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BUSEMANN FUNCTIONS AND GIBBS MEASURES IN DIRECTED POLYMER MODELS ON $\mathbb{Z}^2$

BY CHRISTOPHER JANJIGIAN[*] AND FIRAS RASSOUL-AGHA[**]

Mathematics Department, University of Utah, [*]*janjigia@math.utah.edu;* [**]*firas@math.utah.edu*

We consider random walk in a space-time random potential, also known as directed random polymer measures, on the planar square lattice with nearest-neighbor steps and general i.i.d. weights on the vertices. We construct covariant cocycles and use them to prove new results on existence, uniqueness/nonuniqueness, and asymptotic directions of semi-infinite polymer measures (solutions to the Dobrushin–Lanford–Ruelle equations). We also prove nonexistence of covariant or deterministically directed bi-infinite polymer measures. Along the way, we prove almost sure existence of Busemann function limits in directions where the limiting free energy has some regularity.

1. Introduction. We study a class of probability measures on nearest-neighbor up-right random walk paths in the two-dimensional square lattice. The vertices of the lattice are populated with i.i.d. random variables called *weights* and the energy of a finite path is given by the sum of the weights along the path. We assume that these weights are nondegenerate and have finite $2+\varepsilon$ moments, but they are otherwise general. The *point-to-point quenched polymer measures* are probability measures on admissible paths between two fixed sites in which the probability of a path is proportional to the exponential of its energy. This model is known as the *directed polymer with bulk disorder* and it was introduced in the statistical physics literature by Huse and Henley [38] in 1985 to model the domain wall in the ferromagnetic Ising model with random impurities. It has been the subject of intense study over the past three decades; see the recent surveys [15, 16, 21].

Many of our main results concern *semi-infinite polymer measures*, which we will also call *semi-infinite DLR solutions* or *Gibbs measures* to help connect our results to the usual language of statistical mechanics. Semi-infinite polymer measures are probability measures on infinite length admissible up-right paths emanating from a fixed site which are consistent with the point-to-point quenched polymer measures. Some of the natural questions about such measures include whether all such measures must satisfy a law of large numbers (LLN), whether measures exist which satisfy a LLN with any given direction, and under what conditions such measures are unique. Ideally, one would like to answer these questions for almost every realization of the environment simultaneously for all directions.

This is the third paper to consider these questions in $1+1$ dimensional directed polymer models; the recent [31] and [8] address similar questions in related models which have more structure than the models considered here.

[31] studies the model first introduced in [55], which is a special case of the model studied in this paper where the weights have the log-gamma distribution. The authors use the solvability of the model (i.e., the possibility of exact computations) to introduce semi-infinite polymer measures which satisfy a LLN with any fixed direction for that model. As alluded to

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in the fourth paragraph on page 2283 of [31], the authors expected their structures and conclusions to generalize. We demonstrate that they do, but in addition to studying more general models, the present paper considers a much wider class of problems than [31]; hence most of the results we discuss are new even in this solvable setting.

[8] studies $1 + 1$ dimensional directed polymers in continuous space and discrete time, where the underlying random walk has Gaussian increments. The authors show existence and uniqueness of semi-infinite polymer measures satisfying the law of large numbers with a fixed deterministic direction, but the event on which this holds depends on the direction chosen. While the model considered in [8] is not solvable, a symmetry in the model inherited from the Gaussian walk leads to a quadratic limiting free energy. This is a critical feature of the model, since the method used in that project relies in an essential way on having a curvature bound for the free energy.

Some of our results, specifically ones concerning existence and uniqueness of semi-infinite polymer measures in deterministic directions, can likely be obtained with the techniques of [8] if one assumes or proves a curvature condition on the limiting free energy, which we will denote by Λ. Proving such a condition is a long-standing open problem. We prefer to avoid *a priori* curvature assumptions for two reasons: first, most of our theorems are valid under no assumptions on Λ and second, as we will see in Section 4.1, the stochastic process that is our main tool, the *Busemann process*, is naturally indexed by elements of the superdifferential of Λ, and we believe that understanding the structure of this object without any *a priori* regularity assumptions might provide a path to proving differentiability or strict concavity of Λ.

We now sketch what we can show about semi-infinite polymers in more detail. Before beginning, we remark that the set of semi-infinite polymer measures is convex and it suffices to study the extreme points. Although most of our theorems apply without *a priori* assumptions on Λ, they take their nicest form when Λ is both differentiable and strictly concave. This is conjectured to be the case in general. In this case, our results say that except for a single null set of weights all of the following hold. Every extremal measure satisfies a strong LLN (Corollary 3.6). For every direction in $\mathcal{U} = \{(t, 1-t) : 0 \le t \le 1\}$, there is at least one extremal semi-infinite polymer measure with that asymptotic direction (Corollary 3.3). Except for possibly a random countable set of directions, this measure is unique (Theorem 3.10(e)). The directions of nonuniqueness are precisely the directions at which the Busemann process is discontinuous (Theorem 3.10(e)). This set of directions is either always empty or always infinite (Theorem 3.10(c)). The connection between the nonuniqueness set and discontinuities of the Busemann process has not previously been observed. Moreover, this is the first time the countability of this set has been shown in positive temperature.

We do not resolve the question of whether or not the set of nonuniqueness directions is actually empty almost surely. As mentioned above, this is equivalent to the almost sure continuity of the process of Busemann functions viewed as a function of the direction. This latter question can likely be answered for the log-gamma polymer, where it is natural to expect that the distribution of the Busemann process can be described explicitly using positive temperature analogues of the ideas in [25]. It is known that this set is not empty in last-passage percolation (LPP), the zero-temperature version of the polymer model. See Theorem 2.8 and Lemma 5.2 in [29].

Aside from the problems discussed above, we study a number of natural related questions. For example, based on analogies to bi-infinite geodesics in percolation, it is natural to expect that nontrivial *bi-infinite polymer measures* should not exist. We are able to prove nonexistence of shift-covariant bi-infinite polymer measures and of bi-infinite polymer measures satisfying a LLN with a given fixed direction, but do not otherwise address noncovariant measures. We further study the *competition interface*, introduced in [31] as a positive-temperature analogue of the object from last-passage percolation [26]. In particular, we prove that the interface satisfies a LLN and characterize its random direction in terms of the Busemann process.

Our results can also be interpreted in terms of existence and uniqueness of global stationary solutions and pull-back attractors of a discrete viscous stochastic Burgers equation. This is the main focus of our companion paper [42]. See also [7] and the discussion in [8], which focuses on this viewpoint.

1.1. *Related works.* In his seminal paper [56], Sinai proved existence and uniqueness of stationary global solutions to the stochastic viscous Burgers equation with a forcing that is periodic in space and either also periodic in time or a white noise in time. Later, [32] extended Sinai's results to the multidimensional setting using a stochastic control approach and [22] used PDE methods to prove similar results for both viscous and inviscid Hamilton–Jacobi equations with periodic spatial dependence. Periodicity was relaxed in [6, 57], where the random potential was assumed to have a special form (not stationary in space) that ensures localization of the reference random walk near the origin and makes the situation essentially compact so the arguments from [56] could be used. A similar multidimensional model is treated in [6]. See also [3, 24, 33, 39] for zero temperature results using similar methods.

The connection between solving the stochastic viscous Burgers equation and the existence of Busemann limits in related directed polymer models was observed in [43] where they treated the case of strong forcing (high viscosity) or, in statistical mechanics terms, weak disorder (high temperature). See also the Markov chains constructed by Comets–Yoshida [18], Yilmaz [60], Section 6 in [53] and Example 7.7 in [28]. The model we consider is in $1+1$ space-time dimensions, which is known to be always in strong disorder [17, 44].

The recent papers [8] and [31], mentioned earlier, are more closely related to this work as both study strictly positive temperature polymers in a noncompact setting and in the strong disorder regime.

Currently, there are two major approaches to studying the general structure of infinite and semi-infinite directed polymers in zero or positive temperature. The first approach was introduced by Newman and coauthors [36, 37, 45, 49] in the context of first-passage percolation (FPP). This approach requires control of the curvature of Λ. This property is used to prove straightness estimates for the quenched point-to-point polymer measures. Existence and uniqueness results then come as consequences, as well as existence of *Busemann functions*, which are defined through limits of ratios of partition functions. This is the approach taken by [8]. See also [4, 5, 12–14, 26, 59] for other papers following this approach in zero temperature.

In this paper, we take the other, more recent, approach in which Busemann functions are the fundamental object. The use of Busemann functions to study the structure of semi-infinite geodesics traces back to the seminal work of Hoffman [34, 35] on FPP. Here, we construct covariant cocycles which are consistent with the weights on an extension of our probability space and then use a coupling argument and planarity to prove existence and properties of Busemann functions. The bulk of the work then goes toward using this process of Busemann functions to prove the results about infinite and semi-infinite polymer measures. This program was first achieved in zero temperature by [19, 20] in FPP and [29, 30] in LPP. [11] also takes this approach to construct correctors, which are the counterparts of Busemann functions, in their study of stochastic homogenization of viscous Hamilton–Jacobi equations.

In [31], the desired cocycles were constructed using the solvability of the model. In the present paper, we build cocycles using weak subsequential Cesàro limits of ratios of partition functions, which is a version of the method Damron and Hanson [19] used in their study of FPP. Our situation requires overcoming some nontrivial technical hurdles not encountered there which arise due to the path directedness in our model. An alternative approach to producing cocycles based on lifting the queueing theoretic arguments of [47] to positive temperature is also possible. These queueing theoretic results furnished the desired cocycles in [29, 30]. It is noteworthy that the queuing results rely on a specific choice of admissible path increments, while the weak convergence idea seems to work more generally.

1.2. *Organization.* Our paper is structured as follows. We start with some notation in Section 2.1 then introduce the model in Section 2.2. Section 2.4 introduces semi-infinite and bi-infinite polymer measures (DLR solutions). Our main results are stated in Section 3. In Section 4, we address existence of covariant cocycles and Busemann functions. Using these cocycles, we prove (more general versions of) our main results on semi-infinite DLR solutions in Section 5. In Section 6, we use these results to show nonexistence of covariant or deterministically directed bi-infinite DLR solutions. A number of technical results are deferred to Appendices A and B, and some technical results are stated without proof. The reader is referred to our preprint [40] for the relatively easy proofs of such results. One such result on almost sure coalescence of coupled random walks in a common random environment, Theorem A.2, may be of independent interest to some readers.

2. Setting. We begin by introducing some notation.

2.1. *Notation.* Throughout the paper, $(\Omega, \mathcal{F}, \mathbb{P})$ is a Polish probability space equipped with a group of $\mathcal{F}$-measurable $\mathbb{P}$-preserving transformations $T_x : \Omega \to \Omega$, $x \in \mathbb{Z}^2$, such that T_0 is the identity map and $T_x T_y = T_{x+y}$ for all $x, y \in \mathbb{Z}^2$. $\mathbb{E}$ is expectation relative to $\mathbb{P}$. A generic point in this space will be denoted by $\omega \in \Omega$. We assume that there exists a family $\{\omega_x(\omega) : x \in \mathbb{Z}^2\}$ of real-valued random variables called *weights* such that

$$\{\omega_x\} \text{ are i.i.d. under } \mathbb{P}, \exists p > 2: \quad \mathbb{E}\big[|\omega_0|^p\big] < \infty \quad \text{and} \quad \mathbb{V}\mathrm{ar}(\omega_0) > 0. \tag{2.1}$$

We assume further that $\omega_y(T_x\omega) = \omega_{x+y}(\omega)$ for all $x, y \in \mathbb{Z}^2$. An example is the canonical setting of a product space $\Gamma = \mathbb{R}^{\mathbb{Z}^2}$ equipped with the product topology, product Borel σ-algebra $\mathfrak{S}$, the product measure $\mathbb{P}_0^{\otimes \mathbb{Z}^2}$ with $\mathbb{P}_0$ a probability measure on $\mathbb{R}$, the natural shift maps, and with ω_x denoting the natural coordinate projection.

We study probability measures on paths with increments $\mathcal{R} = \{e_1, e_2\}$, the standard basis of $\mathbb{R}^2$. Let $\mathcal{U}$ denote the convex hull of $\mathcal{R}$ with $\mathrm{ri}\,\mathcal{U}$ its relative interior. Write $\widehat{e} = e_1 + e_2$. For $m \in \mathbb{Z}$, denote by $\mathbb{V}_m = \{x \in \mathbb{Z}^2 : x \cdot \widehat{e} = m\}$. We denote sequences of sites by $x_{m,n} = (x_i : m \le i \le n)$ where $-\infty \le m \le n \le \infty$. We require throughout that $x_i \in \mathbb{V}_i$.

For $x \in \mathbb{V}_m$ and $y \in \mathbb{V}_n$ with $m \le n$, the collection of admissible paths from x to y is denoted $\mathbb{X}_x^y = \{x_{m,n} : x_m = x, x_n = y, x_i - x_{i-1} \in \mathcal{R}\}$. This set is empty unless $x \le y$. ($x \le y$ is understood coordinatewise.) The collection of *admissible* paths from x to *level* n is denoted $\mathbb{X}_x^{(n)} = \{x_{m,n} : x_m = x, x_i - x_{i-1} \in \mathcal{R}\}$. The collection of *semi-infinite* paths *rooted* (*or starting*) *at* x is denoted by $\mathbb{X}_x = \{x_{m,\infty} : x_m = x, x_i - x_{i-1} \in \mathcal{R}\}$ and the collection of *bi-infinite* paths is $\mathbb{X} = \{x_{-\infty,\infty} : x_i - x_{i-1} \in \mathcal{R}\}$. The spaces $\mathbb{X}_x^y$, $\mathbb{X}_x^{(n)}$, and $\mathbb{X}_x$ are compact and, therefore, separable. The space $\mathbb{X}$ can be viewed naturally as $\mathbb{V}_0 \times \{e_1, e_2\}^{\mathbb{Z}}$ which is separable but not compact. We equip these spaces with the associated Borel σ-algebras $\mathcal{X}^{x,y}$, $\mathcal{X}^{x,(n)}$, $\mathcal{X}^x$ and $\mathcal{X}$. Given a subset of indices A, we denote by $\mathcal{X}_A^{x,y}$, $\mathcal{X}_A^{x,(n)}$, $\mathcal{X}_A^x$ and $\mathcal{X}_A$ the associated sub σ-algebra generated by the coordinate projections $\{x_i : i \in A\}$.

The space of probability measures on a metric measure space $(\Gamma, \mathcal{B})$, equipped with the topology of weak convergence, is denoted $\mathcal{M}_1(\Gamma, \mathcal{B})$. Expectation with respect to a measure μ is denoted E^μ. For $u, v \in \mathbb{R}^2$, we use the notation $[u, v] = \{su + (1-s)v : s \in [0, 1]\}$ and $]u, v[= \{su + (1-s)v : s \in (0, 1)\}$. The set of extreme points of a convex set C is denoted by $\mathrm{ext}\, C$.

2.2. *Finite polymer measures.* For an *inverse temperature* $\beta \in (0, \infty)$, $x \in \mathbb{V}_m$, and $y \in \mathbb{V}_n$, with $m, n \in \mathbb{Z}$, and $x \le y$, the *quenched point-to-point partition function* and *free energy* are

$$Z^{\beta}_{x,y} = \sum_{x_{m,n} \in \mathbb{X}^y_x} e^{\beta \sum_{i=m}^{n-1} \omega_{x_i}} \quad \text{and} \quad F^{\beta}_{x,y} = \frac{1}{\beta} \log Z^{\beta}_{x,y}.$$

We take the convention that $Z^{\beta}_{x,x} = 1$ and $F^{\beta}_{x,x} = 0$ while $Z^{\beta}_{x,y} = 0$ and $F^{\beta}_{x,y} = -\infty$ whenever we do not have $x \le y$. Similarly, we define the *last passage time* to be the zero temperature ($\beta = \infty$) free energy:

$$G_{x,y} = F^{\infty}_{x,y} = \max_{x_{m,n} \in \mathbb{X}^y_x} \left\{ \sum_{i=m}^{n-1} \omega_{x_i} \right\}.$$

The *quenched point-to-point polymer measure* is the probability measure on $(\mathbb{X}^y_x, \mathcal{X}^{x,y})$ given by

$$Q^{\omega,\beta}_{x,y}(A) = \frac{1}{Z^{\beta}_{x,y}} \sum_{x_{m,n} \in A} e^{\beta \sum_{i=m}^{n-1} \omega_{x_i}}$$

for a subset $A \subset \mathbb{X}^y_x$, with the convention that an empty sum is 0.

For a *tilt* (or *external field*) $h \in \mathbb{R}^2$, $n \in \mathbb{Z}$ and $x \in \mathbb{V}_m$ with $m \le n$, the quenched *tilted point-to-line* partition function and free energy are

$$Z^{\beta,h}_{x,(n)} = \sum_{x_{m,n} \in \mathbb{X}^{(n)}_x} e^{\beta \sum_{i=m}^{n-1} \omega_{x_i} + \beta h \cdot (x_n - x_m)} \quad \text{and} \quad F^{\beta,h}_{x,(n)} = \frac{1}{\beta} \log Z^{\beta,h}_{x,(n)}.$$

We take the convention that $Z^{\beta,h}_{x,(m)} = 1$ and $F^{\beta,h}_{x,(m)} = 0$ while $Z^{\beta,h}_{x,(n)} = 0$ and $F^{\beta,h}_{x,(n)} = -\infty$ if $n < m$. Again, we define the point-to-line last passage time to be the zero temperature free energy:

$$G^h_{x,(n)} = F^{\infty,h}_{x,(n)} = \max_{x_{m,n} \in \mathbb{X}^{(n)}_x} \left\{ \sum_{i=m}^{n-1} \omega_{x_i} + h \cdot (x_n - x_m) \right\}.$$

The quenched tilted point-to-line polymer measure is

$$Q^{\omega,\beta,h}_{x,(n)}(A) = \frac{1}{Z^{\beta,h}_{x,(n)}} \sum_{x_{m,n} \in A} e^{\beta \sum_{i=m}^{n-1} \omega_{x_i} + \beta h \cdot (x_n - x_m)} \quad \text{for } A \subset \mathbb{X}^{(n)}_x.$$

We will denote by $E^{\omega,\beta}_{x,y}$ the expectation with respect to $Q^{\omega,\beta}_{x,y}$ and similarly $E^{\omega,\beta,h}_{x,(n)}$ will denote the expectation with respect to $Q^{\omega,\beta,h}_{x,(n)}$. The random variable given by the natural coordinate projection to level i is denoted by X_i. We will frequently abbreviate the event $\{X_{m,n} = x_{m,n}\}$ by $\{x_{m,n}\}$.

2.3. *Limiting free energy.* For $\beta \in (0, \infty]$, there are deterministic functions $\Lambda^{\beta} : \mathbb{R}^2_+ \to \mathbb{R}$ and $\Lambda^{\beta}_{\mathrm{pl}} : \mathbb{R}^2 \to \mathbb{R}$ such that $\mathbb{P}$-a.s. for all $0 < C < \infty$,

$$\lim_{n\to\infty} \max_{\substack{x \in \mathbb{Z}_+ \\ |x|_1 \le Cn}} \frac{|F^{\beta}_{0,x} - \Lambda^{\beta}(x)|}{n} = \lim_{n\to\infty} \sup_{|h|_1 \le C} \frac{|F^{\beta,h}_{0,(n)} - \Lambda^{\beta}_{\mathrm{pl}}(h)|}{n} = 0. \tag{2.2}$$

These are called *shape theorems*. The first limit comes from the point-to-point free energy limit (2.3) in [53] and the now standard argument in [48]. The second equality comes from the point-to-line free energy limit (2.4) in [53] and $|F^{\beta,h}_{0,(n)} - F^{\beta,h'}_{0,(n)}| \le |h - h'|_1$.

Λ^β is concave, 1-homogenous, and continuous on $\mathbb{R}^2_+$. $\Lambda^\beta_{\mathrm{pl}}$ is convex and Lipschitz on $\mathbb{R}^2$. Lattice symmetry and i.i.d. weights imply that

$$\Lambda^\beta(\xi_1 e_1 + \xi_2 e_2) = \Lambda^\beta(\xi_2 e_1 + \xi_1 e_2).$$

By (4.3)–(4.4) in [28] Λ^β and $\Lambda^\beta_{\mathrm{pl}}$ are related via the duality

$$\Lambda^\beta_{\mathrm{pl}}(h) = \sup_{\xi\in\mathcal{U}}\{\Lambda^\beta(\xi) + h\cdot\xi\} \quad \text{and} \quad \Lambda^\beta(\xi) = \inf_{h\in\mathbb{R}^2}\{\Lambda^\beta_{\mathrm{pl}}(h) - h\cdot\xi\}. \tag{2.3}$$

$h \in \mathbb{R}^2$ and $\xi \in \operatorname{ri}\mathcal{U}$ are said to be in duality if

$$\Lambda^\beta_{\mathrm{pl}}(h) = h\cdot\xi + \Lambda^\beta(\xi).$$

We denote the set of directions dual to h by $\mathcal{U}^\beta_h \subset \operatorname{ri}\mathcal{U}$. In the arguments that follow, the superdifferential of Λ^β at $\xi \in \mathbb{R}^2_+$,

$$\partial\Lambda^\beta(\xi) = \{v \in \mathbb{R}^2 : v\cdot(\xi - \zeta) \le \Lambda^\beta(\xi) - \Lambda^\beta(\zeta)\ \forall\zeta \in \mathbb{R}^2_+\}, \tag{2.4}$$

will play a key role. We also introduce notation for the image of $\mathcal{U}$ under the superdifferential map via

$$\partial\Lambda^\beta(\mathcal{U}) = \{v \in \mathbb{R}^2 : \exists\xi \in \operatorname{ri}\mathcal{U} : v \in \partial\Lambda^\beta(\xi)\}.$$

The following lemma gives a useful characterization of $\partial\Lambda^\beta(\mathcal{U})$. The proof is a straightforward exercise in convex analysis and is included in [40].

LEMMA 2.1. *For $h \in \mathbb{R}^2$, $-h \in \partial\Lambda^\beta(\mathcal{U})$ if and only if $\Lambda^\beta_{\mathrm{pl}}(h) = 0$. Moreover, if $-h \in \partial\Lambda^\beta(\xi)$ for $\xi \in \operatorname{ri}\mathcal{U}$, then $\xi\cdot h + \Lambda^\beta(\xi) = 0$.*

Concavity implies the existence of one-sided derivatives:

$$\nabla\Lambda^\beta(\xi\pm)\cdot e_1 = \lim_{\varepsilon\searrow 0}\frac{\Lambda^\beta(\xi \pm \varepsilon e_1) - \Lambda^\beta(\xi)}{\pm\varepsilon} \quad \text{and}$$

$$\nabla\Lambda^\beta(\xi\pm)\cdot e_2 = \lim_{\varepsilon\searrow 0}\frac{\Lambda^\beta(\xi \mp \varepsilon e_2) - \Lambda^\beta(\xi)}{\mp\varepsilon}.$$

These are the two extreme points of the convex set $\partial\Lambda^\beta(\xi)$. The collection of directions of differentiability of Λ^β will be denoted by

$$\mathcal{D}^\beta = \{\xi \in \operatorname{ri}\mathcal{U} : \Lambda^\beta \text{ is differentiable at } \xi\}.$$

[54], Theorem 25.2, shows that $\xi \in \mathcal{D}^\beta$ is the same as $\nabla\Lambda^\beta(\xi+) = \nabla\Lambda^\beta(\xi-)$.

Abusing notation, for $\xi \in \operatorname{ri}\mathcal{U}$ define the maximal linear segments

$$\mathcal{U}^\beta_{\xi\pm} = \{\zeta \in \operatorname{ri}\mathcal{U} : \Lambda^\beta(\zeta) - \Lambda^\beta(\xi) = \nabla\Lambda^\beta(\xi\pm)\cdot(\zeta - \xi)\} = \mathcal{U}^\beta_{-\nabla\Lambda^\beta(\xi\pm)}.$$

Although we abuse notation, it should be clear from context whether we are referring to sets indexed by directions or tilts.

We say Λ^β is *strictly concave* at $\xi \in \operatorname{ri}\mathcal{U}$ if $\mathcal{U}^\beta_{\xi-} = \mathcal{U}^\beta_{\xi+} = \{\xi\}$. The usual notion of strict concavity on an open subinterval of $\mathcal{U}$ is the same as having our strict concavity at ξ for all ξ in the interval. Let

$$\mathcal{U}^\beta_\xi = \mathcal{U}^\beta_{\xi-} \cup \mathcal{U}^\beta_{\xi+} = [\underline{\xi}^\beta, \overline{\xi}^\beta] \quad \text{with } \underline{\xi}^\beta\cdot e_1 \le \overline{\xi}^\beta\cdot e_1.$$

Lemma B.1 justifies setting $\mathcal{U}^\beta_{e_i} = \{e_i\}$ for $i \in \{1, 2\}$, since it implies that the free energy is not locally linear near the boundary.

