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Stochastic Hölder continuity of random fields governed by a system of stochastic PDEs

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Abstract. This paper constructs a solvability theory for a system of stochastic partial differential equations. On account of the Kolmogorov continuity theorem, solutions are looked for in certain Hölder-type classes in which a random field is treated as a space-time function taking values in L^p-space of random variables. A modified stochastic parabolicity condition involving p is proposed to ensure the finiteness of the associated norm of the solution, which is showed to be sharp by examples. The Schauder-type estimates and the solvability theorem are proved.

Résumé. Cet article construit une théorie sur la solvabilité d'un système d'équations différentielles partielles stochastiques. En raison du théorème de continuité de Kolmogorov, les solutions sont recherchées dans certaines classes de Hölder, dans lesquelles un champ aléatoire est considéré comme une fonction spatio-temporelle prenant des valeurs dans l'espace L^p des variables aléatoires. Une condition de parabolicité stochastique modifiée impliquant p est proposée afin d'assurer la finitude de la norme associée de la solution. En étudiant des exemples, cette condition est montrée être optimale. Les estimations de type de Schauder et la solvabilité de l'équation sont démontrées.

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Keywords: Stochastic partial differential system; Stochastic parabolicity condition; Schauder estimate; Stochastic continuity

1. Introduction

Random fields governed by systems of stochastic partial differential equations (SPDEs) have been used to model many physical phenomena in random environments such as the motion of a random string, stochastic fluid mechanic, the precessional motion of magnetisation with random perturbations, and so on; specific models can be founded in [3,7,13,16,29,31] and references therein. This paper concerns the smoothness properties of the random field

$$\boldsymbol{u} = \left(u^1, \dots, u^N\right)' : \mathbf{R}^d \times [0, \infty) \times \Omega \to \mathbf{R}^N$$

described by the following linear system of SPDEs:

$$\mathrm{d}u^\alpha = \left(a^{ij}_{\alpha\beta}\partial_{ij}u^\beta + b^i_{\alpha\beta}\partial_i u^\beta + c_{\alpha\beta}u^\beta + f_\alpha\right)\mathrm{d}t + \left(\sigma^{ik}_{\alpha\beta}\partial_i u^\beta + \nu^k_{\alpha\beta}u^\beta + g^k_\alpha\right)\mathrm{d}w^k_t, \tag{1.1}$$

where $\{w^k\}$ are countable independent Wiener processes defined on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and Einstein's summation convention is used with

$$i, j = 1, 2, \dots, d; \qquad \alpha, \beta = 1, 2, \dots, N; \qquad k = 1, 2, \dots,$$

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and the coefficients and free terms are all random fields. Considering infinitely many Wiener processes enables us to treat systems driven by space-time white noise (see [22]). Regularity theory for system (1.1) can not only directly apply to some concrete models, see for example [13,31,38,39], but also provide with important estimates for solutions of suitable approximation to nonlinear systems in the literature such as [7,21,30] and references therein.

The literature dedicated to SPDEs (not systems) is quite extensive and fruitful. In the framework of Sobolev spaces, a complete L^p-theory ($p \geq 2$) has been developed, see [5,12,20,22,23,32,37] and references therein. However, the L^p-theory for systems of SPDEs is far from complete, though for $p = 2$ it has been fully solved by [18], and for $p > 2$ some results were obtained by [19,28,29,33] in the special case where the matrices $\sigma^{ik} = [\sigma^{ik}_{\alpha\beta}]_{N\times N}$ are diagonal or nearly diagonal. The smoothness properties of random fields follow from Sobolev's embedding theorem in this framework.

The present paper investigates the regularity of random fields from another aspect prompted by Kolmogorov's continuity theorem, which ensures a continuous modification for a random field under some mild conditions. The key point is to derive appropriate estimates on L^p-moments of increments of the random field. This motivates one to consider a random field as a function of (x,t) to the space $L^p_\omega := L^p(\Omega)$ and to introduce appropriate L^p_ω-valued Hölder spaces, for instance, as in [11,25,34]. Correspondingly, in this paper we shall consider the space $\mathcal{C}^\delta_p$ of all jointly measurable random fields u satisfying

$$\|u\|_{\mathcal{C}^\delta_p} := \left[\sup_{t,x}\mathbb{E}\big|u(x,t)\big|^p + \sup_{t,x\neq y}\frac{\mathbb{E}|u(x,t)-u(y,t)|^p}{|x-y|^{\delta p}}\right]^{\frac{1}{p}} < \infty$$

with some constants $\delta \in (0,1)$ and $p \in [2,\infty)$. The space $\mathcal{C}^\delta_p$ can be regarded as a stochastic version of classical Hölder spaces. If $\delta p > d$, for each $u \in \mathcal{C}^\delta_p$, it has a modification with Hölder continuous in space by Kolmogorov's theorem.

For the Cauchy problem for parabolic SPDEs (not systems), a $C^{2+\delta}$-theory was once an open problem proposed by [22]. Based on the Hölder class $\mathcal{C}^\delta_p$, it was partially addressed by [25] and generally solved by [10,11] very recently. The theory says that, *under natural conditions on the coefficients, the solution u and its derivatives ∂u and $\partial^2 u$ belong to a space $\mathcal{C}^\delta_p$, provided that f, g and ∂g belong to the same space*. In addition, [11] also obtained stochastic Hölder continuity in time for $\partial^2 u$. Their established theory is sharp in the sense that the result cannot be improved under the same assumptions. Extensions to the Cauchy–Dirichlet problem of SPDEs can be found in [26,27], and for more related results, we refer the reader to, for instance, [2,6,36]. Nevertheless, $C^{2+\delta}$-theory for systems of SPDEs is not yet known in the literature.

The purpose of this paper is to construct such a $C^{2+\delta}$-theory for systems of type (1.1) under mild conditions. Like the situation in the L^p framework this extension is also nontrivial as some new features emerge in the system of SPDEs comparing with single equations. It is well-known that the well-posedness of a second order SPDE is usually guaranteed by certain coercivity conditions. For system (1.1), [18] recently obtained W^n_2-solutions under the following algebraic condition: there is a constant $\kappa > 0$ such that

$$\big(2a^{ij}_{\alpha\beta} - \sigma^{ik}_{\gamma\alpha}\sigma^{jk}_{\gamma\beta}\big)\xi^\alpha_i\xi^\beta_j \geq \kappa|\xi|^2 \quad \forall \xi \in \mathbf{R}^{d\times N}. \tag{1.2}$$

Although it is a natural extension of the strong ellipticity condition for PDE systems ($\sigma \equiv 0$, see for example [35]) and of the stochastic parabolicity condition for SPDEs ($N = 1$, see for example [22]), the following example constructed by [19] reveals that condition (1.2) is not sufficient to ensure the finiteness of L^p_ω-norm of the solution of some system even the given data are smooth, and some structure condition stronger than (1.2) is indispensable to establish a general L^p or $C^{2+\delta}$ theory for systems of type (1.1).

Example 1.1. Let $d = 1$, $N = 2$ and $p > 2$. Consider the following system:

$$\begin{cases} \mathrm{d}u^{(1)} = u^{(1)}_{xx}\,\mathrm{d}t - \mu u^{(2)}_x\,\mathrm{d}w_t, \\ \mathrm{d}u^{(2)} = u^{(2)}_{xx}\,\mathrm{d}t + \mu u^{(1)}_x\,\mathrm{d}w_t \end{cases} \tag{1.3}$$

with the initial data

$$u^{(1)}(x,0) = \mathrm{e}^{-\frac{x^2}{2}}, \qquad u^{(2)}(x,0) = 0,$$

where μ is a given constant. In this case, condition (1.2) reads $\mu^2 < 2$, but we will see that this is not sufficient to ensure the finiteness of $\mathbb{E}|\boldsymbol{u}(x,t)|^p$ with $p > 2$. Set $v = u^{(1)} + \sqrt{-1}u^{(2)}$, and the above system turns to a single equation:

$$\mathrm{d}v = v_{xx}\,\mathrm{d}t + \sqrt{-1}\mu v_x\,\mathrm{d}w_t \tag{1.4}$$

with $v(x,0)=u^{(1)}(x,0)$. It can be verified directly by Itô's formula that

$$v(x,t)=\frac{1}{\sqrt{1+(2+\mu^2)t}}\exp\left\{-\frac{(x+\sqrt{-1}\mu w_t)^2}{2[1+(2+\mu^2)t]}\right\}$$

solves (1.4) with the given initial condition. So we can compute

$$\begin{aligned}\mathbb{E}\left|\boldsymbol{u}(x,t)\right|^p&=\mathbb{E}\left|v(x,t)\right|^p\\&=\frac{1}{\sqrt{2\pi t}}\frac{1}{[1+(2+\mu^2)t]^{p/2}}\mathrm{e}^{-\frac{px^2}{2[1+(2+\mu^2)t]}}\int_{\mathbf{R}}\mathrm{e}^{-\frac{y^2}{2t}[1-\frac{p\mu^2t}{1+(2+\mu^2)t}]}\,\mathrm{d}y.\end{aligned}\tag{1.5}$$

It is noticed that

$$1-\frac{p\mu^2t}{1+(2+\mu^2)t}\to\frac{2-(p-1)\mu^2}{2+\mu^2}\quad\text{as } t\to\infty,$$

which implies that if

$$\mu^2>\frac{2}{p-1},\tag{1.6}$$

the integral in (1.5) diverges for large t, and $\mathbb{E}|\boldsymbol{u}(x,t)|^p=\infty$ for every x.

A major contribution of this paper is the finding of a general coercivity condition that ensures us to construct a general $C^{2+\delta}$-theory for system (1.1). The basic idea is to impose an appropriate correction term involving p to the left-hand side of (1.2). More specifically, we introduce

Definition 1.2 (MSP condition). Let $p\in[2,\infty)$. The coefficients $a=(a^{ij}_{\alpha\beta})$ and $\sigma=(\sigma^{ik}_{\alpha\beta})$ are said to satisfy the *modified stochastic parabolicity (MSP) condition* if there are measurable functions $\lambda^{ik}_{\alpha\beta}:\mathbf{R}^d\times[0,\infty)\times\Omega\to\mathbf{R}$ with $\lambda^{ik}_{\alpha\beta}=\lambda^{ik}_{\beta\alpha}$, such that

$$\mathcal{A}^{ij}_{\alpha\beta}(p,\lambda):=2a^{ij}_{\alpha\beta}-\sigma^{ik}_{\gamma\alpha}\sigma^{jk}_{\gamma\beta}-(p-2)\big(\sigma^{ik}_{\gamma\alpha}-\lambda^{ik}_{\gamma\alpha}\big)\big(\sigma^{jk}_{\gamma\beta}-\lambda^{jk}_{\gamma\beta}\big)\tag{1.7}$$

satisfy the *Legendre–Hadamard condition*: there is a constant $\kappa>0$ such that

$$\mathcal{A}^{ij}_{\alpha\beta}(p,\lambda)\xi_i\xi_j\eta^\alpha\eta^\beta\ge\kappa|\xi|^2|\eta|^2\quad\forall\xi\in\mathbf{R}^d,\eta\in\mathbf{R}^N\tag{1.8}$$

everywhere on $\mathbf{R}^d\times[0,\infty)\times\Omega$.

In particular, the following criteria for the MSP condition, simplified by taking $\lambda^{ik}_{\alpha\beta}=0$ and $\lambda^{ik}_{\alpha\beta}=(\sigma^{ik}_{\alpha\beta}+\sigma^{ik}_{\beta\alpha})/2$ respectively in (1.7), could be very convenient in applications.

Lemma 1.3. *The MSP condition is satisfied if either*

(i) $2a^{ij}_{\alpha\beta}-(p-1)\sigma^{ik}_{\gamma\alpha}\sigma^{jk}_{\gamma\beta}$ *or*
(ii) $2a^{ij}_{\alpha\beta}-\sigma^{ik}_{\gamma\alpha}\sigma^{jk}_{\gamma\beta}-(p-2)\widehat{\sigma}^{ik}_{\gamma\alpha}\widehat{\sigma}^{jk}_{\gamma\beta}$ *with* $\widehat{\sigma}^{ik}_{\alpha\beta}:=(\sigma^{ik}_{\alpha\beta}-\sigma^{ik}_{\beta\alpha})/2$

satisfies the Legendre–Hadamard condition.

Evidently, the MSP condition is *invariant* under change of basis of $\mathbf{R}^d$ or under orthogonal transformation of $\mathbf{R}^N$. Also the Legendre–Hadamard condition (see for example [14]) is more general than the strong ellipticity condition. The MSP condition coincides with the Legendre–Hadamard condition for PDE systems and the stochastic parabolicity condition for SPDEs. Besides when $p=2$ it becomes

$$\big(2a^{ij}_{\alpha\beta}-\sigma^{ik}_{\gamma\alpha}\sigma^{jk}_{\gamma\beta}\big)\xi_i\xi_j\eta^\alpha\eta^\beta\ge\kappa|\xi|^2|\eta|^2\quad\forall\xi\in\mathbf{R}^d,\eta\in\mathbf{R}^N\tag{1.9}$$

which is weaker than (1.2) used in [18]. Moreover, the case (ii) in Lemma 1.3 shows that the MSP condition is also reduced to (1.9) if the matrices $B^{ik}:=[\sigma^{ik}_{\alpha\beta}]_{N\times N}$ are close to be symmetric. Nevertheless, the generality of the MSP condition cannot be covered by these cases in Lemma 1.3, which is illustrated by Example 6.5 in the final section.

