

# The asymptotic equivalence of the sample trispectrum and the nodal length for random spherical harmonics

Domenico Marinucci<sup>a</sup>, Maurizia Rossi<sup>b</sup> and Igor Wigman<sup>c</sup>

<sup>a</sup>*Department of Mathematics, Università di Roma Tor Vergata, Rome, Italy. E-mail: [marinucc@mat.uniroma2.it](mailto:marinucc@mat.uniroma2.it)*

<sup>b</sup>*Department of Mathematics, Università di Pisa, Pisa, Italy. E-mail: [maurizia.rossi@unipi.it](mailto:maurizia.rossi@unipi.it)*

<sup>c</sup>*Department of Mathematics, King's College London, London, United Kingdom. E-mail: [igor.wigman@kcl.ac.uk](mailto:igor.wigman@kcl.ac.uk)*

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**Abstract.** We study the asymptotic behaviour of the nodal length of random  $2d$ -spherical harmonics  $f_\ell$  of high degree  $\ell \rightarrow \infty$ , i.e. the length of their zero set  $f_\ell^{-1}(0)$ . It is found that the nodal lengths are asymptotically equivalent, in the  $L^2$ -sense, to the “sample trispectrum”, i.e., the integral of  $H_4(f_\ell(x))$ , the fourth-order Hermite polynomial of the values of  $f_\ell$ . A particular by-product of this is a Quantitative Central Limit Theorem (in Wasserstein distance) for the nodal length, in the high energy limit.

**Résumé.** Nous étudions le comportement asymptotique de la longueur nodale de fonctions propres aléatoires  $f_\ell$  du Laplacien sphérique pour valeurs propres très élevés  $\ell \rightarrow +\infty$ , c'est-à-dire la longueur de leur ensemble de niveau zéro  $f_\ell^{-1}(0)$ . Nous démontrons que la longueur nodale est asymptotiquement équivalente, au sens de  $L^2$ , au « sample trispectrum », c'est-à-dire l'intégral de  $H_4(f_\ell(x))$ , le polynôme de Hermite d'ordre quatre évalué en  $f_\ell$ . Une conséquence de ce résultat est un Théorème Central Limite quantitatif (dans le sens de la distance de Wasserstein) pour la longueur nodale, quand l'énergie tend vers l'infini.

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## 1. Introduction and main results

### 1.1. Background

Let  $\mathbb{S}^2$  be the unit  $2d$  sphere and  $\Delta_{\mathbb{S}^2}$  be the Laplace-Beltrami operator on  $\mathbb{S}^2$ . It is well-known that the spectrum of  $\Delta_{\mathbb{S}^2}$  consists of the numbers  $\lambda_\ell = \ell(\ell + 1)$  with  $\ell \in \mathbb{Z}_{\geq 0}$ , and the eigenspace corresponding to  $\lambda_\ell$  is the  $(2\ell + 1)$ -dimensional linear space of degree  $\ell$  spherical harmonics. For  $\ell \geq 0$  let  $\{Y_{\ell m}(\cdot)\}_{m=-\ell, \dots, \ell}$  be an arbitrary  $L^2$ -orthonormal basis of real valued spherical harmonics

$$Y_{\ell m} : \mathbb{S}^2 \rightarrow \mathbb{R}$$

satisfying

$$\Delta_{\mathbb{S}^2} Y_{\ell m} + \lambda_\ell Y_{\ell m} = 0, \quad Y_{\ell m} : \mathbb{S}^2 \rightarrow \mathbb{R}.$$

On  $\mathbb{S}^2$  we consider a family of Gaussian random fields (defined on a suitable probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ )

$$f_\ell(x) = \sqrt{\frac{4\pi}{2\ell + 1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \tag{1.1}$$

where the coefficients  $\{a_{\ell m}\}_{m=-\ell, \dots, \ell}$  are i.i.d. standard Gaussian random variables (zero mean and unit variance); it is immediate to see that the law of the process  $\{f_\ell(\cdot)\}$  is invariant with respect to the choice of a  $L^2$ -orthonormal basis

$\{Y_{\ell m}\}$ . The random fields  $\{f_\ell(x), x \in \mathbb{S}^2\}$  are centred, Gaussian and isotropic, satisfying

$$\Delta_{\mathbb{S}^2} f_\ell + \lambda_\ell f_\ell = 0;$$

these are the *random* degree- $\ell$  spherical harmonics. From the addition formula for spherical harmonics [13, (3.42)], the covariance function of  $f_\ell$  is given by

$$\mathbb{E}[f_\ell(x) \cdot f_\ell(y)] = P_\ell(\cos d(x, y)),$$

where  $P_\ell$  are the Legendre polynomials, and  $d(x, y)$  is the spherical geodesic distance between  $x$  and  $y$ ,  $d(x, y) = \arccos(\langle x, y \rangle)$ . The random spherical harmonics naturally arise from the spectral analysis of isotropic spherical random fields (e.g. [4,6,7]), and Quantum Chaos (e.g. [13,27]); their geometry is of significant interest.

In this paper, we shall focus on the nodal length of the random fields  $\{f_\ell(\cdot)\}$ , i.e. the length of the nodal line:

$$\mathcal{L}_\ell := \text{len}\{f_\ell^{-1}(0)\}.$$

Here  $\{\mathcal{L}_\ell\}_{\ell \geq 0}$  is a sequence of random variables; Yau’s conjecture [28], asserts that the nodal volume of Laplace eigenfunctions on *smooth*  $n$ -manifolds is commensurable with the square root of the eigenvalue. An application of Yau’s conjecture, established [9] for all *analytic* manifolds, on the sample functions  $f_\ell$  implies that one has, for some *absolute* constants  $C \geq c > 0$

$$c\sqrt{\lambda_\ell} \leq \mathcal{L}_\ell \leq C\sqrt{\lambda_\ell} \quad \text{for all } \ell \geq 1. \tag{1.2}$$

The lower bound in (1.2) was recently established [12] for all smooth manifolds.

While the expected value of  $\mathcal{L}_\ell$  was computed [18] by a standard application of the Kac–Rice formula to be

$$\mathbb{E}[\mathcal{L}_\ell] = \left\{ \frac{\lambda_\ell}{2} \right\}^{1/2} \times 2\pi;$$

evaluating the variance proved to be more subtle, and was shown [26] to be asymptotic to

$$\text{Var}\{\mathcal{L}_\ell\} = \frac{\log \ell}{32} + O(1). \tag{1.3}$$

It follows that the “generic” (Gaussian) spherical eigenfunctions obey a stronger law than (1.2), with normalised nodal length  $\frac{\mathcal{L}_\ell}{\sqrt{\lambda_\ell}}$  converging (in mean square and hence in probability) to a positive constant.

### 1.2. Main results

In this work, we take our random eigenfunctions to be defined on a suitable probability space  $\{\Omega, \mathfrak{F}, \mathbb{P}\}$  and we are interested in the analysis of the fluctuations of the nodal length around its expected value; in particular a (quantitative) central limit theorem will be established for the (centred and standardized) fluctuations of  $\mathcal{L}_\ell$ . This convergence is a rather straightforward corollary of a deeper result, namely the asymptotic equivalence (in the  $L^2(\Omega)$  sense) of the nodal length and the sample trispectrum of  $\{f_\ell\}$ , i.e., the integral of  $H_4(f_\ell(x))$ , where  $H_4$  is the fourth-order Hermite polynomial; we recall that

$$H_4(u) = u^4 - 6u^2 + 3.$$

More precisely, let us define the sequence of centred random variables

$$\mathcal{M}_\ell := -\frac{1}{4} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} \int_{\mathbb{S}^2} H_4(f_\ell(x)) dx = -\frac{1}{4} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} h_{\ell;4}, \tag{1.4}$$

$$h_{\ell;4} := \int_{\mathbb{S}^2} H_4(f_\ell(x)) dx, \quad \ell = 1, 2, \dots; \tag{1.5}$$

the sequence  $\{h_{\ell;4}\}$  (which we call the sample trispectrum of  $f_\ell$ ) was studied earlier, and indeed building upon [15, Lemma 3.2], it is immediate to establish the following:

**Lemma 1.1.** *As  $\ell \rightarrow \infty$ , we have*

$$\text{Var}\{\mathcal{M}_\ell\} = \frac{1}{32} \log \ell + O(1). \quad (1.6)$$

By means of Kac–Rice formula [1,2,26], the spherical nodal length  $\mathcal{L}_\ell$  can be formally written as

$$\mathcal{L}_\ell = \int_{\mathbb{S}^2} \|\nabla f_\ell(x)\| \delta(f_\ell(x)) dx, \quad (1.7)$$

where  $\delta(\cdot)$  denotes the Dirac delta function and  $\|\cdot\|$  the standard Euclidean norm in  $\mathbb{R}^2$ ; this representation can be shown to hold almost surely in  $\Omega$ , and it is shown in Appendices A and B that it also holds in  $L^2(\Omega)$ . Denote by  $\tilde{\mathcal{L}}_\ell$  the standardized nodal length, i.e.,

$$\tilde{\mathcal{L}}_\ell := \frac{\mathcal{L}_\ell - \mathbb{E}\mathcal{L}_\ell}{\sqrt{\text{Var}(\mathcal{L}_\ell)}}. \quad (1.8)$$

Note that the variance of  $\mathcal{M}_\ell$  is asymptotic to the one of  $\mathcal{L}_\ell$ , i.e.

