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A BERRY–ESSEEN THEOREM FOR PITMAN'S α-DIVERSITY

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This paper contributes to the study of the random number K_n of blocks in the random partition of $\{1,\dots,n\}$ induced by random sampling from the celebrated two parameter Poisson–Dirichlet process. For any $\alpha \in (0,1)$ and $\theta > -\alpha$ Pitman (*Combinatorial Stochastic Processes* (2006) Springer, Berlin) showed that $n^{-\alpha}K_n \xrightarrow{\text{a.s.}} S_{\alpha,\theta}$ as $n \to +\infty$, where the limiting random variable, referred to as Pitman's α-diversity, is distributed according to a polynomially scaled Mittag–Leffler distribution function. Our main result is a Berry–Esseen theorem for Pitman's α-diversity $S_{\alpha,\theta}$, namely we show that

$$\sup_{x\geq 0}\left|\mathsf{P}\left[\frac{K_n}{n^\alpha}\leq x\right]-\mathsf{P}[S_{\alpha,\theta}\leq x]\right|\leq\frac{C(\alpha,\theta)}{n^\alpha}$$

holds for every $n\in\mathbb{N}$ with an explicit constant term $C(\alpha,\theta)$, for $\alpha\in(0,1)$ and $\theta>0$. The proof relies on three intermediate novel results which are of independent interest: (i) a probabilistic representation of the distribution of K_n in terms of a compound distribution; (ii) a quantitative version of the Laplace's approximation method for integrals; (iii) a refined quantitative bound for Poisson approximation. An application of our Berry–Esseen theorem is presented in the context of Bayesian nonparametric inference for species sampling problems, quantifying the error of a posterior approximation that has been extensively applied to infer the number of unseen species in a population.

1. Introduction. The two parameter Poisson–Dirichlet process is a discrete random probability measure introduced by Perman et al. [23]. For any $\alpha\in[0,1)$ and $\theta>-\alpha$, let $\{V_i\}_{i\geq 1}$ be independent random variables such that V_i is distributed as a Beta distribution with parameter $(1-\alpha,\theta+i\alpha)$, for $i\geq 1$. Set $P_1:=V_1$ and $P_i:=V_i\prod_{1\leq j\leq i-1}(1-V_j)$ for $i\geq 2$, then $\sum_{i\geq 1}P_i=1$ almost surely. The two parameter Poisson–Dirichlet process is defined as the random probability measure $\tilde{\mathfrak{p}}_{\alpha,\theta}$ on $(\mathbb{N},2^{\mathbb{N}})$ such that $\tilde{\mathfrak{p}}_{\alpha,\theta}(\{i\})=P_i$ for all $i\in\mathbb{N}$. A random sample $(X_1,\dots,X_n)$ from $\tilde{\mathfrak{p}}_{\alpha,\theta}$ is the first n-segment of the $\mathbb{N}$-valued exchangeable sequence $\{X_i\}_{i\geq 1}$ having $\tilde{\mathfrak{p}}_{\alpha,\theta}$ as directing measure. Due to the discreteness of $\tilde{\mathfrak{p}}_{\alpha,\theta}$, $(X_1,\dots,X_n)$ induces a random partition Π_n of $\{1,\dots,n\}$ by means of the equivalence relation $i\sim j \iff X_i=X_j$ (Aldous [2] and Pitman [25]). Let $K_n:=K_n(X_1,\dots,X_n)$, where $K_n(X_1,\dots,X_n)$ is the (random) number of blocks of Π_n, that is, the random number of distinct elements in $(X_1,\dots,X_n)$. Let $\mathbf{N}_n:=(N_{1,n},\dots,N_{K_n,n})$, with $N_{j,n}:=N_{j,n}(X_1,\dots,X_n)$, be the sizes of the blocks of Π_n. Pitman [24] showed that

$$\mathsf{P}\big[K_n=j,\mathbf{N}_n=(n_1,\dots,n_j)\big]=\frac{1}{j!}\binom{n}{n_1,\dots,n_j}\frac{[\theta]_{(j,\alpha)}}{[\theta]_{(n)}}\prod_{i=1}^{j}[1-\alpha]_{(n_i-1)},\tag{1}$$

for any $j\in\{1,\dots,n\}$ and $(n_1,\dots,n_j)\in\mathbb{N}^j$ such that $\sum_{1\leq i\leq j}n_i=n$, where $[x]_{(n,a)}$ denotes the rising factorial of x of order n and increment a, that is, $[x]_{(n,a)}:=\prod_{0\leq i\leq n-1}(x+ia)$,