Example 1.1 illustrates that in (1.7) the coefficient of the correction term $p-2$ is *optimal* to guarantee the Schauder regularity for the SPDEs (1.1). Indeed, if $p>2$ is fixed and the coefficient $p-2$ in (1.7) drops down a bit to $p-2-\varepsilon>0$, we can choose the value of μ satisfying

$$\frac{2}{p-1}<\mu^2<\frac{2}{p-1-\varepsilon},$$

then it is easily verified that system (1.3) satisfies (1.8) in this setting by taking $\lambda^{ik}_{\alpha\beta}=0$ and $p-2$ replaced by $p-2-\varepsilon$. However, Example 1.1 has showed that when t is large enough $\mathbb{E}|\boldsymbol{u}(x,t)|^p$ becomes infinite for such a choice of μ, let alone the $\mathcal{C}^\delta_p$-norm of the solution. More examples in this respect are discussed in the final section.

Technically speaking, the MSP condition is explicitly used to derive a class of mixed norm estimates for the model system in the space $L^p(\Omega;W^n_2)$. Owing to Sobolev's embedding the mixed norm estimates lead to the local boundedness of $\mathbb{E}|\partial^m\boldsymbol{u}(x,t)|^p$, which plays a key role in the derivation of the fundamental interior estimate of Schauder-type for system (1.1). A similar issue was addressed in [4] for an abstract stochastic equation on torus, in which the authors give a stochastic parabolic condition depending on p for the well-posedness of some $L^p(X)$-solution ($p\ge 2$ for general cases, $1<p<2$ with a stronger restriction for a special case), and show the sharpness of their condition. Very recently, [33] establish an L^p-theory for a class of complex valued stochastic PDE systems, see the last remark in Section 2.

As in the classic PDE theory, the results for linear equations are the building base for the study of nonlinear equations. We point out that it is not hard to generalize our linear theory to the nonlinear case in which f_α is Lipschitz continuous w.r.t. u and ∂u, and g^k_α is Lipschitz continuous w.r.t. u as well.

The paper is organized as follows. In Section 2 we introduce some notations and state our main results. In Sections 3 and 4 we consider the model system

$$\mathrm{d}u^\alpha=\left(a^{ij}_{\alpha\beta}\partial_{ij}u^\beta+f_\alpha\right)\mathrm{d}t+\left(\sigma^{ik}_{\alpha\beta}\partial_i u^\beta+g^k_\alpha\right)\mathrm{d}w^k_t,$$

where the coefficients a and σ are *random* but independent of x. We prove the crucial mixed norm estimates in Section 3, and then establish the interior Hölder estimate in Section 4. In Section 5 we complete the proofs of our main results. The final section is devoted to more comments and examples on the sharpness and flexibility of the MSP condition.

2. Main results

Let us first introduce our working spaces and associated notations. A Banach-space valued Hölder continuous function is defined analogously to the classical Hölder continuous function. Let E be a Banach space, $\mathcal{O}$ a domain in $\mathbf{R}^d$ and I an interval. We define the parabolic modulus

$$|X|_{\mathrm{p}}=|x|+\sqrt{|t|}\quad\text{for } X=(x,t)\in Q:=\mathcal{O}\times I.$$

For a space-time function $\boldsymbol{u}:Q\to E$, we define

$$\begin{aligned}
&[\boldsymbol{u}]^E_{m;Q}:=\sup\left\{\left\|\partial^{\mathfrak{s}}\boldsymbol{u}(X)\right\|_E : X=(x,t)\in Q,|\mathfrak{s}|=m\right\},\\
&|\boldsymbol{u}|^E_{m;Q}:=\max\left\{[\boldsymbol{u}]^E_{k;Q}:k\le m\right\},\\
&[\boldsymbol{u}]^E_{m+\delta;Q}:=\sup_{|\mathfrak{s}|=m}\sup_{t\in I}\sup_{x,y\in\mathcal{O}}\frac{\|\partial^{\mathfrak{s}}\boldsymbol{u}(x,t)-\partial^{\mathfrak{s}}\boldsymbol{u}(y,t)\|_E}{|x-y|^\delta},\\
&|\boldsymbol{u}|^E_{m+\delta;Q}:=|\boldsymbol{u}|^E_{m;Q}+[\boldsymbol{u}]^E_{m+\delta;Q},\\
&[\boldsymbol{u}]^E_{(m+\delta,\delta/2);Q}:=\sup_{|\mathfrak{s}|=m}\sup_{X,Y\in Q}\frac{\|\partial^{\mathfrak{s}}\boldsymbol{u}(X)-\partial^{\mathfrak{s}}\boldsymbol{u}(Y)\|_E}{|X-Y|^\delta_{\mathrm{p}}},\\
&|\boldsymbol{u}|^E_{(m+\delta,\delta/2);Q}:=|\boldsymbol{u}|^E_{m;Q}+[\boldsymbol{u}]^E_{(m+\delta,\delta/2);Q}
\end{aligned}$$

with $m\in\mathbf{N}:=\{0,1,2,\dots\}$ and $\delta\in(0,1)$, where $\mathfrak{s}=(\mathfrak{s}_1,\dots,\mathfrak{s}_d)\in\mathbf{N}^d$ with $|\mathfrak{s}|=\sum_{i=1}^d\mathfrak{s}_i$, and all the derivatives of an E-valued function are defined with respect to the *spatial variable* in the strong sense, see [17]. In the following context, the space E is either (i) an Euclidean space, (ii) the space ℓ^2, or (iii) $L^p_\omega:=L^p(\Omega)$ (abbreviation for L^p_ω for both $L^p(\Omega;\mathbf{R}^N)$

or $L^p(\Omega;\ell^2)$). We omit the superscript in cases (i) and (ii), and in case (iii), we introduce some new notations:

$$[\![\boldsymbol{u}]\!]_{m+\delta,p;Q} := [\boldsymbol{u}]^{L^p_\omega}_{m+\delta;Q}, \qquad [\![\boldsymbol{u}]\!]_{(m+\delta,\delta/2),p;Q} := [\boldsymbol{u}]^{L^p_\omega}_{(m+\delta,\delta/2);Q},$$

$$|\!|\!|\boldsymbol{u}|\!|\!|_{m+\delta,p;Q} := |\boldsymbol{u}|^{L^p_\omega}_{m+\delta;Q}, \qquad |\!|\!|\boldsymbol{u}|\!|\!|_{(m+\delta,\delta/2),p;Q} := |\boldsymbol{u}|^{L^p_\omega}_{(m+\delta,\delta/2);Q}.$$

As the random fields in this paper take values in different spaces like $\mathbf{R}^N$ (say, $\boldsymbol{u}$ and $\boldsymbol{f}$) or ℓ^2 (say, $\boldsymbol{g}$), we shall use $|\cdot|$ uniformly for the standard norms in Euclidean spaces and in ℓ^2, and L^p_ω for both $L^p(\Omega;\mathbf{R}^N)$ and $L^p(\Omega;\ell^2)$. The specific meaning of the notation can be easily understood in context.

Definition. The Hölder classes $C^{m+\delta}_x(Q;L^p_\omega)$ and $C^{m+\delta,\delta/2}_{x,t}(Q;L^p_\omega)$ are defined as the sets of all *predictable* random fields $\boldsymbol{u}$ defined on $Q\times\Omega$ and taking values in an Euclidean space or ℓ^2 such that $|\!|\!|\boldsymbol{u}|\!|\!|_{m+\delta,p;Q}$ and $|\!|\!|\boldsymbol{u}|\!|\!|_{(m+\delta,\delta/2),p;Q}$ are finite, respectively.

The following notations for special domains are frequently used:

$$B_r(x) = \{y\in\mathbf{R}^d : |y-x|<r\}, \qquad Q_r(x,t) = B_r(x)\times(t-r^2,t],$$

and $B_r = B_r(0)$, $Q_r = Q_r(0,0)$, and also

$$\mathcal{Q}_{r,T}(x) := B_r(x)\times(0,T], \qquad \mathcal{Q}_{r,T} = \mathcal{Q}_{r,T}(0) \quad \text{and} \quad \mathcal{Q}_T := \mathbf{R}^d\times(0,T].$$

Assumption. The following conditions are used throughout the paper unless otherwise stated:

(i) For all $i,j=1,\dots,d$ and $\alpha,\beta=1,\dots,N$, the random fields $a^{ij}_{\alpha\beta}$, $b^i_{\alpha\beta}$, $c_{\alpha\beta}$ and f_α are real-valued, and $\sigma^i_{\alpha\beta}$, $\nu_{\alpha\beta}$ and g_α are ℓ^2-valued; all of them are predictable.

(ii) $a^{ij}_{\alpha\beta}$ and $\sigma^i_{\alpha\beta}$ satisfy the MSP condition with some $p\in[2,\infty)$.

(iii) For some $\delta\in(0,1)$, the classical C^δ_x-norms of $a^{ij}_{\alpha\beta}$, $b^i_{\alpha\beta}$ and $c_{\alpha\beta}$, and the $C^{1+\delta}_x$-norms of $\sigma^i_{\alpha\beta}$ and $\nu_{\alpha\beta}$ are all dominated by a constant K.

We are ready to state the main results of the paper. The first result is the a priori *interior Hölder estimates* for system (1.1).

Theorem 2.1. *Under the above setting, there exist two constants $\rho_0\in(0,1)$ and $C>0$, both depending only on d, N, κ, K, p and δ, such that if $\boldsymbol{u}\in C^{2+\delta}_x(Q_1(X);L^p_\omega)$ satisfies (1.1) in $Q_1(X)$ with $X=(x,t)\in\mathbf{R}^d\times[1,\infty)$, then*

$$\rho^{2+\delta}[\![\partial^2\boldsymbol{u}]\!]_{(\delta,\delta/2),p;Q_{\rho/2}(X)} \\ \le C\bigl\{\rho^2|\!|\!|\boldsymbol{f}|\!|\!|_{\delta,p;Q_\rho(X)} + \rho|\!|\!|\boldsymbol{g}|\!|\!|_{1+\delta,p;Q_\rho(X)} + \rho^{-\frac{d}{2}}\bigl[\mathbb{E}\|\boldsymbol{u}\|^p_{L^2(Q_\rho(X))}\bigr]^{\frac{1}{p}}\bigr\} \tag{2.1}$$

for any $\rho\in(0,\rho_0]$, provided the right-hand side is finite.

By rescaling one can obtain the local estimate around any point $X\in\mathbf{R}^d\times(0,\infty)$.

The second theorem is regarding the global Hölder estimate and solvability for the Cauchy problem for system (1.1) with zero initial condition.

Theorem 2.2. *Under the above setting, if $\boldsymbol{f}\in C^\delta_x(\mathcal{Q}_T;L^p_\omega)$ and $\boldsymbol{g}\in C^{1+\delta}_x(\mathcal{Q}_T;L^p_\omega)$ with $T>0$, then system (1.1) with the initial condition*

$$\boldsymbol{u}(x,0) = \mathbf{0} \quad \forall x\in\mathbf{R}^d$$

admits a unique solution $\boldsymbol{u}\in C^{2+\delta,\delta/2}_{x,t}(\mathcal{Q}_T;L^p_\omega)$, and it satisfies the estimate

$$|\!|\!|\boldsymbol{u}|\!|\!|_{(2+\delta,\delta/2),p;\mathcal{Q}_T} \le C\mathrm{e}^{CT}\bigl(|\!|\!|\boldsymbol{f}|\!|\!|_{\delta,p;\mathcal{Q}_T} + |\!|\!|\boldsymbol{g}|\!|\!|_{1+\delta,p;\mathcal{Q}_T}\bigr), \tag{2.2}$$

where the constant C depends only on d, N, κ, K, p and δ.

Remark. Theorem 2.2 still holds true if the system is considered on the torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$ instead of $\mathbf{R}^d$.

Remark. The above theorems show that the solutions possess the Hölder continuity in time even for time-irregular coefficients and free terms. A similar property of classical PDEs is well-known in the literature, see for example [9, 24] and references therein. In view of an anisotropic Kolmogorov continuity theorem (see [8]) the solution obtained in Theorem 2.2 has a modification that is Hölder continuous jointly in space and time.

Remark. Although we only consider the real valued SPDE systems in this paper, our method can also be applied to the complex valued case, as long as the complex valued coefficients $a = (a^{ij}_{\alpha\beta})$ and $\sigma = (\sigma^{ik}_{\alpha\beta})$ satisfy the following *complex version MSP condition*:

There are measurable complex valued functions $\lambda^{ik}_{\alpha\beta} : \mathbf{R}^d \times [0,\infty) \times \Omega \to \mathbf{C}$ with $\lambda^{ik}_{\alpha\beta} = \bar{\lambda}^{ik}_{\beta\alpha}$, such that

$$\mathcal{A}^{ij}_{\alpha\beta}(p,\lambda) := 2a^{ij}_{\alpha\beta} - \sigma^{ik}_{\gamma\alpha}\bar{\sigma}^{jk}_{\gamma\beta} - (p-2)\big(\sigma^{ik}_{\gamma\alpha} - \lambda^{ik}_{\gamma\alpha}\big)\big(\bar{\sigma}^{jk}_{\gamma\beta} - \bar{\lambda}^{jk}_{\gamma\beta}\big)$$

satisfy the *Legendre–Hadamard condition*: there is a constant $\kappa > 0$ such that

$$\mathrm{Re}\big[\mathcal{A}^{ij}_{\alpha\beta}(p,\lambda)\xi_i\bar{\xi}_j\eta^\alpha\bar{\eta}^\beta\big] \geq \kappa|\xi|^2|\eta|^2 \quad \forall \xi \in \mathbf{C}^d, \eta \in \mathbf{C}^N$$

everywhere on $\mathbf{R}^d \times [0,\infty) \times \Omega$.

