QUENCHED INVARIANCE PRINCIPLE FOR RANDOM WALKS AMONG RANDOM DEGENERATE CONDUCTANCES

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We consider the random conductance model in a stationary and ergodic environment. Under suitable moment conditions on the conductances and their inverse, we prove a quenched invariance principle for the random walk among the random conductances. The moment conditions improve earlier results of Andres, Deuschel and Slowik (*Ann. Probab.* **43** (2015) 1866–1891) and are the minimal requirement to ensure that the corrector is sublinear everywhere. The key ingredient is an essentially optimal deterministic local boundedness result for finite difference equations in divergence form.

1. Introduction.

1.1. *Setting of the problem and the main result.* In this paper, we study the nearest neighbor random conductance model on the d-dimensional Euclidean lattice $(\mathbb{Z}^d, \mathbb{B}^d)$, for $d \geq 3$. Here, $\mathbb{B}^d$ is given by the set of nonoriented nearest neighbor bonds, that is, $\mathbb{B}^d := \{\{x, y\} \mid x, y \in \mathbb{Z}^d, |x - y| = 1\}$.

We set $\Omega := (0, \infty)^{\mathbb{B}^d}$ and call $\omega(e)$ the *conductance* of the bond $e \in \mathbb{B}^d$ for every $\omega = \{\omega(e) \mid e \in \mathbb{B}^d\} \in \Omega$. To lighten the notation, for any $x, y \in \mathbb{Z}^d$, we set

$$\omega(x, y) = \omega(y, x) := \omega(\{x, y\}) \quad \forall \{x, y\} \in \mathbb{B}^d,$$

$$\omega(\{x, y\}) = 0 \quad \forall \{x, y\} \notin \mathbb{B}^d.$$

In what follows, we consider random conductances that are distributed according to a probability measure $\mathbb{P}$ on Ω equipped with the σ-algebra $\mathcal{F} := \mathcal{B}((0, \infty))^{\otimes \mathbb{B}^d}$ and we write $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$.

We introduce the family of *space shifts* $\{\tau_x : \Omega \to \Omega \mid x \in \mathbb{Z}^d\}$ defined by

$$\tau_x \omega(\cdot) := \omega(\cdot + x) \quad \text{where for any } e = \{\underline{e}, \overline{e}\} \in \mathbb{B}^d, e + x := \{\underline{e} + x, \overline{e} + x\}.$$

For any fixed realization ω, we study the reversible continuous time Markov chain, $X = \{X_t : t \geq 0\}$, on $\mathbb{Z}^d$ with generator $\mathcal{L}^\omega$ acting on bounded functions $f : \mathbb{Z}^d \to \mathbb{R}$ as

$$(1) \qquad (\mathcal{L}^\omega f)(x) := \sum_{y \in \mathbb{Z}^d} \omega(x, y)(f(y) - f(x)).$$

We emphasize at this point that $\mathcal{L}^\omega$ is in fact a finite-difference operator in divergence form; see (14) below. Following [3], we denote by $\mathbf{P}_x^\omega$ the law of the process starting at the vertex $x \in \mathbb{Z}^d$ and by $\mathbf{E}_x^\omega$ the corresponding expectation. X is called the *variable speed random walk* (VSRW) since it waits at $x \in \mathbb{Z}^d$ an exponential time with mean $1/\mu^\omega(x)$, where $\mu^\omega(x) = \sum_{y \in \mathbb{Z}^d} \omega(x, y)$ and chooses its next position y with probability $p^\omega(x, y) := \omega(x, y)/\mu^\omega(x)$.

ASSUMPTION 1. Assume that $\mathbb{P}$ satisfies the following conditions:

(i) (Stationarity) $\mathbb{P}$ is stationary with respect to shifts, that is, $\mathbb{P} \circ \tau_x^{-1} = \mathbb{P}$ for all $x \in \mathbb{Z}^d$.

(ii) (Ergodicity) $\mathbb{P}$ is ergodic, that is, $\mathbb{P}[A] \in \{0, 1\}$ for any $A \in \mathcal{F}$ such that $\tau_x(A) = A$ for all $x \in \mathbb{Z}^d$

(iii) (Moment condition) There exists $p, q \in (1, \infty]$ satisfying

$$(2) \qquad \frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$$

such that

$$(3) \qquad \mathbb{E}[\omega(e)^p] < \infty, \qquad \mathbb{E}[\omega(e)^{-q}] < \infty \quad \text{for any } e \in \mathbb{B}^d.$$

The main result of the present paper is a quenched invariance principle for the process X under Assumption 1.

DEFINITION 1. Set $X_t^{(n)} := \frac{1}{n} X_{n^2 t}$, $t \geq 0$. We say that a *quenched functional CLT* (QF-CLT) or *quenched invariance principle* holds for X if for $\mathbb{P}$-a.e. ω under $\mathbf{P}_0^\omega$, $X^{(n)}$ converges in law to a Brownian motion on $\mathbb{R}^d$ with covariance matrix $\Sigma^2 = \Sigma \cdot \Sigma^t$. That is, for every $T > 0$ and every bounded continuous function F on the Skorokhod space $D([0, T], \mathbb{R}^d)$, setting $\psi_n = \mathbf{E}_0^\omega[F(X^{(n)})]$ and $\psi_\infty = \mathbf{E}_0^{BM}[F(\Sigma \cdot W)]$ with $(W, \mathbf{P}_0^{BM})$ being a Brownian motion started at 0, we have that $\psi_n \to \psi_\infty$, $\mathbb{P}$-a.s.

THEOREM 1 (Quenched invariance principle). *Suppose $d \geq 3$ and that Assumption 1 is satisfied. Then the QFCLT holds for X with a deterministic nondegenerate covariance matrix Σ^2.*

REMARK 1. Another natural process is given by the so-called *constant speed random walk (CSRW) Y*, which is defined via the generator $\mathcal{L}_Y^\omega$,

$$(\mathcal{L}_Y^\omega f)(x) := \frac{1}{\mu^\omega(x)} \sum_{y \in \mathbb{Z}^d} \omega(x, y)(f(y) - f(x)),$$

where $\mu^\omega(x) = \sum_{y \in \mathbb{Z}^d} \omega(x, y)$. In contrast to the VSRW, the CSRW waits on each vertex $x \in \mathbb{Z}^d$ an exponential time with mean 1. The invariance principle for the VSRW X and Assumption 1 implies also a QFCLT for Y with a covariance matrix given by $[\mu^\omega(0)]^{-1}\Sigma^2$ (where Σ is as in Theorem 1); see [3], Remark 1.5.

Random walks among random conductances are one of the most studied examples of random walks in a random environment; see [13, 27] for relatively recent overviews of the field. In [20] (see also [25]), a weak FCLT, that is, the convergence of ψ_n to ψ_∞ in Definition 1 holds in $\mathbb{P}$-probability, for stationary and ergodic laws $\mathbb{P}$ with $\mathbb{E}[\omega(e)] < \infty$ is established. In the last two decades, much attention has been devoted to obtain quenched FCLT. In [35], the quenched invariance principle is proven in the uniformly elliptic case, that is, with the assumption that there exists $c \in (0, 1]$ such that $\mathbb{P}[c \leq \omega(e) \leq c^{-1}] = 1$ for all $e \in \mathbb{B}^d$, which corresponds to the case $p = q = \infty$ (see also an earlier result [17] valid only in $d = 2$). Recently, there is an increasing interest to relax the uniform ellipticity assumption. In the special case of i.i.d. conductances, that is, when $\mathbb{P}$ is the product measure which includes, for example, percolation models, it is shown in [1] (building on previous works [9, 12, 15, 28, 29, 35]) that a QFCLT holds provided that $\mathbb{P}[\omega(e) > 0] > p_c$ with $p_c = p_c(d)$ being the bond percolation threshold. In particular, no moment conditions such as (3) are needed. In

the general ergodic situation, it is known that at least first moments of ω and ω^{-1} are necessary for a QFCLT to hold (see [8] for an example where the QFCLT fails but (3) holds for any $p, q \in (0, 1)$). In [3], Andres, Deuschel and Slowik proved the conclusion of Theorem 1 under the moment condition (3) with the more restrictive relation

$$(4) \qquad \frac{1}{p} + \frac{1}{q} < \frac{2}{d}.$$

The result of [3] was already extended in several directions: to the continuum case [19] (for an earlier contribution with $q = \infty$, see [22]), random walks on more general graphs [21] and to dynamic situations [2]; see also [16, 32]. Previous to [3], Biskup [13] proved QFCLT under the minimal moment condition $p = q = 1$ in two dimensions, and thus we focus our attention to the case $d \geq 3$. To the best of our knowledge Theorem 1 is the first quenched invariance principle in the general stationary and ergodic setting under a less restrictive moment condition compared to (4) valid in $d \geq 3$. Optimality of condition (2) in Theorem 1 is not clear to us, since in particular in [7] a quenched invariance principle for diffusion in $\mathbb{R}^d$ with a locally integrable periodic potential is proven. However, we emphasize that condition (2) is essentially optimal for the everywhere sublinearity of the corrector; see Proposition 2 and Remark 4. The latter is of independent interest for stochastic homogenization of elliptic operators in divergence form with degenerate coefficients; for further recent results in that direction, see [5, 6, 10, 34].

1.2. *Strategy.* The proof of Theorem 1 follows the classical approach to show an invariance principle and relies on a decomposition of the process X into a martingale part and a remainder (see, e.g., [25]). General martingale theory (in particular [24]) yields a quenched invariance principle for the martingale part and it remains to show that the remainder is negligible. A key insight in [3] was to apply deterministic elliptic regularity theory, in particular Moser's iteration argument [30, 31], to control the remainder term. The main effort in the present contribution is to improve the deterministic part of the argument. Let us now be more precise (in what follows we use the notation introduced in Section 1.3 below). Following, for example, [3, 13], we introduce harmonic coordinates, that is, we construct a corrector field $\chi : \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$ such that

$$\Phi(\omega, x) = x - \chi(\omega, x)$$

is $\mathcal{L}^\omega$-harmonic in the sense that for every $x \in \mathbb{Z}^d$ and $j \in \{1, \ldots, d\}$

$$(5) \qquad 0 = \mathcal{L}^\omega(\Pi_j - \chi_j)(x) = -\nabla^*(\omega \nabla(\Pi_j - \chi_j))(x),$$

where $\Pi_j(y) = y \cdot e_j$ and $\chi_j(y) = \chi(y) \cdot e_j$ for every $y \in \mathbb{Z}^d$. The $\mathcal{L}^\omega$-harmonicity of Φ implies that

$$M_t := \Phi(\omega, X_t) = X_t - \chi(\omega, X_t)$$

is a martingale under $\mathbf{P}_0^\omega$ for $\mathbb{P}$-a.e. ω. The QFCLT of M can, for example, be found in [3] under less restrictive assumptions compared to Assumption 1; see Proposition 1 below. In order to establish the QFCLT for X, we show that for any $T > 0$ and $\mathbb{P}$-a.e. ω,

$$\sup_{t \in [0,T]} \frac{1}{n} |\chi(\omega, n X_t^{(n)})| \to 0 \quad \text{in } \mathbf{P}_0^\omega\text{-probability as } n \to \infty;$$

see Proposition 3 below. In fact, we establish a much stronger statement: instead of proving sublinearity of χ along the path of the process X we show sublinearity everywhere

$$(6) \qquad \lim_{n \to \infty} \max_{x \in B(n)} \frac{1}{n} |\chi(\omega, x)| = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega;$$

see Proposition 2 below. The proof of (6) relies on the following deterministic regularity result for $\mathcal{L}^\omega$-harmonic functions.

