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EXCHANGEABLE INTERVAL HYPERGRAPHS AND LIMITS OF ORDERED DISCRETE STRUCTURES

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A hypergraph (V, E) is called an interval hypergraph if there exists a linear order l on V such that every edge $e \in E$ is an interval w.r.t. l; we also assume that $\{j\} \in E$ for every $j \in V$. Our main result is a de Finetti-type representation of random exchangeable interval hypergraphs on $\mathbb{N}$ (EIHs): the law of every EIH can be obtained by sampling from some random compact subset K of the triangle $\{(x, y) : 0 \le x \le y \le 1\}$ at i.i.d. uniform positions $U_1, U_2, \dots$, in the sense that, restricted to the node set $[n] := \{1, \dots, n\}$ every nonsingleton edge is of the form $e = \{i \in [n] : x < U_i < y\}$ for some $(x, y) \in K$. We obtain this result via the study of a related class of stochastic objects: erased-interval processes (EIPs). These are certain transient Markov chains $(I_n, \eta_n)_{n\in\mathbb{N}}$ such that I_n is an interval hypergraph on $V = [n]$ w.r.t. the usual linear order (called interval system). We present an almost sure representation result for EIPs. Attached to each transient Markov chain is the notion of Martin boundary. The points in the boundary of EIPs can be seen as limits of growing interval systems. We obtain a one-to-one correspondence between these limits and compact subsets K of the triangle with $(x, x) \in K$ for all $x \in [0, 1]$.

Interval hypergraphs are a generalizations of hierarchies and as a consequence we obtain a representation result for exchangeable hierarchies, which is close to a result of Forman, Haulk and Pitman in (*Probab. Theory Related Fields* **172** (2018) 1–29). Several ordered discrete structures can be seen as interval systems with additional properties, that is, Schröder trees (rooted, ordered, no node has outdegree one) or even more special: binary trees. We describe limits of Schröder trees as certain tree-like compact sets. These can be seen as an ordered counterpart to real trees, which are widely used to describe limits of discrete unordered trees. Considering binary trees, we thus obtain a homeomorphic description of the Martin boundary of Rémy's tree growth chain, which has been analyzed by Evans, Grübel and Wakolbinger in (*Ann. Probab.* **45** (2017) 225–277).

1. Introduction. The classical de Finetti representation theorem can be stated as follows: *the law of every exchangeable* $\{0, 1\}$*-valued stochastic processes can be expressed as a mixture of laws of i.i.d. processes*. More precisely, for any law P of exchangeable $\{0, 1\}$-valued processes there exists a unique Borel probability measure μ on $[0, 1]$ such that $P = \int_{[0,1]} \mathrm{Bin}(1, p)^{\otimes\mathbb{N}}\, d\mu(p)$. Here, $\mathrm{Bin}(n, p)$ denotes the binomial distribution and $Q^{\otimes\mathbb{N}} = Q \otimes Q \otimes \cdots$ is the law of an i.i.d. sequence with marginal distribution Q. De Finetti's theorem has been generalized in many different directions. With the present paper, we contribute to the growing list of de Finetti-type representation theorems by studying *exchangeable interval hypergraphs on* $\mathbb{N}$. Up to now, the list of (combinatorial) structures whose attached exchangeability structure have been analyzed includes sequences, partitions, graphs, general arrays (see [19]) and more recently, hierarchies (total partitions) by Forman,

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Haulk and Pitman [10]. A more complete list concerning exchangeability in combinatorial structures can be found in [10], Section 1.2.

We are also interested in topological aspects of representation results, and we illustrate this in the context of de Finettis's theorem: every $\{0, 1\}$-valued exchangeable process is a mixture of i.i.d. processes and the laws of i.i.d. processes on $\{0, 1\}$ are parametrized by the set $[0, 1]$, where the parameter $p \in [0, 1]$ is the success probability. The unit interval can be considered, as usual, as a topological space and this topology is the right one: the map $p \mapsto \mathrm{Bin}(1, p)^{\otimes \mathbb{N}}$ from the unit interval to the set of all laws of i.i.d. $\{0, 1\}$-valued processes is not only bijective, but a *homeomorphism*, if the latter space is equipped with its natural topology of weak convergence. Topological considerations may not be very complex in this case, but in other cases (e.g., graphs, hierarchies or interval hypergraphs instead of sequences) $[0, 1]$ gets replaced with more complex spaces and accurate topological descriptions become a more challenging task. We not only present a de Finetti-type representation result but also analyze topological properties of the space of laws of exchangeable interval hypergraphs and use the language of *Choquet simplex theory* in our studies, to which we give a short introduction in Section 1.7.

1.1. *Interval hypergraphs and interval systems.* For a set V, let $\mathcal{P}(V)$ be the power set of V. A *hypergraph* is a pair $\mathtt{H} = (V, E)$ where V is the set of nodes and $E \subseteq \mathcal{P}(V)$ is the set of (hyper)edges. We only consider the case where $\{j\} \in E$ for each $j \in V$. In such a case, one can identify $\mathtt{H}$ with its set of edges E. For $n \in \mathbb{N} = \{1, 2, \dots\}$, we write $[n] := \{1, \dots, n\}$. For a linear order l on a set V, we write xly if x is smaller than y w.r.t. l. An interval with respect to l is a subset $e \subseteq V$ such that for every $x, y \in e$ and $z \in V$ the implication $xlz \wedge zly \Rightarrow z \in e$ holds.

DEFINITION 1.1. A set $\mathtt{H} \subseteq \mathcal{P}([n])$ is called *interval hypergraph on* $[n]$ if:

(i) $\varnothing \in \mathtt{H}$ and $\{j\} \in \mathtt{H}$ for all $j \in [n]$,
(ii) there exists a linear order l on $[n]$ such that every $e \in \mathtt{H}$ is an interval with respect to l.

For a linear order l on $[n]$, we write $\mathrm{InHy}(n, l)$ for the set of all interval hypergraphs $\mathtt{H}$ on $[n]$ such that each $e \in \mathtt{H}$ is an interval with respect to l. If $\mathtt{H} \in \mathrm{InHy}(n, l)$, we say that $\mathtt{H}$ is an interval hypergraph with respect to l. Let

$$\mathrm{InHy}(n) := \bigcup_l \mathrm{InHy}(n, l) \quad \text{and} \quad \mathrm{InSy}(n) := \mathrm{InHy}(n, <),$$

where $1 < 2 < \cdots < n$ is the usual linear order. $\mathrm{InHy}(n)$ is the set of all interval hypergraphs on $[n]$. Elements in $\mathrm{InSy}(n)$ are called *interval systems* and we use the variable $\mathtt{I} \in \mathrm{InSy}(n) \subset \mathrm{InHy}(n)$ instead of $\mathtt{H}$ when talking about interval systems.

Note that every nonempty set $e \in \mathtt{I}$ in an interval system $\mathtt{I} \in \mathrm{InSy}(n)$ is of the form $e = [a, b] = \{a, a+1, a+2, \dots, b\}$ for some $1 \le a \le b \le n$. See Figure 1 for an example of an interval hypergraph on [5] (only edges with $|e| \ge 2$ are shown).

Interval hypergraphs have been studied from a combinatorial point of view before by Moore [27]. We think that interval hypergraphs and interval systems are interesting to study because they generalize classical combinatorial structures: Interval hypergraphs are a generalization both of hierarchies and of partitions and interval systems—the ordered version of interval hypergraphs—are a generalization of Schröder trees (and hence of rooted plane binary trees) and of compositions. We explore these connections in Section 5 and direct the reader to page 1158, Figure 10 to see how one can represent a Schröder tree as a interval system.

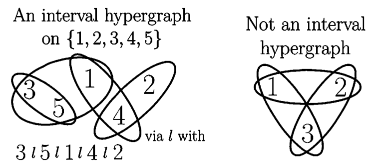

FIG. 1. *On the left an interval hypergraph on* [7], *on the right a subset of* $\mathcal{P}([3])$ *that is not an interval hypergraph. In both examples singleton sets* $\{j\}$ *are meant to be included.*

REMARK 1.2. One can identify interval systems on $[n]$ and simple graphs with node set $[n]$ by mapping discrete intervals $[a, b]$ with $1 \le a < b \le n$ to edges (=two-element sets) $\{a, b\}$. In particular, $|\operatorname{InSy}(n)| = 2^{\binom{n}{2}}$. However, the way we deal with interval systems is not very graph-like, but rather tree-like. We therefore think that the objects should be understood as a generalization of ordered trees rather than as graphs.

REMARK 1.3. A graph (V, E) is called *interval graph*, if there exists a collection of intervals $I_j \subset (0, 1)$, $j \in V$ such that $\{i, j\} \in E$ iff $I_j \cap I_i \neq \varnothing$. Interval graph limits have been studied by Diaconis, Holmes and Janson [5]. They used an identification of points $(x, y) \in \mathbb{R}^2$ and intervals $(x, y) \subset \mathbb{R}$ in a similar way that we will do later. Nevertheless, the questions and answers concerning interval graph limits and our studies do not seem to overlap very much. This can be seen in the way one measures the sizes of the discrete objects: With interval hypergraphs, the size equals the number of atoms, whereas in interval graphs the size equals the total number of intervals.

1.2. *Motivation and related work.* Before we introduce exchangeable interval hypergraphs (Definition 1.6), we mention two important publications, since our results can be seen as an extension and topological refinement of these works. We think that this serves as a main motivation for our studies, in particular as the structural connections we are about to explore can be found in other combinatorial structures as well.

I. Forman, Haulk and Pitman [10] investigate *exchangeable hierarchies*, that is, random projective sequences $(H_n)_{n\in\mathbb{N}}$ such that each H_n is a hierarchy on $[n]$ (total partition) that has an exchangeable law, and present two versions of de Finetti-type representation results. The first involves a procedure in which one samples from random real trees at random leaves and induce combinatorial subtrees. Using that real trees can be encoded by excursion functions they obtained a different version of that procedure in which one samples i.i.d. uniforms and a certain random set of intervals first. Our representation result for exchangeable interval hypergraphs generalizes this, since every hierarchy is an interval hypergraph as well. We prove our representation theorem through the study of a related class of stochastic processes, so-called *erased-interval processes* (EIPs), which are certain transient Markov chains $(I_n, \eta_n)_{n\in\mathbb{N}}$ such that $I_n \in \operatorname{InSy}(n)$ for each n (Definition 1.8).

II. Evans, Grübel and Wakolbinger [7] investigate the Martin boundary of *Rémy's tree growth chain* (RTGC), a famous Markov chain $(T_n)_{n\in\mathbb{N}}$ producing growing uniform rooted plane binary trees (counted by Catalan numbers). RTGC is relevant because it yields a practical algorithm for sampling uniform binary trees and because the normalized trees converge *almost surely* toward the plane Brownian CRT (i.e., the normalized exploration paths converge a.s. toward the Brownian excursion). In some sense, the Martin boundary of a transient Markov chain can be seen as the (topological) space of all possible ways the Markov chain "can go to ∞." The authors show that one can describe all limits of RTGC by a certain class of exchangeable random objects and offer a de Finetti-type representation theorem involving sampling from real trees very similar to [10]. Additionally, their description involves some kernel functions to introduce the *plane embedding* of the sampled combinatorial trees. We extend their studies beyond binary trees to the case of interval systems by introducing the formerly mentioned EIPs. We present an explicit homeomorphic description of Martin boundaries attached to EIPs and in particular, we present a homeomorphic description of the Martin boundary of RTGC involving a topological space of certain binary tree-like compact subsets of the triangle

$$\triangledown := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\}.$$

Applying these results to RTGC, we obtain a new representation of the plane Brownian CRT as a random compact binary tree-like subset of $\triangledown$ (Section 5.5). The set $\triangledown$ will play an important part throughout this paper.

In Section 5, we explain the connections of [7, 10] and our work in detail. These connections are of the same nature as the connections between *exchangeable partitions* and *composition structures*, whose attached representation theorems have been proven by Kingman [20] and Gnedin [15]. We explain how to deduce these results from our main statements in Section 5 as well.

1.3. *Exchangeable interval hypergraphs.* Let $\mathbb{S}_n$ be the group of bijections $\pi : [n] \to [n]$. The *one-line notation* of $\pi \in \mathbb{S}_n$ is the vector $(\pi(1), \dots, \pi(n))$ and π^{-1} is the *inverse* of π. For $\pi \in \mathbb{S}_n$ and $\mathtt{H} \in \mathrm{InHy}(n)$, we define the *relabeled* interval hypergraph

$$\pi(\mathtt{H}) := \{\pi(e) : e \in \mathtt{H}\}. \tag{1.1}$$

This yields an interval hypergraph again: if $\mathtt{H}$ is an interval hypergraph w.r.t. l, then $\pi(\mathtt{H})$ is an interval hypergraph w.r.t. l^π, where $il^\pi j :\Leftrightarrow \pi^{-1}(i)l\pi^{-1}(j)$. A *finite* exchangeable interval hypergraphs on $[n]$ is a $\mathrm{InHy}(n)$-valued random variable H_n such that H_n has the same law as $\pi(H_n)$ for each $\pi \in \mathbb{S}_n$. From a de Finetti point of view, finite exchangeable random structures can be understood quite easily. Things become much more interesting when investigating the associated *infinite* exchangeable random structures. We define an infinite interval hypergraph (on $\mathbb{N}$) as a *projective sequence of finite interval hypergraphs*, more concretely: Given some $k \leq n$ and some $\mathtt{H} \in \mathrm{InHy}(n)$, we first introduce the *restricted* interval hypergraph

$$\mathtt{H}_{|k} := [k] \cap \mathtt{H} = \{\{1, \dots, k\} \cap e : e \in \mathtt{H}\}. \tag{1.2}$$

This really yields an interval hypergraph: if $\mathtt{H}$ is an interval hypergraph w.r.t. l, then $\mathtt{H}_{|k}$ is an interval hypergraph w.r.t. $l_{|k}$, where $l_{|k}$ is the restriction of the linear order l to the set $[k]$. See Figure 2 for a visualization of $\pi(\mathtt{H})$ and $\mathtt{H}_{|k}$.

DEFINITION 1.4. An *interval hypergraph on* $\mathbb{N}$ is a sequence $\mathtt{H} = (\mathtt{H}_n)_{n\in\mathbb{N}}$ such that $\mathtt{H}_n \in \mathrm{InHy}(n)$ and $\mathtt{H}_n = (\mathtt{H}_{n+1})_{|n}$ for each n. Let $\mathrm{InHy}(\mathbb{N})$ be the set of all interval hypergraphs on $\mathbb{N}$.

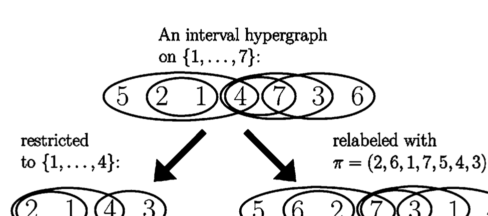

FIG. 2. *Example of restricting and relabeling an interval hypergraphs on* [7].

See Remark 1.5 for why interval hypergraphs on $\mathbb{N}$ have not been defined like in Definition 1.1 with $\mathbb{N}$ instead of $[n]$.

REMARK 1.5. Consider subsets $\mathtt{H}', \mathtt{H} \subseteq \mathcal{P}(\mathbb{N})$ that both satisfy (i) and (ii) as in Definition 1.1 but with $\mathbb{N}$ instead of $[n]$. It is possible that $\mathtt{H} \neq \mathtt{H}'$ but $\mathtt{H}_{|n} = \mathtt{H}'_{|n}$ for all n, where $\mathtt{H}_{|n} = \{[n] \cap e : e \in \mathtt{H}\}$. The set of all $\mathtt{H} \subseteq \mathcal{P}(\mathbb{N})$ satisfying (i) and (ii) has a cardinality higher than that of the continuum. Thus it cannot be equipped with a σ-field that turns it into a Borel space, which would be a desirable technical feature when dealing with exchangeable random objects. In [10], hierarchies on $\mathbb{N}$ are defined as projective sequences of finite hierarchies as well.

A *random interval hypergraph on* $\mathbb{N}$ is a stochastic process $H = (H_n)_{n \in \mathbb{N}}$ such that $H_n \in \mathrm{InHy}(n)$ and $H_n = (H_{n+1})_{|n}$ almost surely for all n. Let $\mathrm{Law}(H)$ be the law of a random interval hypergraph H on $\mathbb{N}$. Now we can introduce one of our main objects of interest.

DEFINITION 1.6. An *exchangeable interval hypergraph on* $\mathbb{N}$ (EIH) is a random interval hypergraph $H = (H_n)_{n \in \mathbb{N}}$ on $\mathbb{N}$ such that all H_n have *exchangeable laws*, that is, for every n it holds that $\pi(H_n) \overset{\mathcal{D}}{=} H_n$ for every $\pi \in \mathbb{S}_n$. Let

$$\mathtt{ExInHy} = \{\mathrm{Law}(H) : H \text{ is an exchangeable interval hypergraph on } \mathbb{N}\}$$

be the collection of all possible laws of EIHs.

Our main result concerning EIHs reads as follows. We ignore some measurability issues for the moment.

THEOREM 1.7. *For every exchangeable interval hypergraph* $(H_n)_{n \in \mathbb{N}}$, *there exists a random compact set* $K \subseteq \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\}$ *with* $(x, x) \in K$ *for every* $x \in [0, 1]$ *such that, given an i.i.d. sequence* $U_1, U_2, \ldots$ *of uniform RVs independent of* K, *it holds that* $(H_n)_{n \in \mathbb{N}}$ *has the same distribution as* $(H'_n)_{n \in \mathbb{N}}$, *where*

$$H'_n := \{\{i \in [n] : x < U_i < y\} : (x, y) \in K\} \cup \{\{j\} : j \in [n]\} \cup \{\varnothing\}.$$

Our aim is not only to describe the laws of EIHs as in Theorem 1.7, but also to describe some topological aspects of the space $\mathtt{ExInHy}$. In particular, $\mathtt{ExInHy}$ is naturally equipped with the structure of a *metrizable Choquet simplex*, that is, it is a compact convex set in which every point can be expressed as a mixture of extreme points in a unique way. We not only describe the convexity structure $\mathtt{ExInHy}$, but also describe its topology. In particular, we show that $\mathtt{ExInHy}$ is a Bauer simplex, that is, the extreme points (which are precisely the ergodic exchangeable laws) form a closed set. More details about Choquet simplices can be found in Section 1.7.

As noted above, we obtain our result about EIHs through the study of erased-interval processes, which we introduce next.

1.4. *Erased-interval processes.* The definition of exchangeable interval hypergraphs on $\mathbb{N}$ was built upon restricting and relabeling interval hypergraphs. For erased-interval processes, certain transient Markov chains $(I_n, \eta_n)_{n \in \mathbb{N}}$ with $I_n \in \mathrm{InSy}(n)$, we introduce a different operation: *removing elements from intervals* $[a, b]$ *and then relabeling them in a strictly monotone fashion*. Note that every nonempty edge $e \in \mathtt{I}$ in an interval system $\mathtt{I} \in \mathrm{InSy}(n) =$

$\mathrm{InHy}(n,<)$ is of the form $e=[a,b]:=\{i\in[n]:a\le i\le b\}$ for some $1\le a\le b\le n$. Let $n\in\mathbb{N}$, $[a,b]\subseteq[n+1]$ and $k\in[n+1]$. We define an interval $[a,b]-\{k\}\subseteq[n]$ by

$$[a,b]-\{k\}:=\begin{cases}[a-1,b-1] & \text{if } k<a\le b,\\ [a,b-1] & \text{if } a\le k\le b,\\ [a,b] & \text{if } a\le b<k.\end{cases} \tag{1.3}$$

This operation can be lifted to interval systems: If $\mathtt{I}\in\mathrm{InSy}(n+1)$ is an interval system on $[n+1]$ and $k\in[n+1]$, then $\phi_n^{n+1}(\mathtt{I},k)$ is defined by removing k from every $[a,b]\in\mathtt{I}$ in the sense of (1.3). So ϕ_n^{n+1} is formally defined as

$$\phi_n^{n+1}:\mathrm{InSy}(n+1)\times[n+1]\longrightarrow\mathrm{InSy}(n),$$
$$\phi_n^{n+1}(\mathtt{I},k):=\{[a,b]-\{k\}:[a,b]\in\mathtt{I}\}.$$

DEFINITION 1.8. An *erased-interval process* (EIP) is a stochastic process $(I,\eta)=(I_n,\eta_n)_{n\in\mathbb{N}}$ such that for every $n\in\mathbb{N}$:

(i) (I_n,η_n) takes values in $\mathrm{InSy}(n)\times[n+1]$ almost surely.
(ii) The random variable η_n is uniformly distributed on $[n+1]$ and independent of the σ-field $\mathcal{F}_{n+1}$, where

$$\mathcal{F}_n:=\sigma(I_m,\eta_m:m\ge n)$$

is the σ-field generated by the future of (I,η) after time n.
(iii) $I_n=\phi_n^{n+1}(I_{n+1},\eta_n)$ almost surely.

Let $\mathrm{Law}((I,\eta))$ be the law of the erased-interval process (I,η) and let

$$\mathtt{ErInPr}:=\{\mathrm{Law}((I,\eta)):(I,\eta)\text{ is an erased-interval process}\}$$

be the space of all possible laws of erased-interval processes. See Figure 3 for a visualization of property (iii).

Our main result about EIPs, an *almost sure* de Finetti-type representation theorem, is stated in Section 2, Theorem 2.4.

REMARK 1.9. Erased-type processes occur in the literature concerning *polyadic filtrations*; see [21] and [11]. Certain results in these papers later yield an explanation for why we consider only hypergraphs in which all singleton sets $\{j\}$ are part of the edge set. Our almost-sure representation result for EIPs, as a by-product, also clarifies the isomorphism structure of the polyadic backward filtrations generated by those processes: such a backward filtration $\mathcal{F}=(\mathcal{F}_n)_{n\in\mathbb{N}}$ is of product-type iff it is Kolmogorovian, that is, if the terminal σ-field $\mathcal{F}_\infty=\bigcap_{n\in\mathbb{N}}\mathcal{F}_n$ is a.s. trivial.

1.5. *Method of proof.* Our method of proof differs strongly from the proofs presented in [7, 10]: both of these works deal with trees. This fact is used to define certain random $\{0,1\}$-valued sequences that inherit exchangeability. Then the classical de Finetti theorem mentioned in the beginning is applied.