If $\xi \in \mathcal{D}^\beta$, then $\mathcal{U}^\beta_{\xi-} = \mathcal{U}^\beta_{\xi+} = \mathcal{U}^\beta_\xi$ while if $\xi \notin \mathcal{D}^\beta$ then $\mathcal{U}^\beta_{\xi-} \cap \mathcal{U}^\beta_{\xi+} = \{\xi\}$. For $h \in \mathbb{R}^2$, the set $\mathcal{U}^\beta_h$ is either a singleton $\{\xi\}$ or equals $\mathcal{U}^\beta_{\xi-}$ or $\mathcal{U}^\beta_{\xi+}$, for some $\xi \in \operatorname{ri}\mathcal{U}$ dual to h. In particular, it is a closed nonempty interval.

With the exception of Section 4.1, our results are for a fixed $\beta < \infty$. Therefore, in the rest of the paper, except in Section 4.1, we will assume without loss of generality that $\beta = 1$ and will omit the β from our notation.

2.4. *Infinite polymer measures and DLR solutions.*

DEFINITION 2.2. Given $\omega \in \Omega$ and $x \in \mathbb{V}_m$, $m \in \mathbb{Z}$, a probability measure Π on $(\mathbb{X}_x, \mathcal{X}^x)$ is a *semi-infinite quenched polymer measure* in environment ω rooted at $x_m = x$ if for all $n \ge m$ and all up-right paths $x_{m,n} \in \mathbb{X}^{(n)}_x$:

$$\Pi(X_{m,n} = x_{m,n}) = \Pi(X_n = x_n)\, Q^\omega_{x_m,x_n}(X_{m,n} = x_{m,n}). \tag{2.5}$$

DEFINITION 2.3. Given $\omega \in \Omega$, a probability measure $\Pi \in \mathcal{M}_1(\mathbb{X}, \mathcal{X})$ is said to be a *bi-infinite quenched polymer measure* in environment ω if for all $n \ge m$ and any up-right path $x_{m,n}$ the following holds:

$$\Pi(X_{m,n} = x_{m,n}) = \Pi(X_m = x_m, X_n = x_n)\, Q^\omega_{x_m,x_n}(X_{m,n} = x_{m,n}). \tag{2.6}$$

Equations (2.5) and (2.6) are the positive-temperature analogues of the definitions of semi-infinite and bi-infinite geodesics in percolation.

Sections 2.4 and 2.5 in our preprint [40] explain how these definitions fit into the standard formalism of Gibbs measures and how semi-infinite and bi-infinite polymer measures are precisely the semi-infinite-volume and infinite-volume solutions to the familiar *Dobrushin–Lanford–Ruelle* (*DLR*) *equations*. Hence, we write DLR^ω_x to denote the set of semi-infinite quenched polymer measures in environment ω rooted at x and we denote the set of bi-infinite quenched polymer measures in environment ω by $\overleftrightarrow{\mathrm{DLR}}^\omega$.

Equations (2.5) show that DLR^ω_x is a closed convex subset of the compact space $\mathcal{M}_1(\mathbb{X}_x, \mathcal{X}_x)$, which we view as a subspace of the complex Radon measures on $\mathbb{X}_x$. By Choquet's theorem [51], Section 3, each element in DLR^ω_x is a convex integral mixture of extremal elements of DLR^ω_x.

Naturally, conditioning bi-infinite DLR solutions on passing through a point produces rooted DLR solutions. See [40], Lemma 2.9, for a proof.

LEMMA 2.4. *Fix $\omega \in \Omega$ and $\Pi \in \overleftrightarrow{\mathrm{DLR}}^\omega$. Fix $x \in \mathbb{V}_m$, $m \in \mathbb{Z}$, such that $\Pi(X_m = x) > 0$. Let Π_x be the probability measure on $(\mathbb{X}_x, \mathcal{X}^x)$ defined by*

$$\Pi_x(X_{m,n} = x_{m,n}) = \Pi(X_{m,n} = x_{m,n} \mid X_m = x), \tag{2.7}$$

for any up-right path $x_{m,n}$ with $x_m = x$. Then $\Pi_x \in \mathrm{DLR}^\omega_x$.

We also study consistent and covariant families of DLR solutions, in the sense of the following two definitions.

DEFINITION 2.5. Given $\omega \in \Omega$, we say $\{\Pi_x : x \in \mathbb{Z}^2\}$ is a family of consistent rooted (or semi-infinite) DLR solutions (in environment ω) if for all $x \in \mathbb{Z}^2$, $\Pi_x \in \mathrm{DLR}^\omega_x$ and the following holds: For each $y \in \mathbb{V}_m$, $m \in \mathbb{Z}$, $x \le y$, $n \ge m$ and for each up-right path $x_{m,n}$ with $x_m = y$,

$$\Pi_x(X_{m,n} = x_{m,n} \mid X_m = y) = \Pi_y(X_{m,n} = x_{m,n}).$$

We will denote the set of such families by $\overrightarrow{\mathrm{DLR}}^\omega$.

Define the shift θ_z acting on up-right paths by $\theta_z x_{m,n} = z + x_{m,n}$.

DEFINITION 2.6. A family $\{\Pi_x^\omega : x \in \mathbb{Z}^2, \omega \in \Omega\}$ is said to be a T-covariant family of consistent rooted (or semi-infinite) DLR solutions if for each $x \in \mathbb{Z}^2$, $\omega \mapsto \Pi_x^\omega$ is measurable, there exists a full-measure T-invariant event $\Omega' \subset \Omega$ such that for each $\omega \in \Omega'$, $\{\Pi_x^\omega : x \in \mathbb{Z}^2\}$ is consistent in environment ω, and for all $z \in \mathbb{Z}^2$, $\Pi_{x-z}^{T_z\omega} \circ \theta_{-z} = \Pi_x^\omega$.

3. Main results.

3.1. *Semi-infinite polymer measures.* We begin with a definition of directedness. For $A \subset \mathbb{R}^2$ and $\xi \in \mathbb{R}^2$, let $\operatorname{dist}(\xi, A) = \inf_{\zeta \in A} |\xi - \zeta|_1$.

DEFINITION 3.1. For a set $A \subset \mathcal{U}$, a sequence $x_n \in \mathbb{Z}^2$ is said to be *A-directed* if $|x_n|_1 \to \infty$ and the set of limit points of $x_n/|x_n|_1$ is included in A. We say that Π is *strongly* A-directed if

$$\Pi\{(X_n) \text{ is } A\text{-directed}\} = 1.$$

We say that Π is *weakly* A-directed if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \Pi\{\operatorname{dist}(X_n/n, A) > \varepsilon\} = 0.$$

A family of probability measures is said to be weakly/strongly A-directed if each member of the family is. Sometimes we say *directed into A* instead of A-directed, almost surely directed instead of strongly directed and directed in probability instead of weakly directed.

When $A = \{\xi\}$ is a singleton, weak directedness into A means Π satisfies the weak law of large numbers (WLLN) while strong directedness means the strong law of large numbers (SLLN) holds, with asymptotic direction ξ in either case. We then say that Π satisfies WLLN_ξ and SLLN_ξ, respectively.

First, we address the existence of directed DLR solutions. Recall at this point that we set $\beta = 1$ throughout this section.

THEOREM 3.2. *There exists an event Ω_{exist} such that $\mathbb{P}(\Omega_{\text{exist}}) = 1$ and for every $\omega \in \Omega_{\text{exist}}$ and every $\xi \in \mathcal{U}$ there exists a consistent family in $\overrightarrow{\mathrm{DLR}}^\omega$ that is strongly $\mathcal{U}_{\xi-}$-directed and a consistent family in $\overrightarrow{\mathrm{DLR}}^\omega$ that is strongly $\mathcal{U}_{\xi+}$-directed. If $\xi \notin \mathcal{D}$ then for each $x \in \mathbb{Z}^2$ the members rooted at x, from each family, are different.*

The following is an immediate corollary.

COROLLARY 3.3. *For any $\omega \in \Omega_{\text{exist}}$ and for any $\xi \in \operatorname{ri}\mathcal{U}$ at which Λ is strictly concave, there exists at least one consistent family in $\overrightarrow{\mathrm{DLR}}^\omega$ satisfying SLLN_ξ. If, furthermore, $\xi \notin \mathcal{D}$, then there exist at least two such families.*

For $x \in \mathbb{V}_m$, $m \in \mathbb{Z}$, two trivial (and degenerate) elements of DLR_x^ω are given by $\Pi_x^{e_i} = \delta_{x_{m,\infty}}$ with $x_k = x + (k - m)e_i$, $k \geq m$, $i \in \{1, 2\}$. These two solutions are clearly extreme in DLR_x^ω.

We say that $\Pi_x \in \mathrm{DLR}_x^\omega$ is *nondegenerate* if it satisfies

$$\Pi_x(x_{m,n}) > 0 \quad \text{for all admissible finite paths with } x_m = x. \tag{3.1}$$

The next lemma states that outside one null set of weights ω, convex combinations of $\Pi_x^{e_i}$ are the only degenerate DLR solutions.

LEMMA 3.4. *There exists an event Ω_{nondeg} such that $\mathbb{P}(\Omega_{\text{nondeg}}) = 1$ and for all $\omega \in \Omega_{\text{nondeg}}$ and $x \in \mathbb{Z}^2$, any solution $\Pi_x \in \text{DLR}_x^\omega$ that is not a convex combination of $\Pi_x^{e_i}$, $i \in \{1, 2\}$, is nondegenerate.*

The next result is on directedness of DLR solutions.

THEOREM 3.5. *There exists an event Ω_{dir} such $\mathbb{P}(\Omega_{\text{dir}}) = 1$ and for all $\omega \in \Omega_{\text{dir}}$, all $x \in \mathbb{Z}^2$ and any extreme nondegenerate solution $\Pi_x \in \text{DLR}_x^\omega$ there exists a $\xi \in \text{ri}\,\mathcal{U}$ such that one of the following three holds*:

(a) *Π_x satisfies WLLN_ξ and is strongly $\mathcal{U}_{\overline{\xi}}$-directed or strongly $\mathcal{U}_{\underline{\xi}}$-directed,*
(b) *Π_x is strongly $\mathcal{U}_\xi$-directed, or*
(c) *$\xi \in \mathcal{D}$ and Π_x is weakly $\mathcal{U}_\xi$-directed and strongly directed into $\mathcal{U}_{\underline{\xi}} \cup \mathcal{U}_{\overline{\xi}}$.*

If $\omega \in \Omega_{\text{nondeg}}$, then Lemma 3.4 says the only extreme degenerate solutions of the DLR equations are $\Pi_x^{e_i}$, $i \in \{1, 2\}$, which are $\{e_i\}$-directed. Theorem 3.5 shows that if $\omega \in \Omega_{\text{dir}}$, then there are no nondegenerate extreme DLR solutions directed weakly into $\{e_1\}$ or $\{e_2\}$.

Note that when Λ is differentiable on $\text{ri}\,\mathcal{U}$ we have $\mathcal{U}_\xi = \mathcal{U}_{\xi\pm} = \mathcal{U}_{\underline{\xi}} = \mathcal{U}_{\overline{\xi}}$ for all $\xi \in \mathcal{U}$. When Λ is strictly concave at a point ξ we have $\mathcal{U}_\xi = \mathcal{U}_{\xi\pm} = \mathcal{U}_{\underline{\xi}} = \mathcal{U}_{\overline{\xi}} = \{\xi\}$. Thus, the following is an immediate corollary.

COROLLARY 3.6. *The following hold*:

(a) *Assume Λ is differentiable on $\text{ri}\,\mathcal{U}$. For any $\omega \in \Omega_{\text{dir}}$, for all $x \in \mathbb{Z}^2$, any extreme solution in DLR_x^ω is strongly $\mathcal{U}_\xi$-directed for some $\xi \in \mathcal{U}$.*

(b) *Assume Λ is strictly concave on $\text{ri}\,\mathcal{U}$. Then for any $\omega \in \Omega_{\text{dir}}$, for all $x \in \mathbb{Z}^2$, any extreme solution in DLR_x^ω satisfies SLLN_ξ for some $\xi \in \mathcal{U}$.*

We next show existence and uniqueness of DLR solutions.

THEOREM 3.7. *Fix $\xi \in \mathcal{D}$ such that $\underline{\xi}, \overline{\xi} \in \mathcal{D}$. There exists a T-invariant event $\Omega_{[\underline{\xi},\overline{\xi}]} \subset \Omega$ such that $\mathbb{P}(\Omega_{[\underline{\xi},\overline{\xi}]}) = 1$ and for every $\omega \in \Omega_{[\underline{\xi},\overline{\xi}]}$ and $x \in \mathbb{Z}^2$, there exists a unique weakly $\mathcal{U}_\xi$-directed solution $\Pi_x^{\xi,\omega} \in \text{DLR}_x^\omega$. This $\Pi_x^{\xi,\omega}$ is strongly $\mathcal{U}_\xi$-directed and for any $\mathcal{U}_\xi$-directed sequence (x_n) the sequence of quenched point-to-point polymer measures Q_{x,x_n}^ω converges weakly to $\Pi_x^{\xi,\omega}$. The family $\{\Pi_x^{\xi,\omega} : x \in \mathbb{Z}^2, \omega \in \Omega\}$ is consistent.*

Our next result shows existence of Busemann functions in directions ξ with $\xi, \underline{\xi}, \overline{\xi} \in \mathcal{D}$ or, equivalently, $\partial\Lambda(\zeta) = \{h\}$ for some h and all $\zeta \in \mathcal{U}_\xi$.

THEOREM 3.8. *Fix $\xi \in \mathcal{D}$ such that $\underline{\xi}, \overline{\xi} \in \mathcal{D}$ and let $\{h\} = \partial\Lambda(\xi)$. There exists a T-invariant event $\Omega'_{[\underline{\xi},\overline{\xi}]}$ with $\mathbb{P}(\Omega'_{[\underline{\xi},\overline{\xi}]}) = 1$ such that for all $\omega \in \Omega'_{[\underline{\xi},\overline{\xi}]}$, $x, y \in \mathbb{Z}^2$, and all $\mathcal{U}_\xi$-directed sequences $x_n \in \mathbb{V}_n$, the following limits exist and are equal*:

$$B^\xi(x, y, \omega) = \lim_{n\to\infty} (\log Z_{x,x_n} - \log Z_{y,x_n}) \tag{3.2}$$

$$= \lim_{n\to\infty} \bigl(\log Z_{x,(n)}^h - \log Z_{y,(n)}^h\bigr) - h \cdot (y - x). \tag{3.3}$$

Additionally, if $\zeta \in \mathcal{D}$ is such that $\underline{\zeta}, \overline{\zeta} \in \mathcal{D}$ and $\xi \cdot e_1 < \zeta \cdot e_1$, then for $\omega \in \Omega'_{[\underline{\xi},\overline{\xi}]} \cap \Omega'_{[\underline{\zeta},\overline{\zeta}]}$ and $x \in \mathbb{Z}^2$, we have

$$\begin{aligned} &B^\xi(x, x + e_1, \omega) \geq B^\zeta(x, x + e_1, \omega) \quad \textit{and} \\ &B^\xi(x, x + e_2, \omega) \leq B^\zeta(x, x + e_2, \omega). \end{aligned} \tag{3.4}$$

As a consequence of the above theorem, the unique DLR measures from Theorem 3.7 have a concrete structure, as the next corollary shows.

COROLLARY 3.9. *Fix $\xi \in \mathcal{D}$ such that $\underline{\xi}, \overline{\xi} \in \mathcal{D}$ and $\omega \in \Omega_{[\underline{\xi},\overline{\xi}]} \cap \Omega'_{[\underline{\xi},\overline{\xi}]}$. Then $\Pi_x^{\xi,\omega}$ is a Markov chain starting at x, with transition probabilities $\pi_{y,y+e_i}^{\xi,\omega} = e^{\omega_y - B^{\xi}(y,y+e_i,\omega)}$, $y \in \mathbb{Z}^2$, $i \in \{1,2\}$. The family $\{\Pi_x^{\xi,\omega} : x \in \mathbb{Z}^2, \omega \in \Omega_{[\underline{\xi},\overline{\xi}]} \cap \Omega'_{[\underline{\xi},\overline{\xi}]}\}$ is T-covariant.*

In contrast to Theorem 3.7, Theorem 3.2 demonstrated nonuniqueness at points of nondifferentiability of Λ. It is conjectured that $\mathcal{D} = \mathrm{ri}\,\mathcal{U}$; if true, then Theorem 3.7 would cover all directions in $\mathrm{ri}\,\mathcal{U}$ and there would not exist directions to which the nonuniqueness claim in Theorem 3.2 would apply. The event on which Theorem 3.7 holds, however, depends on the direction chosen. It leaves open the possibility of random directions of nonuniqueness. Our next result says that under a mild regularity assumption, with the exception of one null set of environments, uniqueness holds for all but countably many points in $\mathcal{U}$. The assumption we need for this is:

$$\begin{aligned} &\Lambda \text{ is strictly concave at all } \xi \notin \mathcal{D}, \text{ or equivalently} \\ &\Lambda \text{ is differentiable at the endpoints of its linear segments.} \end{aligned} \tag{3.5}$$

The above condition is also equivalent to the existence of a countable dense set $\mathcal{D}_0 \subset \mathcal{D}$ such that for each $\zeta \in \mathcal{D}_0$ we also have $\underline{\zeta}, \overline{\zeta} \in \mathcal{D}$.

Assume (3.5) and fix such a set $\mathcal{D}_0$. Using monotonicity (3.4) we define processes $B^{\xi\pm}(x, x+e_i, \omega)$ for $\xi \in \mathrm{ri}\,\mathcal{U}$ and $\omega \in \Omega_1 = \bigcap_{\xi \in \mathcal{D}_0} \Omega'_{[\underline{\xi},\overline{\xi}]}$:

$$\begin{aligned} B^{\xi+}(x, x+e_i) &= \lim_{\substack{\zeta\cdot e_1 \searrow \xi\cdot e_1 \\ \zeta \in \mathcal{D}_0}} B^{\zeta}(x, x+e_i) \quad \text{and} \\ B^{\xi-}(x, x+e_i) &= \lim_{\substack{\zeta\cdot e_1 \nearrow \xi\cdot e_1 \\ \zeta \in \mathcal{D}_0}} B^{\zeta}(x, x+e_i). \end{aligned} \tag{3.6}$$

For $\omega \in \Omega_1$, let

$$\mathcal{U}_x^{\omega} = \{\xi \in \mathrm{ri}\,\mathcal{U} : \exists y \geq x : B^{\xi-}(y, y+e_1, \omega) \neq B^{\xi+}(y, y+e_1, \omega)\}. \tag{3.7}$$

For $\omega \notin \Omega_1$, set $\mathcal{U}_x^{\omega} = \varnothing$. Note that for any $\omega \in \Omega$, $\mathcal{U}_x^{\omega}$ is countable.

The following theorem can be viewed as our main result. Its primary content is contained in part III, which shows that the discontinuity set of the Busemann processes ahead of x defined in (3.7) is exactly the set of directions for which uniqueness of DLR solutions rooted at x fails. This connection has not been observed before in the positive or zero temperature literature. As a consequence, we obtain that the set of directions for which uniqueness may fail is countable, which is new in positive temperature. As noted in the Introduction, this connection also provides an avenue for answering the question of whether or not on a single event of full measure uniqueness holds simultaneously in all directions.

THEOREM 3.10. *Assume (3.5). There exists a measurable set Ω_{uniq} with $\mathbb{P}(\Omega_{\mathrm{uniq}}) = 1$ such that the following hold for all $x \in \mathbb{Z}^2$.*

I. Structure of $\mathcal{U}_x^{\omega}$:

(a) *For any $\omega \in \Omega_{\mathrm{uniq}}$, $(\mathrm{ri}\,\mathcal{U}) \setminus \mathcal{D} \subset \mathcal{U}_x^{\omega}$. For each $\xi \in \mathcal{D}$, $\mathbb{P}\{\xi \in \mathcal{U}_x^{\omega}\} = 0$.*

(b) *For any $\omega \in \Omega_{\mathrm{uniq}}$, $\mathcal{U}_x^{\omega}$ is supported outside the linear segments of Λ: For any $\xi \in \mathrm{ri}\,\mathcal{U}$ with $\underline{\xi} \neq \overline{\xi}$, $[\underline{\xi}, \overline{\xi}] \cap \mathcal{U}_x^{\omega} = \varnothing$.*

(c) *For any distinct* $\eta, \zeta \in \mathcal{U}$, $\mathbb{P}([\eta, \zeta] \cap \mathcal{U}_0^\omega \neq \varnothing) \in \{0, 1\}$. *If* $[\eta, \zeta] \cap \operatorname{ri}\mathcal{U} \subset \mathcal{D}$ *and* $\mathbb{P}\{[\eta, \zeta] \cap \mathcal{U}_x^\omega \neq \varnothing\} = 1$, *then the set of* $\xi \in [\eta, \zeta]$ *satisfying* $\mathbb{P}\{\xi$ *is an accumulation point of* $\mathcal{U}_0^\omega\} = 1$ *is infinite and has no isolated points.*

II. Directedness of DLR solutions:

(d) *For any* $\omega \in \Omega_{\text{uniq}}$, *every nondegenerate extreme solution is strongly* $\mathcal{U}_\xi$*-directed for some* $\xi \in \operatorname{ri}\mathcal{U}$. *The only degenerate extreme solutions are* $\Pi_x^{e_i}$, $i \in \{1, 2\}$.

III. $\mathcal{U}_x^\omega$ and the uniqueness of DLR solutions:

(e) *For any* $\omega \in \Omega_{\text{uniq}}$ *and* $\xi \in \mathcal{U} \setminus \mathcal{U}_x^\omega$, *there exists a unique strongly* $\mathcal{U}_\xi$*-directed solution* $\Pi_x^{\xi,\omega} \in \mathrm{DLR}_x^\omega$. *Moreover,* $\Pi_x^{\xi,\omega}$ *is an extreme point of* DLR_x^ω *and for any* $\mathcal{U}_\xi$*-directed sequence* (x_n) *the sequence* Q_{x,x_n}^ω *converges weakly to* $\Pi_x^{\xi,\omega}$. *The family* $\{\Pi_x^{\xi,\omega} : x \in \mathbb{Z}^2\}$ *is consistent.*

(f) *For any* $\omega \in \Omega_{\text{uniq}}$ *and* $\xi \in \mathcal{U}_x^\omega$, *there exist at least two extreme strongly* $\mathcal{U}_\xi$*-directed solutions in* DLR_x^ω.

When Λ is strictly concave, that is, $\mathcal{U}_\xi = \{\xi\}$ for all $\xi \in \mathcal{U}$, the above theorem states that outside one null set of weights ω, and except for an ω-dependent set of directions (countable and possibly empty), there is a unique DLR solution in environment ω satisfying WLLN_ξ (and in fact SLLN_ξ).

3.2. *The competition interface.* An easy computation checks the following.

LEMMA 3.11. *For* $x \le y$, *the quenched polymer measure* $Q_{x,y}^\omega$ *is the same as the distribution of the backward Markov chain starting at* y *and taking steps in* $\{-e_1, -e_2\}$ *with transition probabilities*

$$\overleftarrow{\pi}^{\,x}_{u,u-e_i}(\omega) = \frac{e^{\omega_{u-e_i}} Z_{x,u-e_i}}{Z_{x,u}}, \quad u \ge x.$$

Couple the backward Markov chains $\{Q_{x,y}^\omega : y \ge x\}$ by a quenched probability measure Q_x^ω on the space $\mathbb{T}_x$ of trees that span $x + \mathbb{Z}_+^2$. Precisely, for each $y \in x + \mathbb{Z}_+^2 \setminus \{0\}$ choose a parent $\gamma(y) = y - e_i$ with probability $\overleftarrow{\pi}^{\,x}_{y,y-e_i}(\omega)$, $i \in \{1, 2\}$. We denote the random tree by $\mathcal{T}_x^\omega \in \mathbb{T}_x$. For any $y \ge x$, there is a unique up-right path from x to y on $\mathcal{T}_x^\omega$. Lemma 3.11 implies that the distribution of this path under Q_x^ω is exactly the polymer measure $Q_{x,y}^\omega$.

