.

First of all, we need the following generalization to the abstract Wiener space context of the finite dimensional Sard lemma.

**PROPOSITION 52.** *Let  $T(w) = w + U(w)$  be a  $H - C^1$  map and let  $M \subset W$  be the set of the zeros of  $\det_2(I + \nabla U(w))$ , then the  $\mu$  measure of the set  $T(M)$  is zero, that is,  $\mu(T(M)) = 0$ .*

**PROOF.** See Theorem 4.4.1 [59].  $\square$

The following is the change of variable theorem for (generally not invertible)  $H - C^1$  maps.

**THEOREM 53.** *Let  $T(w) = w + U(w)$  be a  $H - C^1$  map and let  $f, g$  be two positive measurable functions, then*

$$(70) \quad \int_W f \circ T(w)g(w)|\Lambda_U(w)| d\mu(w) = \int_W f(w)\left(\sum_{y \in T^{-1}(w)} g(y)\right) d\mu(w).$$

*In particular, if  $K \subset W$  is a measurable subset where  $T|_K$  is invertible, we have*

$$\int_K f \circ T(w)|\Lambda_U(w)| d\mu(w) = \int_{T(K)} f(w) d\mu(w).$$

**PROOF.** See Theorem 4.4.1 [59].  $\square$

In order to prove Theorem 17, and so the relationship between the weak solutions to equation (6) and the integrals with respect to the signed measure  $\Lambda_U d\mu$ , it is not enough to consider Theorem 53 but we need a relationship analogous to (70) with  $|\Lambda_U|$  replaced by  $\Lambda_U$ . In order to achieve this result, we need some more hypotheses on the map  $U$ :

*Hypothesis DEG1* The map  $U : W \rightarrow H \hookrightarrow W$  is a Fréchet differentiable map from  $W$  into itself and, furthermore, it is  $C^1$  with respect to the usual topology of  $W$ .

*Hypothesis DEG2* The map  $T$  is proper (i.e., inverse images of compact subsets are compact) and the equation  $T^{-1}(y) = w$  has a finite number of solution  $y$  for  $\mu$  almost every  $w \in W$ .



Under the Hypothesis DEG1 and DEG2, we can define the following number:

$$\text{DEG}(w, T) := \sum_{y \in T^{-1}(w)} \text{sign}(\det_2(I_W + \nabla U(y))).$$

This index is a suitable modification of the Leray–Schauder degree of a Fredholm nonlinear operator described, for example, in [11] Section 5.3C, where the following definition is given: if  $B$  is a bounded set of  $W$  such that  $T^{-1}(w) \cap \partial B = \emptyset$  and  $\nabla T(y) \neq 0$  for  $y \in T^{-1}(w) \cap B$  we have

$$\text{DEG}_B(w, T) = \sum_{y \in T^{-1}(w) \cap B} (-1)^{(\text{number of negative eigenvalues of } \nabla T(y))}.$$

It is evident that under the Hypothesis DEG2 and, as a consequence of Proposition 52, we have

$$\lim_{B \rightarrow W} \text{DEG}_B(w, T) = \text{DEG}(w, T)$$

for almost all  $w \in W$ .

**THEOREM 54.** *Under the Hypotheses DEG1 and DEG2, we have that  $\text{DEG}(w, T)$  is  $\mu$  almost surely equal to the constant  $\text{DEG}(T) \in \mathbb{Z}$  independent of  $w$  and for any bounded function  $f$  such that  $f \circ T \cdot \Lambda_U \in L^1(\mu)$  we have*

$$\int_W f \circ T(w) \Lambda_U(w) \, d\mu(w) = \text{DEG}(T) \cdot \int_W f(w) \, d\mu(w).$$

**PROOF.** The proof can be found in [59] Theorem 9.4.1 and Theorem 9.4.6.  $\square$

In general, it is not simple to compute  $\text{DEG}(T)$  but this computation is simplified under the following hypothesis.