THEOREM 2. *Fix $d \geq 3$, $\omega \in \Omega$ and let $p, q \in (1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$. Then there exists $c = c(d, p, q) \in [1, \infty)$ such that solutions of $\nabla^*(\omega \nabla u) = 0$ in $\mathbb{Z}^d$ satisfy for every $y \in \mathbb{Z}^d$ and every $n \in \mathbb{N}$,*

$$\tag{7} \max_{x \in B(y,n)} |u(x)| \leq c \Lambda^\omega \big(B(y, 2n)\big)^{p' \frac{\delta+1}{\delta}} \|u\|_{\underline{L}^1(B(y,2n))},$$

where $\delta := \frac{1}{d-1} - \frac{1}{2p} - \frac{1}{2q} > 0$, $p' := \frac{p}{p-1}$ and for every bounded set $S \subset \mathbb{Z}^d$,

$$\tag{8} \Lambda^\omega(S) := \|\omega\|_{\underline{L}^p(S)} \|\omega^{-1}\|_{\underline{L}^q(S)}.$$

REMARK 2. A continuum version of Theorem 2 was recently proven by the authors of the present paper in [11]. In the continuum case, relation $\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$ is essentially optimal for local boundedness (see [23]) and so it is in the discrete setting considered here; see Remark 4 below. In [3], a version of Theorem 2 is proven for solutions of the Poisson equation

$$\tag{9} \nabla^*(\omega \nabla u) = \nabla^*(\omega \nabla f)$$

on rather general weighted graphs but under the more restrictive relation $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$; see [3], Theorem 3.7 (for related classical results in the continuum see [33, 36, 37]). This regularity statement is then applied in [3] to the corrector equation (5) to ensure (6). Our method does not directly apply to solutions of (9) but due to the specific form of the right-hand side in the corrector equation (5), that is, $f(x) = x \cdot e_j$, we are able to deduce from Theorem 2 the needed sublinearity of the corrector. In Proposition 4 below, we present a version of Theorem 2 in the case $d = 2$ with the minimal requirement $p = q = 1$.

REMARK 3. In [11], we also establish Harnack inequality for nonnegative solutions u and we expect that this can be extended to the discrete case, too. In [4], Andres, Deuschel and Slowik establish elliptic and parabolic versions of Harnack inequality for the CSRW; see Remark 1, on weighted graphs under moment conditions (3) with $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$. From the parabolic version, they deduced a quenched local limit theorem and showed that condition $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$ is essentially optimal for that result. It is an interesting question if the methods developed here can be used to derive parabolic Harnack inequality and local limit theorems for the VSRW under less restrictive relations between the exponents p and q compared to the CSRW.

We provide the proof of Theorem 2 in Section 3.2. Similar as in [3, 36], the argument relies on a modification of the Moser iteration method [30, 31]. The argument in [3], Theorem 3.7, relies on two key observations: By a combination of the Hölder and Sobolev inequalities, we find $\kappa = \kappa(d, q) > 1$ (in fact $\frac{1}{\kappa} = 1 + \frac{1}{q} - \frac{2}{d}$) such that for any u with compact support in $B(n)$

$$\tag{10} \|u^2\|_{\underline{L}^\kappa(B(n))} \leq c n^2 \|\omega^{-1}\|_{\underline{L}^q(B(n))} \|\omega(\nabla u)^2\|_{\underline{L}^1(B(n))}.$$

Moreover, by testing the equation $\mathcal{L}^\omega u = 0$ with $u\eta^2$, where η is a suitable cut-off function satisfying $\eta = 1$ in $B(n)$ with compact support in $B(2n)$, we can bound the right-hand side of (10) with help of the following Caccioppoli inequality:

$$\tag{11} \|\omega(\nabla u)^2\|_{\underline{L}^1(B(n))} \leq c \|\nabla \eta\|_{L^\infty(B(2n))}^2 \|\omega\|_{\underline{L}^p(B(2n))} \|u^2\|_{\underline{L}^{\frac{p}{p-1}}(B(2n))}.$$

Choosing a linear cut-off function, that is, $\|\nabla \eta\|_{L^\infty(B(2n))} \leq cn^{-1}$, estimates (10) and (11) contain an improvement in summability of u if and only if $\kappa > \frac{p}{p-1}$ which is equivalent to $\frac{1}{p} +$

$\frac{1}{q} < \frac{2}{d}$. This tiny improvement suffices to obtain local boundedness of u by Moser's iteration argument; see [3, 36]. In the situation of Theorem 2, that is, with the relaxed assumption $\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$, we do not have a weighted Poincaré inequality in the form of (10) at hand. However, a version of (10) is valid if we replace the d-dimensional box $B(n)$ by $(d-1)$-dimensional "spheres" $\partial B(n)$. In order to exploit this observation, we need an additional optimization step compared to the Caccioppoli inequality (11) which is gathered in Lemma 2 given in Section 3.1.

1.3. *Notation.*

- (Sets and L^p spaces) For $y \in \mathbb{Z}^d$, $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, we set $B(y, n) := y + ([-n, n] \cap \mathbb{Z})^d$ with the shorthand $B(n) = B(0, n)$. For any $S \subset \mathbb{Z}^d$, we denote by $S_{\mathbb{B}^d} \subset \mathbb{B}^d$ the set of bonds for which both end-points are contained in S, that is, $S_{\mathbb{B}^d} := \{e = \{\underline{e}, \overline{e}\} \in \mathbb{B}^d \mid \underline{e}, \overline{e} \in S\}$. For any $S \subset \mathbb{Z}^d$, we set $\partial S := \{x \in S \mid \exists y \in \mathbb{Z}^d \setminus S \text{ s.t. } \{x, y\} \in \mathbb{B}^d\}$. Given $p \in (0, \infty)$, $S \subset \mathbb{Z}^d$, we set for any $f : \mathbb{Z}^d \to \mathbb{R}^d$ and $F : \mathbb{B}^d \to \mathbb{R}$,

$$\|f\|_{L^p(S)} := \left(\sum_{x \in S} |f(x)|^p \right)^{\frac{1}{p}}, \qquad \|F\|_{L^p(S_{\mathbb{B}^d})} := \left(\sum_{e \in S_{\mathbb{B}^d}} |F(e)|^p \right)^{\frac{1}{p}},$$

and $\|f\|_{L^\infty(S)} = \sup_{x \in S} |f(x)|$. Moreover, normalized versions of $\| \cdot \|_{L^p}$ are defined for any finite subset $S \subset \mathbb{Z}^d$ and $p \in (0, \infty)$ by

$$\|f\|_{\underline{L}^p(S)} := |S|^{-\frac{1}{p}} \|f\|_{L^p(S)}, \qquad \|F\|_{\underline{L}^p(S_{\mathbb{B}^d})} := |S_{\mathbb{B}^d}|^{-\frac{1}{p}} \|F\|_{L^p(S_{\mathbb{B}^d})},$$

where $|S|$ and $|S_{\mathbb{B}^d}|$ denote the cardinality of S and $S_{\mathbb{B}^d}$, respectively. Throughout the paper, we drop the subscript in $S_{\mathbb{B}^d}$ if the context is clear.
- (Discrete calculus) For any bond $e \in \mathbb{B}^d$, we denote by $\underline{e}, \overline{e} \in \mathbb{Z}^d$ the (unique) vertices satisfying $e = \{\underline{e}, \overline{e}\}$ and $\overline{e} - \underline{e} \in \{e_1, \ldots, e_d\}$. For $f : \mathbb{Z}^d \to \mathbb{R}$, we define its *discrete derivative* as

$$\nabla f : \mathbb{B}^d \to \mathbb{R}, \qquad \nabla f(e) := f(\overline{e}) - f(\underline{e}).$$

For $f, g : \mathbb{Z}^d \to \mathbb{R}$, the following discrete product rule is valid:

$$(12) \qquad \nabla(fg)(e) = f(\overline{e})\nabla g(e) + g(\underline{e})\nabla f(e) = f(e)\nabla g(e) + g(e)\nabla f(e),$$

where we use for the last equality the convenient identification of a function $h : \mathbb{Z}^d \to \mathbb{R}$ with the function $h : \mathbb{B}^d \to \mathbb{R}$ defined by the corresponding arithmetic mean

$$h(e) := \frac{1}{2}(h(\overline{e}) + h(\underline{e})).$$

The *discrete divergence* is defined for every $F : \mathbb{B}^d \to \mathbb{R}$ as

$$\nabla^* F(x) := \sum_{\substack{e \in \mathbb{B}^d \\ \overline{e} = x}} F(e) - \sum_{\substack{e \in \mathbb{B}^d \\ \underline{e} = x}} F(e) = \sum_{i=1}^{d} (F(\{x - e_i, x\}) - F(\{x, x + e_i\})).$$

Note that for every $f : \mathbb{Z}^d \to \mathbb{R}$, that is, nonzero only on finitely many vertices and every $F : \mathbb{B}^d \to \mathbb{R}$ it holds

$$(13) \qquad \sum_{e \in \mathbb{B}^d} \nabla f(e) F(e) = \sum_{x \in \mathbb{Z}^d} f(x) \nabla^* F(x).$$

Finally, we observe that the generator $\mathcal{L}^\omega$ defined in (1) can be written as a second-order finite-difference operator in divergence form, in particular,

$$(14) \qquad \forall u : \mathbb{Z}^d \to \mathbb{R} \quad \mathcal{L}^\omega u(x) = -\nabla^*(\omega \nabla u)(x) \quad \text{for all } x \in \mathbb{Z}^d.$$

2. The quenched invariance principle. In this section, we prove Theorem 1. As mentioned above, we follow a well-established strategy and decompose the process X such that $M_t = X_t - \chi(\omega, X_t)$ is a martingale under $\mathbf{P}_0^\omega$ for $\mathbb{P}$-a.e. ω. It is already known that under Assumption 1 the martingale part M satisfies a QFCLT and it is left to show that the remainder $\chi(\omega, X_t)$ vanishes in a suitable sense. In Section 2.1, we recall the construction of the corrector from [3] and state the needed known results for M and χ. In Section 2.2, we use Theorem 2 to prove that the corrector is sublinear everywhere.

2.1. *Harmonic embedding and the corrector.* The construction of the corrector and the invariance principle for the martingale part can be found in the literature; see, for example, [3, 13]. For convenience, we recall the needed results.

DEFINITION 2. A random field $\Psi : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ satisfies the *cocycle property* if for $\mathbb{P}$-a.e. ω,

$$\Psi(\tau_x \omega, y - x) = \Psi(\omega, y) - \Psi(\omega, x) \quad \text{for all } x, y \in \mathbb{Z}^d.$$

We denote by L_{cov}^2 the set of functions $\Psi : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ satisfying the cocycle property such that

$$\|\Psi\|_{L_{\mathrm{cov}}^2}^2 := \mathbb{E}\left[\sum_{x \in \mathbb{Z}^d} \omega(0, x) \Psi(\omega, x)^2\right] < \infty.$$

Note that:

LEMMA 1. L_{cov}^2 *is a Hilbert-space.*

A function $\phi : \Omega \to \mathbb{R}$ is called *local* if it depends on the value of $\omega \in \Omega$ (recall $\Omega = (0, \infty)^{\mathbb{B}^d}$) at finitely many bonds $\mathrm{e} \in \mathbb{B}^d$. The horizontal derivative $D\phi : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ of ϕ is defined by

$$D\phi(\omega, x) = \phi(\tau_x \omega) - \phi(\omega), \quad x \in \mathbb{Z}^d.$$

We define the subspace L_{pot}^2 of potential random fields as

$$L_{\mathrm{pot}}^2 := \mathrm{cl}\{D\phi \mid \phi : \Omega \to \mathbb{R} \text{ local}\} \quad \text{in } L_{\mathrm{cov}}^2$$

and the subspace L_{sol}^2, of solinoidal random fields, as the orthogonal complement of L_{pot}^2 in L_{cov}^2.

