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A SIMPLE PROOF OF THE DPRZ THEOREM FOR 2D COVER TIMES

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We give a simple proof of the theorem by Dembo, Peres, Rosen and Zeitouni (DPRZ) regarding the time Brownian motion needs to cover every ε ball on the two-dimensional unit torus in the $\varepsilon \searrow 0$ limit.

The *ε-cover time* of the two-dimensional unit torus $\mathbb{T}_2$ by Brownian motion (BM) is the time for the process to come within distance $\varepsilon > 0$ from any point. Denoting by $T_\varepsilon(x)$ the first time BM hits the ε-ball centered in $x \in \mathbb{T}_2$, the ε-cover time is thus given by

$$T_\varepsilon \equiv \sup_{x \in \mathbb{T}_2} T_\varepsilon(x). \tag{1}$$

The purpose of these short notes is to provide a concise proof of a celebrated theorem by Dembo, Peres, Rosen and Zeitouni, DPRZ for short, which settles the leading order in the small-ε regime:

THEOREM 1 (The DPRZ theorem, [3]). *Almost surely*,

$$\lim_{\varepsilon \downarrow 0} \frac{T_\varepsilon}{(\ln \varepsilon)^2} = \frac{2}{\pi}. \tag{2}$$

A key idea in the DPRZ approach is to relate hitting times of ε-balls on $\mathbb{T}_2$ to excursion counts between circles of mesoscopic sizes around these balls [6]; the DPRZ proof of the theorem goes then through an involved multiscale analysis in the form of a second moment computation with truncation. We take here a similar point of view but with a number of twists which altogether lead to a considerable streamlining of the arguments. In particular, we implement the multiscale refinement of the second moment method emerged in the recent studies of Derrida's GREM and branching Brownian motion [5]. This tool brings to the fore the *true* process of covering [1] with the help of minimal infrastructure only; it also efficiently replaces the delicate tracking of points which DPRZ refer to as "n-successful," and requires the use of finitely many scales only. All of these features simplify substantially the proof of the DPRZ theorem.

We believe the route taken here would also streamline the deep DPRZ results on late and thin/thick points of BM [2], and what is perhaps more, that it might be useful in the study of the *finer* properties. In fact, our approach carries over, *mutatis mutandis*, to these issues as well: when backed with [1], the present notes suggest that in order to address lower order corrections, one "simply" needs to increase the number of scales.

These notes are self-contained. Although, as mentioned, some key insights are taken from [3]; no knowledge of the latter is assumed and detailed proofs to all statements are given.

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1. The (new) road to the DPRZ theorem. We identify the unit torus $\mathbb{T}_2$ with $[0,1) \times [0,1) \subset \mathbb{R}^2$, endowed with the metric

$$d_{\mathbb{T}_2}(x,y) = \min\{\|x - y + (e_1, e_2)\| : e_1, e_2 \in \{-1, 0, 1\}\}.$$

We construct BM on $\mathbb{T}_2$ by $W_t \equiv (\hat{W}_1(t) \mod 1, \hat{W}_2(t) \mod 1)$, where $\hat{W}$ is standard BM on $\mathbb{R}^2$.

By the Borel–Cantelli lemma and monotonicity of T_ε, the DPRZ theorem follows from the following.

THEOREM 2. *For $\delta > 0$ small enough, there exist constants $c(\delta), c'(\delta) > 0$ such that the following bounds hold for any $0 < \varepsilon < c'(\delta)$:*

(1) (*upper bound*)

$$\mathbb{P}\left(T_\varepsilon > (1+\delta)\frac{2}{\pi}(\ln \varepsilon)^2\right) \le \varepsilon^{c(\delta)}, \tag{3}$$

(2) (*lower bound*)

$$\mathbb{P}\left(T_\varepsilon < (1-\delta)\frac{2}{\pi}(\ln \varepsilon)^2\right) \le \varepsilon^{c(\delta)}. \tag{4}$$

Theorem 2 will be proved by relating the natural timescale of the covering process to the excursion counts of an embedded random walk. We will then perform a multiscale analysis of the latter which exploits some underlying, approximate hierarchical structure in the spirit of [1]. More precisely, we will argue in Section 1.1 that it suffices to (1) count excursions between concentric circles and (2) to consider only finitely many centers and radii. In the process, we will however heavily rely on the picture/tools which are somewhat classical in the analysis of hierarchical models [5], Chapter 2.2.1 and 3. For the readers convenience, here is a "dictionary "allowing to translate the picture of hierarchical models to the cover time setting": the number of excursions in the latter plays the role of what is called "energy" in the former; the circle centers then correspond to the leaves (or configurations/particles). Excursion counts between larger circles are thus "closer to the root," whereas those between smaller circles are "further away from the root." It turns out that the first moment computed in Section 1.2 is a sharp upper bound (to leading order). To prove this, we identify in Section 1.3 the strategy employed by a circle center to have its ε-ball avoided, that is, we compute the expected excursion counts conditioned on avoidance. We then proceed to check that there exists (at least) a circle center following this strategy by showing that the number of these has growing mean and is concentrated around it: this will be done via the multiscale refinement of the second moment method from [5]. In general, we refer the reader to these lecture notes for an introduction to this approach to hierarchical fields in an exactly hierarchical setting.

1.1. *Scales, embedded random walks and excursion counts.* For $R \in (0, \frac{1}{2})$ and $K \ge 1$, we consider *scales* $i = 0, 1, .., K$ and associate to each such scale a radius

$$r_i \equiv R\left(\frac{\varepsilon}{R}\right)^{i/K}. \tag{5}$$

BM started on ∂B_{r_i} hits $\partial B_{r_{i+1}}$ before $\partial B_{r_{i-1}}$ with probability 1/2: by the strong Markovianity and rotational invariance, it follows that the process obtained by tracking the order in which BM visits the scales (with respect to one fixed center point and not counting multiple consecutive hits to the same scale) during one excursion from scale 1 to scale 0 is a simple random walk (SRW) started at 1 and stopped in 0. Keeping track of all BM excursions up to