$$\frac{\text{Var}\{\mathcal{L}_\ell\}}{\text{Var}\{\mathcal{M}_\ell\}} = 1 + O\left(\frac{1}{\log \ell}\right), \quad \text{as } \ell \rightarrow \infty;$$

we shall also standardize the zero-mean sequence  $\{\mathcal{M}_\ell\}$ , writing

$$\tilde{\mathcal{M}}_\ell := \frac{\mathcal{M}_\ell}{\sqrt{\text{Var}(\mathcal{M}_\ell)}}. \quad (1.9)$$

The main contribution of the present manuscript is establishing the following asymptotic representation for  $\tilde{\mathcal{L}}_\ell$ :

**Theorem 1.2.** *As  $\ell \rightarrow \infty$ , we have that*

$$\mathbb{E}[\{\tilde{\mathcal{L}}_\ell - \tilde{\mathcal{M}}_\ell\}^2] = O\left(\frac{1}{\log \ell}\right),$$

and thus in particular

$$\tilde{\mathcal{L}}_\ell = \tilde{\mathcal{M}}_\ell + O_p\left(\frac{1}{\sqrt{\log \ell}}\right).$$

In other words, after centering and normalization the spherical nodal lengths (1.8) and the sample trispectrum (1.9) are asymptotically equivalent in  $L^2(\Omega)$  (and thus in probability and in law). Now recall that the Wasserstein distance between two random variables  $X$  and  $Y$  is given by (see e.g. [19, Appendix C])

$$d_W(X, Y) = \sup_{h: \|h\|_{\text{Lip}} \leq 1} |Eh(X) - Eh(Y)|;$$

convergence in mean-square implies convergence in Wasserstein distance, and both imply convergence in distribution. Let  $\mathcal{N}(0, 1)$  denote a standard Gaussian random variable; in view of the aforementioned CLT [15] on  $\{\tilde{\mathcal{M}}_\ell\}$ , it follows directly from Theorem 1.2 that:

**Corollary 1.3.** *As  $\ell \rightarrow \infty$ , we have that*

$$d_W(\tilde{\mathcal{L}}_\ell, \mathcal{N}(0, 1)) = O\left(\frac{1}{\sqrt{\log \ell}}\right).$$

Hence we obtain here a new Quantitative Central Limit Theorem (in Wasserstein distance) for the spherical nodal length.

1.3. Discussion and overview of some related literature

Theorem 1.2 is closely related to the recent characterization of the asymptotic distribution for the nodal length of arithmetic random waves, i.e. Gaussian eigenfunctions on the two-dimensional torus  $\mathbb{T}^2$ , which was established in [17]. The approach in the latter paper can be summarized as follows: the nodal length can be decomposed into so-called Wiener-Chaos components, i.e., it can be projected on the orthogonal subspaces of  $L^2(\Omega)$  spanned by linear combinations of multiple Hermite polynomials of degree  $q$ ; more precisely, we have the orthogonal decomposition

$$L^2(\Omega) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q,$$

where  $\mathcal{H}_q$  denotes the  $q$ th Wiener chaos, i.e., the linear span of Hermite polynomials of order  $q$ , and hence

$$\mathcal{L}_n = \sum_{q=0}^{\infty} \text{Proj}[\mathcal{L}_n|q], \tag{1.10}$$

where  $\mathcal{L}_n$  denotes the nodal length of Gaussian arithmetic random waves of degree  $n = a^2 + b^2$ , where  $n, a, b \in \mathbb{N}$ , and  $\text{Proj}[\cdot|q]$  projection on  $\mathcal{H}_q$ , see [11,17] for details. For arithmetic random waves, all the terms  $\{\text{Proj}[\mathcal{L}_n|q], q \text{ odd}\}$  in the expansion (1.10) vanish for symmetry reasons, and so does the term corresponding to  $q = 2$ . The latter phenomenon is one interpretation of the so-called *Berry's cancellation* (see [3]), i.e. the fact that the nodal length variance is of order of magnitude smaller than the natural scaling. Indeed it has been shown [17,20] that the term corresponding to  $q = 2$  dominates the fluctuations of the boundary length of excursion sets for arithmetic random waves for any threshold value  $z \neq 0$ ; for  $z = 0$  it vanishes, and the dominating term is the projection onto the 4th order chaos.

The asymptotic domination of the second-order chaos for  $z \neq 0$ , and its disappearance for  $z = 0$ , have been shown recently to occur for other geometric functionals of excursion sets of random eigenfunctions in a variety of circumstances, such as the excursion area and the Defect ([14–16,21] covering all dimensions  $d \geq 2$ ), and the Euler-Poincaré characteristic [5,6] (see also [8]). The fact that a single chaos dominates clearly allows for a much neater derivation of asymptotic distribution results; in particular, quantitative central limit theorems have been given [6,14–16] for various geometric functionals of random spherical harmonics, in the high-energy limit where  $\lambda_\ell \rightarrow \infty$ ; on the torus the asymptotic behaviour is more complicated, depending on the different subsequences as  $n$  grows [11,17,22] for the nodal length of arithmetic random waves, and [23] for nodal intersections of arithmetic random waves against a fixed curve.

The results we shall give here confirm the asymptotic dominance of the fourth-order component; in this sense, they are analogous to those in [17] for the case of the torus. On the other hand, here we are able to obtain a neater expression for the leading term, which is of independent interest, and makes the derivation of a Quantitative Central Limit Theorem much more elegant. In fact, rather than studying the asymptotic behaviour of the fourth-order chaos (which is a sum of six terms involving the eigenfunctions and their gradients), we establish the asymptotically full correlation of the nodal length with a term which can be evaluated in terms of the eigenfunctions themselves, and not their gradient components. The resulting approximation (valid in the mean square sense) is therefore surprisingly simple, and the quantitative central limit theorem follows as an immediate consequence. We believe that this technique can be applicable to other examples of geometric functionals for random spherical harmonics.

As mentioned earlier, the approach we use in this paper does not require to study directly the asymptotic behaviour of the full components in the fourth-order chaos, as it was done earlier in [17] for eigenfunctions on the torus. On the other hand it is obvious that the random variables  $\mathcal{M}_\ell \in \mathcal{H}_4$ , and a quick inspection to the proof of our main result reveals that we have also the following asymptotic equivalence: As  $\ell \rightarrow \infty$ , we have that

$$\text{Corr}\{\text{Proj}[\mathcal{L}_\ell|4], \mathcal{M}_\ell\} = 1 + O\left(\frac{1}{\log \ell}\right),$$

and hence

$$\mathbb{E}[\{\text{Proj}[\tilde{\mathcal{L}}_\ell|4] - \tilde{\mathcal{M}}_\ell\}^2] = O\left(\frac{1}{\log \ell}\right) \implies \text{Proj}[\tilde{\mathcal{L}}_\ell|4] = \tilde{\mathcal{M}}_\ell + O_p\left(\frac{1}{\sqrt{\log \ell}}\right).$$

Likewise, as  $\ell \rightarrow \infty$ , we also have that

$$\text{Var}\{\text{Proj}[\mathcal{L}_\ell|4]\} = \frac{\log \ell}{32} + O(1) \implies \frac{\text{Var}\{\text{Proj}[\mathcal{L}_\ell|4]\}}{\text{Var}\{\mathcal{L}_\ell\}} = 1 + O\left(\frac{1}{\log \ell}\right),$$

and hence

$$\mathbb{E}[\{\tilde{\mathcal{L}}_\ell - \text{Proj}[\tilde{\mathcal{L}}_\ell|4]\}^2] = O\left(\frac{1}{\log \ell}\right) \implies \tilde{\mathcal{L}}_\ell = \text{Proj}[\tilde{\mathcal{L}}_\ell|4] + O_p\left(\frac{1}{\sqrt{\log \ell}}\right).$$

In other words, it does follow from our results that the fourth-order chaos projection dominates the high-frequency behaviour of the spherical nodal length, as for the two-dimensional toroidal eigenfunctions.

As discussed before, the nodal length of random spherical harmonics can be viewed as the special case (for  $z = 0$ ) of the boundary length of excursion sets ( $\mathcal{L}_\ell(z)$ , say, with  $\mathcal{L}_\ell := \mathcal{L}_\ell(0)$ ). For  $z \neq 0$ , it was shown in [20, Proposition 7.3.1] (see also [6, Section 1.2.2] and [17, Remark 2.4]) that the dominant term corresponds to the second order chaos, which can be expressed as

$$\text{Proj}[\mathcal{L}_\ell(z)|2] = 2\left\{\frac{\lambda_\ell}{2}\right\}^{1/2} \sqrt{\frac{\pi}{8}} \{z^2 \phi(z)\} \frac{1}{2!} \int_{\mathbb{S}^2} H_2(f_\ell(x)) dx,$$

a component that vanishes identically for  $z = 0$  (here, as usual,  $\phi(z)$  denotes the density function of a standard Gaussian variable). On the other hand, for the nodal case from the results in this paper one obtains the related expression

$$\text{Proj}[\mathcal{L}_\ell(0)|4] = -\left\{\frac{\lambda_\ell}{2}\right\}^{1/2} \sqrt{\frac{\pi}{8}} \phi(0) \frac{1}{4!} \left\{ \int_{\mathbb{S}^2} H_4(f_\ell(x)) dx + O_p\left(\frac{1}{\ell}\right) \right\}.$$

It is instructive to compare these expressions with the results provided by the Gaussian Kinematic Formula (see e.g., [1,25]) for the expected value of the boundary length, which in terms of Wiener-chaos projections can be written in this framework as

$$\mathbb{E}[\mathcal{L}_\ell(z)] = \text{Proj}[\mathcal{L}_\ell(z)|0] = 2\left\{\frac{\lambda_\ell}{2}\right\}^{1/2} \sqrt{\frac{\pi}{8}} \phi(z) \int_{\mathbb{S}^2} H_0(f_\ell(x)) dx.$$

We leave as an issue for further research to determine whether similarly neat expressions can be shown to hold in greater generality, i.e. in dimension greater than two, for higher-order chaos projections, or for different geometric functionals.