Our random objects do not have tree-like structure, hence we have to follow a different route. Our arguments involve the Glivenko–Cantelli theorem, some folklore knowledge about the *exchangeable linear order* and a property of the intersection behavior of random rectangles: almost surely, a compact set intersects a random closed rectangle in its interior or not at all, provided the corners have a.s. conditional diffuse laws. We show this using some topological properties of the Sorgenfrey plane. Finally, we use some convex-geometric arguments to transfer the results about EIPs to EIHs. We do not use any version of a de Finetti-type representation theorem to obtain our results.

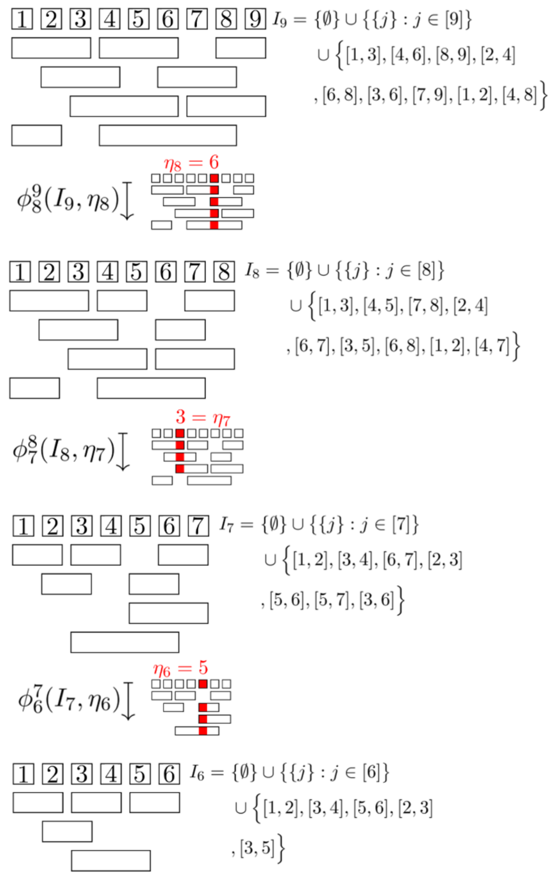

FIG. 3. *Example for the backwards evolution of an EIP.*

1.6. *Outline of the paper.* In Section 1.7, we present a short introduction to Choquet simplex theory and in Section 1.8 we briefly discuss some very important and well-known properties of the exchangeable linear order.

In Section 2, we present our main result, a de Finetti-type representation result for EIPs in a very "strong" form: we not only describe every possible law of EIPs in a certain way, but we express *any given* EIP in a certain *almost sure* way. We describe $\mathtt{ErInPr}$, which is a Choquet simplex, up to affine homeomorphism. We then deduce our representation result for exchangeable interval hypergraphs by providing an explicit affine surjective continuous map $\mathtt{ErInPr} \to \mathtt{ExInHy}$.

In Section 3, we prove our main theorem and in Section 4 we proof an important proposition about random rectangles used in the proof.

In Section 5, we discuss the relations of our work to the papers [7, 10, 15, 20], analyze Martin boundaries, that is, limits of discrete structures, of several ordered discrete structures (compositions, binary trees, Schröder trees, interval systems) and present the formerly mentioned representation of the plane Brownian CRT as a random compact set.

Connections of Martin boundary theory, limit theories and exchangeability in discrete structures have been analyzed before: [2, 6, 16] discuss graphs, [13] deals with the classical situation of $\{0, 1\}$-valued sequences, [18] analyzes pattern densities in permutations, [11] subsequence densities in words and [4] subsequence densities in words in which all letters of the alphabet occur equally often.

1.7. *Choquet simplices.* A metrizablel Choquet simplex (just "simplex" for short) is a metrizable compact convex set $\mathcal{M}$ in which every point $x \in \mathcal{M}$ can be expressed in a unique way as a mixture of extreme points. Let $\mathrm{ex}(\mathcal{M}) \subseteq \mathcal{M}$ be the set of extreme points of $\mathcal{M}$. Mixtures of extreme points are directed by Borel probability measures μ concentrated on $\mathrm{ex}(\mathcal{M})$: To every μ, there corresponds a unique point $x_\mu = \int_{\mathrm{ex}(\mathcal{M})} y \, d\mu(y)$ and the map $\mu \mapsto x_\mu$ is an affine continuous bijection from $\mathcal{M}_1(\mathrm{ex}(\mathcal{M}))$ to $\mathcal{M}$. We direct the reader to [14, 19, 28] for an introduction and additional material concerning Choquet theory and simplices. Next, we explain why and in what sense both `ExInHy` and `ErInPr` can be seen as simplices.

Consider the already introduced set

$$\mathrm{InHy}(\mathbb{N}) = \{\mathtt{H} = (\mathtt{H}_n)_{n\in\mathbb{N}} : \mathtt{H}_n \in \mathrm{InHy}(n) \text{ and } \mathtt{H}_n = (\mathtt{H}_{n+1})_{|n} \text{ for all } n\}$$

of interval hypergraphs on $\mathbb{N}$. The set $\mathrm{InHy}(\mathbb{N})$ is a compact subset of the compact metrizable discrete product space $\prod_n \mathrm{InHy}(n)$. Hence the space of all Borel probability measures on $\mathrm{InHy}(\mathbb{N})$, denoted by $\mathcal{M}_1(\mathrm{InHy}(\mathbb{N}))$, is a compact metrizable space under the topology of weak convergence. It is easily checked that $\mathtt{ExInHy} \subseteq \mathcal{M}_1(\mathrm{InHy}(\mathbb{N}))$ is a compact and convex subset of that space. One can observe that `ExInHy` is a simplex by use of some basic facts from ergodic theory: Denote by $\mathbb{S}_\infty$ the countable amenable group of finite bijections of $\mathbb{N}$. One can introduce a group action from $\mathbb{S}_\infty$ to $\mathrm{InHy}(\mathbb{N})$ such that `ExInHy` is precisely the set of $\mathbb{S}_\infty$-invariant laws on $\mathrm{InHy}(\mathbb{N})$: For some $\pi \in \mathbb{S}_\infty$, let $|\pi| := \min\{n \in \mathbb{N} : \pi(i) = i \text{ for all } i \geq n\}$ be the size of π. Now given some $\mathtt{H} = (\mathtt{H}_n)_n \in \mathrm{InHy}(\mathbb{N})$ and some $\pi \in \mathbb{S}_\infty$ with $m := |\pi|$ define $\pi(\mathtt{H}) := \mathtt{H}' = (\mathtt{H}'_n)_n$ by

$$\mathtt{H}'_n := \{\pi(e) : e \in \mathtt{H}_n\} \quad \text{for } n \geq m \quad \text{and} \quad \mathtt{H}'_n := (\mathtt{H}'_m)_{|n} \quad \text{for } n < m.$$

It is easy to see that $\pi(\mathtt{H}) \in \mathrm{InHy}(\mathbb{N})$ and that $(\pi, \mathtt{H}) \mapsto \pi(\mathtt{H})$ is a group action of $\mathbb{S}_\infty$ on $\mathrm{InHy}(\mathbb{N})$ such that $\mathtt{H} \mapsto \pi(\mathtt{H})$ is a homeomorphism on $\mathrm{InHy}(\mathbb{N})$ for every π. Now `ExInHy` are precisely the $\mathbb{S}_\infty$-invariant Borel probability measures on the compact metrizable space $\mathrm{InHy}(\mathbb{N})$ and so `ExInHy` is a nonempty simplex (see [14], Chapter 4). Moreover, the extreme points of the convex set `ExInHy` are precisely the ergodic $\mathbb{S}_\infty$-invariant probability measures on $\mathrm{InHy}(\mathbb{N})$, where some $P \in \mathtt{ExInHy}$ is called ergodic iff for every $\mathbb{S}_\infty$-invariant event $E \subseteq \mathrm{InHy}(\mathbb{N})$ it holds that $P(E) \in \{0, 1\}$. Introduce

$$\mathrm{erg}(\mathtt{ExInHy}) := \{\mathrm{Law}(H) : H \in \mathrm{InHy}(\mathbb{N}) \text{ has a } \mathbb{S}_\infty\text{-invariant ergodic law}\}.$$

The famous *ergodic decomposition theorem* (see, e.g., [14]) yields

$$P \in \mathrm{erg}(\mathtt{ExInHy}) \quad \Longleftrightarrow \quad P \text{ is an extreme point of the convex set } \mathtt{ExInHy}.$$

Moreover, the general theory concerning compact convex sets yields that $\mathrm{erg}(\mathtt{ExInHy})$ is a G_δ-subset of `ExInHy`. The uniqueness of the ergodic decomposition can now be stated in the following explicit form: For every Borel probability measure μ on $\mathrm{erg}(\mathtt{ExInHy})$, the *integration of* μ, given by P_μ with

$$P_\mu(A) := \int_{\mathrm{erg}(\mathtt{ExInHy})} Q(A) \, d\mu(Q) \quad \text{for every event } A \subseteq \mathrm{InHy}(\mathbb{N}), \tag{1.4}$$

is the law of some exchangeable interval hypergraph, so $P_\mu \in \mathtt{ExInHy}$ and the map

$$\mathcal{M}_1(\mathrm{erg}(\mathtt{ExInHy})) \to \mathtt{ExInHy}, \qquad \mu \mapsto P_\mu$$

is an affine continuous bijection. The inverse of $\mu \mapsto P_\mu$ is given as follows: For $P \in \mathtt{ExInHy}$, let μ^P be the law of the conditional distribution of P given the $\mathbb{S}_\infty$-invariant σ-field (law under P itself). With this, it holds that $\mu^P(\mathrm{erg}(\mathtt{ExInHy})) = 1$, $P_{\mu^P} = P$ and $\mu = \mu^{P_\mu}$ for every $\mu \in \mathcal{M}_1(\mathrm{erg}(\mathtt{ExInHy}))$. We will describe the ergodic laws not only as a set but give some insight into the intrinsic topology as well. In particular, we will see that $\mathrm{erg}(\mathtt{ExInHy})$

is a closed, hence compact, space and by that we identify ExInHy as a so-called *Bauer simplex*. In view of Theorem 1.7, the ergodic exchangeable interval hypergraphs are precisely those that can be represented by some deterministic compact set K.

As it is the case with the space of laws of exchangeable interval hypergraphs on $\mathbb{N}$, the space ErInPr of laws of erased-interval processes is a simplex as well: Every $\text{Law}((I,\eta)) \in$ ErInPr can be considered as a Borel probability measure on the path space $\prod_{n\in\mathbb{N}} \text{InSy}(n) \times [n+1]$ with its product topology. Now ErInPr is a compact convex subset of the space of all Borel probability measures on that path space, and in fact, it is a simplex: ErInPr is equal to the set of all *Markov laws with prescribed co-transition probabilities* $\theta = (\theta_n^{n+1}, n \in \mathbb{N})$, where θ_n^{n+1} is given by

$$(1.5)\qquad \begin{aligned} &\theta_n^{n+1} : \big(\text{InSy}(n) \times [n+1]\big) \times \big(\text{InSy}(n+1) \times [n+2]\big) \longrightarrow [0,1], \\ &\theta\big((\mathtt{I}_n, k_n), (\mathtt{I}_{n+1}, k_{n+1})\big) := \frac{1}{n+1} 1_{\{\mathtt{I}_n\}}\big(\phi_n^{n+1}(\mathtt{I}_{n+1}, k_n)\big). \end{aligned}$$

General theory implies that ErInPr is a metrizable Choquet simplex (see [29]) and that some $\text{Law}((I,\eta)) \in$ ErInPr is an extreme point of the convex set ErInPr iff the terminal σ-field $\mathcal{F}_\infty = \bigcap_{n\in\mathbb{N}} \sigma(I_m, \eta_m : m \geq n)$ generated by (I,η) is trivial almost surely, that is every terminal event has probability either zero or one. Introduce

$$\text{erg}(\texttt{ErInPr}) := \left\{ \text{Law}\big((I,\eta)\big) : \begin{array}{l} (I,\eta) \text{ is an EIP generating an} \\ \text{almost surely trivial terminal } \sigma\text{-field} \end{array} \right\}.$$

As above, some P is an extreme point of the convex set ErInPr iff $P \in \text{erg}(\texttt{ErInPr})$. The ergodic decomposition for ErInPr can now be stated in the exact same form as in (1.4) and the corresponding map $\mathcal{M}_1(\text{erg}(\texttt{ErInPr})) \to \texttt{ErInPr}$, $\mu \mapsto P_\mu$ is again continuous affine and bijective. We show that erg(ErInPr) is compact by providing an explicit homeomorphism to a compact metric space.

1.8. *The exchangeable linear order.* In this section, we introduce and briefly discuss the random exchangeable linear order on $\mathbb{N}$. The statements seem to be "folklore"; as we use them extensively, we think it pays to present them in a condensed form.

Consider the set $\mathbb{L}$ of all linear orders on $\mathbb{N}$. Given some $l \in \mathbb{L}$ and some $n \in \mathbb{N}$, let $l_{|n}$ be the restriction of l to the set $[n]$. Equip $\mathbb{L}$ with the σ-field generated by all these restriction maps. One defines a group action from $\mathbb{S}_\infty$ to $\mathbb{L}$ in the following way: If $l \in \mathbb{L}$ and $\pi \in \mathbb{S}_\infty$, then $\pi(l) \in \mathbb{L}$ is defined by

$$i\ \pi(l)\ j \quad :\Longleftrightarrow \quad \pi^{-1}(i)\ l\ \pi^{-1}(j) \quad \text{for all } i,j \in \mathbb{N}.$$

Let L be a random linear order on $\mathbb{N}$ with an exchangeable law, that is, $\pi(L) \stackrel{\mathcal{D}}{=} L$ for every $\pi \in \mathbb{S}_\infty$. The law of such an object is unique, for every $n \in \mathbb{N}$ the restriction $L_{|n}$ is uniform on the finite set of all possible linear orders on $[n]$. Such an exchangeable linear order L naturally occurs in the context of exchangeability, but often not directly in the form of a linear order: There are other types of stochastic objects that are in some sense equivalent to an exchangeable linear order. We now introduce some notation that is needed throughout the entire paper.

NOTATION 1. For $k \in \mathbb{N}$, we define:

– The set $[0,1]^k_{\neq}$ consisting of all $(u_1,\dots,u_k) \in [0,1]^k$ with $u_i \neq u_j$ for all $i \neq j$.

– The set $[0,1]^k_{<}$ consisting of all $(u_1,\dots,u_k) \in [0,1]^k$ that are strictly increasing, so $0 \leq u_1 < u_2 < \cdots < u_{k-1} < u_k \leq 1$.

– Given some $(u_1, \dots, u_k) \in [0, 1]^k_<$ define $u_0 := -1$ and $u_{k+1} := 2$ (to avoid tedious case studies in some of the following definitions).

– For $(u_1, \dots, u_k) \in [0, 1]^k_{\neq}$, let $\pi := \pi(u_1, \dots, u_k) \in \mathbb{S}_k$ be the unique permutation of $[k]$ such that $u_{\pi(1)} < \cdots < u_{\pi(k)}$. Define $u_{i:k} := u_{\pi(i)}$. In particular, $u_{0:k} = -1$ and $u_{k+1:k} = 2$.

DEFINITION 1.10. We define three types of stochastic processes indexed by $\mathbb{N}$:

- A process $U = (U_i)_{i\in\mathbb{N}}$ such that
  - $U_1, U_2, \dots$ are independent identically distributed,
  - each U_i is uniform on the unit interval, $U_i \sim \mathrm{unif}([0, 1])$,

  is called a *U-process*.
- A process $S = (S_n)_{n\in\mathbb{N}}$ such that for each n
  - S_n is a uniform random permutation of $[n]$, so $S_n \sim \mathrm{unif}(\mathbb{S}_n)$,
  - the one-line notation of S_n is almost surely obtained from S_{n+1} by erasing "$n + 1$" in the one-line notation of S_{n+1}

  is called a *permutation process*.
- A process $\eta = (\eta_n)_{n\in\mathbb{N}}$ such that
  - $\eta_1, \eta_2, \dots$ are independent,
  - η_n is uniformly distributed on the finite set $[n + 1] = \{1, \dots, n + 1\}$ for each n,

  is called an *eraser process*.

If (I, η) is an erased-interval process, then η is an eraser process. Next, we explain in what sense the four introduced objects—exchangeable linear order L, U-process, permutation process S, eraser process η—can be considered to be equivalent, that is given any one of the four types of stochastic objects one can pass to any other in an almost surely defined functional way without losing probabilistic information:

$\underline{U \to L}$: Given some U-process $U = (U_i)_{i\in\mathbb{N}}$, one can define a random linear order L on $\mathbb{N}$ by

$$i \; L \; j \quad :\Longleftrightarrow \quad U_i < U_j.$$

This random linear order is exchangeable by the exchangeability of U.

$\underline{L \to U}$: Given some exchangeable linear order L and some $i \in \mathbb{N}$, the above construction of L from a U-process directly yields that the limit

$$U_i = \lim_{n\to\infty} \frac{\#\{k \in [n] : k \; L \; i\}}{n}$$

exists almost surely for all i and yields a U-process $U = (U_i)_{i\in\mathbb{N}}$.

$\underline{U \to \eta}$: Given some U-process $U = (U_i)_{i\in\mathbb{N}}$, one can define

$$\eta_n := \#\{i \in [n + 1] : U_i \le U_{n+1}\},$$

that is, η_n is the rank of U_{n+1} in $U_1, \dots, U_{n+1}$. Obviously, $\eta = (\eta_n)_{n\in\mathbb{N}}$ is an eraser process.

$\underline{\eta \to S}$: We introduce, for every $n \ge 2$, the bijection

$$b_n : [2] \times [3] \times \cdots \times [n] \to \mathbb{S}_n, \tag{1.6}$$

where b_n is defined inductively: b_1 is the unique permutation of $[1]$ and the one-line notation of $\pi = b_n(i_1, \dots, i_{n-1})$ is obtained from the one-line notation of $\pi' = b_{n-1}(i_1, \dots, i_{n-2})$ by

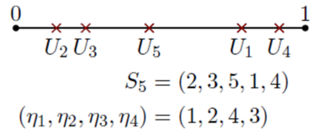

FIG. 4. *Possible beginning of a corresponding triple* (U, S, η). *Note that* $S_5^{-1}(5) = 3 = \eta_4$.

placing "n" in the i_{n-1}th gap of $\pi' = (\Box\pi'(1)\Box\pi'(2)\Box\cdots\Box\pi'(n-1)\Box)$. Now given some eraser process η we define $S_1 := (1)$ and for $n \geq 2$

$$S_n := b_n(\eta_1, \dots, \eta_{n-1}).$$

$S = (S_n)_{n\in\mathbb{N}}$ is a permutation process.

$\underline{S \to U}$: Given some permutation process S and some $i \in \mathbb{N}$ the limit

$$U_i = \lim_{n\to\infty} \frac{S_n^{-1}(i)}{n}$$

exists almost surely and these limits form a U-process $U = (U_i)_{i\in\mathbb{N}}$. This can be seen by letting η be constructed from some U-process U' like explained before. With this, $S_n^{-1}(i) = \#\{j \in [n] : U'_j \leq U'_i\}$ and the claim follows from strong law of large numbers.

In particular, $U \to \eta \to S \to U'$ implies $U' = U$ almost surely. We have presented a minimal set of constructions to go from each of the four objects under consideration to any other type of object in a way that does not lose any probabilistic information. We briefly describe some of the constructions obtained by connecting the previous ones:

$\underline{U \to \eta \to S}$: S_n is the permutation statistics of $U_1, \dots, U_n$, that is, the unique random permutation with $U_{S_n(1)} < \cdots < U_{S_n(n)}$.

$\underline{S \to U \to \eta}$: $\eta_n = S_{n+1}^{-1}(n+1)$.

$\underline{L \to U \to S}$: S_n is the unique permutation of $[n]$ with $S_n(1)L\cdots LS_n(n)$.

$\underline{S \to U \to L}$: iLj iff $S_n^{-1}(i) < S_n^{-1}(j)$ where $n = \max\{i, j\}$.

We omit presenting the missing constructions. If one starts with any of the four objects under consideration, there are almost surely uniquely defined objects of the other three types. We will refer to them as *corresponding objects*. Figure 4 shows the first steps of some realization of a corresponding triple (U, S, η).

In particular, for any erased-interval process (I, η) there are a U-process and a permutation process both corresponding to the eraser process η, and thus defined on the same probability space as (I, η). These processes will play an important role in our representation result, since the corresponding U-process serves as the randomization used to *sample from infinity* and the permutation process S is used to pass from (I, η) to an exchangeable interval hypergraph.

2. Main results. Our first main theorem will be the characterization of erased-interval processes. At first, we introduce a compact metric space that turns out to be homeomorphic to the space of ergodic EIPs, that is, erg(ErInPr). The elements of this space are *limits of scaled interval systems as* $n \to \infty$. We need to recall some topological definitions: Given any metric space (M, d), we introduce

$$\mathcal{K}(M) = \{\text{all nonempty compact subsets of } M\}.$$

On $\mathcal{K}(M)$ we will consider the *Hausdorff distance* defined by

$$d_{\text{haus}}(K_1, K_2) := \max\{\max\{d(x, K_2) : x \in K_1\}, \max\{d(x, K_1) : x \in K_2\}\}. \tag{2.1}$$

From [3], Chapter 7, if (M, d) is a compact metric space, then so is $(\mathcal{K}(M), d_{\text{haus}})$. If we talk about random compact sets, we always mean random variables taking values in the space

$\mathcal{K}(M)$ equipped with the Borel σ-field corresponding to d_{haus}. We need the following characterization of convergence in $(\mathcal{K}(M), d_{\text{haus}})$ (see [3], Exercise 7.3.4):

LEMMA 2.1. *Let* (M, d) *be a compact metric space and* $K_n, K \in \mathcal{K}(M)$ *be such that* $d_{\text{haus}}(K_n, K) \to 0$. *Then for every* $x \in K$ *there exists a sequence* (x_n) *such that* $x_n \in K_n$ *for every* n *and* $d(x_n, x) \to 0$. *If* (x_n) *is a sequence with* $x_n \in K_n$ *for every* n *and* $d(x_n, x) \to 0$ *for some* $x \in M$, *then* $x \in K$.