Fix the starting point to be $x = 0$. Consider the two (random) subtrees $\mathcal{T}_{0,e_i}^\omega$ of $\mathcal{T}_0^\omega$, rooted at e_i, $i \in \{1, 2\}$. Following [31], define the path ϕ_n^ω such that $\phi_0^\omega = 0$ and for each $n \in \mathbb{N}$ and $i \in \{1, 2\}$, $\phi_n^\omega - \phi_{n-1}^\omega \in \{e_1, e_2\}$ and $\{\phi_n^\omega + ke_i : k \in \mathbb{N}\} \subset \mathcal{T}_{0,e_i}^\omega$. The path $\{(1/2, 1/2) + \phi_n^\omega : n \in \mathbb{Z}_+\}$ threads in between the two trees $\mathcal{T}_{0,e_i}^\omega$, $i \in \{1, 2\}$, and is hence called the *competition interface*. See Figure 1.

By Lemma 2.2 in [31], there exists a unique such path and its distribution under Q_0^ω is that of a Markov chain that starts at 0 and has transitions

$$\pi^{\text{cif}}_{y,y+e_i} = \frac{e^{-\omega_{y+e_i}}/Z_{0,y+e_i}}{e^{-\omega_{y+e_1}}/Z_{0,y+e_1} + e^{-\omega_{y+e_2}}/Z_{0,y+e_2}}.$$

The partition functions $Z_{0,y}$ in [31] include the weight ω_y and exclude ω_0, while we do the opposite. This is the reason for which our formula for π^{cif} is not as clean as the one in [31].

The above says that ϕ_n^ω is in fact a random walk in random environment, but with highly correlated transition probabilities. Our next result concerns the law of large numbers.

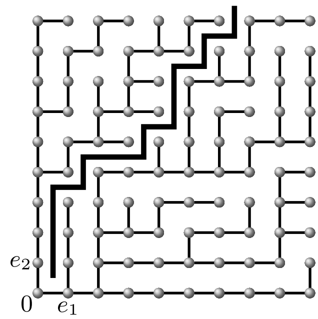

FIG. 1. *The competition interface shifted by* $(1/2, 1/2)$ (*solid line*) *separating the subtrees* $\mathcal{T}^{\omega}_{0,e_1}$ *and* $\mathcal{T}^{\omega}_{0,e_2}$.

THEOREM 3.12. *Assume* (3.5). *There exists a measurable* $\xi_* : \Omega \times \mathbb{T}_0 \to \mathrm{ri}\,\mathcal{U}$ *and an event* Ω_{cif} *such that* $\mathbb{P}(\Omega_{\mathrm{cif}}) = 1$ *and for every* $\omega \in \Omega_{\mathrm{cif}}$:

(a) *The competition interface has a strong law of large numbers*:

$$Q_0^{\omega}\{\phi_n^{\omega}/n \to \xi_*\} = 1.$$

(b) ξ_* *has cumulative distribution function*

$$Q_0^{\omega}\{\xi_* \cdot e_1 \le \xi \cdot e_1\} = e^{\omega_0 - B^{\xi+}(0,e_1,\omega)} \quad \text{for } \xi \in \mathrm{ri}\,\mathcal{U}. \tag{3.8}$$

Thus, $Q_0^{\omega}(\xi_* = \xi) > 0$ *if and only if* $B^{\xi-}(0, e_1, \omega) \neq B^{\xi+}(0, e_1, \omega)$.

(c) *If* Λ *is linear on* $]\eta, \zeta[$, *then* $Q_0^{\omega}(\eta \cdot e_1 < \xi_* \cdot e_1 < \zeta \cdot e_1) = 0$.

(d) *For any* $\xi \in \mathrm{ri}\,\mathcal{U}$, $\mathbb{E}Q_0^{\omega}(\xi_* = \xi) > 0$ *if and only if* $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \mathcal{D}$.

Part (d) in the above theorem says that if Λ is differentiable at all points then the distribution of ξ_* induced by the averaged measure $Q_0^{\omega}(d\mathcal{T}_0^{\omega})\mathbb{P}(d\omega)$ is continuous. Even in this case, part (b) leaves open the possibility that for a fixed ω the distribution of ξ_* under the quenched measure Q_0^{ω} has atoms.

3.3. *Bi-infinite polymer measures*. Theorem 3.7 and a variant of the Burton–Keane lack of space argument [10] allow us to prove that deterministically $\mathcal{U}_\xi$-directed bi-infinite Gibbs measures do not exist if $\mathcal{U}_\xi \subset \mathcal{D}$.

THEOREM 3.13. *Suppose that* $\xi, \underline{\xi}, \overline{\xi} \in \mathcal{D}$. *Then there exists an event* $\Omega_{\mathrm{bi},[\underline{\xi},\overline{\xi}]}$ *with* $\mathbb{P}(\Omega_{\mathrm{bi},[\underline{\xi},\overline{\xi}]}) = 1$ *such that for all* $\omega \in \Omega_{\mathrm{bi},[\underline{\xi},\overline{\xi}]}$ *there is no weakly* $\mathcal{U}_\xi$*-directed measure* $\Pi \in \overleftrightarrow{\mathrm{DLR}}^{\omega}$.

We now turn to nonexistence of covariant bi-infinite Gibbs measures. A similar question has been studied for spin systems including the random field Ising model; see [1, 2, 50, 58].

DEFINITION 3.14. A *T-covariant bi-infinite Gibbs measure* or *metastate* is a $\mathcal{M}_1(\mathbb{X}, \mathcal{X})$-valued random variable Π^{ω} satisfying the following:

(a) The map $\Omega \to \mathcal{M}_1(\mathbb{X}, \mathcal{X}) : \omega \mapsto \Pi^{\omega}$ is measurable.

(b) $\mathbb{P}(\Pi^{\omega} \in \overleftrightarrow{\mathrm{DLR}}^{\omega}) = 1$.

(c) For each $z \in \mathbb{Z}^2$, $\mathbb{P}(\Pi^{T_z\omega} \circ \theta_{-z} = \Pi^{\omega}) = 1$.

A quick proof checks that not only do metastates not exist, but in fact there are no shift-covariant measures on $\mathbb{X}$. This can be compared to the corresponding result showing nonexistence of metastates for the random field Ising model, proven in [58], where the mechanism is different.

LEMMA 3.15. *There does not exist a random variable satisfying Definition* 3.14(a) *and Definition* 3.14(c).

4. Shift-covariant cocycles. We now introduce our main tools, cocycles and correctors, and address their existence and regularity properties.

DEFINITION 4.1. A *shift-covariant cocycle* is a Borel-measurable function $B : \mathbb{Z}^2 \times \mathbb{Z}^2 \times \Omega \to \mathbb{R}$ which satisfies the following for all $x, y, z \in \mathbb{Z}^2$:

(a) (Shift-covariance) $\mathbb{P}\{B(x+z, y+z, \omega) = B(x, y, T_z\omega)\} = 1$.
(b) (Cocycle property) $\mathbb{P}\{B(x, y) + B(y, z) = B(x, z)\} = 1$.

REMARK 4.2. We will also use the term cocycle to denote a function satisfying Definition 4.1(b) only when $x, y, z \geq u$ for some $u \in \mathbb{Z}^2$.

As has already been done in the above definition, we will typically suppress the ω from the arguments unless it adds clarity. A shift-covariant cocycle is said to be $L^p(\mathbb{P})$ if $\mathbb{E}[|B(0, e_i)|^p] < \infty$ for $i \in \{1, 2\}$.

We are interested in cocycles that are consistent with the weights $\omega_x(\omega)$ in the following sense.

DEFINITION 4.3. For $\beta \in (0, \infty]$, a shift-covariant cocycle B satisfies β*-recovery* if for all $x \in \mathbb{Z}^2$ and $\mathbb{P}$-almost every ω:

$$\begin{aligned} e^{-\beta B(x,x+e_1,\omega)} + e^{-\beta B(x,x+e_2,\omega)} &= e^{-\beta\omega_x(\omega)} \quad \text{if } 0 < \beta < \infty, \\ \min\{B(x, x+e_1, \omega), B(x, x+e_2, \omega)\} &= \omega_x(\omega) \quad \text{if } \beta = \infty. \end{aligned} \tag{4.1}$$

Such cocycles are called *correctors*.

For a shift-covariant $L^1(\mathbb{P})$ cocycle, define the random vector $h(B) \in \mathbb{R}^2$ via

$$h(B) \cdot e_i = -\mathbb{E}[B(0, e_i) \mid \mathcal{I}],$$

where $\mathcal{I}$ is the σ-algebra generated by T-invariant events.

The next result is a special case of an extension of Theorem A.3 of [31] to the stationary setting. The proof of the extension is identical once one alters the definitions appropriately and can be found in the preprint [40]. Alternatively, one could pass through the ergodic decomposition theorem.

THEOREM 4.4. *Fix $\beta \in (0, \infty]$. Suppose B is a shift-covariant $L^1(\mathbb{P})$ β-recovering cocycle. Then*

$$\lim_{n\to\infty} \max_{x \in n\mathcal{U}\cap\mathbb{Z}_+^2} \frac{|B(0, x) + h(B) \cdot x|}{n} = 0 \quad \mathbb{P}\text{-almost surely}. \tag{4.2}$$

The next lemma shows that β-recovering covariant cocycles are naturally indexed by elements of the superdifferential $\partial\Lambda^\beta(\mathcal{U})$. This explains why we only consider cocycles with mean vectors lying in the superdifferential when we construct recovering cocycles in the next subsection. A similar observation in FPP appears in [19], Theorem 4.6.

LEMMA 4.5. *Assume the setting of Theorem* 4.4. *The following hold*:

(a) $-h(B) \in \partial\Lambda^\beta(\mathcal{U})$ *almost surely.*

(b) *If* $-\mathbb{E}[h(B)] \in \partial\Lambda^{\beta}(\xi)$ *for* $\xi \in \mathcal{U}$ *then* $-h(B) \in \partial\Lambda^{\beta}(\xi)$ *almost surely.*
(c) *If* $-\mathbb{E}[h(B)] \in \operatorname{ext}\partial\Lambda^{\beta}(\xi)$ *for some* $\xi \in \mathcal{U}$ *then* $h(B) = \mathbb{E}[h(B)]$ $\mathbb{P}$*-a.s.*

PROOF. Iterating the recovery property shows that almost surely

$$1 = \sum_{x \in n\mathcal{U} \cap \mathbb{Z}_+^2} Z_{0,x}^{\beta} e^{-\beta B(0,x)} \quad \text{if } \beta < \infty \quad \text{and}$$

$$0 = \max_{x \in n\mathcal{U} \cap \mathbb{Z}_+^2} \{G_{0,x} - B(0,x)\} \quad \text{if } \beta = \infty.$$

Take logs, divide by $n\beta$ if $\beta < \infty$ and n if $\beta = \infty$ then send $n \to \infty$ to get

$$0 = \max_{\xi \in \mathcal{U}} \{\Lambda^{\beta}(\xi) + h(B) \cdot \xi\} = \Lambda_{\mathrm{pl}}^{\beta}(h(B)) \quad \mathbb{P}\text{-almost surely.}$$

The first equality comes by an application of (2.2) and Theorem 4.4 and a fairly standard argument (e.g., the proof of Lemma 2.9 in [53]). The second equality is (2.3). By Lemma 2.1, the above implies $-h(B) \in \partial\Lambda^{\beta}(\mathcal{U})$.

Since $\Lambda_{\mathrm{pl}}^{\beta}(h(B)) = 0$, we have almost surely $\xi \cdot h(B) + \Lambda^{\beta}(\xi) \le 0$ for any $\xi \in \mathcal{U}$. If now ξ is such that $-\mathbb{E}[h(B)] \in \partial\Lambda^{\beta}(\xi)$, then again by Lemma 2.1 $\xi \cdot \mathbb{E}[h(B)] + \Lambda^{\beta}(\xi) = 0$ and, therefore, we must have $\xi \cdot h(B) + \Lambda^{\beta}(\xi) = 0$ almost surely. Again, we deduce that $-h(B) \in \partial\Lambda^{\beta}(\xi)$ almost surely.

If in addition we know that $-\mathbb{E}[h(B)] \in \operatorname{ext}\partial\Lambda^{\beta}(\xi)$, then we must have $h(B) = \mathbb{E}[h(B)]$ almost surely by definition of an extreme point. □

Before discussing existence of shift-covariant cocycles, we mention a few more basic properties of the superdifferential $\partial\Lambda^{\beta}(\mathcal{U})$. The proofs are easy exercises in convex analysis and are included in the preprint [40].

LEMMA 4.6. *The superdifferential map has the following properties*:

(a) *Let* $\xi, \xi' \in \operatorname{ri}\mathcal{U}$, $h \in -\partial\Lambda^{\beta}(\xi)$, *and* $h' \in -\partial\Lambda^{\beta}(\xi')$. *If for some* $i \in \{1,2\}$, $h \cdot e_i < h' \cdot e_i$, *then* $h \cdot e_{3-i} > h' \cdot e_{3-i}$. *Consequently, if* $h \cdot e_i = h' \cdot e_i$ *then* $h = h'$. *If for some* $i \in \{1,2\}$, $\xi \cdot e_i < \xi' \cdot e_i$, *then* $h \cdot e_i \le h' \cdot e_i$.

(b) $\partial\Lambda^{\beta}(\mathcal{U})$ *is closed in* $\mathbb{R}^2$; *if* $\xi_n \to \xi \in \operatorname{ri}\mathcal{U}$ *and* $h_n \to h$ *with* $-h_n \in \partial\Lambda^{\beta}(\xi_n)$, *then* $-h \in \partial\Lambda^{\beta}(\xi)$. *If* $h \in -\partial\Lambda^{\beta}(\mathcal{U})$ *and* $\varepsilon > 0$, *there exist* $h', h'' \in -\partial\Lambda^{\beta}(\mathcal{U})$ *with* $h \cdot e_1 - \varepsilon < h' \cdot e_1 < h \cdot e_1 < h'' \cdot e_1 < h \cdot e_1 + \varepsilon$.

(c) *For each* $\xi \in \operatorname{ri}\mathcal{U}$, $\partial\Lambda^{\beta}(\xi) = [\nabla\Lambda^{\beta}(\xi+), \nabla\Lambda^{\beta}(\xi-)]$. *This line segment is nontrivial for countably many* $\xi \in \operatorname{ri}\mathcal{U}$.

4.1. *Existence and regularity of shift-covariant correctors.* Fix any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as in Section 2.1. Let $\mathcal{B}_0$ be the union of $\{\infty\}$ and a dense countable subset of $(0,\infty)$. For $\beta \in (0,\infty]$ recall the limiting free energy Λ^{β} from Section 2.3 and let $\mathcal{H}^{\beta} = -\partial\Lambda^{\beta}(\mathcal{U})$. Let $\mathcal{H}_0^{\beta}$ be a countable dense subset of $\mathcal{H}^{\beta}$. Let $\mathcal{B}_0 \times \mathcal{H}_0^{\bullet} = \{(\beta, h) : \beta \in \mathcal{B}_0, h \in \mathcal{H}_0^{\beta}\}$ and define $\mathcal{B}_0 \times \mathcal{H}^{\bullet}$ similarly. Let $\widehat{\Omega} = \Omega \times \mathbb{R}^{\mathbb{Z}^2 \times \{1,2\} \times (\mathcal{B}_0 \times \mathcal{H}_0^{\bullet})}$ be equipped with the product topology and product Borel σ-algebra, $\widehat{\mathcal{G}}$. This space satisfies the conditions in Section 2.1 if Ω does. Let $\widehat{T} = \{\widehat{T}_z : z \in \mathbb{Z}^2\}$ be the $\widehat{\mathcal{G}}$-measurable group of transformations that map $(\omega, \{t_{x,i,\beta,h} : (x,i,\beta,h) \in \mathbb{Z}^2 \times \{1,2\} \times (\mathcal{B}_0 \times \mathcal{H}_0^{\bullet})\})$ to $(T_z\omega, \{t_{x+z,i,\beta,h} : (x,i,\beta,h) \in \mathbb{Z}^2 \times \{1,2\} \times (\mathcal{B}_0 \times \mathcal{H}_0^{\bullet})\})$. Denote by π_{Ω} the projection map to the Ω coordinate. We will write ω for $\pi_{\Omega}(\widehat{\omega})$ and the usual ω_x for $\omega_x(\omega)$.

The next theorem furnishes the covariant, recovering cocycles used in [29, 30] without the condition $\mathbb{P}(\omega_0 \ge c) = 1$ which was inherited from queueing theory; see [30], (2.1). In [30], the authors also prove ergodicity of these cocycles. As one can see from the proofs in this paper, ergodicity can be replaced by stationarity without losing the conclusions of [30]. We do not need ergodicity in the present project and so do not prove it here. These questions are addressed in our companion paper [41].

Our construction of cocycles follows ideas from [19]. However, there is a novel technical difficulty stemming from the directedness of the paths, boiling down to a lack of uniform integrability of pre-limit Busemann functions. Essentially the same issue is resolved in the zero temperature queueing literature by an argument which relies on Prabhakar's [52] rather involved result showing that ergodic fixed points of the corresponding $\cdot/G/1/\infty$ queue are attractive. Instead, we handle this problem by appealing to the variational formulas for the free energy derived in [28].

For a subset $I \subset \mathbb{Z}^2$, let $I^< = \{x \in \mathbb{Z}^2 : x \not\geq z \ \forall z \in I\}$.

THEOREM 4.7. *Assume* (2.1). *There are a $\widehat{T}$-invariant probability measure $\widehat{\mathbb{P}}$ on $(\widehat{\Omega}, \widehat{\mathcal{G}})$ and real-valued measurable functions $B^{\beta,h+}(x, y, \widehat{\omega})$ and $B^{\beta,h-}(x, y, \widehat{\omega})$ of $(\beta, h, x, y, \widehat{\omega}) \in (\mathcal{B}_0 \times \mathcal{H}^\bullet) \times \mathbb{Z}^2 \times \mathbb{Z}^2 \times \widehat{\Omega}$ such that*:

(a) *For any event $A \in \mathcal{F}$, $\widehat{\mathbb{P}}(\pi_\Omega(\widehat{\omega}) \in A) = \mathbb{P}(A)$.*

(b) *For any $I \subset \mathbb{Z}^2$, the variables*

$$\big\{\big(\omega_x, B^{\beta,h\pm}(\widehat{\omega}, x, y)\big) : x \in I, y \geq x, \beta \in \mathcal{B}_0, h \in \mathcal{H}^\beta\big\}$$

are independent of $\{\omega_x : x \in I^<\}$.

(c) *For each $\beta \in \mathcal{B}_0$, $h \in \mathcal{H}^\beta$ and $x, y \in \mathbb{Z}^2$, $B^{\beta,h\pm}(x, y)$ are integrable and*

$$\widehat{\mathbb{E}}\big[B^{\beta,h\pm}(x, x + e_i)\big] = -h \cdot e_i. \tag{4.3}$$

(d) *There exists a $\widehat{T}$-invariant event $\widehat{\Omega}_{\mathrm{coc}}$ with $\widehat{\mathbb{P}}(\widehat{\Omega}_{\mathrm{coc}}) = 1$ such that for each $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{coc}}$, $x, y, z \in \mathbb{Z}^2$, $\beta \in \mathcal{B}_0$, $h \in \mathcal{H}^\beta$ and $\epsilon \in \{-, +\}$*

$$B^{\beta,h\epsilon}(x + z, y + z, \widehat{\omega}) = B^{\beta,h\epsilon}(x, y, \widehat{T}_z\widehat{\omega}), \tag{4.4}$$

$$B^{\beta,h\epsilon}(x, y, \widehat{\omega}) + B^{\beta,h\epsilon}(y, z, \widehat{\omega}) = B^{\beta,h\epsilon}(x, z, \widehat{\omega}) \quad \text{and} \tag{4.5}$$

$$e^{-\beta B^{\beta,h\epsilon}(x,x+e_1,\widehat{\omega})} + e^{-\beta B^{\beta,h\epsilon}(x,x+e_2,\widehat{\omega})} = e^{-\beta\omega_x} \quad \text{if } \beta < \infty, \tag{4.6}$$

$$\min\big\{B^{\beta,h\epsilon}(x, x + e_1, \widehat{\omega}), B^{\beta,h\epsilon}(x, x + e_2, \widehat{\omega})\big\} = \omega_x \quad \text{if } \beta = \infty. \tag{4.7}$$

(e) *For each $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{coc}}$, $x \in \mathbb{Z}^2$, $\beta \in \mathcal{B}_0$, and $h, h' \in \mathcal{H}^\beta$ with $h \cdot e_1 \leq h' \cdot e_1$,*

$$\begin{aligned}
&B^{\beta,h-}(x, x + e_1, \widehat{\omega}) \geq B^{\beta,h+}(x, x + e_1, \widehat{\omega}) \\
&\qquad\qquad \geq B^{\beta,h'-}(x, x + e_1, \widehat{\omega}) \geq B^{\beta,h'+}(x, x + e_1, \widehat{\omega}) \quad \text{and} \\
&B^{\beta,h-}(x, x + e_2, \widehat{\omega}) \leq B^{\beta,h+}(x, x + e_2, \widehat{\omega}) \\
&\qquad\qquad \leq B^{\beta,h'-}(x, x + e_2, \widehat{\omega}) \leq B^{\beta,h'+}(x, x + e_2, \widehat{\omega}).
\end{aligned} \tag{4.8}$$

(f) *For each $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{coc}}$, $\beta \in \mathcal{B}_0$, $h \in \mathcal{H}^\beta$ and $x, y \in \mathbb{Z}^2$,*

$$\begin{aligned}
B^{\beta,h-}(x, y, \widehat{\omega}) &= \lim_{\substack{\mathcal{H}^\beta \ni h' \to h \\ h'\cdot e_1 \nearrow h\cdot e_1}} B^{\beta,h'\pm}(x, y, \widehat{\omega}) \quad \text{and} \\
B^{\beta,h+}(x, y, \widehat{\omega}) &= \lim_{\substack{\mathcal{H}^\beta \ni h' \to h \\ h'\cdot e_1 \searrow h\cdot e_1}} B^{\beta,h'\pm}(x, y, \widehat{\omega}).
\end{aligned} \tag{4.9}$$

When $B^{\beta,h+}(x, y, \widehat{\omega}) = B^{\beta,h-}(x, y, \widehat{\omega})$, we drop the $+/-$ and write $B^{\beta,h}(x, y, \widehat{\omega})$ and then for any $\epsilon \in \{-, +\}$

$$\lim_{\mathcal{H}^\beta \ni h' \to h} B^{\beta,h'\epsilon}(x, y, \widehat{\omega}) = B^{\beta,h}(x, y, \widehat{\omega}). \tag{4.10}$$

(g) *For each $\beta \in \mathcal{B}_0$ and $h \in \mathcal{H}^\beta$, there exists an event $\widehat{\Omega}_{\mathrm{cont},\beta,h} \subset \widehat{\Omega}_{\mathrm{coc}}$ with $\widehat{\mathbb{P}}(\widehat{\Omega}_{\mathrm{cont},\beta,h}) = 1$ and for all $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{cont},\beta,h}$ and all $x, y \in \mathbb{Z}^2$*

$$B^{\beta,h+}(x, y, \widehat{\omega}) = B^{\beta,h-}(x, y, \widehat{\omega}) = B^{\beta,h}(x, y, \widehat{\omega}).$$

REMARK 4.8. The proofs of parts (a) and (c)–(f) work word-for-word if the distribution of $\{\omega_x(\omega) : x \in \mathbb{Z}^2\}$ induced by $\mathbb{P}$ is T-ergodic and $\omega_0(\omega)$ belongs to class $\mathcal{L}$, defined in [28], Definition 2.1.

REMARK 4.9. In the rest of the paper, we will construct various full-measure events. By shift-invariance of $\mathbb{P}$ and $\widehat{\mathbb{P}}$, replacing any such event with the intersection of all its shifts we can assume these full-measure events to also be shift-invariant. This will be implicit in the proofs that follow.