### 1.4. Outline of the paper

In Section 2 we discuss some issues concerning the  $L^2$  expansion of the spherical nodal length into Wiener chaos components, and we present the analytic expression for the fourth-order chaos (which corresponds to the leading non-deterministic term); in Section 3 we give the proofs of the two main Theorems, which are largely based on a Key Proposition whose proof is collected in Section 4. Appendices A and B collect the justification for the  $L^2$  expansion of the nodal length into Hermite polynomials and some elementary facts about the covariances of random spherical harmonics and their derivatives.

## 2. The $L^2$ expansion of nodal length

In this section, we present the Wiener chaos expansion of the nodal length  $\mathcal{L}_\ell$  as in (1.7). The details of this derivation are similar to those given in [17] (see also [10]). Let us first recall the expression for the projection coefficients in the Hermite expansions of the two-dimensional norm and the Dirac  $\delta$ -function. For independent, standard Gaussian variables  $\zeta, \eta$  the expansion of the Euclidean norms has been established to be (see i.e., [10,17])

$$\|(\zeta, \eta)\| = \sum_{q=0}^{\infty} \sum_{n,m:2n+2m=q} \frac{\alpha_{2n,2m}}{(2n)!(2m)!} H_{2n}(\zeta) H_{2m}(\eta),$$

where  $(\zeta, \eta) \in \mathbb{R}^2$  and

$$\alpha_{2n,2m} := \sqrt{\frac{\pi}{2}} \frac{(2n)!(2m)!}{n!m!} \frac{1}{2^{n+m}} p_{n+m}\left(\frac{1}{4}\right),$$

and  $p_N$  is the swinging factorial coefficient

$$p_N(x) := \sum_{j=0}^N (-1)^j (-1)^N \binom{N}{j} \frac{(2j+1)!}{(j!)^2} x^j.$$

For the first few terms we have

$$\alpha_{00} = \sqrt{\frac{\pi}{2}}; \quad \alpha_{02} = \frac{1}{2}\sqrt{\frac{\pi}{2}}; \quad \alpha_{04} = -\frac{3}{8}\sqrt{\frac{\pi}{2}}.$$

On the other hand, the first few coefficients for the expansion into Wiener chaoses of the Dirac delta function  $\delta$ -function are given by ([10,17]):

$$\beta_0 = \frac{1}{\sqrt{2\pi}}; \quad \beta_2 = -\frac{1}{\sqrt{2\pi}}; \quad \beta_4 = \frac{3}{\sqrt{2\pi}}.$$

The Wiener-chaos decompositions need to be evaluated on variables of unit variance; this requires dividing the derivatives by  $\sqrt{\ell(\ell+1)/2} \sim \ell/\sqrt{2}$  (here and everywhere else  $a_\ell \sim b_\ell$  means that the ratio between the two sequences converges to unity as  $\ell \rightarrow \infty$ ). The  $L^2(\Omega)$  expansion of the nodal length (1.7) then takes the form

$$\begin{aligned} \mathcal{L}_\ell - \mathbb{E}\mathcal{L}_\ell &= \sqrt{\frac{\ell(\ell+1)}{2}} \sum_{q=2}^{\infty} \sum_{u=0}^q \sum_{k=0}^u \frac{\alpha_{k,u-k}\beta_{q-u}}{k!(u-k)!(q-u)!} \\ &\quad \times \int_{\mathbb{S}^2} H_{q-u}(f_\ell(x)) H_k\left(\frac{\partial_{1;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) H_{u-k}\left(\frac{\partial_{2;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) dx \\ &= \sum_{q=2}^{\infty} \int_{\mathbb{S}^2} \Psi_\ell(x; q) dx, \end{aligned}$$

where

$$\Psi_\ell(x; q) := \sqrt{\frac{\ell(\ell+1)}{2}} \sum_{u=0}^q \sum_{k=0}^u \frac{\alpha_{k,u-k}\beta_{q-u}}{k!(u-k)!(q-u)!} H_{q-u}(f_\ell(x)) H_k\left(\frac{\partial_{1;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) H_{u-k}\left(\frac{\partial_{2;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right);$$

here, we are using spherical coordinates (colatitude  $\theta$ , longitude  $\varphi$ ) and for  $x = (\theta_x, \varphi_x)$  we are using the notation

$$\partial_{1;x} = \frac{\partial}{\partial \theta} \Big|_{\theta=\theta_x}, \quad \partial_{2;x} = \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \Big|_{\theta=\theta_x, \varphi=\varphi_x}.$$

In particular, the projection of the nodal length on the fourth-order chaos has the expression

$$\begin{aligned} \text{Proj}[\tilde{\mathcal{L}}_\ell|4] &= \int_{\mathbb{S}^2} \Psi_\ell(x; 4) dx \\ &= \sqrt{\frac{\ell(\ell+1)}{2}} \left\{ \frac{\alpha_{00}\beta_4}{4!} \int_{\mathbb{S}^2} H_4(f_\ell(x)) dx + \frac{\alpha_{20}\beta_2}{2!2!} \int_{\mathbb{S}^2} H_2(f_\ell(x)) H_2\left(\frac{\partial_{1;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) dx \right. \\ &\quad + \frac{\alpha_{40}\beta_0}{4!} \int_{\mathbb{S}^2} H_4\left(\frac{\partial_{1;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) dx + \frac{\alpha_{22}\beta_0}{2!2!} \int_{\mathbb{S}^2} H_2\left(\frac{\partial_{1;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) H_2\left(\frac{\partial_{2;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) dx \\ &\quad \left. + \frac{\alpha_{02}\beta_2}{2!2!} \int_{\mathbb{S}^2} H_2(f_\ell(x)) H_2\left(\frac{\partial_{2;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) dx + \frac{\alpha_{04}\beta_0}{4!} \int_{\mathbb{S}^2} H_4\left(\frac{\partial_{2;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) dx \right\}. \end{aligned} \tag{2.1}$$

### 3. Proof of the main results (Theorem 1.2 and Corollary 1.3)

#### 3.1. Proof of Lemma 1.1

**Proof.** Before we proceed with the proof, we need to introduce some more notation: we shall write  $\bar{x} = (0, 0)$  for the ‘‘North Pole’’ and  $y(\theta) = (0, \theta)$  for the points on the meridian where  $\varphi = 0$ . It is now sufficient to note that

$$\begin{aligned} \text{Var}\{\mathcal{M}_\ell\} &= \frac{\ell(\ell+1)}{2 \times 4^2 \times 24^2} \times \mathbb{E}\left[\left\{\int_{\mathbb{S}^2} H_4(f_\ell) \sin \theta d\theta\right\}^2\right] \\ &= \frac{\ell(\ell+1)}{2 \times 4^2 \times 24^2} \times 576 \frac{\log \ell}{\ell^2} + O(1) = \frac{1}{32} \log \ell + O(1), \end{aligned}$$

where we have used the asymptotic result [15, Lemma 3.2]

$$\mathbb{E}\left[\left\{\int_{\mathbb{S}^2} H_4(f_\ell(x)) dx\right\}^2\right] = 576\frac{\log \ell}{\ell^2} + O\left(\frac{1}{\ell^2}\right), \quad \text{as } \ell \rightarrow \infty. \quad \square$$

Now we transform variables as

$$\psi := L\theta, \quad \text{for } L := \left(\ell + \frac{1}{2}\right), \text{ whence } y(\theta) = y\left(\frac{\psi}{L}\right);$$

also, let us define the following 2-point cross-correlation function

$$\mathcal{J}_\ell(\psi; 4) := \left[-\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{4!}\right] \times \frac{8\pi^2}{L} \mathbb{E}\left\{\Psi_\ell(\bar{x}; 4) H_4\left(f_\ell\left(y\left(\frac{\psi}{L}\right)\right)\right)\right\}. \quad (3.1)$$

Our main result will follow from the following Key Proposition:

**Proposition 3.1.** *For any constant  $C > 0$ , uniformly over  $\ell$  we have, for  $0 < \psi < C$ ,*

$$\mathcal{J}_\ell(\psi; 4) = O(\ell), \quad (3.2)$$

and, for  $C < \psi < L\frac{\pi}{2}$ ,

$$\mathcal{J}_\ell(\psi; 4) = \frac{1}{64} \frac{1}{\psi \sin \frac{\psi}{L}} + \frac{5}{64} \frac{\cos 4\psi}{\psi \sin \frac{\psi}{L}} - \frac{3}{16} \frac{\sin 2\psi}{\psi \sin \frac{\psi}{L}} + O\left(\frac{1}{\psi^2 \sin \frac{\psi}{L}}\right) + O\left(\frac{1}{\ell \psi \sin \frac{\psi}{L}}\right). \quad (3.3)$$

The proof of this Proposition is given later in Section 4; with this result at hand, we can proceed with the proof of Theorem 1.2 as follows.