A major point for the intuition behind our constructions is that one can identify any open subinterval of the open unit interval $(0, 1)$, which is a *set* of the form $(x, y) = \{z \in (0, 1) : x < z < y\}$ with the *point* in $\mathbb{R}^2$ whose coordinates are given by the end points of that interval, $(x, y) \in \mathbb{R}^2$ (note the present overloading of symbols). Consider the *triangle* $\text{◸} = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le y \le 1\}$ introduced before and let

$$\diagup := \{(x, x) \in \mathbb{R}^2 : 0 \le x \le 1\}$$

be the diagonal line from $(0, 0)$ to $(1, 1)$. Consider the metric induced by the 1-norm on $\mathbb{R}^2$, so $d((x_1, y_1), (x_2, y_2)) := |x_1 - x_2| + |y_1 - y_2|$. Define

$$\text{InSy}(\infty) := \{K \subseteq \text{◸} : K \text{ is compact and } \diagup \subseteq K\}. \tag{2.2}$$

The following lemma is very easy to prove, but nevertheless of great importance later on.

LEMMA 2.2. $\text{InSy}(\infty)$ *is a closed subset of* $\mathcal{K}(\text{◸})$ *and* $(\text{InSy}(\infty), d_{\text{haus}})$ *is a compact metric space*. *In particular*, $(\text{InSy}(\infty), d_{\text{haus}})$ *is complete*.

PROOF. We only need to prove the first statement, since $(\mathcal{K}(\text{◸}), d_{\text{haus}})$ is known to be a compact metric space and closed subspaces of compact metric spaces are compact. Furthermore, every compact metric space is complete. So let $K_n \in \text{InSy}(\infty)$ and $K \in \mathcal{K}(\text{◸})$ be such that $d_{\text{haus}}(K_n, K) \to 0$. Since $(x, x) \in K_n$ for every $x \in [0, 1]$ and every $n \in \mathbb{N}$, Lemma 2.1 yields that $(x, x) \in K$. Hence $\diagup \subseteq K$ and $K \in \text{InSy}(\infty)$. $\square$

The space $(\text{InSy}(\infty), d_{\text{haus}})$ will turn out to be homeomorphic to erg(`ErInPr`). As the notation indicates, $\text{InSy}(\infty)$ can be seen, in various ways, as the analogue for interval systems $\text{InSy}(n)$ as $n \to \infty$. Our main theorem says that *every ergodic erased-interval process can be obtained by sampling from some unique* $K \in \text{InSy}(\infty)$, *even in a homeomorphic way*. We will now present the map that describes this "sampling from infinity":

Let $k \in \mathbb{N}$ and define

$$\phi_k^\infty : \text{InSy}(\infty) \times [0, 1]^k_< \longrightarrow \text{InSy}(k),$$

$$\phi_k^\infty(K, u_1, \dots, u_k) := \left\{ [a, b] : \begin{matrix} 1 \le a \le b \le k \text{ s.t. exists } (x, y) \in K \text{ with} \\ u_{a-1} < x < u_a \le u_b < y < u_{b+1} \end{matrix} \right\}$$
$$\cup \big\{\{j\} : j \in [k]\big\} \cup \{\varnothing\}.$$

Let $K \in \text{InSy}(\infty)$, $(u_1, \dots, u_k) \in [0, 1]^k_<$ and $1 \le a < b \le k$. One directly obtains the following very useful description of ϕ_k^∞:

$$[a, b] \in \phi_k^\infty(K, u_1, \dots, u_k) \quad \Longleftrightarrow \quad K \cap (u_{a-1}, u_a) \times (u_b, u_{b+1}) \neq \varnothing, \tag{2.3}$$

where $(u_{a-1}, u_a), (u_b, u_{b+1}) \subset \mathbb{R}$ are open intervals. One may wish to take a look at Figure 5.

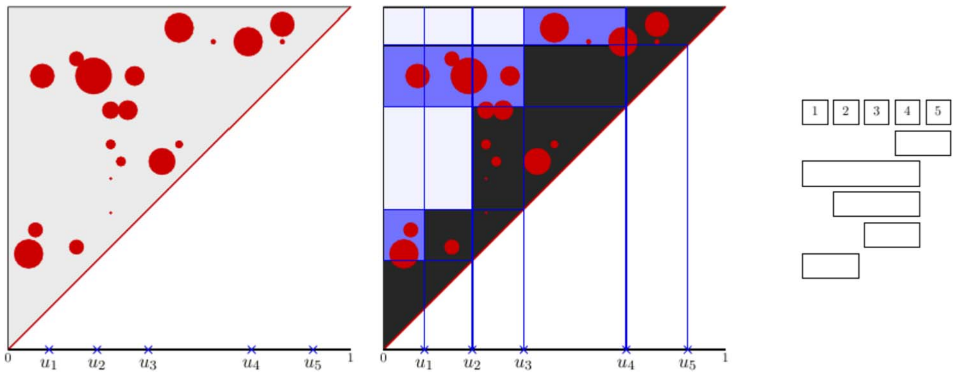

FIG. 5. *On the left a compact set $K \in \mathrm{InSy}(\infty)$ and some points $(u_1, u_2, u_3, u_4, u_5) \in [0,1]^5_<$. These 5 points divide the upper triangle into parts of which $\binom{5}{2}$ are relevant to the definition of $\phi_5^\infty(K, u_1, u_2, u_3, u_4, u_5) \in \mathrm{InSy}(5)$. The latter can be seen on the right: an interval $[a,b]$ with $1 \le a < b \le 5$ is present in the induced interval system iff the compact set K intersects the open rectangle $(u_{a-1}, u_a) \times (u_b, u_{b+1})$; here again $u_0 := -1$ and $u_6 := 2$.*

REMARK 2.3. Let $(u_1, \dots, u_k) \in [0,1]^k_<$. We used the conventions $u_0 := -1$ and $u_{k+1} := 2$ because we want points $(x, y) \in K \in \mathrm{InSy}(\infty)$ with $x = 0$ or $y = 1$ to eventually have an effect on $\phi_k^\infty(K, u_1, \dots, u_k)$. Since we work with open rectangles (see (2.3)) the choices $u_0 := 0$ and $u_{k+1} := 1$ would have failed to achieve this.

We will prove that the map ϕ_k^∞ is measurable for every k with respect to the Borel σ-field on $\mathrm{InSy}(\infty) \times [0,1]^k_<$. Thus one can plug in random elements and obtain $\mathrm{InSy}(k)$-valued random elements. In particular, we will plug in the order statistics $(U_{1:k}, \dots, U_{k:k})$ obtained from the U-processes $(U_i)_{i\in\mathbb{N}}$ corresponding to an eraser process η which stems from an erased-interval process (I, η).

Let $(U_i)_{i\in\mathbb{N}}$ be a U-process and let η be the eraser process corresponding to U. We will prove that for every $K \in \mathrm{InSy}(\infty)$ the stochastic process

$$(\phi_n^\infty(K, U_{1:n}, \dots, U_{n:n}), \eta_n)_{n\in\mathbb{N}} \tag{2.4}$$

is an ergodic erased-interval process and that *every* ergodic erased-interval process is of this form, *not only in law but almost surely*. Denote the law of the process in (2.4) with $\mathrm{Law}(K)$. So in particular, we will show that $\mathrm{Law}(K) \in \mathrm{erg}(\mathtt{ErInPr})$ for every $K \in \mathrm{InSy}(\infty)$. To prove this almost sure representation, we need to explain how to obtain an appropriate compact subset $K \in \mathrm{InSy}(\infty)$ when given an erased-interval process (I, η). This desired interval system is obtained by scaling I_n and then letting $n \to \infty$. We now introduce this scaling procedure. Note that in the following definition, $(x, y) \in$ ◸ $\subset \mathbb{R}^2$ represents a point and not an open interval. For $n \in \mathbb{N}$ and $\mathtt{I} \in \mathrm{InSy}(n)$, let

$$n^{-1}\mathtt{I} := \left\{ \left(\frac{a-1}{n}, \frac{b}{n} \right) : \varnothing \neq [a,b] \in \mathtt{I} \right\} \cup ⟋. \tag{2.5}$$

In particular, $n^{-1}\mathtt{I} \subset$ ◸ and ⟋ $\subseteq n^{-1}\mathtt{I}$ by definition and so $n^{-1}\mathtt{I} \in \mathrm{InSy}(\infty)$, since $n^{-1}\mathtt{I}$ is obviously compact. See Figure 6 for a visualization. We will prove that $n^{-1}I_n$ converges almost surely in the space $(\mathrm{InSy}(\infty), d_{\mathrm{haus}})$ for every erased-interval process $(I_n, \eta_n)_{n\in\mathbb{N}}$.

We are now ready to state our main theorems.

THEOREM 2.4. *For every $K \in \mathrm{InSy}(\infty)$, one has $\mathrm{Law}(K) \in \mathrm{erg}(\mathtt{ErInPr})$ and the map $\mathrm{InSy}(\infty) \to \mathrm{erg}(\mathtt{ErInPr})$, $K \mapsto \mathrm{Law}(K)$ is a homeomorphism. One has the following almost sure representation: Let $(I, \eta) = (I_n, \eta_n)_{n\in\mathbb{N}}$ be an erased-interval process. Then $n^{-1}I_n$*

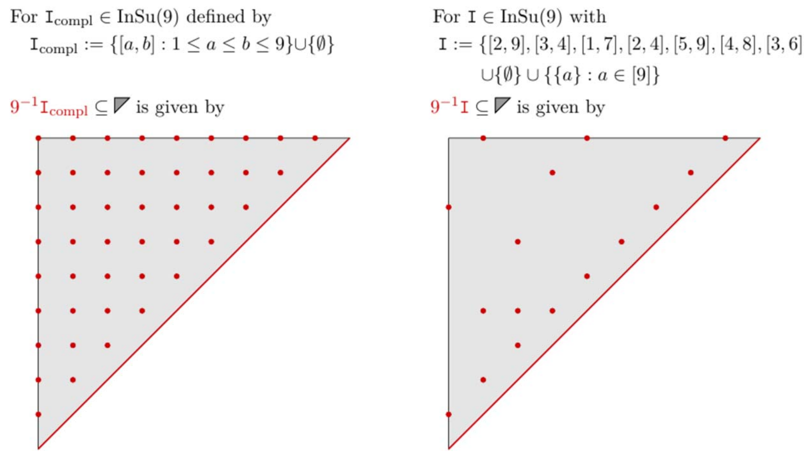

FIG. 6. *Two examples of interval systems* $\mathtt{I}$ *on* [9] *and below in red their scaled versions* $9^{-1}\mathtt{I}$ *which are closed subsets of* ◸ $\subset \mathbb{R}^2$.

converges almost surely as $n \to \infty$ *toward some* $\text{InSy}(\infty)$*-valued random variable* I_∞. *Let* $U = (U_i)_{i\in\mathbb{N}}$ *be the* U*-process corresponding to* η. *Then* I_∞ *and* U *are independent and one has the equality of processes*

$$(I_n, \eta_n)_{n\in\mathbb{N}} = \big(\phi_n^\infty(I_\infty, U_{1:n}, \dots, U_{n:n}), \eta_n\big)_{n\in\mathbb{N}} \quad \textit{almost surely.}$$

In particular, for every erased-interval process (I, η) *the conditional law of* (I, η) *given the terminal* σ*-field* $\mathcal{F}_\infty$ *is* $\text{Law}(I_\infty)$ *almost surely and* I_∞ *generates* $\mathcal{F}_\infty$ *almost surely.*

THEOREM 2.5. *Let* $(I, \eta) = (I_n, \eta_n)_{n\in\mathbb{N}}$ *be an erased-interval process and let* $(S_n)_{n\in\mathbb{N}}$ *be the permutation process corresponding to* η. *Let* $(U_i)_{i\in\mathbb{N}}$ *be the* U*-process corresponding to* η *and let* $I_\infty = \lim_{n\to\infty} n^{-1} I_n$. *Define* $H_n := S_n(I_n)$. *Then it holds that a.s. for every* n

$$H_n = \big\{\{j \in [n] : x < U_j < y\} : (x, y) \in I_\infty\big\} \cup \big\{\{j\} : j \in [n]\big\} \cup \{\varnothing\} \tag{2.6}$$

and $H = (H_n)_{n\in\mathbb{N}}$ *is an exchangeable interval hypergraph on* $\mathbb{N}$. *The map*

$$\mathtt{ErInPr} \longrightarrow \mathtt{ExInHy}, \qquad \text{Law}\big((I, \eta)\big) \longmapsto \text{Law}(H)$$

is a continuous affine surjection.

We will shortly state some corollaries that follow easily from the previous two theorems: Let K_1 and K_2 be convex sets with extreme points $\text{ex}(K_1)$, $\text{ex}(K_2)$ and let $f : K_1 \to K_2$ be an affine surjective map. Then it holds that $f^{-1}(\text{ex}(K_2)) \subseteq \text{ex}(K_1)$. This can be applied to $K_1 = \mathtt{ErInPr}$, $K_2 = \mathtt{ExInHy}$ and f as in Theorem 2.5. One easily sees that the map f in this situation maps extreme points to extreme points: every exchangeable interval hypergraph H that is constructed as in (2.6) with some *deterministic* $K = I_\infty \in \text{InSy}(\infty)$ is ergodic, due to the Hewitt–Savage zero-one law. Hence $f(\text{ex}(K_1)) = \text{erg}(K_2)$. One can summarize these considerations to the following.

COROLLARY 2.6. *Let* $K \in \text{InSy}(\infty)$ *and* $U = (U_i)_{i\in\mathbb{N}}$ *be a* U*-process. Then the process*

$$\big(\big\{\{j \in [n] : x < U_j < y\} : (x, y) \in K\big\} \cup \big\{\{j\} : j \in [n]\big\} \cup \{\varnothing\}\big)_{n\in\mathbb{N}} \tag{2.7}$$

is an ergodic exchangeable interval hypergraph on $\mathbb{N}$ *and the law of* every *ergodic exchangeable interval hypergraph on* $\mathbb{N}$ *can be expressed in this form. Denote the law of* (2.7) *by* $\text{Law}^{ih}(K)$. *The map* $\text{InSy}(\infty) \to \text{erg}(\mathtt{ExInHy})$, $K \mapsto \text{Law}^{ih}(K)$ *is surjective and continuous.*

The next corollary is about the structure of the spaces ErInPr and ExInHy as simplices.

COROLLARY 2.7. *The simplex* ErInPr *is a Bauer simplex affinely homeomorphic to the simplex of all Borel probability measures on* $\mathrm{InSy}(\infty)$ *equipped with the topology of weak convergence. The simplex* ExInHy *is also a Bauer simplex: its extreme points are a continuous image of the compact space* $\mathrm{InSy}(\infty)$.

2.1. *Polyadic filtrations.* Now we shortly explain how one can easily deduce a statement concerning certain polyadic backward filtrations generated by erased-interval processes and explain why all singletons $\{j\}$ are assumed to be part of any interval hypergraph. For an introduction to polyadic filtrations and further references, we refer the reader to [23]. One should emphasize that the properties concerning (backwards) filtrations we are going to state are properties concerning *filtered probability spaces*, so they are, in general, not stable under a change of measure. Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and sub-σ-fields $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, we say that $\mathcal{B} \subseteq \mathcal{C}$ holds almost surely iff for every $B \in \mathcal{B}$ there is a $C \in \mathcal{C}$ such that $\mathbb{P}(B \Delta C) = 0$. Consequently, $\mathcal{B} = \mathcal{C}$ almost surely iff both $\mathcal{B} \subseteq \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{B}$ hold almost surely.

Consider the backward filtration $\mathcal{F}$ generated by an erased-interval process (I, η), so $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{N}}$ with

$$\mathcal{F}_n = \sigma(I_m, \eta_m : m \geq n).$$

Since $I_{n-1} = \phi^n_{n-1}(I_n, \eta_{n-1})$ holds almost surely for every $n \geq 2$, one has that

$$\mathcal{F}_{n-1} = \mathcal{F}_n \vee \sigma(\eta_{n-1}) \quad \text{almost surely for every } n \geq 2.$$

By definition, η_{n-1} is independent of $\mathcal{F}_n$ and uniformly distributed on the finite set $\{1, \ldots, n\}$. The process η is called a process of *local innovations* for $\mathcal{F}$ and the backward filtration $\mathcal{F}$ is an example of a *polyadic* (backward) filtration. Inductively, applying the above almost sure equality of σ-fields yields

$$\mathcal{F}_k = \sigma(\eta_k, \eta_{k+1}, \ldots, \eta_{n-1}) \vee \mathcal{F}_n \quad \text{almost surely for every } 1 \leq k < n.$$

Now $\sigma(\eta_k, \eta_{k+1}, \ldots, \eta_{n-1}) \vee \mathcal{F}_n = \sigma(\eta_m : m \geq k) \vee \mathcal{F}_n$ for all $1 \leq k < n$ holds by definition of $\mathcal{F}_n$. Via this one obtains

$$\mathcal{F}_k = \bigcap_{n>k} [\sigma(\eta_m : m \geq k) \vee \mathcal{F}_n] \quad \text{almost surely for every } k \in \mathbb{N}.$$

Since $\sigma(\eta_m : m \geq k)$ does not depend on n, one may wonder if one can *interchange the order of taking the supremum and taking the intersection* on the right-hand side in the last equation. This is *not always* allowed: In [30], one can find a treatment of such questions in a very general setting. However, Theorem 2.4 shows that this interchange *is allowed* in our concrete situation: For this, one only needs to observe that

$$\begin{aligned}&\sigma(\eta_k, \eta_{k+1}, \eta_{k+2}, \ldots)\\ &\quad = \sigma(U_{1:k}, \ldots, U_{k:k}, U_{k+1}, U_{k+2}, \ldots) \quad \text{a.s. for every } k \geq 1,\end{aligned}$$

where U is the U-process corresponding to η. Of course, I_∞ is $\mathcal{F}_\infty$-measurable. The representation $I_k = \phi^\infty_k(I_\infty, U_{1:k}, \ldots, U_{k:k})$ almost surely thus yields that

$$\begin{aligned}\sigma(I_k) &\subseteq \sigma(U_{1:k}, \ldots, U_{k:k}) \vee \sigma(I_\infty)\\ &\subseteq \sigma(\eta_k, \eta_{k+1}, \ldots) \vee \mathcal{F}_\infty \quad \text{a.s. for every } k \geq 1.\end{aligned}$$

Hence one obtains

$$\mathcal{F}_k = \sigma(\eta_m : m \geq k) \vee \bigcap_{n>k} \mathcal{F}_n \quad \text{almost surely for every } k \geq 1.$$

In particular, if $\mathcal{F}_\infty$ is Kolmogorovian, that is if $\mathcal{F}_\infty$ is a.s. trivial, then $\mathcal{F}$ is almost surely generated by η and so it is of product type. This yields the following.

COROLLARY 2.8. *Let $(I, \eta) = (I_n, \eta_n)_{n\in\mathbb{N}}$ be an erased-interval process and let $\mathcal{F}$ be the backwards filtration generated by (I, η). Then $\mathcal{F}$ is of product type iff it is Kolmogorovian and in particular, η generates $\mathcal{F}$ almost surely.*

In Section 5, we explain in what sense every infinite labeled Rémy bridge can be seen as an erased-interval process. The above statement concerning the filtrations was already formulated in [7], Lemma 5.3., but the proof they give contains errors (see the Annex in [23]). However, our result shows that the lemma formulated in [7] is correct.

In [11] and [21], different erased-type processes and their backward filtrations have been analyzed: *(general) erased-word processes*. A general erased-word process over a finite alphabet Σ is a stochastic process $(W_n, \eta_n)_{n\in\mathbb{N}}$ that is almost like an erased-interval process, but with the following differences: $W_n = (W_{n,1}, \dots, W_{n,n})$ is a random word of length n over the alphabet Σ and W_{n-1} is obtained by erasing the η_{n-1}th letter from W_n. In [11], the following was shown.

THEOREM. *The backward filtration generated by a general erased-word process over some finite alphabet Σ is of product-type iff it is Kolmogorovian*, but it is in this case not always generated by η.

If one had defined interval hypergraphs such that singleton sets may or may not be part of the edge sets, then some erased-interval process $(I_n, \eta_n)_{n\in\mathbb{N}}$ would have included nontrivial erased-word processes over the alphabet $\Sigma = \{0, 1\}$: $W_{n,i} = 1 :\Leftrightarrow \{i\} \in I_n$. Not only would the description of the ergodic laws have become a more challenging task, but an almost sure functional representation in the spirit of Theorem 2.4 would not have been possible and our method of proof would have failed. Some applications of Laurent's results concerning erased-word processes can also be found in [23].

3. Proofs. Now we will prove our main theorems; most of the effort lies in the proof of Theorem 2.4. We will first gather some lemmas and finally put them together. We need to introduce some notation, which are the finite analogues of the ones introduced in Notation 1.

NOTATION 2. Let $n \in \mathbb{N}$ and $1 \le k \le n$.

– $[k : n]$ is the set of all vectors $\vec{j} = (j_1, \dots, j_k) \in [n]^k$ that are strictly increasing, so $1 \le j_1 < j_2 < \dots < j_k \le n$.

– For $\vec{j} = (j_1, \dots, j_k) \in [k : n]$, we define $j_0 := -n$ and $j_{k+1} := 2n$ (to avoid unpleasant case studies in some of the following definitions).

– Given some permutation $\pi \in \mathbb{S}_n$ and some $1 \le k \le n$, we define $\vec{j}_k^\pi \in [k : n]$ to be the increasing enumeration of the set $\pi^{-1}([k]) \subseteq [n]$. So $\vec{j}_k^\pi$ is the unique vector $\vec{j}_k^\pi = (j_{k,1}^\pi, \dots, j_{k,k}^\pi) \in [k : n]$ with $\{j_{k,1}^\pi, \dots, j_{k,k}^\pi\} = \pi^{-1}([k])$. In particular, $j_{k,0}^\pi = -n$ and $j_{k,k+1}^\pi = 2n$.