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and $[x]_{(n)} := [x]_{(n,1)}$. Equation (1) is referred to as Ewens–Pitman sampling formula, and for $\alpha = 0$ it reduces to the celebrated sampling formula of Ewens [7]. The two parameter Poisson–Dirichlet process plays a fundamental role in a variety of research areas, such as mathematical population genetics, Bayesian nonparametric statistics, statistical machine learning, excursion theory, combinatorics and statistical physics. We refer to Pitman [25] and Crane [4] for a comprehensive treatment of this subject.

There have been several studies on the large n behavior of K_n. For $\alpha = 0$ and $\theta > 0$, $K_n = \sum_{1 \le i \le n} Z_i$ where the Z_i's are independent Bernoulli random variables with parameter $\theta/(\theta + i - 1)$, for $i = 1, \dots, n$. Korwar and Hollander [15] showed that $K_n/\log(n)$ converges almost surely to θ as $n \to +\infty$. Also, it follows from Lindberg's theorem that $(K_n - \theta \log(n))/\sqrt{\theta \log(n)}$ converges weakly to a standard Gaussian random variable as $n \to +\infty$. For $\alpha \in (0, 1)$ the large n Gaussian limit for K_n no longer holds. In particular, Theorem 3.8 in Pitman [25] exploited a martingale convergence argument to show that

$$\frac{K_n}{n^\alpha} \xrightarrow{\text{a.s.}} S_{\alpha,\theta} \tag{2}$$

as $n \to +\infty$, where $0 < S_{\alpha,\theta} < +\infty$ is a random variable distributed as a scaled Mittag–Leffler distribution. Precisely, let Γ stand for the Gamma function, and let

$$f_\alpha(z) = \frac{1}{\pi} \sum_{j \ge 1} \frac{(-1)^{j+1}}{j!} \sin(\pi \alpha j) \frac{\Gamma(\alpha j + 1)}{z^{\alpha j + 1}} \mathbb{1}\{z > 0\}, \tag{3}$$

for $\alpha \in (0, 1)$, be the positive α-stable density function. Then $S_{\alpha,\theta}$ has density function

$$f_{S_{\alpha,\theta}}(s) = \frac{\Gamma(\theta + 1)}{\alpha \Gamma(\theta/\alpha + 1)} s^{\frac{\theta - 1}{\alpha} - 1} f_\alpha\big(s^{-1/\alpha}\big) \mathbb{1}\{s > 0\}. \tag{4}$$

One may easily generate random variates from $S_{\alpha,\theta}$ (e.g., Devroye [5]). For $\theta = 0$, equation (4) reduces to the Mittag–Leffler density function. For $\alpha \in (0, 1)$ and $\theta > -\alpha$, the random variable $S_{\alpha,\theta}$ is referred to as Pitman's α-diversity. Large and moderate deviations for K_n are established in Feng and Hoppe [12] and Favaro et al. [9], whereas a concentration inequality for K_n is obtained in Pereira et al. [22] by relying on concentration inequalities for martingales.