PROOF OF THEOREM 4.7. For $\beta \in (0, \infty]$, $h \in \mathbb{R}^2$, $n \in \mathbb{N}$, $x \in \mathbb{Z}^2$ and $i \in \{1, 2\}$ define

$$B_n^{\beta,h}(x, x + e_i) = F_{x,(n)}^{\beta,h} - F_{x+e_i,(n)}^{\beta,h} - h \cdot e_i$$

if $x \cdot \widehat{e} < n$ and $B_n^{\beta,h}(x, x + e_i) = 0$ otherwise. A direct computation shows that if $x \cdot \widehat{e} < n$ then

$$B_n^{\beta,h}(x, x + e_i) = B_{n-x\cdot\widehat{e}}^{\beta,h}(0, e_i) \circ T_x, \tag{4.11}$$

$$e^{-\beta\omega_x} = e^{-\beta B_n^{\beta,h}(x,x+e_1)} + e^{-\beta B_n^{\beta,h}(x,x+e_2)} \quad \text{if } \beta < \infty \quad \text{and} \tag{4.12}$$

$$\omega_x = \min\bigl(B_n^{\beta,h}(x, x + e_1), B_n^{\beta,h}(x, x + e_2)\bigr) \quad \text{if } \beta = \infty. \tag{4.13}$$

Moreover, if $n > x \cdot \widehat{e} + 1$, then

$$\begin{aligned} &B_n^{\beta,h}(x, x + e_1) + B_n^{\beta,h}(x + e_1, x + e_1 + e_2) \\ &\quad = B_n^{\beta,h}(x, x + e_2) + B_n^{\beta,h}(x + e_2, x + e_1 + e_2). \end{aligned} \tag{4.14}$$

We also prove the following in Appendix A.1.

LEMMA 4.10. *Suppose $n > x \cdot \widehat{e}$ and that $h \cdot e_1 \le h' \cdot e_1$ and $h \cdot e_2 \ge h' \cdot e_2$. Then for each $\beta \in (0, \infty]$, each n, each $x \in \mathbb{Z}^2$, $\mathbb{P}$-almost surely*

$$\begin{aligned} &B_n^{\beta,h}(x, x + e_1) \ge B_n^{\beta,h'}(x, x + e_1) \quad \textit{and} \\ &B_n^{\beta,h}(x, x + e_2) \le B_n^{\beta,h'}(x, x + e_2). \end{aligned} \tag{4.15}$$

Next, we employ an averaging procedure previously used by [19, 27, 35, 46], among others. For each $n \in \mathbb{N}$, let N_n be uniformly distributed on $\{1, \dots, n\}$ and independent of everything else. Let $\mathbf{P}_n$ be its distribution and abbreviate $\widetilde{\mathbb{P}}_n = \mathbb{P} \otimes \mathbf{P}_n$ with expectation $\widetilde{\mathbb{E}}_n$. Define

$$\widehat{B}_n^{\beta,h}(x, x + e_i) = B_{N_n}^{\beta,h}(x, x + e_i).$$

Then whenever $n > x \cdot \widehat{e}$,

$$\begin{aligned} \widetilde{\mathbb{E}}_n\bigl[\widehat{B}_n^{\beta,h}(x, x + e_i)\bigr] &= \frac{1}{n} \sum_{j=x\cdot\widehat{e}+1}^{n} \bigl(\mathbb{E}\bigl[F_{0,(j-x\cdot\widehat{e})}^{\beta,h} - F_{0,(j-x\cdot\widehat{e}-1)}^{\beta,h}\bigr] - h \cdot e_i\bigr) \\ &= \frac{1}{n}\mathbb{E}\bigl[F_{0,(n-x\cdot\widehat{e})}^{\beta,h}\bigr] - \Bigl(\frac{n - x \cdot \widehat{e}}{n}\Bigr) h \cdot e_i. \end{aligned} \tag{4.16}$$

By (4.12)–(4.13), we have $\widehat{B}^{\beta,h}_n(x, x+e_i) \ge \omega_x$ on the event $\{N_n > x \cdot \widehat{e}\}$. On the complementary event, we have $\widehat{B}^{\beta,h}_n(x, x+e_i) = 0$. Whenever $n > x \cdot \widehat{e}$,

$$\widetilde{\mathbb{E}}_n\big[\big|\widehat{B}^{\beta,h}_n(x, x+e_i)\big|\big] = \widetilde{\mathbb{E}}_n\big[\widehat{B}^{\beta,h}_n(x, x+e_i)\big] - 2\widetilde{\mathbb{E}}_n\big[\min\big(0, \widehat{B}^{\beta,h}_n(x, x+e_i)\big)\big]$$
$$\le \frac{1}{n}\mathbb{E}\big[F^{\beta,h}_{0,(n-x\cdot\widehat{e})}\big] - \Big(\frac{n - x\cdot\widehat{e}}{n}\Big)h\cdot e_i + 2\mathbb{E}\big[|\omega_0|\big].$$

The first term converges to $\Lambda^{\beta}_{\mathrm{pl}}(h)$, which equals zero if $h \in \mathcal{H}^{\beta}$ by Lemma 2.1. Then the right-hand side is bounded by a finite constant $c(x, \beta, h)$. If we denote by $\mathbb{P}_n$, the law of

$$\big(\omega, \big\{\widehat{B}^{\beta,h}_n(x, x+e_i) : x \in \mathbb{Z}^2, i \in \{1,2\}, \beta \in \mathcal{B}_0, h \in \mathcal{H}^{\beta}_0\big\}\big)$$

induced by $\widetilde{\mathbb{P}}_n$ on $(\widehat{\Omega}, \widehat{\mathcal{G}})$, then the family $\{\mathbb{P}_n : n \in \mathbb{N}\}$ is tight. Let $\widehat{\mathbb{P}}$ denote any weak subsequential limit point of this family of measures. $\widehat{\mathbb{P}}$ is then $\widehat{T}$-invariant because of (4.11) and the T-invariance of $\mathbb{P}$. We prove next that such a measure satisfies all of the conclusions of the theorem.

Let $B^{\beta,h}(x, x+e_i, \widehat{\omega})$ be the (x, i, β, h)-coordinate of $\widehat{\omega} \in \widehat{\Omega}$. Since inequalities (4.15) hold for every n, there exists an event $\widehat{\Omega}'_0$ (which can be assumed to be $\widehat{T}$-invariant) with $\widehat{\mathbb{P}}(\widehat{\Omega}'_0) = 1$ such that for any $\beta \in \mathcal{B}_0$, $h, h' \in \mathcal{H}^{\beta}_0$ with $h \cdot e_1 \le h' \cdot e_1$, $x \in \mathbb{Z}^2$, and $\widehat{\omega} \in \widehat{\Omega}'_0$,

$$\begin{aligned} &B^{\beta,h}(x, x+e_1, \widehat{\omega}) \ge B^{\beta,h'}(x, x+e_1, \widehat{\omega}) \quad \text{and} \\ &B^{\beta,h}(x, x+e_2, \widehat{\omega}) \le B^{\beta,h'} x, x+e_2, \widehat{\omega}). \end{aligned} \tag{4.17}$$

Due to this monotonicity, we can define

$$B^{\beta,h-}(x, x+e_i, \widehat{\omega}) = \lim_{h' \in \mathcal{H}^{\beta}_0, h'\cdot e_1 \nearrow h\cdot e_1} B^{\beta,h}(x, x+e_i) \quad \text{and}$$
$$B^{\beta,h+}(x, x+e_i, \widehat{\omega}) = \lim_{h' \in \mathcal{H}^{\beta}_0, h'\cdot e_1 \searrow h\cdot e_1} B^{\beta,h}(x, x+e_i).$$

Then parts (e) and (f) come immediately.

Since (4.14) holds for every n, we get the existence of a $\widehat{T}$-invariant event $\widehat{\Omega}''_0 \subset \widehat{\Omega}'_0$ with $\widehat{\mathbb{P}}(\widehat{\Omega}''_0) = 1$ and

$$\begin{aligned} &B^{\beta,h}(x, x+e_1, \widehat{\omega}) + B^{\beta,h}(x+e_1, x+e_1+e_2, \widehat{\omega}) \\ &\quad = B^{\beta,h}(x, x+e_2, \widehat{\omega}) + B^{\beta,h}(x+e_2, x+e_1+e_2, \widehat{\omega}) \end{aligned} \tag{4.18}$$

for all $x \in \mathbb{Z}^2$, $\beta \in \mathcal{B}_0$, $h \in \mathcal{H}^{\beta}_0$, and $\widehat{\omega} \in \widehat{\Omega}''_0$. This equality transfers to $B^{\beta,h\pm}$. Set $B^{\beta,h\pm}(x + e_i, x, \widehat{\omega}) = -B^{\beta,h\pm}(x, x+e_i, \widehat{\omega})$ and for $x, y \in \mathbb{Z}^2$ and $\widehat{\omega} \in \widehat{\Omega}''_0$

$$B^{\beta,h\pm}(x, y, \widehat{\omega}) = \sum_{k=0}^{m-1} B^{\beta,h\pm}(x_k, x_{k+1}, \widehat{\omega}),$$

where $x_{0,m}$ is any path from x to y with $|x_{k+1} - x_k|_1 = 1$. The sum does not depend on the path we choose, due to (4.18). Property (4.5) follows.

For each n and each $A \in \mathcal{F}$, $\mathbb{P}_n(\pi_{\Omega}(\widehat{\omega}) \in A) = \mathbb{P}(\omega \in A)$. Moreover, for each n and each $I \subset \mathbb{Z}^2$, the family $\{\omega_x, \widehat{B}^{\beta,h}_n(x, x+e_i) : x \in I, \beta \in \mathcal{B}_0, h \in \mathcal{H}^{\beta}_0, i \in \{1,2\}\}$ is independent of $\{\omega_x : x \in I^{<}\}$. These properties transfer to $\widehat{\mathbb{P}}$ and parts (a) and (b) follow.

Again, since (4.11)–(4.13) hold for each n, there exists a $\widehat{T}$-invariant full $\widehat{\mathbb{P}}$-measure event $\widehat{\Omega}_{\mathrm{coc}} \subset \widehat{\Omega}''_0$ on which (4.4) and (4.6)–(4.7) hold. (d) is proved.

Recall (4.16) and that the right-hand side converges to $\Lambda^\beta(h) - h \cdot e_i = -h \cdot e_i$. We have also seen that

$$\widehat{B}_n^{\beta,h}(x, x+e_i) \geq \omega_x \mathbb{1}\{N_n > x \cdot \widehat{e}\}.$$

Fatou's lemma then implies that $B^{\beta,h}(x, x+e_i)$ is integrable under $\widehat{\mathbb{P}}$ and

$$\widehat{\mathbb{E}}\big[B^{\beta,h}(x, x+e_i)\big] \leq -h \cdot e_i \quad \text{for } \beta \in \mathcal{B}_0, h \in \mathcal{H}_0^\beta. \tag{4.19}$$

The reverse inequality is the nontrivial step in this construction. We spell out the argument in the case $\beta < \infty$, with the $\beta = \infty$ case being similar.

Let $\hbar = -\widehat{\mathbb{E}}[B^{\beta,h}(x, x+e_1)]e_1 - \widehat{\mathbb{E}}[B^{\beta,h}(x, x+e_2)]e_2$, $\mathfrak{S} = \sigma(\omega_x : x \in \mathbb{Z}^2)$, and define $\widetilde{B}^{\beta,h}(x, x+e_i) = \widehat{\mathbb{E}}[B^{\beta,h}(x, x+e_i) \mid \mathfrak{S}]$. Then $\widetilde{B}^{\beta,h}$ satisfies an equation like (4.18) which we can use to define a cocycle $\widetilde{B}^{\beta,h}(x, y)$, $x, y \in \mathbb{Z}^2$. Note that in general this cocycle will not recover the potential, even if $B^{\beta,h}$ does; it does however have the same mean vector $\hbar$ as $B^{\beta,h}$. By Jensen's inequality and recovery,

$$\begin{aligned} e^{-\beta \widetilde{B}^{\beta,h}(0,e_1)} + e^{-\beta \widetilde{B}^{\beta,h}(0,e_2)} &\leq \widehat{\mathbb{E}}\big[e^{-\beta B^{\beta,h}(0,e_1)} + e^{-\beta B^{\beta,h}(0,e_2)} \mid \mathfrak{S}\big] \\ &= \widehat{\mathbb{E}}\big[e^{-\beta\omega_0} \mid \mathfrak{S}\big] = e^{-\beta\omega_0}. \end{aligned} \tag{4.20}$$

For $h \in \mathcal{H}^\beta$, let $\xi \in \operatorname{ri}\mathcal{U}$ be such that $-h \in \partial\Lambda^\beta(\xi)$. Having conditioned on $\mathfrak{S}$, we are back in the canonical setting where $\widetilde{B}^{\beta,h}$ can be viewed as defined on the product space $\mathbb{R}^{\mathbb{Z}^2}$ with its Borel σ-algebra and an i.i.d. probability measure $\mathbb{P}_0^{\otimes\mathbb{Z}^2}$, where $\mathbb{P}_0$ is the distribution of ω_0 under $\mathbb{P}$. This setting is ergodic. Apply the duality of ξ and h, the variational formula of [28], Theorem 4.4, and (4.20) to obtain

$$-h \cdot \xi = \Lambda^\beta(\xi) \leq \mathbb{P}\text{-}\operatorname*{ess\,sup}_\omega \frac{1}{\beta} \log \sum_{i=1,2} e^{\beta\omega_0 - \beta\tilde{B}^{\beta,h}(0,e_i) - \beta\hbar\cdot\xi} \leq -\hbar \cdot \xi.$$

This, inequality (4.19), and the fact that ξ has positive coordinates imply $\hbar \cdot e_i = h \cdot e_i$ for $i = 1, 2$. In other words,

$$\widehat{\mathbb{E}}\big[B^{\beta,h}(x, x+e_i)\big] = -h \cdot e_i \quad \text{for } \beta \in \mathcal{B}_0, h \in \mathcal{H}_0^\beta.$$

Part (c) follows from this, monotonicity (4.17), and the monotone convergence theorem. Then part (g) follows from monotonicity (4.8) and the fact that for $i \in \{1, 2\}$, $B^{\beta,h\pm}(x, x+e_i)$ have the same mean $\hbar$. $\square$

It will be convenient to also define the process indexed by $\xi \in \operatorname{ri}\mathcal{U}$:

$$B^{\beta,\xi\pm}(x, y) = B^{\beta,-\nabla\Lambda^\beta(\xi\pm)\pm}(x, y).$$

REMARK 4.11. Parts (b)–(f) of Theorem 4.7 transfer to this process in the obvious way. For example the first set of inequalities in (4.8) becomes

$$B^{\beta,\xi-}(x, x+e_1) \geq B^{\beta,\xi+}(x, x+e_1) \geq B^{\beta,\zeta-}(x, x+e_1) \geq B^{\beta,\zeta+}(x, x+e_1)$$

for $\xi, \zeta \in \operatorname{ri}\mathcal{U}$ with $\xi \cdot e_1 \leq \zeta \cdot e_1$. (g) becomes the following: for each $\beta \in \mathcal{B}_0$ and $\xi \in \mathcal{D}^\beta$ there exists an event $\widehat{\Omega}_{\text{cont},\beta,\xi} = \widehat{\Omega}_{\text{cont},\beta,-\nabla\Lambda^\beta(\xi)}$ with $B^{\beta,\xi+}(x, y, \widehat{\omega}) = B^{\beta,\xi-}(x, y, \widehat{\omega}) = B^{\beta,\xi}(x, y, \widehat{\omega})$ for all $\widehat{\omega} \in \widehat{\Omega}_{\text{cont},\beta,\xi}$ and all $x, y \in \mathbb{Z}^2$.

We will need two lemmas in what follows.

LEMMA 4.12. *For each $\xi \in \operatorname{ri}\mathcal{U}$, there exists an event $\widehat{\Omega}_{\text{tilt},\xi+}$ such that $\widehat{\mathbb{P}}(\widehat{\Omega}_{\text{tilt},\xi+}) = 1$ and $h(B^{\beta,\xi+}) = -\nabla\Lambda^\beta(\xi+)$ on $\widehat{\Omega}_{\text{tilt},\xi+}$ for all $\beta \in \mathcal{B}_0$. A similar statement holds for $\xi-$.*

PROOF. By (4.3), we have $-\widehat{\mathbb{E}}[h(B^{\beta,\xi\pm})] = \nabla\Lambda^{\beta}(\xi\pm)$. The claim then follows from Lemma 4.5(c). $\square$

LEMMA 4.13. *There exists a $\widehat{T}$-invariant event $\widehat{\Omega}_{e_1,e_2}$ so that for $\widehat{\omega} \in \widehat{\Omega}_{e_1,e_2}$, $x \in \mathbb{Z}^2$, $\beta \in \mathcal{B}_0\backslash\{\infty\}$, and $i \in \{1,2\}$,*

$$\lim_{\mathrm{ri}\,\mathcal{U}\ni\xi\to e_i} B^{\beta,\xi\pm,\widehat{\omega}}(x, x+e_i) = \omega_x \quad \textit{and} \quad \lim_{\mathrm{ri}\,\mathcal{U}\ni\xi\to e_i} B^{\beta,\xi\pm}(x, x+e_{3-i}, \widehat{\omega}) = \infty.$$

PROOF. Take $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{coc}}$. Then the claimed limits exist due to the above monotonicity. The second limit follows from the first by recovery (4.6)–(4.7). Recovery also implies that $B^{\beta,\xi\pm}(x, x+e_i, \widehat{\omega}) - \omega_x \geq 0$. But then

$$\begin{aligned} 0 &\leq \widehat{\mathbb{E}}\Big[\lim_{\xi\to e_i} B^{\beta,\xi\pm}(x, x+e_i) - \omega_x\Big] = \widehat{\mathbb{E}}\Big[\inf_{\xi\in\mathrm{ri}\,\mathcal{U}} B^{\beta,\xi\pm}(x, x+e_i) - \omega_x\Big] \\ &\leq \inf_{\xi\in\mathrm{ri}\,\mathcal{U}} \widehat{\mathbb{E}}\big[B^{\beta,\xi\pm}(x, x+e_i)\big] - \mathbb{E}[\omega_x] = \inf_{\xi\in\mathrm{ri}\,\mathcal{U}} \nabla\Lambda^{\beta}(\xi\pm)\cdot e_i - \mathbb{E}[\omega_0] = 0, \end{aligned}$$

where the last equality follows from Lemma B.1. $\square$

As mentioned earlier, in the rest of the paper we assume $\beta = 1$ and omit it from the notation. In particular, we write Λ and $\mathcal{H}$ instead of Λ^1 and $\mathcal{H}^1$.

4.2. *Ratios of partition functions.* Following similar steps to the proofs of (4.3) of [31] and Theorem 6.1 in [30] we obtain the next theorem. Our more natural definition of the $B^{\xi\pm}$ processes makes the claim hold on one full-measure event, in contrast with [30, 31] where the events depend on ξ.

THEOREM 4.14. *There exists a shift-invariant $\widehat{\Omega}_{\mathrm{Bus}}$ with $\widehat{\mathbb{P}}(\widehat{\Omega}_{\mathrm{Bus}}) = 1$ and for all $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{Bus}}$, any (possibly $\widehat{\omega}$-dependent) $\xi \in \mathrm{ri}\,\mathcal{U}$, $x \in \mathbb{Z}^2$, and $\mathcal{U}_{\xi}$-directed sequence $x_n \in \mathbb{V}_n$:*

$$\begin{aligned} &e^{-B^{\underline{\xi}-}(x,x+e_1,\widehat{\omega})} \leq \varliminf_{n\to\infty} \frac{Z_{x+e_1,x_n}}{Z_{x,x_n}} \leq \varlimsup_{n\to\infty} \frac{Z_{x+e_1,x_n}}{Z_{x,x_n}} \leq e^{-B^{\overline{\xi}+}(x,x+e_1,\widehat{\omega})}, \\ &e^{-B^{\overline{\xi}+}(x,x+e_2,\widehat{\omega})} \leq \varliminf_{n\to\infty} \frac{Z_{x+e_2,x_n}}{Z_{x,x_n}} \leq \varlimsup_{n\to\infty} \frac{Z_{x+e_2,x_n}}{Z_{x,x_n}} \leq e^{-B^{\underline{\xi}-}(x,x+e_2,\widehat{\omega})}. \end{aligned} \tag{4.21}$$

PROOF. Let $\mathcal{D}_0$ be a countable dense subset of $\mathcal{D}$. Let $\xi \in \mathrm{ri}\,\mathcal{U}$ and

$$\widehat{\omega} \in \widehat{\Omega}_{\mathrm{Bus}} = \widehat{\Omega}_{\mathrm{coc}} \cap \bigcap_{\zeta\in\mathcal{D}_0} \widehat{\Omega}_{\mathrm{cont},\zeta}.$$

First, consider $\bar{x}_n = \bar{x}_n(\xi)$ that is the (leftmost) closest point in $\mathbb{V}_n$ to $n\xi$. Then $\bar{x}_n/n \to \xi$ as $n \to \infty$. Let $\zeta \in \mathcal{D}_0$ be such that $\zeta\cdot e_1 > \overline{\xi}\cdot e_1$. Since $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{cont},\zeta}$ we have $B^{\zeta\pm} = B^{\zeta}$. For $x \in \mathbb{V}_k$, $y \in \mathbb{V}_{\ell}$, $k, \ell \in \mathbb{Z}$ and $x \leq y$, define the point-to-point partition function

$$Z^{\mathrm{NE}}_{x,y} = \sum_{x_{k,\ell}\in\mathbb{X}_x^y} e^{\sum_{i=k}^{\ell-1}\bar{\omega}_{x_i}}, \tag{4.22}$$

where $\bar{\omega}_u = B^{\zeta}(u, u+e_i, \widehat{\omega})$ if $y - u \in \mathbb{N}e_i$, $i \in \{1,2\}$ and $\bar{\omega}_u = \omega_u$ otherwise. Recovery and the cocycle property of B^{ζ} imply $Z^{\mathrm{NE}}_{x,y} = e^{B^{\zeta}(x,y)}$, $x \leq y$, because they satisfy the same recursion with the same boundary conditions on $y - x \in \mathbb{N}e_i$, $i \in \{1,2\}$. Then

$$\frac{Z^{\mathrm{NE}}_{x,\bar{x}_n+e_1+e_2}}{Z^{\mathrm{NE}}_{x+e_1,\bar{x}_n+e_1+e_2}} = e^{B^{\zeta}(x,x+e_1,\widehat{\omega})}.$$

For v with $x \le v \le y$, let $Z^{\mathrm{NE}}_{x,y}(v)$ be defined as in (4.22) but with the sum being only over admissible paths that go through v. Apply the first inequality in (B.1) with $\tilde{\omega}$ such that $\omega_y(\tilde{\omega}) = \omega_y$ for $y \le \bar{x}_n$, $\omega_y(\tilde{\omega}) = B^\zeta(y, y + e_i, \widehat{\omega})$ for y with $\bar{x}_n + e_1 + e_2 - y \in \mathbb{N}e_i$, $i \in \{1, 2\}$, $v = \bar{x}_n$ and $u = \bar{x}_n + e_1$ to get

$$\begin{aligned}\frac{Z_{x,\bar{x}_n}}{Z_{x+e_1,\bar{x}_n}} &\ge \frac{Z^{\mathrm{NE}}_{x,\bar{x}_n+e_1+e_2}(\bar{x}_n + e_1)}{Z^{\mathrm{NE}}_{x+e_1,\bar{x}_n+e_1+e_2}(\bar{x}_n + e_1)} \\ &\ge \frac{Z^{\mathrm{NE}}_{x,\bar{x}_n+e_1+e_2}(\bar{x}_n + e_1)}{Z^{\mathrm{NE}}_{x,\bar{x}_n+e_1+e_2}} \cdot e^{B^\zeta(x,x+e_1)}.\end{aligned} \tag{4.23}$$

Using the shape theorems (2.2) and (4.2) and a standard argument, given for example in the proof of [30], Lemma 6.4, we have

$$\begin{aligned}&\lim_{n\to\infty} n^{-1} \log Z^{\mathrm{NE}}_{x,\bar{x}_n+e_1+e_2}(\bar{x}_n + e_1) \\ &\quad = \sup\{\Lambda(\eta) + (\xi - \eta) \cdot \nabla\Lambda(\zeta) : \eta \in [(\xi \cdot e_1)e_1, \xi]\} \quad \text{and} \\ &\lim_{n\to\infty} n^{-1} \log Z^{\mathrm{NE}}_{x,\bar{x}_n+e_1+e_2}(\bar{x}_n + e_2) \\ &\quad = \sup\{\Lambda(\eta) + (\xi - \eta) \cdot \nabla\Lambda(\zeta) : \eta \in [(\xi \cdot e_2)e_2, \xi]\}.\end{aligned}$$

By Lemma 4.6(a) $\nabla\Lambda(\zeta) \cdot e_1 \le \nabla\Lambda(\xi) \cdot e_1$, and hence for $\eta \in [(\xi \cdot e_2)e_2, \xi]$,

$$(\xi - \eta) \cdot \nabla\Lambda(\zeta) \le (\xi - \eta) \cdot \nabla\Lambda(\xi) \le \Lambda(\xi) - \Lambda(\eta).$$

Thus, the second supremum in the above is achieved at $\eta = \xi$ and the limit is equal to $\Lambda(\xi)$. Set $\eta_0 = (\xi \cdot e_1/\zeta \cdot e_1)\zeta \in [(\xi \cdot e_1)e_1, \xi]$. For $\eta \in [(\xi \cdot e_1)e_1, \xi]$,

$$\begin{aligned}(\eta_0 - \eta) \cdot \nabla\Lambda(\zeta) &= \frac{\xi \cdot e_1}{\zeta \cdot e_1}\left(\zeta - \frac{\zeta \cdot e_1}{\xi \cdot e_1}\eta\right) \cdot \nabla\Lambda(\zeta) \\ &\le \frac{\xi \cdot e_1}{\zeta \cdot e_1}\left(\Lambda(\zeta) - \frac{\zeta \cdot e_1}{\xi \cdot e_1}\Lambda(\eta)\right) = \Lambda(\eta_0) - \Lambda(\eta).\end{aligned}$$

Rearranging, we have $\Lambda(\eta) + (\xi - \eta) \cdot \nabla\Lambda(\zeta) \le \Lambda(\eta_0) + (\xi - \eta_0) \cdot \nabla\Lambda(\zeta)$. Hence, the first supremum is achieved at η_0. But if equality also held for $\eta = \xi$, then concavity of Λ would imply that Λ is linear on $[\eta_0, \xi]$, and hence on $[\zeta, \xi]$. This cannot be the case since $\zeta \notin \mathcal{U}_\xi$. We therefore have

$$\Lambda(\eta_0) + (\xi - \eta_0) \cdot \nabla\Lambda(\zeta) > \Lambda(\xi).$$

This implies that $Z^{\mathrm{NE}}_{x,\bar{x}_n+e_1+e_2}(\bar{x}_n + e_2)/Z^{\mathrm{NE}}_{x,\bar{x}_n+e_1+e_2}(\bar{x}_n + e_1) \to 0$ as $n \to \infty$. Since $Z^{\mathrm{NE}}_{x,\bar{x}_n+e_1+e_2} = Z^{\mathrm{NE}}_{x,\bar{x}_n+e_1+e_2}(\bar{x}_n + e_1) + Z^{\mathrm{NE}}_{x,\bar{x}_n+e_1+e_2}(\bar{x}_n + e_2)$ we conclude that the fraction in (4.23) converges to 1. Consequently,

$$\varlimsup_{n\to\infty} \frac{Z_{x+e_1,\bar{x}_n}}{Z_{x,\bar{x}_n}} \le e^{-B^\zeta(x,x+e_1)}.$$

Taking $\zeta \to \overline{\xi}$, we get the right-most inequality in the first line of (4.21). The other inequalities come similarly.