The corrector is now constructed as a suitable projection. For this, we introduce the position field $\Pi : \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$ with $\Pi(\omega, x) = x$ for all $x \in \mathbb{Z}^d$ and $\omega \in \Omega$. Set $\Pi_j := \Pi \cdot e_j$ and observe that Π_j satisfies the cocycle property and $\|\Pi_j\|_{L_{\mathrm{cov}}^2}^2 = 2\mathbb{E}[\omega(0, e_j)] < \infty$. Hence, $\Pi_j \in L_{\mathrm{cov}}^2$ and we define $\chi_j \in L_{\mathrm{pot}}^2$ and $\Phi_j \in L_{\mathrm{sol}}^2$ by

$$(15) \qquad\qquad \Pi_j = \chi_j + \Phi_j \in L_{\mathrm{pot}}^2 \oplus L_{\mathrm{sol}}^2.$$

Finally, we define the corrector $\chi = (\chi_1, \ldots, \chi_d) : \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$ and the process M_t as

$$M_t := \Phi(\omega, X_t) = X_t - \chi(\omega, X_t).$$

The needed properties of M_t, Φ and χ are gathered in the following.

PROPOSITION 1. *Let $d \geq 2$ and suppose that part (i) and (ii) of Assumption 1 are satisfied. Moreover, suppose that $\mathbb{E}[\omega(\mathrm{e})] < \infty$ and $\mathbb{E}[\omega(\mathrm{e})^{-1}] < \infty$ for every $\mathrm{e} \in \mathbb{B}^d$. Then there exists $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that:*

(i) ($\mathcal{L}^\omega$-*harmonicity of* Φ) *for all* $\omega \in \Omega_1$

$$\mathcal{L}^\omega \Phi(x) = \sum_{y \in \mathbb{Z}^d} \omega(x, y)(\Phi(\omega, y) - \Phi(\omega, x)) = 0, \qquad \Phi(\omega, 0) = 0. \tag{16}$$

(ii) (*QFCLT for* M) *Set* $M_t^{(n)} := \frac{1}{n} M_{n^2 t}$, $t \geq 0$. *For all* $\omega \in \Omega_1$, *the sequence* $\{M^{(n)}\}$ *converges in law in the Skorokhod topology to a Brownian motion with a nondegenerate covariance matrix* Σ^2 *given by*

$$\Sigma_{ij}^2 = \mathbb{E}\left[\sum_{x \in \mathbb{Z}^d} \omega(0, x) \Phi_i(\omega, x) \Phi_j(\omega, x) \right].$$

(iii) (L^1-*sublinearity of* χ) *For all* $\omega \in \Omega_1$ *and any* $j \in \{1, \ldots, d\}$,

$$\lim_{n \to \infty} \frac{1}{n} \|\chi_j(\omega, \cdot)\|_{\underline{L}^1(B(n))} = 0. \tag{17}$$

Statement (i) is contained in [3], Proposition 2.3, (ii) is contained [3], Proposition 2.5, and (iii) can be easily deduced from [3], Proposition 2.9.

2.2. L^∞-*Sublinearity of the corrector.* In this section, we improve the sublinearity of the corrector in L^1 (see Proposition 1 part (iii)) to sublinearity in the L^∞-sense. This is content of the following.

PROPOSITION 2. *Let* $d \geq 3$ *and suppose that Assumptions* 1 *is satisfied. Then, for any* $L \in \mathbb{N}$ *and* $j \in \{1, \ldots, d\}$,

$$\lim_{n \to \infty} \frac{1}{n} \max_{x \in B(Ln)} |\chi_j(\omega, x)| = 0 \quad \mathbb{P}\text{-}a.s. \tag{18}$$

REMARK 4. In [3], the sublinearity of the corrector in the form (18) is shown under moment conditions (3) with the more restrictive relation $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$. In two dimensions, (18) is proven in [13] under the minimal assumptions $p = q = 1$, and thus we focus here on $d \geq 3$ (see however Appendix B for a discussion of the case $d = 2$). We emphasize that Assumption 1 is essentially optimal for the conclusion of Proposition 2. Indeed, it was recently shown by Biskup and Kumagai [14] that the statement of Proposition 2 fails if (3) only holds for p, q satisfying $\frac{1}{p} + \frac{1}{q} > \frac{2}{d-1}$ provided $d \geq 4$; see [14], Theorem 2.7. This nonexistence of a sublinear corrector implies that the condition $\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$ in Theorem 2 is essentially sharp. Indeed, if estimate (7) were valid for some $p, q \in [1, \infty]$, then the proof of Proposition 2 together with Proposition 1 yield (18) which contradicts the findings in [14] if $\frac{1}{p} + \frac{1}{q} > \frac{2}{d-1}$.

PROOF OF PROPOSITION 2. Throughout the proof, we write $\lesssim$ if $\leq$ holds up to a positive constant which depends only on d, p and q. Before we give the details of the proof, we briefly explain the idea. We introduce an additional length-scale $\frac{n}{m}$ with $m \in \mathbb{N}$ such that $1 \ll m \ll n$ and compare χ_j on boxes with diameter $\sim \frac{n}{m}$ with $\mathcal{L}^\omega$-harmonic functions $\Phi_j - (e_j \cdot x - c)$ with a suitable chosen $c \in \mathbb{R}$. Using the L^1-sublinearity of χ_j and the fact that the linear part coming from $e_j \cdot x$ can be controlled by $\frac{1}{m}$ on each box of radius $\sim \frac{n}{m}$, we obtain the desired claim.

Step 1. As a preliminarily step, we recall the needed input from ergodic theory. Following [3], we introduce the following measures μ^ω and ν^ω on $\mathbb{Z}^d$:

$$\mu^\omega(x) = \sum_{\substack{y \in \mathbb{Z}^d \\ |y-x|=1}} \omega(x, y) \quad \text{and} \quad \nu^\omega(x) = \sum_{\substack{y \in \mathbb{Z}^d \\ |y-x|=1}} \frac{1}{\omega(x, y)}. \tag{19}$$

In view of the spatial ergodic theorem, we obtain from the moment condition (3) that there exists $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for $\omega \in \Omega'$ and every $z \in \mathbb{Z}^d$

$$
\begin{aligned}
(20) \qquad & \lim_{n \to \infty} \|\mu^\omega\|^p_{\underline{L}^p(B(nz,n))} = \mathbb{E}[\mu(0)^p] \quad \text{and} \\
& \lim_{n \to \infty} \|\nu^\omega\|^q_{\underline{L}^q(B(nz,n))} = \mathbb{E}[\nu(0)^q];
\end{aligned}
$$

see, for example, [26], Section 6.

Step 2. We set $\Omega_2 := \Omega_1 \cap \Omega'$, where Ω' is given as in Step 1 and Ω_1 in Proposition 1. Clearly, Ω_2 has full measure. From now on, we fix $\omega \in \Omega_2$.

Fix $m \in \mathbb{N}$. For n sufficiently large compared to m (the choice $n \geq m(m+1)$ will do), we cover the box $B(n)$ with finitely many boxes $B(\lfloor \frac{n}{m} \rfloor z, \lfloor \frac{n}{m} \rfloor)$, $z \in B(m)$. For $z \in B(m)$, set $u_j^z(\omega, x) := \chi_j(\omega, x) - e_j \cdot (x - \lfloor \frac{n}{m} \rfloor z) = -\Phi_j(\omega, x) + e_j \cdot \lfloor \frac{n}{m} \rfloor z$. Obviously, (16) implies that u_j^z is $\mathcal{L}^\omega$-harmonic. Hence, (7) yields

$$
\begin{aligned}
(21) \qquad & \|u_j^z\|_{L^\infty(B(\lfloor \frac{n}{m} \rfloor z, \lfloor \frac{n}{m} \rfloor))} \\
& \lesssim \Lambda^\omega \left(B\left(\left\lfloor \frac{n}{m} \right\rfloor z, 2 \left\lfloor \frac{n}{m} \right\rfloor \right) \right)^{p'(1+\frac{1}{\delta})} \|u_j^z\|_{\underline{L}^1(B(\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))} \\
& \lesssim \Lambda^\omega \left(B\left(\left\lfloor \frac{n}{m} \right\rfloor z, 2 \left\lfloor \frac{n}{m} \right\rfloor \right) \right)^{p'(1+\frac{1}{\delta})} \left(\|\chi_j\|_{\underline{L}^1(B(\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))} + \left\lfloor \frac{n}{m} \right\rfloor \right),
\end{aligned}
$$

where $p' = \frac{p}{p-1}$, $\delta = \frac{1}{d-1} - \frac{1}{2p} - \frac{1}{2q} > 0$. Estimate (21) implies the following L^∞-estimate on χ_j:

$$
\begin{aligned}
(22) \qquad & \|\chi_j\|_{L^\infty(B(n))} \\
& \leq \sup_{z \in B(m)} \|\chi_j\|_{L^\infty(B(\lfloor \frac{n}{m} \rfloor z, \lfloor \frac{n}{m} \rfloor))} \\
& \lesssim \sup_{z \in B(m)} \|u_j^z\|_{L^\infty(B(\lfloor \frac{n}{m} \rfloor z, \lfloor \frac{n}{m} \rfloor))} + \left\lfloor \frac{n}{m} \right\rfloor \\
& \overset{(21)}{\lesssim} \sup_{z \in B(m)} \Lambda^\omega \left(B\left(\left\lfloor \frac{n}{m} \right\rfloor z, 2 \left\lfloor \frac{n}{m} \right\rfloor \right) \right)^{p'(1+\frac{1}{\delta})} \\
& \qquad \times \left(\|\chi_j\|_{\underline{L}^1(B(\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))} + \left\lfloor \frac{n}{m} \right\rfloor \right) + \left\lfloor \frac{n}{m} \right\rfloor \\
& \lesssim \left(m^d \|\chi_j\|_{\underline{L}^1(B(2n))} + \left\lfloor \frac{n}{m} \right\rfloor \right) \\
& \qquad \times \sup_{z \in B(m)} \Lambda^\omega \left(B\left(\left\lfloor \frac{n}{m} \right\rfloor z, 2 \left\lfloor \frac{n}{m} \right\rfloor \right) \right)^{p'(1+\frac{1}{\delta})} + \left\lfloor \frac{n}{m} \right\rfloor.
\end{aligned}
$$

Since $B(m)$ is a finite set, we obtain from the definition of μ and ν (see (19)), and the spatial ergodic theorem in the form (20) that

$$
\begin{aligned}
(23) \qquad & \limsup_{n \to \infty} \max_{z \in B(m)} \Lambda^\omega \left(B\left(\left\lfloor \frac{n}{m} \right\rfloor z, 2 \left\lfloor \frac{n}{m} \right\rfloor \right) \right) \\
& \lesssim \limsup_{n \to \infty} \max_{z \in B(m)} \|\mu^\omega\|_{\underline{L}^p(B(2\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))} \|\nu^\omega\|_{\underline{L}^q(B(2\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))} \\
& \leq \mathbb{E}[\mu(0)^p]^{\frac{1}{p}} \mathbb{E}[\nu(0)^q]^{\frac{1}{q}} < \infty.
\end{aligned}
$$

Finally, we combine (22) and (23) with the L^1-sublinearity of χ_j (17) and obtain

$$\limsup_{n\to\infty} \frac{1}{n}\|\chi_j\|_{L^\infty(B(n))} \lesssim m^{-1}\big(\mathbb{E}[\mu(0)^p]^{\frac{1}{p}}\mathbb{E}[\nu(0)^q]^{\frac{1}{q}}\big)^{p'(1+\frac{1}{\delta})} + m^{-1}.$$

The arbitrariness of $m \in \mathbb{N}$ implies (18) for $L = 1$ and the trivial identity

$$\lim_{n\to\infty} \frac{1}{n} \max_{x\in B(Ln)} |\chi_j(\omega,x)| = L \lim_{n\to\infty} \frac{1}{n} \max_{x\in B(n)} |\chi_j(\omega,x)| = 0$$

completes the proof. $\square$

2.3. *Proof of Theorem* 1. With the help of Propositions 1 and 2, we can establish Theorem 1 following the argument in [3]. First, we observe that Proposition 2 implies the following.

