

ON BAYESIAN CONSISTENCY FOR FLOWS OBSERVED THROUGH A PASSIVE SCALAR

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We consider the statistical inverse problem of estimating a background fluid flow field \mathbf{v} from the partial, noisy observations of the concentration θ of a substance passively advected by the fluid, so that θ is governed by the partial differential equation

$$\frac{\partial}{\partial t}\theta(t, \mathbf{x}) = -\mathbf{v}(\mathbf{x}) \cdot \nabla\theta(t, \mathbf{x}) + \kappa \Delta\theta(t, \mathbf{x}), \quad \theta(0, \mathbf{x}) = \theta_0(\mathbf{x})$$

for $t \in [0, T]$, $T > 0$ and $\mathbf{x} \in \mathbb{T}^2 = [0, 1]^2$. The initial condition θ_0 and diffusion coefficient κ are assumed to be known and the data consist of point observations of the scalar field θ corrupted by additive, i.i.d. Gaussian noise. We adopt a Bayesian approach to this estimation problem and establish that the inference is consistent, that is, that the posterior measure identifies the true background flow as the number of scalar observations grows large. Since the inverse map is ill-defined for some classes of problems even for perfect, infinite measurements of θ , multiple experiments (initial conditions) are required to resolve the true fluid flow. Under this assumption, suitable conditions on the observation points, and given support and tail conditions on the prior measure, we show that the posterior measure converges to a Dirac measure centered on the true flow as the number of observations goes to infinity.

1. Introduction. In this work we consider the inverse problem of estimating a background fluid flow from partial, noisy observations of a dye, pollutant, or other solute advecting and diffusing within the fluid. The physical model considered is the two-dimensional advection-diffusion equation on the periodic domain $\mathbb{T}^2 = [0, 1]^2$:

$$(1) \quad \frac{\partial}{\partial t}\theta(t, \mathbf{x}) = -\mathbf{v}(\mathbf{x}) \cdot \nabla\theta(t, \mathbf{x}) + \kappa \Delta\theta(t, \mathbf{x}), \quad \theta(0, \mathbf{x}) = \theta_0(\mathbf{x}).$$

Here:

- $\theta : \mathbb{R}^+ \times \mathbb{T}^2 \rightarrow \mathbb{R}$ is a *passive scalar*, typically the concentration of some solute of interest, which is spread by diffusion and by the motion of a (time-stationary) fluid flow \mathbf{v} . This solute is “passive” in that it does not affect the motion of the underlying fluid.
- $\mathbf{v} : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ is an *incompressible background flow*, that is, \mathbf{v} is constant in time and satisfies $\nabla \cdot \mathbf{v} = 0$.
- $\kappa > 0$ is the *diffusion coefficient*, which models the rate at which local concentrations of the solute spread out within the solvent in the absence of advection.

We obtain finite observations $\mathcal{Y} \in \mathbb{R}^N$ subject to additive noise η , that is,

$$(2) \quad \mathcal{Y} = \mathcal{G}(\mathbf{v}) + \eta, \quad \eta \sim \gamma_0$$

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for some measure γ_0 related to the precision of the observations. Here, the forward map $\mathcal{G} : H \rightarrow \mathbb{R}^N$ associates the background flow \mathbf{v} , sitting in a suitable function space H , with a finite collection of measurements (observables) of the resulting solution $\theta = \theta(\mathbf{v})$ of (1). In this work, we are primarily interested in spatial-temporal point observations:

$$(3) \quad \mathcal{G}_j(\mathbf{v}) := \theta(t_j, \mathbf{x}_j, \mathbf{v}) \quad \text{for any } t_j \in [0, T] \text{ and } \mathbf{x}_j \in [0, 1]^2.$$

The goal of the inverse problem is then to estimate the flow \mathbf{v} from data \mathcal{Y} . The initial condition is assumed to be known, so the problem can be interpreted as a controlled experiment, where the solute is added at known locations and then observed as the system evolves to investigate the structure of the underlying flow. This is a common experimental approach to investigating complex fluid flows; see, for example, [Karch et al. \(2012\)](#), [Kellay and Goldburg \(2002\)](#), [Smits \(2012\)](#), [Wolfgang \(1987\)](#).

As we will illustrate, the inverse problem is ill-posed, that is, the flow \mathbf{v} is not uniquely defined by the scalar field θ ; that the observations of θ are both finite-dimensional and polluted by noise exacerbates this problem. We therefore adopt a Bayesian approach to regularize the inverse problem, as described for this problem in our companion work [Borggaard, Glatt-Holtz and Krometis \(2018\)](#) (see also [Krometis \(2018\)](#)) and in a more general setting in, for example, [Dashti and Stuart \(2017\)](#), [Kaipio and Somersalo \(2005\)](#), [Stuart \(2010\)](#). A key component of this approach is the selection of a prior probability measure on the space of divergence-free flows, H . It is then natural to ask to what extent the result of the inference depends on the choice of prior, and in particular whether the Bayesian approach to the inverse problem is *consistent*: That is, under what conditions does the posterior measure concentrate on the true fluid flow as the number of observations N of θ grows large?

In this work, we establish conditions under which the Bayesian inference of \mathbf{v} given data (2) is consistent for i.i.d. observational noise $\eta = (\eta_1, \dots, \eta_N)$, $\eta_j \sim N(0, \sigma_\eta^2)$. We then prove that the posterior measure converges weakly to a Dirac measure centered on the true background flow as the number of scalar observations N grows large; see Section 3 for a full statement of the assumptions and the key result. Here it is a nontrivial task to determine suitable conditions on the structure of the observed data and on the prior measure for which consistency would be expected to hold. As such, as a crucial starting point for the analysis of consistency, one must address difficult experimental design questions.

In our problem, even under the noiseless and complete measurement of θ , essential symmetries can prevent the recovery of \mathbf{v} . For example, a poor choice of θ_0 in (1) makes it impossible to distinguish between (an infinite class of) laminar flows, so multiple experiments (initial conditions) are required to guarantee resolution of the true background flow. A second useful structural condition is that, by picking spatial-temporal observation points at random, we can ensure a sufficiently complete recovery of the solution θ as the number of observation points grows. Third, it is worth emphasizing that we require special conditions on the prior measure. Crucially, we identify a tail condition that ensures that flows are sufficiently smooth—that is, the prior turns out to be critical to the result by restricting consideration to flows of limited roughness (up to a region of low probability).

An important outcome of this experimental design is that it allows us to use compactness to effectively constrain the space of possible divergence-free velocity fields. Indeed, compactness plays an important role in two components of the consistency proof. First, we use it to show the continuity of the inverse map from θ to \mathbf{v} (see Section 4). Second, we use it to develop a suitable uniform version of the law of large numbers in order to show that noisy observations can differentiate between the true and other scalar fields (Section 5).

Consistency of Bayesian estimators has been of interest since at least [Laplace \(1810\)](#), with rigorous proofs of convergence for some problems appearing in the mid-twentieth century ([Doob \(1949\)](#), [Le Cam \(1953\)](#)). The works [Diaconis and Freedman \(1986\)](#), [Freedman \(1963\)](#), [Schwartz \(1965\)](#) identified infinite-dimensional examples where Bayesian estimators are *not*

consistent—that is, there are cases where the data can never guarantee recovery of the true parameter value. See, for example, Wasserman (1998), Le Cam and Yang (2000) or Nickl (2013) for a more detailed description on the history of consistency and the main ideas.

In recent years, there has been interest in extending these consistency results to infinite-dimensional inverse problems, and in particular those constrained by PDEs. Our result is one of the first on consistency in this context. Recent work in this area includes Vollmer (2013), which used an elliptic PDE as the guiding example, and Nickl (2017), which establishes a Bernstein–von Mises theorem—consistency, but also contraction rates in the form of a Gaussian approximation—for Bayesian estimation of parameters of the time-independent Schrödinger equation.

It is worth noting that the related inverse problem of estimating the drift function b from partial observations $\{X_1, \dots, X_N\}$ of the Itô diffusion

$$(4) \quad dX_t = b(X_t) dt + \sigma(X_t) dW_t, \quad t > 0$$

has been studied extensively; see, for example, Papaspiliopoulos et al. (2012) or Gugushvili and Spreij (2014). Consistency has been established in various forms for this problem; see Abraham (2019), Koskela, Spanò and Jenkins (2019), Nickl and Söhl (2017), van der Meulen and van Zanten (2013). However, while the equations (1) and (4) are related by the Kolmogorov equations (see, e.g., Øksendal ((2013), Chapter 8)), the observed data are different: Observations of an individual diffusion provide an approximate measurement of the drift, whereas observations of the concentration θ are less direct—movement of individual particles must be inferred. Our consistency proof therefore, while retaining some similarities with other such arguments, requires an original approach with different assumptions.

The remainder of the paper is organized as follows. Section 2 describes the mathematical framework of the inverse problem and why it is ill-posed in the traditional sense. The main result and key assumptions are stated in Section 3. Continuity of the inverse map is shown in Section 4. Uniform convergence of the log-likelihood is shown in Section 5. Convergence of the posterior to the inverse image of the true scalar field is shown in Section 6. Finally, the proof of the main result is provided in Section 7. Energy estimates for the advection-diffusion problem used to show continuity of the forward and inverse maps are reserved for the Appendix.

2. Preliminaries. In this section, we describe the mathematical framework of the inverse problem (2). We begin by defining the functional analytic setting for the problem, including how we represent divergence-free background flows. We then define the inverse problem, key notation and Bayes’ Theorem for this application.

2.1. Representation of divergence-free background flows. The target of the inference is a divergence-free background flow \mathbf{v} , so we start by describing the space H of such flows that we consider. For this purpose we begin by recalling the Sobolev spaces of (scalar valued) periodic functions on the domain $\mathbb{T}^2 = [0, 1]^2$

$$(5) \quad H^s(\mathbb{T}^2) = \left\{ u : u = \sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus \{0\}} c_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}, \overline{c_{\mathbf{k}}} = c_{-\mathbf{k}}, \|u\|_{H^s} < \infty \right\},$$

where $\|u\|_{H^s}^2 := \sum_{\mathbf{k} \in \mathbb{Z}^2} \|\mathbf{k}\|^{2s} |c_{\mathbf{k}}|^2,$

defined for any $s \in \mathbb{R}$; see, for example, Robinson (2001), Temam (1995). We will abuse notation and use the same notation for periodic divergence-free background flows by replacing the coefficients $c_{\mathbf{k}}$ in (5) as

$$(6) \quad c_{\mathbf{k}} = v_{\mathbf{k}} \frac{\mathbf{k}^\perp}{\|\mathbf{k}\|_2}, \quad \overline{v_{\mathbf{k}}} = -v_{-\mathbf{k}},$$

where for $\mathbf{k} = (k_1, k_2)$ we set $\mathbf{k}^\perp = (-k_2, k_1)$ to ensure $\mathbf{k} \cdot \mathbf{k}^\perp = 0$. Throughout the rest of the paper we fix our parameter space as follows:

NOTATION 2.1 (Parameter space, H). We consider background flows $\mathbf{v} \in H$, where H is the Sobolev space (see (5)),

$$(7) \quad H = H^m(\mathbb{T}^2) \quad \text{for some } m > 1$$

with coefficients $c_{\mathbf{k}}$ given by (6).

Here the exponent m is chosen so that vector fields in H , as well as their corresponding solutions $\theta(\mathbf{v})$, exhibit continuity properties convenient for our analysis below (see Corollary 2.4 below). We take $L^p(\mathbb{T}^2)$ with $p \in [1, \infty]$ for the usual Lebesgue spaces and denote the space of continuous and p th integrable, X -valued functions by $C([0, T]; X)$ and $L^p([0, T]; X)$, respectively, for a given Banach space X . All of these spaces are endowed with their standard topologies unless otherwise specified.

2.2. *Mathematical setting of the advection-diffusion problem.* In this section, we provide a precise definition of solutions θ for the advection-diffusion problem (1). Crucially the setting we choose yields a map from \mathbf{v} to θ and then to observations of θ that is continuous.

PROPOSITION 2.2 (Well-posedness and continuity of the solution map for (1)).