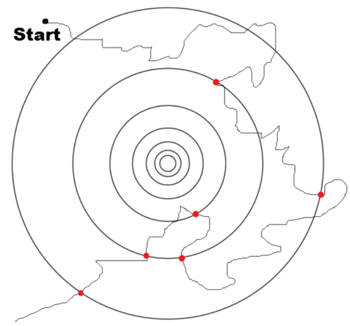

FIG. 1. *Reading off the SRW excursions* $1 \to 0$ *and* $1 \to 2 \to 1 \to 0$.

some time thus yields a collection of independent SRW excursions from 1 to 0. A formally precise statement of this observation is provided by Lemma 5 in Section 2.3. (The evolution of the SRW excursions can be unambiguously read off the BM path; see Figure 1.) For $x \in \mathbb{T}_2$, we set

$$D_n(x) \equiv \text{time at which } W \text{ completes the } n\text{th excursion from } \partial B_{r_1}(x) \text{ to } B_{r_0}^c(x). \tag{6}$$

PROPOSITION 1 (Concentration of excursion counts). *For $\delta, R \in (0, \frac{1}{2})$ and $x \in \mathbb{T}_2$, it holds*

$$\mathbb{P}\left(D_n(x) \geq (1+\delta)n\frac{1}{\pi}\ln\frac{r_0}{r_1}\right) \leq \exp\left(-n\left(\frac{\delta^2}{8} + o_{r_1}(1)\right)\right), \tag{7}$$

$$\mathbb{P}\left(D_n(x) \leq (1-\delta)n\frac{1}{\pi}\ln\frac{r_0}{r_1}\right) \leq \exp\left(-n\left(\frac{\delta^2}{4} + o_{r_1}(1)\right)\right) \tag{8}$$

for all $n \in \mathbb{N}$ as $r_1 \to 0$.

Proposition 1 will bear fruit when combined with the following.

PROPOSITION 2 (First moment of hitting times). *There exists an universal constant $C > 0$, such that*

$$\left|\mathbb{E}_y[\tau_{B_r(x)}] - \frac{1}{\pi}\ln\frac{d_{\mathbb{T}_2}(x,y)}{r}\right| \leq C \tag{9}$$

for all $x \in \mathbb{T}_2$, $r > 0$ and $y \in \mathbb{T}_2 \setminus B_r(x)$. Also,

$$\mathbb{E}_y[\tau_{B_r^c(x)}] = \frac{r^2 - d_{\mathbb{T}_2}(x,y)^2}{2} \tag{10}$$

for all $x \in \mathbb{T}_2$, $r \in (0, \frac{1}{2})$ and $y \in B_r(x)$.

Propositions 1 and 2 make precise the intuition that $D_n(x) \approx n\mathbb{E}_{B_{r_0}}[\tau_{B_{r_1}}]$, allowing in particular to switch from the natural timescale to the excursion counts. Armed with the above results, which will be proved in Section 2.1, we discuss the main steps behind Theorem 2. The upper bound is easy: we address that first.

Here and below, L_ε will denote the square grid of mesh size $\lceil \varepsilon^{-1} \rceil^{-1}$.

1.2. *The upper bound.* We will show that, with overwhelming probability, at time

$$t_\varepsilon(\delta) \equiv (1+\delta)\frac{2}{\pi}(\ln\varepsilon)^2, \tag{11}$$

each ε-ball with center on L_ε has been hit by BM and extend this to the entire torus thereafter.

LEMMA 1. *For $\delta > 0$ small enough, there exist constants $c, c' > 0$ depending on δ only such that*

$$\mathbb{P}\big(\exists x \in L_\varepsilon \text{ such that } T_\varepsilon(x) > t_\varepsilon(\delta)\big) \le \varepsilon^c \tag{12}$$

holds for all $0 < \varepsilon < c'$.

PROOF. We set

$$n_\varepsilon(\delta) = -(1+\delta/2)2K\ln(\varepsilon), \tag{13}$$

which is slightly larger than the typical amount of excursions up to time $t_\varepsilon(\delta)$ from scale 1 to scale 0. For an ε-ball to be avoided up to some time: either *(i)* BM needs to complete less than $n_\varepsilon(\delta)$ excursions from scale 1 to scale 0 in that time or *(ii)* scale K, corresponding to the ε-ball, has to be avoided for at least $n_\varepsilon(\delta)$ many excursions. Therefore, setting

$$\mathcal{T}(x) \equiv \textit{number of the first excursions from } \partial B_{r_1}(x)$$
$$\textit{to } B^c_{r_0}(x) \textit{ that hits } B_{r_K}(x)$$

we have

$$\begin{aligned} &\mathbb{P}\big(\exists x \in L_\varepsilon \text{ s.t. } T_\varepsilon(x) > t_\varepsilon(\delta)\big) \\ &\quad \le \mathbb{P}\big(\exists x \in L_\varepsilon \text{ s.t. } \mathcal{T}(x) > n_\varepsilon(\delta) \text{ or } D_{n_\varepsilon(\delta)}(x) \ge t_\varepsilon(\delta)\big). \end{aligned} \tag{14}$$

By Markov's inequality and union bound,

$$(14) \le \sum_{x\in L_\varepsilon} \mathbb{P}\big(\mathcal{T}(x) > n_\varepsilon(\delta)\big) + \mathbb{P}\big(D_{n_\varepsilon(\delta)}(x) \ge t_\varepsilon(\delta)\big). \tag{15}$$

The probability that $n_\varepsilon(\delta)$ independent excursions of a SRW starting in 1 all hit 0 before K is given by $(1-1/K)^{n_\varepsilon(\delta)}$, while the second probability on the r.h.s of (15) is estimated by Proposition 1. This shows that the above is at most

$$|L_\varepsilon|\left[\left(1-\frac{1}{K}\right)^{n_\varepsilon(\delta)} + \exp\left(-\frac{\delta^2}{72}n_\varepsilon(\delta)\right)\right] \le \varepsilon^\delta\big(1+o_\varepsilon(1)\big) \tag{16}$$

for $K > \frac{144+72\delta}{2\delta^2+\delta^3}$. The the last inequality by estimating $1-\frac{1}{K} \le e^{-1/K}$, and $|L_\varepsilon| \le \varepsilon^{-2}$. The given minimum size of K is then the result of a simple comparison of the exponents of the two summands. □