Next, we prove the full statement of the theorem, namely that (4.21) holds for all sequences $x_n \in \mathbb{V}_n$, directed into $\mathcal{U}_\xi$. To this end, take such a sequence and let $\eta_\ell, \zeta_\ell \in \mathrm{ri}\,\mathcal{U}$ be two sequences such that $\eta_\ell \cdot e_1 < \underline{\xi} \cdot e_1 \le \overline{\xi} \cdot e_1 < \zeta_\ell \cdot e_1$, $\eta_\ell \to \underline{\xi}$, and $\zeta_\ell \to \overline{\xi}$. For a fixed ℓ and a large n, we have

$$\bar{x}_n(\eta_\ell) \cdot e_1 < x_n \cdot e_1 < \bar{x}_n(\zeta_\ell) \cdot e_1 \quad \text{and} \quad \bar{x}_n(\eta_\ell) \cdot e_2 > x_n \cdot e_2 > \bar{x}_n(\zeta_\ell) \cdot e_2.$$

Applying (B.1), we have

$$\frac{Z_{x+e_1,\bar{x}_n(\eta_\ell)}}{Z_{x,\bar{x}_n(\eta_\ell)}} \leq \frac{Z_{x+e_1,x_n}}{Z_{x,x_n}} \leq \frac{Z_{x+e_1,\bar{x}_n(\zeta_\ell)}}{Z_{x,\bar{x}_n(\zeta_\ell)}}.$$

Take $n \to \infty$ and apply the already proved version of (4.21) for the sequences $\bar{x}_n(\eta_\ell)$ and $\bar{x}_n(\zeta_\ell)$ to get for each ℓ,

$$e^{-B^{\underline{\eta}_\ell -}(x,x+e_1,\widehat{\omega})} \leq \varliminf_{n\to\infty} \frac{Z_{x+e_1,x_n}}{Z_{x,x_n}} \leq \varlimsup_{n\to\infty} \frac{Z_{x+e_1,x_n}}{Z_{x,x_n}} \leq e^{-B^{\overline{\zeta}_\ell +}(x,x+e_1,\widehat{\omega})}.$$

Send $\ell \to \infty$ to get the first line of (4.21). The second line is similar. □

5. Semi-infinite polymer measures. In this section, we prove general versions of our main results on rooted solutions, starting with Lemma 3.4.

PROOF OF LEMMA 3.4. Fix $x \in \mathbb{V}_m$, $m \in \mathbb{Z}$. Suppose Π_x is degenerate. By (2.5), there exist $y \geq x$ and $n \geq m$ with $y \in \mathbb{V}_n$ and $\Pi_x(X_n = y) = 0$. Then for $v \geq y$ with $v \cdot \widehat{e} = k$,

$$0 = \Pi_x(X_n = y) \geq \Pi_x(X_n = y, X_k = v) = \Pi_x(X_k = v)\, Q^\omega_{x,v}(X_n = y).$$

Hence, $\Pi_x(X_k = v) = 0$. This means that

$$\Pi_x\{\forall n \geq m : X_n \cdot e_1 \leq y \cdot e_1\} + \Pi_x\{\forall n \geq m : X_n \cdot e_2 \leq y \cdot e_2\} = 1.$$

Denote the first probability by α. We will show that

$$\begin{aligned} &\Pi_x\big\{\forall n \geq m : X_n = x + (n-m)e_2\big\} = \alpha \quad \text{and} \\ &\Pi_x\big\{\forall n \geq m : X_n = x + (n-m)e_1\big\} = 1 - \alpha. \end{aligned} \tag{5.1}$$

If (5.1) holds, then $\Pi_x = \alpha \Pi_x^{e_2} + (1-\alpha)\Pi_x^{e_1}$. Let us now prove (5.1).

If $\alpha = 0$, then also $\Pi_x\{\forall n \geq m : X_n = x + (n-m)e_2\} = 0 = \alpha$. If, on the other hand, $\alpha > 0$, then either again $\Pi_x\{\forall n \geq m : X_n = x + (n-m)e_2\} = \alpha$ or there exist $k \geq m$ and $v \geq x$ with $v \cdot e_1 \in (0, y \cdot e_1]$ such that

$$\Pi_x\big\{\forall n \geq k : X_n = v + (n-k)e_2\big\} = \delta \in (0,\alpha]. \tag{5.2}$$

Let $\ell = (v - x) \cdot e_2$. Then $v - x = (k - m - \ell)e_1 + \ell e_2$. For any $n \geq k$,

$$\begin{aligned} \delta &\leq \Pi_x\big\{X_i = v + (i-k)e_2, k \leq i \leq n\big\} \\ &= \Pi_x\big\{X_n = v + (n-k)e_2\big\}\, Q^\omega_{x,v+(n-k)e_2}\big\{X_i = v + (i-k)e_2, k \leq i \leq n\big\} \\ &\leq \frac{Z_{x,v} e^{\sum_{i=k}^{n-1} \omega_{v+(i-k)e_2}}}{Z_{x,v+(n-k)e_2}} \\ &\leq \frac{Z_{x,v} \exp\{\sum_{i=k}^{n-1} \omega_{v+(i-k)e_2}\}}{\exp\{\sum_{i=0}^{\ell+n-k-1} \omega_{x+ie_2}\} \exp\{\sum_{i=0}^{k-m-\ell-1} \omega_{x+(\ell+n-k)e_2+ie_1}\}} \\ &= \frac{Z_{x,v} \exp\{\sum_{i=k}^{n-1} (\omega_{v+(i-k)e_2} - \omega_{x+(i+\ell-k)e_2})\}}{\exp\{\sum_{i=0}^{\ell-1} \omega_{x+ie_2}\} \exp\{\sum_{i=0}^{k-m-\ell-1} \omega_{x+(\ell+n-k)e_2+ie_1}\}}. \end{aligned}$$

Let Ω_{nondeg} be the intersection of the events

$$\left\{\exists n \geq k : \frac{\exp\{\sum_{i=k}^{n-1} (\omega_{v+(i-k)e_j} - \omega_{x+(i+\ell-k)e_j})\}}{\exp\{\sum_{i=0}^{k-m-\ell-1} \omega_{x+(\ell+n-k)e_j+ie_{3-j}}\}} \leq e^{-r}\right\}$$

over all $x, v \in \mathbb{Z}^2$ such that $v \geq x$, $r \in \mathbb{N}$, $j \in \{1, 2\}$, and integers $k \geq \ell = (v - x) \cdot e_j$ and $m = (v - x) \cdot (e_1 + e_2)$. The event Ω_{nondeg} has full $\mathbb{P}$-probability. Indeed, for each $r \in \mathbb{N}$,

$$\mathbb{P}\biggl(\exists n \geq k : \frac{\exp\{\sum_{i=k}^{n-1} (\omega_{v+(i-k)e_j} - \omega_{x+(i+\ell-k)e_j})\}}{\exp\{\sum_{i=0}^{k-m-\ell-1} \omega_{x+(\ell+n-k)e_j+ie_{3-j}}\}} \leq e^{-r}\biggr)$$
$$= \mathbb{P}\biggl(\exists n \geq 0 : \frac{\exp\{\sum_{i=0}^{n-1} (\omega_{e_1+ie_2} - \omega_{ie_2})\}}{\exp\{\sum_{i=0}^{k-m-\ell-1} \omega_{(i+2)e_1}\}} \leq e^{-r}\biggr) = 1.$$

The first equality is because weights are i.i.d., and hence the distribution of the two ratios is the same. The second equality holds because $\sum_{i=0}^{n-1} (\omega_{e_1+ie_2} - \omega_{ie_2})$ is a sum of i.i.d. centered nondegenerate random variables, and hence has liminf $-\infty$.

For $\omega \in \Omega_{\text{nondeg}}$, we have

$$\delta \leq \frac{Z_{x,v} e^{-r}}{e^{\sum_{i=0}^{\ell-1} \omega_{x+ie_2}}}$$

for all $r \in \mathbb{N}$. Taking $r \to \infty$ gives a contradiction. Therefore, (5.2) cannot hold. The first equality in (5.1) is proved. The other one is similar. $\square$

Since $\{\forall n \geq m : X_n = x + (n - m)e_{3-i}\} \subset \{\forall n \geq m : X_n \cdot e_i \leq y \cdot e_i\}$, (5.1) implies that for any $\omega \in \Omega_{\text{nondeg}}$, $x \in \mathbb{V}_m$, $m \in \mathbb{Z}$, $\Pi_x \in \text{DLR}_x^\omega$, $y \in x + \mathbb{Z}_+^2$, and $i \in \{1, 2\}$ we have

$$\Pi_x\bigl\{\{\forall n \geq m : X_n \cdot e_i \leq y \cdot e_i\} \setminus \{X_{m,\infty} = x + \mathbb{Z}_+ e_{3-i}\}\bigr\} = 0. \tag{5.3}$$

LEMMA 5.1. *Fix $\omega \in \Omega$ and $x \in \mathbb{Z}^2$. Let $\Pi_x \in \text{DLR}_x^\omega$ be a nondegenerate solution. Then Π_x is a Markov chain with transition probabilities*

$$\pi_{y,y+e_i}^x(\omega) = \frac{\Pi_x(y + e_i)\, Z_{x,y}\, e^{\omega_y}}{\Pi_x(y)\, Z_{x,y+e_i}}, \quad y \geq x, i \in \{1, 2\}. \tag{5.4}$$

PROOF. Let $x \in \mathbb{V}_m$ and $y \in \mathbb{V}_n$, $n \geq m$. Fix an admissible path $x_{m,n}$ with $x_m = x$ and $x_n = y$. Compute for $i \in \{1, 2\}$,

$$\begin{aligned}
&\Pi_x(X_{n+1} = y + e_i \mid X_{m,n} = x_{m,n}) \\
&\quad = \frac{\Pi_x(X_{n+1} = y + e_i)\, Z_{x,y}\, e^{\sum_{i=m}^{n} \omega_{x_i}}}{\Pi_x(X_n = y)\, Z_{x,y+e_i}\, e^{\sum_{i=m}^{n-1} \omega_{x_i}}} \\
&\quad = \frac{\Pi_x(X_{n+1} = y + e_i)\, Z_{x,y}\, e^{\omega_y}}{\Pi_x(X_n = y)\, Z_{x,y+e_i}} \\
&\quad = \Pi_x(X_{n+1} = y + e_i \mid X_n = y).
\end{aligned}$$

$\square$

Next, we relate nondegenerate DLR solutions in environment ω and cocycles that recover the potential $\{\omega_x(\omega)\}$.

THEOREM 5.2. *Fix $\omega \in \Omega$ and $x \in \mathbb{V}_m$, $m \in \mathbb{Z}$. Then Π_x is a nondegenerate DLR solution in environment ω if, and only if, there exists a cocycle $\{B(u, v) : u, v \geq x\}$ that satisfies recovery (4.1) and*

$$\Pi_x(X_{m,n} = x_{m,n}) = e^{\sum_{k=m}^{n-1} \omega_{x_k} - B(x,x_n)} \tag{5.5}$$

for every admissible path $x_{m,n}$ starting at $x_m = x$. This cocycle is uniquely determined by the formula

$$e^{-B(u,v)} = \frac{\Pi_x(v)}{\Pi_x(u)} \cdot \frac{Z_{x,u}}{Z_{x,v}}, \quad u, v \geq x. \tag{5.6}$$

It satisfies

$$e^{-B(x,y)} = E^{\Pi_x}\left[\frac{Z_{y,X_n}}{Z_{x,X_n}}\right] \quad \textit{for all } y \geq x \textit{ and } n \geq y \cdot \widehat{e}. \tag{5.7}$$

The transition probabilities of Π_x are then given by

$$\pi^x_{y,y+e_i}(\omega) = e^{\omega_y - B(y,y+e_i)}, \quad y \geq x, i \in \{1, 2\}. \tag{5.8}$$

When Π_x is given, we denote the corresponding cocycle by $B^{\Pi_x}(u, v)$. Conversely, when B is given, we denote the corresponding DLR solution in environment ω (that B recovers) by Π^B_x.

PROOF OF THEOREM 5.2. Given a nondegenerate solution $\Pi_x \in \mathrm{DLR}^\omega_x$ define B via (5.6). Telescoping products check that this is a cocycle. To check the recovery property, write $Q^\omega_{x,u+e_i}(u) = Z_{x,u}e^{\omega_u}(Z_{x,u+e_i})^{-1}$. Hence,

$$\begin{aligned}
&e^{-B(u,u+e_1)} + e^{-B(u,u+e_2)} \\
&\quad= \frac{e^{-\omega_u}}{\Pi_x(u)}\big(\Pi_x(u+e_1)Q^\omega_{x,u+e_1}(u) + \Pi_x(u+e_2)Q^\omega_{0,u+e_2}(u)\big) \\
&\quad= \frac{e^{-\omega_u}}{\Pi_x(u)}\big(\Pi_x(u, u+e_1 \in X_\bullet) + \Pi_x(u, u+e_2 \in X_\bullet)\big) = e^{-\omega_u}.
\end{aligned}$$

(5.8) follows from (5.4) and (5.6) and then (5.5) follows from (5.8), the Markov property of Π_x, and the cocycle property of B.

Conversely, given a cocycle B that recovers the potential, define π^x via (5.8). Recovery implies that π^x are transition probabilities. Let Π_x be the distribution of the Markov chain with these transition probabilities. Again, (5.8) and the cocycle property imply (5.5). In particular, Π_x is not degenerate. For $y \geq x$, adding (5.5) over all admissible paths from x to y gives

$$\Pi_x(y) = Z_{x,y}e^{-B(x,y)}. \tag{5.9}$$

This and the cocycle property of B imply (5.6). Using $y = x_n$ and solving for $e^{-B(x,y)}$ in (5.9), then plugging back into (5.5) gives

$$\Pi_x(x_{m,n}) = e^{\sum_{k=m}^{n-1}\omega_{x_k} - B(x,x_n)} = \Pi_x(y)\, Q^\omega_{x,y}(x_{m,n}),$$

which says Π_x is a DLR solution in environment ω.

Lastly, we prove (5.7). Let $k = y \cdot \widehat{e} \geq m$. Then

$$\begin{aligned}
Z_{x,y}E^{\Pi_x}\left[\frac{Z_{y,X_n}}{Z_{x,X_n}}\right] &= \sum_{\substack{v \geq y \\ v \in \mathbb{V}_n}} \Pi_x(X_n = v)\frac{Z_{x,y}Z_{y,v}}{Z_{x,v}} \\
&= \sum_{\substack{v \geq y \\ v \in \mathbb{V}_n}} \Pi_x(X_n = v)Q^\omega_{x,v}(X_k = y) = \Pi_x(X_k = y).
\end{aligned}$$

Then (5.7) follows from this and (5.9). The theorem is proved. □

The DLR solutions that correspond to cocycles $B^{\xi\pm}$, $\xi \in \operatorname{ri}\mathcal{U}$, will play a key role in what follows. We will denote these by $\Pi_x^{\xi\pm,\widehat{\omega}}$ and the corresponding transition probabilities by $\pi^{\xi\pm,\widehat{\omega}}$. These transition probabilities do not depend on the starting point x. When $B^{\xi-} = B^{\xi+} = B^{\xi}$ we also write $\Pi_x^{\xi,\widehat{\omega}}$ and $\pi^{\xi,\widehat{\omega}}$. In addition to recovering the potential, the $B^{\xi\pm}$ cocycles are also $\widehat{T}$-covariant when ξ is deterministic. We next show how these observations relate to the law of large numbers for the corresponding DLR solution.

THEOREM 5.3. *Let B be an $L^1(\widehat{\Omega}, \widehat{\mathbb{P}})$ $\widehat{T}$-covariant cocycle that recovers the weights (ω_x). There exists an event $\widehat{\Omega}_B \subset \widehat{\Omega}$ such that $\widehat{\mathbb{P}}(\widehat{\Omega}_B) = 1$ and for every $\widehat{\omega} \in \widehat{\Omega}_B$ and $x \in \mathbb{Z}^2$ the distribution of X_n/n under $\Pi_x^{B(\widehat{\omega})}$ satisfies a large deviation principle with convex rate function $I_B(\xi) = -h(B)\cdot\xi - \Lambda(\xi)$, $\xi \in \mathcal{U}$. Consequently, $\Pi_x^{B(\widehat{\omega})}$ is strongly directed into $\mathcal{U}_{h(B)}$.*

PROOF. From equation (5.9) and the shape theorems (2.2) and (4.2) for the free energy and shift-covariant cocycles, we get that $\widehat{\mathbb{P}}$-almost surely, for all $x \in \mathbb{Z}^2$, all $\xi \in \mathcal{U}$, and any sequence $x_n \ge x$ with $x_n \in \mathbb{V}_n$ and $x_n/n \to \xi$,

$$n^{-1}\log \Pi_x^B(X_n = x_n) = n^{-1}\log Z_{x,x_n} - n^{-1}B(x,x_n) \underset{n\to\infty}{\longrightarrow} \Lambda(\xi) + h(B)\cdot\xi.$$

The large deviation principle follows. Then Borel–Cantelli and strict positivity of I_B off of $\mathcal{U}_{h(B)}$ imply the directedness claimed in the theorem. $\square$

Next, couple $\Pi_x^{\xi\pm,\widehat{\omega}}$, $x \in \mathbb{Z}^2$, $\xi \in \operatorname{ri}\mathcal{U}$, pathwise, as described in Appendix A.1. Denote the coupled up-right paths by $X_{m,\infty}^{x,\xi\pm,\widehat{\omega}}$, $x \in \mathbb{V}_m$, $m \in \mathbb{Z}$, $\xi \in \operatorname{ri}\mathcal{U}$. When $B^{\xi-} = B^{\xi+} = B^{\xi}$, we write $X^{x,\xi,\widehat{\omega}}$. For $i \in \{1,2\}$, set $X_k^{x,e_i\pm,\widehat{\omega}} = X_k^{x,e_i,\widehat{\omega}} = x + (k-m)e_i$, $k \ge m$.

When $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{coc}}$, the event from Theorem 4.7 on which (4.8) holds, and $x \in \mathbb{V}_m$, $m \in \mathbb{Z}$, paths $X^{x,\xi\pm,\widehat{\omega}}$ are ordered: For any $\xi, \zeta \in \mathcal{U}$ with $\xi\cdot e_1 < \zeta\cdot e_1$ and any $k \ge m$,

$$X_k^{x,\xi-,\widehat{\omega}}\cdot e_1 \le X_k^{x,\xi+,\widehat{\omega}}\cdot e_1 \le X_k^{x,\zeta-,\widehat{\omega}}\cdot e_1 \le X_k^{x,\zeta+,\widehat{\omega}}\cdot e_1. \tag{5.10}$$

THEOREM 5.4. *There exists an event $\widehat{\Omega}_{\mathrm{exist}} \subset \widehat{\Omega}$ such that $\widehat{\mathbb{P}}(\widehat{\Omega}_{\mathrm{exist}}) = 1$ and for every $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{exist}}$, $x \in \mathbb{Z}^2$ and $\xi \in \operatorname{ri}\mathcal{U}$, $\Pi_x^{\xi\pm,\widehat{\omega}} \in \mathrm{DLR}_x^{\omega}$ and are, respectively, strongly $\mathcal{U}_{\xi\pm}$-directed. For $i \in \{1,2\}$, the trivial polymer measure $\Pi_x^{e_i}$ gives a DLR solution that is strongly $\mathcal{U}_{e_i}$-directed.*

PROOF. Let $\mathcal{U}_0$ be a countable subset of $\operatorname{ri}\mathcal{U}$ that contains all of $(\operatorname{ri}\mathcal{U})\setminus\mathcal{D}$ and a countable dense subset of $\mathcal{D}$. Let

$$\widehat{\Omega}_{\mathrm{exist}} = \widehat{\Omega}_{\mathrm{coc}} \cap \bigcap_{\xi\in\mathcal{U}_0\cap\mathcal{D}} \widehat{\Omega}_{\mathrm{cont},\xi} \cap \bigcap_{\xi\in\mathcal{U}_0} (\widehat{\Omega}_{B^{\xi+}} \cap \widehat{\Omega}_{\mathrm{tilt},\xi+} \cap \widehat{\Omega}_{B^{\xi-}} \cap \widehat{\Omega}_{\mathrm{tilt},\xi-}).$$

When $\xi \in \mathcal{U}_0$ and $\widehat{\omega} \in \widehat{\Omega}_{B^{\xi+}} \cap \widehat{\Omega}_{\mathrm{tilt},\xi+}$ Lemma 4.12 says $h(B^{\xi+}) = -\nabla\Lambda(\xi+)$ and then Theorem 5.3 says that $\Pi_x^{\xi+,\widehat{\omega}}$ is strongly $\mathcal{U}_{\xi+}$-directed, for all $x \in \mathbb{Z}^2$. A similar argument works for $\Pi_x^{\xi-,\widehat{\omega}}$.