The readers can check out that Theorem 3.1 still holds true for the complex SPDE system with the complex version MSP condition in hand, which is the key theorem to guarantee the whole theory workable. Very recently, [33] also proposed a stochastic parabolic condition for complex valued SPDE systems and established an L^p theory.

3. Integral estimates for the model system

Throughout this section we assume that $a^{ij}_{\alpha\beta}$ and $\sigma^{ik}_{\alpha\beta}$ depend only on (t,ω), but are *independent of* x, satisfying the MSP condition (in this case $\lambda^{ik}_{\alpha\beta}$ *is chosen to be independent of* x) and

$$\big|a^{ij}_{\alpha\beta}\big|, \big|\sigma^{ij}_{\alpha\beta}\big| \leq K, \quad \forall t,\omega, \tag{3.1}$$

and we consider the following model system

$$\mathrm{d}u^\alpha = \big(a^{ij}_{\alpha\beta}\partial_{ij}u^\beta + f_\alpha\big)\,\mathrm{d}t + \big(\sigma^{ik}_{\alpha\beta}\partial_i u^\beta + g^k_\alpha\big)\,\mathrm{d}w^k_t. \tag{3.2}$$

The aim of this section is to derive several auxiliary estimates for the model system that will be used to prove the interior Hölder estimate in the next section.

In this section and the next one, *we may consider* (3.2) *in the entire space* $\mathbf{R}^n \times \mathbf{R}$. On the one hand, we can always extend (1.1) and (3.2) to the entire space if we require $u(x,0) = 0$. Indeed, the zero extensions of $\boldsymbol{u}$, $\boldsymbol{f}$ and $\boldsymbol{g}$ (i.e., these functions are defined to be zero for $t < 0$) satisfy the equations in the entire space, where the extension of coefficients and Wiener processes are quite easy; for example, we can define $a^{ij}_{\alpha\beta}(t) = \delta^{ij}$ and $\sigma^{ik}_{\alpha\beta} = 0$ for $t < 0$, and $w_t := \tilde{w}_{-t}$ for $t < 0$ with $\tilde{w}$ being an independent copy of w. On the other hand, we mainly concern the local estimates for the equation (3.2) in the following two sections, so we can only focus on the estimates around the origin on account of a translation. Indeed, we can reduce the estimates around a point (x_0,t_0) to the estimates around the origin by a change of variables $(x,t) \mapsto (x-x_0, t-t_0)$.

Let $\mathcal{O} \in \mathbf{R}^d$ and $H^m(\mathcal{O}) = W^m_2(\mathcal{O})$ be the usual Sobolev spaces. Let $I \subset \mathbf{R}$ be an interval and $Q = \mathcal{O} \times I$. For $p,q \in [1,\infty]$, define

$$L^p_\omega L^q_t H^m_x(Q) := L^p\big(\Omega; L^q\big(I; H^m\big(\mathcal{O};\mathbf{R}^N\big)\big)\big).$$

In what follows, we denote $\partial^m\boldsymbol{u}$ the set of all m-order derivatives of a function $\boldsymbol{u}$. These $\partial^m\boldsymbol{u}(x)$ for each x and (ω,t) are regarded as elements of a Euclidean space of proper dimension.

Our $\mathcal{C}^{2+\delta}$-theory is grounded in the following mixed norm estimates for model system (3.2), in which the modified stochastic parabolicity condition (1.8) plays a key role.

Theorem 3.1. *Let $p \in [2,\infty)$ and $m \geq 0$. Suppose $\boldsymbol{f} \in L^p_\omega L^2_t H^{m-1}_x(\mathcal{Q}_T)$ and $\boldsymbol{g} \in L^p_\omega L^2_t H^m_x(\mathcal{Q}_T)$. Then (3.2) with zero initial value admits a unique solution $\boldsymbol{u} \in L^p_\omega L^\infty_t H^m_x(\mathcal{Q}_T) \cap L^p_\omega L^2_t H^{m+1}_x(\mathcal{Q}_T)$. Moreover, for any multi-index $\mathfrak{s}$ such that $|\mathfrak{s}| \leq m$,*

$$\left\|\partial^{\mathfrak{s}}\boldsymbol{u}\right\|_{L^p_\omega L^\infty_t L^2_x} + \left\|\partial^{\mathfrak{s}}\boldsymbol{u}_x\right\|_{L^p_\omega L^2_t L^2_x} \leq C\big(\left\|\partial^{\mathfrak{s}}\boldsymbol{f}\right\|_{L^p_\omega L^2_t H^{-1}_x} + \left\|\partial^{\mathfrak{s}}\boldsymbol{g}\right\|_{L^p_\omega L^2_t L^2_x}\big), \tag{3.3}$$

where the constant C depends only on d, p, T, N, κ, and K.

The proof of Theorem 3.1 is postponed to the end of this section. A quick consequence of this theorem is the following local estimates for model equations with smooth free terms.

Proposition 3.2. *Let $m \geq 1$, $p \geq 2$, $r > 0$ and $0 < \theta < 1$, and let $\boldsymbol{u} \in L^p_\omega L^\infty_t H^m_x(Q_r) \cap L^p_\omega L^2_t H^{m+1}_x(Q_r)$ solve (3.2) in Q_r with $\boldsymbol{f} \in L^p_\omega L^2_t H^{m-1}_x(Q_r)$ and $\boldsymbol{g} \in L^p_\omega L^2_t H^m_x(Q_r)$. Then there is a constant $C = C(d,p,\kappa,K,m,N,\theta)$ such that*

$$\begin{aligned}\left\|\partial^m\boldsymbol{u}\right\|_{L^p_\omega L^\infty_t L^2_x(Q_{\theta r})} + \left\|\partial^m\boldsymbol{u}_x\right\|_{L^p_\omega L^2_t L^2_x(Q_{\theta r})} \leq{}& Cr^{-m-1}\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(Q_r)} + C\sum_{k=0}^{m-1} r^{-m+k+1}\left\|\partial^k\boldsymbol{f}\right\|_{L^p_\omega L^2_t L^2_x(Q_r)}\\ &+ C\sum_{k=0}^{m} r^{-m+k}\left\|\partial^k\boldsymbol{g}\right\|_{L^p_\omega L^2_t L^2_x(Q_r)}.\end{aligned} \tag{3.4}$$

Consequently, for $2(m-|\mathfrak{s}|) > d$,

$$\begin{aligned}\Big\|\sup_{Q_{\theta r}}\left|\partial^{\mathfrak{s}}\boldsymbol{u}\right|\Big\|_{L^p_\omega} \leq{}& Cr^{-|\mathfrak{s}|-d/2-1}\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(Q_r)}\\ &+ C\sum_{k=0}^{m-1} r^{-|\mathfrak{s}|-d/2+k+1}\left\|\partial^k\boldsymbol{f}\right\|_{L^p_\omega L^2_t L^2_x(Q_r)} + C\sum_{k=0}^{m} r^{-|\mathfrak{s}|-d/2+k}\left\|\partial^k\boldsymbol{g}\right\|_{L^p_\omega L^2_t L^2_x(Q_r)}.\end{aligned} \tag{3.5}$$

Proof. It suffices to prove (3.4) as (3.5) follows from (3.4) immediately by Sobolev's embedding theorem [1, Theorem 4.12]. Moreover for general $r > 0$, we can apply the obtained estimates for $r = 1$ to the rescaled function

$$\boldsymbol{v}(x,t) := \boldsymbol{u}\big(rx, r^2t\big), \quad \forall (x,t) \in \mathbf{R}^d \times \mathbf{R}$$

which solves the equation

$$\mathrm{d}v^\alpha(x,t) = \big(a^{ij}_{\alpha\beta}\big(r^2t\big)\partial_{ij}v^\beta(x,t) + F_\alpha\big)\,\mathrm{d}t + \big(\sigma^{ik}_{\alpha\beta}\big(r^2t\big)\partial_i v^\beta(x,t) + G^k_\alpha\big)\,\mathrm{d}\beta^k_t, \tag{3.6}$$

with

$$F_\alpha(x,t) = r^2 f_\alpha\big(rx, r^2t\big), \qquad G^k_\alpha(x,t) = rg\big(rx, r^2t\big), \qquad \beta^k_t = r^{-1}w^k_{r^2t}.$$

Obviously, β^k are mutually independent standard Wiener processes.

For any $\theta \in (0,1)$, choose cut-off functions $\zeta^\ell \in C^\infty_0(\mathbf{R}^{d+1})$, $\ell = 1,2$, satisfying i) $0 \leq \zeta^\ell \leq 1$, ii) $\zeta^1 = 1$ in $Q_{\sqrt{\theta}}$ and $\zeta^1 = 0$ outside Q_1, and iii) $\zeta^2 = 1$ in Q_θ and $\zeta^2 = 0$ outside $Q_{\sqrt{\theta}}$. Then $\boldsymbol{v}_\ell = \zeta^\ell\boldsymbol{u}$ ($\ell = 1,2$) satisfy

$$\mathrm{d}v^\alpha_l = \big(a^{ij}_{\alpha\beta}\partial_{ij}v^\beta_\ell + \tilde{f}_{\ell,\alpha}\big)\,\mathrm{d}t + \big(\sigma^{ik}_{\alpha\beta}\partial_i v^\beta_\ell + \tilde{g}^k_{\ell,\alpha}\big)\,\mathrm{d}w^k_t, \quad \ell = 1,2, \tag{3.7}$$

where

$$\begin{aligned}\tilde{f}_{\ell,\alpha} &= \zeta^\ell f_\alpha - a^{ij}_{\alpha\beta}\big(\zeta^\ell_{x_i}u^\beta\big)_{x_j} + a^{ij}_{\alpha\beta}\zeta^\ell_{x_ix_j}u^\beta + \big(\partial_t\zeta^\ell\big)u^\alpha,\\ \tilde{g}^k_{\ell,\alpha} &= \zeta^\ell g^k_\alpha - \sigma^{ik}_{\alpha\beta}\zeta^\ell_{x_i}u^\beta, \quad \ell = 1,2.\end{aligned}$$

Applying Theorem 3.1 to (3.7) for $\ell = 1$, $\mathfrak{s} = 0$ and for $\ell = 2$, $|\mathfrak{s}| = 1$, we have

$$\begin{aligned}&\|\boldsymbol{u}\|_{L^p_\omega L^\infty_t L^2_x(Q_{\sqrt{\theta}})} + \|\boldsymbol{u}_x\|_{L^p_\omega L^2_t L^2_x(Q_{\sqrt{\theta}})}\\ &\quad\leq C\big(\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(Q_1)} + \|\boldsymbol{f}\|_{L^p_\omega L^2_t L^2_x(Q_1)} + \|\boldsymbol{g}\|_{L^p_\omega L^2_t L^2_x(Q_1)}\big);\end{aligned}$$

$$
\begin{aligned}
&\|\partial^{\mathfrak{s}}\boldsymbol{u}\|_{L^p_\omega L^\infty_t L^2_x(Q_\theta)}+\|\partial^{\mathfrak{s}}\boldsymbol{u}_x\|_{L^p_\omega L^2_t L^2_x(Q_\theta)}\\
&\quad\le C\big(\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(Q_{\sqrt{\theta}})}+\|\partial^{\mathfrak{s}}\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(Q_{\sqrt{\theta}})}\|+\|\boldsymbol{f}\|_{L^p_\omega L^2_t L^2_x(Q_{\sqrt{\theta}})}+\|\partial^{\mathfrak{s}}\boldsymbol{g}\|_{L^p_\omega L^2_t L^2_x(Q_{\sqrt{\theta}})}\big).
\end{aligned}
$$

Combining these two estimates, we have (3.4) for $m=1$. Higher order estimates follows from induction. The proof is complete. □

Another consequence of Theorem 3.1 is the following lemma concerning the solution for equation (3.2) with the Cauchy–Dirichlet boundary conditions:

$$
\begin{cases}\boldsymbol{u}(x,0)=0, & \forall x\in B_r;\\ \boldsymbol{u}(x,t)=0, & \forall (x,t)\in\partial B_r\times(0,T].\end{cases}\tag{3.8}
$$

Proposition 3.3. *Let $\boldsymbol{f}=\boldsymbol{f}^0+\partial_i\boldsymbol{f}^i$ and $\boldsymbol{f}^0,\boldsymbol{f}^1,\dots,\boldsymbol{f}^d,\boldsymbol{g}\in L^p_\omega L^2_t H^m_x(\mathcal{Q}_{r,r^2})$ for all $m\ge 0$. Then problem (3.2) and (3.8) admits a unique solution $\boldsymbol{u}\in L^2_\omega L^2_t H^1_x(\mathcal{Q}_{r,r^2})$, and for each $t\in(0,r^2)$, $\boldsymbol{u}(\cdot,t)\in L^p(\Omega;C^m(B_\varepsilon;\mathbf{R}^N))$ with any $m\ge 0$ and $\varepsilon\in(0,r)$. Moreover, there is a constant $C=C(n,p)$ such that*

$$
\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(\mathcal{Q}_{r,r^2})}\le C\big(r^2\|\boldsymbol{f}^0\|_{L^p_\omega L^2_t L^2_x(\mathcal{Q}_{r,r^2})}+r\|(\boldsymbol{f}^1,\dots,\boldsymbol{f}^d,\boldsymbol{g})\|_{L^p_\omega L^2_t L^2_x(\mathcal{Q}_{r,r^2})}\big).\tag{3.9}
$$

Proof. The existence, uniqueness and smoothness of the solution of problem (3.2) and (3.8) follow from [18, Theorem 4.8], and (3.9) from (3.3) and rescaling. We remark that, although the results in [18] used condition (1.2), Lemma 3.4 below ensures that those results remain valid for the model equation (3.2) under condition (1.9) that is implied by the MSP condition. □

The following lemma is standard (cf. [14]).