Coming back to the upper bound in Theorem 2,

$$\begin{aligned} \mathbb{P}\big(T_\varepsilon > t_\varepsilon(\delta)\big) &= \mathbb{P}\big(\exists x \in \mathbb{T}_2 : T_\varepsilon(x) > t_\varepsilon(\delta)\big) \\ &\le \mathbb{P}\big(\exists x \in L_{\varepsilon/10} : T_{\varepsilon/10}(x) > t_\varepsilon(\delta)\big), \end{aligned} \tag{17}$$

the last step using that any ε-ball contains a ball of radius $\varepsilon/10$ with center in $L_{\varepsilon/10}$. For $\varepsilon > 0$ small enough depending on δ, we have $t_\varepsilon(\delta) \ge t_{\varepsilon/10}(\delta/2)$, therefore, it holds that

$$(17) \le \mathbb{P}\big(\exists x \in L_{\varepsilon/10} : T_{\varepsilon/10}(x) > t_{\varepsilon/10}(\delta/2)\big). \tag{18}$$

Lemma 1 with $\varepsilon/10$ and $\delta/2$ then yields the upper bound in Theorem 2.

1.3. *The lower bound.* We show that with overwhelming probability there exists $x \in \mathbb{T}_2$ with avoided ε-ball at time

$$\mathfrak{t} = \mathfrak{t}(\varepsilon, \delta) \equiv (1-\delta)^4 \frac{2}{\pi} (\ln \varepsilon)^2. \tag{19}$$

Theorem 2 will then follow immediately by considering[1] $\hat{\delta} \equiv 1 - (1-\delta)^4$. We set

$$\mathfrak{n}(j) = \mathfrak{n}(j; \varepsilon, \delta, K) \equiv -2K(1-\delta)^j \ln \varepsilon \quad (j \in \mathbb{N}), \tag{20}$$

which is for any fixed $j > 0$ slightly smaller than the typical amount of excursions up to time $t_\varepsilon(\delta)$ from scale 1 to scale 0. When ever we need to leave some slight room we drop the exponent j by one. With $\tau_r \equiv \tau_r(x)$ denoting the first time BM hits the r-ball around $x \in \mathbb{T}_2$, we define the events

$$\mathcal{R} \equiv \bigcap_{x \in L_\varepsilon} \{D_{\mathfrak{n}(3)}(x) > \mathfrak{t}\} \quad \text{and} \tag{21}$$

$$\begin{aligned} \mathcal{R}^x \equiv \{\tau_{r_1} < \tau_{r_K}\} \cap \{&\text{at most } \mathfrak{n}(2) \text{ excursions } \lfloor \delta k \rfloor \to \lfloor \delta k \rfloor - 1 \\ &\text{during first } \mathfrak{n}(3) \text{ excursions } 1 \to 0\}. \end{aligned} \tag{22}$$

For $n \in \mathbb{N}$ and $l \in \{1, .., K-1\}$, let

$$\begin{aligned} \mathcal{N}_l^x(n) \equiv\ &\text{number of excursions of } W \text{ from } \partial B_{r_l}(x) \text{ to } \partial B_{r_{l+1}}(x) \\ &\text{within the first } n \text{ excursions from } \partial B_{r_l}(x) \text{ to } \partial B_{r_{l-1}}(x) \\ &\text{after time } \tau_{r_1}. \end{aligned} \tag{23}$$

A very useful property of these counting random variables, which is proved in Section 2.3, is the following.

PROPOSITION 3 ($\mathcal{N}$-Independence). *For $x, y \in \mathbb{T}_2$ with $d_{\mathbb{T}_2}(x, y) \in [r_{i+1}, r_i]$, we have that*

$$\{\mathcal{N}_l^x : l \in \{1, .., K\} \setminus \{i-1, i, i+1, i+2\}\} \cup \{\mathcal{N}_l^y : l \in \{i+3, .., K\}\} \tag{24}$$

is a collection of independent processes.

For $x \in \mathbb{T}_2$, define the events

$$A^x \equiv \bigcap_{l=\lfloor \delta K \rfloor}^{K-1} A_l^x, \tag{25}$$

where

$$A_l^x \equiv \left\{ \mathcal{N}_l^x \left(\mathfrak{n}(0) \left(1 - \frac{l}{K}\right)^2 \right) \le \mathfrak{n}(0) \left(1 - \frac{l+1}{K}\right)^2 \right\}. \tag{26}$$

The events A, $\mathcal{R}$ are motivated by the following observations. First, it can be checked via Doob's h-transform that the expected number of excursions from l to $l+1$ performed by a SRW started at 1 and stopped at 0 and conditioned not to hit K, is approximately $[1 - (l+1)/K]^2$. The events A^x thus describe the natural avoidance strategy of scale K by $\mathfrak{n}(0)$ independent such SRW, which is in turn equivalent to specifying the avoidance strategy of an ε-ball.

[1] This is notationally convenient, but holds no deeper meaning.

Second, we claim that

$$\mathcal{R} \cap \mathcal{R}^x \cap A^x \subset \{B_\varepsilon(x) \text{ is not hit up to time } t\}. \tag{27}$$

Remark in fact that on $\mathcal{R}^x$, the ball $B_\varepsilon(x)$ is not hit before $\partial B_{r_1}(x)$, hence the ε-ball can only be hit in an excursion from B_{r_1} to B_{r_0}. $\mathcal{R}$ ensures that there are at most $\mathfrak{n}(3)$ excursions before time $\mathfrak{t}$. Therefore, on $\mathcal{R}^x \cap \mathcal{R}$, there are at most $\mathfrak{n}(2)$ excursions from scale $\lfloor \delta K \rfloor \to \lfloor \delta K \rfloor - 1$ at time $\mathfrak{t}$. But on A^x, none of these excursions reaches scale K, hence the ε-ball is not hit, and (27) holds.

In light of (27), and in view of the lower bound in Theorem 2, estimates on the probabilities of the $\mathcal{R}$, A-events are needed. This information is provided by Lemma 2 and 3 below, whose proofs are deferred to Section 2.2. Concerning the $\mathcal{R}$-event we state the following.