PROPOSITION 3. *Let* $T > 0$. *For* $\mathbb{P}$-*a.e.* ω,

$$\sup_{t\in[0,T]} \frac{1}{n}|\chi(\omega,nX_t^{(n)})| \to 0 \quad in\ \mathbf{P}_0^\omega\text{-}probability\ as\ n \to \infty.$$

PROOF. Appealing to Proposition 2, we can follow verbatim the argument of the proof given [3], Proposition 2.13. $\square$

PROOF OF THEOREM 1. A combination of Proposition 1 (part (ii)) and Proposition 3 yields the desired claim. $\square$

3. Local boundedness for $\mathcal{L}^\omega$-harmonic functions.

3.1. *An auxiliary lemma.* In this section, we provide a key estimate, formulated in Lemma 2 below, that is central in our proof of Theorem 2. Before we come to this lemma, we recall suitable versions of the Sobolev inequality.

THEOREM 3. *Fix* $d \geq 2$. *For every* $s \in [1, d)$, *set* $s_d^* := \frac{ds}{d-s}$.

(i) *For every* $s \in [1, d)$, *there exists* $c = c(d, s) \in [1, \infty)$ *such that for every* $f : \mathbb{Z}^d \to \mathbb{R}$ *it holds*

$$(24) \qquad \big\|f - (f)_{B(n)}\big\|_{L^{s_d^*}(B(n))} \leq c\|\nabla f\|_{L^s(B(n))},$$

where $(f)_{B(n)} := \frac{1}{|B(n)|}\sum_{x\in B(n)} f(x)$.

(ii) *For every* $s \in [1, d-1)$, *there exists* $c = c(d, s) \in [1, \infty)$ *such that for every* $f : \mathbb{Z}^d \to \mathbb{R}$ *it holds*

$$(25) \qquad \|f\|_{L^{s_{d-1}^*}(\partial B(n))} \leq c\big(\|\nabla f\|_{L^s(\partial B(n))} + n^{-1}\|f\|_{L^s(\partial B(n))}\big).$$

PROOF. The above statements are standard. Since we did not find a textbook reference for the discrete situation considered here we provide the argument for some parts of the statement. In what follows, we write $\lesssim$ if $\leq$ holds up to a positive constant that depends only on the dimension d.

Step 1. Proof of part (i). For $s \in (1, d)$, the proof of the claim can be found in [32], Theorem 2.6. It is left to consider the case $s = 1$. In [16], it is proven that for any $f : \mathbb{Z}^d \to \mathbb{R}$ with finite support it holds

$$(26) \qquad \left(\sum_{x\in\mathbb{Z}^d} |f(x)|^{\frac{d}{d-1}}\right)^{\frac{d-1}{d}} \lesssim \sum_{e\in\mathbb{B}^d} |\nabla f(e)|,$$

and for any $f : \mathbb{Z}^d \to \mathbb{R}$

$$(27) \qquad \big\| f - (f)_{B(n)} \big\|_{L^1(B(n))} \lesssim |B(n)|^{\frac{1}{d}} \|\nabla f\|_{L^1(B(n))};$$

see [16], Lemma 2.1 and 2.2. We deduce (24) (with $s = 1$) from (26) and (27) by a simple extension argument. Indeed, functions defined on a box $B(n)$ can easily extended by successive reflections (see, e.g., [18], Section 9.2). In particular, there exists $k = k(d) \in \mathbb{N} \setminus \{1, 2\}$ such that for every $g : B(n) \to \mathbb{R}$ we find $\bar{g} : B(kn) \to \mathbb{R}$ such that

$$(28) \qquad \bar{g}(x) = g(x) \quad \text{in } B(n), \qquad \|\bar{g}\|_{L^1(B(kn))} \lesssim \|g\|_{L^1(B(n))},$$
$$\|\nabla \bar{g}\|_{L^1(B(kn))} \lesssim \|\nabla g\|_{L^1(B(n))}.$$

Choose $g := f - (f)_{B(n)}$ and consider a cut-off function $\eta : \mathbb{Z}^d \to [0, 1]$ satisfying

$$(29) \qquad \eta(x) = \begin{cases} 1 & \text{if } x \in B(n), \\ 0 & \text{if } x \in \mathbb{Z}^d \setminus B(3n-1), \end{cases} \qquad |\nabla \eta(e)| \lesssim n^{-1} \quad \text{for all } e \in \mathbb{B}^d.$$

Then

$$\big\| f - (f)_{B(n)} \big\|_{L^{\frac{d}{d-1}}(B(n))} \overset{(28),(29)}{\leq} \|\eta \bar{g}\|_{L^{\frac{d}{d-1}}(B(kn))}$$

$$\overset{(26)}{\lesssim} \|\nabla(\eta\bar{g})\|_{L^1(B(kn))}$$

$$\overset{(12),(29)}{\lesssim} \|\nabla\bar{g}\|_{L^1(B(kn))} + n^{-1}\|\bar{g}\|_{L^1(B(kn))}$$

$$\overset{(28)}{\lesssim} \|\nabla f\|_{L^1(B(n))} + n^{-1}\big\| f - (f)_{B(n)} \big\|_{L^1(B(n))}$$

$$\overset{(27)}{\lesssim} \|\nabla f\|_{L^1(B(n))},$$

where in the last estimate we used $|B(n)|^{\frac{1}{d}} \lesssim n$.

Step 2. Proof of part (ii). Consider a facet F of $\partial B(n)$ given by $\{x \in B(n) \mid x \cdot e_j = tn\}$ for some $j \in \{1, \ldots, d\}$ and $t \in \{-1, 1\}$. Then, appealing to part (i), we find $c = c(d, s) \in [1, \infty)$ such that

$$\|f\|_{L^{s^*_{d-1}}(F)} \leq \big\| f - (f)_F \big\|_{L^{s^*_{d-1}}(F)} + \big\| (f)_F \big\|_{L^{s^*_{d-1}}(F)}$$

$$(30) \qquad \qquad \leq c\|\nabla f\|_{L^s(F)} + |F|^{\frac{1}{s} - \frac{1}{d-1}} \big| (f)_F \big|$$

$$\leq c\|\nabla f\|_{L^s(F)} + |F|^{-\frac{1}{d-1}} \|f\|_{L^s(F)}.$$

Summing (30) over all facets F and using $|F| = (2n-1)^{d-1}$, we obtain (25). $\square$

LEMMA 2. *Fix* $d \geq 3$, $\omega \in \Omega$, $\rho, \sigma \in \mathbb{N}$ *with* $\rho < \sigma$ *and* $v : \mathbb{Z}^d \to \mathbb{R}$. *Consider*

$$J(\rho, \sigma, v) := \inf\bigg\{ \sum_{e \in \mathbb{B}^d} \omega(e)\big(|v|(e)\big)^2 \big(\nabla\eta(e)\big)^2 \,\Big|\, \eta : \mathbb{Z}^d \to [0, \infty),$$

$$\eta = 1 \text{ in } B(\rho) \text{ and } \eta = 0 \text{ in } \mathbb{Z}^d \setminus B(\sigma - 1)\bigg\},$$

where $|v|(e) = \frac{1}{2}(|v(\underline{e})| + |v(\overline{e})|)$. *Then there exists* $c = c(d, p) \in [1, \infty)$ *such that*

(31)
$$
\begin{aligned}
&J(\rho, \sigma, v) \\
&\quad \leq c \frac{\|\omega\|_{L^p(B(\sigma)\setminus B(\rho-1))}}{(\sigma - \rho)^{\frac{2d}{d-1}}} \left(\|\nabla v\|_{L^{p*}(B(\sigma)\setminus B(\rho-1))}^2 + \rho^{-2} \|v\|_{L^{p*}(B(\sigma)\setminus B(\rho-1))}^2 \right),
\end{aligned}
$$

where p_* *is given by* $\frac{1}{p_*} = \frac{1}{2} - \frac{1}{2p} + \frac{1}{d-1}$.

PROOF.   *Step 1.* We claim

(32)
$$
J(\rho, \sigma, v) \leq (\sigma - \rho)^{-(1+\frac{1}{\gamma})} \left(\sum_{k=\rho}^{\sigma-1} \left(\sum_{e \in S(k)} \omega(e)|v|(e)^2 \right)^\gamma \right)^{\frac{1}{\gamma}}
$$

for every $\gamma > 0$,

where for every $m \in \mathbb{N}$

$$
S(m) := \left\{ e \in \mathbb{B}^d \mid \underline{e} \in \partial B(m), \overline{e} \in \partial B(m+1) \right\}.
$$

Restricting the class of admissible cut-off functions to those of the form $\eta(x) = \hat{\eta}(\max_{i=1,\dots,d}\{|x \cdot e_i|\})$, we obtain

(33)
$$
\begin{aligned}
J(\rho, \sigma, v) \leq \inf \Bigg\{ &\sum_{k=\rho}^{\sigma-1} \hat{\eta}'(k)^2 \left(\sum_{e \in S(k)} \omega(e)\big(|v|(e)\big)^2 \right) \,\Big|\, \hat{\eta} : \mathbb{N} \to [0, \infty), \\
&\hat{\eta}(\rho) = 1, \hat{\eta}(\sigma) = 0 \Bigg\} =: J_{1d},
\end{aligned}
$$

where $\hat{\eta}'(k) := \hat{\eta}(k+1) - \hat{\eta}(k)$. The minimization problem (33) can be solved explicitly. Indeed, set $f(k) := \sum_{e \in S(k)} \omega(e)\big(|v|(e)\big)^2$ for every $k \in \mathbb{Z}$ and suppose $f(k) > 0$ for every $k \in \{\rho, \dots, \sigma - 1\}$. Then $\hat{\eta} : \mathbb{N} \to [0, \infty)$ with $\hat{\eta}(i) = 1 - (\sum_{k=\rho}^{\sigma-1} f(k)^{-1})^{-1} \sum_{k=\rho}^{i-1} f(k)^{-1}$ is a valid competitor in the minimization problem for J_{1d} and we obtain

$$
J(\rho, \sigma, v) \leq \left(\sum_{k=\rho}^{\sigma-1} \left(\sum_{e \in S(k)} \omega(e)\big(|v|(e)\big)^2 \right)^{-1} \right)^{-1}.
$$

By Hölder's inequality, we obtain for any $s > 1$ that $\sigma - \rho = \sum_{k=\rho}^{\sigma-1} (\frac{f}{f})^{\frac{1}{s'}} \leq (\sum_{k=\rho}^{\sigma-1} f^{\frac{s}{s'}})^{\frac{1}{s}} \times (\sum_{k=\rho}^{\sigma-1} \frac{1}{f})^{\frac{1}{s'}}$ with $s' = \frac{s}{s-1}$, and thus

$$
J_{1d} \leq (\sigma - \rho)^{-s'} \left(\sum_{k=\rho}^{\sigma-1} \left(\sum_{e \in S(k)} \omega(e)\big(|v|(e)\big)^2 \right)^{\frac{s}{s'}} \right)^{\frac{s'}{s}}.
$$

The claim (32) follows with $\gamma = s - 1 > 0$. Finally, in the case that $f(k) = \sum_{e \in S(k)} \omega(e) \times (|v|(e))^2 = 0$ for some $k \in \{\rho, \dots, \sigma - 1\}$, we easily obtain $J_{1d} = 0$ and (32) is trivially satisfied.