Now fix $\xi \in (\operatorname{ri}\mathcal{U})\setminus\mathcal{U}_0$ and $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{exist}}$. If $\xi\cdot e_1 < \overline{\xi}\cdot e_1$, then pick $\zeta \in \mathcal{U}_0\cap\mathcal{D}$ such that $\xi\cdot e_1 < \zeta\cdot e_1 < \overline{\xi}\cdot e_1$. Then $\overline{\xi} = \overline{\zeta}$. The ordering of paths (5.10) implies that $X_k^{x,\xi+,\widehat{\omega}}\cdot e_1 \le X_k^{x,\zeta,\widehat{\omega}}\cdot e_1$ for all $k \ge m$ (there is no need for the $\pm$ distinction for $\zeta \in \mathcal{U}_0\cap\mathcal{D}$). Since the distribution of the latter path is $\Pi_x^{\zeta,\widehat{\omega}}$ and it is strongly $\mathcal{U}_\zeta$-directed, we deduce that

$$\varlimsup_{n\to\infty} n^{-1}X_n\cdot e_1 \le \overline{\zeta}\cdot e_1 = \overline{\xi}\cdot e_1, \quad \Pi_x^{\xi+,\widehat{\omega}}\text{-almost surely.}$$

If $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \mathcal{U}_0$ is such that $\xi = \overline{\xi}$, then let $\varepsilon > 0$ and pick $\zeta \in \mathcal{U}_0 \cap \mathcal{D}$ such that $\xi \cdot e_1 < \zeta \cdot e_1 \le \overline{\zeta} \cdot e_1 < \xi \cdot e_1 + \varepsilon = \overline{\xi} \cdot e_1 + \varepsilon$. This is possible because $\nabla\Lambda(\zeta)$ converges to but never equals $\nabla\Lambda(\xi)$ as $\zeta \cdot e_1 \searrow \xi \cdot e_1$. (Note that $\xi \in \mathcal{D}$.) The same ordering argument as above implies

$$\varlimsup_{n\to\infty} n^{-1} X_n \cdot e_1 \le \overline{\zeta} \cdot e_1 \le \overline{\xi} \cdot e_1 + \varepsilon, \quad \Pi_x^{\xi+,\widehat{\omega}}\text{-almost surely.}$$

Take $\varepsilon \to 0$. Similarly,

$$\varliminf_{n\to\infty} n^{-1} X_n \cdot e_1 \ge \underline{\xi} \cdot e_1, \quad \Pi_x^{\xi-,\widehat{\omega}}\text{-almost surely.}$$

Appealing once again to the path ordering, we see now that both $\Pi_x^{\xi\pm,\widehat{\omega}}$ are strongly directed into $\mathcal{U}_\xi$. Since $\xi \in \mathcal{D}$ we have $\mathcal{U}_{\xi+} = \mathcal{U}_{\xi-} = \mathcal{U}_\xi$. The theorem is proved. □

PROOF OF THEOREM 3.2. Recall the set $\mathcal{U}_0$ from the proof of Theorem 5.4. When $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \mathcal{D}$ and $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{exist}}$, $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{tilt},\xi+} \cap \widehat{\Omega}_{\mathrm{tilt},\xi-}$ and we have by Lemma 4.12

$$\widehat{\mathbb{E}}[B^{\xi-}(0,e_1)\,|\,\mathcal{I}] = e_1 \cdot \nabla\Lambda(\xi-) > e_1 \cdot \nabla\Lambda(\xi+) = \widehat{\mathbb{E}}[B^{\xi+}(0,e_1)\,|\,\mathcal{I}]. \tag{5.11}$$

By the ergodic theorem, there exists a full $\widehat{\mathbb{P}}$-measure event $\widehat{\Omega}_0''' \subset \widehat{\Omega}_{\mathrm{exist}}$ such that for each $\widehat{\omega} \in \widehat{\Omega}_0'''$, $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \mathcal{D}$ and $x \in \mathbb{Z}^2$ there is a $y \ge x$ such that $B^{\xi-}(y, y+e_1) \ne B^{\xi+}(y, y+e_1)$. This implies $\Pi_x^{\xi-,\widehat{\omega}} \ne \Pi_x^{\xi+,\widehat{\omega}}$.

Recall the projection π_Ω from $\widehat{\Omega}$ onto Ω. There exists a family of regular conditional distributions $\mu_\omega(\cdot) = \widehat{\mathbb{P}}(\cdot\,|\,\pi_\Omega^{-1}(\omega))$ and a Borel set $\Omega_{\mathrm{reg}} \subset \Omega$ such that $\mathbb{P}(\Omega_{\mathrm{reg}}) = 1$ and for every $\omega \in \Omega_{\mathrm{reg}}$, $\mu_\omega(\pi_\Omega^{-1}(\omega)) = 1$. See Example 10.4.11 in [9]. Since

$$\int \mu_\omega(\widehat{\Omega}_0''')\,\mathbb{P}(d\omega) = \widehat{\mathbb{P}}(\widehat{\Omega}_0''') = 1$$

we see that $\mu_\omega(\widehat{\Omega}_0''') = 1$, $\mathbb{P}$-almost surely. Set

$$\Omega_{\mathrm{exist}} = \Omega_{\mathrm{reg}} \cap \{\omega \in \Omega : \mu_\omega(\widehat{\Omega}_0''') = 1\}. \tag{5.12}$$

Then $\mathbb{P}(\Omega_{\mathrm{exist}}) = 1$. We take $\omega \in \Omega_{\mathrm{exist}}$ so that $\mu_\omega(\pi_\Omega^{-1}(\omega) \cap \widehat{\Omega}_0''') = \mu_\omega(\widehat{\Omega}_0''') = 1$. There exists $\widehat{\omega} \in \widehat{\Omega}_0'''$ with $\pi_\Omega(\widehat{\omega}) = \omega$. For $\xi \in \mathcal{U}$, the $\mathcal{U}_{\xi\pm}$-directed solutions in the claim are $\Pi_x^{\xi\pm,\widehat{\omega}}$. □

THEOREM 5.5. *Fix $\xi \in \mathcal{D}$. Assume $\underline{\xi}, \overline{\xi} \in \mathcal{D}$. There exists a $\widehat{T}$-invariant event $\widehat{\Omega}_{[\underline{\xi},\overline{\xi}]} \subset \widehat{\Omega}$ such that $\widehat{\mathbb{P}}(\widehat{\Omega}_{[\underline{\xi},\overline{\xi}]}) = 1$ and for every $\widehat{\omega} \in \widehat{\Omega}_{[\underline{\xi},\overline{\xi}]}$ and $x \in \mathbb{Z}^2$, $\Pi_x^{\xi\pm,\widehat{\omega}} = \Pi_x^{\xi,\widehat{\omega}}$ is the unique weakly $\mathcal{U}_\xi$-directed solution in DLR_x^ω. It is also strongly directed into $\mathcal{U}_\xi$ and for any $\mathcal{U}_\xi$-directed sequence (x_n) the sequence of quenched point-to-point polymer measures Q_{x,x_n}^ω converges weakly to $\Pi_x^{\xi,\widehat{\omega}}$. The family $\{\Pi_x^{\xi,\widehat{\omega}} : x \in \mathbb{Z}^2, \widehat{\omega} \in \widehat{\Omega}\}$ is consistent and $\widehat{T}$-covariant.*

PROOF. Let $\eta_k, \zeta_k \in \mathrm{ri}\,\mathcal{U}$ be such that $\eta_k \cdot e_1$ strictly increases to $\underline{\xi} \cdot e_1$ and $\zeta_k \cdot e_1$ strictly decreases to $\overline{\xi} \cdot e_1$. Let

$$\widehat{\Omega}_{[\underline{\xi},\overline{\xi}]} = \pi_\Omega^{-1}(\Omega_{\mathrm{nondeg}}) \cap \widehat{\Omega}_{\mathrm{coc}} \cap \widehat{\Omega}_{\mathrm{cont},\underline{\xi}} \cap \widehat{\Omega}_{\mathrm{cont},\overline{\xi}} \cap \widehat{\Omega}_{\mathrm{Bus}} \cap \widehat{\Omega}_{B^{\xi+}}.$$

Take $\widehat{\omega} \in \widehat{\Omega}_{[\underline{\xi},\overline{\xi}]}$. Since $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{Bus}}$, Theorem 4.14 implies that for all $y \in \mathbb{Z}^2$ and $k \in \mathbb{N}$,

$$\varliminf_{n\to\infty} \frac{Z_{y+e_1,\lfloor n\eta_k\rfloor}}{Z_{y,\lfloor n\eta_k\rfloor}} \ge e^{-B^{\underline{\eta}_k-}(y,y+e_1,\widehat{\omega})} \quad \text{and}$$

$$\varlimsup_{n\to\infty} \frac{Z_{y+e_2,\lfloor n\eta_k\rfloor}}{Z_{y,\lfloor n\eta_k\rfloor}} \le e^{-B^{\underline{\eta}_k-}(y,y+e_2,\widehat{\omega})}.$$

Fix $x \in \mathbb{V}_m$, $m \in \mathbb{Z}$ and $y \geq x$. Fix $\varepsilon > 0$. Since the choice of $\widehat{\omega}$ guarantees continuity as $\underline{\eta}_k \to \underline{\xi}$, we can choose k large then n large so that

$$\frac{Z_{y+e_2,\lfloor n\eta_k\rfloor}}{Z_{y,\lfloor n\eta_k\rfloor}} \leq e^{-B^{\underline{\eta}_k -}(y,y+e_2,\widehat{\omega})} + \varepsilon/2 \leq e^{-B^{\underline{\xi}}(y,y+e_2,\widehat{\omega})} + \varepsilon.$$

Let $\Pi_x \in \mathrm{DLR}_x^{\omega}$ be weakly $\mathcal{U}_\xi$-directed. Since $\eta_k, \zeta_k \in \mathrm{ri}\,\mathcal{U}$, both $n\eta_k \geq y$ and $n\zeta_k \geq y$ for large n. Applying (B.1) in the first inequality, we have

$$\begin{aligned}
&\Pi_x\{X_n \cdot e_1 > \lfloor n\eta_k \cdot e_1\rfloor, X_n \geq y\} \\
&\quad \leq \Pi_x\left\{\frac{Z_{y+e_2,X_n}}{Z_{y,X_n}} \leq \frac{Z_{y+e_2,\lfloor n\eta_k\rfloor}}{Z_{y,\lfloor n\eta_k\rfloor}}, X_n \geq y\right\} \\
&\quad \leq \Pi_x\left\{\frac{Z_{y+e_2,X_n}}{Z_{y,X_n}} \leq e^{-B^{\underline{\xi}}(y,y+e_2,\widehat{\omega})} + \varepsilon, X_n \geq y\right\} \leq 1.
\end{aligned}$$

The weak directedness implies the first probability converges to one. Hence,

$$\lim_{n\to\infty} \Pi_x\left\{\frac{Z_{y+e_2,X_n}}{Z_{y,X_n}} \leq e^{-B^{\underline{\xi}}(y,y+e_2,\widehat{\omega})} + \varepsilon, X_n \geq y\right\} = 1.$$

Using a similar argument with the sequence ζ_k, we also get

$$\lim_{n\to\infty} \Pi_x\left\{\frac{Z_{y+e_2,X_n}}{Z_{y,X_n}} \geq e^{-B^{\overline{\xi}}(y,y+e_2,\widehat{\omega})} - \varepsilon, X_n \geq y\right\} = 1.$$

Since $\xi, \underline{\xi}, \overline{\xi} \in \mathcal{D}$, we have $\nabla\Lambda(\underline{\xi}-) = \nabla\Lambda(\overline{\xi}+) = \nabla\Lambda(\xi)$ and by our choice of $\widehat{\omega}$, $B^{\overline{\xi}} = B^{\underline{\xi}} = B^{\xi\pm} = B^{\xi}$. We have shown that $Z_{y+e_2,X_n}/Z_{y,X_n}$ converges in Π_x-probability to $e^{-B^{\xi}(y,y+e_2,\widehat{\omega})}$, for every $y \geq x$. The case of e_1-increments is similar.

Using any fixed admissible path from x to y and applying the cocycle property of B^{ξ} and the above limit (to the increments of the path) we see that $Z_{y,X_n}/Z_{x,X_n} \to e^{-B^{\xi}(x,y,\widehat{\omega})}$, in Π_x-probability. But if $\ell = y \cdot \widehat{e}$, then

$$0 \leq \frac{Z_{y,X_n}}{Z_{x,X_n}} = \frac{Z_{y,X_n}}{\sum_{\substack{v \geq x \\ v \in \mathbb{V}_\ell}} Z_{x,v} Z_{v,X_n}} \leq \frac{1}{Z_{x,y}} < \infty. \tag{5.13}$$

Since $\omega = \pi_\Omega(\widehat{\omega}) \in \Omega_{\mathrm{nondeg}}$, Lemma 3.4 says Π_x is nondegenerate. Thus, bounded convergence and (5.7) imply that B^{ξ} is the cocycle that corresponds to Π_x. In other words, $\Pi_x = \Pi_x^{\xi,\widehat{\omega}}$. Since $B^{\xi} = B^{\xi+}$ and $\widehat{\omega} \in \widehat{\Omega}_{B^{\xi+}}$ we conclude that Π_x is strongly $\mathcal{U}_\xi$-directed.

For the weak convergence claim, apply Theorem 4.14 to get that for any up-right path $x_{m,k}$ out of x,

$$\begin{aligned}
Q^{\omega}_{x,x_n}(X_{m,k} = x_{m,k}) &= \frac{e^{\sum_{i=m}^{k-1}\omega_{x_i}} Z_{x_k,x_n}}{Z_{x,x_n}} \\
&\underset{n\to\infty}{\longrightarrow} e^{\sum_{i=m}^{k-1}\omega_{x_i} - B^{\xi}(x,x_k,\widehat{\omega})} = \Pi_x^{\xi,\widehat{\omega}}(X_{m,k} = x_{m,k}).
\end{aligned} \tag{5.14}$$

The covariance and consistency claims follow from the covariance of $B^{\xi} = B^{\xi+}$ and the fact that $\Pi_x^{\xi,\widehat{\omega}}$ all use the same transition probabilities $\pi^{\xi,\widehat{\omega}}$, regardless of the starting point x, as noted right before the statement of Theorem 5.3. The theorem is proved. □

PROOF OF THEOREM 3.7. Define $\Omega_{[\underline{\xi},\overline{\xi}]}$ out of $\widehat{\Omega}_{[\underline{\xi},\overline{\xi}]}$, similar to (5.12). Then $\mathbb{P}(\Omega_{[\underline{\xi},\overline{\xi}]}) = 1$ and for each $\omega \in \Omega_{[\underline{\xi},\overline{\xi}]}$ there exists $\widehat{\omega} \in \widehat{\Omega}_{[\underline{\xi},\overline{\xi}]}$ with $\pi_\Omega(\widehat{\omega}) = \omega$. The claim now follows directly from Theorem 5.5. □

PROOF OF THEOREM 3.8. Let $\widehat{\Omega}'_{[\underline{\xi},\overline{\xi}]} = \widehat{\Omega}_{\text{Bus}} \cap \widehat{\Omega}_{[\underline{\xi},\overline{\xi}]}$. Define $\Omega'_{[\underline{\xi},\overline{\xi}]}$ out of $\widehat{\Omega}'_{[\underline{\xi},\overline{\xi}]}$, similar to (5.12). Then $\mathbb{P}(\Omega'_{[\underline{\xi},\overline{\xi}]}) = 1$ and for each $\omega \in \Omega'_{[\underline{\xi},\overline{\xi}]}$ there exists $\widehat{\omega} \in \widehat{\Omega}'_{[\underline{\xi},\overline{\xi}]}$ with $\pi_\Omega(\widehat{\omega}) = \omega$. When $\underline{\xi}, \xi, \overline{\xi} \in \mathcal{D}$, $\nabla\Lambda(\underline{\xi}\pm) = \nabla\Lambda(\overline{\xi}\pm) = \nabla\Lambda(\xi\pm) = \nabla\Lambda(\xi)$. Since $\widehat{\omega} \in \widehat{\Omega}_{\text{Bus}} \subset \widehat{\Omega}_{\text{cont},\underline{\xi}} \cap \widehat{\Omega}_{\text{cont},\overline{\xi}}$, $B^{\underline{\xi}-} = B^{\xi-} = B^{\xi+} = B^{\overline{\xi}+}$. Theorem 4.14 then implies the limit in (3.2) exists and equals the value of the cocycle $B^\xi(x, y, \widehat{\omega})$. Then (3.4) follows from (4.8).

Take $x \in \mathbb{V}_m$ and consider $n > m$. By [53], Theorem 4.1, the distributions of X_n/n under $Q^{\omega,h}_{x,(n)}$ satisfy a large deviation principle with rate function

$$J(\zeta) = -h \cdot \zeta - \Lambda(\zeta) + \Lambda_{\text{pl}}(h), \qquad \zeta \in \mathcal{U}.$$

By duality, $J(\cdot)$ vanishes exactly on $\mathcal{U}_h = [\underline{\xi}, \overline{\xi}]$. Borel–Cantelli and strict positivity of J off of $[\underline{\xi}, \overline{\xi}]$ imply that $\nu^\omega = \bigotimes_{n>m} Q^{\omega,h}_{x,(n)}$ is strongly $[\underline{\xi}, \overline{\xi}]$-directed. For $i \in \{1, 2\}$, use the weak convergence in Theorem 3.7 to find

$$\begin{aligned}\frac{Z^h_{x+e_i,(n)}}{Z^h_{x,(n)}} &= e^{-\omega_x - h\cdot e_i} E^{\nu^\omega}\big[Q^\omega_{x,X_n}(x + e_i)\big]\\ &\underset{n\to\infty}{\longrightarrow} e^{-\omega_x - h\cdot e_i} \Pi^{\xi,\omega}_x(x + e_i) = e^{-B^\xi(x,x+e_i;\omega) - h\cdot e_i}.\end{aligned}$$

Equation (3.3) follows from the above, telescoping products and (4.5). □

PROOF OF COROLLARY 3.9. The claims follow from the observation that the limit in (3.2) is exactly the cocycle B^ξ. □

LEMMA 5.6. *Fix $x, y \in \mathbb{Z}^2$, $\omega \in \Omega$, and $\Pi_x \in \text{DLR}^\omega_x$. Then $Z_{y,X_n}/Z_{x,X_n}$ is a Π_x-backward martingale relative to the filtration $\mathcal{X}_{[n,\infty)}$.*

PROOF. Fix $N > n$ and an up-right path $x_{n+1,N}$ with $\Pi_x(x_{n+1,N}) > 0$. Abbreviate $A = \{x_{n+1} - e_1, x_{n+1} - e_2\}$. Write

$$\begin{aligned}E^{\Pi_x}\Big[\frac{Z_{y,X_n}}{Z_{x,X_n}} \,\Big|\, X_{n+1,N} = x_{n+1,N}\Big] &= \sum_{x_n\in A} \frac{Z_{y,x_n}}{Z_{x,x_n}} \cdot \frac{\Pi_x(x_{n,N})}{\Pi_x(x_{n+1,N})}\\ &= \sum_{x_n\in A} \frac{Z_{y,x_n} e^{\omega_{x_n}}}{Z_{x,x_{n+1}}} = \frac{Z_{y,x_{n+1}}}{Z_{x,x_{n+1}}}.\end{aligned}$$

□

THEOREM 5.7. *Fix $\omega \in \Omega$ and $x \in \mathbb{V}_m$, $m \in \mathbb{Z}$. Let Π_x be a nondegenerate extreme point of* DLR^ω_x. *Then for all $u, v \geq x$,*

$$\frac{Z_{v,X_n}}{Z_{u,X_n}} \underset{n\to\infty}{\longrightarrow} e^{-B^{\Pi_x}(u,v)}, \qquad \Pi_x\text{-almost surely.} \tag{5.15}$$

PROOF. By the backward-martingale convergence theorem [23], Theorem 5.6.1, $Z_{y,X_n}/Z_{x,X_n}$ converges Π_x-almost surely and in $L^1(\Pi_x)$ to a limit $\kappa_{x,y} = \kappa_{x,y}(x_{m,\infty})$. *A priori* $\kappa_{x,y}$ is $\bigcap_n \mathcal{X}_{[n,\infty)}$-measurable. Define

$$\kappa_{y,y+e_i} = \frac{\kappa_{x,y+e_i}}{\kappa_{x,y}} \quad \text{for } i \in \{1, 2\} \text{ and } y \in x + \mathbb{Z}^2_+ \text{ with } \kappa_{x,y} > 0.$$

Note that

$$\frac{Z_{y+e_1,X_n}}{Z_{y,X_n}} + \frac{Z_{y+e_2,X_n}}{Z_{y,X_n}} = e^{-\omega_y}. \tag{5.16}$$

This implies

$$\Pi_x\{\forall y \geq x : \kappa_{x,y} = 0 \text{ or } \kappa_{y,y+e_1} + \kappa_{y,y+e_2} = e^{-\omega_y}\} = 1. \tag{5.17}$$

Telescoping products imply

$$\begin{aligned}\Pi_x\{\forall y \geq x : \kappa_{x,y+e_1+e_2} &= 0 \text{ or} \\ \kappa_{y,y+e_1}\kappa_{y+e_1,y+e_1+e_2} &= \kappa_{y,y+e_2}\kappa_{y+e_2,y+e_1+e_2}\} = 1.\end{aligned} \tag{5.18}$$

On the event in (5.17),

$$\pi_{y,y+e_i} = \begin{cases} \kappa_{y,y+e_i}\, e^{\omega_y}, & i \in \{1,2\} \text{ and } y \in x + \mathbb{Z}_+^2 \text{ such that } \kappa_{x,y} > 0, \\ 1/2, & i \in \{1,2\} \text{ and } y \in x + \mathbb{Z}_+^2 \text{ such that } \kappa_{x,y} = 0 \end{cases}$$

define transition probabilities. Let Π_x^κ be the distribution of the Markov chain $X_{m,\infty}$ starting at $X_m = x$ and using these transition probabilities.

Note that $\kappa_{x,x} = 1$ and if $\kappa_{x,y} > 0$ and $\pi_{y,y+e_i} > 0$, then $\kappa_{y,y+e_i} > 0$ and $\kappa_{x,y+e_i} = \kappa_{x,y}\kappa_{y,y+e_i} > 0$. This means that the Markov chain stays Π_x^κ-almost surely within the set $\{y \geq x : \kappa_{x,y} > 0\}$.

On the intersection of the two events in (5.17) and (5.18), if $x_{m,k}$ is an admissible path starting at x and $\Pi_x^\kappa(x_{m,k}) > 0$, then the above paragraph says $\kappa_{x,x_i} > 0$ for each $i \in \{m, \dots, k\}$ and then

$$\Pi_x^\kappa(x_{m,k}) = \prod_{i=m}^{k-1} \pi_{x_i,x_{i+1}} = \prod_{i=m}^{k-1} \kappa_{x_i,x_{i+1}}\, e^{\omega_{x_i}} = \kappa_{x,x_k}\, e^{\sum_{i=m}^{k-1}\omega_{x_i}}. \tag{5.19}$$

Adding over all admissible paths from x to $y \in \mathbb{V}_k$ gives

$$\Pi_x^\kappa(y) = \kappa_{x,y}\, Z_{x,y}. \tag{5.20}$$

Putting the two displays together gives

$$\Pi_x^\kappa(x_{m,k}) = \Pi_x^\kappa(y) \cdot \frac{e^{\sum_{i=m}^{k-1}\omega_{x_i}}}{Z_{x,y}} = \Pi_x^\kappa(y)\, Q_{x,y}^\omega(x_{m,k}).$$

In other words, $\Pi_x^\kappa \in \mathrm{DLR}_x^\omega$, Π_x-almost surely. The L^1-convergence implies

$$E^{\Pi_x}[\kappa_{x,y}] = \lim_{n\to\infty} E^{\Pi_x}\left[\frac{Z_{y,X_n}}{Z_{x,X_n}}\right] = e^{-B^{\Pi_x}(x,y)},$$

where we used (5.7) for the last equality (since Π_x is assumed to be nondegenerate). The above, (5.19) and (5.5) give

$$E^{\Pi_x}\big[\Pi_x^\kappa(x_{m,k})\big] = e^{\sum_{i=m}^{k-1}\omega_{x_i} - B^{\Pi_x}(x,y)} = \Pi_x(x_{m,k}).$$

In other words, $\Pi_x = \int \Pi_x^{\kappa(x_{m,\infty})}\, \Pi_x(dx_{m,\infty})$. Since Π_x was assumed to be an extreme point in DLR_x^ω, we conclude that $\Pi_x(\Pi_x^\kappa = \Pi_x) = 1$. Since Π_x^κ determines κ, this says that $\kappa_{x,y}(x_{m,\infty}) = e^{-B^{\Pi_x}(x,y)}$ for all $y \geq x$ and Π_x-almost every $x_{m,\infty}$. Now (5.15) follows from writing

$$\frac{Z_{v,X_n}}{Z_{u,X_n}} = \frac{Z_{v,X_n}/Z_{x,X_n}}{Z_{u,X_n}/Z_{x,X_n}},$$

taking $n \to \infty$ and applying the cocycle property of B^{Π_x}. $\square$

We now turn to the proof of Theorem 3.5. The full proof requires handling some technical issues, so we begin with a brief sketch of the main idea in the case where Λ is strictly concave to give a sense of how the argument works. By (5.15), $\log Z_{y,X_n} - \log Z_{y+e_1,X_n}$ converges Π_x-almost surely to $B^{\Pi_x}(y, y+e_1)$. On the other hand, (3.2) implies that for nice directions ξ, $\log Z_{y,\lfloor n\xi\rfloor} - \log Z_{y+e_2,\lfloor n\xi\rfloor}$ converges $\mathbb{P}$-almost surely to $B^{\xi}(y, y+e_1)$. This, and the monotonicity from (B.1) imply that if $X_n \cdot e_1 > n\xi \cdot e_1$ happens infinitely often, then $B^{\Pi_x}(y, y+e_1) \le B^{\xi}(y, y+e_1)$ for all $y \ge x$. But then coupling $\Pi_x^{\xi,\omega}$ and Π_x pathwise, as described in Appendix A.1, implies that almost surely the Π_x-path must stay to the right of the $\Pi_x^{\xi,\omega}$-path. A similar argument holds if $X_n \cdot e_1 < n\xi \cdot e_1$ happens infinitely often. In short, this argument shows that if a subsequential limit point of X_n goes to the right of a nice direction ζ with positive probability, then every subsequential limit point must stay to the right of ζ. Similarly, if any subsequential limit goes to the left of a nice direction η, then every subsequential limit point must stay to the left of η. These two statements are only consistent if the path satisfies the strong law of large numbers for some direction $\xi \in \mathrm{ri}\,\mathcal{U}$. The technicalities in the proof arise because we do not assume strict concavity.