LEMMA 2. *For all $\delta > 0$ and large enough $K = K(\delta) \in \mathbb{N}$, there exist constants $\kappa, \kappa' > 0$ depending on δ, K only such that*

$$\inf_{x \in L_\varepsilon \setminus B_{r_1}(W_0), \varepsilon \in (0,\kappa')} \mathbb{P}(\mathcal{R}^x), \mathbb{P}(\mathcal{R}) \geq 1 - \varepsilon^\kappa. \tag{28}$$

Concerning the A-events, we have the following.

LEMMA 3 (One point estimates). *For K large, $\varepsilon > 0$ small enough (depending on δ and K), we have*

$$\varepsilon^{2-1.99\delta} \leq \mathbb{P}(A^x) \leq \varepsilon^{2-2.01\delta} \quad \textit{and} \tag{29}$$

$$\prod_{l=\lfloor \delta K \rfloor, |l-i|>2}^{K-1} \mathbb{P}(A_l^x) \prod_{l=i+3}^{K-1} \mathbb{P}(A_l^y) \leq \varepsilon^{4-2.01\delta-2\frac{i}{K}}. \tag{30}$$

Coming back to the lower bound, restricting to the set $L_\varepsilon^* \equiv L_\varepsilon \setminus B_{r_1}(W_0)$ yields that

$$\begin{aligned}
\mathbb{P}\Big(\sup_{x \in \mathbb{T}_2} T_\varepsilon(x) > \mathfrak{t}\Big) &\geq \mathbb{P}(\exists x \in L_\varepsilon^* \text{ such that } B_\varepsilon(x) \text{ is not hit up to time } \mathfrak{t}) \\
&\overset{(27)}{\geq} \mathbb{P}(\mathcal{R} \text{ and } \exists x \in L_\varepsilon^* \text{ such that } \mathcal{R}^x \cap A^x) \qquad (31)\\
&\geq \frac{\mathbb{E}[\#\{x \in L_\varepsilon^* : \mathcal{R}^x \cap A^x\}]^2}{\mathbb{E}[\#\{x \in L_\varepsilon^* : \mathcal{R}^x \cap A^x\}^2]} - \mathbb{P}(\mathcal{R}^c),
\end{aligned}$$

the last inequality by Paley–Zygmund.

It is intuitively clear (and rigorously proven in Lemma 5 below) that rotational invariance and strong Markovianity imply that $\mathcal{R}^x$ and A^x are, in fact, *independent*: the above is thus *at least*

$$\Big[\sum_{x \in L_\varepsilon^*} \mathbb{P}(\mathcal{R}^x)\mathbb{P}(A^x)\Big]^2 \Big/ \Big[\sum_{x,y \in L_\varepsilon^*} \mathbb{P}(A^x \cap A^y)\Big] - \mathbb{P}(\mathcal{R}^c). \tag{32}$$

We now analyze the denominator. First, remark that for $d_{\mathbb{T}_2}(x, y) > 2r_{\lfloor \delta K \rfloor - 1}$, the A-events decouple: in fact, they are rotationally invariant and depend on disjoint excursions, hence by the strong Markov property,

$$\mathbb{P}(A^x \cap A^y) = \mathbb{P}(A^x)\mathbb{P}(A^y).$$

Shortening

$$\mathcal{A} \equiv \sum_{x \in L_\varepsilon^*} \mathbb{P}(A^x), \qquad \mathcal{B} \equiv \sum_{x,y \in L_\varepsilon} \mathbb{1}_{\{d_{\mathbb{T}_2}(x,y) \leq 2r_{\lfloor \delta K \rfloor - 1}\}} \mathbb{P}(A^x \cap A^y),$$

by Lemma 2 and the exact decoupling we thus have that

$$(32) \geq (1-\varepsilon^{\kappa})^2 \frac{\mathcal{A}^2}{\mathcal{A}^2+\mathcal{B}} - \varepsilon^{\kappa} \geq (1-\varepsilon^{\kappa})^2\left(1-\frac{\mathcal{B}}{\mathcal{A}^2}\right) - \varepsilon^{\kappa} \\ \geq (1-\varepsilon^{\kappa})^2\left(1-\frac{\mathcal{B}}{\varepsilon^{-3.96\delta}}\right) - \varepsilon^{\kappa}, \tag{33}$$

the last step by Lemma 3 and using that $|L_\varepsilon| \geq \varepsilon^{-2+0.01\delta}$. It thus remains to analyze the $\mathcal{B}$-term: by regrouping terms according to the distance,

$$\mathcal{B} \leq \sum_{i=\lfloor \delta K \rfloor -2}^{K} \sum_{x,y \in L_\varepsilon} \mathbb{1}_{\{d_{\mathbb{T}_2}(x,y) \in [r_{i+1}, r_i]\}} \mathbb{P}(A^x \cap A^y). \tag{34}$$

To get a handle on the two points probabilities appearing in (34), we follow the recipe from [5], Section 3.1.1, pages 97–98, exploiting the approximate hierarchical structure which underlies the excursion counts, and which is best explained with the help of a picture; see Figure 2. First, the circles associated to x, y on small scales i (left) are almost identical and so are the excursion counts; this suggests that $A_i^x \cap A_i^y$ is well represented by A_i^x alone. Dropping one of the events is an estimate by the worst case scenario known in this context as “REM approximation.” For larger i (middle), this approximation is not sharp, but only few scales can fall into this case as we can choose ε arbitrarily small for given K. Choosing K large makes the influence of few scales comparatively small. For i large (right), balls are disjoint, which by rotational invariance and strong Markovianity yields independent excursion counts. The technical details of this argument are discussed in the proof of Proposition 3. The approximate tree structure of excursion counts is summarized in the lower pictures, the red box corresponding to the scale at hand. By these considerations, for $i \geq \lfloor \delta K \rfloor - 2$ and $d_{\mathbb{T}_2}(x, y) \in [r_{i+1}, r_i]$, we write

$$\begin{aligned} &\mathbb{P}(A^x \cap A^y) \\ &= \mathbb{P}\left(\bigcap_{l=\lfloor \delta K \rfloor}^{K-1} A_l^x \cap \bigcap_{l=\lfloor \delta K \rfloor}^{K-1} A_l^y \right) \\ &\leq \mathbb{P}\left(\bigcap_{l=\lfloor \delta K \rfloor, |l-i|>2}^{K-1} A_l^x \cap \bigcap_{l=i+3}^{K-1} A_l^y \right) \quad \text{(REM approximation)} \\ &= \prod_{l=\lfloor \delta K \rfloor, |l-i|>2}^{K-1} \mathbb{P}(A_l^x) \prod_{l=i+3}^{K-1} \mathbb{P}(A_l^y) \quad \text{(Proposition 3, } \mathcal{N}\text{-independence)} \\ &\leq \varepsilon^{4-2.01\delta-2\frac{i}{K}} \quad \text{(Lemma 3, one-point estimates).} \end{aligned} \tag{35}$$