Step 2. We estimate the right-hand side of (32) with the help of the Hölder inequality and the Sobolev inequality in the form (25). More precisely, there exists $c = c(p, d) \in [1, \infty)$

(changing from line to line) such that

$$(\sigma - \rho)^{1+\frac{1}{\gamma}} J(\rho, \sigma, v)$$

$$\leq c\left(\sum_{k=\rho}^{\sigma-1}\left(\sum_{e\in S(k)}\omega(e)^p\right)^{\frac{\gamma}{p}}\left(\sum_{\substack{x\in\partial B(k)\\x\in\partial B(k+1)}}|v(x)|^{\frac{2p}{p-1}}\right)^{\frac{(p-1)\gamma}{p}}\right)^{\frac{1}{\gamma}}$$

$$\leq 2^{1-\frac{1}{p}}c\left(\sum_{k=\rho}^{\sigma-1}\left(\sum_{e\in S(k)}\omega(e)^p\right)^{\frac{\gamma}{p}}\left(\|v\|^{2\gamma}_{L^{\frac{2p}{p-1}}(\partial B(k))} + \|v\|^{2\gamma}_{L^{\frac{2p}{p-1}}(\partial B(k+1))}\right)\right)^{\frac{1}{\gamma}}$$

$$\overset{(25)}{\leq} c\left(\sum_{k=\rho}^{\sigma-1}\left(\sum_{e\in S(k)}\omega(e)^p\right)^{\frac{\gamma}{p}}\sum_{i=k}^{k+1}\left(\|\nabla v\|^{2\gamma}_{L^{p*}(\partial B(i))} + i^{-2\gamma}\|v\|^{2\gamma}_{L^{p*}(\partial B(i))}\right)\right)^{\frac{1}{\gamma}},$$

where $\frac{1}{2} - \frac{1}{2p} = \frac{1}{p_*} - \frac{1}{d-1}$ (note that $(p_*)^*_{d-1} = \frac{2p}{p-1}$). The choice $\gamma = \frac{d-1}{d+1}$ yields $\frac{\gamma}{p} + \frac{2\gamma}{p_*} = 1$, and thus by Hölder's inequality we obtain (31) for some $c = c(d, p) \in [1, \infty)$. □

3.2. *Proof of Theorem* 2. We first present a weaker version of Theorem 2 in which the right-hand side of the estimate (7) is replaced by a slightly larger term.

THEOREM 4. *Fix $d \geq 3$, $\omega \in \Omega$ and let $p, q \in (1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$. Then there exists $c = c(d, p, q) \in [1, \infty)$ such that solutions of $\nabla^*(\omega\nabla u) = 0$ in $\mathbb{Z}^d$ satisfy for every $n \in \mathbb{N}$,*

$$\max_{x\in B(n)}|u(x)| \leq c\Lambda^\omega\big(B(4n)\big)^{\frac{\delta+1}{2\delta}}\|u\|_{\underline{L}^{2p'}(B(4n))},$$

where $\delta = \frac{1}{d-1} - \frac{1}{2p} - \frac{1}{2q} > 0$, $p' = \frac{p}{p-1}$ and Λ^ω is defined in (8).

PROOF. Throughout the proof, we write $\lesssim$ if $\leq$ holds up to a positive constant that depends only on d, p and q. For a function $v : \mathbb{Z}^d$ and $\alpha \geq 1$, we set

$$\tilde{v}_\alpha := (\text{sign } v)|v|^\alpha.$$

The proof is divided in three steps: First, by testing $\mathcal{L}^\omega u = 0$ with $\eta^2\tilde{u}_{2\alpha-1}$, where η is a suitable cut-off function, and using Lemma 2 we obtain a crucial a priori estimate for $\|\nabla\tilde{u}_\alpha\|_{L^{\frac{2q}{q+1}}}$. In the second step, we refine this a priori estimate to the one-step improvement (40), where a suitable Sobolev-type norm of $\tilde{u}_{\chi\alpha}$ with $\chi > 1$ is bounded by the corresponding norm of $\tilde{u}_\alpha$ on a slightly larger ball. With help of (40), we obtain the desired claim by an iteration argument similar to [3], proof of Theorem 3.7.

Step 1. Basic energy estimate.
    We claim that for every $\alpha \geq 1$ and $n, \rho, \sigma \in \mathbb{N}$ with $n \leq \rho < \sigma \leq 2n$ it holds

$$(34) \qquad \|\nabla\tilde{u}_\alpha\|_{\underline{L}^{\frac{2q}{q+1}}(B(\rho))} \lesssim \alpha\frac{\Lambda^\omega(B(2n))^{\frac{1}{2}}}{(1-\frac{\rho}{\sigma})^{\frac{d}{d-1}}}\big(\rho^{-1}\|\tilde{u}_\alpha\|_{\underline{L}^{p*}(B(\sigma))} + \|\nabla\tilde{u}_\alpha\|_{\underline{L}^{p*}(B(\sigma))}\big),$$

$$(35) \qquad \|\nabla u\|_{\underline{L}^{\frac{2q}{q+1}}(B(\rho))} \lesssim \frac{\Lambda^\omega(B(2n))^{\frac{1}{2}}}{\sigma - \rho}\|u\|_{\underline{L}^{2p'}(B(\sigma))},$$

where $\frac{1}{p_*} = \frac{1}{2} - \frac{1}{2p} + \frac{1}{d-1}$ and $p' = \frac{p}{p-1}$.

Substep 1.1. Let $\eta : \mathbb{Z}^d \to [0, \infty)$ be such that $\eta = 0$ in $\mathbb{Z}^d \setminus B(2n - 1)$. We claim that for every $\alpha \geq 1$,

$$(36) \qquad \sum_{e \in \mathbb{B}^d} \eta^2(e)\omega(e)\nabla\tilde{u}_\alpha(e)^2 \leq \frac{256\alpha^4}{(2\alpha - 1)^2} \sum_{e \in \mathbb{B}^d} \omega(e)\big(|u|^\alpha(e)\big)^2\big(\nabla\eta(e)\big)^2,$$

where $\eta^2(e) = \frac{1}{2}(\eta^2(\overline{e}) + \eta^2(\underline{e}))$ and $|u|^\alpha(e) = \frac{1}{2}(|u(\overline{e})|^\alpha + |u(\underline{e})|^\alpha)$; see Section 1.3. Using $\mathcal{L}^\omega u \overset{(14)}{=} -\nabla^*(\omega\nabla u) = 0$ in $\mathbb{Z}^d$ and the summation by parts formula (13) with $F = \omega\nabla u$ and $f = \eta^2\tilde{u}_{2\alpha-1}$, we obtain

$$(37) \qquad \begin{aligned} 0 &= \sum_{e \in \mathbb{B}^d} \omega(e)\nabla u(e)\nabla\big(\eta^2\tilde{u}_{2\alpha-1}\big)(e) \\ &= \sum_{e \in \mathbb{B}^d} 2\eta(e)\tilde{u}_{2\alpha-1}(e)\omega(e)\nabla u(e)\nabla\eta(e) \\ &\quad + \sum_{e \in \mathbb{B}^d} \eta^2(e)\omega(e)\nabla u(e)\nabla\tilde{u}_{2\alpha-1}(e), \end{aligned}$$

where we use the discrete chain rule (12) and $\nabla\eta^2(e) = \eta^2(\overline{e}) - \eta^2(\underline{e}) = (\eta(\overline{e}) - \eta(\underline{e}))(\eta(\overline{e}) + \eta(\underline{e})) = 2\nabla\eta(e)\eta(e)$. Estimate (51) implies $\nabla\tilde{u}_\alpha(e)^2 \leq \frac{\alpha^2}{2\alpha-1}(\nabla u(e)\nabla\tilde{u}_{2\alpha-1}(e))$ for all $e \in \mathbb{B}^d$, and thus

$$(38) \qquad \sum_{e \in \mathbb{B}^d} \eta^2(e)\omega(e)\nabla u(e)\nabla\tilde{u}_{2\alpha-1}(e) \geq \frac{2\alpha - 1}{\alpha^2} \sum_{e \in \mathbb{B}^d} \eta^2(e)\omega(e)\nabla\tilde{u}_\alpha(e)^2.$$

To estimate the first term on the right-hand side in (37), we use the pointwise inequality $|\tilde{u}_{2\alpha-1}|(e)|\nabla u(e)| \leq 4|\nabla\tilde{u}_\alpha(e)||u|^\alpha(e)$ (see (52)), and thus by Young's inequality (together with elementary inequality $\eta(e)^2 \leq \eta^2(e)$)

$$(39) \qquad \begin{aligned} 2\bigg|\sum_{e \in \mathbb{B}^d} &\omega(e)\nabla u(e)\eta(e)\nabla\eta(e)\tilde{u}_{2\alpha-1}(e)\bigg| \\ &\leq \frac{2\alpha - 1}{2\alpha^2} \sum_{e \in \mathbb{B}^d} \omega(e)\eta^2(e)\big(\nabla\tilde{u}_\alpha(e)\big)^2 \\ &\quad + \frac{128\alpha^2}{2\alpha - 1} \sum_{e \in \mathbb{B}^d} \omega(e)\big(|u|^\alpha(e)\big)^2\big(\nabla\eta(e)\big)^2. \end{aligned}$$

Combining (37)–(39), we obtain (36).

Substep 1.2. Proof of (34). By minimizing the right-hand side of (36) over all $\eta : \mathbb{Z}^d \to [0, \infty)$ satisfying $\eta = 1$ on $B(\rho)$ and $\eta = 0$ in $\mathbb{Z}^d \setminus B(\sigma - 1)$, we obtain in view of Lemma 2,

$$\sum_{e \in B(\rho)} \omega(e)\big(\nabla\tilde{u}_\alpha(e)\big)^2 \lesssim \frac{\alpha^2\|\omega\|_{L^p(B(\sigma))}}{(\sigma - \rho)^{\frac{2d}{d-1}}}\big(\|\nabla\tilde{u}_\alpha\|^2_{L^{p*}(B(\sigma))} + \rho^{-2}\|\tilde{u}_\alpha\|^2_{L^{p*}(B(\sigma))}\big).$$

By Hölder's inequality, we obtain

$$\begin{aligned} \|\nabla\tilde{u}_\alpha\|^2_{L^{\frac{2q}{q+1}}(B(\rho))} &\leq \|\omega^{-1}\|_{L^q(B(\rho))}\bigg(\sum_{e \in B(\rho)} \omega(e)\big(\nabla\tilde{u}_\alpha(e)\big)^2\bigg) \\ &\lesssim \frac{\alpha^2\|\omega^{-1}\|_{L^q(B(\rho))}\|\omega\|_{L^p(B(\sigma))}}{(\sigma - \rho)^{\frac{2d}{d-1}}}\big(\|\nabla\tilde{u}_\alpha\|^2_{L^{p*}(B(\sigma))} + \rho^{-2}\|\tilde{u}_\alpha\|^2_{L^{p*}(B(\sigma))}\big) \end{aligned}$$

and the claim (34) follows.

Substep 1.3. Estimate (35) is a straightforward consequence of (36) (with $\alpha = 1$ and a "linear" cut-off function η satisfying $\eta(x) = 1$ for $x \in B(\rho)$, $\eta = 0$ on $\mathbb{Z}^d \setminus B(\sigma - 1)$ and $\nabla\eta(e) = (\sigma - \rho)^{-1}$ for all $e \in S(k)$ and $k \in \{\rho, \ldots, \sigma - 1\}$) and an application of Hölder's inequality.

Step 2. One-step improvement.