PROOF OF THEOREM 3.5. Let $\widehat{\Omega}_{\mathrm{dir}} = \widehat{\Omega}_{\mathrm{exist}} \cap \widehat{\Omega}_{\mathrm{Bus}} \cap \widehat{\Omega}_{e_1,e_2}$ and similar to (5.12) let $\Omega_{\mathrm{dir}} = \Omega_{\mathrm{nondeg}} \cap \Omega_{\mathrm{reg}} \cap \{\omega \in \Omega : \mu_\omega(\widehat{\Omega}_{\mathrm{dir}}) = 1\}$. Then $\mathbb{P}(\Omega_{\mathrm{dir}}) = 1$. Fix $\omega \in \Omega_{\mathrm{dir}}$. There exists $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{dir}}$ such that $\pi_\Omega(\widehat{\omega}) = \omega$.

Take $\zeta \in \mathrm{ri}\,\mathcal{U}$. For any $y \ge x$, we have $n\zeta \ge y$ when n is large enough. Applying (B.1), (5.3) and Theorem 4.14 gives that for $\varepsilon > 0$,

$$\begin{aligned}
&\varlimsup_{n\to\infty} \Pi_x\{X_n \cdot e_1 > \lfloor n\zeta \cdot e_1\rfloor\} \\
&\quad\le \varlimsup_{n\to\infty} \Pi_x\left\{\frac{Z_{y+e_2,X_n}}{Z_{y,X_n}} \le \frac{Z_{y+e_2,\lfloor n\zeta\rfloor}}{Z_{y,\lfloor n\zeta\rfloor}}, X_n \ge y\right\} \\
&\quad\le \varlimsup_{n\to\infty} \Pi_x\left\{\frac{Z_{y+e_2,X_n}}{Z_{y,X_n}} < e^{-B^{\underline{\zeta}-}(y,y+e_2,\widehat{\omega})} + \varepsilon, X_n \ge y\right\}.
\end{aligned}$$

If the limsup on the left is positive, then using (5.15) implies $e^{-B^{\Pi_x}(y,y+e_2)} \le e^{-B^{\underline{\zeta}-}(y,y+e_2,\widehat{\omega})} + \varepsilon$. The case of e_1 is similar. Taking $\varepsilon \to 0$, we get

$$\begin{aligned}
&B^{\Pi_x}(y, y+e_1) \le B^{\underline{\zeta}-}(y, y+e_1, \widehat{\omega}) \quad \text{and} \\
&B^{\Pi_x}(y, y+e_2) \ge B^{\underline{\zeta}-}(y, y+e_2, \widehat{\omega})
\end{aligned} \tag{5.21}$$

for each $y \in x + \mathbb{Z}_+^2$ and $\zeta \in \mathrm{ri}\,\mathcal{U}$ such that

$$\varlimsup_{n\to\infty} \Pi_x\{X_n \cdot e_1 > n\zeta \cdot e_1\} > 0. \tag{5.22}$$

Couple $\{\Pi_x, \Pi_x^{\zeta\pm,\widehat{\omega}} : \zeta \in \mathrm{ri}\,\mathcal{U}\}$ as described in Appendix A.1 and denote the coupled paths by $\overline{X}_{m,\infty}^{x,\omega}$ (distribution Π_x) and $X_{m,\infty}^{x,\zeta\pm,\widehat{\omega}}$ (distribution $\Pi_x^{\zeta\pm,\widehat{\omega}}$).

We have already seen that paths $X^{x,\zeta\pm,\widehat{\omega}}$ are monotone in ζ. Similarly, (5.21) implies that for $\zeta \in \mathrm{ri}\,\mathcal{U}$ satisfying (5.22), we have

$$\overline{X}_k^{x,\omega} \cdot e_1 \ge X_k^{x,\underline{\zeta}-,\widehat{\omega}} \cdot e_1 \quad \text{for all } k \in \mathbb{Z}_+.$$

Since the distribution of $X_k^{x,\underline{\zeta}-,\widehat{\omega}}$ is $\Pi_x^{\underline{\zeta}-,\widehat{\omega}}$ and is strongly directed into $\mathcal{U}_{\underline{\zeta}-}$ (because $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{exist}}$), we see that for $\zeta \in \mathrm{ri}\,\mathcal{U}$ satisfying (5.22)

$$\Pi_x\left\{\varliminf_{n\to\infty} n^{-1} X_n \cdot e_1 \ge \underline{\underline{\zeta}} \cdot e_1\right\} = 1. \tag{5.23}$$

Here, $\underline{\underline{\zeta}} \cdot e_1 = \inf\{\eta \cdot e_1 : \eta \in \mathcal{U}_{\underline{\zeta}-}\} = 1 - \underline{\underline{\zeta}} \cdot e_2$. Let $\xi' \in \mathcal{U}$ be such that

$$\xi' \cdot e_1 = \sup\{\zeta \cdot e_1 : \zeta \in \mathrm{ri}\,\mathcal{U} \text{ and (5.22) holds for } \zeta\}.$$

If the above set is empty, then we set $\xi' = e_2$. Let $\xi_1 = \underline{\xi}'$. If $\xi' = e_2$, then $\xi_1 = \underline{\xi}_1 = e_2$ as well and we trivially have

$$\Pi_x\Big\{\varliminf_{n\to\infty} n^{-1}X_n\cdot e_1 \ge \underline{\xi}_1\cdot e_1\Big\} = 1. \tag{5.24}$$

Assume $\xi' \neq e_2$ and take $\zeta \in \operatorname{ri}\mathcal{U}$ with $\zeta\cdot e_1 < \xi'\cdot e_1$. Observe that we can take $\underline{\underline{\zeta}}$ arbitrarily close to $\underline{\xi}_1$. Indeed, if $\xi_1\cdot e_1 < \xi'\cdot e_1$, then take $\xi_1\cdot e_1 < \zeta\cdot e_1 < \xi'\cdot e_1$ to get $\underline{\underline{\zeta}} = \underline{\xi}' = \xi_1$ and $\underline{\underline{\zeta}} = \underline{\xi}_1$. If instead $\xi_1 = \xi'$, then also $\underline{\xi}_1 = \underline{\xi}' = \xi_1$. Now, as $\zeta \to \xi_1$, $\nabla\Lambda(\zeta\pm)$ approach but never equal $\nabla\Lambda(\xi_1-)$ because there is no linear segment of Λ adjacent to ξ_1 on the left. This forces $\underline{\zeta}$ and $\underline{\underline{\zeta}}$ to converge to ξ_1.

Fix $\varepsilon > 0$ and take $\zeta \in \operatorname{ri}\mathcal{U}$ with $\zeta\cdot e_1 < \xi'\cdot e_1$ and $\underline{\underline{\zeta}}\cdot e_1 > \underline{\xi}_1\cdot e_1 - \varepsilon$. Then (5.22) holds and, therefore, (5.23) holds too and we have

$$\Pi_x\Big\{\varliminf_{n\to\infty} n^{-1}X_n\cdot e_1 > \underline{\xi}_1\cdot e_1 - \varepsilon\Big\} \ge \Pi_x\Big\{\varliminf_{n\to\infty} n^{-1}X_n\cdot e_1 \ge \underline{\underline{\zeta}}\cdot e_1\Big\} = 1.$$

Take $\varepsilon \to 0$ to get (5.24) when $\xi' \neq e_2$.

A symmetric argument (e.g., exchanging the roles of e_1 and e_2) gives

$$\Pi_x\Big\{\varlimsup_{n\to\infty} n^{-1}X_n\cdot e_1 \le \overline{\xi}_2\cdot e_1\Big\} = 1, \tag{5.25}$$

where $\xi_2 = \overline{\xi}''$ and $\xi'' \in \mathcal{U}$ is such that

$$\xi''\cdot e_1 = \inf\Big\{\zeta\cdot e_1 : \zeta \in \operatorname{ri}\mathcal{U} \text{ and } \varlimsup_{n\to\infty} \Pi_x\{X_n\cdot e_1 < n\zeta\cdot e_1\} > 0\Big\},$$

with $\xi'' = e_1$ if the set is empty.

Equations (5.24) and (5.25) imply that $\underline{\xi}_1\cdot e_1 \le \xi''\cdot e_1$ and $\xi'\cdot e_1 \le \overline{\xi}_2\cdot e_1$. For example, if $\zeta \in \operatorname{ri}\mathcal{U}$ is such that $\zeta\cdot e_1 > \overline{\xi}_2\cdot e_1$ then Fatou's lemma gives

$$\begin{aligned}1 = \Pi_x\Big\{\varlimsup_{n\to\infty} n^{-1}X_n\cdot e_1 \le \overline{\xi}_2\cdot e_1\Big\} &\le \Pi_x\Big\{\varlimsup_{n\to\infty} n^{-1}X_n\cdot e_1 < \zeta\cdot e_1\Big\}\\ \le E^{\Pi_x}\Big[\varliminf_{n\to\infty} \mathbb{1}\{X_n\cdot e_1 < n\zeta\cdot e_1\}\Big] &\le \varliminf_{n\to\infty} \Pi_x\{X_n\cdot e_1 < n\zeta\cdot e_1\},\end{aligned}$$

and then $\zeta\cdot e_1 \ge \xi'\cdot e_1$.

Also, $\xi'\cdot e_1 \ge \xi''\cdot e_1$. To see this, take $\zeta, \zeta' \in \operatorname{ri}\mathcal{U}$ with $\zeta\cdot e_1 < \zeta'\cdot e_1 < \xi''\cdot e_1$. Then $\Pi_x(X_n\cdot e_1 \ge n\zeta'\cdot e_1) \to 1$, and hence (5.22) holds and $\zeta\cdot e_1 \le \xi'\cdot e_1$. Take $\zeta \to \xi''$. We now consider three cases.

Case (a): If $\xi' = \xi_1$, then $\xi' = \xi_1 = \underline{\xi}_1$, forcing $\xi'' = \xi' = \xi_1$. Let $\xi = \xi'$. Weak $\{\xi\}$-directedness holds by the definitions of ξ' and ξ'', since they equal ξ. Note that $\overline{\xi} = \xi_2$ and $\mathcal{U}_{\overline{\xi}} = [\xi_1, \overline{\xi}_2] = [\underline{\xi}_1, \overline{\xi}_2]$. Then strong directedness into $\mathcal{U}_{\overline{\xi}}$ follows from (5.24) and (5.25). The case $\xi'' = \xi_2$ is similar.

Case (b): Assume $\xi' \neq \xi_1$ and $\xi'' \neq \xi_2$ but $\underline{\xi}_1\cdot e_1 \le \xi''\cdot e_1 \le \xi_1\cdot e_1$. Then set $\xi = \xi_1$. We have $\overline{\xi}'' = \xi$, and thus $\overline{\xi}_2 = \overline{\xi}$. We also have $\underline{\xi}_1 = \underline{\xi}$ and again strong directedness into $\mathcal{U}_\xi$ follows from (5.24) and (5.25). The case $\xi_2\cdot e_1 \le \xi'\cdot e_1 \le \overline{\xi}_2\cdot e_1$ is similar.

Case (c): In the remaining case, $\xi_1\cdot e_1 < \xi''\cdot e_1 \le \xi'\cdot e_1 < \xi_2\cdot e_1$ we have $[\xi_1, \xi_2] = \mathcal{U}_{\xi'} = \mathcal{U}_{\xi''}$. In this case, Λ is linear on $[\xi_1, \xi_2]$ and, therefore, $\xi', \xi'' \in \mathcal{D}$. Let $\xi = \xi'$. The definitions of ξ'' and ξ' give weak directedness into $[\xi'', \xi'] \subset [\xi_1, \xi_2] = \mathcal{U}_\xi$. Strong directedness into $\mathcal{U}_{\underline{\xi}} \cup \mathcal{U}_{\overline{\xi}} = [\underline{\underline{\xi}}, \overline{\overline{\xi}}] = [\underline{\xi}_1, \underline{\xi}_2]$ follows from (5.24) and (5.25).

To finish, note that in all three cases $\xi \in \operatorname{ri}\mathcal{U}$. Indeed, strong directedness into $\mathcal{U}_{e_1}$ would imply (5.22), and thus (5.21) hold for all $\zeta \in \operatorname{ri}\mathcal{U}$. Then Lemma 4.13 would imply $B^{\Pi_x}(y, y + e_2) = \infty$, contradicting nondegeneracy. Strong directedness into $\mathcal{U}_{e_2}$ is argued similarly. $\square$

For the rest of the section, we assume that (3.5) holds. Then, in Theorem 4.7, we can ask that $1 \in \mathcal{B}_0$ and take $\mathcal{H}_0^1$ to be $\{-\nabla\Lambda(\xi) : \xi \in \mathcal{D}_0\}$, where $\mathcal{D}_0$ is the countable dense subset of $\mathrm{ri}\,\mathcal{U}$ from the paragraph following (3.5). Theorem 4.14 then implies that for $\xi \in \mathcal{D}_0$ and $\widehat{\omega} \in \widehat{\Omega}_{\mathrm{coc}} \cap \widehat{\Omega}_{\mathrm{Bus}} \cap \widehat{\Omega}_{\mathrm{cont},\underline{\xi}} \cap \widehat{\Omega}_{\mathrm{cont},\overline{\xi}}$, $B^{\underline{\xi}-} = B^{\overline{\xi}+} = B^{\xi}$ is a function of $\{\omega_x(\widehat{\omega}) : x \in \mathbb{Z}^2\}$. This and (4.9) imply that the whole process $\{B^{h\pm} : h \in \mathcal{B}\}$ is measurable with respect to $\mathfrak{S} = \sigma(\omega_x : x \in \mathbb{Z}^2) \subset \mathcal{F}$. In other words, we do not need the extended space $\widehat{\Omega}$. For the rest of the section, we write ω instead of $\widehat{\omega}$ and more generally drop the hats from our notation.

Recall the definition of the countable random set $\mathcal{U}_x^{\omega} \subset \mathrm{ri}\,\mathcal{U}$ in (3.7). It is a straightforward exercise to check that all the events in the statements of Lemma 5.8 and Theorem 3.10 are measurable; see Lemma 5.11 in [40].

LEMMA 5.8. *Assume* (3.5). *Fix $x \in \mathbb{Z}^2$. Then for any $\eta, \zeta \in \mathcal{U}$, $\mathbb{P}([\eta,\zeta] \cap \mathcal{U}_0^{\omega} \neq \varnothing) \in \{0,1\}$ and $\mathbb{P}([\eta,\zeta] \cap \mathcal{U}_0^{\omega} \neq \varnothing) = 1$ if and only if*

$$\mathbb{P}\{\omega \in \Omega_{\mathrm{coc}} : \exists \xi \in [\eta,\zeta] \cap \mathrm{ri}\,\mathcal{U} : B^{\xi+}(0,e_1,\omega) \neq B^{\xi-}(0,e_1,\omega)\} > 0. \tag{5.26}$$

PROOF. Fix $\eta, \zeta \in \mathcal{U}$. The event

$$\begin{aligned}\mathcal{E} = \{\omega \in \Omega_{\mathrm{coc}} : \exists y \in \mathbb{Z}^2, \exists i \in \{1,2\}, \exists \xi \in [\eta,\zeta] \cap \mathrm{ri}\,\mathcal{U} :\\ B^{\xi+}(y, y+e_i, \omega) \neq B^{\xi-}(y, y+e_i, \omega)\}\end{aligned}$$

is shift-invariant and the ergodicity of the distribution of $\{\omega_x : x \in \mathbb{Z}^2\}$ induced by $\mathbb{P}$ implies that this event has probability either 0 or 1. It has probability 1 if and only if

$$\mathbb{P}\{\exists i \in \{1,2\}, \exists \xi \in [\eta,\zeta] \cap \mathrm{ri}\,\mathcal{U} : B^{\xi+}(0,e_i) \neq B^{\xi-}(0,e_i)\} > 0. \tag{5.27}$$

But recovery (4.6) implies that $B^{\xi+}(0,e_1) \neq B^{\xi-}(0,e_1)$ is equivalent to $B^{\xi+}(0,e_2) \neq B^{\xi-}(0,e_2)$. Therefore, (5.27) holds if and only if (5.26) holds.

If $\mathbb{P}(\mathcal{E}) = 0$, then $\mathbb{P}\{\omega : [\eta,\zeta] \cap \mathcal{U}_0^{\omega} \neq \varnothing\} = 0$, since the latter is a smaller event. On the other hand, if $\mathbb{P}(\mathcal{E}) = 1$ then (5.27) holds and ergodicity implies that with $\mathbb{P}$-probability one there is a positive density of sites y such that there exist $i \in \{1,2\}$ and $\xi \in [\eta,\zeta] \cap \mathrm{ri}\,\mathcal{U}$ with $B^{\xi+}(y, y+e_i) \neq B^{\xi-}(y, y+e_i)$. In particular, there exist such sites in $\mathbb{Z}_+^2$ and so $[\eta,\zeta] \cap \mathcal{U}_0^{\omega} \neq \varnothing$. $\square$

PROOF OF THEOREM 3.10. For $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \mathcal{D}$, let $\eta = \zeta = \xi$. Then (4.3) implies (5.26) holds. The first claim in part (a) follows from applying Lemma 5.8, since there are countably many directions of nondifferentiability. The second claim, about $\xi \in \mathcal{D}$, comes from the continuity in Remark 4.11.

When $\underline{\xi} \neq \overline{\xi}$, condition (3.5) implies that $[\underline{\xi}, \overline{\xi}] \subset \mathcal{D}$, and hence $\nabla\Lambda(\zeta\pm) = \nabla\Lambda(\xi)$ and $B^{\zeta-} = B^{\zeta+} = B^{\xi}$ for all $\zeta \in [\underline{\xi}, \overline{\xi}]$. Part (b) now follows from Lemma 5.8 with $\eta = \underline{\xi}$ and $\zeta = \overline{\xi}$. (There are countably many ξ with $\underline{\xi} \neq \overline{\xi}$.)

The first claim in part (c) is the same as the first claim in Lemma 5.8. Fix η and ζ as in the second claim. Define

$$A = \{\xi \in [\eta,\zeta[: \mathbb{P}(]\xi,\xi'] \cap \mathcal{U}_x^{\omega} \neq \varnothing) = 1\ \forall \xi' \in]\xi,\zeta]\} \subset [\eta,\zeta].$$

Note that any point in A is an almost sure (right) accumulation point of $\mathcal{U}_0^{\omega}$. Let $\xi_0 \in [\eta,\zeta]$ be such that

$$\xi_0 \cdot e_1 = \sup\{\xi' \cdot e_1 : \xi' \in [\eta,\zeta] \text{ and } \mathbb{P}([\eta,\xi'] \cap \mathcal{U}_x^{\omega} \neq \varnothing) = 0\}.$$

We have $\mathbb{P}([\eta,\xi'] \cap \mathcal{U}_x^{\omega} = \varnothing) = 1$ for all $\xi' \in [\eta,\xi_0[$. Taking $\xi' \to \xi_0$ implies the same claim for $[\eta,\xi_0[$. Since $\xi_0 \in \mathcal{D}$, part (a) implies the same holds for $[\eta,\xi_0]$; therefore $\xi_0 \neq \zeta$. The definition of ξ_0, the (already proven) first claim in (c), and $\mathbb{P}(\xi_0 \notin \mathcal{U}_x^\omega) = 1$ now imply that $\xi_0 \in A$ and so A is not empty.

For any $\xi \in A$ and $\xi' \in]\xi, \zeta[$, there exists $\xi'' \in]\xi, \xi']$ such that $\mathbb{P}([\xi'', \xi'] \cap \mathcal{U}_x^\omega \neq \varnothing) = 1$. Otherwise, taking $\xi'' \to \xi$ and using $\mathbb{P}(\xi \in \mathcal{U}_x^\omega) = 0$ we get a contradiction with $\xi \in A$. The previous paragraph shows that there exists $\xi''' \in]\xi'', \xi'[\cap A$. It follows that ξ is an accumulation point of A. (c) is proved.

Let Ω_{uniq} be the intersection of $\Omega_{\mathrm{coc}} \cap \Omega_{\mathrm{Bus}} \cap \Omega_{\mathrm{nondeg}} \cap \Omega_{\mathrm{exist}} \cap \Omega_{\mathrm{dir}}$ with the full-measure event from the already proven parts (a) and (b) and with $\Omega_{\mathrm{cont},\underline{\xi}} \cap \Omega_{\mathrm{cont},\overline{\xi}}$ for all of Λ's linear segments $[\underline{\xi}, \overline{\xi}]$, $\underline{\xi} \neq \overline{\xi}$ (if any). Take $\omega \in \Omega_{\mathrm{uniq}}$.

Since $\omega \in \Omega_{\mathrm{nondeg}}$, uniqueness of degenerate extreme solutions comes from Lemma 3.4. Assumption (3.5) implies that

$$\mathcal{U}_{\underline{\xi}} = \mathcal{U}_{\overline{\xi}} = \mathcal{U}_{\xi-} = \mathcal{U}_{\xi+} = \mathcal{U}_\xi \quad \text{for all } \xi \in \mathcal{U}. \tag{5.28}$$

Then strong directedness of nondegenerate extreme solutions follows from Theorem 3.5 (since $\omega \in \Omega_{\mathrm{dir}}$). This proves part (d).

Now fix $\xi \in \mathcal{U} \setminus \mathcal{U}_x^\omega$. Since $\omega \in \Omega_{\mathrm{exist}}$ and $\mathcal{U}_{\xi-} = \mathcal{U}_{\xi+}$, we already know from Theorem 5.4 that $\Pi_x^{\xi,\omega}$ is a strongly $\mathcal{U}_\xi$-directed DLR solution. Let Π_x be (possibly another) strongly $\mathcal{U}_\xi$-directed DLR solution. If $\underline{\xi} \neq \xi$, then assumption (3.5) implies Λ is linear on $[\underline{\xi}, \overline{\xi}] \subset \mathcal{D}$ and $\omega \in \Omega_{\mathrm{cont},\underline{\xi}}$ implies $B^{\underline{\xi}-} = B^{\underline{\xi}} = B^{\xi} = B^{\xi-}$. Either way, we have $B^{\underline{\xi}-} = B^{\xi-}$. Similarly, $B^{\overline{\xi}+} = B^{\xi+}$. By Theorem 4.14, we have Π_x-a.s., for all $y \in x + \mathbb{Z}_+^2$,

$$e^{-B^{\xi-}(y,y+e_1,\omega)} \le \varliminf_{n\to\infty} \frac{Z_{y+e_1,X_n}}{Z_{y,X_n}} \le \varlimsup_{n\to\infty} \frac{Z_{y+e_1,X_n}}{Z_{y,X_n}} \le e^{-B^{\xi+}(y,y+e_1,\omega)}, \tag{5.29}$$

with similar inequalities for e_2-increments. Consequently, if $\xi \notin \mathcal{U}_x^\omega$, then

$$\lim_{n\to\infty} \frac{Z_{y,X_n}}{Z_{x,X_n}} = e^{-B^\xi(x,y,\omega)}.$$

Due to (5.13) and (5.7), applying bounded convergence we deduce that $\Pi_x = \Pi_x^{\xi,\omega}$. The existence and uniqueness claimed in part (e) have been verified.