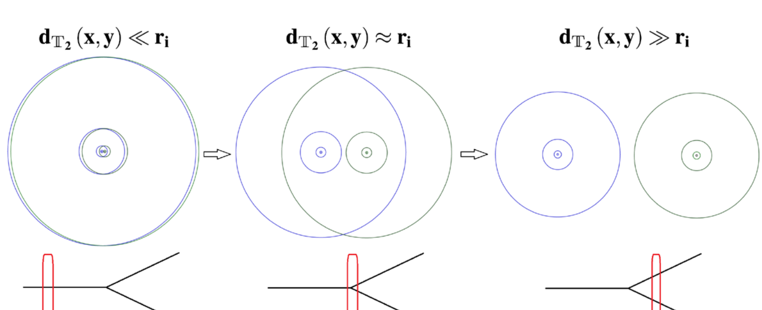

FIG. 2. *Common branch on small scales (left) and decoupling on large scales (right).*

There are at most $2\varepsilon^{-4}\pi r_i^2$ pairs of points on L_ε with distance at most r_i: using that $r_i \le \varepsilon^{i/K}$, and (35) in (34) we get

$$\mathcal{B} \le \sum_{i=\lfloor \delta K\rfloor -2}^{K} 2\pi\varepsilon^{-2.01\delta} \le \varepsilon^{-2.02\delta}. \tag{36}$$

Applying this estimate to (33) and putting $\hat{\delta} \equiv 1-(1-\delta)^4$ we therefore see that

$$\mathbb{P}\left(\sup_{x\in\mathbb{T}_2} T_\varepsilon(x) > (1-\hat{\delta})\frac{2}{\pi}(\ln\varepsilon)^2\right) \ge 1-\varepsilon^{\hat{c}}, \tag{37}$$

for $\hat{c} \equiv \frac{1}{2}\min\{\kappa, 1.94\delta\}$, settling the lower bound of Theorem 2.

2. Proofs.

2.1. *Hitting times and excursion counts.* The study of hitting times for BM is closely related to Green's functions. Estimates on the torus have however proofs which are either opaque or hard to find: we include here an elementary treatment based on Fourier analysis for the reader's convenience.

LEMMA 4. *The function*

$$\begin{aligned} F(x,y) &\equiv G_x(y) - \frac{1}{2\pi}\ln d_{\mathbb{T}_2}(x,y) \\ \textit{where } G_x(y) &\equiv -\sum_{p\in 2\pi\mathbb{Z}^2\setminus\{0\}} \frac{1}{|p|^2}e^{ip(x-y)} \end{aligned} \tag{38}$$

is bounded on $\mathbb{T}_2^2 \setminus \{(x,x): x\in\mathbb{T}_2\}$.

PROOF. It suffices to consider y in a small neighborhood of x, as otherwise the result is trivial. So let $z \equiv x-y$ and assume that $2|z_1| \ge |z|$ (swapping coordinates otherwise). We have

$$\left|\sum_{\substack{p\in 2\pi\mathbb{Z}^2\setminus\{0\}\\ |p|>|z|^{-1}}} \frac{1}{|p|^2}e^{ipz}\right| = \left|\sum_{\substack{p\in 2\pi\mathbb{Z}^2\setminus\{0\}\\ |p|>|z|^{-1}}} \frac{1}{1-e^{i2\pi z_1}}\frac{1}{|p|^2}\left(e^{ipz}-e^{i(p+(2\pi,0))z}\right)\right|. \tag{39}$$

Shifting the difference from the exponential to $|p|^{-2}$ by collecting terms with the same exponent, and by the triangle inequality, one obtains boundedness uniformly over $z \ne 0$ in a small enough neighborhood of 0. The extra terms due to the boundary of the summation domain are easily shown to be bounded. By combining the summand p and $-p$, we see that sums of this form are real valued. Therefore,

$$\sum_{\substack{p\in 2\pi\mathbb{Z}^2\setminus\{0\}\\ |p|\le|z|^{-1}}} \frac{1}{|p|^2}e^{ipz} = \sum_{\substack{p\in 2\pi\mathbb{Z}^2\setminus\{0\}\\ |p|\le|z|^{-1}}} \frac{1}{|p|^2}\cos(pz). \tag{40}$$

Since $|pz| \le 1$ for all summands contained in this sum, we can estimate $\cos(x) \le 1 - x^2/4$. Hence

$$\left|G_x(y) - \sum_{\substack{p\in 2\pi\mathbb{Z}^2\setminus\{0\}\\ |p|\le|z|^{-1}}} \frac{1}{|p|^2}\right| \tag{41}$$

is uniformly bounded for y in a small neighborhood of x. The claim of Lemma 4 then follows by rearranging summands into groups $C_j \equiv \{p \in 2\pi\mathbb{Z}^2 \setminus \{0\} : |p|^2 \in ((j-1)^3, j^3]\}$, estimating $|p|^{-2}$ by the best/worst case scenario within each group, and using that $|C_j| = \frac{3}{4\pi} j^2 + O(j^{3/2})$. □