Lemma 3.4. *If the real numbers $A^{ij}_{\alpha\beta}$ satisfy the Legendre–Hadamard condition, then there exists a constant $\epsilon>0$ depending only on d, N and κ such that*

$$
\int_{\mathbf{R}^d}A^{ij}_{\alpha\beta}\partial_i u^\alpha\partial_j u^\beta\ge\epsilon\int_{\mathbf{R}^d}|\partial\boldsymbol{u}|^2
$$

for any $\boldsymbol{u}\in H^1(\mathbf{R}^d;\mathbf{R}^N)$.

The rest of this section is devoted to the proof of Theorem 3.1.

Proof of Theorem 3.1. According to Theorem 2.3 in [18] the model system (3.2) with zero initial value admits a unique solution

$$
\boldsymbol{u}\in L^2_\omega L^\infty_t H^m_x(\mathcal{Q}_T)\cap L^2_\omega L^2_t H^{m+1}_x(\mathcal{Q}_T).
$$

Noting that $\boldsymbol{u}\in L^p_\omega L^\infty_t H^m_x(\mathcal{Q}_T)\cap L^p_\omega L^2_t H^{m+1}_x(\mathcal{Q}_T)$ follows from estimate (3.3) by approximation, it remains to prove (3.3). As we can differentiate (3.2) with order $\mathfrak{s}$, it suffices to show (3.3) for $m=0$.

By Itô's formula in Hilbert space (see Theorem 4.32 in [7]) for $\|\boldsymbol{u}(\cdot,t)\|^2_{L^2_x}$, we derive

$$
\begin{aligned}
&\mathrm{d}\|\boldsymbol{u}(\cdot,t)\|^2_{L^2_x}\\
&\quad=\int_{\mathbf{R}^d}\big[-\big(2a^{ij}_{\alpha\beta}-\sigma^{ik}_{\gamma\alpha}\sigma^{jk}_{\gamma\beta}\big)\partial_i u^\alpha\partial_j u^\beta+2u^\alpha f_\alpha+2\sigma^{ik}_{\alpha\beta}\partial_i u^\beta g^k_\alpha+|\boldsymbol{g}|^2\big]\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_{\mathbf{R}^d}2\big(\sigma^{ik}_{\alpha\beta}u^\alpha\partial_i u^\beta+u^\alpha g^k_\alpha\big)\,\mathrm{d}x\,\mathrm{d}w^k_t.
\end{aligned}\tag{3.10}
$$

Applying Itô's formula to $\|\boldsymbol{u}(\cdot,t)\|^p_{L^2_x}$, one has

$$
\begin{aligned}
&\mathrm{d}\|\boldsymbol{u}(\cdot,t)\|^p_{L^2_x}\\
&\quad=\frac{p}{2}\|\boldsymbol{u}\|^{p-2}_{L^2_x}\int_{\mathbf{R}^d}\big[-\big(2a^{ij}_{\alpha\beta}-\sigma^{ik}_{\gamma\alpha}\sigma^{jk}_{\gamma\beta}\big)\partial_i u^\alpha\partial_j u^\beta+2u^\alpha f_\alpha+2\sigma^{ik}_{\alpha\beta}\partial_i u^\beta g^k_\alpha+|\boldsymbol{g}|^2\big]\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$+\frac{p(p-2)}{2}\mathbf{1}_{\{\|\boldsymbol{u}\|_{L^2_x}\neq 0\}}\|\boldsymbol{u}\|^{p-4}_{L^2_x}\sum_k\left[\int_{\mathbf{R}^d}\left(\sigma^{ik}_{\alpha\beta}u^\alpha\partial_i u^\beta+u^\alpha g^k_\alpha\right)\mathrm{d}x\right]^2\mathrm{d}t$$
$$+p\|\boldsymbol{u}\|^{p-2}_{L^2_x}\int_{\mathbf{R}^d}\left(\sigma^{ik}_{\alpha\beta}u^\alpha\partial_i u^\beta+u^\alpha g^k_\alpha\right)\mathrm{d}x\,\mathrm{d}w^k_t.$$

Recalling the MSP condition for the definition of $\lambda^{ik}_{\alpha\beta}$ and that $\lambda^{ik}_{\alpha\beta}=\lambda^{ik}_{\beta\alpha}$, we compute

$$\begin{aligned}\sigma^{ik}_{\alpha\beta}u^\alpha\partial_i u^\beta&=\left(\sigma^{ik}_{\alpha\beta}-\lambda^{ik}_{\alpha\beta}\right)u^\alpha\partial_i u^\beta+\lambda^{ik}_{\alpha\beta}u^\alpha\partial_i u^\beta\\&=\left(\sigma^{ik}_{\alpha\beta}-\lambda^{ik}_{\alpha\beta}\right)u^\alpha\partial_i u^\beta+\frac{1}{2}\lambda^{ik}_{\alpha\beta}\partial_i\left(u^\alpha u^\beta\right),\end{aligned}$$

so by the integration by parts,

$$\int_{\mathbf{R}^d}\sigma^{ik}_{\alpha\beta}u^\alpha\partial_i u^\beta\,\mathrm{d}x=\int_{\mathbf{R}^d}\left(\sigma^{ik}_{\alpha\beta}-\lambda^{ik}_{\alpha\beta}\right)u^\alpha\partial_i u^\beta\,\mathrm{d}x.$$

Using the MSP condition and Lemma 3.4, we can dominate the highest order terms:

$$\begin{aligned}&-\|\boldsymbol{u}\|^2_{L^2_x}\int_{\mathbf{R}^d}\left(2a^{ij}_{\alpha\beta}-\sigma^{ik}_{\gamma\alpha}\sigma^{jk}_{\gamma\beta}\right)\partial_i u^\alpha\partial_j u^\beta\,\mathrm{d}x+(p-2)\sum_k\left(\int_{\mathbf{R}^d}\sigma^{ik}_{\alpha\beta}u^\alpha\partial_i u^\beta\,\mathrm{d}x\right)^2\\&\quad\le-\|\boldsymbol{u}\|^2_{L^2_x}\int_{\mathbf{R}^d}\left(2a^{ij}_{\alpha\beta}-\sigma^{ik}_{\gamma\alpha}\sigma^{jk}_{\gamma\beta}\right)\partial_i u^\alpha\partial_j u^\beta\,\mathrm{d}x\\&\qquad+(p-2)\|\boldsymbol{u}\|^2_{L^2_x}\sum_{k,\gamma}\int_{\mathbf{R}^d}\left[\left(\sigma^{ik}_{\gamma\beta}-\lambda^{ik}_{\gamma\beta}\right)\partial_i u^\beta\right]^2\mathrm{d}x\\&\quad=-\|\boldsymbol{u}\|^2_{L^2_x}\int_{\mathbf{R}^d}\left[2a^{ij}_{\alpha\beta}-\sigma^{ik}_{\gamma\alpha}\sigma^{jk}_{\gamma\beta}-(p-2)\left(\sigma^{ik}_{\gamma\alpha}-\lambda^{ik}_{\gamma\alpha}\right)\left(\sigma^{jk}_{\gamma\beta}-\lambda^{jk}_{\gamma\beta}\right)\right]\partial_i u^\alpha\partial_j u^\beta\,\mathrm{d}x\\&\quad\le-\epsilon\|\boldsymbol{u}\|^2_{L^2_x}\|\partial\boldsymbol{u}\|^2_{L^2_x}.\end{aligned}$$

So we have

$$\begin{aligned}&\mathrm{d}\left\|\boldsymbol{u}(\cdot,t)\right\|^p_{L^2_x}\\&\quad\le\frac{p}{2}\|\boldsymbol{u}\|^{p-2}_{L^2_x}\left(-\epsilon\|\partial\boldsymbol{u}\|^2_{L^2_x}+2\|\boldsymbol{u}\|_{H^1_x}\|\boldsymbol{f}\|_{H^{-1}_x}+C\|\boldsymbol{g}\|^2_{L^2_x}+C\|\partial\boldsymbol{u}\|_{L^2_x}\|\boldsymbol{g}\|_{L^2_x}\right)\mathrm{d}t\\&\qquad+p\|\boldsymbol{u}\|^{p-2}_{L^2_x}\int_{\mathbf{R}^d}\left(\sigma^{ik}_{\alpha\beta}u^\alpha\partial_i u^\beta+u^\alpha g^k_\alpha\right)\mathrm{d}x\,\mathrm{d}w^k_t\\&\quad\le\left[-\frac{p\epsilon}{4}\|\boldsymbol{u}\|^{p-2}_{L^2_x}\|\partial\boldsymbol{u}\|^2_{L^2_x}+C\|\boldsymbol{u}\|^p_{L^2_x}+C\|\boldsymbol{u}\|^{p-2}_{L^2_x}\left(\|\boldsymbol{f}\|^2_{H^{-1}_x}+\|\boldsymbol{g}\|^2_{L^2_x}\right)\right]\mathrm{d}t\\&\qquad+p\|\boldsymbol{u}\|^{p-2}_{L^2_x}\int_{\mathbf{R}^d}\left(\sigma^{ik}_{\alpha\beta}u^\alpha\partial_i u^\beta+u^\alpha g^k_\alpha\right)\mathrm{d}x\,\mathrm{d}w^k_t.\end{aligned}\tag{3.11}$$

Integrating with respect to time on $[0,s]$ for any $s\in[0,T]$, and keeping in mind the initial condition $\boldsymbol{u}(x,0)\equiv 0$, we know that

$$\begin{aligned}&\left\|\boldsymbol{u}(s)\right\|^p_{L^2_x}+\frac{p\epsilon}{4}\int_0^s\|\boldsymbol{u}\|^{p-2}_{L^2_x}\|\partial\boldsymbol{u}\|^2_{L^2_x}\,\mathrm{d}t\\&\quad\le C\int_0^s\left[\left\|\boldsymbol{u}(t)\right\|^p_{L^2_x}+\|\boldsymbol{u}\|^{p-2}_{L^2_x}\left(\|\boldsymbol{f}\|^2_{H^{-1}_x}+\|\boldsymbol{g}\|^2_{L^2_x}\right)\right]\mathrm{d}t\\&\qquad+\int_0^s p\|\boldsymbol{u}\|^{p-2}_{L^2_x}\int_{\mathbf{R}^d}\left[\sigma^{ik}_{\alpha\beta}u^\alpha\partial_i u^\beta+u^\alpha g^k_\alpha\right]\mathrm{d}x\,\mathrm{d}w^k_t,\quad\text{a.s.}\end{aligned}\tag{3.12}$$

Let $\tau \in [0,T]$ be a stopping time such that

$$\mathbb{E}\sup_{t\in[0,\tau]}\|\boldsymbol{u}(t)\|_{L_x^2}^p + \mathbb{E}\left(\int_0^\tau \|\partial\boldsymbol{u}(t)\|_{L_x^2}^2\,\mathrm{d}t\right)^{\frac{p}{2}} < \infty.$$

Then it is easily verified that the last term on the right-hand side of (3.12) is a martingale with parameter s. Taking the expectation on both sides of (3.12), and by Young's inequality and Gronwall's inequality, we can obtain

$$\begin{aligned}
&\sup_{t\in[0,T]}\mathbb{E}\|\boldsymbol{u}(t\wedge\tau)\|_{L_x^2}^p + \mathbb{E}\int_0^\tau \|\boldsymbol{u}(t)\|_{L_x^2}^{p-2}\|\partial\boldsymbol{u}(t)\|_{L_x^2}^2\,\mathrm{d}t \\
&\quad\le C\mathbb{E}\int_0^\tau \|\boldsymbol{u}(t)\|_{L_x^2}^{p-2}\big(\|\boldsymbol{f}\|_{H_x^{-1}}^2 + \|\boldsymbol{g}\|_{L_x^2}^2\big)\,\mathrm{d}t. \qquad (3.13)
\end{aligned}$$

On the other hand, by the Burkholder–Davis–Gundy (BDG) inequality (c.f. Theorem 4.36 in [7]), we can derive from (3.12) that