– Given some $\vec{j} = (j_1, \dots, j_k) \in [k : n]$, we define

$$n^{-1}\vec{j} := \left(\frac{2j_1 - 1}{2n}, \dots, \frac{2j_k - 1}{2n}\right) \in [0, 1]_<^k.$$

EXAMPLE 1. The vector $\vec{j} = (j_1, j_2, j_3, j_4) = (3, 4, 6, 11)$ is element of $[4 : 12]$. Given the permutation $\pi \in \mathbb{S}_9$ with one-line-notation $\pi = (5, 3, 7, 1, 9, 8, 2, 4, 6)$, then $\vec{j}_5^\pi = (j_1, \dots, j_5) \in [5 : 9]$ is the increasing vector of positions of $\{1, \dots, 5\}$ in π, that is, $(1, 2, 4, 7, 8)$.

Up to now, the restriction map ϕ_n^{n+1} has only been considered for successive numbers $(n, n+1)$. We will now present the multistep restriction functions ϕ_k^n, $1 \le k \le n$ and show that ϕ_k^∞ are, in some sense, the limiting analogues for fixed k with $n \to \infty$.

Let $\mathtt{I}_n \in \mathrm{InSy}(n)$ and $(i_k, i_{k+1}, \dots, i_{n-1}) \in [k+1] \times [k+2] \times \cdots \times [n]$ be some sequence of erasers. Define inductively $\mathtt{I}_m := \phi_m^{m+1}(\mathtt{I}_{m+1}, i_m)$ for $m = n-1, \dots, k$. The resulting $\mathtt{I}_k \in \mathrm{InSy}(k)$ does not depend on the full information contained in the sequence of erasers $(i_k, \dots, i_{n-1})$ as one can interchange orders of erasing in certain senses and obtain the same result. The relevant information contained in $(i_k, \dots, i_{n-1})$ is described by a vector from $[k:n]$: extend the eraser vector to some vector $(\star, \dots, \star, i_k, \dots, i_{n-1}) \in [2] \times \cdots \times [n]$ and use this vector to define a permutation $\pi \in \mathbb{S}_n$ via $\pi := b_n(\star, \dots, \star, i_k, \dots, i_{n-1})$ (see (1.6)). Now $\mathtt{I}_k$ only depends on $\mathtt{I}_n$ and $\vec{j}_k^\pi \in [k:n]$. This is well defined, since $\vec{j}_k^\pi$ does not depend on the choices of $\star$ that were used to produce π. These functional dependences are given by the following definition.

DEFINITION 3.1. For $n \in \mathbb{N}$ and $1 \le k \le n$, let $\vec{j} = (j_1, \dots, j_k) \in [k:n]$. Define $\phi_k^n : \mathrm{InSy}(n) \times [k:n] \longrightarrow \mathrm{InSy}(k)$ via

$$\phi_k^n(\mathtt{I}, \vec{j}) := \left\{ [a,b] : \begin{array}{l} 1 \le a \le b \le k \text{ s.t. exists } [A,B] \in \mathtt{I} \text{ with} \\ j_{a-1} < A \le j_a \le j_b \le B < j_{b+1} \end{array} \right\} \cup \{\varnothing\}.$$

EXAMPLE 2. Consider $\mathtt{I} = \{[1,3],[2,5],[4,6]\} \cup \{\{j\} \in [7]\} \cup \{\varnothing\} \in \mathrm{InSy}(7)$ and $\vec{j} = (j_1, j_2, j_3) = (2,4,5) \in [3:7]$ and let $\mathtt{I}' := \phi_k^n(\mathtt{I}, \vec{j}) \subseteq \mathcal{P}([3])$. It holds that $[1,3] \in \mathtt{I}'$ because $[A,B] = [2,5] \in \mathtt{I}$ and $-7 = j_0 < 2 \le j_1 \le j_3 \le 5 < j_4 = 14$. It holds that $[2,3] \in \mathtt{I}'$ because $[A,B] = [4,6] \in \mathtt{I}$ and $2 < 4 \le j_2 \le j_3 \le 6 < 14$. It holds that $[1,2] \notin \mathtt{I}'$ because there is no point $[A,B] \in \mathtt{I}$ with $-7 < A \le j_1 = 2 \le j_2 = 4 \le B < j_3 = 5$.

The overloading of symbols when dealing with open intervals (x,y) and points (x,y) in two dimensions can be carried out for finite interval systems $\mathtt{I} \in \mathrm{InSy}(n)$ as well. Given some nonempty interval $[a,b] \in \mathtt{I}$, we can map $[a,b]$ to the point $(a,b) \in [n] \times [n]$ and via this we can interpret each $\mathtt{I} \in \mathrm{InSy}(n)$ as a subset $\mathtt{I} \subset [n] \times [n]$ (ignoring the empty set $\varnothing \in \mathtt{I}$). With this in mind, we can give a description of the map ϕ_k^n that is in direct analogy with the one given for ϕ_k^∞ in (2.3): Let $1 \le k \le n$, $\vec{j} = (j_1, \dots, j_k) \in [k:n]$, $\mathtt{I} \in \mathrm{InSy}(n)$ and some $1 \le a < b \le k$. Then it holds that

$$[a,b] \in \phi_k^n(\mathtt{I}, \vec{j}) \quad \Longleftrightarrow \quad \mathtt{I} \cap [j_{a-1}+1, j_a] \times [j_b, j_{b+1}-1] \neq \varnothing. \tag{3.1}$$

See Figure 7.

We notice that we have defined ϕ_n^{n+1} in two ways: at first in Section 1 as a function

$$\mathrm{InSy}(n+1) \times [n+1] \to \mathrm{InSy}(n)$$

and then in Definition 3.1 as a function

$$\mathrm{InSy}(n+1) \times [n:n+1] \to \mathrm{InSy}(n).$$

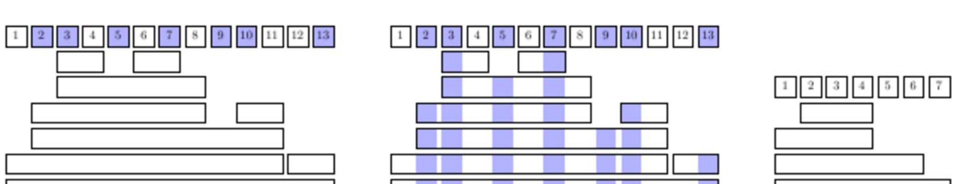

FIG. 7. *On the left, some interval system* $\mathtt{I} \in \mathrm{InSy}(13)$. *Highlighted in blue: the vector* $\vec{j} = (2,3,5,7,9,10,13) \in [7:13]$. *On the right, the interval system* $\phi_7^{13}(\mathtt{I}, \vec{j})$.

This "overloading" of the function symbol ϕ_n^{n+1} is justified by noticing that

$$[n+1] \to [n : n+1], \qquad k \mapsto (1, \dots, k-1, k+1, \dots, n+1)$$

is bijective and it holds that

$$\phi_n^{n+1}(\mathtt{I}, k) = \phi_n^{n+1}\bigl(\mathtt{I}, (1, \dots, k-1, k+1, \dots, n+1)\bigr) \tag{3.2}$$

for any n, $\mathtt{I} \in \mathrm{InSy}(n+1)$ and $k \in [n+1]$. Let $\pi \in \mathbb{S}_n$ and $k \le n$. If one deletes $k+1, \dots, n$ from the one-line notation of π, one obtains the one-line notation of some permutation of $[k]$ which we will call $\pi_{|k} \in \mathbb{S}_k$. This is consistent with building a permutation from a sequence of erasers: Let $(i_1, \dots, i_{n-1}) \in [2] \times \cdots \times [n]$ be a sequence of erasers and $\pi = b_n(i_1, \dots, i_{n-1})$, then $\pi_{|k} = b_k(i_1, \dots, i_{k-1})$. The next lemma shows that ϕ_k^n really describes the multistep deletion operations as claimed above and gives some further algorithmic properties.

LEMMA 3.2. *Let $n \in \mathbb{N}$ and $\mathtt{I} \in \mathrm{InSy}(n)$.*

(a) *If $1 \le k \le m \le n$ and $\vec{j} = (j_1, \dots, j_m) \in [m : n]$, $\vec{h} = (h_1, \dots, h_k) \in [k : m]$, then*

$$\phi_k^m\bigl(\phi_m^n(\mathtt{I}, \vec{j}), \vec{h}\bigr) = \phi_k^n\bigl(\mathtt{I}, (j_{h_1}, \dots, j_{h_k})\bigr).$$

Let $\pi \in \mathbb{S}_n$. For $1 \le k \le n-1$, let $i_k := (\pi_{|k+1})^{-1}(k+1) \in [k+1]$. Define $(\mathtt{I}_n, \mathtt{I}_{n-1}, \dots, \mathtt{I}_1)$ inductively by $\mathtt{I}_n := \mathtt{I}$ and $\mathtt{I}_{k-1} := \phi_{k-1}^k(\mathtt{I}_k, i_{k-1})$ for $2 \le k \le n$. Then for every $1 \le k \le n$:

(b) *If $\vec{j}_k^\pi$ is the enumeration of the set $\pi^{-1}([k])$, it holds that $\mathtt{I}_k = \phi_k^n(\mathtt{I}, \vec{j}_k^\pi)$.*

(c) *It holds that $\mathtt{I}_k = \pi_{|k}^{-1}(\pi(\mathtt{I})_{|k})$, where $\pi(\mathtt{I})$ and $\mathtt{I}_{|k}$ are the operations introduced for interval hypergraphs in Section 1.*

PROOF. (a) Let $1 \le a < b \le k$. Equation (3.1) yields

$$\begin{aligned}
&[a, b] \in \phi_k^m\bigl(\phi_m^n(\mathtt{I}, \vec{j}), \vec{h}\bigr) \\
&\iff \quad \phi_m^n(\mathtt{I}, \vec{j}) \cap [h_{a-1}+1, h_a] \times [h_b, h_{b+1}-1] \ne \varnothing \\
&\iff \quad \exists [a', b'] \subseteq [m] : \mathtt{I} \cap [j_{a'-1}+1, j_{a'}] \times [j_{b'}, j_{b'+1}-1] \ne \varnothing \quad \text{and} \\
&\qquad\quad h_{a-1}+1 \le a' \le h_a < h_b \le b' \le h_{b+1}-1 \\
&\iff \quad \mathtt{I} \cap [j_{h_{a-1}}+1, j_{h_a}] \times [j_{h_b}, j_{h_{b+1}}-1] \ne \varnothing \\
&\iff \quad [a, b] \in \phi_k^n\bigl(\mathtt{I}, (j_{h_1}, \dots, j_{h_k})\bigr).
\end{aligned}$$

(b) The vector $\vec{j}_k^\pi$ is the enumeration of the set $\pi^{-1}([k])$ and $\vec{j}_{k+1}^\pi$ is the enumeration of the set $\pi^{-1}([k+1])$. Now $i_k = (\pi_{|k+1})^{-1}(k+1)$ is that position in $\vec{j}_{k+1}^\pi$, that stems from the preimage of "$k+1$" under π. So if one removes the i_kth value in $\vec{j}_{k+1}^\pi$, one obtains precisely $\vec{j}_k^\pi$. The first equality $\mathtt{I}_k = \phi_k^n(\mathtt{I}, \vec{j}_k^\pi)$ now follows from (3.2) together with (1) by induction.

(c) For the second equation, we first argue that it is enough to prove it for $k = n-1$. Define $f_k^n(\mathtt{I}, \pi) := \pi_{|k}^{-1}(\pi(\mathtt{I})_{|k})$ and $g_k^n(\mathtt{I}, \pi) := \phi_k^n(\mathtt{I}, \vec{j}_k^\pi)$. For every $1 \le k \le m \le n$, one obtains

$$f_k^m\bigl(f_m^n(\mathtt{I}, \pi), \pi_{|m}\bigr) = f_k^n(\mathtt{I}, \pi) \quad \text{and} \quad g_k^m\bigl(g_m^n(\mathtt{I}, \pi), \pi_{|m}\bigr) = g_k^n(\mathtt{I}, \pi),$$

where the first equality follows easily from the definition. The second equality follows from (b). So it is enough to prove the second equality for the case $k = n-1$: First, relabeling $\mathtt{I}$ with π and then restricting $\pi(\mathtt{I})$ to the set $[n-1]$ results in the deletion of $\pi^{-1}(n) = i_{n-1}$. Then relabeling $\pi(\mathtt{I})_{|n-1}$ with the inverse of $\pi_{|n-1}$ results in reordering the set $\{1, 2, \dots, i_{n-1}-1, i_{n-1}+1, \dots, n\}$ in its usual order. So the result is precisely $\phi_{n-1}^n(\mathtt{I}, i_{n-1})$. $\square$

In particular, if $(I,\eta)=(I_n,\eta_n)_{n\in\mathbb{N}}$ is an erased-interval process and $S=(S_n)_{n\in\mathbb{N}}$ is the permutation process associated to η, then

$$I_k=\phi_k^n\big(I_n,\vec{j}_k^{S_n}\big)\quad\text{almost surely for all } 1\le k\le n. \tag{3.3}$$

In the next lemma, we will state and prove some technical features involving the maps ϕ_k^n and ϕ_k^∞. In particular, it shows that one can interpret the maps ϕ_k^∞ to be the *extensions of* ϕ_k^n *as* $n\to\infty$. It also explains why we have chosen our particular method of scaling finite interval systems and vectors from $[k:n]$.

LEMMA 3.3. *Let $n\in\mathbb{N}$ and $1\le k\le n$.*

(i) *For all $\vec{j}\in[k:n]$ and $\mathtt{I}\in\mathrm{InSy}(n)$,*

$$\phi_k^n(\mathtt{I},\vec{j})=\phi_k^\infty\big(n^{-1}\mathtt{I},n^{-1}\vec{j}\big).$$

(ii) *For all $K\in\mathrm{InSy}(\infty)$ and $(u_1,\dots,u_n)\in[0,1]^n_{\neq}$, one has with $\pi:=\pi(u_1,\dots,u_n)$*

$$\phi_k^\infty(K,u_{1:k},\dots,u_{k:k})=\phi_k^n\big(\phi_n^\infty(K,u_{1:n},\dots,u_{n:n}),\vec{j}_k^\pi\big).$$

(iii) *The map ϕ_k^∞ is measurable.*

PROOF. (i) For any $1\le a<b\le k$, it holds that

$$\begin{aligned}
&[a,b]\in\phi_k^n(\mathtt{I},\vec{j})\\
&\iff \mathtt{I}\cap[j_{a-1}+1,j_a]\times[j_b,j_{b+1}-1]\neq\varnothing\\
&\iff \text{there exists some } [A,B]\in\mathtt{I}\text{ such that}\\
&\qquad j_{a-1}+1\le A\le j_a<j_b\le B\le j_{b+1}-1\\
&\iff \text{there exists some } [A,B]\in\mathtt{I}\text{ such that}\\
&\qquad \frac{2j_{a-1}-1}{2n}<\frac{A-1}{n}<\frac{2j_a-1}{2n}<\frac{2j_b-1}{2n}<\frac{B}{n}<\frac{2j_{b+1}-1}{2n}\\
&\iff [a,b]\in\phi_k^\infty\big(n^{-1}\mathtt{I},n^{-1}\vec{j}\big).
\end{aligned}$$

(ii) Let $\mathtt{I}:=\phi_n^\infty(K,u_{1:n},\dots,u_{n:n})$. One obtains

$$\pi(\mathtt{I})=\big\{\{i\in[n]:x<u_i<y\}:(x,y)\in K\big\}\cup\big\{\{j\}:j\in[n]\big\}\cup\{\varnothing\}. \tag{3.4}$$

Thus restricting $\pi(\mathtt{I})$ to the set $\{1,\dots,k\}$ yields

$$\pi(\mathtt{I})_{|k}=\big\{\{i\in[k]:x<u_i<y\}:(x,y)\in K\big\}\cup\big\{\{j\}:j\in[k]\big\}\cup\{\varnothing\}. \tag{3.5}$$

Hence the claimed equality follows from Lemma 3.2(b) and (c).

(iii) Fix some $1\le a<b\le k$. One needs to show that the set

$$A:=\big\{(K,u_1,\dots,u_k)\in\mathrm{InSy}(\infty)\times[0,1]^k_<:K\cap(u_{a-1},u_a)\times(u_b,u_{b+1})\neq\varnothing\big\}$$

is a Borel subset of $\mathrm{InSy}(\infty)\times[0,1]^k_<$. For convenience, we consider only the case $2\le a<b\le k-1$. Define

$$B:=\big\{(K,u_1,u_2,u_3,u_4)\in\mathrm{InSy}(\infty)\times[0,1]^4_<:K\cap[u_1,u_2]\times[u_3,u_4]\neq\varnothing\big\}.$$

This is a closed subset of $\mathrm{InSy}(\infty)\times[0,1]^4_<$, and hence Borel. Let

$$C:=\big\{(K,u_1,u_2,u_3,u_4)\in\mathrm{InSy}(\infty)\times[0,1]^4_<:K\cap(u_1,u_2)\times(u_3,u_4)\neq\varnothing\big\}.$$

One has that

$$(K, u_1, \dots, u_k) \in A \quad \iff \quad (K, u_{a-1}, u_a, u_b, u_{b+1}) \in C$$

and so it is enough to argue that C is Borel. For $n \in \mathbb{N}$, define the set $C_n \subseteq \mathrm{InSy}(\infty) \times [0,1]^4_<$ by

$$(K, u_1, u_2, u_3, u_4) \in C_n \quad :\Leftrightarrow \quad \bigl(K, u_1 + n^{-1}, u_2 - n^{-1}, u_3 + n^{-1}, u_4 - n^{-1}\bigr) \in B.$$

Now every C_n is closed, and hence $B = \bigcup_{n \geq 1} C_n$ is Borel. $\square$

The next lemma shows that the Hausdorff distance between K and $k^{-1}\phi_k^\infty(K, u_1, \dots, u_k)$ can be bounded uniformly in K just depending on $(u_1, \dots, u_k) \in [0,1]^k_<$ in a nontrivial way. We temporarily use the abbreviation $\phi_k^\infty(K, u) := \phi_k^\infty(K, u_1, \dots, u_k)$ given some $u = (u_1, \dots, u_k) \in [0,1]^k_<$.

LEMMA 3.4. *Let $k \in \mathbb{N}$ and $u = (u_1, \dots, u_k) \in [0,1]^k_<$. Let F_u be the empirical distribution function associated to u, so $F_u(x) := k^{-1} \sum_{j=1}^k 1(u_j \leq x)$ for $x \in [0,1]$ and let F_u^- be the left-continuous version of F_u, so $F_u^-(x) = k^{-1} \sum_{j=1}^k 1(u_j < x)$. Let $K \in \mathrm{InSy}(\infty)$. Then the sampled and normalized interval system $k^{-1}\phi_k^\infty(K, u)$ has the concrete representation*

$$k^{-1}\phi_k^\infty(K, u) = \bigl\{\bigl(F_u(x), F_u^-(y)\bigr) : (x, y) \in K, F_u(x) \leq F_u^-(y)\bigr\} \cup \diagup,$$

and the Hausdorff distance to K can be bounded just in terms of u:

$$d_{\mathrm{haus}}\bigl(k^{-1}\phi_k^\infty(K, u), K\bigr) \leq \sup_{x \in [0,1]} \bigl|F_u(x) - x\bigr| + \sup_{y \in [0,1]} \bigl|F_u^-(y) - y\bigr|.$$

PROOF. Since $[a, b] \in \phi_k^\infty(K, u)$ iff $K \cap (u_{a-1}, u_a) \times (u_b, u_{b+1}) \neq \varnothing$, every interval $[a, b] \in \phi_k^\infty(K, u)$ is of the form

$$\left[1 + \sum_{j=1}^k 1(u_j \leq x), \sum_{j=1}^k 1(u_j < y)\right] \quad \text{for some } (x, y) \in K.$$

Hence the scaling by k yields the stated representation of $k^{-1}\phi_k^\infty(K, u)$. The distance bound is an easy consequence: For every $(x', y') \in K$, one has that

$$\min_{(x'', y'') \in K} \bigl|x' - F_u(x'')\bigr| + \bigl|y' - F_u^-(y'')\bigr| \leq \sup_{x \in [0,1]} \bigl|F_u(x) - x\bigr| + \sup_{y \in [0,1]} \bigl|F_u^-(y) - y\bigr|,$$

hence

$$d\bigl((x', y'), k^{-1}\phi_k^\infty(K, u)\bigr) \leq \sup_{x \in [0,1]} \bigl|F_u(x) - x\bigr| + \sup_{y \in [0,1]} \bigl|F_u^-(y) - y\bigr|,$$

and thus

$$\max_{(x', y') \in K} d\bigl((x', y'), k^{-1}\phi_k^\infty(K, u)\bigr) \leq \sup_{x \in [0,1]} \bigl|F_u(x) - x\bigr| + \sup_{y \in [0,1]} \bigl|F_u^-(y) - y\bigr|.$$

In the same way, one can argue that

$$\max_{(x', y') \in K} d\bigl(\bigl(F_u(x'), F_u^-(y')\bigr), K\bigr) \leq \sup_{x \in [0,1]} \bigl|F_u(x) - x\bigr| + \sup_{y \in [0,1]} \bigl|F_u^-(y) - y\bigr|.$$

This yields the distance bound. $\square$

We can now easily deduce that $n^{-1}I_n$ converges almost surely for every erased-interval process $(I_n,\eta_n)_{n\in\mathbb{N}}$. Let S be the permutation process corresponding to an eraser process η. For $1\le k\le n$ define

$$Y_k^n=\left(Y_{k,1}^n,\dots,Y_{k,k}^n\right):=n^{-1}\vec{j}_k^{S_n}, \tag{3.6}$$

where $\vec{j}_k^{S_n}\in[k:n]$ is the enumeration of the random k-set $S_n^{-1}([k])\subseteq[n]$. The scaled vector Y_k^n takes values in $[0,1]^k_<$. Now let $(U_i)_{i\in\mathbb{N}}$ be the U-process corresponding to η. One easily obtains that for every $k\in\mathbb{N}$,

$$Y_k^n=\left(Y_{k,1}^n,\dots,Y_{k,k}^n\right)\longrightarrow(U_{1:k},\dots,U_{k:k})\quad\text{almost surely as } n\to\infty. \tag{3.7}$$

Now we can prove the *strong law of large numbers* for erased-interval processes.