*Hypothesis DEG3* The map  $T_\epsilon(w) = w + \epsilon U(w)$  has bounded level set uniformly in  $\epsilon \in [0, 1]$ , that is, if  $B \subset W$  is bounded  $\bigcup_{\epsilon \in [0,1]} T_\epsilon^{-1}(B)$  is a bounded set in  $W$ .

**THEOREM 55.** *Under the Hypotheses DEG1, DEG2 and DEG3 we have that, for any  $\epsilon \in [0, 1]$ :*

$$\text{DEG}(T) = \text{DEG}(w, T) = \text{DEG}(w, T_\epsilon) = 1.$$

**PROOF.** The proposition follows from the invariance of  $\text{DEG}_B$  under homotopies of the operator  $T$ . In other words for any  $B$ , such that  $T_\epsilon^{-1}(w) \cap \partial B = \emptyset$  we have  $\text{DEG}_B(w, T_\epsilon) = \text{DEG}_B(w, T)$ . Under the Hypothesis DEG3, we can choose  $B$  big enough such that  $\text{DEG}_B(w, T_\epsilon) = \text{DEG}(w, T_\epsilon)$  for any  $\epsilon \in [0, 1]$ . Since  $\text{DEG}(w, T_0) = \text{DEG}(w, \text{id}_W) = 1$ , the thesis follows.  $\square$

### APPENDIX B: FERMIONIC FIELDS

In this Appendix, we introduce the notion of fermionic fields used in Section 4 and Section 5. For a more detailed discussion about this subject, see [15–17].

We consider a quantum probability space  $(\mathfrak{H}, \rho)$ , where  $\mathfrak{H}$  is a separable Hilbert space and  $\rho$  is a positive trace class operator. If  $K \in \mathcal{B}(\mathfrak{H})$  (where  $\mathcal{B}(\mathfrak{H})$  is the Hilbert space of bounded operators defined on  $\mathfrak{H}$ ), we define  $\langle K \rangle = \text{Tr}(K \cdot \rho)$ .

Let  $H$  be a Hilbert space, we consider two continuous linear maps  $\psi, \bar{\psi} : H \rightarrow \mathcal{B}(\mathfrak{H})$  such that for any  $f_1, f_2 \in H$  we have

$$\{\psi(f_1), \psi(f_2)\} = \{\bar{\psi}(f_1), \bar{\psi}(f_2)\} = \{\psi(f_1), \bar{\psi}(f_2)\} = 0,$$

where  $\{K_1, K_2\} = K_1 \cdot K_2 + K_2 \cdot K_1$  is the anticommutator of the operators  $K_1, K_2 \in \mathcal{B}(\mathfrak{H})$ .

DEFINITION 56. Using the previous notation, the two antisymmetric fields  $\psi, \bar{\psi} : H \rightarrow \mathcal{B}(\mathfrak{H})$  are called fermionic fields associated with the Hilbert space  $H$  if we have that

$$(71) \quad \langle \bar{\psi}(f_1)\psi(g_1) \dots \bar{\psi}(f_n)\psi(g_n) \rangle = \det(\langle f_i, g_j \rangle).$$

The following theorem ensure the existence of a pair of fermionic fields for each separable Hilbert space  $H$ .

THEOREM 57. For any separable Hilbert space  $H$ , there exists a quantum probability space  $(\mathfrak{H}, \rho)$  and two continuous linear maps  $\psi, \bar{\psi} : H \rightarrow \mathcal{B}(\mathfrak{H})$  such that  $\psi, \bar{\psi}$  are a pair of fermionic fields associated with  $H$ . Furthermore, we have

$$(72) \quad \|\psi(f)\|_{\mathcal{B}(\mathfrak{H})}, \|\bar{\psi}(f)\|_{\mathcal{B}(\mathfrak{H})} \leq 2\|f\|_H$$

(we use the notation  $\|\cdot\|_{\mathfrak{H}}$  for the norm in a Hilbert space  $\mathfrak{H}$ ).