Fix $\alpha \geq 1$ and $\rho, \sigma \in \mathbb{N}$ with $n \leq \rho < \sigma \leq 2n$. We claim that there exists $c = c(d, p, q) \in [1, \infty)$ such that

$$(40) \qquad \|\tilde{u}_{\chi\alpha}\|_{\underline{W}^{1,p_*}(B(\rho))}^{\frac{1}{\chi\alpha}} \leq \left(\frac{c\alpha\Lambda^\omega(B(2n))^{\frac{1}{2}}}{(1 - \frac{\rho}{\sigma})^{\frac{d}{d-1}}}\right)^{\frac{1}{\chi\alpha}} \|\tilde{u}_\alpha\|_{\underline{W}^{1,p_*}(B(\sigma))}^{\frac{1}{\alpha}},$$

where $\chi := 1 + \delta > 1$ with $\delta = \frac{1}{d-1} - \frac{1}{2q} - \frac{1}{2p} > 0$ and for $p \in [1, \infty)$ and any pair $(y, n) \in \mathbb{Z}^d \times \mathbb{N}$ the (normalized) Sobolev norm $\|\cdot\|_{\underline{W}^{1,p}(B(y,n))}$ is defined for any $v : \mathbb{Z}^d \to \mathbb{R}$ as

$$\|v\|_{\underline{W}^{1,p}(B(y,n))} := \|v\|_{\underline{L}^p(B(y,n))} + n\|\nabla v\|_{\underline{L}^p(B(y,n))}.$$

In order to establish (40), we use Step 1 and the following two estimates:

$$(41) \qquad \|\tilde{u}_{\chi\alpha}\|_{\underline{L}^{p_*}(B(\rho))} = \|\tilde{u}_\alpha\|_{\underline{L}^{\chi p_*}(B(\rho))}^\chi \lesssim \|\tilde{u}_\alpha\|_{\underline{W}^{1,p_*}(B(\rho))}^\chi,$$

$$(42) \qquad \rho\|\nabla\tilde{u}_{\chi\alpha}\|_{\underline{L}^{p_*}(B(\rho))} \lesssim \rho\|\nabla\tilde{u}_\alpha\|_{\underline{L}^{\frac{2q}{q+1}}(B(\rho))} \|\tilde{u}_\alpha\|_{\underline{L}^1(B(\rho))}^\delta.$$

Estimate (41) is a consequence of Sobolev inequality (note that $\chi \in (1, \frac{d}{d-1}]$, and thus $\frac{1}{\chi p_*} \geq \frac{1}{p_*} - \frac{1}{dp_*} \geq \frac{1}{p_*} - \frac{1}{d}$ since $p_* \geq 1$) and (42) follows from

$$\|\nabla\tilde{u}_{\alpha(1+\delta)}\|_{L^{p_*}(B(\rho))} \overset{(50)}{\leq} (1 + \delta)\left(\sum_{e \in B(\rho)} |\nabla\tilde{u}_\alpha(e)|^{p_*}(2|u|^{\alpha\delta}(e))^{p_*}\right)^{\frac{1}{p_*}}$$

$$\leq (1 + \delta)\|\nabla\tilde{u}_\alpha\|_{L^{\frac{2q}{q+1}}(B(\rho))}\left(\sum_{e \in B(\rho)} (2|u|^{\alpha\delta}(e))^{\frac{1}{\delta}}\right)^\delta.$$

Appealing to estimates (34), (42), (41) and Jensen inequality in the form $\|\cdot\|_{\underline{L}^1(B(\rho))} \leq \|\cdot\|_{\underline{L}^{p_*}(B(\rho))}$, we obtain

$$\|\tilde{u}_{\chi\alpha}\|_{\underline{W}^{1,p_*}(B(\rho))} \overset{(41),(42)}{\lesssim} \|\tilde{u}_\alpha\|_{\underline{W}^{1,p_*}(B(\rho))}^\chi$$

$$(43) \qquad\qquad + \rho\|\nabla\tilde{u}_\alpha\|_{\underline{L}^{\frac{2q}{q+1}}(B(\rho))} \|\tilde{u}_\alpha\|_{\underline{L}^{p_*}(B(\rho))}^\delta$$

$$\overset{(34)}{\lesssim} \left(\frac{\alpha\Lambda^\omega(B(2n))^{\frac{1}{2}}}{(1 - \frac{\rho}{\sigma})^{\frac{d}{d-1}}}\right)\|\tilde{u}_\alpha\|_{\underline{W}^{1,p_*}(B(\sigma))}^\chi,$$

where for the last estimate we used that $\alpha \geq 1$, $n \leq \rho < \sigma \leq 2n$, $\Lambda^\omega(B(2n)) \geq 1$ and $\chi = 1 + \delta$. Taking the $\frac{1}{\chi\alpha}$ power, we obtain (40).

Step 3. Iteration.

For $\nu \in \mathbb{N} \cup \{0\}$, set $\alpha_\nu = \chi^{\nu-1}$ and $\rho_\nu = n + \lfloor\frac{n}{2^\nu}\rfloor$. Then for any $\nu \in \mathbb{N}$ satisfying $2^\nu \leq n$, estimate (40) (with $\alpha = \alpha_\nu$, $\rho = \rho_\nu$ and $\sigma = \rho_{\nu-1}$) implies that there exists $c = c(d, p, q) \in [1, \infty)$ such that

$$(44) \qquad \|\tilde{u}_{\chi^\nu}\|_{\underline{W}^{1,p_*}(B(\rho_\nu))}^{\frac{1}{\chi^\nu}} \leq (c\Lambda^\omega(B(2n))^{\frac{1}{2}}(4\chi)^\nu)^{\frac{1}{\chi^\nu}} \|\tilde{u}_{\chi^{\nu-1}}\|_{\underline{W}^{1,p_*}(B(\rho_{\nu-1}))}^{\frac{1}{\chi^{\nu-1}}},$$

where we used the elementary estimate

$$\left(1 - \frac{\rho_v}{\rho_{v-1}}\right)^{-\frac{d}{d-1}} = \left(\frac{n + \lfloor\frac{n}{2^{v-1}}\rfloor}{\lfloor\frac{n}{2^{v-1}}\rfloor - \lfloor\frac{n}{2^v}\rfloor}\right)^{\frac{d}{d-1}} \le \left(2^{2+v}\right)^{\frac{d}{d-1}} \overset{d\ge 2}{\le} 4^{2+v}.$$

Set $\hat{v}(n) := \max\{v \in \mathbb{N} \mid 2^v \le n\}$. Using (44) $\hat{v}(n)$-times, we obtain

$$\|u\|_{\underline{L}^{p_* \chi^{\hat{v}(n)}}(B(n))} \le \prod_{v=1}^{\hat{v}(n)} \left(c\Lambda^\omega(B(2n))^{\frac{1}{2}}(4\chi)^v\right)^{\frac{1}{\chi^v}} \|u\|_{\underline{W}^{1,p_*}(B(2n))}$$

(45)

$$\le \left(c\Lambda^\omega(B(2n))^{\frac{1}{2}}\right)^{\sum_{v=1}^\infty \frac{1}{\chi^v}} (4\chi)^{\sum_{v=1}^\infty \frac{v+1}{\chi^v}} \|u\|_{\underline{W}^{1,p_*}(B(2n))}.$$

To estimate the right-hand side of (45), we use (35), Jensen's inequality and the fact that $p_* < \frac{2q}{q+1} \le 2 \le 2p'$,

$$\|\nabla u\|_{\underline{L}^{p_*}(B(2n))} \le \|\nabla u\|_{\underline{L}^{\frac{2q}{q+1}}(B(2n))} \overset{(35)}{\lesssim} n^{-1}\Lambda^\omega(B(4n))^{\frac{1}{2}} \|u\|_{\underline{L}^{2p'}(B(4n))},$$

$$\|u\|_{\underline{L}^{p_*}(B(2n))} \lesssim \|u\|_{\underline{L}^{2p'}(B(4n))}.$$

Since $\Lambda^\omega \ge 1$ and $\sum_{v=0}^\infty (1+v)\chi^{-v} \lesssim 1$, we obtain

$$\|u\|_{L^\infty(B(n))} \le |B(n)|^{\frac{1}{p_* \chi^{\hat{v}(n)}}} \|u\|_{\underline{L}^{p_* \chi^{\hat{v}(n)}}(B(n))}$$

$$\lesssim |B(n)|^{\frac{1}{p_* \chi^{\hat{v}(n)}}} \Lambda^\omega(B(2n))^{\frac{1}{2}(\frac{1}{1-\chi^{-1}}-1)} \|u\|_{\underline{W}^{1,p_*}(B(2n))}$$

$$\lesssim |B(n)|^{\frac{1}{p_* \chi^{\hat{v}(n)}}} \Lambda^\omega(B(4n))^{\frac{1}{2}\frac{\chi}{\chi-1}} \|u\|_{\underline{L}^{2p'}(B(4n))}.$$

Hence, it is left to show that $|B(n)|^{\frac{1}{\chi^{\hat{v}(n)}}} \lesssim 1$ (recall $\chi = 1+\delta$). Assuming $n \in \mathbb{N}$ is sufficiently large, we have $\hat{v}(n) \ge \frac{1}{2}\log_2 n$, and thus

(46)

$$|B(n)|^{\frac{1}{\chi^{\hat{v}(n)}}} \lesssim n^{d/\sqrt{\chi}^{\log_2 n}} = n^{d/\sqrt{n}^{\log_2 \chi}} = \exp\left(\log\left(n^{d/\sqrt{n}^{\log_2 \chi}}\right)\right)$$

$$= \exp\left(d\log(n)/\sqrt{n}^{\log_2 \chi}\right) \lesssim 1,$$

which completes the proof. □

Using a well-known iteration argument (see, e.g., [3], Corollary 3.9), we refine the statement of Theorem 4 and obtain the following.

COROLLARY 1.   *Fix $d \ge 3$, $\omega \in \Omega$ and let $p, q \in (1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$. For every $\gamma \in (0, 1]$, there exists $c = c(d, p, q, \gamma) \in [1, \infty)$ such that solutions of $\nabla^*(\omega\nabla u) = 0$ in $\mathbb{Z}^d$ satisfy for every $n \in \mathbb{N}$,*

$$\max_{x \in B(n)} |u(x)| \le c\Lambda^\omega(B(2n))^{\frac{\delta+1}{2\delta\gamma}} \|u\|_{\underline{L}^{2p'\gamma}(B(2n))},$$

where $\delta = \frac{1}{d-1} - \frac{1}{2p} - \frac{1}{2q} > 0$, $p' = \frac{p}{p-1}$ and Λ^ω is defined in (8).

PROOF OF THEOREM 2.   The choice $\gamma = \frac{1}{2p'}$ in Corollary 1 yields (7) for $y = 0 \in \mathbb{Z}^d$ and by translation we obtain the general claim. □

PROOF OF COROLLARY 1. Throughout the proof, we write $\lesssim$ if $\leq$ holds up to a positive constant that depends only on d, p and q.

Step 1. We claim that for every $N, N' \in \mathbb{N}$ with $N' < N$,

$$(47) \qquad \max_{x \in B(N')} |u(x)| \lesssim \frac{\Lambda^\omega(B(N))^{\frac{\delta+1}{2\delta}}}{(1 - \frac{N'}{N})^s} \|u\|_{\underline{L}^{\frac{2p}{p-1}}(B(N))},$$

where $s := \frac{d}{2}(1 + \frac{1}{q} + (\frac{1}{p} + \frac{1}{q})\frac{1}{\delta})$. Suppose $N - N' \geq 4$. Theorem 4 and a simple translation argument yield for every $y \in B(N')$,

$$\max_{x \in B(y, \lfloor \frac{N-N'}{4} \rfloor)} |u(x)|$$

$$\lesssim \Lambda^\omega\big(B(y, N - N')\big)^{\frac{\delta+1}{2\delta}} \|u\|_{\underline{L}^{\frac{2p}{p-1}}(B(y, N-N'))}$$

$$\lesssim \Big(\frac{N^d}{(N-N')^d}\Big)^{\frac{p-1}{2p} + (\frac{1}{q} + \frac{1}{p})\frac{\delta+1}{2\delta}} \Lambda^\omega\big(B(N)\big)^{\frac{\delta+1}{2\delta}} \|u\|_{\underline{L}^{\frac{2p}{p-1}}(B(N))}$$

and estimate (47) follows. For the remaining case $1 \leq N - N' \leq 3$, we use the discrete L^∞-L^1-estimate in the form

$$\max_{x \in B(N')} |u(x)| \leq \|u\|_{L^{\frac{2p}{p-1}}(B(N))} \lesssim N^{\frac{d}{2}(1 - \frac{1}{p})} \|u\|_{\underline{L}^{\frac{2p}{p-1}}(B(N))}$$

which implies estimate (47) (using $1 \leq N$, $\Lambda^\omega(B(N))$ and $(1 - \frac{N'}{N})^{-s} \geq (\frac{N}{3})^s$ with $\frac{d}{2}(1 - \frac{1}{p}) \leq s$).