LEMMA 3.5. *Let $(I_n,\eta_n)_{n\in\mathbb{N}}$ be an erased-interval process. Then $n^{-1}I_n$ converges almost surely in the space* $(\mathrm{InSy}(\infty),d_{\mathrm{haus}})$ *toward some random variable I_∞ as $n\to\infty$.*

PROOF. We will prove that $(n^{-1}I_n)_{n\in\mathbb{N}}$ is a Chauchy sequence almost surely, which is sufficient since $(\mathrm{InSy}(\infty),d_{\mathrm{haus}})$ is complete. Let $n\le m$ and $Y_n^m=n^{-1}\vec{j}_n^{S_m}$ be defined like in (3.6), where $(S_n)_{n\in\mathbb{N}}$ is the permutation process corresponding to η. By (3.3) and Lemma 3.3(i), one obtains

$$n^{-1}I_n=n^{-1}\phi_n^\infty(m^{-1}I_m,Y_n^m)\quad\text{almost surely.}$$

Let $F_{Y_n^m}^-$ and $F_{Y_n^m}$ be the functions associated to Y_m^n like in Lemma 3.4 which then yields

$$d_{\mathrm{haus}}(n^{-1}I_n,m^{-1}I_m)\le\sup_{x\in[0,1]}\left|F_{Y_n^m}(x)-x\right|+\sup_{y\in[0,1]}\left|F_{Y_n^m}^-(y)-y\right|.$$

Now for every fixed n the vector Y_n^m converges almost surely toward $(U_{1:n},\dots,U_{n:n})$ as $m\to\infty$, where $U=(U_i)_{i\in\mathbb{N}}$ is the U-process associated to η; see (3.7). Since the uniform distribution on $[0,1]$ is diffuse, the Glivenko–Cantelli theorem yields that both $\sup_{x\in[0,1]}|F_{Y_n^m}(x)-x|$ and $\sup_{y\in[0,1]}|F_{Y_n^m}^-(y)-y|$ converge almost surely toward zero as $n,m\to\infty$. $\square$

Next, we present a lemma which we state and prove in a slightly more general form in Section 4. The proof presented there is based on topological features of isolated points in the *Sorgenfrey plane*.

If A, B are events, we will say that A implies B almost surely iff $A\cap B^C$ has probability zero. We will denote this by $A\overset{\text{a.s.}}{\Rightarrow}B$. Consequently, we will say that A and B are equivalent almost surely iff the symmetric difference $A\Delta B=A\cap B^C\cup A^C\cap B$ has probability zero and we will denote this by $A\overset{\text{a.s.}}{\Leftrightarrow}B$, so $A\overset{\text{a.s.}}{\Leftrightarrow}B\iff(A\overset{\text{a.s.}}{\Rightarrow}B\wedge B\overset{\text{a.s.}}{\Rightarrow}A)$.

LEMMA 3.6. *Let $U=(U_i)_{i\in\mathbb{N}}$ be a U-process, $n\ge 2$ and $0\le j_1<j_2<j_3<j_4\le n+1$. Let $Y_i:=U_{j_i:n}$ for $i\in\{1,2,3,4\}$. Consider the random rectangle*

$$R:=[Y_1,Y_2]\times[Y_3,Y_4]=\left\{(x,y)\in\triangledown:Y_1\le x\le Y_2\text{ and }Y_3\le y\le Y_4\right\}$$

and the interior of of that random rectangle

$$\mathrm{int}(R):=(Y_1,Y_2)\times(Y_3,Y_4)=\left\{(x,y)\in\triangledown:Y_1<x<Y_2\text{ and }Y_3<y<Y_4\right\}.$$

Let I_∞ be a random variable with values in $(\mathrm{InSy}(\infty),d_{\mathrm{haus}})$ *and independent of U. Then the sets $\{I_\infty\cap R\ne\varnothing\}$ and $\{I_\infty\cap\mathrm{int}(R)\ne\varnothing\}$ are events such that*

$$I_\infty\cap R\ne\varnothing\quad\overset{\text{a.s.}}{\Longrightarrow}\quad I_\infty\cap\mathrm{int}(R)\ne\varnothing.$$

In words: *The random compact set I_∞ almost surely either intersects R in its interior or not at all.*

PROOF. See Section 4. □

The next lemma states that one can extend the almost sure equality $I_k = \phi_k^n(I_n, \vec{j}_k^{S_n})$ that holds almost surely for every $1 \le k \le n$ to the case $n = \infty$. Hence in some sense, the maps ϕ_k^∞ are not just the algorithmic extension in the sense of Lemma 3.3, but also the *continuous extension*, continuous with respect to the randomized dynamics given by η.

LEMMA 3.7. *Let (I, η) be an erased-interval process and I_∞ be the a.s. limit of $n^{-1} I_n$ according to Lemma 3.5. Let $(S_n)_{n\in\mathbb{N}}$ and $(U_i)_{i\in\mathbb{N}}$ be the permutation process and the U-process corresponding to η. For $n \ge k$, let $Y_k^n = n^{-1} \vec{j}_k^{S_n}$ be like in (3.7). Then almost surely for every $k \in \mathbb{N}$,*

$$I_k = \lim_{n\to\infty} \phi_k^\infty(n^{-1} I_n, Y_k^n) = \phi_k^\infty\Big(\lim_{n\to\infty}(n^{-1} I_n, Y_k^n)\Big) = \phi_k^\infty(I_\infty, U_{1:k}, \dots, U_{k:k}).$$

PROOF. Let $Y_k^n = (Y_{k,1}^n, \dots, Y_{k,k}^n)$. With $\widehat{I_k} := \phi_k^\infty(I_\infty, U_{1:k}, \dots, U_{k:k})$, we need to prove that $I_k = \widehat{I_k}$ almost surely. Since both I_k and $\widehat{I_k}$ are random sets, we show that $I_k \subseteq \widehat{I_k}$ and $\widehat{I_k} \subseteq I_k$ almost surely. Since singleton sets are part of every interval system by definition, we only need to consider intervals $[a, b]$ with $1 \le a < b \le k$.

'$\subseteq$': We will show that $I_k \subseteq \widehat{I_k}$ almost surely. By Lemma 3.3(i), it holds that $I_k = \phi_k^\infty(n^{-1} I_n, Y_k^n)$ almost surely for every $n \ge k$, and hence by (2.3) one obtains

$$[a, b] \in I_k \quad \overset{\text{a.s.}}{\Leftrightarrow} \quad n^{-1} I_n \cap \big(Y_{k,a-1}^n, Y_{k,a}^n\big) \times \big(Y_{k,b}^n, Y_{k,b+1}^n\big) \neq \varnothing \quad \text{for all } n \ge k.$$

This yields

$$[a, b] \in I_k \quad \overset{\text{a.s.}}{\Rightarrow} \quad n^{-1} I_n \cap \big[Y_{k,a-1}^n, Y_{k,a}^n\big] \times \big[Y_{k,b}^n, Y_{k,b+1}^n\big] \neq \varnothing \quad \text{for all } n \ge k.$$

Now $n^{-1} I_n \to I_\infty$ and $[Y_{k,a-1}^n, Y_{k,a}^n] \times [Y_{k,b}^n, Y_{k,b+1}^n] \to [U_{a-1:k}, U_{a:k}] \times [U_{b-1:k}, U_{b:k}]$ almost surely, where both convergences take place in the space $(\mathcal{K}(\triangle), d_{\text{haus}})$. One can easily check that if $(K_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ are two sequences of compact sets converging toward K and G and if $K_n \cap G_n \neq \varnothing$ for all n, then also $K \cap G \neq \varnothing$. This yields

$$[a, b] \in I_k \quad \overset{\text{a.s.}}{\Rightarrow} \quad I_\infty \cap [U_{a-1:k}, U_{a:k}] \times [U_{b:k}, U_{b+1:k}] \neq \varnothing.$$

Now we apply Lemma 3.6 and obtain

$$[a, b] \in I_k \quad \overset{\text{a.s.}}{\Rightarrow} \quad I_\infty \cap (U_{a-1:k}, U_{a:k}) \times (U_{b:k}, U_{b+1:k}) \neq \varnothing,$$

so $[a, b] \in I_k \overset{\text{a.s.}}{\Rightarrow} [a, b] \in \phi_k^\infty(I_\infty, U_{1:k}, \dots, U_{k:k}) = \widehat{I_k}$, and thus $I_k \subseteq \widehat{I_k}$ almost surely.

'$\supseteq$': We will show that $\widehat{I_k} \subseteq I_k$ almost surely. If $[a, b] \in \widehat{I_k}$, there is by definition some point $z = (x, y) \in I_\infty$ such that $U_{a-1:k} < x < U_{a:k} \le U_{b:k} < y < U_{b+1:k}$, due to (2.3). Now since I_∞ is the almost sure limit of $n^{-1} I_n$ the point z is the limit of some sequence $z_n = (x_n, y_n) \in n^{-1} I_n$. Since $Y_k^n \to (U_{1:k}, \dots, U_{k:k})$ almost surely one obtains that almost surely $Y_{a-1,k}^n < x_n < Y_{a,k}^n \le Y_{b,k}^n < y_n < Y_{b+1,k}^n$ holds for all but finitely many $n \ge k$. In particular,

$$[a, b] \in \widehat{I_k} \quad \overset{\text{a.s.}}{\Rightarrow} \quad n^{-1} I_n \cap \big(Y_{a-1:k}^n, Y_{a,k}^n\big) \times \big(Y_{b:k}^n, Y_{b+1,k}^n\big) \neq \varnothing \quad \text{for some } n \ge k.$$

And so again by (2.3)

$$[a, b] \in \widehat{I_k} \quad \overset{\text{a.s.}}{\Rightarrow} \quad [a, b] \in \phi_k^\infty(n^{-1} I_n, Y_k^n) \quad \text{for some } n \ge k.$$

But because $I_k = \phi_k^\infty(n^{-1} I_n, Y_k^n)$ almost surely for all $n \ge k$ by Lemma 3.3(i), one obtains

$$[a, b] \in \widehat{I_k} \quad \overset{\text{a.s.}}{\Rightarrow} \quad [a, b] \in I_k$$

and so $\widehat{I_k} \subseteq I_k$ almost surely. □

The next lemma is used to obtain the topological description of the space of ergodic erased-interval processes.

LEMMA 3.8. *Let $(U_i)_{i\in\mathbb{N}}$ be a U-process and $(K_n)_{n\in\mathbb{N}} \subset \mathrm{InSy}(\infty)$ be a sequence in $\mathrm{InSy}(\infty)$ that converges toward some $K \in \mathrm{InSy}(\infty)$. Then, for every $k \in \mathbb{N}$, the sequence $\phi_k^\infty(K_n, U_{1:k}, \dots, U_{k:k})$ converges almost surely toward $\phi_k^\infty(K, U_{1:k}, \dots, U_{k:k})$ as $n \to \infty$.*

PROOF. Since the RVs under consideration now take values in a discrete space, convergence of a sequence means that the sequence stays finally constant. We fix some $k \in \mathbb{N}$ and some $1 \le a < b \le k$.

If $[a, b] \in \phi_k^\infty(K, U_{1:k}, \dots, U_{k:k})$, then $K \cap (U_{a-1:k}, U_{a:k}) \cap (U_{b:k}, U_{b+1:k}) \neq \varnothing$ by (2.3). Since $K_n \to K$, there is a sequence $z_n = (x_n, y_n) \in K_n$ that converges toward some point $z = (x, y) \in K$ with $U_{a-1:k} < x < U_{a:k} \le U_{b:k} < y < U_{b+1:k}$. Then for all but finitely many n, the same inequality holds for (x_n, y_n) instead of (x, y) and so $[a, b] \in \phi_k^\infty(K_n, U_{1:k}, \dots, U_{k:k})$ for all but finitely many n. Since interval systems always have finitely many elements, we have established that $\phi_k^\infty(K, U_{1:k}, \dots, U_{k:k}) \subseteq \phi_k^\infty(K_n, U_{1:k}, \dots, U_{k:k})$ almost surely for all but finitely many n.

Now let $[a, b]$ be such that $[a, b] \in \Phi_k^\infty(K_n, U_{1:k}, \dots, U_{k:k})$ for infinitely many n. So there is a subsequence (n_m) of $\mathbb{N}$ and points $z_{n_m} = (x_{n_m}, y_{n_m}) \in K_{n_m}$ with $U_{a-1:k} < x_{n_m} < U_{a:k} \le U_{b:k} < y_{n_m} < U_{b+1:k}$ for all m. Now since ◸ is compact, the sequence (z_{n_m}) has a further subsequence that converges toward some $z = (x, y)$. Since $K_n \to K$, one has that $z \in K$ and, furthermore, $U_{a-1:k} \le x \le U_{a:k} \le U_{b:k} \le y \le U_{b+1:k}$, so $K \cap [U_{a-1:k}, U_{a:k}] \times [U_{b:k}, U_{b+1:k}] \neq \varnothing$. Now with Lemma 3.6 one finally obtains that $\{[a, b] \in \phi_k^\infty(K_n, U_{1:k}, \dots, U_{k:k})$ for infinitely many $n\}$ almost surely implies $\{[a, b] \in \phi_k^\infty(K, U_{1:k}, \dots, U_{k:k})\}$. This completes the proof. □

Now we have all the ingredients we need to prove our first main theorem.

PROOF OF THEOREM 2.4. Let $U = (U_i)_{i\in\mathbb{N}}$ be a U-process and let $K \in \mathrm{InSy}(\infty)$. Let η be the eraser process corresponding to U. Then by the measurability of ϕ_k^∞ for every k (Lemma 3.3(iii)), the object

$$(I, \eta) := \big(\phi_n^\infty(K, U_{1:n}, \dots, U_{n:n}), \eta_n\big)_{n\in\mathbb{N}}$$

introduced in (2.4) is a stochastic process. Let $\mathrm{Law}(K)$ be its law. We will now argue that $\mathrm{Law}(K) \in \mathrm{erg}(\mathtt{ErInPr})$, so that the above defined process is an ergodic EIP. The first defining property of an erased-interval process is obvious. The third property follows from Lemma 3.3(ii). So we need to show that η_n is independent of $\mathcal{F}_{n+1} = \sigma(I_m, \eta_m : m \ge n+1)$ for every n. By definition, I_m is measurable with respect to $\sigma(U_{1:m}, \dots, U_{m:m})$ and the latter is included in $\sigma(U_{1:n+1}, \dots, U_{n+1:n+1}, U_{n+2}, U_{n+3}, \dots)$ for every $m \ge n+1$. One has the almost sure equality of σ-fields

$$\sigma(U_{1:n+1}, \dots, U_{n+1:n+1}) \vee \sigma(U_{n+2}, U_{n+3}, \dots) \quad \text{and} \quad \sigma(\eta_{n+1}, \eta_{n+2}, \dots).$$

Since η consists of independent RVs, η_n is thus independent of $\mathcal{F}_{n+1}$ for every $n \in \mathbb{N}$, so (I, η) really is an erased-interval process.

Now we will show that it is ergodic, so that $\mathcal{F}_\infty = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n$ is trivial almost surely. By elementary arguments one can show that $\mathcal{F}_\infty = \bigcap_{n\in\mathbb{N}} \sigma(I_m, \eta_m : m \ge n)$ is a.s. equal to $\bigcap_{n\in\mathbb{N}} \sigma(I_m : m \ge n)$. Since I_m is measurable with respect to $\sigma(U_{1:m}, \dots, U_{m:m})$ by construction, the latter σ-field is included in the exchangeable σ-field of U, and thus is trivial by Hewitt–Savage zero-one law. So we have proved that $\mathrm{Law}(K) \in \mathrm{erg}(\mathtt{ErInPr})$ for every $K \in \mathrm{InSy}(\infty)$.

Now let $(I, \eta) = (I_n, \eta_n)_{n\in\mathbb{N}}$ be an arbitrary erased-interval process. By Lemma 3.5, $n^{-1} I_n$ converges almost surely toward some $\mathrm{InSy}(\infty)$-valued RV I_∞ as $n \to \infty$. Let U be the U-process corresponding to η and S the corresponding permutation process. For $n \in \mathbb{N}$, define

$$\vec{S}_n := n^{-1}\big(S_n^{-1}(1), S_n^{-1}(2), \dots, S_n^{-1}(n), 0, 0, \dots\big) \in [0, 1]^{\mathbb{N}},$$

so $\vec{S}_n$ is considered to be a $[0, 1]^{\mathbb{N}}$-valued RV. $\vec{S}_n$ converges almost surely toward $(U_1, U_2, \dots)$. Since $\vec{S}_n$ and I_n are independent for every n, so are the a.s. limits U and I_∞.

By Lemma 3.7, it holds that $I_k = \phi^\infty(I_\infty, U_{1:k}, \dots, U_{k:k})$ almost surely for every $k \in \mathbb{N}$. If (I, η) is ergodic, then I_∞ is almost surely constant. This yields that the map $\mathrm{InSy}(\infty) \to \mathrm{erg}(\mathtt{ErInPr})$, $K \mapsto \mathrm{Law}(K)$ is surjective. The map is also injective: For this, it suffice to show that

$$d_{\mathrm{haus}}\big(k^{-1}\phi_k^\infty(K, U_{1:k}, \dots, U_{k:k}), K\big) \to 0$$

almost surely for $k \to \infty$. Since then, as limits are unique, $\mathrm{Law}(K)$ and $\mathrm{Law}(K')$ are clearly different for different $K, K' \in \mathrm{InSy}(\infty)$. That the above Hausdorff distance tends to zero now follows easily with the general bound obtained in Lemma 3.4 and the Glivenko–Cantelli theorem. So we have proven that $K \mapsto \mathrm{Law}(K)$ is bijective and Lemma 3.7 yields that every erased-interval process posses the described a.s. representation.

The last statement in Theorem 2.4 concerning the conditional distributions is immediate from the fact that the random objects I_∞ and U that occur in the a.s. representation are independent.

It remains to show that the map $K \mapsto \mathrm{Law}(K)$ is a homeomorphism. Since it is bijective, $\mathrm{InSy}(\infty)$ is compact and $\mathrm{erg}(\mathtt{ErInPr})$ is Hausdorff, we only need to show that it is continuous. So we need to show that if $K_n \to K$ in $(\mathrm{InSy}(\infty), d_{\mathrm{haus}})$ then $\mathrm{Law}(K_n) \to \mathrm{Law}(K)$. Fix some U-process and consider $(I^n, \eta) = (\phi_k^\infty(K_n, U_{1:k}, \dots, U_{k:k}), \eta_k)_{k\in\mathbb{N}}$, where η is the eraser process corresponding to U. Now for every n one has $\mathrm{Law}(I^n, \eta) = \mathrm{Law}(K_n)$. By Lemma 3.8, one has $(I^n, \eta) \to (I, \eta)$ almost surely as $n \to \infty$, where (I, η) is constructed by sampling from K via U. Now almost sure convergence implies convergence in law. □

PROOF OF THEOREM 2.5. We first argue that $\mathrm{Law}(I, \eta) \mapsto \mathrm{Law}(H)$ is a surjective, affine and continuous map from $\mathtt{ErInPr}$ to $\mathtt{ExInHy}$.

So let $(I_n, \eta_n)_{n\in\mathbb{N}}$ be an erased-interval process and let $S = (S_n)_{n\in\mathbb{N}}$ be the permutation process corresponding to η. Let $H_n := S_n(I_n)$. We first show that $H = (H_n)_{n\in\mathbb{N}}$ is an exchangeable interval hypergraph on $\mathbb{N}$. The exchangeability follows from the fact that S_n is a uniform permutation independent of I_n. Now for every $\pi \in \mathbb{S}_n$ one has $\pi(H_n) = \pi(S_n(I_n)) = \pi \circ S_n(I_n)$. Now $\pi \circ S_n$ is again a uniform permutation independent of I_n, so $\pi(H_n) \sim H_n$ for every $n \in \mathbb{N}$. Now we have that $I_n = \phi_n^{n+1}(I_{n+1}, S_{n+1}^{-1}([n]))$ and the latter term is by Lemma 3.2 equal to $(S_{n+1})_{|n}^{-1}((S_{n+1}(I_{n+1}))_{|n})$. Now since $S_n = (S_{n+1})_{|n}$ applying $S_n(\cdot)$ on both sides of $I_n = \phi_n^{n+1}(I_{n+1}, S_{n+1}^{-1}([n]))$ almost surely yields $H_n = (H_{n+1})_{|n}$. Hence H is an exchangeable interval hypergraph on $\mathbb{N}$. The map $\mathrm{Law}(I, \eta) \mapsto \mathrm{Law}(H)$ is clearly affine and continuous, so we need to argue that it is surjective.

Take some arbitrary exchangeable interval hypergraph $H = (H_n)_{n\in\mathbb{N}}$ and perform the following steps:

1. As explained in Section 1, given some interval hypergraph $\mathtt{H} \in \mathrm{InHy}(n)$ there exists some $\mathtt{I} \in \mathrm{InSy}(n)$ and a permutation $\pi \in \mathbb{S}_n$ such that $\mathtt{H} = \pi(\mathtt{I})$. For every $\mathtt{H}$, fix some $\mathtt{I}_{\mathtt{H}} \in \mathrm{InSy}(n)$ and a permutation $\pi_{\mathtt{H}} \in \mathbb{S}_n$ with $\mathtt{H} = \pi_{\mathtt{H}}(\mathtt{I}_{\mathtt{H}})$.
2. Consider the sequence $\mathtt{I}_{H_1}, \mathtt{I}_{H_2}, \dots$ of random interval systems, so $\mathtt{I}_{H_n}$ is a $\mathrm{InSy}(n)$-valued RV for every n. If S_n is a uniform random permutation of $[n]$ independent of H_n, then $S_n(\mathtt{I}_{H_n})$ has the same law as H_n: One has that $\mathtt{I}_{H_n} = \pi_{H_n}^{-1}(H_n)$ and so $S_n(\mathtt{I}_{H_n}) = S_n \circ \pi_{H_n}^{-1}(H_n)$. The random permutation $S_n \circ \pi_{H_n}^{-1}$ is again uniform and independent of H_n and the claim follows by exchangeability of H_n.
3. Let $(S_n)_{n\in\mathbb{N}}$ be a permutation process independent of H. For all $k, n \in \mathbb{N}$, define

$$I_k^n := \begin{cases} \phi_k^n(\mathtt{I}_{H_n}, \vec{j}_k^{S_n}) & \text{if } 1 \le k \le n, \\ \text{arbitrary element of } \mathrm{InSy}(k) & \text{else.} \end{cases}$$

Now we have defined, for each n, a stochastic process $(I^n, \eta) = (I^n_k, \eta_k)_{k\in\mathbb{N}}$ such that $I^n_k \in \mathrm{InSy}(k)$ for every k, where η is the eraser process corresponding to S. The law of each process (I^n, η) is a member of the compact metrizable space $\mathcal{M}_1(\prod_{k\in\mathbb{N}} \mathrm{InSy}(k) \times [k+1])$. Denote the law of the nth process by L_n.