PROOF. By standard results of quantum fields theory (see, e.g., [8], Chapter 2), there are four operators  $a, a^*, b, b^* : H \rightarrow \mathcal{B}(\mathfrak{H})$  (formed by two independent pairs of anticommuting creation  $a, b$  and anticommuting adjoint annihilation  $a^*, b^*$  operators) such that

$$\begin{aligned} \{a(f), a(g)\} &= \{b(f), b(g)\} = 0, \\ \{a(f), b(g)\} &= \{a^*(f), b(g)\} = 0, \\ \{a^*(g), a(f)\} &= \{b^*(g), b(f)\} = \langle f, g \rangle_H I_{\mathfrak{H}}, \end{aligned}$$

and such that

$$\langle a(f)K \rangle = \langle Ka^*(f) \rangle = \langle b(f)K \rangle = \langle Kb^*(f) \rangle = 0$$

for any  $f, g \in H$  and any bounded operator  $K \in \mathcal{B}(\mathfrak{H})$ . Consider now

$$\psi(f) = a^*(f) + b(f), \quad \bar{\psi}(f) = b^*(f) - a(f),$$

where  $f \in H$ . We want to prove that  $\psi, \bar{\psi}$  are the two fermionic fields associated with  $H$  fields of the thesis of the theorem. Obviously,  $\{\psi(f), \bar{\psi}(g)\} = \{\psi(f), \psi(g)\} = \{\bar{\psi}(f), \bar{\psi}(g)\} = 0$ , so we have only to prove that  $\psi, \bar{\psi}$  satisfy equality (71) and inequality (72).

We prove equality (71) by induction on  $n$ . By the properties of  $a, a^*, b, b^*$ , we have

$$\begin{aligned} \langle \bar{\psi}(f_1)\psi(g_1) \rangle &= \langle b^*(f_1)a^*(g_1) \rangle + \langle b^*(f_1)b(g_1) \rangle - \langle a(f_1)a^*(g_1) \rangle \\ &\quad - \langle a(f_1)b(g_1) \rangle = \langle f_1, g_1 \rangle_H. \end{aligned}$$

Suppose that  $\langle \bar{\psi}(f_1)\psi(g_1) \dots \bar{\psi}(f_{n-1})\psi(g_{n-1}) \rangle = \det(\langle f_i, g_j \rangle_H)$  and we want to prove the same equality for  $n$  operators. We have

$$\begin{aligned} &\langle \bar{\psi}(f_1)\psi(g_1) \dots \bar{\psi}(f_n)\psi(g_n) \rangle \\ &= \langle b^*(f_1)\psi(g_1) \dots \bar{\psi}(f_n)\psi(g_n) \rangle \\ &= \sum_{k=1}^n (-1)^k \langle b^*(f_1)b(g_k) \rangle \langle \bar{\psi}(f_2)\psi(g_1) \dots \bar{\psi}(f_k)\psi(g_k) \dots \bar{\psi}(f_n)\psi(g_n) \rangle \\ &= \sum_{k=1}^n (-1)^k \langle f_1, g_k \rangle_H \det(\langle f_i, g_j \rangle_{i \neq 1, j \neq k}) = \det(\langle f_i, g_j \rangle), \end{aligned}$$

where we use the commutation relations of  $a^*$  with  $a, b, b^*$ , the induction hypothesis and the properties of determinant. Since

$$\|a(f)\|_{\mathcal{B}(\mathfrak{H})} = \|a^*(f)\|_{\mathcal{B}(\mathfrak{H})} = \|b(f)\|_{\mathcal{B}(\mathfrak{H})} = \|b^*(f)\|_{\mathcal{B}(\mathfrak{H})} = \|f\|_H,$$