(i) Fix any $s \geq 0$ and $m \geq s$ with $m > 0$ and suppose that $\mathbf{v} \in H^m(\mathbb{T}^2)$ and $\theta_0 \in H^s(\mathbb{T}^2)$. Then there exists a unique $\theta = \theta(\mathbf{v}, \theta_0)$ such that

$$\theta \in L^2_{\text{loc}}([0, \infty); H^{s+1}(\mathbb{T}^2)) \cap L^\infty([0, \infty); H^s(\mathbb{T}^2))$$

$$\text{with } \frac{\partial \theta}{\partial t} \in L^2_{\text{loc}}([0, \infty); H^{s-1}(\mathbb{T}^2))$$

so that in particular

$$\theta \in C([0, \infty); H^s(\mathbb{T}^2))$$

solves (1) at least weakly. In other words, θ satisfies

$$(8) \quad \left\langle \frac{\partial \theta}{\partial t}, \phi \right\rangle_{H^{-1}(\mathbb{T}^2) \times H^1(\mathbb{T}^2)} + \langle \mathbf{v} \cdot \nabla \theta, \phi \rangle_{L^2(\mathbb{T}^2)} + \kappa \langle \nabla \theta, \nabla \phi \rangle_{L^2(\mathbb{T}^2)} = 0$$

for all $\phi \in H^1(\mathbb{T}^2)$ and almost all times $t \in [0, \infty)$.

(ii) For any $T > 0$ the map that associates $\mathbf{v} \in H^m(\mathbb{T}^2)$ and $\theta_0 \in H^s(\mathbb{T}^2)$ to the corresponding $\theta(\mathbf{v}, \theta_0)$ is continuous relative to the standard topologies on $H^m(\mathbb{T}^2) \times H^s(\mathbb{T}^2)$ and $C([0, T]; H^s(\mathbb{T}^2))$.

This result can be proven using energy methods; similar results can be found for example in Evans (2010), Lieberman (1996). In the case of a smooth solution where $s > 3$ one may also establish Proposition 2.2 using particle methods as in for example, Øksendal (2013) by observing that (1) is the Kolmogorov equation corresponding to a stochastic differential equation with the drift given by \mathbf{v} ; see Krometis (2018) for details in our setting. For completeness, we provide the a priori estimates leading to Proposition 2.2 in the Appendix.

DEFINITION 2.3 (Solution operator \mathcal{S} , observation operator \mathcal{O}). Fix $\theta_0 \in H^s(\mathbb{T}^2)$ and a time $T > 0$ and consider the phase space H defined as (7). The forward map \mathcal{G} as in (2) is interpreted as the composition $\mathcal{G}(\mathbf{v}) = \mathcal{O} \circ \mathcal{S}(\mathbf{v})$, where:

1. The *solution operator* $\mathcal{S} : H \rightarrow C([0, T]; H^s(\mathbb{T}^2))$ maps a given \mathbf{v} to the corresponding solution $\theta(\mathbf{v}, \theta_0)$ of (1) (in the sense of Proposition 2.2).

2. The *observation operator* $\mathcal{O} : C([0, T]; H^s(\mathbb{T}^2)) \rightarrow \mathbb{R}^N$ measures point observations $\mathcal{O}(\theta) = (\mathcal{O}_1(\theta), \dots, \mathcal{O}_N(\theta))$ defined by $\mathcal{O}_j(\theta) = \theta(t_j, \mathbf{x}_j)$ for $t_j \in [0, T]$ and $\mathbf{x}_j \in [0, 1]^2$.

We now note assumptions on \mathbf{v} and θ_0 under which these observations are well defined and vary continuously with \mathbf{v} .

COROLLARY 2.4 (Continuity of θ). *Let $\mathbf{v} \in H$ with associated exponent $m > 1$ (see (7)) and let $\theta_0 \in H^s$, for $m \geq s > 1$. Recalling that $H^s(\mathbb{T}^2)$, $s > 1$ embeds continuously in $C(\mathbb{T}^2)$ in dimension 2 (see, e.g., Robinson (2001), Theorem A.1) we have that $C([0, T]; H^s) \subset C([0, T] \times \mathbb{T}^2)$ again with the embedding continuous. Thus, with Proposition 2.2, we have that*

$$\mathcal{S} : H \rightarrow C([0, T] \times \mathbb{T}^2)$$

continuously. In particular this justifies that \mathcal{G} is well defined and continuous in the case of point observations as in Definition 2.3.

2.3. Bayesian setting of the inverse problem. In this subsection, we define the setting of the statistical inverse problem and note cases where the inverse map is ill-posed. This will inform the assumptions required for the consistency argument. We close with a definition of Bayes’ theorem for this problem.

We begin by fixing some notation used in the remainder of the paper.

DEFINITION 2.5 (\mathbf{v}^* , \mathcal{Y} , \mathcal{G} , η). We frequently fix a “true” background flow by $\mathbf{v}^* \in H$. For the given \mathbf{v}^* , the observed data \mathcal{Y} is given by

$$\mathcal{Y} = \mathcal{G}(\mathbf{v}^*) + \eta,$$

where

- the forward map $\mathcal{G} : H \rightarrow \mathbb{R}^N$ with $\mathcal{G}_j(\mathbf{v}) := \theta(t_j, \mathbf{x}_j; \mathbf{v})$ corresponding to the observation point $(t_j, \mathbf{x}_j) \in [0, T] \times [0, 1]^2$;
- the observational noise $\eta = \{\eta_1, \dots, \eta_N\} \in \mathbb{R}^N$ for i.i.d. $\eta_j \sim N(0, \sigma_\eta^2)$.

We emphasize, however, that \mathbf{v}^* is not necessarily the only \mathbf{v} that could produce such data, as we describe in the next remark.

REMARK 2.6. Since the background flow \mathbf{v} enters (1) through the $\mathbf{v} \cdot \nabla \theta$ term, the inverse problem of recovering \mathbf{v} from $\theta(\mathbf{v})$ can be ill-posed. One important class of examples illustrating this difficulty arises when $\mathbf{v} \cdot \nabla \theta$ is zero everywhere, in which case the fluid flow does not have any effect on θ . Two such examples are as follows:

(i) *Ill-posedness: Laminar flow:* Let $\theta_0(\mathbf{x})$ be independent of x_2 and $\mathbf{v}^* = (0, f(x_1))$. Then $\theta(\mathbf{v}^*) = \theta(\mathbf{v})$ for any $\mathbf{v} = (0, g(x_1))$.

(ii) *Ill-posedness: Radial symmetry:* Set $\theta_0(\mathbf{x}) \propto \sin(\pi x_1) + \sin(\pi x_2)$ and $\mathbf{v}^* = (\cos(\pi x_2), -\cos(\pi x_1))$. Then $\theta(\mathbf{v}^*) = \theta(\mathbf{v})$ for any $\mathbf{v} = c\mathbf{v}^*$, $c \in \mathbb{R}$.

In these cases, even noiseless and complete spatial/temporal observations of θ have no way to discriminate between a range of background flows, making it impossible to uniquely identify a true background flow \mathbf{v}^* in general.

We have following adaptation of Bayes' Theorem to the advection-diffusion problem; see the derivation in [Borggaard, Glatt-Holtz and Krometis \(\(2018\), Appendix C\)](#) or [Dashti and Stuart \(2017\)](#) for additional information.

THEOREM 2.7 (Bayes' theorem). *Fix a prior distribution $\mu_0 \in \text{Pr}(H)$ and let forward maps \mathcal{G}_j , data \mathcal{Y}_j , and associated i.i.d. observational noise $\eta_j \sim N(0, \sigma_\eta^2)$ be as defined in Definition 2.5. Then the posterior measure $\mu_{\mathcal{Y}}$ associated with the random variable $\mathbf{v}|\mathcal{Y}$ is absolutely continuous with respect to μ_0 and given by*

$$(9) \quad \mu_{\mathcal{Y}}(d\mathbf{v}) = \frac{1}{Z_{\mathcal{Y}}} \exp\left[-\frac{1}{2\sigma_\eta^2} \sum_{j=1}^N (\mathcal{Y}_j - \mathcal{G}_j(\mathbf{v}))^2\right] \mu_0(d\mathbf{v}),$$

where $Z_{\mathcal{Y}}$ is the normalization

$$(10) \quad Z_{\mathcal{Y}} = \int_H \exp\left[-\frac{1}{2\sigma_\eta^2} \sum_{j=1}^N (\mathcal{Y}_j - \mathcal{G}_j(\mathbf{v}))^2\right] \mu_0(d\mathbf{v}).$$

3. Statement of the main result. With the mathematical preliminaries in Section 2 in hand, we are now ready to provide a precise formulation of the main result of the paper. Referring back to Remark 2.6 we do not expect consistency to hold without delicate assumptions on the initial conditions in (1) and on the observation points in our forward function \mathcal{G} in (2). Moreover our result relies on the selection of an appropriate prior μ_0 . In particular this μ_0 should distinguish the regularity of the 'true' background flow \mathbf{v}^* for which we assume there is greater degree of spatial smoothness than for generic elements in the ambient parameter space H (although a slight generalization is described in Remark 3.9). We therefore define an additional smaller space used throughout.

DEFINITION 3.1 (Higher regularity space). Define the space

$$(11) \quad V = H^{m^*}(\mathbb{T}^2), \quad m^* > m,$$

where m is the exponent associated with the parameter space H defined according to (7). We denote $\|\cdot\|_V$ for the associated norm and take

$$(12) \quad B_V^r(\mathbf{v}_0) = \{\mathbf{v} \in V : \|\mathbf{v} - \mathbf{v}_0\|_V \leq r\},$$

that is, the ball about $\mathbf{v}_0 \in V$ of radius $r > 0$ in the V -norm.

Our main result is as follows:

THEOREM 3.2 (Convergence of posterior to a Dirac). *Let $\{(t_j, \mathbf{x}_j)\}_{j=1}^\infty$ be a sequence of observation points that we assume are i.i.d. uniform random variables in $[0, T] \times \mathbb{T}^2$. Fix any $\theta_0^{(1)}, \theta_0^{(2)} \in H^m$, with $m > 1$ determined from (7), such that*

$$(13) \quad (\nabla\theta_0^{(1)}(\mathbf{x}))^\perp \cdot \nabla\theta_0^{(2)}(\mathbf{x}) \neq 0 \quad \text{for almost all } \mathbf{x} \in \mathbb{T}^2.$$

Define the parameter-to-observable (forward) maps \mathcal{G}_j for $\{(t_j, \mathbf{x}_j)\}_{j=1}^\infty$ and the initial conditions $\theta_0^{(1)}, \theta_0^{(2)}$ by

$$(14) \quad \begin{aligned} \mathcal{G}_{2j-1}(\mathbf{v}) &:= \theta(t_j, \mathbf{x}_j, \mathbf{v}, \theta_0^{(1)}), \\ \mathcal{G}_{2j}(\mathbf{v}) &:= \theta(t_j, \mathbf{x}_j, \mathbf{v}, \theta_0^{(2)}) \end{aligned}$$

for $j = 1, 2, \dots$. As in Definition 2.5, we fix any $\mathbf{v}^* \in V$ and draw data points $\{\mathcal{Y}_j\}_{j=1}^\infty$, where

$$(15) \quad \mathcal{Y}_j = \mathcal{G}_j(\mathbf{v}^*) + \eta_j$$

for i.i.d. observational noises $\eta_j \sim N(0, \sigma_\eta^2)$ that are independent of the observation points $\{(t_j, \mathbf{x}_j)\}_{j=1}^\infty$.