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau]}\|\boldsymbol{u}(t)\|_{L_x^2}^p + \mathbb{E}\int_0^\tau \|\boldsymbol{u}(t)\|_{L_x^2}^{p-2}\|\partial\boldsymbol{u}(t)\|_{L_x^2}^2\,\mathrm{d}t \\
&\quad\le C\mathbb{E}\int_0^\tau \big[\|\boldsymbol{u}(t)\|_{L_x^2}^p + \|\boldsymbol{u}(t)\|_{L_x^2}^{p-2}\big(\|\boldsymbol{f}\|_{H_x^{-1}}^2 + \|\boldsymbol{g}\|_{L_x^2}^2\big)\big]\,\mathrm{d}t \\
&\qquad + C\mathbb{E}\left\{\int_0^\tau \|\boldsymbol{u}\|_{L_x^2}^{2(p-2)}\sum_k\left[\int_{\mathbf{R}^d}\big(\sigma_{\alpha\beta}^{ik}u^\alpha\partial_i u^\beta + u^\alpha g_\alpha^k\big)\,\mathrm{d}x\right]^2\mathrm{d}t\right\}^{\frac{1}{2}}, \qquad (3.14)
\end{aligned}$$

and by Hölder's inequality, the last term on the right-hand side of the above inequality is dominated by

$$\begin{aligned}
&C\mathbb{E}\left[\int_0^\tau \|\boldsymbol{u}\|_{L_x^2}^{2(p-2)}\big(\|\boldsymbol{u}\|_{L_x^2}^2\|\partial\boldsymbol{u}\|_{L_x^2}^2 + \|\boldsymbol{u}\|_{L_x^2}^2\|\boldsymbol{g}\|_{L_x^2}^2\big)\,\mathrm{d}t\right]^{\frac{1}{2}} \\
&\quad\le C\mathbb{E}\left\{\sup_{t\in[0,\tau]}\|\boldsymbol{u}(t)\|_{L_x^2}^{p/2}\left[\int_0^\tau \big(\|\boldsymbol{u}\|_{L_x^2}^{p-2}\|\partial\boldsymbol{u}\|_{L_x^2}^2 + \|\boldsymbol{u}\|_{L_x^2}^{p-2}\|\boldsymbol{g}\|_{L_x^2}^2\big)\,\mathrm{d}t\right]^{\frac{1}{2}}\right\} \\
&\quad\le \frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau]}\|\boldsymbol{u}(t)\|_{L_x^2}^p + C\mathbb{E}\int_0^\tau \|\boldsymbol{u}\|_{L_x^2}^{p-2}\|\partial\boldsymbol{u}\|_{L_x^2}^2\,\mathrm{d}t + C\int_0^\tau \|\boldsymbol{u}\|_{L_x^2}^{p-2}\|\boldsymbol{g}\|_{L_x^2}^2\,\mathrm{d}t,
\end{aligned}$$

which along with (3.13) and (3.14) yields that

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,\tau]}\|\boldsymbol{u}(t)\|_{L_x^2}^p &\le C\mathbb{E}\int_0^\tau \|\boldsymbol{u}(t)\|_{L_x^2}^{p-2}\big(\|\boldsymbol{f}\|_{H_x^{-1}}^2 + \|\boldsymbol{g}\|_{L_x^2}^2\big)\,\mathrm{d}t \\
&\le C\mathbb{E}\left[\sup_{t\in[0,\tau]}\|\boldsymbol{u}(t)\|_{L_x^2}^{p-2}\int_0^\tau \big(\|\boldsymbol{f}\|_{H_x^{-1}}^2 + \|\boldsymbol{g}\|_{L_x^2}^2\big)\,\mathrm{d}t\right] \\
&\le \frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau]}\|\boldsymbol{u}(t)\|_{L_x^2}^p + C\mathbb{E}\left[\int_0^T \big(\|\boldsymbol{f}\|_{H_x^{-1}}^2 + \|\boldsymbol{g}\|_{L_x^2}^2\big)\,\mathrm{d}t\right]^{\frac{p}{2}}.
\end{aligned}$$

Thus we obtain the estimate

$$\frac{1}{C}\mathbb{E}\sup_{t\in[0,\tau]}\|\boldsymbol{u}(t)\|_{L_x^2}^p \le \mathbb{E}\left[\int_0^T \big(\|\boldsymbol{f}\|_{H_x^{-1}}^2 + \|\boldsymbol{g}\|_{L_x^2}^2\big)\,\mathrm{d}t\right]^{\frac{p}{2}} =: F \qquad (3.15)$$

with $C = C(d, N, \kappa, K, p, T)$.

In order to estimate $\|\partial \boldsymbol{u}_x\|_{L^p_\omega L^2_t L^2_x}$, we go back to (3.10). Bearing in mind Condition (1.8) (actually here we only need the weaker one (1.9)) we can easily get that

$$\begin{aligned}\|\boldsymbol{u}(\tau)\|^2_{L^2_x} + \epsilon \int_0^\tau \|\partial \boldsymbol{u}(t)\|^2_{L^2_x}\,\mathrm{d}t \le \int_0^\tau \int_{\mathbf{R}^d} \big(2u^\alpha f_\alpha + 2\sigma^{ik}_{\alpha\beta}\partial_i u^\beta g^k_\alpha + |\boldsymbol{g}|^2\big)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^\tau \int_{\mathbf{R}^d} 2\big(\sigma^{ik}_{\alpha\beta}u^\alpha \partial_i u^\beta + u^\alpha g^k_\alpha\big)\,\mathrm{d}x\,\mathrm{d}w^k_t,\end{aligned}$$

where ϵ is the constant in Lemma 3.4. Computing $\mathbb{E}[\cdot]^{p/2}$ on both sides of the above inequality and by Hölder's inequality and the BDG inequality, we derive that

$$\begin{aligned}&\mathbb{E}\bigg(\int_0^\tau \|\partial \boldsymbol{u}(t)\|^2_{L^2_x}\,\mathrm{d}t\bigg)^{\frac{p}{2}} \\ &\le \frac{1}{4}\mathbb{E}\bigg(\int_0^\tau \|\boldsymbol{u}(t)\|^2_{H^1_x}\,\mathrm{d}t\bigg)^{\frac{p}{2}} + CF + C\mathbb{E}\bigg|\int_0^\tau \int_{\mathbf{R}^d}\big(\sigma^{ik}_{\alpha\beta}u^\alpha\partial_i u^\beta + u^\alpha g^k_\alpha\big)\,\mathrm{d}x\,\mathrm{d}w^k_t\bigg|^{\frac{p}{2}} \\ &\le \frac{1}{4}\mathbb{E}\bigg(\int_0^\tau \|\boldsymbol{u}(t)\|^2_{H^1_x}\,\mathrm{d}t\bigg)^{\frac{p}{2}} + CF + C\mathbb{E}\bigg[\sum_k \int_0^\tau \bigg\{\int_{\mathbf{R}^d}\big(\sigma^{ik}_{\alpha\beta}u^\alpha\partial_i u^\beta + u^\alpha g^k_\alpha\big)\,\mathrm{d}x\bigg\}^2\,\mathrm{d}t\bigg]^{\frac{p}{4}} \\ &\le \frac{1}{4}\mathbb{E}\bigg(\int_0^\tau \|\boldsymbol{u}(t)\|^2_{H^1_x}\,\mathrm{d}t\bigg)^{\frac{p}{2}} + CF + C\mathbb{E}\bigg[\int_0^\tau \|\boldsymbol{u}(t)\|^2_{L^2_x}\big(\|\partial\boldsymbol{u}(t)\|^2_{L^2_x} + \|\boldsymbol{g}(t)\|^2_{L^2_x}\big)\,\mathrm{d}t\bigg]^{\frac{p}{4}} \\ &\le \frac{1}{2}\mathbb{E}\bigg(\int_0^\tau \|\partial\boldsymbol{u}(t)\|^2_{L^2_x}\,\mathrm{d}t\bigg)^{\frac{p}{2}} + C\mathbb{E}\sup_{t\in[0,\tau]}\|\boldsymbol{u}(t)\|^p_{L^2_x} + CF,\end{aligned}$$

which along with (3.15) implies

$$\mathbb{E}\sup_{t\in[0,\tau]}\|\boldsymbol{u}(t)\|^p_{L^2_x} + \mathbb{E}\bigg(\int_0^\tau \|\partial\boldsymbol{u}(t)\|^2_{L^2_x}\,\mathrm{d}t\bigg)^{\frac{p}{2}} \le CF,$$

where the constant C depends only on d, p, T, κ, K, N, but is independent of τ. Note that ϵ that depends on d, N, and κ has been absorbed into the constant C. Finally, by taking the stopping time τ to be

$$\tau_n := \inf\bigg\{s \ge 0 : \sup_{t\in[0,s]}\|\boldsymbol{u}(t)\|^2_{L^2_x} + \int_0^s \|\partial\boldsymbol{u}(t)\|^2_{L^2_x}\,\mathrm{d}t \ge n\bigg\} \wedge T,$$

and letting n tend to infinity we obtain the estimate (3.3) with $m = 0$. Theorem 3.1 is proved. □

4. Interior Hölder estimates for the model system

The aim of this section is to prove the interior Hölder estimates for the model equation (3.2). The conditions (1.8) and (3.1) are also assumed throughout this section. Take $\boldsymbol{f} \in C^0_x(\mathbf{R}^d \times \mathbf{R}; L^p_\omega)$ and $\boldsymbol{g} \in C^1_x(\mathbf{R}^d \times \mathbf{R}; L^p_\omega)$ such that the modulus of continuity

$$\varpi(r) := \operatorname*{ess\,sup}_{t\in\mathbf{R},|x-y|\le r}\big(\|\boldsymbol{f}(x,t) - \boldsymbol{f}(y,t)\|_{L^p_\omega} + \|\partial\boldsymbol{g}(x,t) - \partial\boldsymbol{g}(y,t)\|_{L^p_\omega}\big)$$

satisfies the Dini condition:

$$\int_0^1 \frac{\varpi(r)}{r}\,\mathrm{d}r < \infty.$$

Theorem 4.1. *Let $\boldsymbol{u} \in \mathcal{C}^{2,1}_{x,t}(Q_2; L^p_\omega)$ satisfy (3.2). Under the above setting, there is a positive constant C, depending only on d, N, K, κ, and p, such that for any $X, Y \in Q_{1/4}$,*

$$\|\partial^2\boldsymbol{u}(X) - \partial^2\boldsymbol{u}(Y)\|_{L^p_\omega} \le C\bigg[\Delta M + \int_0^\Delta \frac{\varpi(r)}{r}\,\mathrm{d}r + \Delta\int_\Delta^1 \frac{\varpi(r)}{r^2}\,\mathrm{d}r\bigg],$$

where $\Delta := |X - Y|_{\mathrm{p}}$ *and*

$$M := \|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(Q_1)} + |||\boldsymbol{f}|||_{0,p;Q_1} + |||\boldsymbol{g}|||_{1,p;Q_1}.$$

Then the interior Hölder estimates are straightforward:

Corollary 4.2. *Under the same setting of Theorem* (4.1) *and given* $\delta \in (0,1)$, *there is a constant* $C > 0$, *depending only on* d, N, K, κ *and* p, *such that*

$$[\![\partial^2 \boldsymbol{u}]\!]_{(\delta,\delta/2),p;Q_{1/4}} \le C\left[\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(Q_1)} + \frac{|||\boldsymbol{f}|||_{\delta,p;Q_1} + |||\boldsymbol{g}|||_{1+\delta,p;Q_1}}{\delta(1-\delta)}\right],$$

provided the right-hand side is finite.

Proof of Theorem 4.1. Letting $\varphi : \mathbf{R}^n \to \mathbf{R}$ be a nonnegative and symmetric mollifier and $\varphi^\varepsilon(x) = \varepsilon^n \varphi(x/\varepsilon)$, we define $u^{\alpha,\varepsilon} = \varphi^\varepsilon * u^\alpha$, $f^\varepsilon_\alpha = \varphi^\varepsilon * f_\alpha$ and $g^\varepsilon_\alpha = \varphi^\varepsilon * g_\alpha$. Then it is easily checked that $\boldsymbol{f}^\varepsilon$ and $\partial \boldsymbol{g}^\varepsilon$ are also Dini continuous and has the same continuity modulus ϖ with $\boldsymbol{f}$ and $\partial \boldsymbol{g}$, and

$$\left|\left|\left|\boldsymbol{f}^\varepsilon - \boldsymbol{f}\right|\right|\right|_{0,p;\mathbf{R}^n} + \left|\left|\left|\boldsymbol{g}^\varepsilon - \boldsymbol{g}\right|\right|\right|_{1,p;\mathbf{R}^n} \to 0,$$
$$\left\|\partial^2 \boldsymbol{u}^\varepsilon(X) - \partial^2 \boldsymbol{u}(X)\right\|_{L^p_\omega} \to 0, \quad \forall X \in \mathbf{R}^n \times \mathbf{R},$$

as $\varepsilon \to 0$. On the other hand, from Fubini's theorem one can check that $\boldsymbol{u}^\varepsilon$ satisfies the model equation (3.2) in the classical sense with free terms $\boldsymbol{f}^\varepsilon$ and $\boldsymbol{g}^\varepsilon$. Therefore, it suffices to prove the theorem for the mollified functions, and the general case is straightforward by passing the limits.

Based on the above analysis and the smoothness of mollified functions, we may suppose that (cf. [11])

(A) $\boldsymbol{f}, \boldsymbol{g} \in L^p_\omega L^2_t H^k_x(Q_R) \cap C^k_x(Q_R; L^p_\omega)$ for all $k \in \mathbf{Z}_+$ and $R > 0$.

We can also set $X = 0$ without loss of generality. With $\rho = 1/2$, we define

$$Q^\ell := Q_{\rho^\ell} = Q_{\rho^\ell}(0,0), \quad \ell \in \mathbf{N} = \{0, 1, 2, \ldots\},$$

and introduce the following boundary value problems:

$$\begin{cases} \mathrm{d}u^{\alpha,\ell} = [a^{ij}_{\alpha\beta}\partial_{ij}u^{\beta,\ell} + f_\alpha(0,t)]\,\mathrm{d}t + [\sigma^{ik}_{\alpha\beta}\partial_i u^{\beta,\ell} + g^k_\alpha(0,t) + x^i\partial_i g^k_\alpha(0,t)]\,\mathrm{d}w^k_t \\ u^{\alpha,\ell} = u^\alpha \quad \text{on } \partial_{\mathrm{p}} Q^\ell, \end{cases} \tag{4.1}$$

where $\partial_{\mathrm{p}} Q^\ell$ denotes the parabolic boundary of the cylinder Q^ℓ. The existence and interior regularity of $\boldsymbol{u}^\ell$ can be direct yielded by Proposition 3.3.