PROOF OF PROPOSITION 2: FIRST MOMENT OF HITTING TIMES. Let $\mu(y) \equiv \mathbb{E}_y[\tau_{B_r(x)}]$. For Δ, the Laplacian with periodic boundary condition on $\mathbb{T}_2$, we have Poisson's equation $\Delta\mu = -2$ on $\mathbb{T}_2 \setminus B_r(x)$ with $\mu = 0$ on $B_r(x)$. Plainly,

$$G_x(y) \equiv -\sum_{p \in 2\pi\mathbb{Z}^2 \setminus \{0\}} \frac{1}{|p|^2} e^{ip(x-y)} \tag{42}$$

is a Green function, that is, solution of $\Delta G_x = 1 - \delta_x$ on the torus. In particular, $\mu + 2G_x$ is harmonic on $\mathbb{T}_2 \setminus B_r(x)$. By the maximum principle, and since $\mu \equiv 0$ on $\partial B_r(x)$,

$$2 \inf_{z \in \partial B_r(x)} G_x(z) \le \mu(y) + 2G_x(y) \le 2 \sup_{z \in \partial B_r(x)} G_x(z) \tag{43}$$

holds. It follows from Lemma 4 that $\mu(y) - \frac{1}{\pi} \ln[d_{\mathbb{T}_2}(x, y)/r]$ is bounded, and the first claim (9) is proved. The second claim (10) is elementary as we can identify the ball on $\mathbb{T}_2$ with the ball in $\mathbb{R}^2$ and exploit rotational invariance to solve Poisson's equation explicitly. □

PROOF OF PROPOSITION 1: CONCENTRATION OF EXCURSION COUNTS. By Kac's moment formula [4],

$$\mathbb{E}_x[\tau_A^i] \le i! \sup_{x \in \mathbb{T}} \mathbb{E}_x[\tau_A]^i, \qquad A \subset \mathbb{T} \text{ closed.} \tag{44}$$

By monotone convergence, Taylor expanding the exponential function and by the above estimate,

$$\begin{aligned} \mathbb{E}_x[e^{\theta\tau_A}] &\le 1 + \theta\mathbb{E}_x[\tau_A] + \sum_{i=2}^{\infty} \Big(\theta \sup_{x \in \mathbb{T}} \mathbb{E}_x[\tau_A]\Big)^i \\ &\le \exp\Big(\theta\mathbb{E}_x[\tau_A] + 2\theta^2 \sup_{x \in \mathbb{T}} \mathbb{E}_x[\tau_A]^2\Big) \end{aligned} \tag{45}$$

for $0 < \theta < \frac{1}{2}(\sup_{x \in \mathbb{T}} \mathbb{E}_x[\tau_A])^{-1}$. Using $e^{-x} \le 1 - x + x^2$ for positive x gives

$$\begin{aligned} \mathbb{E}_x[e^{-\theta\tau_A}] &\le 1 - \theta\mathbb{E}_x[\tau_A] + \theta^2 \sup_{x \in \mathbb{T}} \mathbb{E}_x[\tau_A]^2 \\ &\le \exp\Big(-\theta\mathbb{E}_x[\tau_A] + \theta^2 \sup_{x \in \mathbb{T}} \mathbb{E}_x[\tau_A]^2\Big). \end{aligned} \tag{46}$$

Consider $\tau^{(i\leftarrow)}$ the time it takes W to get from $\partial B_{r_1}(x)$ to $B_{r_0}^c(x)$ the ith time; $\tau^{i\rightarrow}$ the time W needs to get from $\partial B_{r_0}(x)$ to $B_{r_1}(x)$ the ith time after $B_{r_1}(x)$ has been hit the first time and τ_{r_1} the time it takes W to get from the starting point to $\partial B_{r_1}(x)$. Now by definition we have

$$D_n(x) = \tau_{r_1} + \sum_{i=1}^{n-1} \tau^{(i\rightarrow)} + \sum_{i=1}^{n} \tau^{(i\leftarrow)}. \tag{47}$$

Exponential Markov inequality gives for any $t, \theta > 0$,

$$\mathbb{P}(D_n(x) \ge t) \le e^{-\theta t} \mathbb{E}[e^{\theta D_n(x)}]. \tag{48}$$

Using (47), the strong Markov property, and estimating by worst starting points the right-hand side above is at most

$$e^{-\theta t}\Big(\sup_{z\in\mathbb{T}_2}\mathbb{E}_z\big[e^{\theta\tau_{r_1}}\big]\Big)\Big(\sup_{z\in B_{r_0}(x)}\mathbb{E}_z\big[e^{\theta\tau^{(1\to)}}\big]\Big)^{n-1}\Big(\sup_{z\in B_{r_1}(x)}\mathbb{E}_z\big[e^{\theta\tau^{(1\leftarrow)}}\big]\Big)^{n}. \tag{49}$$

Using (45) with $\theta=-\frac{\pi\delta}{4\ln r_1}$, and applying Proposition 2, we obtain

$$\begin{aligned}\sup_{z\in\mathbb{T}_2}\mathbb{E}_z\big[e^{\theta\tau_{r_1}}\big]&\le e^{\frac{\delta}{4}+\frac{\delta^2}{8}+o_{r_1}(1)},\\ \sup_{z\in B_{r_0}(x)}\mathbb{E}_z\big[e^{\theta\tau^{(1\to)}}\big]^{n-1}&\le e^{(n-1)(\frac{\delta}{4}+\frac{\delta^2}{8}+o_{r_1}(1))},\\ \sup_{z\in B_{r_1}(x)}\mathbb{E}_z\big[e^{\theta\tau^{(1\leftarrow)}}\big]^{n}&\le e^{no_{r_1}(1)}.\end{aligned} \tag{50}$$

With $t=(1+\delta)n\frac{1}{\pi}\ln\frac{r_0}{r_1}$, and by the above estimates, (49) reads

$$\mathbb{P}\bigg(D_n(x)\ge(1+\delta)n\frac{1}{\pi}\ln\frac{r_0}{r_1}\bigg)\le e^{-n(\frac{\delta}{4}+\frac{\delta^2}{4}+o_{r_1}(1))}e^{n(\frac{\delta}{4}+\frac{\delta^2}{8}+o_{r_1}(1))}, \tag{51}$$

settling (7). As for (8), for any $n\in\mathbb{N}$ and $\theta>0$, we have

$$\mathbb{P}\big(D_n(x)\le t\big)\le e^{\theta t}\mathbb{E}e^{-\theta D_n(x)}\le e^{\theta t}\mathbb{E}\big[e^{-\theta\tau^{(1\to)}}\big]^{n-1}. \tag{52}$$