Fix a prior distribution $\mu_0 \in \text{Pr}(H)$ and for $N \geq 1$ observations, let $\mu_{\mathcal{Y}}^N$ be the Bayesian posterior measure on H , given by (cf. Theorem 2.7)

$$(16) \quad \mu_{\mathcal{Y}}^N(d\mathbf{v}) = \frac{1}{Z_{\mathcal{Y}}^N} \exp\left[-\frac{1}{2\sigma_\eta^2} \sum_{j=1}^N (\mathcal{Y}_j - \mathcal{G}_j(\mathbf{v}))^2\right] \mu_0(d\mathbf{v}),$$

where $Z_{\mathcal{Y}}^N$ is the normalization

$$Z_{\mathcal{Y}}^N = \int_H \exp\left[-\frac{1}{2\sigma_\eta^2} \sum_{j=1}^N (\mathcal{Y}_j - \mathcal{G}_j(\mathbf{v}))^2\right] \mu_0(d\mathbf{v}).$$

Suppose that

$$(17) \quad \text{for any } r > 0, \quad \mu_0(B_V^r(\mathbf{v}^*)) > 0.$$

Additionally assume that there exists an $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that f is monotone increasing with $\lim_{r \rightarrow \infty} f(r) = \infty$ and

$$(18) \quad \sup_N \int_H f(\|\mathbf{v}\|_V) \mu_{\mathcal{Y}}^N(d\mathbf{v}) < \infty \quad \text{a.s.}$$

Then $\mu_{\mathcal{Y}}^N \rightharpoonup \delta_{\mathbf{v}^*}$ (weakly in H) almost surely. In other words, on a set of full measure,

$$(19) \quad \int_H \phi(\mathbf{v}) \mu_{\mathcal{Y}}^N(d\mathbf{v}) \rightarrow \phi(\mathbf{v}^*) \quad \text{as } N \rightarrow \infty \text{ for any } \phi \in C_b(H).$$

REMARK 3.3 (Support of the prior). We note that the assumption (17) is a classic assumption in posterior consistency, cf. Ghosal and van der Vaart ((2017), Chapter 6); if the prior “rules out” the true flow, the posterior cannot recover it.

REMARK 3.4 (Equivalent prior tail condition). Condition (18) is equivalent to the assumption that for all $\epsilon > 0$ there exists an R such that

$$(20) \quad \sup_N \mu_{\mathcal{Y}}^N((B_V^R(\mathbf{0}))^c) < \epsilon.$$

To establish that (18) implies (20), let $\epsilon > 0$ and choose R such that

$$\sup_N \int_H f(\|\mathbf{v}\|_V) \mu_{\mathcal{Y}}^N(d\mathbf{v}) < \epsilon f(R).$$

Then for any $N \geq 1$, Markov’s inequality yields

$$\mu_{\mathcal{Y}}^N((B_V^R(\mathbf{0}))^c) = \int_{(B_V^R(\mathbf{0}))^c} \mu_{\mathcal{Y}}^N(d\mathbf{v}) \leq \frac{1}{f(R)} \int_H f(\|\mathbf{v}\|_V) \mu_{\mathcal{Y}}^N(d\mathbf{v}) < \epsilon.$$

Thus, (18) implies (20). For the converse direction, use (20) to select an increasing sequence $\{R_j\}_{j=1}^\infty$ such that

$$\sup_N \mu_{\mathcal{Y}}^N((B_V^{R_j}(\mathbf{0}))^c) < \frac{1}{4^j}.$$

Now define $f(r) = \sum_{j=1}^{\infty} 2^j \mathbb{1}_{r \in [R_j, R_{j+1})}$. Then

$$\begin{aligned} \sup_N \mathbb{E}_{\mu_Y^N} f(\|\mathbf{v}\|_V) &= \sup_N \mathbb{E}_{\mu_Y^N} \sum_{j=1}^{\infty} 2^j \mathbb{1}_{\|\mathbf{v}\|_V \in [R_j, R_{j+1})} \leq \sup_N \mathbb{E}_{\mu_Y^N} \sum_{j=1}^{\infty} 2^j \mathbb{1}_{\mathbf{v} \in (B_V^{R_j}(0))^c} \\ &= \sum_{j=1}^{\infty} 2^j \sup_N \mu_Y^N((B_V^{R_j}(0))^c) < \sum_{j=1}^{\infty} 2^j \frac{1}{4^j} = \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty \end{aligned}$$

so that (20) implies (18).

REMARK 3.5 (Sufficient conditions on the prior). Suppose that

$$(21) \quad \mu_0(B_V^r(\mathbf{0})) = 1$$

for some $r > 0$. In this case, (20) and therefore (18) (see Remark 3.4) are clearly satisfied. Thus we can guarantee the existence of a class of nontrivial priors such that Theorem 3.2 holds. On the other hand the reverse implication is not to be expected to hold and thus the general significance of (18) for the admissible classes of μ_0 is not immediately clear. In particular μ_0 having bounded support is a strong restriction and indeed we conjecture that there is a class of Gaussian measures on V such that (18) still holds. We will investigate this question in future work.

REMARK 3.6 (Poincaré inequality, support of μ_0). Since we are assuming that elements in H are mean-free (see (5)) we have the Poincaré-type inequality

$$(22) \quad \|\mathbf{v}\|_H \leq C \|\mathbf{v}\|_V$$

for a constant C independent of \mathbf{v} . As such, for any $\epsilon > 0$, $B_V^\epsilon \subset B_H^{C\epsilon}$ where C is the constant appearing in (22). In particular under the condition (17) in Theorem 3.2 we have that $\mathbf{v}^* \in \text{supp}(\mu_0) = \{\mathbf{v} \in H : \mu_0(B_H^r(\mathbf{v}^*)) > 0, \text{ for all } r > 0\}$.

REMARK 3.7 (Restrictions on the initial conditions). It unavoidable that we impose a condition such as (13) on the initial data in Theorem 3.2. In Remark 2.6 we provide two examples where the observations have no way to discriminate between a range of background flows. In these two examples as well as many other classes of initial conditions, the posterior fails to concentrate on \mathbf{v}^* as the number of observations $N \rightarrow \infty$ (except for very particular priors). An interesting question for future work is to characterize the support of the limiting measure for the analogue of μ_Y^N as $N \rightarrow \infty$ as a function of a single initial condition θ_0 .

REMARK 3.8 (The role of T). It is worth noting that any $T > 0$ suffices for consistency. This is because the two initial conditions have been chosen so that any \mathbf{v} will have an immediate effect on at least one of $\theta^{(1)}, \theta^{(2)}$. As a result, we need only observe the evolution of the two systems for some nonzero time interval to identify the effect of \mathbf{v}^* (as the number of observations grows large). The question of how $T > 0$ (and hence the placement of observation points) affects the rate at which μ_Y^N converges to $\delta_{\mathbf{v}^*}$ is a much more delicate question for future work.

REMARK 3.9 (Case where $\mathbf{v}^* \notin V$). As pointed out by a helpful reviewer, Theorem 3.2 can be extended to the case where $\mathbf{v}^* \notin V$ by replacing $B_V^r(\mathbf{v}^*)$ with $\mathbf{v}^* + B_V^r(0)$ throughout the proof, as long as the assumption (18) is appropriately recentered on \mathbf{v}^* , that is,

$$\sup_N \int_H f(\|\mathbf{v} - \mathbf{v}^*\|_V) \mu_Y^N(d\mathbf{v}) < \infty \quad \text{a.s.}$$

or, as in Remark 3.4, for all $\epsilon > 0$ there exists R such that

$$\sup_N \mu_{\mathcal{Y}}^N((\mathbf{v}^* + B_V^R(0))^c) < \epsilon.$$

That is, the prior needs to help rule out flows that are far from \mathbf{v}^* as measured by $\|\cdot\|_V$.

Before turning to the technical details let us provide an overview of the method of the proof of Theorem 3.2. Our starting point is based on two basic observations. First, according to Portmanteau’s Theorem, condition (19) can be established with the equivalent condition that

$$(23) \quad \liminf_{N \geq 1} \mu_{\mathcal{Y}}^N(B_H^\epsilon(\mathbf{v}^*)) \geq 1$$

for any $\epsilon > 0$. See for example, Billingsley (2013) for further details on such generalities concerning the weak convergence of probability measures.

Our second observation concerns using the law of large numbers to identify the approximate character of the potential terms in (16) for large N . Referring back to (15) and (16), we have

$$(\mathcal{Y}_j - \mathcal{G}_j(\mathbf{v}))^2 = \eta_j^2 + 2\eta_j(\mathcal{G}_j(\mathbf{v}^*) - \mathcal{G}_j(\mathbf{v})) + (\mathcal{G}_j(\mathbf{v}^*) - \mathcal{G}_j(\mathbf{v}))^2.$$

Invoking the law of large numbers, using assumed statistical properties of $\{\eta_j\}_{j \geq 1}$ and $\{(t_j, x_j)\}_{j \geq 1}$, we have

$$(24) \quad \begin{aligned} & \frac{1}{N} \sum_{j=1}^N (\mathcal{Y}_j - \mathcal{G}_j(\mathbf{v}))^2 \\ & \approx \sigma_\eta^2 + \frac{1}{T} \sum_{l=1}^2 \int_0^T \int_{\mathbb{T}^2} (\theta(t, x, \mathbf{v}, \theta_0^{(l)}) - \theta(t, x, \mathbf{v}^*, \theta_0^{(l)}))^2 d\mathbf{x} dt \end{aligned}$$

for all N sufficiently large.¹ For $\delta > 0$, take

$$(25) \quad \mathcal{X}_\delta = \left\{ \mathbf{v} \in H : \sum_{l=1}^2 \int_0^T \int_{\mathbb{T}^2} (\theta(t, x, \mathbf{v}, \theta_0^{(l)}) - \theta(t, x, \mathbf{v}^*, \theta_0^{(l)}))^2 d\mathbf{x} dt < \delta^2 \right\}.$$

Invoking (24), we observe that

$$(26) \quad \begin{aligned} \mu_{\mathcal{Y}}^N(\mathcal{X}_\delta^c) & \approx \frac{\int_{\mathcal{X}_\delta^c} \exp[-\frac{N}{4\sigma_\eta^2 T} \sum_{l=1}^2 \int_0^T \int_{\mathbb{T}^2} (\theta(\theta_0^{(l)}, \mathbf{v}) - \theta(\theta_0^{(l)}, \mathbf{v}^*))^2 d\mathbf{x} dt] \mu_0(d\mathbf{v})}{\int \exp[-\frac{N}{4\sigma_\eta^2 T} \sum_{l=1}^2 \int_0^T \int_{\mathbb{T}^2} (\theta(\theta_0^{(l)}, \mathbf{v}) - \theta(\theta_0^{(l)}, \mathbf{v}^*))^2 d\mathbf{x} dt] \mu_0(d\mathbf{v})} \\ & \leq \frac{\exp(-\frac{N\delta^2}{4\sigma_\eta^2 T}) \mu_0(\mathcal{X}_\delta^c)}{\int_{\mathcal{X}_{\delta/2}} \exp[-\frac{N}{4\sigma_\eta^2 T} \sum_{l=1}^2 \int_0^T \int_{\mathbb{T}^2} (\theta(\theta_0^{(l)}, \mathbf{v}) - \theta(\theta_0^{(l)}, \mathbf{v}^*))^2 d\mathbf{x} dt] \mu_0(d\mathbf{v})} \\ & \leq \frac{\exp(-\frac{3N\delta^2}{16\sigma_\eta^2 T})}{\mu_0(\mathcal{X}_{\delta/2})}. \end{aligned}$$

Here note that (cf. Remark 3.6) $\mathbf{v}^* \in \text{supp}(\mu_0)$ so that we are not dividing by zero in the final upper bound.

¹Referring back to Section 2.1 we are assuming that \mathbb{T}^2 is unit length.

One is thus tempted to now combine (23) and (26) and find for every $\epsilon > 0$ a corresponding $\delta > 0$ such that

$$(27) \quad \mathcal{X}_\delta \subset B_H^\epsilon$$

so that

$$\mu_{\mathcal{Y}}^N(B_H^\epsilon) \geq \mu_{\mathcal{Y}}^N(\mathcal{X}_\delta) \geq 1 - \frac{\exp(-\frac{3N\delta^2}{16\sigma_\eta^2 T})}{\mu_0(\mathcal{X}_{\delta/2})},$$

yielding the desired weak convergence (19). However this naïve argument runs up against two fundamental flaws:

(i) Although, as we establish below in Lemma 4.3, the condition (24) ensures that the map $\mathbf{v} \mapsto (\theta(\cdot, \theta_0^{(1)}, \mathbf{v}), \theta(\cdot, \theta_0^{(2)}, \mathbf{v}))$ is injective into $L^2([0, T] \times \mathbb{T}^2)$ it is not clear if this map has a continuous inverse, which we would need for (27).

(ii) It is not obvious that we have sufficient uniformity over $\mathbf{v} \in H$ in our invocation of the LLN in (24). In particular this means that the approximation in the first line in (26) would be unjustified.