### 3.2. Proof of Theorem 1.2

**Proof.** To establish Theorem 1.2, it is clearly sufficient to show that, as  $\ell \rightarrow \infty$ ,

$$\text{Corr}\{\mathcal{L}_\ell, \mathcal{M}_\ell\} = 1 + O\left(\frac{1}{\log \ell}\right),$$

and to this end we will prove the equivalent

$$\text{Cov}\{\mathcal{L}_\ell, \mathcal{M}_\ell\} = \frac{\log \ell}{32} + O(1)$$

(cf. (1.3) and (1.6)); here, as usual Corr and Cov denote correlation and covariance (respectively), while the  $O(1)$  term is uniform in  $\varepsilon$ . By continuity of the inner product in  $L^2$  spaces, we need to prove that

$$\text{Cov}\{\mathcal{L}_\ell, \mathcal{M}_\ell\} = \lim_{\varepsilon \rightarrow 0} \text{Cov}\{\mathcal{L}_{\ell;\varepsilon}, \mathcal{M}_\ell\} = \frac{\log \ell}{32} + O(1),$$

where

$$\mathcal{L}_{\ell;\varepsilon} := \int_{\mathbb{S}^2} \|\nabla f_\ell(x)\| \chi_\varepsilon(f_\ell(x)) dx, \quad \chi_\varepsilon(\cdot) := \frac{1}{2\varepsilon} \mathbb{I}_{[-\varepsilon, \varepsilon]}(\cdot).$$

Now define the ‘‘approximate local nodal length’’

$$\Psi_{\ell;\varepsilon}(x) := \|\nabla f_\ell(x)\| \chi_\varepsilon(f_\ell(x)),$$

where  $\mathbb{I}_A(\cdot)$  denotes the characteristic function of the set  $A$ . The newly defined  $\Psi_{\ell;\varepsilon}(x)$  is an isotropic random field on  $\mathbb{S}^2$  admitting the  $L^2(\Omega)$  expansion

$$\Psi_{\ell;\varepsilon}(x) = \mathbb{E}\Psi_{\ell;\varepsilon}(x) + \sum_{q=2}^{\infty} \Psi_{\ell;\varepsilon}(x; q);$$

moreover, as established in Appendices A and B, we have the  $L^2(\Omega)$  convergence

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{S}^2} \Psi_{\ell; \varepsilon}(x) dx = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{S}^2} \{ \|\nabla f_\ell(x)\| \chi_\varepsilon(f_\ell(x)) \} = \mathcal{L}_\ell.$$

Note also that  $\Psi_\varepsilon(x)$ ,  $H_4(f_\ell(y))$  are both in  $L^2(\mathbb{S}^2 \times \Omega)$  and they are isotropic, and thus

$$\begin{aligned} \text{Cov}\{\mathcal{L}_{\ell; \varepsilon}, \mathcal{M}_\ell\} &= -\frac{1}{4} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} \text{Cov} \left\{ \int_{\mathbb{S}^2} \Psi_\varepsilon(x), \int_{\mathbb{S}^2} H_4(f_\ell(y)) dy \right\} \\ &= -\frac{1}{4} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \mathbb{E} \{ \Psi_\varepsilon(x) H_4(f_\ell(y)) \} dx dy \\ &= -\frac{1}{4} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \mathbb{E} \left\{ \sum_{q=2}^{\infty} \Psi_{\ell; \varepsilon}(x; q) H_4(f_\ell(y)) \right\} dx dy \\ &= \left[ -\frac{1}{4} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} \right] \times 8\pi^2 \int_0^\pi \mathbb{E} \left\{ \sum_{q=2}^{\infty} \Psi_{\ell; \varepsilon}(\bar{x}; q) H_4(f_\ell(y(\theta))) \right\} \sin \theta d\theta \\ &= \left[ -\frac{1}{4} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} \right] \times 8\pi^2 \int_0^\pi \mathbb{E} \{ \Psi_{\ell; \varepsilon}(\bar{x}; 4) H_4(f_\ell(y(\theta))) \} \sin \theta d\theta. \end{aligned}$$

The integrand  $\mathbb{E}\{\Psi_{\ell; \varepsilon}(\bar{x}; 4)H_4(f_\ell(y(\theta)))\}$  can be computed explicitly and it is easily seen to be absolutely bounded for fixed  $\ell$ , uniformly over  $\varepsilon$ , see Proposition 3.1 above. Hence by the Dominated Convergence Theorem we may exchange the limit and the integral, and we have that

$$\begin{aligned} \text{Cov}\{\mathcal{L}_\ell, \mathcal{M}_\ell\} &= \lim_{\varepsilon \rightarrow 0} \text{Cov}\{\mathcal{L}_\ell^\varepsilon, \mathcal{M}_\ell\} \\ &= \left[ -\frac{1}{4} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} \right] \times 8\pi^2 \lim_{\varepsilon \rightarrow 0} \int_0^\pi \mathbb{E} \{ \Psi_{\ell; \varepsilon}(\bar{x}; 4) H_4(f_\ell(y(\theta))) \} \sin \theta d\theta \\ &= \left[ -\frac{1}{4} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} \right] \times 8\pi^2 \int_0^\pi \lim_{\varepsilon \rightarrow 0} \mathbb{E} \{ \Psi_{\ell; \varepsilon}(\bar{x}; 4) H_4(f_\ell(y(\theta))) \} \sin \theta d\theta \\ &= \left[ -\frac{1}{4} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} \right] \times 8\pi^2 \int_0^\pi \mathbb{E} \{ \Psi_\ell(\bar{x}; 4) H_4(f_\ell(y(\theta))) \} \sin \theta d\theta. \end{aligned}$$

We can now rewrite, using (3.1)

$$\text{Cov}\{\mathcal{L}_\ell, \mathcal{M}_\ell\} = \int_0^{L\pi} \mathcal{J}_\ell(\psi; 4) \sin \frac{\psi}{L} d\psi, \tag{3.4}$$

where we recall that  $L/\ell = 1 + o(1)$ , as  $\ell \rightarrow \infty$ . It is now sufficient to notice that

$$\text{Cov}\{\mathcal{L}_\ell, \mathcal{M}_\ell\} = \int_0^C \mathcal{J}_\ell(\psi; 4) \sin \frac{\psi}{L} d\psi + 2 \int_C^{L\pi/2} \mathcal{J}_\ell(\psi; 4) \sin \frac{\psi}{L} d\psi. \tag{3.5}$$

For the first summand in (3.5) we have easily

$$\left| \int_0^C \mathcal{J}_\ell(\psi; 4) \sin \frac{\psi}{L} d\psi \right| \leq \text{const} \times \ell \int_0^C \left| \sin \frac{\psi}{L} \right| d\psi \leq \frac{\ell}{L} \int_0^C \psi d\psi = O(1), \quad \text{as } \ell \rightarrow \infty.$$

For the second sum in (3.5), using Proposition 3.1 and integrating we obtain

$$\begin{aligned} &2 \int_C^{L\pi/2} \mathcal{J}_\ell(\psi; 4) \sin \frac{\psi}{L} d\psi \\ &= \frac{1}{L} \int_C^{L\pi/2} \frac{1}{\sin^2 \frac{\psi}{L}} \left\{ \frac{1}{32} + \frac{5}{32} \cos 4\psi - \frac{3}{8} \sin 2\psi \right\} \sin \frac{\psi}{L} d\psi \end{aligned}$$



$$\begin{aligned}
& + O\left(\frac{1}{L} \int_C^{L\pi/2} \frac{1}{\psi} \frac{1}{\sin^2 \frac{\psi}{L}} \sin \frac{\psi}{L} d\psi\right) + O\left(\frac{1}{L} \int_C^{L\pi/2} \frac{1}{\ell} \frac{1}{\sin^2 \frac{\psi}{L}} \sin \frac{\psi}{L} d\psi\right) \\
& = \int_C^{L\pi/2} \frac{1}{\psi} \left\{ \frac{1}{32} + \frac{5}{32} \cos 4\psi - \frac{3}{8} \sin 2\psi \right\} d\psi \\
& + O\left(\int_C^{L\pi/2} \frac{1}{\psi^2} d\psi\right) + O\left(\frac{1}{L} \int_C^{L\pi/2} \frac{1}{\ell} \frac{1}{\sin \frac{\psi}{L}} d\psi\right) \\
& = \frac{\log \ell}{32} + O(1) + O\left(\frac{\log \ell}{\ell}\right),
\end{aligned}$$

as claimed. □

**Remark 3.2.** As mentioned in the [Introduction](#), our main result could equivalently be stated as

$$\text{Corr}\{\mathcal{L}_\ell, \text{Proj}[\mathcal{L}_\ell|4]\}, \text{Corr}\{\text{Proj}[\mathcal{L}_\ell|4], \mathcal{M}_\ell\} \rightarrow 1$$

and thus

$$\begin{aligned}
\text{Proj}[\tilde{\mathcal{L}}_\ell|4] &= -\sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4 \times 24} \int_{\mathbb{S}^2} H_4(f_\ell(x)) dx + O(1), \\
\tilde{\mathcal{L}}_\ell &= -\sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4 \times 24} \int_{\mathbb{S}^2} H_4(f_\ell(x)) dx + O(1),
\end{aligned}$$

both equivalences holding in the  $L^2(\Omega)$  sense.