In this paper, we formulate a Berry–Esseen theorem for Pitman's α-diversity $S_{\alpha,\theta}$. In particular, let F_n and $\mathsf{F}_{\alpha,\theta}$ stand for the distribution functions of K_n/n^α and $S_{\alpha,\theta}$, respectively, that is, $\mathsf{F}_n(x) := \mathsf{P}[K_n/n^\alpha \le x]$ and $\mathsf{F}_{\alpha,\theta}(x) := \mathsf{P}[S_{\alpha,\theta} \le x]$, for any $x > 0$. To measure the discrepancy between F_n and $\mathsf{F}_{\alpha,\theta}$, we consider the Kolmogorov distance which, for any pair of distribution functions F_1 and F_2 supported in $[0, +\infty)$, is defined as $\mathrm{d}_K(\mathsf{F}_1; \mathsf{F}_2) := \sup_{x \ge 0} |\mathsf{F}_1(x) - \mathsf{F}_2(x)|$. The next theorem states our Berry–Esseen bound.

THEOREM 1. *For any $\alpha \in (0, 1)$ and $\theta > 0$, there exists a positive constant $C_{\alpha,\theta}$, depending solely on α and θ, such that $\mathrm{d}_K(\mathsf{F}_n; \mathsf{F}_{\alpha,\theta}) \le n^{-\alpha} C_{\alpha,\theta}$ for every $n \in \mathbb{N}$.*

Theorem 1 is the first quantitative version of Theorem 3.8 in Pitman [25]. The constant $C_{\alpha,\theta}$ is obtained constructively, and its value can be made explicit by gathering equations displayed in Section 2. The proof of Theorem 1 relies on three intermediate novel results which are of independent interest: (i) a probabilistic representation of the distribution of K_n as a compound distribution where the latent term is the distribution of the random number of blocks in Poisson compound random partition model, and the mixing term is the law of a scale mixture between Pitman's α-diversity and a Gamma random variable; (ii) a quantitative version of the asymptotic expansion, in the sense of Poincaré, of a Laplace-type integral;

(iii) a refined quantitative bound for Poisson approximation which improves results obtained in Hwang [13].

We present an application of Theorem 1 in Bayesian nonparametric inference for species sampling. Consider a population of individuals belonging to an infinite number of species with unknown proportions. Given an initial (observable) random sample of size n from the population, a classical statistical problem is to infer the number $K_m^{(n)}$ of hitherto unseen species that would be observed in m additional (unobservable) samples (Orlitsky et al. [21]). A Bayesian approach to this problem was proposed in Lijoi et al. [18], and it relies on the law of $\tilde{\mathfrak{p}}_{\alpha,\theta}$ as a prior distribution for the unknown species composition of the population. That is, the Ewens–Pitman sampling formula (1) models the species composition of the initial (observable) random sample $(X_1, \ldots, X_n)$ from the population, that is, the number K_n of species and their frequencies $\mathbf{N}_n$. Lijoi et al. [18] first derived the posterior distribution, given $\{K_n, \mathbf{N}_n\}$, of $K_m^{(n)}$. Then, Favaro et al. [10] showed that there exists a random variable $S_{\alpha,\theta}(n, j)$, referred to as Pitman's posterior α-diversity, such that

$$\mathsf{P}\left[\frac{K_m^{(n)}}{m^\alpha} \to S_{\alpha,\theta}(n, j) \text{ as } m \to +\infty \middle| K_n = j, \mathbf{N}_n = (n_1, \ldots, n_j)\right] = 1 \tag{5}$$

holds for any $j \in \{1, \ldots, n\}$ and $(n_1, \ldots, n_j) \in \mathbb{N}^j$ with $\sum_{1 \le i \le j} n_i = n$. Also, $\mathsf{P}[S_{\alpha,\theta}(n, j) \in \cdot | K_n = j, \mathbf{N}_n = (n_1, \ldots, n_j)] = \mathsf{P}[B_{j+\theta/\alpha, n/\alpha - j} S_{\alpha,\theta+n} \in \cdot]$ where, under P: (i) $S_{\alpha,\theta+n}$ has a density given by (4); (ii) $B_{j+\theta/\alpha,n/\alpha-j}$ is a Beta random variable with parameter $(j + \theta/\alpha, n/\alpha - j)$; (iii) $S_{\alpha,\theta+n}$ and $B_{j+\theta/\alpha,n/\alpha-j}$ are independent. The importance of (5) is motivated by the fact that the computational burden for evaluating the posterior distribution of $K_m^{(n)}$ becomes overwhelming for large m. This is common in genomics, where DNA libraries consists of millions of genes. In such a context, (5) has been extensively applied to obtain large m approximated posterior inferences for $K_m^{(n)}$ (Favaro et al. [10]). We show that Theorem 1 leads to formulate a Berry–Esseen theorem for Pitman's posterior α-diversity, thus quantifying the error in approximating the posterior distribution of $K_m^{(n)}$ with the law of $S_{\alpha,\theta}(n, j)$.