Choosing $\theta=\frac{\pi\delta}{2\ln r_1}$ and $t=(1-\delta)n\frac{1}{\pi}\ln\frac{r_0}{r_1}$, applying (46) together with Proposition 2 yields the second claim and concludes the proof of Proposition 1. □

2.2. *Estimates for $\mathcal{R}$ and A.*

PROOF OF LEMMA 2. For $x\in L^*_\varepsilon$, $\{\tau_{r_1}<\tau_{r_K}\}$ almost surely. By rotational invariance and strong Markovianity, the number of excursions from scale $\lfloor\delta K\rfloor$ to scale $\lfloor\delta K\rfloor-1$ in different excursions from scale 1 to scale 0 are independent of each other. The number of excursions from scale $\lfloor\delta K\rfloor$ to scale $\lfloor\delta K\rfloor-1$ in one excursion from scale 1 to scale 0 is distributed like the product of a Bernoulli distributed and an independent geometrically distributed random variable, both with parameter $\lfloor\delta K\rfloor^{-1}$. (This product has expectation 1.) By Cramér's theorem,

$$\begin{aligned}&\mathbb{P}\big(\text{more than }\mathfrak{n}(2)\text{ times }\lfloor\delta K\rfloor\to\lfloor\delta K\rfloor-1\text{ in the first}\\ &\qquad\mathfrak{n}(3)\text{ excursions }1\to 0\big)\\ &\quad\le\exp\bigg(-\mathfrak{n}(3)I\Big(\frac{1}{1-\delta}\Big)\bigg)=\varepsilon^{2K(1-\delta)^3I(\frac{1}{1-\delta})},\end{aligned} \tag{53}$$

with I the rate function of a Bernoulli$(1/\lfloor\delta K\rfloor)\times$ geometric$(1/\lfloor\delta K\rfloor)$. It follows that $\mathbb{P}((\mathcal{R}^x)^c)$ vanishes polynomially in ε for fixed δ and K. Taking the complement yields the first claim.

By Proposition 1, we have

$$\mathbb{P}\big(D_{\mathfrak{n}(3)}(x)\le\mathfrak{t}\big)\le\varepsilon^{2K(1-\delta)^3(\delta^2/4+o_{r_1}(1))}, \tag{54}$$

which vanishes faster then, say, ε^3 for K sufficiently large. The second claim thus follows by union bound over all $x\in L_\varepsilon$ on the complements. □

PROOF OF LEMMA 3. The number of times a SRW goes from l to $l+1$ before going from l to $l-1$ is geo(1/2)-distributed. Therefore, $\mathcal{N}_l^x(n)$ is, by strong Markovianity and rotational invariance, the sum of n independent geo(1/2)-distributed r.v.'s. Hence by Cramér's theorem,

$$\begin{aligned}\mathbb{P}(A^x) &= \prod_{l=\lfloor\delta K\rfloor}^{K-1} \mathbb{P}(A_l^x)\\ &= \prod_{l=\lfloor\delta K\rfloor}^{K-1} \exp\biggl(-\mathfrak{n}(0)\Bigl(1-\frac{l}{K}\Bigr)^2 I\Bigl(\frac{(1-\frac{l+1}{K})^2}{(1-\frac{l}{K})^2}\Bigr)+o_\varepsilon(\mathfrak{n}(0))\biggr) \qquad (55)\\ &= \exp\biggl(-\frac{\mathfrak{n}(0)}{K^2}\sum_{l=\lfloor\delta K\rfloor}^{K-1}(K-l)^2 I\Bigl(\Bigl(1-\frac{1}{K-l}\Bigr)^2\Bigr)+o_\varepsilon(\mathfrak{n}(0))\biggr),\end{aligned}$$

where $I(x) = x\ln(x) - (1+x)\ln(\frac{1+x}{2})$ is the geo(1/2)-rate function. Using $I(1) = I'(1) = 0$ and $I''(1) = \frac{1}{2}$, one quickly obtains

$$j^2 I\bigl((1-1/j)^2\bigr) = 1 + o_j(1) \quad (j \to \infty),$$

and, therefore,

$$\begin{aligned}\mathbb{P}(A^x) &= \exp\biggl(-\frac{\mathfrak{n}(0)}{K}(1-\delta)\bigl(1+o_K(1)\bigr)+o_\varepsilon(\mathfrak{n}(0))\biggr) \qquad (56)\\ &= \varepsilon^{2(1-\delta)(1+o_K(1))+o_\varepsilon(1)},\end{aligned}$$

concluding the proof of the first claim. The second claim is analogous. □

2.3. *Independencies and distributional identities*. Fixing a reference point $x \in \mathbb{T}_2$ we define the function

$$s_x(y) \equiv K\frac{\ln d_{\mathbb{T}_2}(x,y) - \ln R}{\ln\varepsilon - \ln R} \qquad (57)$$

for $y \in \mathbb{T}_2$, which maps a point y to the scale i around x it belongs to (i.e., $s_x(y) = i \Leftrightarrow y \in B_{r_i}(x)$). Additionally, we define stopping times:

$$\begin{aligned}\tau_1^{(1)}(x) &\equiv \inf\{t > 0 | W_r \in \partial B_{r_1}(x)\},\\ \tau_0^{(i)}(x) &\equiv \inf\{t > \tau_1^{(i)} | W_r \in \partial B_{r_0}(x)\} \quad \text{for } i \in \mathbb{N},\\ \tau_1^{(i)}(x) &\equiv \inf\{t > \tau_0^{(i)} | W_r \in \partial B_{r_1}(x)\} \quad \text{for } i > 1, i \in \mathbb{N},\\ \sigma_1^{(i)}(x) &\equiv \tau_1^{(i)}(x)\end{aligned}$$

and

$$\sigma_j^{(i)}(x) \equiv \inf\{t > \sigma_{j-1}^{(i)} | W_t \in \partial B_{r_i}(x) \text{ for some } i \in \mathbb{N}_0 \setminus \{s_x(W_{\sigma_{j-1}^{(i)}})\}\},$$

for $j > 1$, $i \in \mathbb{N}$. With these stopping times, we can precisely describe the simple random walk structures embedded in W with respect to a fixed reference point $x \in \mathbb{T}_2$.