Step 2. Iteration. Fix $\gamma \in (0, 1)$. For $\nu \in \mathbb{N} \cup \{0\}$, we set $\rho_\nu = 2n - \lfloor \frac{n}{2^\nu} \rfloor$. Combining the elementary interpolation inequality,

$$(48) \qquad \|u\|_{\underline{L}^{2p'}(B(\rho_\nu))} \leq \|u\|_{\underline{L}^{2p'\gamma}(B(\rho_\nu))}^{\gamma} \|u\|_{L^\infty(B(\rho_\nu))}^{1-\gamma}$$

with the estimate (47), we obtain for every $\nu \in \mathbb{N}$,

$$(49) \qquad \begin{aligned} \|u\|_{L^\infty(B_{\rho_{\nu-1}})} &\overset{(47)}{\lesssim} \Lambda^\omega\big(B(\rho_\nu)\big)^{\frac{\delta+1}{2\delta}} \Big(1 - \frac{\rho_{\nu-1}}{\rho_\nu}\Big)^{-s} \|u\|_{\underline{L}^{2p'}(B(\rho_\nu))} \\ &\overset{(48)}{\leq} \Lambda^\omega\big(B(\rho_\nu)\big)^{\frac{\delta+1}{2\delta}} \Big(1 - \frac{\rho_{\nu-1}}{\rho_\nu}\Big)^{-s} \\ &\quad \times \|u\|_{\underline{L}^{2p'\gamma}(B(\rho_\nu))}^{\gamma} \|u\|_{L^\infty(B(\rho_\nu))}^{1-\gamma} \\ &\leq 2^{\nu s} C \|u\|_{\underline{L}^{2p'\gamma}(B(2n))}^{\gamma} \|u\|_{L^\infty(B(\rho_\nu))}^{1-\gamma} \end{aligned}$$

with $C = c \Lambda^\omega(B(2n))^{\frac{\delta+1}{2\delta}}$ and a suitable constant $c = c(d, p, q) \in [1, \infty)$, where we used for the last estimate $\rho_\nu \geq n$ for all $\nu \in \mathbb{N}$ and $(1 - \frac{\rho_{\nu-1}}{\rho_\nu})^{-s} \leq (2^{2+\nu})^s$.

Iterating (49) from $\nu = 1$ to $\hat{\nu}(n) := \max\{\nu \in \mathbb{N} \mid 2^\nu \leq n\}$, we obtain

$$\begin{aligned} \|u\|_{L^\infty(B(n))} &= \|u\|_{L^\infty(B(\rho_0))} \\ &\overset{(49)}{\leq} 4^{s \sum_{\nu=0}^{\hat{\nu}(n)-1} (\nu+1)(1-\gamma)^\nu} \big(C \|u\|_{\underline{L}^{2p'\gamma}(B(2n))}^{\gamma}\big)^{\sum_{\nu=0}^{\hat{\nu}(n)-1}(1-\gamma)^\nu} \|u\|_{L^\infty(B(\rho_{\hat{\nu}(n)}))}^{(1-\gamma)^{\hat{\nu}(n)}}. \end{aligned}$$

Using $\sum_{\nu=0}^{\infty}(\nu + 1)(1 - \gamma)^{\nu} \lesssim 1$, $\sum_{\nu=0}^{\hat{v}(n)-1}(1 - \gamma)^{\nu} = \frac{1}{\gamma}(1 - (1 - \gamma)^{\hat{v}(n)})$, $C \geq 1$ and the discrete L^{∞}-L^1-estimate in the form $\|u\|_{L^{\infty}(B(\rho_{\hat{v}(n)}))} \leq \|u\|_{L^{2p'\gamma}(B(2n))}$, we obtain

$$\|u\|_{L^{\infty}(B(n))} \leq c\Lambda^{\omega}(B(2n))^{\frac{\delta+1}{2\delta\gamma}} \|u\|_{\underline{L}^{2p'\gamma}(B(2n))} |B(2n)|^{\frac{(1-\gamma)\hat{v}(n)}{2p'\gamma}},$$

where $c = c(\gamma, d, p, q) \in [1, \infty)$. Finally, a similar calculation as in (46) yields $|B(2n)|^{\frac{(1-\gamma)\hat{v}(n)}{2p'\gamma}} \lesssim c(\gamma) \in [1, \infty)$, which completes the proof. $\quad\square$

APPENDIX A: TECHNICAL ESTIMATES

We recall some estimates proven in [3], Lemma A.1, that we used in the proof of Theorem 2.

LEMMA 3 ([3], Lemma A.1). *For $a \in \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$, set $\tilde{a}_{\alpha} = |a|^{\alpha} \operatorname{sign} a$.*

(i) *For all $a, b \in \mathbb{R}$ and any $\alpha, \beta \neq 0$,*

$$(50) \qquad |\tilde{a}_{\alpha} - \tilde{b}_{\alpha}| \leq \left(1 \vee \left|\frac{\alpha}{\beta}\right|\right)|\tilde{a}_{\beta} - \tilde{b}_{\beta}|(|a|^{\alpha-\beta} + |b|^{\alpha-\beta}).$$

(ii) *For all $a, b \in \mathbb{R}$ and $\alpha > \frac{1}{2}$,*

$$(51) \qquad (\tilde{a}_{\alpha} - \tilde{b}_{\alpha})^2 \leq \frac{\alpha^2}{2\alpha - 1}(a - b)(\tilde{a}_{2\alpha-1} - \tilde{b}_{2\alpha-1}).$$

(iii) *For all $a, b \in \mathbb{R}$ and $\alpha \geq \frac{1}{2}$,*

$$(52) \qquad (|a|^{2\alpha-1} + |b|^{2\alpha-1})(a - b) \leq 4|\tilde{a}_{\alpha} - \tilde{b}_{\alpha}|(|a|^{\alpha} + |b|^{\alpha}).$$

APPENDIX B: THE TWO-DIMENSIONAL CASE

In two dimensions, Biskup [13] proved sublinearity of the corrector and the QFCLT under the minimal moment condition $p = q = 1$ in (3). The reasoning in [13] (which has its origins in [12]) combines geometric, analytical and probabilistic arguments. In this section, we sketch a proof of Biskups result that relies only on deterministic regularity theory and the spatial ergodic theorem. The main ingredient is the following local boundedness result.

PROPOSITION 4. *Fix $\omega \in (0, \infty)^{\mathbb{B}^2}$. Then there exists $c \in [1, \infty)$ such that solutions of $\nabla^*(\omega\nabla u) = 0$ in $\mathbb{Z}^2$ satisfy for every $n \in \mathbb{N}$,*

$$(53) \qquad \max_{x \in B(n)}|u(x)| \leq c\left(n\|\omega^{-1}\|_{\underline{L}^1(B(2n))}^{\frac{1}{2}}\|\omega(\nabla u)^2\|_{\underline{L}^1(B(2n))}^{\frac{1}{2}} + \|u\|_{\underline{L}^1(B(2n))}\right).$$

PROOF. Throughout the proof, we write $\lesssim$ if $\leq$ holds up to a generic positive constant.

The proof is elementary and relies on three ingredients: First, $\nabla^*(\omega\nabla u) = 0$ in $\mathbb{Z}^2$ implies a maximum principle in the form

$$(54) \qquad \max_{x \in B(n)}|u(x)| \leq \max_{x \in B(k)}|u(x)| \leq \max_{x \in \partial B(k)}|u(x)| \quad \text{for all } k \in \{n, \ldots, 2n\}.$$

Second, since

$$\sum_{k=n}^{2n}\left(\|\nabla u\|_{L^1(\partial B(k))} + \frac{1}{n}\|u\|_{L^1(\partial B(k))}\right) \leq \|\nabla u\|_{L^1(B(2n))} + \frac{1}{n}\|u\|_{L^1(B(2n))},$$

we can choose a "good" $\tilde{k} \in \{n, \ldots, 2n\}$ satisfying

$$(55) \qquad \|\nabla u\|_{L^1(\partial B(\tilde{k}))} + \frac{1}{n}\|u\|_{L^1(\partial B(\tilde{k}))} \leq \frac{1}{n}\left(\|\nabla u\|_{L^1(B(2n))} + \frac{1}{n}\|u\|_{L^1(B(2n))}\right).$$

The last ingredient is a one-dimensional Sobolev inequality (which follows simply by the discrete version of the fundamental theorem of calculus)

$$(56) \qquad \max_{x \in \partial B(k)}|u(x)| \lesssim \|\nabla u\|_{L^1(\partial B(k))} + \frac{1}{k}\|u\|_{L^1(\partial B(k))} \quad \text{for all } k \in \{n, \ldots, 2n\}.$$

Combining (54)–(56) and $\tilde{k} \in \{n, \ldots, 2n\}$, we obtain

$$\max_{x \in B(n)}|u(x)| \stackrel{(54)}{\leq} \max_{x \in \partial B(\tilde{k})}|u(x)| \stackrel{(56)}{\lesssim} \|\nabla u\|_{L^1(\partial B(\tilde{k}))} + \frac{1}{n}\|u\|_{L^1(\partial B(\tilde{k}))}$$

$$\stackrel{(55)}{\lesssim} n\|\nabla u\|_{\underline{L}^1(B(2n))} + \|u\|_{\underline{L}^1(B(2n))},$$

where we used in the last inequality also the fact $|B(2n)| \lesssim n^2$. Clearly, (53) follows from the last displayed formula and Hölder's inequality. $\square$

PROPOSITION 5. *Let $d = 2$ and suppose that parts* (i) *and* (ii) *of Assumptions* 1 *are satisfied. Moreover, suppose that $\mathbb{E}[\omega(e)] < \infty$ and $\mathbb{E}[\omega(e)^{-1}] < \infty$ for every $e \in \mathbb{B}^2$. Then, for every $j \in \{1, \ldots, d\}$,*

$$\lim_{n \to \infty} \frac{1}{n} \max_{x \in B(n)}|\chi_j(\omega, x)| = 0 \quad \mathbb{P}\text{-}a.s.$$

PROOF. Throughout the proof, we write $\lesssim$ if $\leq$ holds up to a generic positive constant.

Step 1. More ergodic theory. In contrast to (7), the right-hand side of (53) depends on the discrete gradient of the $\mathcal{L}^\omega$-harmonic function. In the application to the corrector equation $\nabla^*(\omega\nabla\Phi_j) = 0$, we use the ergodic theorem to control terms coming from $\nabla\Phi_j$. For this, it is convenient to introduce the following measure on $\mathbb{Z}^d$:

$$\iota_j^\omega(x) := \sum_{y \in \mathbb{Z}^d} \omega(x, y)\big(\chi_j(\omega, y) - \Pi_j(\omega, y) - (\chi_j(\omega, x) - \Pi_j(\omega, x))\big)^2$$

$$\stackrel{(15)}{=} \sum_{y \in \mathbb{Z}^d} \omega(x, y)\big(\Phi_j(\omega, y) - \Phi_j(\omega, x)\big)^2,$$

where $j \in \{1, \ldots, d\}$ and Π_j denotes the position field introduced in Section 2.1. Since Φ_j is defined as a projection of Π_j on L^2_{sol} in L^2_{cov}, we have

$$(57) \qquad \mathbb{E}[\iota_j(0)] = \mathbb{E}\left[\sum_{y \in \mathbb{Z}^d} \omega(0, y)\Phi_j(y)^2\right] = \|\Phi_j\|^2_{L^2_{\text{cov}}} \leq \|\Pi_j\|^2_{L^2_{\text{cov}}} \leq \mathbb{E}[\mu(0)],$$

where we use $\Phi_j(\omega, 0) = 0$ for every ω (which follows directly from the cocycle property). Moreover, appealing to the cocycle property of Φ_j, we have $\iota^{\tau_z\omega}(x) = \iota^\omega(x + z)$ for every $x, z \in \mathbb{Z}^d$, and thus, by (57) and the spatial ergodic theorem, we find a set $\Omega' \in \Omega$ with $\mathbb{P}[\Omega'] = 1$ such that for all $\omega \in \Omega'$ and for every $z \in \mathbb{Z}^d$,

$$(58) \qquad \lim_{n \to \infty}\|\iota_j^\omega\|_{\underline{L}^1(B(nz,n))} = \mathbb{E}[\iota_j^\omega(0)] \leq \mathbb{E}[\mu(0)] < \infty.$$