We address both of these concerns by assuming a little bit of extra regularity for our “true vector field” taking $\mathbf{v}^* \in V$ and by making effective use of the prior to identify this regularity for \mathbf{v}^* (see assumptions (17), (18)). With the Rellich–Kondrachov theorem we are thus able to use “compactness” to address both concerns. Indeed although an injective, continuous map ψ does not have a continuous inverse in general, this property does hold true when the domain of ψ is compact; see Lemma 4.4 below. Regarding the second concern, we establish a uniform version of the LLN Proposition 5.1 below (and see also Newey and McFadden (1994), Nickl (2013)) but our proof makes essential use of the fact that the “parameter” (which for us is $\mathbf{v} \in H$) lies in a compact set.

The precise proof of Theorem 3.2 is presented in a series of sections as follows. Firstly in Section 4 we address the injectivity of the forward map under (13) as well as continuity of the inverse map. In Section 5 we introduce a uniform version of the Law of Large Numbers, Proposition 5.1 and use this Proposition to obtain a quantitative version of (24). Section 6 establishes that $\mu_{\mathcal{Y}}^N$ converges on the “true scalar field” $\theta(\mathbf{v}^*)$ as $N \rightarrow \infty$. Finally Section 7 uses the machinery now in place to complete the proof of Theorem 3.2.

4. Continuity of inverse map. In this section, we lay out conditions under which the inverse solution map $\theta \mapsto \mathbf{v}$ is continuous. This requires some care. Indeed it is not true in general that the forward map \mathcal{S} is injective, as illustrated in Remark 2.6. As such, counterexamples to Theorem 3.2 exist (cf. Remark 3.7) if we fail to impose a suitable assumption on the initial condition(s) for (1) à la (13).

With this in mind we now define the solution map associated with the solution of (1) for the multiple initial conditions.

NOTATION 4.1 (Paired solution map $\tilde{\mathcal{S}}$). Fix any $\theta_0^{(1)}, \theta_0^{(2)} \in H^m$ for $m > 1$ as in (5) and let $\theta^{(1)}(\mathbf{v}), \theta^{(2)}(\mathbf{v})$ be the associated solutions of (1) corresponding to $\mathbf{v} \in H$ defined according to Proposition 2.2. We denote

$$\tilde{\mathcal{S}}(\mathbf{v}) = (\theta(\cdot, \mathbf{v}, \theta_0^{(1)}), \theta(\cdot, \mathbf{v}, \theta_0^{(2)})),$$

regarding $\tilde{\mathcal{S}}$ as a map $\tilde{\mathcal{S}} : H \rightarrow L^2([0, T] \times \mathbb{T}^2)^2$.

We now observe that the the paired solution map $\tilde{\mathcal{S}}$ is continuous (Corollary 4.2) and that under condition (13), $\tilde{\mathcal{S}}$ is 1-to-1 (Lemma 4.3).

COROLLARY 4.2 ($\tilde{\mathcal{S}}$ continuous). *The paired solution map $\tilde{\mathcal{S}} : H \rightarrow L^2([0, T] \times \mathbb{T}^2)^2$ (see Notation 4.1) is continuous.*

PROOF. For any $\theta_0 \in H^m$ (with m as in (7)) the associated solution map $\mathcal{S} : H \rightarrow L^2([0, T] \times \mathbb{T}^2)$ given by $\mathcal{S}(\mathbf{v}) = \theta(\cdot, \mathbf{v}, \theta_0)$ is continuous by Corollary 2.4 so that the map $\tilde{\mathcal{S}}$ is also continuous. \square

LEMMA 4.3 ($\tilde{\mathcal{S}}$ injective). *Let $\tilde{\mathcal{S}}$ be the paired solution map given in Notation 4.1 with initial conditions satisfying (13). Suppose that $\mathbf{v}, \tilde{\mathbf{v}} \in H$ such that*

$$(28) \quad \|\tilde{\mathcal{S}}(\mathbf{v}) - \tilde{\mathcal{S}}(\tilde{\mathbf{v}})\|_{L^2([0, T] \times \mathbb{T}^2)^2} = 0.$$

Then $\mathbf{v} = \tilde{\mathbf{v}}$, or in other words, $\tilde{\mathcal{S}}$ is injective.

PROOF. Let $\mathbf{v}, \tilde{\mathbf{v}} \in H$ satisfy (28), that is,

$$\|\theta^{(i)}(\cdot, \mathbf{v}) - \theta^{(i)}(\cdot, \tilde{\mathbf{v}})\|_{L^2([0, T] \times \mathbb{T}^2)} = 0, \quad i = 1, 2.$$

Then $\theta^{(i)}(t, \mathbf{x}, \mathbf{v}) = \theta^{(i)}(t, \mathbf{x}, \tilde{\mathbf{v}})$ for almost all t, \mathbf{x} and $i = 1, 2$. Denote $\theta^{(i)}(t, \mathbf{x}) := \theta^{(i)}(t, \mathbf{x}, \mathbf{v}) = \theta^{(i)}(t, \mathbf{x}, \tilde{\mathbf{v}})$. Then from Proposition 2.2, $\theta^{(i)}(t, \mathbf{x})$ satisfies

$$(29) \quad \left\langle \frac{\partial \theta^{(i)}}{\partial t}, \phi \right\rangle_{H^{-1}(\mathbb{T}^2) \times H^1(\mathbb{T}^2)} + \langle \mathbf{u} \cdot \nabla \theta^{(i)}, \phi \rangle_{L^2(\mathbb{T}^2)} + \kappa \langle \nabla \theta^{(i)}, \nabla \phi \rangle_{L^2(\mathbb{T}^2)} = 0$$

for all $\phi \in H^1(\mathbb{T}^2)$, almost all time $t \in [0, \infty)$, $i = 1, 2$, and $\mathbf{u} = \mathbf{v}, \tilde{\mathbf{v}}$. Subtraction leads to

$$(30) \quad g(t) = \langle (\mathbf{v} - \tilde{\mathbf{v}}) \cdot \nabla \theta^{(i)}(t), \phi \rangle_{L^2(\mathbb{T}^2)} = 0$$

for all $\phi \in H^1(\mathbb{T}^2)$, almost all time $t \in [0, \infty)$, and $i = 1, 2$. Since we also have

$$(31) \quad g(t) = -\langle \theta^{(i)}(t), (\mathbf{v} - \tilde{\mathbf{v}}) \cdot \nabla \phi \rangle_{L^2(\mathbb{T}^2)}$$

and $\theta^{(i)} \in C([0, \infty); L^2(\mathbb{T}^2))$ by Proposition 2.2 we infer that $g(t) = 0$ for all $t \geq 0$ and in particular $g(0) = 0$. Therefore

$$(32) \quad (\mathbf{v}(\mathbf{x}) - \tilde{\mathbf{v}}(\mathbf{x})) \cdot \nabla \theta_0^{(i)}(\mathbf{x}) = 0$$

for $i = 1, 2$ and almost all $\mathbf{x} \in \mathbb{T}^2$. However, under the assumption (13), $\nabla \theta_0^{(1)}(\mathbf{x}), \nabla \theta_0^{(2)}(\mathbf{x})$ span \mathbb{R}^2 at almost all \mathbf{x} . Therefore $\tilde{\mathbf{v}}(\mathbf{x}) = \mathbf{v}(\mathbf{x})$ for almost all \mathbf{x} and hence $\|\mathbf{v} - \tilde{\mathbf{v}}\|_H = 0$, completing the proof. \square

Even under the conditions of Lemma 4.3 it remains unclear if $\tilde{\mathcal{S}}$ has a continuous inverse. To remedy this we recall the following elementary fact from real analysis suggesting we further restrict the domain of $\tilde{\mathcal{S}}$.

LEMMA 4.4. *Let Y, Z be metric spaces and suppose that $B \subset Y$ is compact. Let $f : Y \rightarrow Z$ be injective and continuous. Then $f^{-1} : f(B) \rightarrow Y$ is also continuous.²*

PROOF. See, for example, Rudin ((1964), Theorem 4.17). \square

From Lemma 4.4 we draw the following two conclusions.

²Here, we denote $f(B) := \{f(y) \in Z : y \in B\}$.

COROLLARY 4.5 (\tilde{S}^{-1} continuous). *Let $\tilde{S} : H \rightarrow L^2([0, T] \times \mathbb{T}^2)^2$ be the paired solution map given in Notation 4.1 with initial conditions meeting (13). Then, for any $r > 0$ and $\mathbf{v}_0 \in V$, $\tilde{S}^{-1} : \tilde{S}(B_V^r(\mathbf{v}_0)) \rightarrow H$ is continuous.*

PROOF. We have $\tilde{S} : H \rightarrow L^2([0, T] \times \mathbb{T}^2)^2$ continuous by Corollary 4.2 and injective by Lemma 4.3. We also have $B_V^r(\mathbf{v}_0)$ compact in H by the Rellich–Kondrachov Theorem; see, for example, Corollary A.5 of Robinson (2001). Therefore, $\tilde{S}^{-1} : \tilde{S}(B_V^r(\mathbf{v}_0)) \rightarrow H$ is continuous by Lemma 4.4. \square

COROLLARY 4.6. *Let $r > 0$ and $\mathbf{v}_0 \in V$. For all $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\{\mathbf{v} \in H : \|\tilde{S}(\mathbf{v}) - \tilde{S}(\mathbf{v}_0)\|_{L^2([0, T] \times \mathbb{T}^2)^2} < \delta\} \cap B_V^r(\mathbf{v}_0) \subset B_H^\epsilon(\mathbf{v}_0).$$

5. Concentration of normalized potentials, uniform law of large numbers. The next step in our analysis is to prove a rigorous and more quantitative version of (24), Proposition 5.2, which yields the asymptotics of the potential functions (log-likelihoods) appearing in the posterior measures μ_Y^N defined as in (16). As a preliminary step we introduce a uniform version of the law of large numbers. See also Newey and McFadden (1994), Nickl (2013) for previous related results.

PROPOSITION 5.1 (Uniform law of large numbers). *Let (X, ρ) be a metric space with $B \subset X$ compact and $f : \mathbb{R}^n \times X \rightarrow \mathbb{R}$ (Borel) measurable. Take $\{Z_j\}_{j=1}^\infty \in \mathbb{R}^n$ to be an i.i.d. sequence of random variables and let Z be any random variable with this distribution. Assume that*

$$(33) \quad \mathbb{E}f(Z, x)^2 < \infty \quad \text{for all } x \in B$$

and that there exists a deterministic function $d : \mathbb{R}^n \rightarrow \mathbb{R}^+$ with $\mathbb{E}d(Z)^2 < \infty$ such that for all $\epsilon > 0$ and $x \in B$, there exists a $\delta = \delta(x, \epsilon) > 0$ such that

$$(34) \quad \rho(x, \tilde{x}) < \delta \implies |f(z, x) - f(z, \tilde{x})| \leq d(z)\epsilon \quad \text{for all } z \in \mathbb{R}^n.$$

Then

$$(35) \quad \lim_{N \rightarrow \infty} \sup_{x \in B} \left| \frac{1}{N} \sum_{j=1}^N f(Z_j, x) - \mathbb{E}f(Z, x) \right| = 0 \quad \text{a.s.}$$

PROOF. Note that, since d is nonnegative, $\mathbb{E}d(Z) = 0$ implies that

$$\tilde{\Omega} = \bigcap_{j=1}^\infty \{\omega \in \Omega : d(Z_j(\omega)) = 0\}$$

is a set of full measure in which case the random functions $x \mapsto f(Z_j, x)$, $j = 1, 2, \dots$ are all constant on $\tilde{\Omega}$ and the result (35) follows for this special case.