### 3.3. Proof of the Central Limit Theorem (Corollary 1.3)

Recall that  $h_{\ell;4}$  is defined in (1.5). It was shown [15, Lemma 3.3] that the so-called fourth-order cumulant of  $h_{\ell;4}$

$$\text{cum}_4\{h_{\ell;4}\} := \mathbb{E}[h_{\ell;4}^4] - 3\{\mathbb{E}[h_{\ell;4}^2]\}^2$$

satisfies  $\text{cum}_4\{h_{\ell;4}\} \approx \ell^{-4}$ , i.e. the ratio between the left and right-hand sides is bounded above and below by finite, strictly positive constants. Taking into account the normalizing factors, it means that  $\tilde{\mathcal{M}}_\ell$  satisfies

$$\text{cum}_4\{\tilde{\mathcal{M}}_\ell\} = \frac{\text{cum}_4\{\mathcal{M}_\ell\}}{\text{Var}^2(\mathcal{M}_\ell)} = \frac{32^2}{\log^2 \ell} \left(-\frac{1}{4} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!}\right)^4 \text{cum}_4\{h_{\ell;4}\} = O\left(\frac{1}{\log^2 \ell}\right),$$

where we have exploited definitions of the sequences  $\{\mathcal{M}_\ell, \tilde{\mathcal{M}}_\ell\}$  and standard properties of the cumulants. Let us now recall the so-called *Stein-Malliavin bound* by Nourdin–Peccati, stating that for a standardized random variable  $F$  which belong to the  $q$ -th order Wiener chaos  $\mathcal{H}_q$  we have the bound (see [19, Theorem 5.2.6])

$$d_W(F, \mathcal{N}(0, 1)) \leq \sqrt{\frac{2q-2}{3\pi q} \text{cum}_4\{F\}}.$$

Now the sequence  $\{\tilde{\mathcal{M}}_\ell\}$  is indeed standardized and belongs to the Wiener chaos for  $q = 4$ , so that we have

$$d_W(\tilde{\mathcal{M}}_\ell, \mathcal{N}(0, 1)) \leq \sqrt{\frac{1}{2\pi} \{\text{cum}_4\{\tilde{\mathcal{M}}_\ell\}\}} = O\left(\frac{1}{\log \ell}\right).$$

As a simple application of the triangle inequality for  $d_W$  (see [19, Appendix C]), for  $\ell \rightarrow \infty$

$$d_W(\tilde{\mathcal{L}}_\ell, \mathcal{N}(0, 1)) \leq d_W(\tilde{\mathcal{M}}_\ell, \mathcal{N}(0, 1)) + \sqrt{\mathbb{E}[\tilde{\mathcal{L}}_\ell - \tilde{\mathcal{M}}_\ell]^2} = O\left(\frac{1}{\sqrt{\log \ell}}\right),$$

and the statement of Corollary 1.3 follows.

### 4. Proof of Proposition 3.1

**Proof.** It is convenient to introduce some further notation, recalling (2.1) and writing

$$\Psi_\ell(x; 4) = A_\ell(x) + B_\ell(x) + C_\ell(x) + D_\ell(x) + E_\ell(x) + F_\ell(x),$$

where

$$\sqrt{\frac{\ell(\ell+1)}{2}} \frac{3}{2} \frac{1}{4!} H_4(f_\ell(x)) =: A_\ell(x), \tag{4.1}$$

$$-\sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4} \frac{1}{2!2!} H_2(f_\ell) H_2\left(\frac{\partial_{1;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) =: B_\ell(x), \tag{4.2}$$

$$-\sqrt{\frac{\ell(\ell+1)}{2}} \frac{3}{16} \frac{1}{4!} H_4\left(\frac{\partial_{1;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) =: C_\ell(x), \tag{4.3}$$

$$+\frac{3}{2} \frac{1}{2!2!} H_2\left(\frac{\partial_{1;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) H_2\left(\frac{\partial_{2;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) =: D_\ell(x), \tag{4.4}$$

$$-\frac{1}{4} \frac{1}{2!2!} H_2(f_\ell(x)) H_2\left(\frac{\partial_{2;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) =: E_\ell(x), \tag{4.5}$$

$$-\frac{3}{16} \frac{1}{4!} H_4\left(\frac{\partial_{2;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) =: F_\ell(x), \tag{4.6}$$

and also

$$\mathcal{M}_\ell := -\frac{1}{4} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} \int_{\mathbb{S}^2} H_4(f_\ell(x)) dx = \int_{\mathbb{S}^2} M_\ell(x) dx, \tag{4.7}$$

$$M_\ell(x) := -\frac{1}{4} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} H_4(f_\ell(x)).$$

For the computations to follow, recall that we focus on  $\bar{x} = (0, 0)$  (the ‘‘North Pole’’) and  $y(\theta) = (\theta, 0)$  (the ‘‘Greenwich meridian’’). By repeated application of the well-known Diagram Formula (see e.g. [13, Section 4.3.1]), we have

$$\begin{aligned} &\mathbb{E} \left[ H_2\left(\frac{\partial_{1;x} f_\ell(\bar{x})}{\sqrt{\ell(\ell+1)/2}}\right) H_2\left(\frac{\partial_{2;x} f_\ell(\bar{x})}{\sqrt{\ell(\ell+1)/2}}\right) H_4(f_\ell(y(\theta))) \right] \\ &= 4! \frac{4}{\ell^2(\ell+1)^2} \{ \mathbb{E}[\partial_{1;x} f_\ell(\bar{x}) f_\ell(y(\theta))] \}^2 \{ \mathbb{E}[\partial_{2;x} f_\ell(\bar{x}) f_\ell(y(\theta))] \}^2 = 0, \end{aligned}$$

in view of (A.3); likewise

$$\begin{aligned} &\mathbb{E} \left[ H_2(f_\ell(\bar{x})) H_2\left(\frac{\partial_{2;x} f_\ell(\bar{x})}{\sqrt{\ell(\ell+1)/2}}\right) H_4(f_\ell(y(\theta))) \right] \\ &= 4! \frac{2}{\ell(\ell+1)} \{ \mathbb{E}[f_\ell(\bar{x}) f_\ell(y(\theta))] \}^2 \{ \mathbb{E}[\partial_{2;x} f_\ell(\bar{x}) f_\ell(y(\theta))] \}^2 = 0, \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E} \left[ H_4\left(\frac{\partial_{2;x} f_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) H_4(f_\ell(y(\theta))) \right] \\ &= 4! \frac{4}{\ell^2(\ell+1)^2} \{ \mathbb{E}[\partial_{2;x} f_\ell(\bar{x}) f_\ell(y(\theta))] \}^4 = 0. \end{aligned}$$

As a consequence, using definitions (4.4), (4.5), (4.6) and (4.7) we have that

$$\mathbb{E}[D_\ell(\bar{x}) M_\ell(y(\theta))] = \mathbb{E}[E_\ell(\bar{x}) M_\ell(y(\theta))] = \mathbb{E}[F_\ell(\bar{x}) M_\ell(y(\theta))] \equiv 0$$

for all  $\theta \in [0, \pi]$ . In the sequel, it is sufficient to focus on  $A_\ell(\cdot)$ ,  $B_\ell(\cdot)$  and  $C_\ell(\cdot)$ . The proof of (3.2) is rather straightforward; indeed, as a simple application of the Cauchy–Schwartz inequality, we have that

$$\begin{aligned} \mathbb{E}[H_4(f_\ell(\bar{x}))H_4(f_\ell(y(\theta)))] &\leq \mathbb{E}[\{H_4(f_\ell(\bar{x}))\}^2] = 24, \\ \mathbb{E}\left[H_2(f_\ell(\bar{x}))H_2\left(\frac{\partial_{1;x}f_\ell(\bar{x})}{\sqrt{\ell(\ell+1)/2}}\right)H_4(f_\ell(y(\theta)))\right] \\ &\leq \sqrt{\mathbb{E}\left[\left\{H_2(f_\ell(\bar{x}))H_2\left(\frac{\partial_{1;x}f_\ell(\bar{x})}{\sqrt{\ell(\ell+1)/2}}\right)\right\}^2\right]\mathbb{E}[\{H_4(f_\ell(\bar{x}))\}^2]} = 24 \end{aligned}$$

and analogously

$$\begin{aligned} \mathbb{E}\left[H_4\left(\frac{\partial_{1;x}f_\ell(\bar{x})}{\sqrt{\ell(\ell+1)/2}}\right)H_4(f_\ell(y(\theta)))\right] \\ \leq \sqrt{\mathbb{E}\left[\left\{H_4\left(\frac{\partial_{1;x}f_\ell(\bar{x})}{\sqrt{\ell(\ell+1)/2}}\right)\right\}^2\right]\mathbb{E}[\{H_4(f_\ell(\bar{x}))\}^2]} = 24. \end{aligned}$$

It then follows that

$$\begin{aligned} |\mathcal{J}_\ell(\psi; 4)| &= 8\pi^2 \left| \mathbb{E}\left[\{A_\ell(\bar{x}) + B_\ell(\bar{x}) + C_\ell(\bar{x})\}M_\ell\left(y\left(\frac{\psi}{L}\right)\right)\right] \right| \\ &\leq \frac{8\pi^2}{L} \left\{ \left| \mathbb{E}\left[A_\ell(\bar{x})M_\ell\left(y\left(\frac{\psi}{L}\right)\right)\right] \right| + \left| \mathbb{E}\left[B_\ell(\bar{x})M_\ell\left(y\left(\frac{\psi}{L}\right)\right)\right] \right| + \left| \mathbb{E}\left[C_\ell(\bar{x})M_\ell\left(y\left(\frac{\psi}{L}\right)\right)\right] \right| \right\} \\ &\leq 24 \frac{\ell(\ell+1)}{2L} 8\pi^2 \left\{ \frac{3}{2} \frac{1}{4!} \frac{1}{4!} \frac{1}{4!} + \frac{1}{4} \frac{1}{2!2!} \frac{1}{4!} \frac{1}{4!} + \frac{3}{16} \frac{1}{4!} \frac{1}{4!} \frac{1}{4!} \right\} = O(\ell), \end{aligned}$$

as claimed.