Given a point $Y = (y,s) \in Q_{1/4}$, there is an $\ell_0 \in \mathbf{N}$ such that

$$\Delta := |Y|_{\mathrm{p}} \in \left[\rho^{\ell_0+2}, \rho^{\ell_0+1}\right).$$

So we have

$$\begin{aligned} &\left\|\partial^2 \boldsymbol{u}(Y) - \partial^2 \boldsymbol{u}(0)\right\|_{L^p_\omega} \\ &\quad \le \left\|\partial^2 \boldsymbol{u}^{\ell_0}(0) - \partial^2 \boldsymbol{u}(0)\right\|_{L^p_\omega} + \left\|\partial^2 \boldsymbol{u}^{\ell_0}(Y) - \partial^2 \boldsymbol{u}(Y)\right\|_{L^p_\omega} + \left\|\partial^2 \boldsymbol{u}^{\ell_0}(Y) - \partial^2 \boldsymbol{u}^{\ell_0}(0)\right\|_{L^p_\omega} \\ &\quad =: N_1 + N_1' + N_2. \end{aligned} \tag{4.2}$$

As N_1 and N_1' are similar, we are going to derive the estimates for N_1 and N_2. The constant C in the following claims only depends on d, N, K, κ, and p

Claim 4.3. $|||\partial^m(\boldsymbol{u}^\ell - \boldsymbol{u}^{\ell+1})|||_{0,p;Q^{\ell+2}} \le C\rho^{(2-m)\ell-m}\varpi(\rho^\ell)$, *where* $m \in \mathbf{N}$.

Proof. Applying Proposition 3.2 to (4.1), we have

$$\left|\left|\left|\partial^m\left(\boldsymbol{u}^\ell - \boldsymbol{u}^{\ell+1}\right)\right|\right|\right|_{0,p;Q^{\ell+2}} \le C\rho^{-m\ell-m}\left\|\fint_{Q^{\ell+1}}\left|\boldsymbol{u}^\ell - \boldsymbol{u}^{\ell+1}\right|^2\right\|^{1/2}_{L^{p/2}_\omega} =: I_{\ell,m}$$

(hereafter we denote $\fint_Q = \frac{1}{|Q|}\int_Q$ with $|Q|$ being the Lebesgue measure of the set $Q \subset \mathbf{R}^{n+1}$), and by Proposition 3.3,

$$J_\ell := \left\| \fint_{Q^{\ell+1}} \left|\boldsymbol{u}^\ell - \boldsymbol{u}\right|^2 \right\|_{L^{p/2}_\omega}^{1/2} \le C\rho^{2\ell}\varpi\left(\rho^\ell\right).$$

So we gain that

$$I_{\ell,m} \le C\rho^{-m\ell-m}(J_\ell + J_{\ell+1}) \le C\rho^{(2-m)\ell-m}\varpi\left(\rho^\ell\right).$$

The claim is proved. □

Claim 4.4. $N_1 \le C\int_0^{\rho^{\ell_0}} \frac{\varpi(r)}{r}\,\mathrm{d}r$.

Proof. It follows from Claim 4.3 that

$$\sum_{\ell\ge\ell_0} \left\|\partial^2\boldsymbol{u}^\ell(0) - \partial^2\boldsymbol{u}^{\ell+1}(0)\right\|_{L^p_\omega} \le C\sum_{\ell\ge\ell_0}\varpi\left(\rho^\ell\right) \le C\int_0^{\rho^{\ell_0}} \frac{\varpi(r)}{r}\,\mathrm{d}r,$$

which implies that $\partial^2\boldsymbol{u}^\ell(0)$ converges in L^p_ω as $\ell\to\infty$, if the limit is $\partial^2\boldsymbol{u}(0)$, then

$$N_1 = \left\|\partial^2\boldsymbol{u}^{\ell_0}(0) - \partial^2\boldsymbol{u}(0)\right\|_{L^p_\omega} \le \sum_{\ell\ge\ell_0}\left\|\partial^2\boldsymbol{u}^\ell(0) - \partial^2\boldsymbol{u}^{\ell+1}(0)\right\|_{L^p_\omega} \le C\int_0^{\rho^{\ell_0}}\frac{\varpi(r)}{r}\,\mathrm{d}r.$$

So it suffices to show that $\lim_{\ell\to\infty}\|\partial^2\boldsymbol{u}^\ell(0) - \partial^2\boldsymbol{u}(0)\|_{L^2_\omega} = 0$. From Proposition 3.2 with $p = 2$, we have

$$\begin{aligned}\sup_{Q^{\ell+1}}\left\|\partial^2\boldsymbol{u}^\ell - \partial^2\boldsymbol{u}\right\|_{L^2_\omega}^2 \le{}& C\rho^{-4\ell}\mathbb{E}\fint_{Q^\ell}\left|\boldsymbol{u}^\ell - \boldsymbol{u}\right|^2 + C\mathbb{E}\fint_{Q^\ell}\Big(\left|\boldsymbol{f}(x,t) - \boldsymbol{f}(0,t)\right|^2 \\ &+ \rho^{-2l}\left|\boldsymbol{g}(x,t) - \boldsymbol{g}(0,t) - x^i\partial_i\boldsymbol{g}(0,t)\right|^2 + \left|\partial\boldsymbol{g}(x,t) - \partial\boldsymbol{g}(0,t)\right|^2\Big)\,\mathrm{d}X \\ &+ C\sum_{k=1}^{[\frac{d+1}{2}]+1}\rho^{2\ell k}\mathbb{E}\fint_{Q^\ell}\left(\left|\partial^k\boldsymbol{f}\right|^2 + \left|\partial^{k+1}\boldsymbol{g}\right|^2\right).\end{aligned} \tag{4.3}$$

The additional assumption (A) on $\boldsymbol{f}$ and $\boldsymbol{g}$ together with Proposition 3.3 implies

$$\begin{aligned}&\rho^{-4\ell}\mathbb{E}\fint_{Q^\ell}\left|\boldsymbol{u}^\ell - \boldsymbol{u}\right|^2 \\ &\quad\le C\mathbb{E}\fint_{Q^\ell}\left(\left|\boldsymbol{f}(x,t) - \boldsymbol{f}(0,t)\right|^2 + \rho^{-2\ell}\left|\boldsymbol{g}(x,t) - \boldsymbol{g}(0,t) - x^i\partial_i\boldsymbol{g}(0,t)\right|^2\right)\mathrm{d}X \\ &\quad\le C\varpi\left(\rho^\ell\right)^2 \to 0, \quad \text{as } \ell\to\infty.\end{aligned}$$

And it is easier to obtain that the last two terms on the right-hand side of (4.3) tend to zero as $\ell\to\infty$. Thus, $\lim_{\ell\to\infty}\|\partial^2\boldsymbol{u}^\ell(0) - \partial^2\boldsymbol{u}(0)\|_{L^2_\omega} = 0$. The claim is proved. □

Claim 4.5. $N_2 \le C\rho^{\ell_0}(M + \int_{\rho^{\ell_0}}^1 \frac{\varpi(r)}{r^2}\,\mathrm{d}r)$.

Proof. Define $\boldsymbol{h}^\ell = \boldsymbol{u}^\ell - \boldsymbol{u}^{\ell-1}$ for $\ell = 1, 2, \dots, \ell_0$, then

$$\begin{aligned}N_2 &= \left\|\partial^2\boldsymbol{u}^{\ell_0}(Y) - \partial^2\boldsymbol{u}^{\ell_0}(0)\right\|_{L^p_\omega} \\ &\le \left\|\partial^2\boldsymbol{u}^0(Y) - \partial^2\boldsymbol{u}^0(0)\right\|_{L^p_\omega} + \sum_{\ell=1}^{\ell_0}\left\|\partial^2\boldsymbol{h}^\ell(Y) - \partial^2\boldsymbol{h}^\ell(0)\right\|_{L^p_\omega}.\end{aligned}$$

As $\partial_{ij}\boldsymbol{u}^0$ satisfies a homogeneous system in Q_1 for any $i, j = 1, \dots, d$, it follows from Proposition 3.2 that, for $m = 1, 2$,

$$\begin{aligned}\big\|\!\big|\partial^m\big(\partial_{ij}\boldsymbol{u}^0\big)\big|\!\big\|_{0,p;Q_{1/4}} &\le C\big\|\partial_{ij}\boldsymbol{u}^0\big\|_{L^p_\omega L^2_t L^2_x(Q_{1/2})}\\ &\le C\big(\big\|\partial_{ij}\boldsymbol{u}^0-\partial_{ij}\boldsymbol{u}\big\|_{L^p_\omega L^2_t L^2_x(Q_{1/2})}+\|\partial_{ij}\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(Q_{1/2})}\big)\\ &\le C\big(\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(Q_1)}+|\!|\!|\boldsymbol{f}|\!|\!|_{0,p;Q_1}+|\!|\!|\boldsymbol{g}|\!|\!|_{1,p;Q_1}\big)=CM,\end{aligned}$$

and for $-1/16 < s < t \le 0$ and $x \in B_{1/4}$,

$$\begin{aligned}\big\|\partial^2 u^{\alpha,0}(x,t)-\partial^2 u^{\alpha,0}(x,s)\big\|_{L^p_\omega} &= \bigg\|\int_s^t a^{ij}_{\alpha\beta}\partial_{ij}\big(\partial^2 u^{\beta,0}\big)\,\mathrm{d}\tau+\int_s^t \sigma^{ik}_{\alpha\beta}\partial_i\big(\partial^2 u^{\beta,0}\big)\,\mathrm{d}w^k_\tau\bigg\|_{L^p_\omega}\\ &\le C\sqrt{t-s}\big(|\!|\!|\partial^3\boldsymbol{u}^0|\!|\!|_{0,p;Q_{1/4}}+|\!|\!|\partial^4\boldsymbol{u}^0|\!|\!|_{0,p;Q_{1/4}}\big)\\ &\le CM\sqrt{t-s}.\end{aligned}$$

So combining above two inequalities we have

$$\big\|\partial^2\boldsymbol{u}^0(Y)-\partial^2\boldsymbol{u}^0(0)\big\|_{L^p_\omega}\le CM|Y|_{\mathrm{p}}\le CM\rho^{\ell_0}.$$

Next, by Claim 4.3,

$$\rho^{-\ell}|\!|\!|\partial^3\boldsymbol{h}^\ell|\!|\!|_{0,p;Q^{\ell+1}}+|\!|\!|\partial^4\boldsymbol{h}^\ell|\!|\!|_{0,p;Q^{\ell+1}}\le C\rho^{-2\ell}\varpi\big(\rho^{\ell-1}\big),$$

thus, for $-\rho^{2(\ell_0+1)} \le t \le 0$ and $|x| \le \rho^{\ell_0+1}$,

$$\big\|\partial^2\boldsymbol{h}^\ell(x,0)-\partial^2\boldsymbol{h}^\ell(0,0)\big\|_{L^p_\omega}\le C\rho^{\ell_0-\ell}\varpi\big(\rho^{\ell-1}\big)$$

and

$$\begin{aligned}\big\|\partial^2 h^{\alpha,\ell}(x,t)-\partial^2 h^{\alpha,\ell}(x,0)\big\|_{L^p_\omega} &= \bigg\|\int_0^t a^{ij}_{\alpha\beta}\partial_{ij}\big(\partial^2 h^{\beta,\ell}\big)\,\mathrm{d}\tau+\int_s^t \sigma^{ik}_{\alpha\beta}\partial_i\big(\partial^2 h^{\beta,\ell}\big)\,\mathrm{d}w^k_\tau\bigg\|_{L^p_\omega}\\ &\le C\big(\rho^{\ell_0}|\!|\!|\partial^3\boldsymbol{h}^\ell|\!|\!|_{0,p;Q^{l+1}}+\rho^{2\ell_0}|\!|\!|\partial^4\boldsymbol{h}^\ell|\!|\!|_{0,p;Q^{l+1}}\big)\\ &\le C\rho^{\ell_0-\ell}\varpi\big(\rho^{\ell-1}\big).\end{aligned}$$

Therefore,

$$N_2\le CM\rho^{\ell_0}+C\sum_{\ell=1}^{\ell_0}\rho^{\ell_0-\ell}\varpi\big(\rho^{\ell-1}\big)\le C\rho^{\ell_0}\bigg(M+\int_{\rho^{\ell_0}}^1\frac{\varpi(r)}{r^2}\,\mathrm{d}r\bigg).$$

The claim is proved. □

Combining (4.2) and Claims 4.4 and 4.5, we conclude Theorem 4.1. □

5. Hölder estimates for general systems

This section is devoted to the proofs of Theorems 2.1 and 2.2. We need two technical lemmas whose proofs can be found in, for example, [11].

Lemma 5.1. *Let $\varphi : [0, T] \to [0, \infty)$ satisfy*

$$\varphi(t)\le\theta\varphi(s)+\sum_{i=1}^m A_i(s-t)^{-\eta_i}\quad\forall 0\le t<s\le T$$

for some nonnegative constants θ, η_i and A_i $(i = 1, \dots m)$, where $\theta < 1$. Then

$$\varphi(0) \le C \sum_{i=1}^{m} A_i T^{-\eta_i},$$

where C depends only on $\eta_1, \dots, \eta_n$ and θ.