We turn to the nontrivial case where $\mathbb{E}d(Z) \neq 0$. Define $g(z, x) := f(z, x) - \mathbb{E}f(Z, x)$, $z \in \mathbb{R}^n, x \in X$. Then by our assumptions on f , $\mathbb{E}g(Z, x)^2 < \infty$ for every $x \in B$. Note also that for any $x, \tilde{x} \in X$, $\epsilon > 0$, and $z \in \mathbb{R}^n$,

$$(36) \quad |f(z, x) - f(z, \tilde{x})| \leq d(z)\epsilon \implies |g(z, x) - g(z, \tilde{x})| \leq [d(z) + \mathbb{E}d(Z)]\epsilon.$$

Fix any $\epsilon > 0$. Then by (34) and (36), for each $x \in B$ there exists a $\delta(x, \epsilon) > 0$ such that $\rho(\tilde{x}, x) < \delta(x, \epsilon)$ implies $|g(z, \tilde{x}) - g(z, x)| < \frac{d(z) + \mathbb{E}d(Z)}{2\mathbb{E}d(Z)}\epsilon$. Let $B^{\delta(x, \epsilon)}(x) = \{\tilde{x} \in X :$

$\rho(\tilde{x}, x) < \delta(x, \epsilon)$ and note that $\bigcup_{x \in B} B^{\delta(x, \epsilon)}(x) \supset B$. Then since B is compact, there exists a finite subcovering $\{B^{\delta_i}(x_i)\}_{i=1}^m$, $\delta_i := \delta(x_i, \epsilon)$ such that

$$\bigcup_{i=1}^m B^{\delta_i}(x_i) \supset B.$$

Let $x \in B$ and let i be an index such that $x \in B^{\delta_i}(x_i)$. Then

$$\begin{aligned} \left| \frac{1}{N} \sum_{j=1}^N g(Z_j, x) \right| &\leq \left| \frac{1}{N} \sum_{j=1}^N (g(Z_j, x) - g(Z_j, x_i)) \right| + \left| \frac{1}{N} \sum_{j=1}^N g(Z_j, x_i) \right| \\ &\leq \frac{\epsilon}{2\mathbb{E}d(Z)} \left| \frac{1}{N} \sum_{j=1}^N d(Z_j) + \mathbb{E}d(Z) \right| + \left| \frac{1}{N} \sum_{j=1}^N g(Z_j, x_i) \right|. \end{aligned}$$

Taking the supremum over x and using the subcovering yields

$$\begin{aligned} \sup_{x \in B} \left| \frac{1}{N} \sum_{j=1}^N g(Z_j, x) \right| &\leq \max_{i=1, \dots, m} \sup_{x \in B^{\delta_i}(x_i)} \left[\frac{\epsilon}{2\mathbb{E}d(Z)} \left| \frac{1}{N} \sum_{j=1}^N d(Z_j) + \mathbb{E}d(Z) \right| + \left| \frac{1}{N} \sum_{j=1}^N g(Z_j, x_i) \right| \right] \\ &\leq \frac{\epsilon}{2\mathbb{E}d(Z)} \left| \frac{1}{N} \sum_{j=1}^N d(Z_j) + \mathbb{E}d(Z) \right| + \max_{i=1, \dots, m} \left| \frac{1}{N} \sum_{j=1}^N g(Z_j, x_i) \right|. \end{aligned}$$

Then the strong law of large numbers gives

$$\begin{aligned} \limsup_{N \rightarrow \infty} \left[\sup_{x \in B} \left| \frac{1}{N} \sum_{j=1}^N g(Z_j, x) \right| \right] &\leq \limsup_{N \rightarrow \infty} \left(\frac{\epsilon}{2\mathbb{E}d(Z)} \left| \frac{1}{N} \sum_{j=1}^N d(Z_j) + \mathbb{E}d(Z) \right| \right. \\ &\quad \left. + \max_{i=1, \dots, m} \left| \frac{1}{N} \sum_{j=1}^N g(Z_j, x_i) \right| \right) \\ &\leq \epsilon \frac{2\mathbb{E}d(Z)}{2\mathbb{E}d(Z)} + \max_{i=1, \dots, m} \mathbb{E}g(Z, x_i) = \epsilon \quad \text{a.s.,} \end{aligned}$$

where the last equality follows from the fact that $\mathbb{E}g(Z, x) = 0$ for all x . Thus, we have

$$\Omega_\epsilon := \left\{ \limsup_{N \rightarrow \infty} \sup_{x \in B} \left| \frac{1}{N} \sum_{j=1}^N g(Z_j, x) \right| < \epsilon \right\}$$

has probability 1 for all $\epsilon > 0$. Then taking $\Omega_0 = \bigcap_{k=1}^\infty \Omega_{\frac{1}{k}}$ and invoking the continuity of measures,

$$\mathbb{P}\{\Omega_0\} = \mathbb{P}\left\{ \limsup_{N \rightarrow \infty} \sup_{x \in B} \left| \frac{1}{N} \sum_{j=1}^N g(Z_j, x) \right| = 0 \right\} = \lim_{K \rightarrow \infty} \mathbb{P}\left\{ \bigcap_{k=1}^K \Omega_{\frac{1}{k}} \right\} = 1,$$

which is the desired result. \square

We now use this uniform law of large numbers to show that for large N , the growth in the log-likelihood (normalized by $\frac{1}{N}$) for a vector field \mathbf{v} can be written in terms of the observation error and the difference between the scalar fields associated with \mathbf{v} and \mathbf{v}^* .

PROPOSITION 5.2. *Let $\{(t_j, \mathbf{x}_j)\}_{j=1}^\infty$ be a sequence of observation points independently and identically uniformly distributed in $[0, T] \times \mathbb{T}^2$. Fix a $\mathbf{v}^* \in V$ and draw associated data points $\{\mathcal{Y}_j\}_{j=1}^\infty$ according to*

$$(37) \quad \mathcal{Y}_j = \mathcal{G}_j(\mathbf{v}^*) + \eta_j$$

for i.i.d. observational noise $\eta_j \sim N(0, \sigma_\eta^2)$ independent of $\{\mathcal{Y}_j\}_{j=1}^\infty$ and the parameter-to-observable (forward) maps \mathcal{G}_j given by (14). Then, for any $r > 0$,

$$(38) \quad \limsup_{N \rightarrow \infty} \sup_{\mathbf{v} \in B'_V(\mathbf{v}^*)} \left| \frac{1}{N} \sum_{j=1}^N (\mathcal{Y}_j - \mathcal{G}_j(\mathbf{v}))^2 - \left(\sigma_\eta^2 + \frac{1}{2T} \|\tilde{\mathcal{S}}(\mathbf{v}^*) - \tilde{\mathcal{S}}(\mathbf{v})\|_{L^2([0, T] \times \mathbb{T}^2)}^2 \right) \right| = 0,$$

almost surely, where $\tilde{\mathcal{S}}$ is the paired solution operator as in Notation 4.1.

PROOF. Referring to (37) and expanding we have

$$(39) \quad \frac{1}{N} \sum_{j=1}^N (\mathcal{Y}_j - \mathcal{G}_j(\mathbf{v}))^2 = \frac{1}{N} \sum_{j=1}^N \eta_j^2 + \frac{2}{N} \sum_{j=1}^N \eta_j (\mathcal{G}_j(\mathbf{v}^*) - \mathcal{G}_j(\mathbf{v})) + \frac{1}{N} \sum_{j=1}^N (\mathcal{G}_j(\mathbf{v}^*) - \mathcal{G}_j(\mathbf{v}))^2$$

$$(40) \quad := \frac{1}{N} \sum_{j=1}^N (T_{1,j} + 2T_{2,j} + T_{3,j}),$$

for any $N \geq 1$. We will now focus on each of the three terms on the right hand side.

Terms involving $T_{1,j}$: For this first term, the law of large numbers yields

$$(41) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N T_{1,j} = \mathbb{E}\eta_j^2 = \sigma_\eta^2 \quad \text{a.s.}$$

Also, these terms are independent of, and therefore uniform in, $\mathbf{v} \in H$.

Terms involving $T_{2,j}$: Here we establish uniform convergence using Proposition 5.1. Denote $z = (z_\eta, z_t, z_{x_1}, z_{x_2}) \in \mathbb{R}^4$ and define

$$(42) \quad f_i(z, \mathbf{v}) = z_\eta (\theta(z_t, (z_{x_1}, z_{x_2}), \mathbf{v}^*, \theta_0^{(i)}) - \theta(z_t, (z_{x_1}, z_{x_2}), \mathbf{v}, \theta_0^{(i)}))$$

for $i = 1, 2$. Let us verify the conditions required by Proposition 5.1 for f_i . Note that by our assumption on η , $\mathbb{E}\eta^2 = \sigma_\eta^2 < \infty$. Thus, by the maximum principle,

$$(43) \quad \mathbb{E}f_i((\eta, t, \mathbf{x}), \mathbf{v})^2 < \|\theta_0^{(i)}\|_{L^\infty}^2 \mathbb{E}|\eta|^2 < \infty, \quad i = 1, 2,$$

which corresponds to (33). Moreover by the continuity identified in Corollary 2.4, for all $\epsilon > 0$, $\mathbf{v} \in H$, there exists a $\delta = \delta(\mathbf{v})$ such that

$$(44) \quad \|\mathbf{v} - \tilde{\mathbf{v}}\| < \delta(\mathbf{v}) \implies |f(z, \mathbf{v}) - f(z, \tilde{\mathbf{v}})| \leq |z_\eta| \epsilon \quad \text{for } i = 1, 2,$$

thus verifying (34). Finally observe that since $\{\eta_j\}$ and $\{t_j, \mathbf{x}_j\}$ are independent, so are $\{\eta_j\}$ and $\{\mathcal{G}_j(\mathbf{v}^*) - \mathcal{G}_j(\mathbf{v})\}$. Furthermore, using $\mathbb{E}\eta_j = 0$, we have for any $j \geq 1$, $\mathbb{E}\eta_j(\mathcal{G}_j(\mathbf{v}^*) - \mathcal{G}_j(\mathbf{v})) = \mathbb{E}\eta_j \mathbb{E}(\mathcal{G}_j(\mathbf{v}^*) - \mathcal{G}_j(\mathbf{v})) = 0$, thus

$$(45) \quad \mathbb{E}f_i(Z, \mathbf{v}) = 0 \quad \text{for } i = 1, 2.$$

Fix any $r > 0$. Since $B_V^r(\mathbf{v}^*)$ is compact in H by the Rellich–Kondrachov Theorem (Robinson ((2001), Corollary A.5)), and using (43)–(45), Proposition 5.1 yields

$$\begin{aligned}
 \limsup_{N \rightarrow \infty} \sup_{\mathbf{v} \in B_V^r(\mathbf{v}^*)} \left| \frac{2}{N} \sum_{j=1}^N T_{2,j}(\mathbf{v}) \right| &\leq \limsup_{N \rightarrow \infty} \sup_{\mathbf{v} \in B_V^r(\mathbf{v}^*)} \left| \frac{2}{N} \sum_{l=0}^{\lceil N/2 \rceil - 1} T_{2,2l+1}(\mathbf{v}) \right| \\
 (46) \qquad &+ \limsup_{N \rightarrow \infty} \sup_{\mathbf{v} \in B_V^r(\mathbf{v}^*)} \left| \frac{2}{N} \sum_{l=1}^{\lfloor N/2 \rfloor} T_{2,2l}(\mathbf{v}) \right| \\
 &= 0.
 \end{aligned}$$

Terms involving $T_{3,j}$: Here we begin by observing that, since the observations (t_j, \mathbf{x}_j) are uniformly distributed on $[0, T] \times \mathbb{T}^2$,

$$\mathbb{E}T_{3,j} := \begin{cases} \frac{1}{T} \|\theta(\cdot, \mathbf{v}^*, \theta_0^{(1)}) - \theta(\cdot, \mathbf{v}, \theta_0^{(1)})\|_{L^2([0,T] \times \mathbb{T}^2)}^2 & \text{if } j \text{ is even,} \\ \frac{1}{T} \|\theta(\cdot, \mathbf{v}^*, \theta_0^{(2)}) - \theta(\cdot, \mathbf{v}, \theta_0^{(2)})\|_{L^2([0,T] \times \mathbb{T}^2)}^2 & \text{if } j \text{ is odd.} \end{cases}$$

To show the uniform convergence of these terms, denote $z = (z_t, z_{x_1}, z_{x_2}) \in \mathbb{R}^3$ and define

$$f_i(z, \mathbf{v}) = (\theta(z_t, z_{x_1}, z_{x_2}, \mathbf{v}^*, \theta_0^{(i)}) - \theta(z_t, z_{x_1}, z_{x_2}, \mathbf{v}, \theta_0^{(i)}))^2$$

for $i = 1, 2$. Invoking the the maximum principle as in (43) we have,

$$(47) \qquad \mathbb{E}f_i(Z, \mathbf{v})^2 < 16\|\theta_0^{(i)}\|_{L^\infty}^4 < \infty, \quad i = 1, 2,$$

where here Z is distributed uniformly as (t_j, \mathbf{x}_j) . Also, by Corollary 2.4, for all $\epsilon > 0$ there exists a δ such that

$$(48) \qquad \|\mathbf{v} - \tilde{\mathbf{v}}\| < \delta(\mathbf{v}) \implies |f(z, \mathbf{v}) - f(z, \tilde{\mathbf{v}})| \leq \epsilon.$$