The paper is structured as follows. In Section 2 we prove Theorem 1, whereas in Section 3 we state and prove a Berry–Esseen theorem for Pitman's posterior α-diversity.

2. Proof of Theorem 1. The proof is divided into four main parts, developed in the Sections 2.1–2.4. Along these subsections, we make use of the notion of probability generating function (PGF) of a random variable X with values in $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, that is, $\mathsf{G}_X(s) := \sum_{x \ge 0} \mathsf{P}[X = x] s^x$.

2.1. *A new probabilistic representation for K_n.* Consider a population of individuals containing a random number N_λ of types, where N_λ has a Poisson distribution with parameter $\lambda = z[1 - (1 - q)^\alpha]$, for $\alpha \in (0, 1)$, $q \in (0, 1)$ and $z > 0$. For any $j \in \mathbb{N}$, let $Q_j(\alpha, q)$ be the random number of individuals of type j in the population. Assume the $Q_j(\alpha, q)$'s are independent of N_λ and independent of each other, with the same probability law

$$\begin{aligned} \mathsf{P}\big[Q_1(\alpha, q) = x\big] &= -\frac{1}{[1 - (1 - q)^\alpha]} \binom{\alpha}{x} (-q)^x \\ &= \frac{\Gamma(\alpha + 1) \sin \pi\alpha}{\pi [1 - (1 - q)^\alpha]} \frac{\Gamma(x - \alpha)}{x!} q^x \end{aligned} \tag{6}$$

for any $x \in \mathbb{N}$. In the next lemma we derive the conditional distribution of N_λ given the size of the random sample $S(\alpha, q, z) := \sum_{j=0}^{N_\lambda} Q_j(\alpha, q)$, with the proviso that $\mathsf{P}[Q_0(\alpha, q) = 0] = 1$.

This distribution relies on a noteworthy probability distribution $\rho(\cdot;\alpha,n,z)$ on $\{1,\dots,n\}$ involving the generalized factorial coefficients, namely $\mathscr{C}(n,k;\alpha) := \frac{1}{k!}\sum_{i=1}^{k}(-1)^i\binom{k}{i}[-i\alpha]_{(n)}$ (Charalambides [3]).

LEMMA 1. *Let $\{Q_j(\alpha,q)\}_{j\geq 1}$ be i.i.d. random variables distributed according to (6), and define $S(\alpha,q,z) := \sum_{j=0}^{N_\lambda} Q_j(\alpha,q)$, with the proviso that $\mathsf{P}[Q_0(\alpha,q)=0]=1$. Then, for every $n\in\mathbb{N}$ and $k\in\{1,\dots,n\}$, it holds*

$$\rho(\{k\};\alpha,n,z) := \mathsf{P}\big[N_\lambda = k | S(\alpha,q,z)=n\big] = \frac{\mathscr{C}(n,k;\alpha)z^k}{\sum_{j=1}^{n}\mathscr{C}(n,j;\alpha)z^j}. \tag{7}$$