Lemma 5.2. *Let $p \ge 1$, $R > 0$ and $0 \le s < r$. There exists a constant $C > 0$, depending only on d and p, such that*

$$[\![\boldsymbol{u}]\!]_{s,p;Q_R} \le C\varepsilon^{r-s}[\![\boldsymbol{u}]\!]_{r,p;Q_R} + C\varepsilon^{-s-d/2}\big[\mathbb{E}\|\boldsymbol{u}\|^p_{L^2(Q_R)}\big]^{\frac{1}{p}}$$

for any $\boldsymbol{u} \in C^r(Q_R; L^p_\omega)$ and $\varepsilon \in (0, R)$.

Now we prove the a priori *interior Hölder estimates* for system (1.1).

Proof of Theorem 2.1. With a change of variable, we may move the point X to the origin. Let $\rho/2 \le r < R \le \rho$ with $\rho \in (0, 1/4)$ to be defined. Take a nonnegative cut-off function $\zeta \in C_0^\infty(\mathbf{R}^{d+1})$ such that $\zeta = 1$ on Q_r, $\zeta = 0$ outside Q_R, and for $\gamma \ge 0$,

$$[\zeta]_{(\gamma,\gamma/2);\mathbf{R}^{d+1}} \le C(d)(R-r)^{-\gamma}.$$

Set $\boldsymbol{v} = \zeta\boldsymbol{u}$, and

$$\tilde{a}^{ij}_{\alpha\beta}(t) = a^{ij}_{\alpha\beta}(0,t), \qquad \tilde{\sigma}^{ik}_{\alpha\beta}(t) = \sigma^{ik}_{\alpha\beta}(0,t),$$

then $\boldsymbol{v} = (v^1, \dots, v^N)$ satisfies

$$\mathrm{d}v^\alpha = \big(\tilde{a}^{ij}_{\alpha\beta}\partial_{ij}v^\beta + \tilde{f}_\alpha\big)\,\mathrm{d}t + \big(\tilde{\sigma}^{ik}_{\alpha\beta}\partial_i v^\beta + \tilde{g}^k_\alpha\big)\,\mathrm{d}w^k_t$$

where

$$\begin{aligned}
\tilde{f}_\alpha &= \big(a^{ij}_{\alpha\beta} - \tilde{a}^{ij}_{\alpha\beta}\big)\zeta\,\partial_{ij}u^\beta + \big(b^i_{\alpha\beta}\zeta - 2\tilde{a}^{ij}_{\alpha\beta}\partial_j\zeta\big)\partial_i u^\beta \\
&\quad + \big(c_{\alpha\beta}\zeta - \tilde{a}^{ij}_{\alpha\beta}\partial_{ij}\zeta\big)u^\beta - \zeta_t u^\alpha + \zeta f^\alpha, \\
\tilde{g}^k_\alpha &= \big(\sigma^{ik}_{\alpha\beta} - \tilde{\sigma}^{ik}_{\alpha\beta}\big)\zeta\,\partial_i u^\beta + \big(\nu^k_{\alpha\beta}\zeta - \tilde{\sigma}^{ik}_{\alpha\beta}\partial_i\zeta\big)u^\beta + \zeta g^\alpha.
\end{aligned}$$

Obviously, $\tilde{a}^{ij}_{\alpha\beta}$ and $\tilde{\sigma}^{ik}_{\alpha\beta}$ satisfy the MSP condition with $\lambda = \lambda(0,t)$. So by Lemma 5.2,

$$\begin{aligned}
|\!|\!|\tilde{\boldsymbol{f}}|\!|\!|_{\delta,p;Q_R} &\le \big(\varepsilon + K\rho^\delta\big)[\![\partial^2\boldsymbol{u}]\!]_{\delta,p;Q_R} + C_1(R-r)^{-2-\delta-d/2}\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(Q_R)} \\
&\quad + [\![\boldsymbol{f}]\!]_{\delta,p;Q_R} + C_1(R-r)^{-\delta}|\!|\!|\boldsymbol{f}|\!|\!|_{0,p;Q_R}, \\
|\!|\!|\tilde{\boldsymbol{g}}|\!|\!|_{1+\delta,p;Q_R} &\le \big(\varepsilon + K\rho^\delta\big)[\![\boldsymbol{u}]\!]_{2+\delta,p;Q_R} + C_1(R-r)^{-2-\delta-d/2}\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(Q_R)} \\
&\quad + [\![\boldsymbol{g}]\!]_{1+\delta,p;Q_R} + C_1(R-r)^{-1-\delta}|\!|\!|\boldsymbol{g}|\!|\!|_{0,p;Q_R},
\end{aligned}$$

where $C_1 = C_1(d, K, p, \varepsilon)$. Applying Corollary 4.2, we gain that

$$\begin{aligned}
&[\![\partial^2\boldsymbol{u}]\!]_{(\delta,\delta/2),p;Q_r} \\
&\quad \le C_2\Big[\big(\varepsilon + K\rho^\delta\big)[\![\partial^2\boldsymbol{u}]\!]_{(\delta,\delta/2),p;Q_R} + C_1(R-r)^{-2-\delta-d/2}\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(Q_R)} \\
&\qquad + [\![\boldsymbol{f}]\!]_{\delta,p;Q_R} + C_1(R-r)^{-\delta}|\!|\!|\boldsymbol{f}|\!|\!|_{0,p;Q_R} + [\![\boldsymbol{g}]\!]_{1+\delta,p;Q_R} + C_1(R-r)^{-1-\delta}|\!|\!|\boldsymbol{g}|\!|\!|_{0,p;Q_R}\Big],
\end{aligned}$$

where $C_2 = C_2(d, N, \kappa, K, p, \delta)$. Set $\varepsilon = (4C_2)^{-1}$, then

$$C_2\big(\varepsilon + K\rho^\delta\big) \le \frac{1}{2} \quad \text{for any } \rho \le (4C_2K)^{-1/\delta} =: \rho_0.$$

Thus, by Lemma 5.1 we have

$$[\![\partial^2 \boldsymbol{u}]\!]_{(\delta,\delta/2),p;Q_{\rho/2}} \le C\big(\rho^{-2-\delta-d/2}\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(Q_\rho)} + \rho^{-\delta}|\!|\!|\boldsymbol{f}|\!|\!|_{\delta,p;Q_\rho} + \rho^{-1-\delta}|\!|\!|\boldsymbol{g}|\!|\!|_{1+\delta,p;Q_\rho}\big),$$

where the constant C depends only on d, N, κ, K, p, and δ. The proof is complete. □

Proof of Theorem 2.2. The solvability of the Cauchy problem follows from the *a priori* estimate (2.2) by the standard method of continuity (see [15, Theorem 5.2]), so it suffices to prove the *a priori* estimate (2.2).

We may extend the equations to $\mathbf{R}^d \times (-\infty, T] \times \Omega$ by letting $\boldsymbol{u}(x,t)$, $\boldsymbol{f}(x,t)$ and $\boldsymbol{g}(x,t)$ be zero if $t \le 0$. Take $\tau \in (0,T]$ and $R = \rho_0/2$, where ρ_0 is determined in Theorem 2.1. Applying the estimate (2.1) on the cylinders centered at (x,s) for all $s \in (-1,\tau]$, we can obtain that

$$\begin{aligned}
[\![\partial^2 \boldsymbol{u}]\!]_{(\delta,\delta/2),p;\mathcal{Q}_{R,\tau}(x)} &\le C\big(\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(\mathcal{Q}_{2R,\tau}(x))} + |\!|\!|\boldsymbol{f}|\!|\!|_{\delta,p;\mathcal{Q}_{2R,\tau}(x)} + |\!|\!|\boldsymbol{g}|\!|\!|_{1+\delta,p;\mathcal{Q}_{2R,\tau}(x)}\big) \\
&\le C\big(\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(\mathcal{Q}_{2R,\tau}(x))} + |\!|\!|\boldsymbol{f}|\!|\!|_{\delta,p;\mathcal{Q}_\tau} + |\!|\!|\boldsymbol{g}|\!|\!|_{1+\delta,p;\mathcal{Q}_\tau}\big),
\end{aligned}$$

then by Lemma 5.2,

$$|\!|\!|\boldsymbol{u}|\!|\!|_{(2+\delta,\delta/2),p;\mathcal{Q}_{R,\tau}(x)} \le C\big(\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(\mathcal{Q}_{2R,\tau}(x))} + |\!|\!|\boldsymbol{f}|\!|\!|_{\delta,p;\mathcal{Q}_\tau} + |\!|\!|\boldsymbol{g}|\!|\!|_{1+\delta,p;\mathcal{Q}_\tau}\big). \tag{5.1}$$

Define

$$M^\tau_{x,R}(\boldsymbol{u}) = \sup_{0\le t\le\tau}\left(\fint_{B_R(x)} \mathbb{E}\big|\boldsymbol{u}(y,t)\big|^p\,\mathrm{d}y\right)^{\frac{1}{p}}, \qquad M^\tau_R(\boldsymbol{u}) = \sup_{x\in\mathbf{R}^d} M^\tau_{x,R}(\boldsymbol{u}).$$

Obviously, $\|\boldsymbol{u}\|_{L^p_\omega L^2_t L^2_x(\mathcal{Q}_{2R,\tau}(x))} \le C(d,p,R)M^\tau_R(\boldsymbol{u})$. So (5.1) implies

$$\sup_{x\in\mathbf{R}^d} |\!|\!|\boldsymbol{u}|\!|\!|_{(2+\delta,\delta/2),p;\mathcal{Q}_{R,\tau}(x)} \le C_3\big(M^\tau_R(\boldsymbol{u}) + |\!|\!|\boldsymbol{f}|\!|\!|_{\delta,p;\mathcal{Q}_\tau} + |\!|\!|\boldsymbol{g}|\!|\!|_{1+\delta,p;\mathcal{Q}_\tau}\big). \tag{5.2}$$

To get rid of $M^\tau_R(\boldsymbol{u})$, we apply Itô's formula to $|\boldsymbol{u}|^p$:

$$\begin{aligned}
\mathrm{d}|\boldsymbol{u}|^p &= p|\boldsymbol{u}|^{p-2}\Big[u^\alpha\big(a^{ij}_{\alpha\beta}\partial_{ij}u^\beta + b^{ij}_{\alpha\beta}\partial_i u^\beta + c_{\alpha\beta}u^\beta + f_\alpha\big) + \frac{1}{2}\sum_k\big(\sigma^{ik}_{\alpha\beta}\partial_i u^\beta + g^k_\alpha\big)^2\Big]\mathrm{d}t \\
&\quad + \frac{p(p-2)}{2}\mathbf{1}_{\{|\boldsymbol{u}|\ne 0\}}|\boldsymbol{u}|^{p-4}\sum_k\big(\sigma^{ik}_{\alpha\beta}u^\alpha\partial_i u^\beta + u^\alpha g^k_\alpha\big)^2\,\mathrm{d}t + \mathrm{d}M_t,
\end{aligned}$$

where M_t is a martingale. Integrating on $\mathcal{Q}_{R,\tau}(x)\times\Omega$ and by the Hölder inequality, we can derive that

$$\sup_{t\in[0,\tau]}\mathbb{E}\int_{B_R(x)}\big|\boldsymbol{u}(y,t)\big|^p\,\mathrm{d}y \le C_4\mathbb{E}\int_{\mathcal{Q}_{R,\tau}(x)}\big(\big|\partial^2\boldsymbol{u}\big|^p + |\boldsymbol{u}|^p + |\boldsymbol{f}|^p + |\boldsymbol{g}|^p\big)\,\mathrm{d}X$$

with $C_4 = C_4(d,N,K,p)$, which implies that

$$\begin{aligned}
M^\tau_{x,R}(\boldsymbol{u}) &\le C_4\tau\big(|\!|\!|\boldsymbol{u}|\!|\!|_{2,p;\mathcal{Q}_{R,\tau}(x)} + |\!|\!|\boldsymbol{f}|\!|\!|_{0,p;\mathcal{Q}_\tau} + |\!|\!|\boldsymbol{g}|\!|\!|_{0,p;\mathcal{Q}_\tau}\big) \\
&\le C_4\tau\Big(\sup_{x\in\mathbf{R}^d}|\!|\!|\boldsymbol{u}|\!|\!|_{(2+\delta,\delta/2),p;\mathcal{Q}_{R,\tau}(x)} + |\!|\!|\boldsymbol{f}|\!|\!|_{0,p;\mathcal{Q}_\tau} + |\!|\!|\boldsymbol{g}|\!|\!|_{0,p;\mathcal{Q}_\tau}\Big),
\end{aligned}$$

Substituting the last relation into (5.2) and taking $\tau = (2C_3C_4)^{-1}$, we get

$$\sup_{x\in\mathbf{R}^d}|\!|\!|\boldsymbol{u}|\!|\!|_{(2+\delta,\delta/2),p;\mathcal{Q}_{R,\tau}(x)} \le C\big(|\!|\!|\boldsymbol{f}|\!|\!|_{\delta,p;\mathcal{Q}_\tau} + |\!|\!|\boldsymbol{g}|\!|\!|_{1+\delta,p;\mathcal{Q}_\tau}\big),$$

and equivalently,

$$|\!|\!|\boldsymbol{u}|\!|\!|_{(2+\delta,\delta/2),p;\mathcal{Q}_\tau} \le C_{(\tau)}\big(|\!|\!|\boldsymbol{f}|\!|\!|_{\delta,p;\mathcal{Q}_\tau} + |\!|\!|\boldsymbol{g}|\!|\!|_{1+\delta,p;\mathcal{Q}_\tau}\big) \tag{5.3}$$

with $C_{(\tau)} = C_{(\tau)}(d,N,\kappa,K,p,\delta) \ge 1$.