4. The sequence $(L_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(L_{n_k})_{k\in\mathbb{N}}$. Let L be its limit and $(I, \eta) = (I_k, \eta_k)_{k\in\mathbb{N}}$ be a stochastic process with law L.
5. We claim that (I, η) is an erased-interval process and that its law, namely L, serves as the desired preimage for $\mathrm{Law}(H)$ with respect to the map under consideration:

(a) $L \in \mathtt{ErInPr}$: For every fixed $n \in \mathbb{N}$, the finite sequence of RVs $(\phi^n_k(\mathtt{I}_{H_n}, \vec{j}^{S_n}_k), \eta_k)_{1\le k\le n}$ is a finite Markov chain with co-transition probabilities θ introduced in (1.5). Now for every subsequence n_k tending to infinity, the first n_k-component part of the law of L_{n_k} is such a Markov chain with co-transitions given by θ. Elementary arguments show that the limit law L is thus in total the law of a Markov chain with co-transitions given by θ, thus $L \in \mathtt{ErInPr}$.

(b) By the algorithmic expression of ϕ^n_k presented in Lemma 3.2 for every $1 \le k \le n$, one obtains $S_k(I^n_k) = (S_n(\mathtt{I}_{H_n}))_{|k}$. Now in (2) it was explained that $S_n(\mathtt{I}_{H_n})$ has the same law as H_n and so $S_k(I^n_k)$ has the same law as H_k. This proves that $\mathtt{ErInPr} \to \mathtt{ExInHy}$, $\mathrm{Law}(I, \eta) \mapsto \mathrm{Law}(H)$ is surjective.

The concrete representation of $H_n = S_n(I_n)$ follows directly from the definitions; it holds that $I_n = \phi^\infty_n(I_\infty, U_{1:n}, \ldots, U_{n:n})$ almost surely and by that

$$S_n(I_n) = \big\{\{j \in [n] : x < U_j < y\} : (x, y) \in I_\infty\big\} \cup \big\{\{j\} : j \in [n]\big\} \cup \{\varnothing\}$$

almost surely. This completes the proof. □

4. Intersections of random sets. In this section, we will prove Lemma 3.6 which is used at two crucial points in the proof of Theorem 2.4. For this, we will establish the following.

PROPOSITION 4.1. *Let (Y_1, Y_2, Y_3, Y_4) be a $[0, 1]^4_<$-valued random vector such that for all $i \neq j$ the conditional law of Y_i given Y_j is almost surely diffuse. Consider the random rectangle*

$$R := [Y_1, Y_2] \times [Y_3, Y_4] = \big\{(x, y) \in \text{◸} : Y_1 \le x \le Y_2 \text{ and } Y_3 \le y \le Y_4\big\}$$

and its interior

$$\mathrm{int}(R) := (Y_1, Y_2) \times (Y_3, Y_4) = \big\{(x, y) \in \text{◸} : Y_1 < x < Y_2 \text{ and } Y_3 < y < Y_4\big\}.$$

Let $K \subseteq$ ◸ be any nonempty compact subset. Then K almost surely intersects the random rectangle R in its interior or not at all, more formally: For every $K \in \mathcal{K}(\text{◸})$, the set $\{K \cap R \neq \varnothing, K \cap \mathrm{int}(R) = \varnothing\}$ is an event with $\mathbb{P}(K \cap R \neq \varnothing, K \cap \mathrm{int}(R) = \varnothing) = 0$.

See Figure 8 for a visualization.

Our proof of this theorem relies on a topological feature of the *Sorgenfrey plane*. The Sorgenfrey plane is a topological space $\mathcal{S}^2$ on the set of points $\mathbb{R}^2$ where we choose the set of rectangles of the form $[a, b) \times [c, d)$ with $a < b$ and $c < d$ as a basis for the topology. This topology really refines the usual topology on $\mathbb{R}^2$, thus *more subsets of* $\mathbb{R}^2$ *are open in* $\mathcal{S}^2$. In fact, the topological space $\mathcal{S}^2$ is no longer "nice": Although it is separable, it is not metrizable. Because there are more open sets, it could happen that in a given subset $A \subseteq \mathbb{R}^2$ more points $x \in A$ are *isolated* than in the Euclidean case. A point $x \in A$ is called isolated if there is an open set $U \subseteq \mathbb{R}^2$ with $U \cap A = \{x\}$. Denote the set of isolated points of $A \subset \mathbb{R}^2$ by $\mathrm{iso}(A)$.

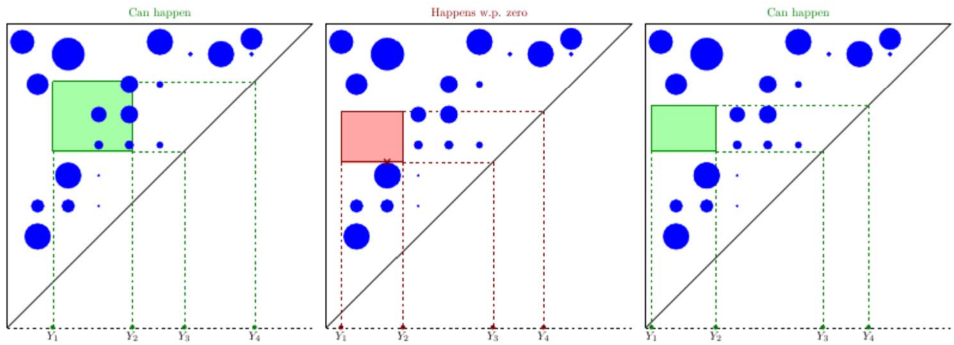

FIG. 8. *The compact set K in blue. With probability one the left or the right case appears, provided the random rectangle has the above stated properties. The middle case, intersection only on the boundary, does not appear almost surely.*

In the usual euclidean topology on $\mathbb{R}^2$ sets of isolated points are at most countable. This feature is lost in $\mathcal{S}^2$, in fact the set of isolated points may well be uncountable. For example, in the uncountable set $\{(x, -x) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ *every point* is isolated: for some point $(x, -x)$ just take the isolating and open neighborhood $[x, x+1) \times [-x, -x+1)$. However, we will show that the set of isolated points for any given subset in the Sorgenfrey plane is "small enough for our purposes": The set $\{(x, -x) : x \in \mathbb{R}\}$, although uncountable, is just the graph of the strictly decreasing function $f(x) = -x$ and a similar feature holds for any given set of isolated points in the Sorgenfrey plane. We do not know if the following proposition is a new result, hence we prove it.

PROPOSITION 4.2. *For every subset $A \subseteq \mathbb{R}^2$ of the Sorgenfrey plane, the set of isolated points* $\mathrm{iso}(A)$ *can be covered by the union of countably many graphs of strictly decreasing functions $f_i : \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{N}$.*

PROOF. Let $\varepsilon > 0$ and define the subset $\mathrm{iso}^\varepsilon(A)$ of A by

$$z \in \mathrm{iso}^\varepsilon(A) \quad :\Longleftrightarrow \quad z = (x, y) \in A \quad \text{and} \quad [x, x+\varepsilon) \times [y, y+\varepsilon) \cap A = \{z\}.$$

Now $z \in A$ is isolated iff there is some $\varepsilon > 0$ such that $z \in \mathrm{iso}^\varepsilon(A)$, in particular

$$\mathrm{iso}(A) = \bigcup_{n \in \mathbb{N}} \mathrm{iso}^{1/n}(A).$$

Fix $\varepsilon > 0$. For some point $z = (x, y) \in \mathbb{R}^2$, define

$$U(z, \epsilon) := \left[x - \frac{\varepsilon}{4}, x + \frac{\varepsilon}{4}\right] \times \left[y - \frac{\varepsilon}{4}, y + \frac{\varepsilon}{4}\right],$$

so $U(z, \epsilon)$ is a closed square that has z as its midpoint and whose edges are of length $\varepsilon/2$. With these definitions, one obtains that for every $z \in \mathbb{R}^2$ and every two different points $(a_1, a_2), (b_1, b_2) \in \mathrm{iso}^\varepsilon(A) \cap U(z, \varepsilon)$:

$$a_1 < b_1 \quad \text{and} \quad a_2 > b_2 \quad \text{or} \quad b_1 < a_1 \quad \text{and} \quad b_2 > a_2.$$

So the set $\mathrm{iso}^\varepsilon(A) \cap U(z, \varepsilon)$ can be covered by the graph of some strictly decreasing function $f_{z,\varepsilon} : [x - \frac{\varepsilon}{4}, x + \frac{\varepsilon}{4}] \to [y - \frac{\varepsilon}{4}, y + \frac{\varepsilon}{4}]$. Now for every $\varepsilon > 0$ one can choose countably many $z_1^\varepsilon, z_2^\varepsilon, \ldots \in \mathbb{R}^2$ such that $\mathbb{R}^2 = \bigcup_{i \in \mathbb{N}} U(z_i^\varepsilon, \varepsilon)$. This implies that $\mathrm{iso}^\epsilon(A)$ is covered by the union of the graphs of the functions $f_{z_i^\varepsilon, \varepsilon}$, $i \in \mathbb{N}$. Consequently, the whole of $\mathrm{iso}(A)$ is contained in the union of all graphs of the functions $f_{z_i^{1/n}, 1/n}$, $n, i \in \mathbb{N}$. $\square$

FIG. 9. *A closed donut A in black and the subset of points* $\mathrm{iso}(A) \subset A$ *that are isolated w.r.t. Sorgenfrey topology in red.* $\mathrm{iso}(A)$ *can be covered by the graphs of two strictly decreasing functions.*

We could introduce "tilted" Sorgenfrey planes by choosing rectangles of the form $(a, b] \times (c, d]$ or $[a, b) \times (c, d]$ or $(a, b] \times [c, d)$ as a basis for the topology. Of course, the analogue statement of Proposition 4.2 would be true for these as well, one would only need to interchange "strictly decreasing" with "strictly increasing" in the latter two cases. In Figure 9, one can see a subset $K \subseteq \mathbb{R}^2$ where the isolated points of K are highlighted in red. Here, two strictly decreasing functions are sufficient to cover the isolated points. The union of the isolated points w.r.t. to all four tilted Sorgenfrey planes would cover the whole (Euclidean) boundary of this set K.

PROOF OF PROPOSITION 4.1. Fix some $K \in \mathcal{K}(◸)$. We first prove that $\{K \cap R \neq \varnothing, K \cap \mathrm{int}(R) = \varnothing\}$ is an event. For this, we introduce

$$A := \{u = (u_1, u_2, u_3, u_4) \in [0, 1]^4_< : K \cap [u_1, u_2] \times [u_3, u_4] \neq \varnothing\}$$

and

$$B := \{u = (u_1, u_2, u_3, u_4) \in [0, 1]^4_< : K \subseteq ((u_1, u_2) \times (u_3, u_4))^C\}.$$

Then

$$\{K \cap R \neq \varnothing, K \cap \mathrm{int}(R) = \varnothing\} = \{(Y_1, Y_2, Y_3, Y_4) \in A \cap B\}.$$

Now both A and B are closed subsets of $[0, 1]^4_<$: Suppose $u^n = (u^n_1, u^n_2, u^n_3, u^n_4)$ is a sequence in A converging toward some $u = (u_1, u_2, u_3, u_4) \in [0, 1]^4_<$. By definition of A for every n, there is a point $y^n = (y^n_1, y^n_2) \in K$ such that $u^n_1 \le y^n_1 \le u^n_2$ and $u^n_3 \le y^n_2 \le u^n_4$. Since K is compact there exists a converging subsequence y_{n_k} with limit $y = (y_1, y_2) \in K$. Since $u^n \to u$ it holds that $u_1 \le y_1 \le u_2$ and $u_3 \le y_2 \le u_4$. So $y \in K \cap [u_1, u_2] \times [u_3, u_4]$, and thus $u \in A$. With the same basic considerations, one can prove that B is closed. Hence $\{K \cap R \neq \varnothing, K \cap \mathrm{int}(R) = \varnothing\}$ is an event.

We will now detect the points in K that can be hit by rectangles on the boundary but not in the interior. Let us introduce the subset W of K as the set of all points $(x, y) \in K$ that are "isolated on the west," meaning there exists some open rectangle r where the closure of the "west" side of r contains (x, y) away from its corners and that is disjoint from K. Formally,

$$W := \left\{(x, y) \in K : \begin{array}{l} \text{there are } (b, c, d) \text{ s.t. } 0 \le x < b < c < y < d \le 1 \\ \text{and } (x, b) \times (c, d) \cap K = \varnothing \end{array}\right\}.$$

In the same way, we introduce the set of points of K that are isolated on the east, north or to the south (denoted by E, N and S). The points in K that can be hit by rectangles on the interior of the four boundary sides but not in the interior of the rectangle are given by $E \cup W \cup N \cup S$.

Rectangles could hit points in K on the corners but not on the interior. We define the set of points that could be hit by the southwest corner of a rectangle but not in its interior to be $SW \subseteq K$, formally:

$$SW := \left\{(x, y) \in K : \begin{array}{l} \text{there are } (b, d) \text{ s.t. } 0 < x < b < y < d < 1 \\ \text{and } [x, b) \times [y, d) \cap K = \{(x, y)\} \end{array}\right\}.$$

In the same way, we introduce the sets SE, NW and NE. Now we can characterize the event under consideration: for that we introduce the projections $\pi_1 : \triangledown \to [0,1]$ and $\pi_2 : \triangledown \to [0,1]$ by $\pi_1((x,y)) = x$ and $\pi_2((x,y)) = y$. Now we claim:

($\star$) For every $(u_1,u_2,u_3,u_4) \in [0,1]^4_<$ such that $[u_1,u_2]\times[u_3,u_4]\cap K \neq \varnothing$ and $(u_1,u_2)\times(u_3,u_4)\cap K = \varnothing$ at least one of the following eight statements is true:

$$u_1 \in \pi_1(W), \qquad u_2 \in \pi_1(E), \qquad u_3 \in \pi_2(S), \qquad u_4 \in \pi_2(N),$$
$$(u_1,u_3) \in SW, \qquad (u_2,u_3) \in SE, \qquad (u_1,u_4) \in NW, \qquad (u_2,u_4) \in NE.$$

The sets $\pi_1(W)$, $\pi_1(E)$, $\pi_2(S)$ and $\pi_2(N)$ are all at most countably infinite, which can be seen quite easily.

Now observe that $SW \subseteq \mathrm{iso}(K)$ so by Proposition 4.2 there are countably many strictly monotone functions $f_i^{\mathrm{sw}} : \mathbb{R} \to \mathbb{R}$ such that $SW \subseteq \mathrm{iso}(K) \subseteq \bigcup_{i\in\mathbb{N}} \mathrm{graph}(f_i^{\mathrm{sw}})$. The sets SE, NW and NE are contained in the isolated points of K with respect to the above mentioned tilted Sorgenfrey planes, so in each case there are countably many strictly monotone functions $f_i^{\mathrm{se}}, f_i^{\mathrm{nw}}, f_i^{\mathrm{ne}}, \in \mathbb{N}$ whose graphs cover the corresponding sets. Since any monotone function $f : \mathbb{R} \to \mathbb{R}$ is measurable, the graph $\mathrm{graph}(f)$ of any monotone function f is a Borel subset of $\mathbb{R}^2$. Thus by using ($\star$) and the union bound for probabilities we arrive at the following upper bound for the probability we are interested in:

$$\begin{aligned}
&\mathbb{P}(K \cap R \neq \varnothing, K \cap \mathrm{int}(R) = \varnothing)\\
&\quad \le \mathbb{P}(Y_1 \in \pi_1(W)) + \mathbb{P}(Y_2 \in \pi_1(E)) + \mathbb{P}(Y_3 \in \pi_2(S)) + \mathbb{P}(Y_4 \in \pi_2(N))\\
&\qquad + \sum_{i=1}^{\infty} \bigl[\mathbb{P}(Y_3 = f_i^{\mathrm{sw}}(Y_1)) + \mathbb{P}(Y_3 = f_i^{\mathrm{se}}(Y_2))\\
&\qquad + \mathbb{P}(Y_4 = f_i^{\mathrm{nw}}(Y_1)) + \mathbb{P}(Y_4 = f_i^{\mathrm{ne}}(Y_2))\bigr].
\end{aligned}$$

Now our assumptions on the law of (Y_1,Y_2,Y_3,Y_4) are needed to conclude that each of the probabilities occurring above is zero: Since the conditional law of Y_i given Y_j is almost surely diffuse, the unconditional law of each Y_i is diffuse. Since the projection sets are countable, the first four probabilities are zero. Now we take a look at $\mathbb{P}(Y_3 = f_i^{\mathrm{sw}}(Y_1))$:

$$\mathbb{P}(Y_3 = f_i^{\mathrm{sw}}(Y_1)) = \int_{[0,1]} \mathbb{P}(Y_3 = f_i^{\mathrm{sw}}(y) | Y_1 = y)\, d\mathbb{P}^{Y_1}(y) = 0,$$

since the conditional law of Y_3 given $Y_1 = y$ is diffuse for $\mathbb{P}^{Y_1}$-almost all $y \in [0,1]$. The same reasoning holds for every other remaining term. □

The lemma used in Section 3 was a little different from Proposition 4.1, but can now be easily deduced from it.

PROOF OF LEMMA 3.6. First, assume that $n \ge 4$ and $1 \le j_1 < j_2 < j_3 < j_4 \le n$. In this case, the random vector (Y_1,Y_2,Y_3,Y_4) satisfies the assumptions of Proposition 4.1. Now the only difference is that the compact set under consideration may be random, so we need to make sure that we really deal with an event. Let

$$A := \{(K,u_1,u_2,u_3,u_4) \in \mathcal{K}(\triangledown) \times [0,1]^4_< : K \cap [u_1,u_2]\times[u_3,u_4] \neq \varnothing\}$$

and

$$B := \{(K,u_1,u_2,u_3,u_4) \in \mathcal{K}(\triangledown) \times [0,1]^4_< : K \cap (u_1,u_2)\times(u_3,u_4) \neq \varnothing\}.$$

As in the proof of Theorem 4.1, one easily obtains that both A and B^C are closed subsets of $\mathcal{K}(\triangledown) \times [0,1]^4_<$, hence we really deal with events and the result follows easily from Theorem 4.1 and the assumed independence of I_∞ and U, using Fubini's theorem.

Now to the case $j_1 = 0$ and/or $j_4 = n + 1$. Assume that $j_1 = 0$ and $j_4 \leq n$. Since we have defined $U_{0:n} = -1$ the random rectangle R cannot intersect I_∞ at its left side or at one of its two left corners, they do not belong to ◸. So if an intersection on the boundary of the rectangle takes place, coordinates have to be involved that satisfy the assumptions of the almost sure diffuseness and we refer to the arguments presented in the proof of Proposition 4.1. The same strategy succeeds in the cases $j_1 \geq 1$ and $j_4 = n + 1$ or $j_1 = 0$ and $j_4 = n + 1$. In the latter case, one only needs to argue with the RVs U_{j_2} and U_{j_3} which always fulfill the almost sure-diffuseness assumption. □

5. Applications. In this section, we will connect our results for exchangeable interval hypergraphs and erased-interval processes to exchangeable hierarchies on $\mathbb{N}$ in the sense of [10], to the Martin boundary of Rémy's tree growth chain in the sense of [7] and to composition structures in the sense of [15]. At the end, we will present an outlook for future research.

5.1. *Hierarchies and Schröder trees.*

DEFINITION 5.1. A *hierarchy* on $[n]$ is a subset $\mathtt{H} \subseteq \mathcal{P}([n])$ such that $\varnothing \in \mathtt{H}$, $\{j\} \in \mathtt{H}$ for every $j \in [n]$, $[n] \in \mathtt{H}$ and such that for all $e, f \in \mathtt{H}$ it holds that $e \cap f \in \{e, f, \varnothing\}$. Let $\mathrm{Hier}(n)$ be the set of all hierarchies on $[n]$.

Hierarchies on $[n]$ are equivalent to leaf-labeled unordered rooted trees in which every internal node has at least two descendants (see [10]). Every such tree can be embedded into the plane: for every internal node one chooses an ordering on the descendants. Now the nodes of that ordered tree get equipped with the canonical lexicographic ordering. This yields a linear order l on $[n]$: i is smaller then j w.r.t. l if and only if the leaf labeled with i is smaller than the leaf labeled with j with respect to the lexicographic ordering. Thereby the hierarchy $\mathtt{H}$ becomes an interval hypergraph. So $\mathrm{Hier}(n) \subseteq \mathrm{InHy}(n)$ for every n. Hierarchies are closed under restriction and relabeling; this is immediate from the definition.

DEFINITION 5.2. An exchangeable hierarchy on $\mathbb{N}$ is an exchangeable interval hypergraph $(H_n)_{n\in\mathbb{N}}$ such that $H_n \in \mathrm{Hier}(n)$ for every n. Let $\mathtt{ExHier}$ be the space of all possible laws of exchangeable hierarchies on $\mathbb{N}$.