We now turn to proving (3.3). Using (A.1), (A.2) and the Diagram Formula we can write explicitly

$$\begin{aligned} \mathbb{E}[A_\ell(\bar{x})M_\ell(y(\theta))] &= -\frac{\ell(\ell+1)}{2} \frac{3}{2} \frac{1}{4!} \frac{1}{4!} \frac{1}{4!} \times 4! P_\ell^4(\cos \theta) \\ &= -\frac{\ell(\ell+1)}{2} \frac{1}{64} P_\ell^4(\cos \theta), \end{aligned} \tag{4.8}$$

$$\begin{aligned} \mathbb{E}[B_\ell(\bar{x})M_\ell(y(\theta))] &= \frac{\ell(\ell+1)}{2} \frac{1}{4} \frac{1}{2!2!} \frac{1}{4!} \frac{1}{4!} \times 4! \frac{2}{\ell(\ell+1)} P_\ell^2(\cos \theta) \{P'_\ell(\cos \theta)\}^2 \\ &= \frac{\ell(\ell+1)}{2} \frac{1}{64} \frac{2}{\ell(\ell+1)} P_\ell^2(\cos \theta) \{P'_\ell(\cos \theta)\}^2, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \mathbb{E}[C_\ell(\bar{x})M_\ell(y(\theta))] &= \frac{\ell(\ell+1)}{2} \frac{3}{16} \frac{1}{4!} \frac{1}{4!} \frac{1}{4!} \times \frac{4}{\ell^2(\ell+1)^2} 4! \{P'_\ell(\cos \theta) \sin \theta\}^4 \\ &= \frac{\ell(\ell+1)}{2} \frac{3}{16} \frac{1}{64} \frac{1}{\ell^2(\ell+1)^2} \{P'_\ell(\cos \theta) \sin \theta\}^4. \end{aligned} \tag{4.10}$$

Now recall that (see Appendices A and B)

$$\begin{aligned} P_\ell\left(\cos \frac{\psi}{L}\right) &= \sqrt{\frac{2}{\pi \ell \sin \frac{\psi}{L}}} \left( \sin\left(\psi + \frac{\pi}{4}\right) + O\left(\frac{1}{\psi}\right) \right), \\ P'_\ell\left(\cos \frac{\psi}{L}\right) &= \sqrt{\frac{2}{\pi \ell \sin^3 \frac{\psi}{L}}} \left( \ell \sin\left(\psi - \frac{\pi}{4}\right) + O(1) \right). \end{aligned}$$

Let us also mention the following standard trigonometric identities that are used repeatedly in our arguments:

$$\left\{ \sin\left(\psi - \frac{\pi}{4}\right) \right\}^4 = \left\{ \frac{\sqrt{2}}{2} \sin \psi - \frac{\sqrt{2}}{2} \cos \psi \right\}^4 = \frac{3}{8} - \frac{1}{8} \cos 4\psi - \frac{1}{2} \sin 2\psi,$$

$$\left\{ \sin\left(\psi + \frac{\pi}{4}\right) \right\}^4 = \frac{3}{8} - \frac{1}{8} \cos 4\psi + \frac{1}{2} \sin 2\psi,$$

and

$$(1 + \sin 2\psi)(1 - \sin 2\psi) = \frac{1 + \cos 4\psi}{2}.$$

Substituting the latter expressions into (4.8), we obtain that

$$\begin{aligned} 8\pi^2 \mathbb{E}[A_\ell(\bar{x})M_\ell(y(\theta))] &= -\frac{\ell(\ell+1)}{2} \frac{1}{8} \pi^2 P_\ell^4(\cos\theta) \\ &= -\frac{\ell(\ell+1)}{2} \frac{1}{8} \pi^2 \left[ \sqrt{\frac{2}{\pi \ell \sin \frac{\psi}{L}}} \left( \sin\left(\psi + \frac{\pi}{4}\right) + O\left(\frac{1}{\psi}\right) \right) \right]^4 \\ &= -\frac{\ell(\ell+1)}{2} \frac{1}{8} \pi^2 \frac{2^2}{\pi^2 \ell^2 \sin^2 \frac{\psi}{L}} \left\{ \sin\left(\psi + \frac{\pi}{4}\right) \right\}^4 + O\left(\frac{1}{\psi \sin^2 \frac{\psi}{L}}\right) \\ &= -\frac{\ell(\ell+1)}{2} \frac{1}{2\ell^2 \sin^2 \frac{\psi}{L}} \left\{ \frac{3}{8} - \frac{1}{8} \cos 4\psi + \frac{1}{2} \sin 2\psi \right\} + O\left(\frac{1}{\psi \sin^2 \frac{\psi}{L}}\right) \\ &= -\frac{1}{4 \sin^2 \frac{\psi}{L}} \left\{ \frac{3}{8} - \frac{1}{8} \cos 4\psi + \frac{1}{2} \sin 2\psi \right\} + O\left(\frac{1}{\psi \sin^2 \frac{\psi}{L}}\right) + O\left(\frac{1}{\ell \sin^2 \frac{\psi}{L}}\right). \end{aligned}$$

Likewise for (4.9)

$$\begin{aligned} 8\pi^2 \mathbb{E}[B_\ell(\bar{x})M_\ell(y(\theta))] &= \frac{\ell(\ell+1)}{2} \frac{1}{8} \pi^2 \frac{2}{\ell(\ell+1)} P_\ell^2(\cos\theta) \{P'_\ell(\cos\theta) \sin\theta\}^2 \\ &= \frac{1}{8} \pi^2 \left[ \sqrt{\frac{2}{\pi \ell \sin \frac{\psi}{L}}} \left( \sin\left(\psi + \frac{\pi}{4}\right) + O\left(\frac{1}{\psi}\right) \right) \right]^2 \left[ \sqrt{\frac{2}{\pi \ell \sin^3 \frac{\psi}{L}}} \left( \ell \sin\left(\psi - \frac{\pi}{4}\right) + O(1) \right) \sin \frac{\psi}{L} \right]^2 \\ &= \frac{1}{8} \pi^2 \frac{2}{\pi \ell \sin \frac{\psi}{L}} \sin^2\left(\psi + \frac{\pi}{4}\right) \frac{2}{\pi \ell \sin \frac{\psi}{L}} \ell^2 \sin^2\left(\psi - \frac{\pi}{4}\right) + O\left(\frac{1}{\ell \sin^2 \frac{\psi}{L}}\right) \\ &= \frac{1}{2} \frac{1}{\sin \frac{\psi}{L}} \sin^2\left(\psi + \frac{\pi}{4}\right) \frac{1}{\sin \frac{\psi}{L}} \sin^2\left(\psi - \frac{\pi}{4}\right) + O\left(\frac{1}{\ell \sin^2 \frac{\psi}{L}}\right) \\ &= \frac{1}{2} \frac{1}{\sin \frac{\psi}{L}} \left\{ \frac{\sqrt{2}}{2} \sin \psi - \frac{\sqrt{2}}{2} \cos \psi \right\}^2 \frac{1}{\sin \frac{\psi}{L}} \left\{ \frac{\sqrt{2}}{2} \sin \psi - \frac{\sqrt{2}}{2} \cos \psi \right\}^2 + O\left(\frac{1}{\ell \sin^2 \frac{\psi}{L}}\right) \\ &= \frac{1}{2} \frac{1}{\sin \frac{\psi}{L}} \frac{1}{4} \{ \sin \psi + \cos \psi \}^2 \frac{1}{\sin \frac{\psi}{L}} \{ \sin \psi - \cos \psi \}^2 + O\left(\frac{1}{\ell \sin^2 \frac{\psi}{L}}\right) \\ &= \frac{1}{2} \frac{1}{\sin^2 \frac{\psi}{L}} \frac{1}{4} \{ \sin^2 \psi - \cos^2 \psi \}^2 + O\left(\frac{1}{\ell \sin^2 \frac{\psi}{L}}\right) = \frac{1}{2} \frac{1}{\sin^2 \frac{\psi}{L}} \frac{1}{4} \{ 1 - 2 \cos^2 \psi \}^2 + O\left(\frac{1}{\ell \sin^2 \frac{\psi}{L}}\right) \\ &= \frac{1}{2} \frac{1}{\sin^2 \frac{\psi}{L}} \frac{1 + \cos 4\psi}{8} + O\left(\frac{1}{\ell \sin^2 \frac{\psi}{L}}\right). \end{aligned}$$

Finally, for (4.10)

$$\begin{aligned} 8\pi^2 \mathbb{E}[C_\ell(\bar{x})M_\ell(y(\theta))] &= \frac{\ell(\ell+1)}{2} \frac{3}{2} \pi^2 \frac{1}{4! \ell^2 (\ell+1)^2} \{P'_\ell(\cos\theta) \sin\theta\}^4 \end{aligned}$$

$$\begin{aligned}
 &= \frac{3}{4} \frac{1}{4!} \pi^2 \frac{1}{\ell(\ell+1)} \left\{ \sqrt{\frac{2}{\pi \ell \sin^3 \frac{\psi}{L}}} \left( \ell \sin \left( \psi - \frac{\pi}{4} \right) + O(1) \right) \sin \frac{\psi}{L} \right\}^4 \\
 &= \frac{3}{4} \frac{1}{4!} \pi^2 \frac{1}{\ell(\ell+1)} \frac{2^2}{\pi^2 \ell^2 \sin^2 \frac{\psi}{L}} \ell^4 \sin^4 \left( \psi - \frac{\pi}{4} \right) + O \left( \frac{1}{\ell \sin^2 \frac{\psi}{L}} \right) \\
 &= \frac{1}{8} \frac{1}{\sin^2 \frac{\psi}{L}} \left[ \frac{3}{8} - \frac{1}{8} \cos 4\psi - \frac{1}{2} \sin 2\psi \right] + O \left( \frac{1}{\ell \sin^2 \frac{\psi}{L}} \right) \\
 &= \frac{1}{8} \frac{1}{\sin^2 \frac{\psi}{L}} \left[ \frac{3}{8} - \frac{1}{8} \cos 4\psi - \frac{1}{2} \sin 2\psi \right] + O \left( \frac{1}{\ell \sin^2 \frac{\psi}{L}} \right).
 \end{aligned}$$