Note that in this case the bound is independent of z . Noting once again that $B_V^r(\mathbf{v}^*)$ is a compact subset of H and that (47), (48) yield the conditions (33), (34) we find with Proposition 5.1 that

$$\begin{aligned}
 \limsup_{N \rightarrow \infty} \sup_{\mathbf{v} \in B_V^r(\mathbf{v}^*)} \left| \frac{1}{N} \sum_{j=1}^N (\mathcal{G}_j(\mathbf{v}^*) - \mathcal{G}_j(\mathbf{v}))^2 - \frac{1}{2T} \|\tilde{\mathcal{S}}(\mathbf{v}^*) - \tilde{\mathcal{S}}(\mathbf{v})\|_{L^2([0,T] \times \mathbb{T}^2)}^2 \right| \\
 (49) \qquad \leq \limsup_{N \rightarrow \infty} \sup_{\mathbf{v} \in B_V^r(\mathbf{v}^*)} \left| \frac{1}{N} \sum_{l=0}^{\lceil N/2 \rceil - 1} T_{3,2l+1}(\mathbf{v}) - \frac{\mathbb{E}f_1(Z, \mathbf{v})}{2} \right| \\
 \qquad \qquad + \limsup_{N \rightarrow \infty} \sup_{\mathbf{v} \in B_V^r(\mathbf{v}^*)} \left| \frac{1}{N} \sum_{l=1}^{\lfloor N/2 \rfloor} T_{3,2l}(\mathbf{v}) - \frac{\mathbb{E}f_2(Z, \mathbf{v})}{2} \right| \\
 = 0.
 \end{aligned}$$

Referring back to (40) and assembling the three estimates (41), (46), and (49), we arrive at (38). The proof is complete. \square

6. Identification of the scalar field. In this section, we show that the Bayesian posterior measure for N point observations μ_N^N converges to background flows that closely match the true scalar field $\theta(\mathbf{v}^*)$. The idea is to use the decomposition of the log-likelihood given in Proposition 5.2 along with the assumptions (17), (18) to gain control of tail events.

The main result is as follows

PROPOSITION 6.1 (Identification of true θ). *Take $\{(t_j, \mathbf{x}_j)\}$, \mathbf{v}^* , \mathcal{G}_j , and $\{\mathcal{Y}_j\}$ to be the observation points, the “true vector field”, the forward map, and the data, respectively that are defined as in and satisfy the conditions of Theorem 3.2. Let $\mu_{\mathcal{Y}}^N$ be the associated posterior measures for N observations given by (16), where we assume that the conditions (17), (18) are enforced. Then, for any $\delta > 0$,*

$$(50) \quad \mu_{\mathcal{Y}}^N(\mathcal{X}_\delta) \rightarrow 1 \quad \text{as } N \rightarrow \infty,$$

on a set of full measure, where, cf. (25),

$$(51) \quad \mathcal{X}_\delta = \{\mathbf{v} \in H : \|\tilde{\mathcal{S}}(\mathbf{v}^*) - \tilde{\mathcal{S}}(\mathbf{v})\|_{L^2([0,T] \times \mathbb{T}^2)} < \delta\}.$$

REMARK 6.2. Note that Corollary 4.6 with $\mathbf{v}_0 = \mathbf{v}^*$ also has implications for \mathcal{X}_δ . Indeed, these two characterizations will be combined in Section 7 to prove the main result.

Before turning directly to the proof of Proposition 6.1 we first establish a lemma that derives some simple but useful consequences of the assumptions (17), (18). We recycle this lemma again for later use in Section 7.

LEMMA 6.3. *Suppose that $\mathbf{v}^* \in V$, and that μ_0 satisfies (17). Define the measures $\mu_{\mathcal{Y}}^N$ as in (16) and assume that the condition (18) is maintained.³ Then, on a set $\tilde{\Omega}$ of full measure, for any $\delta, \epsilon > 0$, there exists an $R = R(\delta, \epsilon, \omega) > 0$ (but independent of N) so that both*

$$(52) \quad \mu_0(\mathcal{X}_\delta \cap B_V^R(\mathbf{v}^*)) > 0 \quad \text{and} \quad \mu_{\mathcal{Y}}^N((B_V^R(\mathbf{v}^*))^c) < \epsilon$$

for every $N \geq 1$.

PROOF. Let $\delta > 0$ and $\epsilon > 0$. By Corollary 4.2, there exists an $r > 0$ such that $B_V^r(\mathbf{v}^*) \subset \mathcal{X}_\delta$. Thus for any $R > r$, we observe that

$$\mu_0(\mathcal{X}_\delta \cap B_V^R(\mathbf{v}^*)) \geq \mu_0(\mathcal{X}_\delta \cap B_V^r(\mathbf{v}^*)) = \mu_0(B_V^r(\mathbf{v}^*)) > 0$$

by (17).

To establish the other condition in (52), use Remark 3.4 to choose $\tilde{R} > r$ such that

$$\mu_{\mathcal{Y}}^N((B_V^{\tilde{R}}(\mathbf{0}))^c) < \epsilon.$$

Then selecting $R = \tilde{R} + \|\mathbf{v}^*\|_V$ ensures that $R > r$, maintaining the first condition in (52), and further guaranteeing that $(B_V^R(\mathbf{v}^*))^c \subset (B_V^{\tilde{R}}(\mathbf{0}))^c$, and thus $\mu_{\mathcal{Y}}^N((B_V^R(\mathbf{v}^*))^c) < \epsilon$, as desired for the second condition in (52). The proof is now complete. \square

With Lemma 6.3 in hand we now turn to the proof of the main result of this section.

PROOF OF PROPOSITION 6.1. We begin by specifying an event on which (50) will be established. Take

$$\tilde{\Omega} = \bigcap_{n=1}^{\infty} \{\omega \in \Omega : (38) \text{ holds for } r = n\}.$$

According to Proposition 5.2 this is a set of full measure. Fix any $\omega \in \tilde{\Omega}$. All of the constants and statements that follow will implicitly depend on this sample ω .

³See also (21) in Remark 3.5.

Take arbitrary $\delta, \epsilon > 0$. As in Lemma 6.3, select $R > 0$ so that both

$$\mu_0(\mathcal{X}_\delta \cap B_V^R(\mathbf{v}^*)) > 0 \quad \text{and} \quad \mu_Y^N((B_V^R(\mathbf{v}^*))^c) < \frac{\epsilon}{2}.$$

For these values of R and δ , we invoke Proposition 5.2 and infer that there exists an $N_1 > 0$ such that for all $N \geq N_1$ and every $\mathbf{v} \in B_V^R(\mathbf{v}^*)$,

$$\left| \frac{1}{N} \sum_{j=1}^N (\mathcal{Y}_j - \mathcal{G}_j(\mathbf{v}))^2 - \left(\sigma_\eta^2 + \frac{1}{2T} \|\tilde{\mathcal{S}}(\mathbf{v}^*) - \tilde{\mathcal{S}}(\mathbf{v})\|_{L^2([0,T] \times \mathbb{T}^2)}^2 \right) \right| < \frac{\delta^2}{8T}.$$

Then for every $\mathbf{v} \in \mathcal{X}_\delta \cap B_V^R(\mathbf{v}^*)$ and $N \geq N_1$, we have

$$\begin{aligned} \frac{1}{N} \sum_{j=1}^N (\mathcal{Y}_j - \mathcal{G}_j(\mathbf{v}))^2 &< \sigma_\eta^2 + \frac{1}{2T} \|\tilde{\mathcal{S}}(\mathbf{v}^*) - \tilde{\mathcal{S}}(\mathbf{v})\|_{L^2([0,T] \times \mathbb{T}^2)}^2 + \frac{\delta^2}{8T} \\ &< \sigma_\eta^2 + \frac{\delta^2}{4T}. \end{aligned}$$

Similarly, for every $\mathbf{v} \in \mathcal{X}_\delta^c \cap B_V^R(\mathbf{v}^*)$ and $N \geq N_1$, we have

$$\begin{aligned} \frac{1}{N} \sum_{j=1}^N (\mathcal{Y}_j - \mathcal{G}_j(\mathbf{v}))^2 &> \sigma_\eta^2 + \frac{1}{2T} \|\tilde{\mathcal{S}}(\mathbf{v}^*) - \tilde{\mathcal{S}}(\mathbf{v})\|_{L^2([0,T] \times \mathbb{T}^2)}^2 - \frac{\delta^2}{8T} \\ &\geq \sigma_\eta^2 + \frac{3\delta^2}{8T}. \end{aligned}$$

Now, leveraging $\mu_0(\mathcal{X}_\delta \cap B_V^R(\mathbf{v}^*)) > 0$, we choose N_2 such that

$$\frac{1}{\mu_0(\mathcal{X}_\delta \cap B_V^R(\mathbf{v}^*))} \exp\left[-\frac{\delta^2}{16T\sigma_\eta^2} N_2\right] < \frac{\epsilon}{2}.$$

Then, for all $N \geq \max\{N_1, N_2\}$, we have (cf. (16))

$$\begin{aligned} \mu_Y^N(\mathcal{X}_\delta^c \cap B_V^R(\mathbf{v}^*)) &\leq \frac{\int_{\mathcal{X}_\delta^c \cap B_V^R(\mathbf{v}^*)} \exp[-\frac{1}{2\sigma_\eta^2} \sum_{j=1}^N (\mathcal{Y}_j - \mathcal{G}_j(\mathbf{v}))^2] \mu_0(d\mathbf{v})}{\int_{\mathcal{X}_\delta \cap B_V^R(\mathbf{v}^*)} \exp[-\frac{1}{2\sigma_\eta^2} \sum_{j=1}^N (\mathcal{Y}_j - \mathcal{G}_j(\mathbf{v}))^2] \mu_0(d\mathbf{v})} \\ &< \frac{\int_{\mathcal{X}_\delta^c \cap B_V^R(\mathbf{v}^*)} \exp[-\frac{1}{2\sigma_\eta^2} N(\sigma_\eta^2 + \frac{3\delta^2}{8T})] \mu_0(d\mathbf{v})}{\int_{\mathcal{X}_\delta \cap B_V^R(\mathbf{v}^*)} \exp[-\frac{1}{2\sigma_\eta^2} N(\sigma_\eta^2 + \frac{\delta^2}{4T})] \mu_0(d\mathbf{v})} \\ &= \exp\left[-\frac{N\delta^2}{16T\sigma_\eta^2}\right] \frac{\mu_0(\mathcal{X}_\delta^c \cap B_V^R(\mathbf{v}^*))}{\mu_0(\mathcal{X}_\delta \cap B_V^R(\mathbf{v}^*))} \\ &\leq \frac{\exp[-\frac{N\delta^2}{16T\sigma_\eta^2}]}{\mu_0(\mathcal{X}_\delta \cap B_V^R(\mathbf{v}^*))} < \frac{\epsilon}{2}. \end{aligned}$$

Then

$$\mu_Y^N(\mathcal{X}_\delta^c) \leq \mu_Y^N(\mathcal{X}_\delta^c \cap B_V^R(\mathbf{v}^*)) + \mu_Y^N((B_V^R(\mathbf{v}^*))^c) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, since ϵ and $\omega \in \tilde{\Omega}$ are arbitrary we conclude that for any $\delta > 0$, $\mu_Y^N(\mathcal{X}_\delta^c) \rightarrow 0$ as $N \rightarrow \infty$ a.s. The desired result (50) follows, completing the proof of Proposition 6.1. \square

7. Convergence of posterior measures to the true vector field. We now combine the continuity of the inverse map (Corollary 4.5) and the convergence of the posterior measure to $\theta(\mathbf{v}^*)$ (Proposition 6.1) to finally prove our main result Theorem 3.2, that is, to show that as the number of observations goes to infinity, the posterior converges weakly to a Dirac measure centered at \mathbf{v}^* .