In [10], the authors provided two de Finetti-type characterization theorems for exchangeable hierarchies on $\mathbb{N}$: At first, they worked out a description via *sampling from real trees*. Given any exchangeable hierarchy on $\mathbb{N}$, they constructed a real tree and a probability measure concentrated on the leafs of that tree and then that they proved that the law of the exchangeable hierarchy is the same as the law of the sequence of finite combinatorial subtrees obtained by sampling at i.i.d. position according to the probability measure on that tree. We will not give further details here and refer the reader to [10], Theorem 5. From this result, they obtained a second representation result, sampling from interval hierarchies on $[0, 1)$: these are subsets $\mathcal{H} \subseteq \mathcal{P}([0, 1))$ such that every $e \in \mathcal{H}$ is an interval (w.r.t. to the usual linear order on $[0, 1)$), such that $\{x\} \in \mathcal{H}$ for all $x \in [0, 1)$, $[0, 1) \in \mathcal{H}$ and such that $e, f \in \mathcal{H}$ implies $e \cap F \in \{\varnothing, e, f\}$. Denote the space of all interval hierarchies on $[0, 1)$ by $\mathrm{IHier}([0, 1))$. This space was then equipped with a measurability structure: The authors considered the σ-field generated by restriction to finite sets, that is given some $\mathcal{H} \in \mathrm{IHier}([0, 1))$ and a finite subset $A \subset [0, 1)$ they considered $\mathcal{H}_{|A} := \{A \cap e : e \in \mathcal{H}\}$. Their representation result reads as follows, where we restate it in a slightly different but equivalent form in which tail σ-fields are replaced by exchangeable σ-fields.

THEOREM ([10], Theorem 4). *Let $(U_i)_{i\in\mathbb{N}}$ be a U-process and let $\mathcal{H} \in \mathrm{IHier}([0,1))$. Let* $\mathrm{Law}^{ih}(\mathcal{H})$ *be the distribution of*

$$\big(\big\{\{i \in [n] : U_i \in e\} : e \in \mathcal{H}\big\} \cup \{\varnothing\}\big)_{n\in\mathbb{N}}.$$

Then:

(a) $\mathrm{Law}^{ih}(\mathcal{H}) \in \mathrm{erg}(\mathtt{ExHier})$ *for every* $\mathcal{H} \in \mathrm{IHier}([0,1))$ *and the map*

$$\mathrm{IHier}\big([0,1)\big) \to \mathrm{erg}(\mathtt{ExHier}), \qquad H \mapsto \mathrm{Law}^{ih}(\mathcal{H})$$

is surjective.

(b) *For any exchangeable hierarchy* $H = (H_n)_{n\in\mathbb{N}}$ *on* $\mathbb{N}$, *there is a random* H-*measurable interval hierarchy* $\mathcal{H}$ *on* $[0,1)$ *such that the conditional law of* H *given the exchangeable* σ-*field of* H *is almost surely equal to* $\mathrm{Law}^{ih}(\mathcal{H})$.

The map in (a) is far from being injective. In fact, the cardinality of $\mathrm{IHier}([0,1))$ is strictly larger than the cardinality of $\mathrm{erg}(\mathtt{ExHier})$, which already is uncountable. As a consequence of Kuratowski's theorem, it is not possible to introduce a metric on $\mathrm{IHier}([0,1))$ that would turn it into a complete separable metric space.

We will offer an improvement of statement (a) below that avoids this. For this, we observe that $\mathtt{ExHier}$ is a simplex and by definition, it is a subset of $\mathtt{ExInHy}$. But it is not just included: $\mathtt{ExHier}$ is a *closed face* in the simplex $\mathtt{ExInHy}$, in particular $\mathrm{erg}(\mathtt{ExHier}) \subseteq \mathrm{erg}(\mathtt{ExInHy})$. We will use this fact to deduce a representation result concerning exchangeable hierarchies from our representation result concerning $\mathtt{ExInHy}$. We will perform this deduction by passing to erased-type objects at first: For some linear order l on $[n]$, let $\mathrm{Hier}(n,l)$ be the set of all hierarchies that are interval hypergraphs w.r.t. l. As it is the case with interval hypergraphs, for every n and every $\mathtt{H} \in \mathrm{Hier}(n)$ there is some bijection π such that $\pi(\mathtt{H})$ is an interval hypergraph w.r.t. the usual linear order $<$.

DEFINITION 5.3. A hierarchy $\mathtt{T}$ on $[n]$ that is an interval hypergraph w.r.t. to the usual linear order $<$ is called a *Schröder tree*. Let $\mathrm{STree}(n) := \mathrm{Hier}(n,<)$ be the set of Schröder trees on $[n]$.

Schröder trees on $[n]$ are usually introduced as rooted ordered trees with exactly n leafs in which every internal node has at least two descendants (see [1]). Our definition is equivalent: Given any rooted ordered tree with exactly n leafs, one can enumerate the leafs from $1,\dots,n$ in the lexicographic ordering. Now to every node of the tree we attach the set of numbers of those leaves that are descendants of that node. Every such set of nodes is an interval. We collect all these intervals into a set and include the empty set. The result is an element of $\mathrm{STree}(n)$ that determines the tree-structure in a unique way. By definition, every Schröder tree on $[n]$ is an interval system on $[n]$ as well, so $\mathrm{STree}(n) \subseteq \mathrm{InSy}(n)$. One can directly see whether an element $\mathtt{I} \in \mathrm{InSy}(n)$ is a Schröder tree: this is the case if and only if for every $[a_1,b_1],[a_2,b_2] \in \mathtt{I}$ with $a_1 < a_2 \le b_1$ it holds that $b_2 \le b_1$, that is, iff intervals do not overlap. This reflects the property that, in any tree, different subtrees are either disjoint or included. Schröder trees are stable under removing elements according to ϕ_n^{n+1}. If $\mathtt{T} \in \mathrm{STree}(n)$ and $\vec{j} \in [k:n]$, then $\phi_k^n(\mathtt{T},\vec{j}) \in \mathrm{InSy}(k)$ is the ordered subtree induced at the leaves $\vec{j}$. The leaves are then renamed by $[k]$ in a strictly increasing manner; see Figure 10 for a visualization of some Schröder tree.

DEFINITION 5.4. An *erased-Schröder tree process* is an erased-interval process $(T_n, \eta_n)_{n\in\mathbb{N}}$ such that $T_n \in \mathrm{STree}(n)$ for every n. Let $\mathtt{ErSTPr}$ be the space of all possible laws of erased-Schröder tree processes.

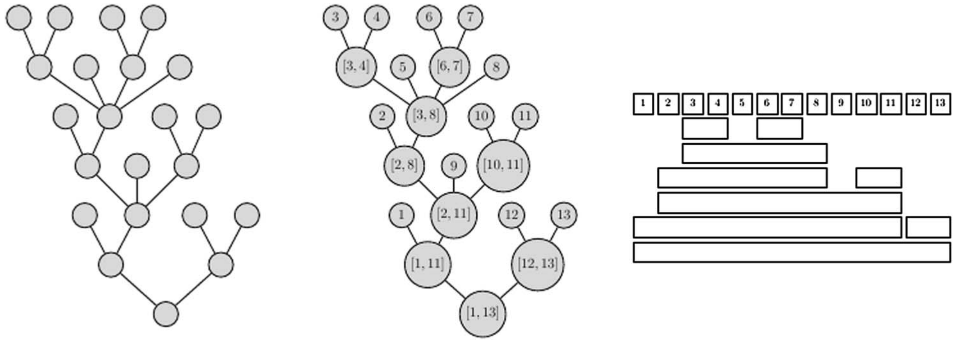

FIG. 10. *On the left, a Schröder tree with* 13 *leafs. Next, the canonical labeling of that tree obtained from the lexicographic order of the leafs. On the right, the representation as an interval system.*

As in the case above, `ErSTPr` is not just a simplex and a subset of `ErInPr`, but also a closed face in `ErInPr`; in particular, erg(`ErSTPr`) $\subseteq$ erg(`ErInPr`). Since we have identified erg(`ErInPr`) with the space $(\mathrm{InSy}(\infty), d_{\mathrm{haus}})$, we only have to find that subspace of $\mathrm{InSy}(\infty)$ that yields Schröder trees. Since $\mathrm{InSy}(\infty)$ was the analogue of $\mathrm{InSy}(n)$ with $n \to \infty$, the analogue for $\mathrm{STree}(n)$ with $n \to \infty$ is straightforward to obtain.

DEFINITION 5.5. $K \in \mathrm{InSy}(\infty)$ is called a *Schröder tree on* $(0, 1)$ iff $(0, 1) \in K$ and for every $(x_1, y_1), (x_2, y_2) \in K$ with $x_1 < x_2 < y_1$ it holds that $y_2 \le y_1$. Denote by $\mathrm{STree}(\infty)$ the set of all Schröder trees on $(0, 1)$.

In particular, for each $K \in \mathrm{STree}(\infty)$, it holds that

$$(x, y) \in K \quad \Longrightarrow \quad K \cap (x, y) \times (y, 1] \cup [0, x) \times (x, y) = \varnothing; \tag{5.1}$$

see Figure 11 for a visualization of this property.

LEMMA 5.6. *Schröder trees form a substructure of interval systems that satisfy the following consistency properties*:

(i) $\mathrm{STree}(\infty)$ *is a closed subset of* $\mathrm{InSy}(\infty)$.
(ii) *If* $K \in \mathrm{STree}(\infty)$, $k \in \mathbb{N}$, $(u_1, \dots, u_k) \in [0, 1]^k_<$ *then* $\phi_k^\infty(K, u_1, \dots, u_k) \in \mathrm{STree}(k)$.
(iii) *For* $n \in \mathbb{N}$ *and* $\mathtt{T} \in \mathrm{STree}(n)$, *it holds that* $n^{-1}\mathtt{T} \in \mathrm{STree}(\infty)$.

PROOF. (i) Let $(K_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathrm{STree}(\infty)$ with limit $K \in \mathrm{InSy}(\infty)$. We need to show that K is a Schröder tree on $(0, 1)$. It is obvious that $(0, 1) \in K$. Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in K$ with $x_1 < x_2 < y_1$. Since $d_{\mathrm{haus}}(K_n, K) \to 0$, there are $z_1^n = (x_1^n, y_1^n)$, $z_2^n = (x_2^n, y_2^n) \in K_n$ such that $z_1^n \to z_1$ and $z_2^n \to z_2$ as $n \to \infty$. Since

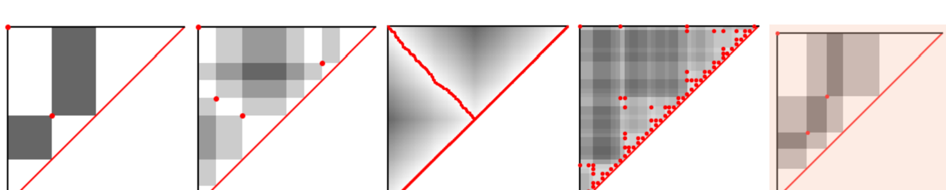

FIG. 11. *The first four pictures show some* $K \in \mathrm{STree}(\infty)$. *For every point* $(x, y) \in K$, *we shaded in the "forbidden" rectangles* $(x, y) \times (y, 1] \cup [0, x) \times (x, y)$ *to illustrate property* (5.1) *using opacity to emphasize that rectangles can overlap. The opacity is normalized so that the darkest part has a constant color. The fifth picture shows some* $K \notin \mathrm{STree}(\infty)$.

$x_1 < x_2 < y_1$, it holds that $x_1^n < x_2^n < y_1^n$ for all but finitely many n. Since all K_n are Schröder trees, it follows that $y_2^n \le y_1^n$ for all but finitely many n. This implies $y_2 \le y_1$. Hence K is a Schröder tree.

(ii) Let $\mathtt{T} := \phi_k^\infty(K, u_1, \dots, u_k)$ and $[a_1, b_1], [a_2, b_2] \in \mathtt{T}$ with $a_1 < a_2 \le b_1$. Hence there are some $(x_1, y_1), (x_2, y_2) \in K$ such that

$$u_{a_1-1} < x_1 < u_{a_1} < u_{b_1} < y_1 < u_{b_1+1} \quad \text{and}$$
$$u_{a_2-1} < x_2 < u_{a_2} < u_{b_2} < y_2 < u_{b_2+1}.$$

Since $a_1 \le a_2 - 1$, it holds that $u_{a_1} \le u_{a_2-1}$ and so $x_1 < x_2$. Further, since $a_2 \le b_1$ it holds that $u_{a_2} \le u_{b_1}$ and so $x_2 < y_1$. So $x_1 < x_2 < y_1$ and because K is assumed to be a Schröder tree it holds that $y_2 \le y_1$. Hence $u_{b_2} < u_{b_1+1}$ and so $b_2 \le b_1$. Furthermore, it holds that $[n] \in \mathtt{T}$, since $(0, 1) \in K$. This shows that $\mathtt{T}$ is a Schröder tree.

(iii) Let $(x_1, y_1), (x_2, y_2) \in n^{-1}\mathtt{T}$ with $x_1 < x_2 < y_1$. By definition of $n^{-1}\mathtt{T}$, there are some $[a_1, b_1], [a_2, b_2] \in \mathtt{T}$ with $(x_i, y_i) = ((a_i - 1)/n, b_i/n)$. One obtains $a_1 < a_2 \le b_1$. Since $\mathtt{T}$ is a Schröder $b_2 \le b_1$, and hence $y_2 \le y_1$. Furthermore, because $[n] \in \mathtt{T}$ also $(0, 1) \in n^{-1}\mathtt{T}$. Hence $n^{-1}\mathtt{T} \in \mathrm{STree}(\infty)$. □

Lemma 5.6 and Theorem 2.4 directly yield a concrete description of erased-Schröder tree processes and in that way a description of exchangeable hierarchies on $\mathbb{N}$. The latter serves as an improvement of part (a) of the theorem given in [10].

COROLLARY 5.7. *$K \mapsto \mathrm{Law}(K)$ is a homeomorphism from $\mathrm{STree}(\infty)$ to $\mathrm{erg}(\mathtt{ErSTPr})$. One has the following concrete representation: Let $(T, \eta) = (T_n, \eta_n)_{n\in\mathbb{N}}$ be an erased-Schröder tree process. Then $n^{-1}T_n$ converges almost surely as $n \to \infty$ toward some $\mathrm{STree}(\infty)$-valued random variable T_∞. Let $U = (U_i)_{i\in\mathbb{N}}$ be the U-process corresponding to η. Then T_∞ and U are independent and one has the equality of processes*

$$(T_n, \eta_n)_{n\in\mathbb{N}} = \big(\phi_n^\infty(T_\infty, U_{1:n}, \dots, U_{n:n}), \eta_n\big)_{n\in\mathbb{N}} \quad \textit{almost surely.}$$

In particular, for every erased-Schöder tree process (T, η) the conditional law of (T, η) given the terminal σ-field $\mathcal{F}_\infty$ is $\mathrm{Law}(T_\infty)$ almost surely and T_∞ generates $\mathcal{F}_\infty$ almost surely.

See Figure 12 for an illustration of the overall procedure.

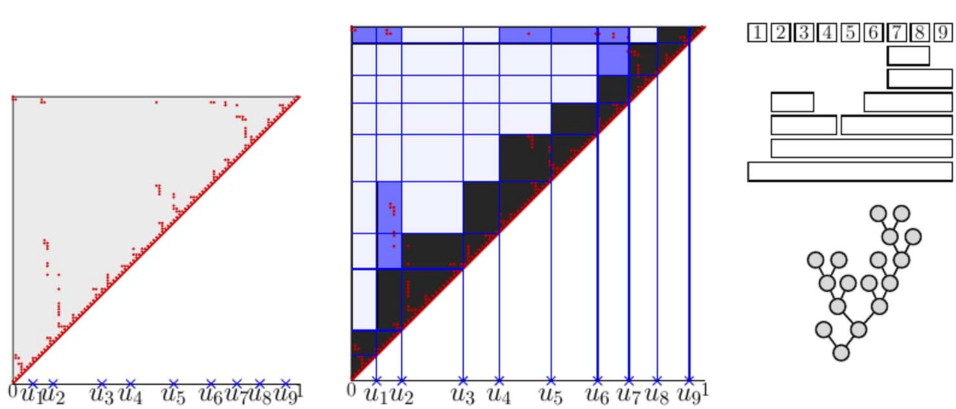

FIG. 12. *On the left, a realization of $n^{-1}I_n$ for $I_n \sim \mathrm{unif}(\mathrm{BinTree}(n))$ with $n = 100$. On the right, the binary tree $\phi_9^n(n^{-1}I_n, u_1, \dots, u_9)$ for some $(u_1, \dots, u_9) \in [0, 1]^9_<$, once pictured as a set of intervals and once in in the usual way as a tree.*

COROLLARY 5.8. *Let $(U_i)_{i\in\mathbb{N}}$ be a U-process and let $K \in \mathrm{STree}(\infty)$. Let $\mathrm{Law}^{ih}(K)$ be the distribution of*

$$\big(\big\{\{j \in [n] : x < U_j < y\} : (x, y) \in K\big\} \cup \big\{\{j\} : j \in [n]\big\} \cup \{\varnothing\}\big)_{n\in\mathbb{N}}.$$

Then the following holds:

(a′) $\mathrm{Law}^{ih}(K) \in \mathrm{erg}(\mathtt{ExHier})$ *for every* $K \in \mathrm{STree}(\infty)$ *and the map*

$$\mathrm{STree}(\infty) \to \mathrm{erg}(\mathtt{ExHier}), \qquad K \mapsto \mathrm{Law}^{ih}(K)$$

is surjective and continuous.

If one identifies every point $(x, y) \in K \in \mathrm{STree}(\infty)$ with the open interval $(x, y) \subseteq (0, 1)$ and the diagonal points with singletons, then one can regard every Schröder tree on $(0, 1)$ as a interval hierarchy on $(0, 1)$, and hence $\mathrm{STree}(\infty) \subseteq \mathrm{IHier}([0, 1))$. The space $\mathrm{STree}(\infty)$ is much “smaller” and more structured then the large space $\mathrm{IHier}([0, 1))$, since $(\mathrm{STree}(\infty), d_{\mathrm{haus}})$ is a compact metric space. Although we reduced the cardinality of the space used to describe all ergodic exchangeable laws, our representation is far from unique as well; many different elements in $\mathrm{InSy}(\infty)$ describe the same ergodic exchangeable hierarchy. Note that we obtained this result without using real trees.

REMARK 5.9. Forman [9] introduces a certain class of rooted weighted real trees, so-called *interval partition trees*, and obtains an improvement of Theorem 4 from [10] by showing a one-to-one correspondence between laws of ergodic exchangeable hierarchies and mass-structural equivalence classes of interval partition trees. As interval partition trees are different from Schröder trees on $(0, 1)$, we think it would pay to explore the connection between the two approaches describing exchangeable hierarchies in the future.

5.2. *Binary trees*. A binary tree on $[n]$ is a rooted ordered trees with exactly n leafs in which every internal node has exactly two descendants (as a consequence, there are $n - 1$ internal nodes). Thus we can introduce binary trees as subsets of Schröder trees: A tree is binary iff for all choices of three disjoint subtrees there exists a fourth subtree that includes exactly two of the former and is disjoint to the third. This can be checked for leaves, which are the subtrees of size one. We present the following equivalent definition for binary trees, first in the finite case and then in the limit.

DEFINITION 5.10. A Schröder tree $\mathtt{T} \in \mathrm{STree}(n)$ is called a *binary tree*, if for every $1 \le j_1 < j_2 < j_3 \le n$ there is some $[a, b] \in \mathtt{T}$ with either $a \le j_1 < j_2 \le b < j_3$ or $j_1 < a \le j_2 < j_3 \le b$. Let $\mathrm{BinTree}(n) \subseteq \mathrm{STree}(n)$ be the set of all binary trees on $[n]$.

DEFINITION 5.11. Let U_1, U_2, U_3 be i.i.d. uniform RVs. An element $K \in \mathrm{STree}(\infty)$ is called a *binary tree on* $(0, 1)$ if $\phi_3^\infty(K, U_{1:3}, U_{2:3}, U_{3:3})$ is almost surely a binary tree on [3].

LEMMA 5.12. *Binary trees form a substructure of Schröder trees that satisfy the following consistency properties*:

(i) $\mathrm{BinTree}(\infty)$ *is a closed subset of* $\mathrm{STree}(\infty)$.

(ii) *If* $K \in \mathrm{BinTree}(\infty)$ *and* $(U_i)_{i\in\mathbb{N}}$ *is a* U*-process and* $k \in \mathbb{N}$, *then* $\phi_k^\infty(K, U_{1:k}, \dots, U_{k:k}) \in \mathrm{BinTree}(k)$ *almost surely.*

(iii) *If* $(\mathtt{T}_n)_{n\in\mathbb{N}}$ *is a sequence with* $\mathtt{T}_n \in \mathrm{BinTree}(n)$ *and such that* $n^{-1}\mathtt{T}_n$ *converges toward some* K, *then* $K \in \mathrm{BinTree}(\infty)$.

PROOF. (i) Let K_n be a sequence in BinTree(∞) converging toward some $K \in$ STree(∞). By Lemma 3.8, the sequence $T_n = \phi_3^\infty(K_n, U_{1:3}, U_{2:3}, U_{3:3})$ converges almost surely toward $T = \phi_3^\infty(K, U_{1:3}, U_{2:3}, U_{3:3})$. Since all T_n are almost surely binary by definition, so is T, and hence $K \in$ BinTree(∞).

(ii) This follows from the fact that a finite tree $\mathtt{T} \in$ STree(k) is binary iff $\phi_3^k(\mathtt{T}, \vec{j})$ is binary for every $\vec{j} \in [3:k]$.

(iii) Given any $\mathtt{T} \in$ BinTree(n) define the set of points $A = \{(a-1)/n : [a,b] \in \mathtt{T}\} \cup \{b/n : [a,b] \in \mathtt{T}\}$. Now if $(u_1, u_2, u_3) \in [0,1]^3_<$ is such that $u_i \notin A$ for $i = 1,2,3$ and $|u_1 - u_3| > 2/n$, then $\phi_3^\infty(\mathtt{T}, u_1, u_2, u_3) \in$ BinTree(3). Now let $\mathtt{T}_n \in$ BinTree(n) and U_1, U_2, U_3 be i.i.d. uniform on [0, 1]. Let A_n be the A-set corresponding to $\mathtt{T}_n$ and $B = \bigcup_n A_n$. Since B is countable, $U_{i:3} \notin B$ almost surely for all $i = 1,2,3$. Almost surely there is an N such that $|U_{1:3} - U_{3:3}| > 2/n$ for all $n \geq N$. Consequently, $\phi_3^\infty(n^{-1}\mathtt{T}_n, U_{1:3}, U_{2:3}, U_{3:3})$ is almost surely binary for all but finitely many n. Hence with Lemma 3.8 the limit $\phi_3^\infty(K, U_{1:3}, U_{2:3}, U_{3:3})$ is almost surely binary and so $K \in$ BinTree(∞). □

Again one obtains as a special case of Theorem 2.4 an almost sure characterization of erased-binary tree processes.