$\psi, \bar{\psi}$  satisfy inequality (72).  $\square$

Suppose that  $i : H \hookrightarrow C^0(\mathbb{R}^2)$  for some continuous injection  $i$ , then by the identification of  $H$  with its dual we have that  $i^*(\delta_x) \in H$ , where  $\delta_x \in (C^0(\mathbb{R}^2))^*$  is the Dirac delta with mass in  $x \in \mathbb{R}^2$ . In this way, we can define  $\psi, \bar{\psi}$  as continuous functions of the point  $\mathbb{R}^2$  in the following way:

$$\psi(x) := \psi(i^*(\delta_x)) \bar{\psi}(x) := \bar{\psi}(i^*(\delta_x))$$

and the corresponding covariance function as

$$S(x; x') = \langle \bar{\psi}(x') \psi(x) \rangle.$$

Hereafter, we suppose that  $S(x; x')$  is a continuous function of the form  $S(x; x') = S(x - x') \geq 0$ . In this case, if  $g \in L^1(\mathbb{R}^2)$ , by Theorem 57 we have  $\|\psi(x) \bar{\psi}(x)\|_{\mathcal{B}(\mathfrak{H})} \leq 2S(0)$ , and thus  $\int_{\mathbb{R}^2} g(x) \bar{\psi}(x) \psi(x) dx$  is a bounded well-defined operator.

Under the previous condition, the operator  $\mathfrak{K}_g : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ , defined as  $\mathfrak{K}_g(h)(x) = \int g(x) S(x - x') h(x') dx'$ , is trace class since

$$\text{Tr}(|\mathfrak{K}_g|) \leq \int_{\mathbb{R}^2} |g(x)| \text{Tr}(|\bar{\psi}(x) \psi(x) \rho|) dx \leq 2S(0) \|g\|_{L^1(\mathbb{R}^2)} < +\infty.$$

This means that the Fredholm determinant (see [56], Chapter 3)  $\det(I + \mathfrak{K}_g)$  is well defined and finite. Furthermore, we have the following representation.

**THEOREM 58.** *Under the previous hypotheses and notation, we have*

$$\left\langle \exp\left(\int_{\mathbb{R}^2} g(x) \bar{\psi}(x) \psi(x) dx\right) \right\rangle = \det(I + \mathfrak{K}_g).$$

**PROOF.** By Definition 56 and the definition of the function  $S$ , we have that

$$\begin{aligned} & \left\langle \left(\int_{\mathbb{R}^2} g(x) \bar{\psi}(x) \psi(x) dx\right)^n \right\rangle \\ &= \int_{\mathbb{R}^{2n}} g(x_1) \dots g(x_n) \det(S(x_i - x_j)) dx_1 \dots dx_n \\ &= \int_{\mathbb{R}^{2n}} \det \begin{pmatrix} g(x_1)S(x_1 - x_1) & \dots & g(x_1)S(x_1 - x_n) \\ \vdots & \ddots & \vdots \\ g(x_n)S(x_n - x_1) & \dots & g(x_n)S(x_n - x_n) \end{pmatrix} dx_1 \dots dx_n. \end{aligned}$$

On the other hand, when  $S$  is continuous, by Theorem 3.10 of [56], we have that

$$\begin{aligned} & \det(I + \mathfrak{K}_g) \\ &= \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{\mathbb{R}^{2n}} \det \begin{pmatrix} g(x_1)S(x_1 - x_1) & \dots & g(x_1)S(x_1 - x_n) \\ \vdots & \ddots & \vdots \\ g(x_n)S(x_n - x_1) & \dots & g(x_n)S(x_n - x_n) \end{pmatrix} dx_1 \dots dx_n. \end{aligned}$$

The thesis follows.  $\square$

**REMARK 59.** The fermionic fields considered in Section 4 and Section 5, where  $S = \varpi \mathcal{G}_{1+2\chi}(x - x')$ ,  $H = W^{1+2\chi, 2}(\mathbb{R}^2)$  with norm  $\|f\|_H^2 = \int_{\mathbb{R}^2} (-\Delta + m^2)^{1+2\chi}(f)(x) f(x) dx$ , satisfies all the hypotheses of Theorem 58.

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