As explained above Lemma 2.4, one can write Π_x as a convex integral mixture of extreme measures from DLR_x^ω. This mixture will then have to be supported on DLR solutions that are all strongly $\mathcal{U}_\xi$-directed. Uniqueness then implies that they are all equal to Π_x and, therefore, Π_x is extreme.

The weak convergence claim comes similarly to (5.14). The argument for consistency is similar to the one below (5.14). Part (e) is proved.

When $\xi \in \mathcal{U}_x^\omega$, $\Pi_x^{\xi\pm,\omega}$ are two DLR solutions which, by Theorem 5.4 and (5.28), are both strongly $\mathcal{U}_\xi$-directed. The two are different because they are nondegenerate and so if $y \in x + \mathbb{Z}_+^2$ and $i \in \{1, 2\}$ are such that $B^{\xi-}(y, y+e_i, \omega) \neq B^{\xi+}(y, y+e_i, \omega)$, then passing through y has a positive probability under both $\Pi_x^{\xi\pm,\omega}$, and the transitions out of y are different.

Since $\Pi_x^{\xi\pm,\omega}$ are two different $\mathcal{U}_\xi$-directed solutions, there must exist at least two different extreme ones. Part (f) is proved and we are done. □

REMARK 5.9. We can in fact prove that in Theorem 3.10(f) $\Pi_x^{\xi\pm,\omega}$ are extreme. See Lemma 5.12 in [40].

PROOF OF THEOREM 3.12. Let $\mathcal{D}_0$ be a countable dense subset of $\mathcal{D}$ containing the endpoints of all linear segments of Λ (if any). We define a coupling of certain paths on the tree $\mathcal{T}_0^\omega$. Set $\Omega_{\mathrm{cif}} = \bigcap_{\zeta \in \mathcal{D}_0} \Omega_{[\underline{\zeta},\overline{\zeta}]}$ and take $\omega \in \Omega_{\mathrm{cif}}$. For $n \in \mathbb{N}$ and $\zeta \in \mathcal{D}_0$, let $\widehat{X}_{0,\infty}^{\zeta,\omega,(n)}$

be the up-right path on $\mathcal{T}_0^\omega$ that goes from 0 to $\lfloor n\zeta \rfloor$ and then continues by taking, say, e_1 steps. Let $\widehat{Q}_{0,(n)}^\omega$ be the joint distribution of $\mathcal{T}_0^\omega$ and $\{\widehat{X}_{0,\infty}^{\zeta,\omega,(n)} : \zeta \in \mathcal{D}_0\}$, induced by Q_0^ω. By compactness, the sequence $\widehat{Q}_{0,(n)}^\omega$ has a subsequence that converges weakly to a probability measure. Let $\widehat{Q}_0^\omega$ be a weak limit. This is a probability measure on trees spanning $\mathbb{Z}_+^2$ and infinite up-right paths on these trees, rooted at 0 and indexed by $\zeta \in \mathcal{D}_0$. We denote the tree by $\widehat{\mathcal{T}}_0^\omega$ and the paths by $\widehat{X}_{0,\infty}^{\zeta,\omega}$. The distribution of $\widehat{\mathcal{T}}_0^\omega$ under $\widehat{Q}_0^\omega$ is the same as that of $\mathcal{T}_0^\omega$ under Q_0^ω. Furthermore, since by Lemma 3.11 for each $n \in \mathbb{N}$ and $\xi \in \mathcal{D}_0$ the distribution of $\widehat{X}_{0,n}^{\zeta,\omega,(n)}$ under Q_0^ω is exactly $Q_{0,\lfloor n\zeta \rfloor}^\omega$, Theorem 3.7 implies that the distribution of $\widehat{X}_{0,\infty}^{\zeta,\omega}$ under $\widehat{Q}_0^\omega$ is exactly $\Pi_0^{\zeta,\omega}$. One consequence is that $\widehat{X}^{\zeta,\omega}$ is $\mathcal{U}_\zeta$-directed, $\widehat{Q}_0^\omega$-almost surely and for all $\zeta \in \mathcal{D}_0$.

We can define a competition interface $\widehat{\phi}_n^\omega$ between the subtrees of $\widehat{\mathcal{T}}_0^\omega$ rooted at e_1 and e_2, and its distribution under $\widehat{Q}_0^\omega$ is then the same as the distribution of the original competition interface ϕ_n^ω under Q_0^ω. Since $\widehat{X}^{\zeta,\omega}$ is a path on the spanning tree $\widehat{\mathcal{T}}_0^\omega$, $\{\widehat{X}_1^{\zeta,\omega} = e_2\}$ implies that $\widehat{\phi}_n^\omega \cdot e_1 \geq \widehat{X}_n^{\zeta,\omega} \cdot e_1$ for all $n \in \mathbb{Z}_+$. This in turn implies the event $\{\underline{\lim}\, \widehat{\phi}_n^\omega \cdot e_1/n \geq \underline{\zeta} \cdot e_1\}$. Consequently, for all $\zeta \in \mathcal{D}_0$,

$$Q_0^\omega\Big\{\varliminf_{n\to\infty} \phi_n^\omega \cdot e_1/n < \underline{\zeta} \cdot e_1\Big\} \leq \Pi_0^{\zeta,\omega}(X_1 = e_1) = e^{\omega_0 - B^\zeta(0,e_1,\omega)}.$$

A similar argument gives

$$Q_0^\omega\Big\{\varlimsup_{n\to\infty} \phi_n^\omega \cdot e_1/n \leq \overline{\zeta} \cdot e_1\Big\} \geq e^{\omega_0 - B^\zeta(0,e_1,\omega)}. \tag{5.30}$$

For $\xi \in \operatorname{ri}\mathcal{U}$ with $\overline{\xi} \in \mathcal{D}_0$ taking $\mathcal{D}_0 \ni \zeta \to \overline{\xi}$ with $\zeta \cdot e_1$ strictly decreasing makes $\underline{\zeta} \to \overline{\xi}$. Recall that $\omega \in \Omega_{\text{cont},\overline{\xi}}$. Applying the above, we get

$$Q_0^\omega\Big\{\varliminf_{n\to\infty} \phi_n^\omega \cdot e_1/n \leq \overline{\xi} \cdot e_1\Big\} \leq e^{\omega_0 - B^{\overline{\xi}}(0,e_1,\omega)}.$$

Applying (5.30) with $\zeta = \overline{\xi}$, we get

$$Q_0^\omega\Big\{\varlimsup_{n\to\infty} \phi_n^\omega \cdot e_1/n \leq \overline{\xi} \cdot e_1\Big\} \geq e^{\omega_0 - B^{\overline{\xi}}(0,e_1,\omega)}.$$

Since the liminf is always bounded above by the limsup, we get

$$Q_0^\omega\Big\{\varliminf_{n\to\infty} \phi_n^\omega \cdot e_1/n \leq \overline{\xi} \cdot e_1\Big\} = Q_0^\omega\Big\{\varlimsup_{n\to\infty} \phi_n^\omega \cdot e_1/n \leq \overline{\xi} \cdot e_1\Big\} = e^{\omega_0 - B^{\overline{\xi}}(0,e_1,\omega)}.$$

A similar argument, starting by taking $\zeta \to \underline{\xi}$ and applying (5.30), gives

$$Q_0^\omega\Big\{\varliminf_{n\to\infty} \phi_n^\omega \cdot e_1/n < \underline{\xi} \cdot e_1\Big\} = Q_0^\omega\Big\{\varlimsup_{n\to\infty} \phi_n^\omega \cdot e_1/n < \underline{\xi} \cdot e_1\Big\} = e^{\omega_0 - B^{\underline{\xi}}(0,e_1,\omega)}$$

for all $\xi \in \operatorname{ri}\mathcal{U}$ with $\underline{\xi} \in \mathcal{D}_0$. But for $\xi \in \mathcal{D}_0$ we have $B^\xi(\omega) = B^{\underline{\xi}}(\omega) = B^{\overline{\xi}}(\omega)$. Hence, all four probabilities in the above two displays equal $e^{\omega_0 - B^\xi(0,e_1,\omega)}$. We conclude that for any $\xi \in \mathcal{D}_0$,

$$Q_0^\omega\Big\{\varliminf_{n\to\infty} \phi_n^\omega \cdot e_1/n \leq \xi \cdot e_1\Big\} = Q_0^\omega\Big\{\varlimsup_{n\to\infty} \phi_n^\omega \cdot e_1/n \leq \xi \cdot e_1\Big\} = e^{\omega_0 - B^\xi(0,e_1,\omega)}.$$

This implies that $\xi_* = \lim_{n\to\infty} \phi_n^\omega \cdot e_1/n$ exists Q_0^ω-almost surely and its cumulative distribution function is given by (3.8). Parts (a) and (b) are proved. Part (c) follows because $B^{\xi+}$ is constant on the linear segments of Λ. For (d), observe that

$$\mathbb{E} Q_0^\omega\{\xi_* = \xi\} = \mathbb{E}\big[e^{\omega_0}\big(e^{-B^{\xi+}(0,e_1,\omega)} - e^{-B^{\xi-}(0,e_1,\omega)}\big)\big],$$

which vanishes if and only if $\mathbb{P}\{B^{\xi+}(0,e_1) = B^{\xi-}(0,e_2)\} = 1$, which holds if and only if $\xi \in \mathcal{D}$. $\square$

6. Bi-infinite polymer measures. We now prove Theorem 3.13 and Lemma 3.15, showing nonexistence of two classes of bi-infinite polymer measures. The following is the key step in the proof of Theorem 3.13.

LEMMA 6.1. *Let B be a shift-covariant cocycle which recovers the potential. Then there is a Borel set $\Omega_{B,\downarrow 0} \subset \Omega$ with $\mathbb{P}(\Omega_{B,\downarrow 0}) = 1$ so that for all $\omega \in \Omega_{B,\downarrow 0}$ and for all $x \in \mathbb{Z}^2$,*

$$\lim_{n\to\infty} \max_{\substack{y\le x\\ |x-y|_1=n}} \Pi_y^{B(\omega)}(x) = 0.$$

PROOF. By shift-covariance of B, it is enough to deal with the case $x = 0$. Couple $\{\Pi_y^{B(\omega)} : y \in \mathbb{Z}^2\}$ as described in Appendix A.1 and denote the coupled paths by $X_{m,\infty}^{y,\omega}$, or X^y for short, $y \in \mathbb{V}_m$, $m \in \mathbb{Z}$. Let $N_v = \{y \le v : v \in X^y\}$. We will call a point $z \in \mathbb{Z}^2$ a *junction point* if there exist distinct $u, v \in \mathbb{Z}^2$ such that $|N_u| = |N_v| = \infty$ and X^u and X^v coalesce precisely at z.

Suppose now $\mathbf{P} \otimes \mathbb{P}(|N_0| = \infty) > 0$. The shift-covariance of B implies $N_u(\tau_v\vartheta, T_v\omega) = N_{u+v}(\vartheta, \omega)$. Hence, by the ergodic theorem, with positive $\mathbf{P}\otimes\mathbb{P}$-probability there is a positive density of sites $v \in \mathbb{Z}^2$ with $|N_v| = \infty$.

By Theorem A.2, for $\mathbf{P} \otimes \mathbb{P}$-almost every (ϑ, ω) and all $u, v \in \mathbb{Z}^2$, X^u and X^v coalesce. It follows from this and the previous paragraph that with positive $\mathbf{P} \otimes \mathbb{P}$-probability there is a positive density of junction points.

For $L \in \mathbb{N}$, let J_L denote the union of the junction points in $[1, L]^2$ together with the vertices of the south-west boundary of $[1, L]^2$, $\{ke_i : 1 \le k \le L, i \in \{1, 2\}\}$, with the property that one of the junction points lies on X^{ke_i}. For each junction point z, there are at least two such points on the southwest boundary. Decompose J_L into finite binary trees by declaring that the two immediate descendants of a junction point z are the two closest points $u, v \in J_L$ with the property that $z \in X^u \cap X^v$. The leaves of these trees are points in J_L which lie on the boundary and the junction points are the interior points of the trees. This tree cannot have more than $2L + 1$ leaves, but this contradicts that there are on the order of L^2 junction points, since a binary tree has more leaves than interior points. Thus $\mathbf{P} \otimes \mathbb{P}(N_0 < \infty) = 1$.

Fix $\varepsilon > 0$. We now know that $\mathbf{P}(|N_0(\vartheta, \omega)| < \infty) = 1$ for $\mathbb{P}$-almost all ω. Then there exists an integer $n_0 = n_0(\omega)$ such that $\mathbf{P}(|N_0(\vartheta, \omega)| \ge n) < \varepsilon$ for $n \ge n_0$. The claim follows from the observation that $\Pi_y^B(0) = \mathbf{P}(0 \in X^y) \le \mathbf{P}(|N_0(\vartheta, \omega)| \ge n)$ for $y \le 0$ with $|y|_1 = n$. $\square$

We can now rule out the existence of polymer Gibbs measures satisfying the law of large numbers in a fixed direction and of metastates.

PROOF OF THEOREM 3.13. Let $\Omega_{\text{bi},[\underline{\xi},\overline{\xi}]} = \Omega_{B^\xi,\downarrow 0} \cap \Omega_{[\underline{\xi},\overline{\xi}]}$ and take $\omega \in \Omega_{\text{bi},[\underline{\xi},\overline{\xi}]}$. Suppose there exists a weakly $\mathcal{U}_\xi$-directed $\Pi \in \overleftrightarrow{\text{DLR}}^\omega$. Take any $x \in \mathbb{Z}^2$ such that $c = \Pi(x) > 0$. Fix $n \le x \cdot \widehat{e}$. If $\Pi(x \mid y) \le c/2$ for all $y \in \mathbb{V}_n$ with $\Pi(y) > 0$, then $\Pi(y, x) \le c\Pi(y)/2$ for all $y \in \mathbb{V}_n$ and adding over y we get $c = \Pi(x) \le c/2$, which contradicts $c > 0$. Hence, there exists a $y_n \le x$ such that $y_n \in \mathbb{V}_n$, $\Pi(y_n) > 0$, and $\Pi(x \mid y_n) > c/2$. But, by Lemma 2.4, $\Pi(\cdot \mid y_n)$ is a weakly $\mathcal{U}_\xi$-directed element of $\text{DLR}_{y_n}^\omega$ and, by Theorem 3.7, it must be that $\Pi(x \mid y_n) = \Pi_{y_n}^{\xi,\omega}(x)$. But then $\Pi_{y_n}^{\xi,\omega}(x) > c/2$ for all n, which contradicts Lemma 6.1 since Theorem 5.5 says $\Pi_{y_n}^{\xi,\omega} = \Pi_{y_n}^{B^\xi(\omega)}$. $\square$

PROOF OF LEMMA 3.15. Suppose that Π is a measure satisfying Definition 3.14(a) and Definition 3.14(c). Then for each $z \in \mathbb{V}_0$

$$\mathbb{E}[\Pi^\omega(X_0 = z)] = \mathbb{E}[\Pi^{T_z\omega}(X_0 = 0)] = \mathbb{E}[\Pi^\omega(X_0 = 0)].$$

This is a contradiction since $\{X_0 = z\}$, $z \in \mathbb{V}_0$, form a partition of $\mathbb{X}$. $\square$

APPENDIX A: COUPLED RWRE PATHS WITH $\{e_1, e_2\}$ STEPS

A.1. Path coupling. In this section, we construct a coupling of a family of random walks in a random environment (RWRE) with admissible steps $\{e_1, e_2\}$ that several arguments in this paper rely on.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ satisfy the assumptions of Section 2. Let $\mathbf{P}$ denote the law of i.i.d. Uniform[0, 1] random variables $\vartheta = \{\vartheta(y) : y \in \mathbb{Z}^2\}$ on $[0,1]^{\mathbb{Z}^2}$, equipped with the Borel σ-algebra and the natural group of coordinate shifts τ_x. Define a family of shifts on the product space $[0,1]^{\mathbb{Z}^2} \times \Omega$ indexed by $x \in \mathbb{Z}^2$ in the natural way, via $\widetilde{T}_x(\nu, \omega) = (\tau_x \nu, T_x \omega)$. This shift preserves $\mathbf{P} \otimes \mathbb{P}$.

Let $\mathcal{A}$ be some index set and let $\{p_x^\alpha : x \in \mathbb{Z}^2, \alpha \in \mathcal{A}\}$ be a collection of $[0,1]$-valued $\mathcal{F}$-measurable random variables. Abbreviate $\mathfrak{G} = \{e_1, e_2\}^{\mathbb{Z}^2}$. For $\alpha \in \mathcal{A}$, construct a random graph $\mathfrak{g}^\alpha(\vartheta, \omega) = \mathfrak{g}^\alpha \in \mathfrak{G}$, via

$$\mathfrak{g}_x^\alpha = \begin{cases} e_1 & \text{if } \vartheta(x) < p_x^\alpha(\omega), \\ e_2 & \text{if } \vartheta(x) \geq p_x^\alpha(\omega). \end{cases}$$

For each $x \in \mathbb{V}_m$, $m \in \mathbb{Z}$, let $X_{m,\infty}^{x,\alpha,\omega} = X_{m,\infty}^{x,\alpha,\omega}(\vartheta)$ denote the random path defined via $X_m^{x,\alpha,\omega} = x$ and $X_k^{x,\alpha,\omega} = X_{k-1}^{x,\alpha,\omega} + \mathfrak{g}^\alpha_{X_{k-1}^{x,\alpha,\omega}}(\vartheta, \omega)$ for $k > m$. We observe that under $\mathbf{P}$, for fixed α, $X_{m,\infty}^{x,\alpha}$ has the law of a quenched RWRE with admissible steps $\{e_1, e_2\}$ started from x and taking the step e_1 at site y with probability $p_y^\alpha(\omega)$. Two properties follow immediately.

COROLLARY A.1. *The following hold for any $\omega \in \Omega$ and $\vartheta \in [0,1]^{\mathbb{Z}^2}$:*

(a) (*Coalescence*) *If for some $\alpha \in \mathcal{A}$, $x, y \in \mathbb{Z}^2$, and $n \geq \max(x \cdot \widehat{e}, y \cdot \widehat{e})$ we have $X_n^{x,\alpha,\omega}(\vartheta) = X_n^{y,\alpha,\omega}(\vartheta)$, then $X_{n,\infty}^{x,\alpha,\omega}(\vartheta) = X_{n,\infty}^{y,\alpha,\omega}(\vartheta)$.*

(b) (*Monotonicity*) *Fix $x \in \mathbb{V}_m$, $m \in \mathbb{Z}$, and $\alpha_1, \alpha_2 \in \mathcal{A}$. If $p_y^{\alpha_1}(\omega) \leq p_y^{\alpha_2}(\omega)$ for all $y \geq x$ then $X_n^{x,\alpha_1,\omega}(\vartheta) \cdot e_1 \leq X_n^{x,\alpha_2,\omega}(\vartheta) \cdot e_1$ for all $n \geq m$.*

The proof of Lemma 4.10 is an example of how we use this coupling.

PROOF OF LEMMA 4.10. It suffices to work with a fixed $\beta \in (0, \infty)$. The case $\beta = \infty$ comes by taking a limit. Fix $n \in \mathbb{Z}$ and construct the coupled paths $X_{m,\infty}^{x,\beta,h,\omega}(\vartheta)$, $x \in \mathbb{V}_m$, $m \in \mathbb{Z}$, as above, with

$$p_x(\omega) = \begin{cases} e^{\beta\omega_{x+e_1} + \beta h \cdot e_1} \dfrac{Z_{x+e_1,(n)}^{\beta,h}}{Z_{x,(n)}^{\beta,h}} & \text{if } |x|_1 < n, x \geq 0, \\ 1/2 & \text{otherwise.} \end{cases}$$

Note that for $x \in \mathbb{V}_m$, $m + 1 < n$, and $i, j \in \{1, 2\}$,

$$\partial_{h_i} F_{x,(n)}^{\beta,h} = E_{x,(n)}^{\omega,\beta,h}[e_i \cdot (X_n - x)] \quad \text{and}$$
$$\partial_{h_i} F_{x+e_j,(n)}^{\beta,h} = E_{x+e_j,(n)}^{\omega,\beta,h}[e_i \cdot (X_n - x - e_j)].$$

It follows that whenever $x \in \mathbb{V}_m$, $m < n$ and $i, j \in \{1, 2\}$,

$$\begin{aligned} \partial_{h_i} B_n^{\beta,h}(x, x + e_j) &= E_{x,(n)}^{\omega,\beta,h}[e_i \cdot X_n] - E_{x+e_j,(n)}^{\omega,\beta,h}[e_i \cdot (X_n - e_j)] - e_i \cdot e_j \\ &= \mathbf{E}\big[e_i \cdot \big(X_n^{x,\beta,h,\omega} - X_n^{x+e_j,\beta,h,\omega}\big)\big]. \end{aligned}$$

Then Corollary A.1(a) and planarity imply that

$$\partial_{h_i} B_n^{\beta,h}(x, x + e_i) \leq 0 \quad \text{and} \quad \partial_{h_{3-i}} B_n^{\beta,h}(x, x + e_i) \geq 0. \qquad \square$$

A.2. Coalescence of RWRE paths. We record the following result showing that quenched measures of a general $1+1$-dimensional random walk with $\{e_1, e_2\}$ steps in a stationary weakly elliptic random environment can be coupled so that the paths coalesce. The proof is an easier version of the well-known Licea–Newman [45] argument for coalescence of first-passage percolation geodesics. Notably, the measurability issues which make the Licea–Newman argument somewhat involved in zero temperature vanish in positive temperature due to the extra layer of randomness coming from the coupling. Since the proof is an easier version of a standard argument, we omit the details. The full proof is available in the preprint [40].

THEOREM A.2. *Let $p : \Omega \to [0, 1]$ be $\mathcal{F}$-measurable. Assume that $\mathbb{P}(0 < p < 1) = 1$ and construct random variables $\{X^{x,\omega}_{m,\infty} : x \in \mathbb{V}_m, m \in \mathbb{Z}\}$ via the coupling in Appendix A.1 with $p_x(\omega) = p(T_x\omega)$. Then for $\mathbf{P} \otimes \mathbb{P}$-almost every (ϑ, ω) and any $u, v \in \mathbb{Z}^2$ there exists an $n \in \mathbb{Z}$ with $X^{u,\omega}_{n,\infty}(\vartheta) = X^{v,\omega}_{n,\infty}(\vartheta)$.*

APPENDIX B: AUXILIARY LEMMAS

The following lemma gives an analogue of J. B. Martin's result [48], Theorem 2.4, on the boundary behavior of the shape function for LPP, in the positive temperature setting. It follows immediately from that result by bounding Λ^β using Λ^∞ and counting paths. The proof is included in [40].

LEMMA B.1. *For each $\beta > 0$, as $s \searrow 0$*

$$\Lambda^\beta(se_1 + e_2) = \Lambda^\beta(e_1 + se_2) = \mathbb{E}[\omega_0] + 2\sqrt{s\,\mathbb{V}\mathrm{ar}(\omega_0)} + o(\sqrt{s}).$$

Next is a lemma that allows us to compare ratios of partition functions.

LEMMA B.2. *For any $\omega \in \Omega$, $x \in \mathbb{Z}^2$, and $u, v \in x + e_1 + e_2 + \mathbb{Z}^2_+$ with $u \cdot e_1 \ge v \cdot e_1$ and $u \cdot e_2 \le v \cdot e_2$*

$$\frac{Z^\beta_{x+e_1,u}}{Z^\beta_{x,u}} \ge \frac{Z^\beta_{x+e_1,v}}{Z^\beta_{x,v}} \quad \text{and} \quad \frac{Z^\beta_{x+e_2,u}}{Z^\beta_{x,u}} \le \frac{Z^\beta_{x+e_2,v}}{Z^\beta_{x,v}}. \tag{B.1}$$

PROOF. Reversing the picture in [31], Lemma A.1, via $x \mapsto -x$ gives

$$\frac{Z^\beta_{x+e_1,y}}{Z^\beta_{x,y}} \ge \frac{Z^\beta_{x+e_1,y-e_1}}{Z^\beta_{x,y-e_1}} \ge \frac{Z^\beta_{x+e_1,y-e_1+e_2}}{Z^\beta_{x,y-e_1+e_2}}$$

for all $x, y \in \mathbb{Z}^2$ with $y \ge x$ and any choice of $\omega \in \Omega$. The first equality in (B.1) comes by applying this repeatedly with y on any up-left path from u to v. The second equality is similar. $\square$

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