PROOF. For fixed $n\in\mathbb{N}$ and $k\in\{1,\dots,n\}$, one has $\mathsf{P}[S(\alpha,q,z)=n|N_\lambda=k] = \mathsf{P}[\sum_{j=1}^{k}Q_j(\alpha,q)=n]$. By virtue of the binomial series, the PGF $\mathsf{G}(\cdot;k,\alpha,q)$ of $\sum_{j=1}^{k}Q_j(\alpha,q)$ reads

$$\mathsf{G}(s;k,\alpha,q) = \left\{-\sum_{x=1}^{\infty}\frac{1}{[1-(1-q)^\alpha]}\binom{\alpha}{x}(-sq)^x\right\}^k = \left[\frac{1-(1-sq)^\alpha}{1-(1-q)^\alpha}\right]^k$$

for $|s|<1$. Since $[1-(1-u)^\alpha]^k = k!\sum_{n\geq k}\mathscr{C}(n,k;\alpha)\frac{u^n}{n!}$ holds whenever $|u|<1$ (see Theorem 8.14 in Charalambides [3]), conclude that

$$\mathsf{P}\big[S(\alpha,q,z)=n|N_\lambda=k\big] = \frac{k!}{[1-(1-q)^\alpha]^k}\mathscr{C}(n,k;\alpha)\frac{q^n}{n!} \tag{8}$$

for $n\geq k$. Moreover, (8) holds also for $k>n$ since $\mathscr{C}(n,k;\alpha)=0$. Hence, using the explicit expression of $\mathsf{P}[N_\lambda=k]$, (7) ensues from (8) by means of the Bayes formula. □

Let $G_{\tau,\lambda}$ be a Gamma random variable with scale parameter $\lambda>0$ and shape parameter $\tau>0$, and let $\{R(\alpha,n,z)\}_{z>0}$ be a family of random variables such that $\mathsf{P}[R(\alpha,n,z)=k] = \rho(\{k\};\alpha,n,z)$, for any $n\in\mathbb{N}$, $k\in\{1,\dots,n\}$, $\alpha\in(0,1)$ and $z>0$.

PROPOSITION 1. *For fixed $n\in\mathbb{N}$, $\alpha\in(0,1)$ and $\theta>-\alpha$, there holds the following (distributional) identity*:

$$K_n \overset{d}{=} R\big(\alpha,n,S_{\alpha,\theta}G^{\alpha}_{\theta+n,1}\big), \tag{9}$$

where $S_{\alpha,\theta}$, $G_{\theta+n,1}$ and $\{R(\alpha,n,z)\}_{z>0}$ are independent random elements.

PROOF. Start from the well-known identity (e.g., Pitman [25])

$$\mathsf{P}[K_n=k] = \frac{[\theta]_{(k,\alpha)}}{[\theta]_{(n)}}\frac{\mathscr{C}(n,k;\alpha)}{\alpha^k} = \frac{\Gamma(\theta/\alpha+k)}{\Gamma(\theta/\alpha)}\frac{\Gamma(\theta)}{\Gamma(\theta+n)}\mathscr{C}(n,k;\alpha) \tag{10}$$

for any $k\in\{1,\dots,n\}$. Due to the identity $\int_0^{+\infty}x^{-\theta}f_\alpha(x)\,\mathrm{d}x = \frac{\Gamma(\theta/\alpha)}{\alpha\Gamma(\theta)}$, it is easily checked that

$$f(z;\alpha,\theta,n) = \frac{z^{\theta/\alpha+n/\alpha-1}}{\Gamma(\theta/\alpha)[\theta]_{(n)}}\left(\int_0^{+\infty}x^n e^{-xz^{1/\alpha}}f_\alpha(x)\,\mathrm{d}x\right)\mathbb{1}\{z>0\} \tag{11}$$

is a probability density function. Thus, one has the following identities:

$$\begin{aligned}&\mathsf{P}[K_n=k]\\ &\quad= \frac{\mathscr{C}(n,k;\alpha)}{\Gamma(\theta/\alpha)[\theta]_{(n)}}\int_0^{+\infty}z^{k+\theta/\alpha-1}e^{-z}\,\mathrm{d}z\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\theta/\alpha)[\theta]_{(n)}} \int_0^{+\infty} z^{\theta/\alpha-1} e^{-z} \left(\sum_{j=1}^{n} \mathscr{C}(n, j; \alpha) z^j \right) \\
&\quad \times \frac{\mathscr{C}(n, k; \alpha) z^k}{\sum_{j=1}^{n} \mathscr{C}(n, j; \alpha) z^j} \, \mathrm{d}z \\
&= \frac{1}{\Gamma(\theta/\alpha)[\theta]_{(n)}} \int_0^{+\infty} z^{\theta/\alpha+n/\alpha-1} \left(\int_0^{+\infty} x^n e^{-xz^{1/\alpha}} f_\alpha(x) \, \mathrm{d}x \right) \\
&\quad \times \frac{\mathscr{C}(n, k; \alpha) z^k}{\sum_{j=1}^{n} \mathscr{C}(n, j; \alpha) z^j} \, \mathrm{d}z \\
&= \int_0^{+\infty} \mathsf{P}\big[R(\alpha, n, z) = k\big] f(z; \alpha, \theta, n) \, \mathrm{d}z,
\end{aligned}
$$

where: (i) the first identity follows from (10) upon noticing that $\alpha^{-k}[\theta]_{(k,\alpha)} = \Gamma(k + \theta/\alpha)/\Gamma(\theta/\alpha)$; (ii) the third exploits the following identity:

$$
\sum_{j=1}^{n} \mathscr{C}(n, j; \alpha) z^j = e^z z^{n/\alpha} \int_0^{+\infty} x^n e^{-xz^{1/\alpha}} f_\alpha(x) \, \mathrm{d}x \tag{12}
$$

displayed in equation 13 of Favaro et al. [11]; the fourth follows from a combination of (7) with (11). To conclude, it is enough to show that the probability distribution of $S_{\alpha,\theta} G^{\alpha}_{\theta+n,1}$ possesses a density coinciding with $f(\cdot; \alpha, \theta, n)$. In fact, one has

$$
\begin{aligned}
&\mathsf{P}\big[S_{\alpha,\theta} G^{\alpha}_{\theta+n,1} \le u\big] \\
&= \int_0^{+\infty} \mathsf{P}\left[G_{\theta+n,1} \le \left(\frac{u}{s}\right)^{1/\alpha} \right] \frac{\Gamma(\theta+1)}{\alpha\Gamma(\theta/\alpha+1)} s^{\frac{\theta-1}{\alpha}-1} f_\alpha\big(s^{-1/\alpha}\big) \, \mathrm{d}s \\
&= \int_0^{+\infty} \left(\int_0^{u^{1/\alpha} x} \frac{t^{\theta+n-1} e^{-t}}{\Gamma(\theta+n)} \, \mathrm{d}t \right) \frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)} x^{-\theta} f_\alpha(x) \, \mathrm{d}x \\
&= \int_0^{+\infty} \left(\int_0^{u} \frac{1}{\Gamma(\theta+n)} e^{-xz^{1/\alpha}} z^{\frac{\theta+n}{\alpha}-1} \, \mathrm{d}z \right) \frac{\Gamma(\theta+1)}{\alpha\Gamma(\theta/\alpha+1)} x^n f_\alpha(x) \, \mathrm{d}x,
\end{aligned}
$$

where: (i) the first identity follows from conditioning; (ii) the second and the third ensue from the changes of variable $x = s^{-1/\alpha}$ and $t = xz^{1/\alpha}$, respectively. □

Observe that the representation (9) highlights the central role of the probability distribution $\rho(\cdot; \alpha, n, z)$. The next result describes the asymptotic behavior of such distribution for large values of n, to be used later on.