PROOF OF THEOREM 3.2. Let $\epsilon > 0$. Let A be an open subset of H . To show weak convergence, according to Portmanteau’s Theorem (see, for example, Billingsley ((2013), Section 2)) we need to show

$$\liminf_{N \rightarrow \infty} \mu_{\mathcal{Y}}^N(A) \geq \delta_{\mathbf{v}^*}(A).$$

If $\mathbf{v}^* \notin A$, then $\delta_{\mathbf{v}^*}(A) = 0$ so the result is trivial in this case.

Now consider $\mathbf{v}^* \in A$ and fix any sample ω on the set $\tilde{\Omega}$ of full measure for which (50) in Proposition 6.1 holds. Fix any $\epsilon > 0$. As guaranteed by Lemma 6.3, we can choose $R > 0$ so that

$$\mu_{\mathcal{Y}}^N((B_V^R(\mathbf{v}^*))^c) < \frac{\epsilon}{2}.$$

Since A is open there exists an $\gamma > 0$ such that $B_H^\gamma(\mathbf{v}^*) \subset A$. Then, by Corollary 4.6, there exists a $\delta > 0$ such that $\mathcal{X}_\delta \cap B_V^R(\mathbf{v}^*) \subset B_H^\gamma(\mathbf{v}^*) \subset A$. As such

$$\begin{aligned} \mu_{\mathcal{Y}}^N(A) &\geq \mu_{\mathcal{Y}}^N(B_H^\gamma(\mathbf{v}^*)) \geq \mu_{\mathcal{Y}}^N(\mathcal{X}_\delta \cap B_V^R(\mathbf{v}^*)) \geq \mu_{\mathcal{Y}}^N(\mathcal{X}_\delta) - \mu_{\mathcal{Y}}^N((B_V^R(\mathbf{v}^*))^c) \\ &\geq \mu_{\mathcal{Y}}^N(\mathcal{X}_\delta) - \frac{\epsilon}{2}. \end{aligned}$$

However, Proposition 6.1 ensures that there exists an N^* such that for all $N > N^*$,

$$\mu_{\mathcal{Y}}^N(\mathcal{X}_\delta) > 1 - \frac{\epsilon}{2}.$$

Therefore for all $N > N^*$,

$$\mu_{\mathcal{Y}}^N(A) \geq \mu_{\mathcal{Y}}^N(\mathcal{X}_\delta) - \frac{\epsilon}{2} > 1 - \epsilon = \delta_{\mathbf{v}^*}(A) - \epsilon.$$

Since $\epsilon > 0$ and ω were arbitrary to begin with, $\liminf_{N \rightarrow \infty} \mu_{\mathcal{Y}}^N(A) \geq \delta_{\mathbf{v}^*}(A)$ with probability 1 completing the proof of Theorem 3.2. \square

APPENDIX: ENERGY ESTIMATES FOR CONTINUITY OF THE SOLUTION MAP

In this appendix we provide some of the a priori estimates leading to Proposition 2.2. As noted above, a suitable Galerkin approximation of (1) can be implemented to provide rigorous justification for the forthcoming formal manipulations.

Let us begin with the L^2 -based estimates. Since \mathbf{v} is divergence free, we have that $\frac{d}{dt} \|\theta\|^2 + 2\kappa \|\nabla\theta\|^2 = 0$ so that for any $T > 0$, $\theta \in L^2([0, T]; H^1(\mathbb{T}^2)) \cap L^\infty([0, T]; L^2(\mathbb{T}^2))$. Turning to the estimate for $\partial_t\theta$ we have

$$(53) \quad \|\partial_t\theta\|_{H^{-1}} \leq \kappa \|\theta\|_{H^1} + \|\mathbf{v} \cdot \nabla\theta\|_{H^{-1}}.$$

Regarding the second term on the right hand, using that \mathbf{v} is divergence free and Hölder’s inequality

$$(54) \quad \|\mathbf{v} \cdot \nabla\theta\|_{H^{-1}} = \sup_{\|\phi\|_{H^1}=1} \left| \int \mathbf{v} \cdot \nabla\theta\phi \, d\mathbf{x} \right| \leq C \|\mathbf{v}\|_{L^p} \|\theta\|_{L^q},$$

where $p^{-1} + q^{-1} = 2^{-1}$. Let us now recall the Sobolev embedding in spatial dimension $d = 2$ which entails the bound

$$(55) \quad \|f\|_{L^p} \leq C \|f\|_{H^r} \quad \text{for any } r \geq 1 - \frac{2}{p}, \text{ with } 2 \leq p < \infty$$

for any sufficiently smooth f and where the constant C depends only on the size of the periodic box, p , and r . Thus, with our assumption that $\mathbf{v} \in H^s$ for some $s > 0$ it now follows from (53), (54), and (55) that $\partial_t \theta \in L^2([0, T]; H^{-1})$.

Regarding the claimed continuity in L^2 we consider any $\theta^{(1)}, \theta^{(2)}$ solving (1) and corresponding to divergence free $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$. Taking $\psi = \theta^{(2)} - \theta^{(1)}$ and $\mathbf{u} = \mathbf{v}^{(2)} - \mathbf{v}^{(1)}$ we have

$$(56) \quad \partial_t \psi = \kappa \Delta \psi - \mathbf{u} \cdot \nabla \theta^{(2)} - \mathbf{v}^{(1)} \cdot \nabla \psi.$$

Multiplying (56) by ψ , integrating and using that both $\mathbf{v}^{(1)}, \mathbf{u}$ are divergence free, we obtain

$$(57) \quad \frac{1}{2} \frac{d}{dt} \|\psi\|^2 + \kappa \|\nabla \psi\|^2 = \int \mathbf{u} \cdot \nabla \psi \theta^{(2)} \, d\mathbf{x}.$$

Now, Hölder’s inequality yields

$$\left| \int \mathbf{u} \cdot \nabla \psi \theta^{(2)} \, d\mathbf{x} \right| \leq \|\mathbf{u}\|_{L^p} \|\nabla \psi\| \|\theta^{(2)}\|_{L^q},$$

which holds for any $2 \leq p, q \leq \infty$ such that $p^{-1} + q^{-1} = 2^{-1}$. Observe that, by choosing $2 < q < \infty$ sufficiently large, obtain a $p = \frac{2q}{q-2}$ such that, according to (55) $\|\mathbf{u}\|_{L^p} \leq C \|\mathbf{u}\|_{H^s}$ where $s > 0$ is the given degree of regularity for $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$. With this observation, another application of (55), this time for $\|\theta^{(2)}\|_{L^q}$, and Young’s inequality we have

$$\left| \int \mathbf{u} \cdot \nabla \psi \theta^{(2)} \, d\mathbf{x} \right| \leq C \|\mathbf{u}\|_{H^s} \|\psi\|_{H^1} \|\theta^{(2)}\|_{H^1} \leq \frac{\kappa}{2} \|\psi\|_{H^1}^2 + C \|\mathbf{u}\|_{H^s}^2 \|\theta^{(2)}\|_{H^1}^2.$$

Combining this bound with (57) we find, for any $T > 0$,

$$\sup_{t \in [0, T]} \|\theta^{(1)}(t) - \theta^{(2)}(t)\|^2 \leq \|\theta_0^{(1)} - \theta_0^{(2)}\|^2 + C \|\mathbf{u}\|_{H^s}^2 \int_0^T \|\theta^{(2)}(t')\|_{H^1}^2 \, dt'$$

from which the desired continuity in the L^2 case now follows.

Before proceeding to the higher order, $s > 0$, estimates, let us introduce further notations and recall some fundamental inequalities. For any $r \geq 0$, we take $\Lambda^r := (-\Delta)^{r/2}$ acting on elements in $H^r(\mathbb{T}^2)$. In other words

$$\Lambda^r f = \sum_{\mathbf{k} \in \mathbb{Z}^2} (2\pi)^r \|\mathbf{k}\|^r c_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \quad \text{for any } f = \sum_{\mathbf{k} \in \mathbb{Z}^2} c_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}$$

and we have $\|\Lambda^r f\| = \|f\|_{H^r}$. We have the following useful interpolation inequality

$$(58) \quad \|\Lambda^r f\| \leq \|\Lambda^{\gamma_l} f\|^{\frac{\gamma_u - r}{\gamma_u - \gamma_l}} \|\Lambda^{\gamma_u} f\|^{\frac{r - \gamma_l}{\gamma_u - \gamma_l}}$$

valid for any $0 \leq \gamma_l < r < \gamma_u$; see, for example, [Robinson \(2001\)](#). We will also make use of the fractional Leibniz inequality or Kato–Ponce inequality:

$$(59) \quad \|\Lambda^r (fg)\|_{L^m} \leq C (\|\Lambda^r f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|\Lambda^r g\|_{L^{q_2}}),$$

which is valid for any $r \geq 0$, $1 < m < \infty$ and $1 < p_i, q_i \leq \infty$ with $m^{-1} = p_j^{-1} + q_j^{-1}$ for $j = 1, 2$ and where the constant C is independent of any suitably smooth f, g . See [Grafakos and Oh \(2014\)](#), [Muscalu and Schlag \(2013\)](#) for further details.

With these preliminaries in hand, now suppose θ solves (1). Applying the operator Λ^s to (1), multiplying by $\Lambda^s \theta$ and integrating over \mathbb{T}^2 we obtain

$$(60) \quad \frac{1}{2} \frac{d}{dt} \|\Lambda^s \theta\|^2 + \kappa \|\Lambda^{s+1} \theta\|^2 = \int \Lambda^s (\mathbf{v} \cdot \nabla \theta) \Lambda^s \theta \, d\mathbf{x}.$$

With Hölder’s inequality and (59) we find

$$(61) \quad \left| \int \Lambda^s (\mathbf{v} \cdot \nabla \theta) \Lambda^s \theta \, d\mathbf{x} \right| \leq C \|\Lambda^s \theta\|_{L^p} (\|\Lambda^s \mathbf{v}\| \|\Lambda^1 \theta\|_{L^q} + \|\mathbf{v}\|_{L^q} \|\Lambda^{s+1} \theta\|),$$

which holds for any $1 < p, q < \infty$ such that $1 - \frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. Noting that $q = \frac{2p}{2p-2-p} \rightarrow 2$ as $p \rightarrow \infty$, using the Sobolev embedding (55) and then the interpolation inequality (58), we thus find, for some $0 < s' < s \wedge 1$,

$$(62) \quad \begin{aligned} \left| \int \Lambda^s (\mathbf{v} \cdot \nabla \theta) \Lambda^s \theta \, d\mathbf{x} \right| &\leq C \|\Lambda^{1+s'} \theta\| \|\Lambda^{1+s} \theta\| \|\Lambda^s \mathbf{v}\| \\ &\leq C \|\Lambda^{1+s} \theta\|^{2-(s-s')} \|\Lambda^s \theta\|^{s-s'} \|\Lambda^s \mathbf{v}\| \end{aligned}$$

$$(63) \quad \leq \kappa \|\Lambda^{1+s} \theta\|^2 + C \|\Lambda^s \theta\|^2 \|\Lambda^s \mathbf{v}\|^{\frac{2}{(s-s')}}.$$

Combining (60), (63) and rearranging we obtain

$$\frac{d}{dt} \|\Lambda^s \theta\|^2 + \kappa \|\Lambda^{s+1} \theta\|^2 \leq C \|\Lambda^s \theta\|^2 \|\Lambda^s \mathbf{v}\|^{\frac{2}{(s-s')}}.$$

This bound and Grönwall’s inequality reveals

$$\sup_{t \in [0, T]} \|\theta(t)\|_{H^s}^2 \leq \exp(TC \|\Lambda^s \mathbf{v}\|^{\frac{2}{(s-s')}}) \|\theta_0\|_{H^s}^2.$$

Using this bound and integrating in time yields

$$\int_0^T \|\Lambda^{s+1} \theta\|^2 \leq 2 \exp(T \|\Lambda^s \mathbf{v}\|^{\frac{2}{(s-s')}}) \|\theta_0\|_{H^s}^2,$$

which indeed shows that for any $T > 0$, $\theta \in L^2([0, T]; H^{s+1}(\mathbb{T}^2)) \cap L^\infty([0, T]; H^s(\mathbb{T}^2))$. We turn next to the estimates for $\partial_t \theta$. Here analogous to (53) we just need a suitable estimate for $\|\mathbf{v} \cdot \nabla \theta\|_{H^{s-1}}$. For any $s > 0$ this amounts to

$$\|\mathbf{v} \cdot \nabla \theta\|_{H^{s-1}} := \sup_{\|\phi\|_{H^{s+1}}=1} \left| \int \mathbf{v} \cdot \nabla \theta \phi \, d\mathbf{x} \right| \leq C \|\mathbf{v}\|_{L^p} \|\theta\|_{L^q} \sup_{\|\phi\|_{H^{s+1}}=1} \|\nabla \phi\|_{L^r}$$

for any $1 \leq p, q, r \leq \infty$ with $p^{-1} + q^{-1} + r^{-1} = 1$ and where again we have used that \mathbf{v} is divergence free. By picking $p = r > 2$ such that $L^p \subset H^s$ according to (55) we finally obtain

$$\|\mathbf{v} \cdot \nabla \theta\|_{H^{s-1}} \leq C \|\mathbf{v}\|_{H^s} \|\theta\|_{H^{s+1}}$$

and thus conclude that $\partial_t \theta \in L^2([0, T]; H^{s-1})$ for any $T > 0$.