Thus, summing (4.8), (4.9) and (4.10) we obtain

$$\begin{aligned}
 \mathcal{J}_\ell(\psi; 4) &= -\frac{1}{4L \sin^2 \frac{\psi}{L}} \left\{ \frac{3}{8} - \frac{1}{8} \cos 4\psi + \frac{1}{2} \sin 2\psi \right\} + O \left( \frac{1}{\psi \sin^2 \frac{\psi}{L}} \right) + O \left( \frac{1}{\ell \sin^2 \frac{\psi}{L}} \right) \\
 &\quad + \frac{1}{2} \frac{1}{L \sin^2 \frac{\psi}{L}} \frac{1 + \cos 4\psi}{8} + O \left( \frac{1}{\ell \sin^2 \frac{\psi}{L}} \right) \\
 &\quad + \frac{1}{8} \frac{1}{L \sin^2 \frac{\psi}{L}} \left[ \frac{3}{8} - \frac{1}{8} \cos 4\psi - \frac{1}{2} \sin 2\psi \right] + O \left( \frac{1}{\ell \sin^2 \frac{\psi}{L}} \right) \\
 &= \frac{1}{64} \frac{1}{L \sin^2 \frac{\psi}{L}} + \frac{5}{64} \frac{\cos 4\psi}{L \sin^2 \frac{\psi}{L}} - \frac{3}{16} \frac{\sin 2\psi}{L \sin^2 \frac{\psi}{L}} + O \left( \frac{1}{\psi L \sin^2 \frac{\psi}{L}} \right) + O \left( \frac{1}{\ell L \sin^2 \frac{\psi}{L}} \right) \\
 &= \frac{1}{64} \frac{1}{\psi \sin \frac{\psi}{L}} + \frac{5}{64} \frac{\cos 4\psi}{\psi \sin \frac{\psi}{L}} - \frac{3}{16} \frac{\sin 2\psi}{\psi \sin \frac{\psi}{L}} + O \left( \frac{1}{\psi^2 \sin \frac{\psi}{L}} \right) + O \left( \frac{1}{\ell \psi \sin \frac{\psi}{L}} \right), \tag{4.11}
 \end{aligned}$$

as claimed. □

**Remark 4.1.** The variance of the spherical nodal length is written [26, Proposition 2.7] as

$$\text{Var}\{\mathcal{L}_\ell\} = \int_0^{L\pi} 4\pi^2 \frac{\ell(\ell+1)}{L} \left\{ \mathcal{K}_\ell(\psi) - \frac{1}{4} \right\} \sin \left( \frac{\psi}{L} \right) d\psi.$$

Here  $\mathcal{K}_\ell(\cdot)$  represents the two-point correlation function of the nodal length, defined as

$$\mathcal{K}_\ell(\psi) = \frac{1}{2\pi \sqrt{1 - P_\ell^2(\cos \frac{\psi}{L})}} \mathbb{E} \left[ \left\| \nabla f_\ell(N) \right\| \cdot \left\| \nabla f_\ell \left( \frac{\psi}{L} \right) \right\| \mid f_\ell(N) = f_\ell \left( \frac{\psi}{L} \right) = 0 \right].$$

It was shown [26] that one has

$$\begin{aligned}
 \mathcal{K}_\ell(\psi) - \frac{1}{4} &= \frac{1}{2} \frac{\sin 2\psi}{\pi \ell \sin \frac{\psi}{L}} + \frac{9}{32} \frac{\cos 2\psi}{\pi \ell \psi \sin \frac{\psi}{L}} + \frac{1}{256} \frac{1}{\pi^2 \ell \psi \sin \frac{\psi}{L}} \\
 &\quad + \frac{\frac{27}{64} \sin 2\psi - \frac{75}{256} \cos 4\psi}{\pi^2 \ell \psi \sin \frac{\psi}{L}} + O \left( \frac{1}{\psi^3} + \frac{1}{\ell \psi} \right).
 \end{aligned}$$

To compare this result with those in the present paper, let us note that

$$4\pi^2 \frac{\ell(\ell+1)}{L} \left\{ \mathcal{K}_\ell(\psi) - \frac{1}{4} \right\} = \frac{1}{64} \cdot \frac{1}{\psi \sin \frac{\psi}{L}} + \text{oscillatory or lower order terms,}$$

in perfect analogy with the two-point cross-correlation function (4.11), used to compute the covariance

$$\text{Cov}\{\mathcal{L}_\ell, \mathcal{M}_\ell\} = \int_0^{L\pi} \mathcal{J}_\ell(\psi; 4) \sin \left( \frac{\psi}{L} \right) d\psi,$$

satisfying

$$\mathcal{J}_\ell(\psi; 4) = \frac{1}{64} \frac{1}{\psi \sin \frac{\psi}{L}} + \text{oscillatory or lower order terms.}$$

## Appendix A: Some background material

For completeness, in this Appendix we record some basic facts about covariances of random spherical harmonics and their derivatives; all the expressions to follow are rather standard and have been repeatedly exploited in the literature. Let us first recall that for arbitrary coordinates  $x = (\theta_x, \varphi_x)$ ,  $y = (\theta_y, \varphi_y)$  we have

$$\langle x, y \rangle = \cos \theta_x \cos \theta_y + \sin \theta_x \sin \theta_y \cos(\varphi_x - \varphi_y).$$

It is then elementary to show that

$$\mathbb{E}[f_\ell(x) \partial_{1;y} f_\ell(y)] = P'_\ell(\langle x, y \rangle) \{-\cos \theta_x \sin \theta_y + \sin \theta_x \cos \theta_y \cos(\varphi_x - \varphi_y)\},$$

$$\mathbb{E}[f_\ell(x) \partial_{2;y} f_\ell(y)] = P'_\ell(\langle x, y \rangle) \sin \theta_x \sin(\varphi_x - \varphi_y),$$

$$\mathbb{E}[\partial_{1;x} f_\ell(x) \partial_{1;y} f_\ell(y)]$$

$$= P''_\ell(\langle x, y \rangle) \{-\cos \theta_x \sin \theta_y + \sin \theta_x \cos \theta_y \cos(\varphi_x - \varphi_y)\} \{-\sin \theta_x \cos \theta_y + \cos \theta_x \sin \theta_y \cos(\varphi_x - \varphi_y)\} \\ + P'_\ell(\langle x, y \rangle) \{\sin \theta_x \sin \theta_y + \cos \theta_x \cos \theta_y \cos(\varphi_x - \varphi_y)\},$$

$$\mathbb{E}[\partial_{1;x} f_\ell(x) \partial_{2;y} f_\ell(y)]$$

$$= -P''_\ell(\langle x, y \rangle) \{\sin \theta_x \cos \theta_y + \cos \theta_x \sin \theta_y \cos(\varphi_x - \varphi_y)\} \sin \theta_x \sin(\varphi_x - \varphi_y) \\ - P'_\ell(\langle x, y \rangle) \cos \theta_x \sin(\varphi_x - \varphi_y),$$

$$\mathbb{E}[\partial_{2;x} f_\ell(x) \partial_{2;y} f_\ell(y)] = -P''_\ell(\langle x, y \rangle) \sin \theta_x \sin \theta_y \sin^2(\varphi_x - \varphi_y) + P'_\ell(\langle x, y \rangle) \cos(\varphi_x - \varphi_y).$$

In particular, the result we exploited several times in this paper are obtained setting  $x = (0, 0)$ ,  $y = (\theta, 0)$ :

$$\mathbb{E}[f_\ell(x) f_\ell(y)] = P_\ell(\cos \theta), \tag{A.1}$$

$$\mathbb{E}[f_\ell(x) \partial_{1;y} f_\ell(y)] = -P'_\ell(\cos \theta) \sin \theta, \tag{A.2}$$

$$\mathbb{E}[f_\ell(x) \partial_{2;y} f_\ell(y)] = \mathbb{E}[f_\ell(y) \partial_{2;y} f_\ell(x)] = 0, \tag{A.3}$$

$$\mathbb{E}[\partial_{1;x} f_\ell(x) \partial_{1;y} f_\ell(y)] = P'_\ell(\cos \theta) \cos \theta - P''_\ell(\cos \theta) \sin^2 \theta, \tag{A.4}$$

$$\mathbb{E}[\partial_{1;x} f_\ell(x) \partial_{2;y} f_\ell(y)] = \mathbb{E}[\partial_{1;x} f_\ell(y) \partial_{2;y} f_\ell(x)] = 0, \tag{A.5}$$

$$\mathbb{E}[\partial_{2;x} f_\ell(x) \partial_{2;y} f_\ell(y)] = P'_\ell(\cos \theta). \tag{A.6}$$