Let us conclude the proof by induction. Assume that there is a constant $C_{(S)} \geq 1$ for some $S > 0$ such that

$$|\!|\!|\boldsymbol{u}|\!|\!|_{(2+\delta,\delta/2),p;\mathcal{Q}_S} \leq C_{(S)}\big(|\!|\!|\boldsymbol{f}|\!|\!|_{\delta,p;\mathcal{Q}_S} + |\!|\!|\boldsymbol{g}|\!|\!|_{1+\delta,p;\mathcal{Q}_S}\big).$$

Then applying (5.3) to $\boldsymbol{v}(x,t) = \mathbf{1}_{\{t\geq 0\}} \cdot [\boldsymbol{u}(x,t+S) - \boldsymbol{u}(x,S)]$, one can easily derive that

$$\begin{aligned}|\!|\!|\boldsymbol{v}|\!|\!|_{(2+\delta,\delta/2),p;\mathcal{Q}_\tau} &\leq C_{(\tau)}\big(|\!|\!|\boldsymbol{f}|\!|\!|_{\delta,p;\mathcal{Q}_{S+\tau}} + |\!|\!|\boldsymbol{g}|\!|\!|_{1+\delta,p;\mathcal{Q}_{S+\tau}} + \tilde{C}\big|\!\big|\!\big|\boldsymbol{u}(\cdot,S)\big|\!\big|\!\big|_{2+\delta,p;\mathbf{R}^d}\big)\\ &\leq 2C_{(\tau)}\tilde{C}C_{(S)}\big(|\!|\!|\boldsymbol{f}|\!|\!|_{\delta,p;\mathcal{Q}_{S+\tau}} + |\!|\!|\boldsymbol{g}|\!|\!|_{1+\delta,p;\mathcal{Q}_{S+\tau}}\big),\end{aligned}$$

with $\tilde{C} = \tilde{C}(N,K) \geq 1$, so

$$\begin{aligned}|\!|\!|\boldsymbol{u}|\!|\!|_{(2+\delta,\delta/2),p;\mathcal{Q}_{S+\tau}} &\leq |\!|\!|\boldsymbol{v}|\!|\!|_{(2+\delta,\delta/2),p;\mathcal{Q}_\tau} + 2|\!|\!|\boldsymbol{u}|\!|\!|_{(2+\delta,\delta/2),p;\mathcal{Q}_S}\\ &\leq 4C_{(\tau)}\tilde{C}C_{(S)}\big(|\!|\!|\boldsymbol{f}|\!|\!|_{\delta,p;\mathcal{Q}_{S+\tau}} + |\!|\!|\boldsymbol{g}|\!|\!|_{1+\delta,p;\mathcal{Q}_{S+\tau}}\big),\end{aligned}$$

that means $C_{(S+\tau)} \leq 4C_{(\tau)}\tilde{C}C_{(S)}$. By iteration we have $C_S \leq C\mathrm{e}^{CS}$ with $C = C(d,N,\kappa,K,p,\delta)$, and the theorem is proved. □

6. More comments on the MSP condition

In this section we discuss more examples on the sharpness and flexibility of the MSP condition (Definition 1.2). We always let $d = 1$ and assume that the coefficient matrices $A = [a_{\alpha\beta}]$ and $B = [\sigma_{\alpha\beta}]$ are *constant*. We write $M \gg 0$ if the matrix M is positive definite.

Under the above setting the MSP condition can be written into the following form if we set $[\lambda^{ik}_{\alpha\beta}] = (B + B')/2 - \Lambda$ in (1.7).

Condition 6.1. There is a symmetric $N \times N$ real matrix Λ such that

$$A + A' - B'B - (p-2)(T_B + \Lambda)'(T_B + \Lambda) \gg 0 \tag{6.1}$$

where $T_B := (B - B')/2$ is the skew-symmetric component of B.

Example 6.2. consider the following system

$$\begin{cases}\mathrm{d}u^{(1)} = u^{(1)}_{xx}\,\mathrm{d}t + (\lambda u^{(1)}_x - \mu u^{(2)}_x)\,\mathrm{d}w_t,\\ \mathrm{d}u^{(2)} = u^{(2)}_{xx}\,\mathrm{d}t + (\mu u^{(1)}_x + \lambda u^{(2)}_x)\,\mathrm{d}w_t\end{cases} \tag{6.2}$$

with $x \in \mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$, real constants λ and μ, and with the initial data

$$u^{(1)}(x,0) + \sqrt{-1}u^{(2)}(x,0) = \sum_{n\in\mathbf{Z}} \mathrm{e}^{-n^2} \cdot \mathrm{e}^{\sqrt{-1}nx}. \tag{6.3}$$

Evidently, if $\lambda^2 + \mu^2 < 2$, then system (6.2) satisfies the condition (1.2), and from the result of [18], it has a unique solution $\boldsymbol{u} = (u^{(1)}, u^{(2)})'$ in the space $L^2(\Omega; C([0,T]; H^m(\mathbf{T}))$ with any $m \geq 0$ and $T > 0$.

To apply our results to (6.2), we should assume it to satisfy Condition 6.1. In the next two lemma, we first simplify the condition into a specific constraint on λ and μ, and then prove it to be optimal.

Lemma 6.3. *Let $p \geq 2$. The coefficients of system (6.2) satisfy Condition 6.1 if and only if they satisfy (6.1) with $\Lambda = 0$, namely,*

$$\lambda^2 + (p-1)\mu^2 < 2. \tag{6.4}$$

Proof. By orthogonal transform, $A + A' - B'B - (p-2)(T_B + \Lambda)'(T_B + \Lambda)$ is positive definite if and only if

$$2 - \big(\lambda^2 + \mu^2\big) - (p-2)\lambda_{\max} > 0, \tag{6.5}$$

where $\lambda_{\max}$ is the larger eigenvalue of $(T_B+\Lambda)'(T_B+\Lambda)$. For $\Lambda=\mu\left[\begin{smallmatrix} a & c\\ c & b\end{smallmatrix}\right]$, we have

$$(T_B+\Lambda)'(T_B+\Lambda)=\mu^2\begin{bmatrix} a^2+(c-1)^2 & ac+bc+a-b\\ ac+bc+a-b & b^2+(c+1)^2\end{bmatrix}$$

whose larger eigenvalue is

$$\lambda_{\max}=\frac{\mu^2}{2}\left(a^2+b^2+2c^2+2\right)+\frac{\mu^2}{2}\sqrt{\left(a^2-b^2-4c\right)^2+4(ac+bc+a-b)^2}.$$

Obviously, $\lambda_{\max}\geq\mu^2$.

Once (6.5) holds for some Λ, we get (6.4), namely (6.1) holds for $\Lambda=0$. Now we prove the *only if* part. The proof of *if* part is trivial. □

Therefore, if (6.4) is satisfied, then $\sup_{x\in\mathbf{T}}\mathbb{E}\|\boldsymbol{u}(x,t)\|^p<\infty$ for any $t\geq 0$; if it is not, even some weaker norm of $\boldsymbol{u}(\cdot,t)$ is infinite for large t as showed in the following lemma.

Lemma 6.4. *Let $p>2$ and $\lambda^2+\mu^2<2$. If $\varepsilon:=\lambda^2+(p-1)\mu^2-2>0$, then*

$$\mathbb{E}\left\|\boldsymbol{u}(\cdot,t)\right\|^p_{L^2(\mathbf{T})}=\infty$$

for any $t>2/\varepsilon$.

Proof. Denote $v=u^{(1)}+\sqrt{-1}u^{(2)}$ that can be verified to satisfy

$$\mathrm{d}v=v_{xx}\,\mathrm{d}t+(\lambda+\sqrt{-1}\mu)v_x\,\mathrm{d}w_t$$

with the initial condition $v(x,0)=\sum_{n\in\mathbf{Z}}\mathrm{e}^{-n^2}\mathrm{e}^{\sqrt{-1}nx}$ for $x\in\mathbf{T}$. By Fourier analysis, we can express

$$v(x,t)=\sum_{n\in\mathbf{Z}}v_n(t)\mathrm{e}^{\sqrt{-1}nx},$$

where $v_n(\cdot)$ satisfies the following SDE:

$$\mathrm{d}v_n=v_n\left[-n^2\,\mathrm{d}t+(-\mu+\sqrt{-1}\lambda)n\,\mathrm{d}w_t\right],\quad v_n(0)=\mathrm{e}^{-n^2}.$$

From the theory of SDEs, we have

$$v_n(t)=\mathrm{e}^{-\frac{1}{2}f(t)n^2-\mu n w_t}\cdot\mathrm{e}^{\sqrt{-1}(\lambda\mu n^2t+\lambda n w_t)},$$

where $f(t):=2+(2+\mu^2-\lambda^2)t$. So we derive

$$\begin{aligned}\left|v_n(t)\right|^2&=\exp\left\{-f(t)n^2-2\mu n w_t\right\}\\&=\exp\left\{-f(t)\left(n+\frac{\mu w_t}{f(t)}\right)^2+\frac{\mu^2|w_t|^2}{f(t)}\right\},\end{aligned}$$

and by Parseval's identity,

$$\begin{aligned}\left\|v(\cdot,t)\right\|^2_{L^2(\mathbf{T})}&=2\pi\sum_{n\in\mathbf{Z}}\left|v_n(t)\right|^2\\&=2\pi\sum_{n\in\mathbf{Z}}\exp\left\{-f(t)\left(n+\frac{\mu w_t}{f(t)}\right)^2+\frac{\mu^2|w_t|^2}{f(t)}\right\}\\&\geq 2\pi\exp\left\{-f(t)+\frac{\mu^2|w_t|^2}{f(t)}\right\}.\end{aligned}$$

Thus, we have

$$
\begin{aligned}
\mathbb{E}\|\boldsymbol{u}(\cdot,t)\|^p_{L^2(\mathbf{T})} &= \mathbb{E}\|v(\cdot,t)\|^p_{L^2(\mathbf{T})}\\
&\geq (2\pi)^p\mathbb{E}\exp\left\{-\frac{pf(t)}{2}+\frac{p\mu^2|w_t|^2}{2f(t)}\right\}\\
&= (2\pi)^p\mathrm{e}^{-pf(t)/2}\mathbb{E}\exp\left\{\frac{p\mu^2|w_1|^2}{2f(t)/t}\right\}\\
&= (2\pi)^p\mathrm{e}^{-pf(t)/2}\mathbb{E}\exp\left\{\frac{p\mu^2|w_1|^2}{2[2+\mu^2-\lambda^2+2t^{-1}]}\right\}\\
&= (2\pi)^{p-1/2}\mathrm{e}^{-pf(t)/2}\int_{\mathbf{R}}\exp\left\{-\frac{y^2}{2}\left[1-\frac{p\mu^2}{2+\mu^2-\lambda^2+2t^{-1}}\right]\right\}\mathrm{d}y.
\end{aligned}
$$

The last integral diverges if

$$1-\frac{p\mu^2}{2+\mu^2-\lambda^2+2t^{-1}}<0.$$

This immediately concludes the lemma. □

Indeed, some specific choices of Λ in Condition 6.1 like $\Lambda=0$ usually lead to a class of convenient and even optimal criteria in applications. For instance, the above discussion shows how the skew-symmetric component of B substantially affects the L^p-norm of the solution of system (6.2). But in general, the choice of Λ still heavily depends on the structure of the concrete problem.

Example 6.5. Let $p\geq 3$ and $\lambda>\mu>0$. Consider

$$A=\begin{bmatrix}1+\lambda^2 & 0\\ 0 & 1+\mu^2\end{bmatrix}\quad\text{and}\quad B=\begin{bmatrix}0 & -\mu\\ \lambda & 0\end{bmatrix}.$$

For the sake of simplicity, we restrict the choice of Λ in the form $\left[\begin{smallmatrix}0 & c\\ c & 0\end{smallmatrix}\right]$. Then we have

$$
\begin{aligned}
&A+A'-B'B-(p-2)(T_B+\Lambda)'(T_B+\Lambda)\\
&\quad=\operatorname{diag}\left\{2+\lambda^2-(p-2)\left(c+\frac{\lambda+\mu}{2}\right)^2, 2+\mu^2-(p-2)\left(c-\frac{\lambda+\mu}{2}\right)^2\right\}\\
&\quad=:\operatorname{diag}\{g(c),h(c)\}.
\end{aligned}
$$

As $p\geq 3$ and $\lambda>\mu>0$, it is easily to check that

$$\max_{c\in\mathbf{R}}\{g(c)\wedge h(c)\}=2+\frac{\lambda^2+\mu^2}{2}-\frac{(p-2)(\lambda+\mu)^2}{4}-\frac{(\lambda-\mu)^2}{4(p-2)},$$

where the maximum is attained when $g(c)=h(c)$, i.e.,

$$c=\frac{\lambda-\mu}{2(p-2)}.$$

So one can easily assign some specific values to p, λ and μ to let A and B satisfy Condition 6.1 but not with $\Lambda=0$, for example, $(p,\lambda,\mu)=(3,3,1)$. This shows that the choice $\Lambda=0$ does not always lead to the minimal requirements.

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