COROLLARY 5.13. *$K \mapsto$ Law(K) is a homeomorphism from* BinTree(∞) *to* erg(ErBTPr). *One has the following concrete representation: Let $(T, \eta) = (T_n, \eta_n)_{n\in\mathbb{N}}$ be an erased-binary tree process. Then $n^{-1}T_n$ converges almost surely as $n \to \infty$ toward some* BinTree(∞)*-valued random variable T_∞. Let $U = (U_i)_{i\in\mathbb{N}}$ be the U-process corresponding to η. Then T_∞ and U are independent and one has the equality of processes*

$$(T_n, \eta_n)_{n\in\mathbb{N}} = \big(\phi_n^\infty(T_\infty, U_{1:n}, \dots, U_{n:n}), \eta_n\big)_{n\in\mathbb{N}} \quad \textit{almost surely.}$$

In particular, for every erased-binary tree process (T, η), the conditional law of (T, η) given the terminal σ-field $\mathcal{F}_\infty$ is Law(T_∞) *almost surely and T_∞ generates $\mathcal{F}_\infty$ almost surely.*

REMARK 5.14. One could introduced exchangeable *binary* hierarchies on $\mathbb{N}$: some exchangeable hierarchy $H = (H_n)_{n\in\mathbb{N}}$ on $\mathbb{N}$ is called binary if every H_n is binary as a tree. One could have obtained as a consequence of Corollary 5.13 the exact analogue of Corollary 5.8 with BinTree(∞) instead of STree(∞).

5.3. *Martin boundaries and limits of ordered discrete structures.* We will give a very short definition of Martin boundary that is adapted best to our already used choice of symbols and refer the reader to [4, 7, 11, 29] for more details. We introduce this concept for interval systems first and then relate this to Martin boundaries associated with Schröder trees and finally with binary trees.

For any $1 \leq k \leq n$, let $\vec{J}$ be a random vector uniformly distributed on $[k:n]$. For any $\mathtt{I}_k \in$ InSy(k) and $\mathtt{I}_n \in$ InSy(n), define

$$\gamma(\mathtt{I}_k, \mathtt{I}_n) := \mathbb{P}\big(\phi_k^n(\mathtt{I}_n, \vec{J}) = \mathtt{I}_k\big) = \frac{1}{\binom{n}{k}} \#\{\vec{j} \in [k:n] : \phi_k^n(\mathtt{I}_n, \vec{j}) = \mathtt{I}_k\} \tag{5.2}$$

and for $k > n$ set $\gamma(\mathtt{I}_k, \mathtt{I}_n) := 0$. The value $\gamma(\mathtt{I}_k, \mathtt{I}_n)$ is obtained by counting how often the smaller interval system $\mathtt{I}_k$ is *embedded* into the larger interval system $\mathtt{I}_n$ and divides this amount by the maximal possible number of such embeddings. One can think of $\gamma(\mathtt{I}_k, \mathtt{I}_n)$ to be the *density of the small $\mathtt{I}_k$ in the large $\mathtt{I}_n$*. This interpretation is in line with a emerging field in the area of limits of combinatorial objects, most famously discussed for graph limits. The connection to exchangeability and related areas is a commonly used tool that helps to understand the limiting behaviors of such density numbers as the size of the large object tends to infinity; see [2, 6, 18, 24].

The object γ and erased-interval processes are linked as follows: For any erased-interval process $(I_n, \eta_n)_{n\in\mathbb{N}}$ the first coordinate process $(I_n)_{n\in\mathbb{N}}$ is a *Markov chain with co-transition probabilities* γ, that is, $\mathbb{P}(I_k = \mathtt{I}_k | I_n = \mathtt{I}_n) = \gamma(\mathtt{I}_k, \mathtt{I}_n)$ for all $1 \le k \le n$, $\mathtt{I}_k \in \mathrm{InSy}(k)$ and $\mathtt{I}_n \in \mathrm{InSy}(n)$ with $\mathbb{P}(I_n = \mathtt{I}_n) > 0$. By Kolmogorov's extension theorem, the reverse is true in the following sense: To any Markov chain $(\hat{I}_n)_{n\in\mathbb{N}}$ with $\hat{I}_n \in \mathrm{InSy}(n)$ and co-transition probabilities given by γ, there is a unique (in law) erased-interval process $(I_n, \eta_n)_{n\in\mathbb{N}}$ such that $(\hat{I}_n)_{n\in\mathbb{N}}$ and $(I_n)_{n\in\mathbb{N}}$ have the same distribution.

Given any sequence $(\mathtt{I}_n)_{n\in\mathbb{N}}$ of interval systems with $\mathtt{I}_n \in \mathrm{InSy}(m_n)$ for some sequence $m_n \to \infty$, one says that this sequence is *γ-convergent* iff $\gamma(\mathtt{I}, \mathtt{I}_n)$ converges as $n \to \infty$ for every $k, \mathtt{I} \in \mathrm{InSy}(k)$. We think of the pointwise defined functions $\lim_{n\to\infty} \gamma(\cdot, \mathtt{I}_n) : \bigcup_{k\ge 1} \mathrm{InSy}(k) \to [0, 1]$ to be the limit objects associated to γ-convergent sequences. The set of all functions $\bigcup_{k\ge 1} \mathrm{InSy}(k) \to [0, 1]$ obtainable in this way constitutes the *Martin boundary associated to* γ. This Martin boundary can be described equivalently as a set of laws: For any γ-convergent sequence $(\mathtt{I}_n)_{n\in\mathbb{N}}$, there exists a unique (in law) erased-interval process $(I_n, \eta_n)_{n\in\mathbb{N}}$ such that

$$\mathbb{P}(I_k = \mathtt{I}_k) = \lim_{n\to\infty} \gamma(\mathtt{I}_k, \mathtt{I}_n) \quad \text{for all } k \in \mathbb{N}, \mathtt{I}_k \in \mathrm{InSy}(k). \tag{5.3}$$

This again is a consequence of Kolmogorov's existence theorem. We identify the Martin boundary associated to γ with the set of all laws of EIPs that fulfill (5.3) for some γ-convergent sequence and we define this set as $\partial(\mathtt{ErInPr})$. So in particular, $\partial(\mathtt{ErInPr}) \subseteq \mathtt{ErInPr}$. General theory yields that $\partial(\mathtt{ErInPr})$ is always a closed subset of $\mathtt{ErInPr}$ and that every extreme point of $\mathtt{ErInPr}$ is a point in the Martin boundary as well, so $\mathrm{erg}(\mathtt{ErInPr}) \subseteq \partial(\mathtt{ErInPr}) \subseteq \mathtt{ErInPr}$. It is often the case that extreme points and Martin boundary coincide and this is also the case here: We will not present a proof of this fact here but direct the reader to [12], Satz 3.4.12, where it was shown that Martin boundary and extreme points generally coincide in the context of exchangeability in discrete structures. The proof presented there was largely inspired by the proof of [7], Corollary 5.21, which shows that Martin boundary and extreme points coincide in the context of Rémy's tree growth chain. Given the equality of extreme points and Martin boundaries, we directly obtain the following corollary to Theorem 2.4.

COROLLARY 5.15. *Let $(U_i)_{i\in\mathbb{N}}$ be a U-process. For any γ-convergent sequence $(\mathtt{I}_n)_{n\in\mathbb{N}}$ of interval systems, there exists a unique $K \in \mathrm{InSy}(\infty)$ such that*

$$\lim_{n\to\infty} \gamma(\mathtt{I}, \mathtt{I}_n) = \mathbb{P}\big(\phi_k^{\infty}(K, U_{1:k}, \dots, U_{k:k}) = \mathtt{I}\big)$$

holds for every $k \in \mathbb{N}$, $\mathtt{I} \in \mathrm{InSy}(k)$. This map $\mathrm{InSy}(\infty) \to \partial(\mathtt{ErInPr})$ yields a homeomorphic description of the Martin boundary of interval systems with respect to γ.

Since Schröder trees and binary trees are both stable under removing elements according to ϕ_k^n, one directly obtains the following.

COROLLARY 5.16. *Let $(U_i)_{i\in\mathbb{N}}$ be a U-process. For any γ-convergent sequence $(\mathtt{T}_n)_{n\in\mathbb{N}}$ of Schröder trees, there exists a unique $K \in \mathrm{STree}(\infty)$ such that*

$$\lim_{n\to\infty} \gamma(\mathtt{T}, \mathtt{T}_n) = \mathbb{P}\big(\phi_k^{\infty}(K, U_{1:k}, \dots, U_{k:k}) = \mathtt{T}\big)$$

holds for every $k \in \mathbb{N}$, $\mathtt{T} \in \mathrm{STree}(k)$. This map $\mathrm{STree}(\infty) \to \partial(\mathtt{ErSTPr})$ yields a homeomorphic description of the Martin boundary of Schröder trees with respect to γ.

COROLLARY 5.17. *Let* $(U_i)_{i\in\mathbb{N}}$ *be a* U*-process. For any* γ*-convergent sequence* $(\mathtt{T}_n)_{n\in\mathbb{N}}$ *of binary trees, there exists a unique* $K \in \mathrm{BinTree}(\infty)$ *such that*

$$\lim_{n\to\infty} \gamma(\mathtt{T}, \mathtt{T}_n) = \mathbb{P}\big(\phi_k^{\infty}(K, U_{1:k}, \dots, U_{k:k}) = \mathtt{T}\big)$$

holds for every $k \in \mathbb{N}$, $\mathtt{T} \in \mathrm{BinTree}(k)$. *This map* $\mathrm{BinTree}(\infty) \to \partial(\mathtt{ErBTPr})$ *yields a homeomorphic description of the Martin boundary of binary trees with respect to* γ.

EXAMPLE 3. In [7], two examples of γ-convergent sequences of binary trees $(\mathtt{T}_n)_{n\in\mathbb{N}}$ were considered:

1. Spine trees: $\mathtt{T}_n \in \mathrm{BinTree}(n)$ is binary of hight $n-1$ that grows from the root left-right-left-right-$\cdots$.
2. Complete trees: $\mathtt{T}_n \in \mathrm{BinTree}(2^n)$ is the complete binary tree of hight n.

Both sequences are γ-convergent. The limit of spine trees is given by

$$K = \big\{(x, 1-x) : 0 \le x \le 0.5\big\} \cup \diagup$$

and the limit of complete trees is given by

$$K = \bigcup_{n\ge 0} \left\{ \left(0, \frac{1}{2^n}\right), \left(\frac{1}{2^n}, \frac{2}{2^n}\right), \left(\frac{2}{2^n}, \frac{3}{2^n}\right), \dots, \left(\frac{2^n-1}{2^n}, 1\right) \right\} \cup \diagup.$$

REMARK 5.18. A general theory of exchangeability in discrete structures that can be applied to prove the equality of Martin boundaries and extreme points is presented in the author's Ph.D. thesis [12].

REMARK 5.19. Putting together all the topological properties presented in this paper, one can deduce that a sequence $(\mathtt{I}_n)_{n\in\mathbb{N}}$ with $\mathtt{I}_n \in \mathrm{InSy}(m_n)$ and $m_n \to \infty$ is γ-convergent iff $m_n^{-1}\mathtt{I}_n$ converges in $(\mathrm{InSy}(\infty), d_{\mathrm{haus}})$.

Next, we present Rémy's tree growth chain (RTGC) and translate the notion of Martin boundary used in [7] to our situation. RTGC is a Markov chain $(T_n)_{n\in\mathbb{N}}$ with $T_n \in \mathrm{BinTree}(n)$ for every n that can be obtained as follows: $T_1 \in \mathrm{BinTree}(1)$ is the unique binary tree consisting only of a root vertex. The transitions are as follows: Given some binary tree $\mathtt{T} \in \mathrm{BinTree}(n)$, one chooses one of the $2n-1$ nodes in $\mathtt{T}$ uniform at random. Let v be that node. Then one cuts the subtree with root v off and puts it aside. At the position of v, one places the unique binary tree with two leaves (the "cherry"). The put-aside subtree then is placed at one of the two leaves of the cherry, chosen with equal probability. The resulting tree is binary by construction and has $n+1$ leaves. The resulting process $T = (T_n)_{n\in\mathbb{N}}$ is Rémy's tree growth chain. In [7], it was shown that T has co-transition probabilities γ and each T_n is *uniform* on $\mathrm{BinTree}(n)$. The Martin boundary associated to Rémy's tree growth chain described in [7] is equivalent to the above introduced Martin boundary of binary trees associated to γ. To obtain a description of that Martin boundary, the authors first used the Kolmogorov existence theorem to construct *labeled infinite Rémy bridges*. Such an object is basically a process $(T_n, S_n)_{n\in\mathbb{N}}$ according to an erased-binary tree process $(T_n, \eta_n)_{n\in\mathbb{N}}$ in which S is the permutation process according to η. The variables η appeared in their work as well, but were called "L_i" instead of "η_i"; see [7], Lemma 5.3. Arriving at labeled infinite Rémy bridges (equivalent to erased-binary tree processes), they constructed what they called *exchangeable didendritic systems*, certain infinite combinatorial exchangeable objects; see [7], Definition 5.8. They then reduced the problem of describing the Martin boundary to the task of describing ergodic didendritic systems by showing that Martin boundary and extreme points coincide. For every ergodic didendritic system, they introduce certain indicator arrays that inherit exchangeability and then employ the classical de Finetti theorem to obtain almost sure convergence. The resulting almost sure limits were then used to construct a *binary real tree with a probability measure concentrated on the leafs* ("Rt" for short). The construction of that Rt basically depends on $(S_n(T_n))_{n\in\mathbb{N}}$ alone, which according to Theorem 2.5 is an exchangeable (binary) hierarchy on $\mathbb{N}$. Consequently, the construction of the Rt takes place in the same situation as it has been done in [10], which explains the resemblances of the two papers. However, in contrast to [10], the random finite objects under consideration are now *ordered* trees, the transition from $\mathrm{Law}((T_n, S_n)_{n\in\mathbb{N}})$ to $\mathrm{Law}((S_n(T_n))_{n\in\mathbb{N}})$ is not injective. As a consequence, the constructed Rt is not sufficient to distinguish all points in the Martin boundary associated to binary trees. The missing information was described in [7], Section 8, and used a higher order Aldous–Hoover–Kallenberg representation result. However, higher-order randomization is not really needed; see [7], Lemma 8.1. The final representation result they obtain is a surjective description of the Martin boundary of RTGC; see [7], Theorem 8.2. A similar approach is applied in [4, 8] to describe Martin boundaries in different situations.

5.4. *Compositions.* We present a further examples of a nicely embedded substructure: compositions. In [15], *exchangeable compositions of* $\mathbb{N}$ have been analyzed. A composition of $\mathbb{N}$ is a tuple $C = (P, l)$, where $P = \{e_1, e_2, \dots\}$ is a partition of $\mathbb{N}$ and l is a linear order on P. Given any finite bijection $\pi \in \mathbb{S}_\infty$, one can relabel the composition: $\pi(C) := (\pi(P), \pi(l))$ where $\pi(P) = \{\pi(e_1), \pi(e_2), \dots\}$ and $\pi(e_i)\pi(l)\pi(e_j) :\Leftrightarrow e_i l e_j$. Now $\pi(C)$ is a composition as well. One can topologize the space of compositions via finite restrictions: Given any $n \in \mathbb{N}$, one introduces $C_{|n}$ where the partition is restricted to $[n]$ and the linear order is restricted to the images of the remaining partition blocks. An *exchangeable composition of* $\mathbb{N}$ is a random composition Π of $\mathbb{N}$ such that $\pi(\Pi) \sim \Pi$ for every $\pi \in \mathbb{S}_\infty$. Let $\mathtt{ExComp}$ be the simplex of laws of exchangeable compositions. Gnedin [15] obtains a homeomorphic description of $\mathrm{erg}(\mathtt{ExComp})$ in terms of open subsets of the unit interval. The set of open subsets was topologized in an explicit way that turns the description into a homeomorphic one.

This is linked to erased-interval processes as follows: An *interval partition* of $[n]$ is an element $\mathtt{P} \in \mathrm{InSy}(n)$ such that all nonsingleton intervals $[a, b], [c, d] \in \mathtt{P}$ are disjoint. Let $\mathrm{InPar}(n)$ be the set of all interval partitions of $[n]$. As above, the family $\mathrm{InPar}(n), n \in \mathbb{N}$ is stable under sampling via ϕ_k^n. In [15], p. 1439, it is explained that describing the simplex $\mathtt{ExComp}$ is equivalent (affinely homeomorphic) to describing the simplex of all laws of Markov chains of growing interval-partitions $(P_n)_{n\in\mathbb{N}}$ with co-transition probabilities γ. Now this simplex of laws is equivalent to the simplex of laws of erased-interval partition processes. The latter can be described by the subspace $\mathrm{InPar}(\infty) \subseteq \mathrm{InSy}(\infty)$ consisting of all K such that for all $(x, y), (x', y') \in K$ with $x < y$ and $x' < y'$ it holds that either $y \le x'$ or $y' \le x$. The homeomorphism is analogous to the Corollaries 5.7 and 5.13. Now every open subset U of the unit interval is a countable union of disjoint open intervals, $U = \bigcup_i (x_i, y_i)$. If one reads every (x_i, y_i) as a point in ◸ and one then passes from U to $\{(x_i, y_i) : i\} \cup \diagup$, one obtains a homeomorphism between the space describing $\mathrm{erg}(\mathtt{ExComp})$ in [15] and the space $(\mathrm{InPar}(\infty), d_{\mathrm{haus}})$.

5.5. *Outlook.* Let $T = (T_n)_{n\in\mathbb{N}}$ be Rémy's tree growth chain. It is a well-known fact that RTGC converges almost surely toward a plane Brownian continuum random tree, to be more precise: the normalized *exploration paths* associated with $(T_n)_{n\in\mathbb{N}}$ converge almost surely toward a (the same for all) Brownian excursion E; see [25] and [26]. Moreover, E generates the terminal σ-field of $(T_n)_{n\in\mathbb{N}}$ almost surely, as was noted in [7].

It is part of our main theorem that $n^{-1}T_n$ converges almost surely in the space $(\mathrm{STree}(\infty), d_{\mathrm{haus}})$ toward some $\mathrm{BinTree}(\infty)$-valued random variable T_∞ and that T_∞ generates the terminal σ-field of $(T_n)_{n\in\mathbb{N}}$ almost surely. As a consequence, $\sigma(T_\infty) = \sigma(E)$ almost

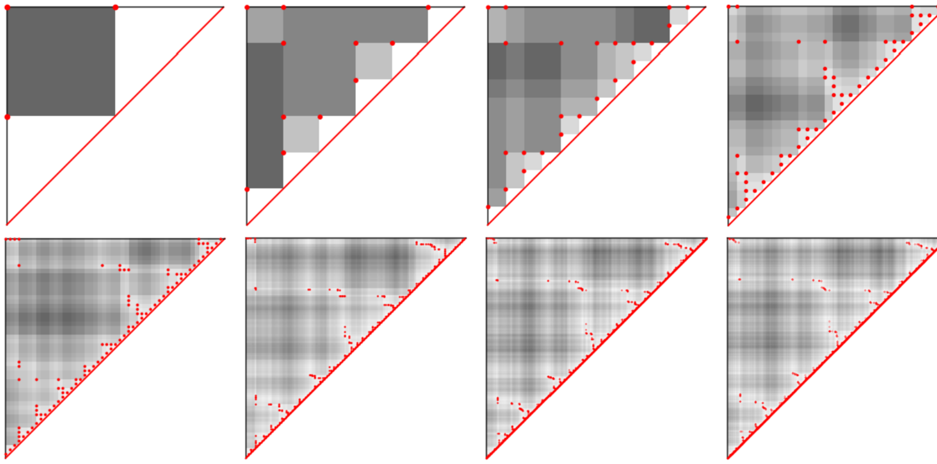

FIG. 13. *A simulation of Rémy's tree growth chain* $(T_n)_{n\in\mathbb{N}}$. *For* $n = 2, 6, 12, 25, 50, 200, 1000, 3000$ *we plotted* $n^{-1}T_n$ *and shaded in the "forbidden rectangles" using opacity as explained below Figure* 11.

surely, that is, one can express T_∞ as a measurable function of E and vice versa, almost surely.

One can think of T_∞ as a particular *representation of the plane Brownian CRT* taking values in the compact metric space $(\mathrm{BinTree}(\infty), d_{\mathrm{haus}})$. See Figure 13 for a simulation of $n^{-1}T_n$ as $n \to \infty$.

In future work, we aim to describe how to express E through T_∞ (and vice versa) in detail. We plan to compare the map $\phi_k^\infty : \mathrm{STree}(\infty) \times [0, 1]^k_< \to \mathrm{STree}(k)$ with maps used to build trees from *excursions* (see [22], Chapter III, Section 3). Excursions are commonly used to describe real trees; we aim to present a way to represent certain Schröder trees on $(0, 1)$ by excursion and compare these to the real tree obtained from the same excursion.

We finish with a conjecture regarding the connection of T_∞ and large uniform pattern-avoiding permutations: There are well-known connections of large uniform pattern avoiding permutations and Brownian excursion (see [17]), which may be explained with the help of T_∞. The following is motivated by a comparison of the last picture in Figure 13 and [17], Figure 3.

CONJECTURE 5.20. *Let* $n \in \mathbb{N}$ *and* S_n *be a uniform* 231*-avoiding permutation of* $[n]$. *Consider the normalized graph of* S_n, *defined as* $gr(S_n) := \{(i/n, S_n(i)/n) : i \in [n]\}$. *We consider* $gr(S_n)$ *as a random compact set, thus taking values in the space* $(\mathcal{K}([0, 1]^2), d_{\mathrm{haus}})$. *As* $n \to \infty$, *the sequence* $gr(S_n)$ *converges in law toward* T_∞, *the plane BRCT represented as a random compact set, where convergence in law is with respect to the Hausdorff topology* d_{haus} *on* $\mathcal{K}([0, 1]^2)$.

We were able to deduce certain results for subclasses of interval systems, since they are included in a "nice way" that is somehow consistent with our most important operations; see Lemmas 5.6 and 5.12. In future research, one could look for more such "nicely embedded" substructures of interval systems.

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