On the other hand, the following very useful expansions are proved [26, LemmaB.3], and hold uniformly for  $C < \psi < L \frac{\pi}{2}$  (recall that  $L := \ell + \frac{1}{2}$ ):

$$P_\ell\left(\cos \frac{\psi}{L}\right) = \sqrt{\frac{2}{\pi \ell \sin \frac{\psi}{L}}} \left( \sin\left(\psi + \frac{\pi}{4}\right) + O\left(\frac{1}{\psi}\right) \right),$$

$$P'_\ell\left(\cos \frac{\psi}{L}\right) = \sqrt{\frac{2}{\pi \ell \sin^3 \frac{\psi}{L}}} \left( \ell \sin\left(\psi - \frac{\pi}{4}\right) + O(1) \right),$$

$$P''_\ell\left(\cos \frac{\psi}{L}\right) = -\frac{\ell^2}{\sin^2 \frac{\psi}{L}} P_\ell\left(\cos \frac{\psi}{L}\right) + \frac{2}{\sin^2 \frac{\psi}{L}} P'_\ell\left(\cos \frac{\psi}{L}\right) + O\left(\frac{\ell^3}{\psi^{5/2}}\right).$$

## Appendix B: The $L^2$ approximation

We know that the nodal length is defined almost-surely by

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{S}^2} \chi_\varepsilon(f_\ell(x)) \|\nabla f_\ell(x)\| dx;$$

the almost-sure convergence follows from the standard argument ([24, Lemma 3.1]), as  $\chi_\varepsilon(\cdot) = \frac{1}{2\varepsilon} \mathbb{I}_{[-\varepsilon, \varepsilon]}(\cdot)$  is integrable and  $f_\ell$  is smooth we have, using the co-area formula [1, p. 169]

$$\int_{\mathbb{S}^2} \chi_\varepsilon(f_\ell(x)) \|\nabla f_\ell(x)\| dx = \int_{\mathbb{R}} \left\{ \int_{f_\ell^{-1}(s)} \chi_\varepsilon(f_\ell(x)) \right\} ds.$$

Since

$$\chi_\varepsilon(f_\ell(x)) = \begin{cases} 0 & \text{for } x : f_\ell(x) > \varepsilon, \\ \frac{1}{2\varepsilon} & \text{for } x : f_\ell(x) \leq \varepsilon \end{cases}$$

we obtain

$$\int_{\mathbb{R}} \left\{ \int_{f_\ell^{-1}(s)} \chi_\varepsilon(f_\ell(x)) \right\} ds = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \text{Vol}[f_\ell^{-1}(s)] ds \rightarrow \text{Vol}[f_\ell^{-1}(0)], \quad \text{as } \varepsilon \rightarrow 0,$$

since the function  $s \rightarrow \text{Vol}[f_\ell^{-1}(s)]$  is continuous for regular (Morse) functions. We now want to show that the convergence occurs also in the  $L^2$  sense; as convergence holds almost surely, it is sufficient to show that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}[\mathcal{L}_{\ell; \varepsilon}^2] = \mathbb{E}[\mathcal{L}_\ell^2].$$

Note that

$$\begin{aligned} \mathbb{E}[\mathcal{L}_{\ell; \varepsilon}^2] &= \mathbb{E} \left[ \left\{ \int_{\mathbb{S}^2} \chi_\varepsilon(f_\ell(x)) \|\nabla f_\ell(x)\| dx \right\}^2 \right] \\ &= \mathbb{E} \left[ \left\{ \int_{\mathbb{R}} \int_{f_\ell(x)=u} \chi_\varepsilon(f_\ell(x)) dx du \right\}^2 \right] \\ &= \mathbb{E} \left[ \left\{ \int_{\mathbb{R}} \mathcal{L}_\ell(u) \chi_\varepsilon(u) du \right\}^2 \right]. \end{aligned}$$

It is easy to see that the application  $u \rightarrow \mathbb{E}[\{\mathcal{L}_\ell(u)\}^2]$  is continuous, where

$$\begin{aligned} \mathbb{E}[\mathcal{L}_\ell^2(u)] &= \int_{\mathbb{S}^2 \times \mathbb{S}^2} \mathbb{E}[\|\nabla f_\ell(x_1)\| \|\nabla f_\ell(x_2)\| | f_\ell(x_1) = u, f_\ell(x_2) = u] \phi_{f_\ell(x_1), f_\ell(x_2)}(u, u) dx_1 dx_2 \\ &= 8\pi^2 \int_0^\pi \mathbb{E}[\|\nabla f_\ell(N)\| \|\nabla f_\ell(y(\theta))\| | f_\ell(N) = u, f_\ell(y(\theta)) = u] \phi_{f_\ell(N), f_\ell(y(\theta))}(u, u) \sin \theta d\theta. \end{aligned}$$

To check continuity, it is enough to show that the Dominated Convergence Theorem holds; we first note that

$$\begin{aligned} \phi_{f_\ell(N), f_\ell(y(\theta))}(u, u) \sin \theta &\leq \phi_{f_\ell(N), f_\ell(y(\theta))}(0, 0) \sin \theta \\ &= \frac{1}{2\pi \sqrt{1 - P_\ell^2(\cos \theta)}} \sin \theta = O(1), \end{aligned}$$

uniformly over  $\theta$ . On the other hand, to evaluate

$$\mathbb{E}[\|\nabla f_\ell(x_1)\| \|\nabla f_\ell(x_2)\| | f_\ell(N) = u, f_\ell(y(\theta)) = u]$$

we can use Cauchy–Schwartz inequality, and bound

$$\mathbb{E}[w_i^2 | f_\ell(N) = u, f_\ell(y(\theta)) = u] = \text{Var}[w_i | f_\ell(N) = u, f_\ell(y(\theta)) = u] + \{\mathbb{E}[w_i | f_\ell(N) = u, f_\ell(y(\theta)) = u]\}^2,$$

for  $i = 1, 2, 3, 4$ , where

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} := \begin{pmatrix} \nabla f_\ell(x_1) \\ \nabla f_\ell(x_2) \end{pmatrix}.$$

Note first that, by standard properties of Gaussian conditional distributions

$$\text{Var}[w_i | f_\ell(N) = u, f_\ell(y(\theta)) = u] = \text{Var}[w_i | f_\ell(N) = 0, f_\ell(y(\theta)) = 0],$$

and the quantities on the right-hand sides have been shown to be uniformly bounded over  $\theta$  in [26]. On the other hand, a direct computation along the same lines as in [26, Appendix A] shows that

$$\mathbb{E} \left[ \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix} \middle| f_\ell(N) = u, f_\ell(y(\theta)) = u \right] = B_\ell^T(\theta) A_\ell^{-1}(\theta) \begin{pmatrix} u \\ u \end{pmatrix},$$

where

$$B_\ell^T(\theta) = \begin{pmatrix} -P'_\ell(\cos \theta) \sin \theta & 0 \\ 0 & 0 \\ 0 & P'_\ell(\cos \theta) \sin \theta \\ 0 & 0 \end{pmatrix},$$

$$A_\ell^{-1}(\theta) = \frac{1}{1 - P_\ell^2(\cos \theta)} \begin{pmatrix} 1 & -P_\ell(\cos \theta) \\ -P_\ell(\cos \theta) & 1 \end{pmatrix},$$

so that the conditional expected value can be written as

$$\begin{aligned} & \frac{1}{1 - P_\ell^2(\cos \theta)} \begin{pmatrix} -P'_\ell(\cos \theta) \sin \theta & P'_\ell(\cos \theta) P_\ell(\cos \theta) \sin \theta \\ 0 & 0 \\ -P'_\ell(\cos \theta) P_\ell(\cos \theta) \sin \theta & P'_\ell(\cos \theta) \sin \theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ u \end{pmatrix} \\ &= \frac{1}{1 - P_\ell^2(\cos \theta)} \begin{pmatrix} u P'_\ell(\cos \theta) \sin \theta (P_\ell(\cos \theta) - 1) \\ 0 \\ u P'_\ell(\cos \theta) \sin \theta (1 - P_\ell(\cos \theta)) \\ 0 \end{pmatrix} = \frac{1}{1 + P_\ell(\cos \theta)} \begin{pmatrix} -u P'_\ell(\cos \theta) \sin \theta \\ 0 \\ u P'_\ell(\cos \theta) \sin \theta \\ 0 \end{pmatrix}. \end{aligned}$$

This vector function is immediately seen to be uniformly bounded over  $\theta$ , whence the Dominated Convergence Theorem holds. Hence

$$\begin{aligned} \mathbb{E}[\mathcal{L}_\ell^2] &\leq \liminf_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \left\{ \int_{\mathbb{S}^2} \{ \chi_\varepsilon(f_\ell(x)) \|\nabla f_\ell(x)\| \} dx \right\}^2 \right] \\ &= \liminf_{\varepsilon \rightarrow 0} \mathbb{E}[\mathcal{L}_{\ell;\varepsilon}^2] \\ &\leq \limsup_{\varepsilon \rightarrow 0} \mathbb{E}[\mathcal{L}_{\ell;\varepsilon}^2] \\ &= \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \left\{ \int_{\mathbb{S}^2} \{ \chi_\varepsilon(f_\ell(x)) \|\nabla f_\ell(x)\| \} dx \right\}^2 \right] \\ &= \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \left\{ \int_{\mathbb{R}} \mathcal{L}_\ell(u) \chi_\varepsilon(u) du \right\}^2 \right] \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \mathbb{E}[\mathcal{L}_\ell^2(u)] \chi_\varepsilon(u) du = \mathbb{E}[\mathcal{L}_\ell^2]. \end{aligned}$$

We have thus shown that  $\mathbb{E}[\mathcal{L}_{\ell;\varepsilon}^2] \rightarrow \mathbb{E}[\mathcal{L}_\ell^2]$ , and the proof is complete.



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