LEMMA 5. *Setting* $J_i \equiv \inf\{j \in \mathbb{N} : \sigma_j^{(i)} = \tau_0^{(i)}\} = \inf\{j \in \mathbb{N} : S_j^{(i)} = 0\}$, *we have that*

$$\bigl(S_j^{(i)}\bigr)_{j\in\{1,..,J_i\}} \equiv \bigl(s_x(W_{\sigma_j^{(i)}(x)})\bigr)_{j\in\{1,..,J_i\}}, \quad i \in \mathbb{N} \qquad (58)$$

are independent excursions of simple random walks on $\mathbb{Z}$ *from* 1 *to* 0.

PROOF. By construction of σ and continuity of W, we have that

$$\left(s_x(W_{\sigma_j^{(i)}(x)})\right)_{j\in\{1,..,J_i\}}$$

has values in $\mathbb{N}_0$ and each step changes by exactly one almost surely. If we furthermore identify the circles with circles in $\mathbb{R}^2$ and solve the Dirichlet problem explicitly, we see that a "+1" step and a "−1" step are equally likely due to the constant multiplicative growth rate the r_i. As the position of W on some scale $\partial B_{r_i}(x)$ by rotational invariance around x has no influence on the distribution of the future of the considered excursions the claim follows by the strong Markov property. □

We now turn our attention to the independencies when considering two reference points $x, y \in \mathbb{T}_2$.

PROOF OF PROPOSITION 3 ($\mathcal{N}$-INDEPENDENCE). We define the set of (viable) indices by

$$\mathcal{C} \equiv \{(x,l,n) : l \in \{1,..,K\} \setminus \{i-1,i,i+1,i+2\}, j \in \mathbb{N}\} \cup \{(y,l,n) : l \in \{i+3,..,K\}, j \in \mathbb{N}\}. \tag{59}$$

For $(z,l,n) \in \mathcal{C}$, let $\gamma_l^z(n)$ be the nth smallest $\sigma_j^{(i)}(z)$ with $j \in \{1,..,J_i\}$, $i \in \mathbb{N}$ and $W_{\sigma_j^{(i)}(x)} \in \partial B_{r_l}(x)$. $\gamma_l^z(n)$ is simply the nth time after the first hit of $\partial B_{r_1}(x)$ that W visits scale l after hitting another scale directly prior. Additionally, set $\tilde{\gamma}_l^x(n)$ the first time at which W visits a scale different from l after $\gamma_l^x(n)$. Clearly, both are stopping times. First, we make sure that the intervals $[\gamma_l^z(n), \tilde{\gamma}_l^z(n))$, $(z,l,n) \in \mathcal{C}$ are disjointed, which is the essential observation for independence. If two such intervals correspond both to x or both to y then they are disjoint by construction, simply by the fact that the nth trip from scale j to the next hit of another scale cannot happen simultaneously to another such trip. Considering now the other case namely $(x,l,n), (y,l',n') \in \mathcal{C}$ the corresponding intervals cannot intersect as on one hand W is within $B_{r_{i+2}}(y) \subset B_{r_{i-1}}(x) \setminus B_{r_{i+2}}(x)$ during the time $[\gamma_{l'}^y(n'), \tilde{\gamma}_{l'}^y(n'))$ for $(y,l',n') \in \mathcal{C}$, but on the other hand during $[\gamma_l^x(n), \tilde{\gamma}_l^x(n))$ we have that W is on $(B_{r_{i-1}}(x) \setminus B_{r_{i+2}}(x))^c$ for $(x,l,n) \in \mathcal{C}$. Consider the indicators

$$I_l^x(n) \equiv \mathbb{1}\{W_{\tilde{\gamma}_l^x(n)} \in \partial B_{r_{l+1}}(x)\}, \tag{60}$$

which track whether $l+1$ or $l-1$ is visited next after one particular hit of scale l. Clearly, the process $\mathcal{N}_l^x$ is a deterministic functional of $(I_l^x(n))_{n\in\mathbb{N}}$. Therefore, it is sufficient to show that

$$\{I_l^z(n) : (z,l,n) \in \mathcal{C}\} \tag{61}$$

is a collection of independent random variables, as this property is inherited by the $\mathcal{N}$. As product measures are uniquely determined by the finite dimensional distributions, we pick a finite $A \subset \mathcal{C}$. Consider $\{u_1,..,u_{|A|}\} = A$ such that (by slight abuse of notation) $\gamma(u_i)$, $i \in \{1,..,|A|\}$ is sorted increasingly. One easily checks that $\gamma(u_i)$ are stopping times, as A is a finite deterministic set. For $v \in \{0,1\}^{|A|}$, we have

$$\mathbb{P}(I(u_i) = v_i, \forall i \le |A|) = \mathbb{E}[\mathbb{1}_{\{I(u_i)=v_i, \forall i \le |A|-1\}} \mathbb{P}(I(u_{|A|}) = v_{|A|} | \mathcal{F}_{\gamma_{u_{|A|}}})], \tag{62}$$

because $I(u_i) = v_i$ are $\mathcal{F}_{\gamma_{u_{|A|}}}$ measurable for $i < |A|$, since $\gamma_{u_{|A|}} > \tilde{\gamma}_{u_i}$ in this case by the disjointness property we showed in the beginning of the proof. By rotational invariance of the construction the probability of going up or down a scale is $1/2$ regardless of the starting point. Hence

$$\mathbb{P}(I(u_{|A|}) = v_{|A|} | W_{\gamma_{u_{|A|}}}) = \mathbb{P}(I(u_{|A|}) = v_{|A|}) = 1/2, \tag{63}$$

holds, which together with the strong Markov property of W allows to reformulate (62) to

$$\begin{aligned}\mathbb{P}\big(I(u_i)=v_i,\forall i\le|A|\big)\\=\mathbb{P}\big(I(u_i)=v_i,\forall i\le|A|-1\big)\mathbb{P}\big(I(u_{|A|})=v_{|A|}\big).\end{aligned}\tag{64}$$

Conditioning on $\mathcal{F}_{\gamma_{u_i}}$ for the largest i that is not yet split off and repeating the argument until product form is established yields independence of the indicators and, therefore, the claim. □

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