We finally address the claimed continuity of the data to solution map in H^s . Adopting the same notations as in (56), we have

$$(64) \quad \frac{1}{2} \frac{d}{dt} \|\Lambda^s \psi\|^2 + \kappa \|\Lambda^{s+1} \psi\|^2 = - \int \Lambda^s (\mathbf{u} \cdot \nabla \theta^{(2)} - \mathbf{v}^{(1)} \cdot \nabla \psi) \Lambda^s \psi \, d\mathbf{x} := T_1 + T_2.$$

Regarding T_1 , we estimate as in (61), (63) and find,

$$(65) \quad \begin{aligned} |T_1| &\leq C \|\Lambda^s \psi\|_{L^p} (\|\Lambda^s \mathbf{u}\| \|\Lambda^1 \theta^{(2)}\|_{L^q} + \|\mathbf{u}\|_{L^q} \|\Lambda^{s+1} \theta^{(2)}\|) \\ &\leq C \|\Lambda^{1+s} \psi\| \|\Lambda^s \mathbf{u}\| \|\Lambda^{s+1} \theta^{(2)}\| \leq \kappa \|\Lambda^{1+s} \psi\|^2 + C \|\Lambda^s \mathbf{u}\|^2 \|\Lambda^{s+1} \theta^{(2)}\|^2. \end{aligned}$$

For T_2 we proceed in precisely the same fashion as (63) and find

$$(66) \quad |T_2| \leq \kappa \|\Lambda^{1+s} \psi\|^2 + C \|\Lambda^s \psi\|^2 \|\Lambda^s \mathbf{v}^{(1)}\|_{\frac{2}{(s-s')}}.$$

Combining the identity (64) with the estimates (65), (66) and rearranging appropriately we obtain

$$\frac{d}{dt} \|\Lambda^s \psi\|^2 \leq C \|\Lambda^s \psi\|^2 \|\Lambda^s \mathbf{v}^{(1)}\|_{\frac{2}{(s-s')}} + C \|\Lambda^s \mathbf{u}\|^2 \|\Lambda^{s+1} \theta^{(2)}\|^2,$$

and hence, with Grönwall's inequality,

$$\begin{aligned} \|\Lambda^s \psi(t)\|^2 &\leq \exp(C \|\Lambda^s \mathbf{v}^{(1)}\|_{\frac{2}{(s-s')}} t) \|\psi(0)\|^2 \\ &\quad + C \|\Lambda^s \mathbf{u}\|^2 \int_0^t \exp(C \|\Lambda^s \mathbf{v}^{(1)}\|_{\frac{2}{(s-s')}} (t-t')) \|\Lambda^{s+1} \theta^{(2)}(t')\|^2 dt'. \end{aligned}$$

Thus given the already established a priori bounds on $\theta^{(2)}$ in H^s and our standing assumption concerning the regularity of $\mathbf{v}^{(1)}$ we have

$$\sup_{t \in [0, T]} \|\theta^{(1)}(t) - \theta^{(2)}(t)\|_{H^s}^2 \leq C (\|\theta_0^{(1)} - \theta_0^{(2)}\|_{H^s}^2 + \|\mathbf{v}^{(1)} - \mathbf{v}^{(2)}\|_{H^s}^2),$$

from which the desired continuity in H^s now follows.

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REFERENCES

- ABRAHAM, K. (2019). Nonparametric Bayesian posterior contraction rates for scalar diffusions with high-frequency data. *Bernoulli* **25** 2696–2728. MR4003562 <https://doi.org/10.3150/18-BEJ1067>
- BILLINGSLEY, P. (2013). *Convergence of Probability Measures*. Wiley, New York. MR0233396
- BORGGGAARD, J., GLATT-HOLTZ, N. and KROMETIS, J. (2018). A Bayesian approach to estimating background flows from a passive scalar. Preprint. Available at [arXiv:1808.01084](https://arxiv.org/abs/1808.01084).
- DASHTI, M. and STUART, A. M. (2017). The Bayesian approach to inverse problems. In *Handbook of Uncertainty Quantification* 311–428. Springer, Cham. MR3839555
- DIACONIS, P. and FREEDMAN, D. (1986). On the consistency of Bayes estimates. *Ann. Statist.* **14** 1–67. MR0829555 <https://doi.org/10.1214/aos/1176349830>
- DOOB, J. L. (1949). Application of the theory of martingales. In *Le Calcul des Probabilités et Ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique* **13** 23–27. Centre National de la Recherche Scientifique, Paris. MR0033460
- EVANS, L. C. (2010). *Partial Differential Equations*, 2nd ed. *Graduate Studies in Mathematics* **19**. Amer. Math. Soc., Providence, RI. MR2597943 <https://doi.org/10.1090/gsm/019>
- FREEDMAN, D. A. (1963). On the asymptotic behavior of Bayes' estimates in the discrete case. *Ann. Math. Stat.* **34** 1386–1403. MR0158483 <https://doi.org/10.1214/aoms/1177703871>
- GHOSAL, S. and VAN DER VAART, A. (2017). *Fundamentals of Nonparametric Bayesian Inference. Cambridge Series in Statistical and Probabilistic Mathematics* **44**. Cambridge Univ. Press, Cambridge. MR3587782 <https://doi.org/10.1017/9781139029834>
- GRAFAKOS, L. and OH, S. (2014). The Kato–Ponce inequality. *Comm. Partial Differential Equations* **39** 1128–1157. MR3200091 <https://doi.org/10.1080/03605302.2013.822885>

- GUGUSHVILI, S. and SPREIJ, P. (2014). Nonparametric Bayesian drift estimation for multidimensional stochastic differential equations. *Lith. Math. J.* **54** 127–141. MR3212631 <https://doi.org/10.1007/s10986-014-9232-1>
- KAIPIO, J. and SOMERSALO, E. (2005). *Statistical and Computational Inverse Problems. Applied Mathematical Sciences* **160**. Springer, New York. MR2102218
- KARCH, G. K., SADLO, F., WEISKOPF, D., HANSEN, C. D., LI, G. S. and ERTL, T. (2012). Dye-based flow visualization. *Comput. Sci. Eng.* **14** 80–86. <https://doi.org/10.1109/MCSE.2012.118>
- KELLAY, H. and GOLDBURG, W. I. (2002). Two-dimensional turbulence: A review of some recent experiments. *Rep. Progr. Phys.* **65** 845–894.
- KOSKELA, J., SPANÒ, D. and JENKINS, P. A. (2019). Consistency of Bayesian nonparametric inference for discretely observed jump diffusions. *Bernoulli* **25** 2183–2205. MR3961245 <https://doi.org/10.3150/18-BEJ1050>
- KROMETIS, J. (2018). A Bayesian approach to estimating background flows from a passive scalar. Ph.D. thesis, Virginia Polytechnic Institute and State Univ.
- LAPLACE, P.-S. (1810). Mémoire sur les approximations des formules qui sont fonctions de très grands nombres et sur leur applications aux probabilités. *Mém. Acad. Sci. Paris*.
- LE CAM, L. (1953). On some asymptotic properties of maximum likelihood estimates and related Bayes' estimates. *Univ. Calif. Publ. Statist.* **1** 277–329. MR0054913
- LE CAM, L. and YANG, G. L. (2000). *Asymptotics in Statistics: Some Basic Concepts*, 2nd ed. *Springer Series in Statistics*. Springer, New York. MR1784901 <https://doi.org/10.1007/978-1-4612-1166-2>
- LIEBERMAN, G. M. (1996). *Second Order Parabolic Differential Equations*. World Scientific, River Edge, NJ. MR1465184 <https://doi.org/10.1142/3302>
- MUSCALU, C. and SCHLAG, W. (2013). *Classical and Multilinear Harmonic Analysis, Vol. II. Cambridge Studies in Advanced Mathematics* **138**. Cambridge Univ. Press, Cambridge. MR3052499
- NEWBY, W. K. and MCFADDEN, D. (1994). Large sample estimation and hypothesis testing. In *Handbook of Econometrics, Vol. IV. Handbooks in Econom.* **2** 2111–2245. North-Holland, Amsterdam. MR1315971
- NICKL, R. (2013). Statistical theory. Lecture notes.
- NICKL, R. (2017). Bernstein–von Mises theorems for statistical inverse problems I: Schrödinger equation. Preprint. Available at arXiv:1707.01764.
- NICKL, R. and SÖHL, J. (2017). Nonparametric Bayesian posterior contraction rates for discretely observed scalar diffusions. *Ann. Statist.* **45** 1664–1693. MR3670192 <https://doi.org/10.1214/16-AOS1504>
- ØKSENDAL, B. (2013). *Stochastic Differential Equations: An Introduction with Applications*, 6th ed. *Universitext*. Springer, Berlin. MR2001996 <https://doi.org/10.1007/978-3-642-14394-6>
- PAPASPILIOPOULOS, O., POKERN, Y., ROBERTS, G. O. and STUART, A. M. (2012). Nonparametric estimation of diffusions: A differential equations approach. *Biometrika* **99** 511–531. MR2966767 <https://doi.org/10.1093/biomet/ass034>
- ROBINSON, J. C. (2001). *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors. Cambridge Texts in Applied Mathematics*. Cambridge Univ. Press, Cambridge. MR1881888 <https://doi.org/10.1007/978-94-010-0732-0>
- RUDIN, W. (1964). *Principles of Mathematical Analysis*. 2nd ed. McGraw-Hill, New York. MR0166310
- SCHWARTZ, L. (1965). On Bayes procedures. *Z. Wahrsch. Verw. Gebiete* **4** 10–26. MR0184378 <https://doi.org/10.1007/BF00535479>
- SMITS, A. J. (2012). *Flow Visualization: Techniques and Examples*. World Scientific, Singapore.
- STUART, A. M. (2010). Inverse problems: A Bayesian perspective. *Acta Numer.* **19** 451–559. MR2652785 <https://doi.org/10.1017/S0962492910000061>
- TEMAM, R. (1995). *Navier–Stokes Equations and Nonlinear Functional Analysis*, 2nd ed. *CBMS–NSF Regional Conference Series in Applied Mathematics* **66**. SIAM, Philadelphia, PA. MR1318914 <https://doi.org/10.1137/1.9781611970050>
- VAN DER MEULEN, F. and VAN ZANTEN, H. (2013). Consistent nonparametric Bayesian inference for discretely observed scalar diffusions. *Bernoulli* **19** 44–63. MR3019485 <https://doi.org/10.3150/11-BEJ385>
- VOLLMER, S. J. (2013). Posterior consistency for Bayesian inverse problems through stability and regression results. *Inverse Probl.* **29** Art. ID 125011. MR3141858 <https://doi.org/10.1088/0266-5611/29/12/125011>
- WASSERMAN, L. (1998). Asymptotic properties of nonparametric Bayesian procedures. In *Practical Nonparametric and Semiparametric Bayesian Statistics. Lect. Notes Stat.* **133** 293–304. Springer, New York. MR1630088 https://doi.org/10.1007/978-1-4612-1732-9_16
- WOLFGANG, M. (1987). *Flow Visualization*. Academic Press, London.