Step 2. From now on, we use the notation of Step 2 in the proof of Proposition 2. Using estimate (53) instead of (7), we obtain

$$
\begin{aligned}
(59) \quad &\|u_j^z\|_{L^\infty(B(\lfloor \frac{n}{m} \rfloor z, \lfloor \frac{n}{m} \rfloor))} \\
&\lesssim \left\lfloor \frac{n}{m} \right\rfloor \|\omega^{-1}\|_{\underline{L}^1(B(\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))}^{\frac{1}{2}} \|\omega(\nabla \chi_j - \nabla \Pi_j)^2\|_{\underline{L}^1(B(\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))}^{\frac{1}{2}} \\
&\quad + \|\chi_j\|_{\underline{L}^1(B(\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))} + \left\lfloor \frac{n}{m} \right\rfloor,
\end{aligned}
$$

instead of (21). Estimate (59) implies the following L^∞-estimate on χ_j:

$$
\begin{aligned}
(60) \quad &\|\chi_j\|_{L^\infty(B(n))} \\
&\lesssim \sup_{z \in B(m)} \|u_j^z\|_{L^\infty(B(\lfloor \frac{n}{m} \rfloor z, \lfloor \frac{n}{m} \rfloor))} + \left\lfloor \frac{n}{m} \right\rfloor \\
&\overset{(59)}{\lesssim} \left\lfloor \frac{n}{m} \right\rfloor \sup_{z \in B(m)} \left(\|\omega^{-1}\|_{\underline{L}^1(B(\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))}^{\frac{1}{2}} \|\omega(\nabla \Phi_j)^2\|_{\underline{L}^1(B(\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))}^{\frac{1}{2}} \right) \\
&\quad + \sup_{z \in B(m)} \|\chi_j\|_{\underline{L}^1(B(\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))} + \left\lfloor \frac{n}{m} \right\rfloor \\
&\lesssim \left\lfloor \frac{n}{m} \right\rfloor \sup_{z \in B(m)} \|\omega^{-1}\|_{\underline{L}^1(B(\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))}^{\frac{1}{2}} \|\omega(\nabla \Phi_j)^2\|_{\underline{L}^1(B(\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))}^{\frac{1}{2}} \\
&\quad + m^d \|\chi_j\|_{\underline{L}^1(B(2n))} + \left\lfloor \frac{n}{m} \right\rfloor.
\end{aligned}
$$

The ergodic theorem in the versions (20) and (58) implies that $\mathbb{P}$-a.s.:

$$
\begin{aligned}
(61) \quad &\limsup_{n \to \infty} \max_{z \in B(m)} \|\omega^{-1}\|_{\underline{L}^1(B(\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))}^{\frac{1}{2}} \|\omega(\nabla \Phi_j)^2\|_{\underline{L}^1(B(\lfloor \frac{n}{m} \rfloor z, 2\lfloor \frac{n}{m} \rfloor))}^{\frac{1}{2}} \\
&\lesssim \mathbb{E}[\nu(0)]^{\frac{1}{2}} \mathbb{E}[\iota_j(0)]^{\frac{1}{2}} \le \mathbb{E}[\nu(0)]^{\frac{1}{2}} \mathbb{E}[\mu(0)]^{\frac{1}{2}}.
\end{aligned}
$$

Combining (60), (61) and the L^1-sublinearity of χ_j (17), we obtain

$$
\limsup_{n \to \infty} \frac{1}{n} \|\chi_j\|_{L^\infty(B(n))} \lesssim \frac{1}{m} \left(1 + \mathbb{E}[\nu(0)]^{\frac{1}{2}} \mathbb{E}|\mu(0)|^{\frac{1}{2}} \right).
$$

The arbitrariness of $m \in \mathbb{N}$ yields the desired claim. $\quad\square$

REFERENCES

[1] ANDRES, S., BARLOW, M. T., DEUSCHEL, J.-D. and HAMBLY, B. M. (2013). Invariance principle for the random conductance model. *Probab. Theory Related Fields* **156** 535–580. MR3078279 https://doi.org/10.1007/s00440-012-0435-2

[2] ANDRES, S., CHIARINI, A., DEUSCHEL, J.-D. and SLOWIK, M. (2018). Quenched invariance principle for random walks with time-dependent ergodic degenerate weights. *Ann. Probab.* **46** 302–336. MR3758732 https://doi.org/10.1214/17-AOP1186

[3] ANDRES, S., DEUSCHEL, J.-D. and SLOWIK, M. (2015). Invariance principle for the random conductance model in a degenerate ergodic environment. *Ann. Probab.* **43** 1866–1891. MR3353817 https://doi.org/10.1214/14-AOP921

[4] ANDRES, S., DEUSCHEL, J.-D. and SLOWIK, M. (2016). Harnack inequalities on weighted graphs and some applications to the random conductance model. *Probab. Theory Related Fields* **164** 931–977. MR3477784 https://doi.org/10.1007/s00440-015-0623-y

[5] ANDRES, S. and NEUKAMM, S. (2018). Berry–Esseen theorem and quantitative homogenization for the random conductance model with degenerate conductances. *Stoch. Partial Differ. Equ. Anal. Comput.* **7** 240–296. MR3949105 https://doi.org/10.1007/s40072-018-0127-8

[6] ARMSTRONG, S. and DARIO, P. (2018). Elliptic regularity and quantitative homogenization on percolation clusters. *Comm. Pure Appl. Math.* **71** 1717–1849. MR3847767 https://doi.org/10.1002/cpa.21726

[7] BA, M. and MATHIEU, P. (2015). A Sobolev inequality and the individual invariance principle for diffusions in a periodic potential. *SIAM J. Math. Anal.* **47** 2022–2043. MR3348123 https://doi.org/10.1137/130949683

[8] BARLOW, M., BURDZY, K. and TIMÁR, Á. (2016). Comparison of quenched and annealed invariance principles for random conductance model. *Probab. Theory Related Fields* **164** 741–770. MR3477779 https://doi.org/10.1007/s00440-015-0618-8

[9] BARLOW, M. T. and DEUSCHEL, J.-D. (2010). Invariance principle for the random conductance model with unbounded conductances. *Ann. Probab.* **38** 234–276. MR2599199 https://doi.org/10.1214/09-AOP481

[10] BELLA, P., FEHRMAN, B. and OTTO, F. (2018). A Liouville theorem for elliptic systems with degenerate ergodic coefficients. *Ann. Appl. Probab.* **28** 1379–1422. MR3809467 https://doi.org/10.1214/17-AAP1332

[11] BELLA, P. and SCHÄFFNER, M. (2019). Local boundedness and Harnack inequality for solutions of linear non-uniformly elliptic equations. Preprint. Available at arXiv:1901.07958.

[12] BERGER, N. and BISKUP, M. (2007). Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields* **137** 83–120. MR2278453 https://doi.org/10.1007/s00440-006-0498-z

[13] BISKUP, M. (2011). Recent progress on the random conductance model. *Probab. Surv.* **8** 294–373. MR2861133 https://doi.org/10.1214/11-PS190

[14] BISKUP, M. and KUMAGAI, T. (2014). Quenched invariance principle for a class of random conductance models with long-range jumps. Preprint. Available at arXiv:1412.0175.

[15] BISKUP, M. and PRESCOTT, T. M. (2007). Functional CLT for random walk among bounded random conductances. *Electron. J. Probab.* **12** 1323–1348. MR2354160 https://doi.org/10.1214/EJP.v12-456

[16] BISKUP, M. and RODRIGUEZ, P.-F. (2018). Limit theory for random walks in degenerate time-dependent random environments. *J. Funct. Anal.* **274** 985–1046. MR3743189 https://doi.org/10.1016/j.jfa.2017.12.002

[17] BOIVIN, D. (1993). Weak convergence for reversible random walks in a random environment. *Ann. Probab.* **21** 1427–1440. MR1235423

[18] BREZIS, H. (2011). *Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext.* Springer, New York. MR2759829

[19] CHIARINI, A. and DEUSCHEL, J.-D. (2016). Invariance principle for symmetric diffusions in a degenerate and unbounded stationary and ergodic random medium. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 1535–1563. MR3573286 https://doi.org/10.1214/15-AIHP688

[20] DE MASI, A., FERRARI, P. A., GOLDSTEIN, S. and WICK, W. D. (1989). An invariance principle for reversible Markov processes. Applications to random motions in random environments. *J. Stat. Phys.* **55** 787–855. MR1003538 https://doi.org/10.1007/BF01041608

[21] DEUSCHEL, J.-D., NGUYEN, T. A. and SLOWIK, M. (2018). Quenched invariance principles for the random conductance model on a random graph with degenerate ergodic weights. *Probab. Theory Related Fields* **170** 363–386. MR3748327 https://doi.org/10.1007/s00440-017-0759-z

[22] FANNJIANG, A. and KOMOROWSKI, T. (1997). A martingale approach to homogenization of unbounded random flows. *Ann. Probab.* **25** 1872–1894. MR1487440 https://doi.org/10.1214/aop/1023481115

[23] FRANCHI, B., SERAPIONI, R. and SERRA CASSANO, F. (1998). Irregular solutions of linear degenerate elliptic equations. *Potential Anal.* **9** 201–216. MR1666899 https://doi.org/10.1023/A:1008684127989

[24] HELLAND, I. S. (1982). Central limit theorems for martingales with discrete or continuous time. *Scand. J. Stat.* **9** 79–94. MR0668684

[25] KIPNIS, C. and VARADHAN, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* **104** 1–19. MR0834478

[26] KRENGEL, U. (1985). *Ergodic Theorems. De Gruyter Studies in Mathematics* **6**. de Gruyter, Berlin. MR0797411 https://doi.org/10.1515/9783110844641

[27] KUMAGAI, T. (2014). Random walks on disordered media and their scaling limits. *Lecture Notes from the 40th Probability Summer School Held in Saint-Flour 2010. Lecture Notes in Math.* Springer, Cham. MR3156983 https://doi.org/10.1007/978-3-319-03152-1

[28] MATHIEU, P. (2008). Quenched invariance principles for random walks with random conductances. *J. Stat. Phys.* **130** 1025–1046. MR2384074 https://doi.org/10.1007/s10955-007-9465-z

[29] MATHIEU, P. and PIATNITSKI, A. (2007). Quenched invariance principles for random walks on percolation clusters. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **463** 2287–2307. MR2345229 https://doi.org/10.1098/rspa.2007.1876

[30] MOSER, J. (1960). A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. *Comm. Pure Appl. Math.* **13** 457–468. MR0170091 https://doi.org/10.1002/cpa.3160130308

[31] MOSER, J. (1961). On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.* **14** 577–591. MR0159138 https://doi.org/10.1002/cpa.3160140329

[32] MOURRAT, J.-C. and OTTO, F. (2016). Anchored Nash inequalities and heat kernel bounds for static and dynamic degenerate environments. *J. Funct. Anal.* **270** 201–228. MR3419760 https://doi.org/10.1016/j.jfa.2015.09.020

[33] MURTHY, M. K. V. and STAMPACCHIA, G. (1968). Boundary value problems for some degenerate-elliptic operators. *Ann. Mat. Pura Appl.* (4) **80** 1–122. MR0249828 https://doi.org/10.1007/BF02413623

[34] NEUKAMM, S., SCHÄFFNER, M. and SCHLÖMERKEMPER, A. (2017). Stochastic homogenization of nonconvex discrete energies with degenerate growth. *SIAM J. Math. Anal.* **49** 1761–1809. MR3650427 https://doi.org/10.1137/16M1097705

[35] SIDORAVICIUS, V. and SZNITMAN, A.-S. (2004). Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probab. Theory Related Fields* **129** 219–244. MR2063376 https://doi.org/10.1007/s00440-004-0336-0

[36] TRUDINGER, N. S. (1971). On the regularity of generalized solutions of linear, non-uniformly elliptic equations. *Arch. Ration. Mech. Anal.* **42** 50–62. MR0344656 https://doi.org/10.1007/BF00282317

[37] TRUDINGER, N. S. (1973). Linear elliptic operators with measurable coefficients. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (3) **27** 265–308. MR0369884