LEMMA 2. *For fixed $\alpha \in (0, 1)$ and $z > 0$, $\lim_{n\to+\infty} \rho(\{k\}; \alpha, n, z) = e^{-z} \frac{z^{k-1}}{(k-1)!}$ for any $k \in \mathbb{N}$. This is tantamount to saying that $R(\alpha, n, z) \longrightarrow 1 + N_z$ in distribution, as $n \to +\infty$.*

PROOF. Letting $\mathsf{G}(\cdot; \alpha, n, z)$ be the PGF of the random variable $R(\alpha, n, z)$, we show that $\mathsf{G}(s; \alpha, n, z) \to s \exp\{z(s-1)\}$ as $n \to +\infty$, for any $s > 0$. By using the definition of $\mathscr{C}(n, k; \alpha)$, for every $u \in \mathbb{C}$ we write

$$
\begin{aligned}
\sum_{k=1}^{n} \mathscr{C}(n, k; \alpha) u^k &= \sum_{i=1}^{n} (-1)^i [-i\alpha]_{(n)} \sum_{k=i}^{n} \frac{1}{k!} \binom{k}{i} u^k \\
&= \sum_{i=1}^{n} (-1)^i [-i\alpha]_{(n)} e^u u^i \frac{\Gamma(n-i+1, u)}{i!\Gamma(n-i+1)},
\end{aligned}
$$

where $\Gamma(a,x) := \int_x^{+\infty} t^{a-1}e^{-t}\,\mathrm{d}t$ denotes the incomplete gamma function for $a,x>0$. Hence, $\mathsf{G}(\cdot;\alpha,n,z)$ has the following expression:

$$\mathsf{G}(s;\alpha,n,z) = e^{z(s-1)}\frac{-zs\frac{\Gamma(n,zs)}{\Gamma(n)} + \sum_{i=2}^{n}(-1)^i\frac{[-i\alpha]_{(n)}}{[-\alpha]_{(n)}}(zs)^i\frac{\Gamma(n-i+1,zs)}{i!\Gamma(n-i+1)}}{-z\frac{\Gamma(n,z)}{\Gamma(n)} + \sum_{i=2}^{n}(-1)^i\frac{[-i\alpha]_{(n)}}{[-\alpha]_{(n)}}z^i\frac{\Gamma(n-i+1,z)}{i!\Gamma(n-i+1)}}.$$

Since $\lim_{n\to+\infty}\frac{\Gamma(n,x)}{\Gamma(n)} = 1$ for any $x>0$, the proof ends by showing that

$$\lim_{n\to+\infty}\sum_{i=2}^{n}(-1)^i\frac{[-i\alpha]_{(n)}}{[-\alpha]_{(n)}}\frac{\Gamma(n-i+1,t)}{\Gamma(n-i+1)}\frac{t^i}{i!} = 0$$

for any $t>0$. Upon noticing that $\frac{\Gamma(n,x)}{\Gamma(n)}\le 1$, we have the following relations:

$$\begin{aligned}\frac{1}{i!}\left|\frac{[-i\alpha]_{(n)}}{[-\alpha]_{(n)}}\right| &= \frac{1}{i!}\left|\frac{\Gamma(n-i\alpha)}{\Gamma(-i\alpha)}\frac{\Gamma(-\alpha)}{\Gamma(n-\alpha)}\right| \\ &= \frac{\Gamma(n-i\alpha)}{i!\Gamma(n-\alpha)}\frac{|\sin i\pi\alpha|}{\pi}\Gamma(i\alpha+1)\left|\Gamma(-\alpha)\right| \\ &\le \left|\Gamma(-\alpha)\right|\frac{\Gamma(n-i\alpha)}{\Gamma(n-\alpha)}\frac{\Gamma(i\alpha+1)}{i!} \le \left|\Gamma(-\alpha)\right|\frac{\Gamma(n-i\alpha)}{\Gamma(n-\alpha)}\end{aligned}$$

for all $n\in\mathbb{N}$, $x>0$ and $i\in\{1,\dots,n\}$. Then, we can write

$$\begin{aligned}&\left|\sum_{i=2}^{n}(-1)^i\frac{[-i\alpha]_{(n)}}{[-\alpha]_{(n)}}\frac{\Gamma(n-i+1,t)}{\Gamma(n-i+1)}\frac{t^i}{i!}\right| \\ &\le \sum_{i=2}^{n}\frac{t^i}{i!}\left|\frac{[-i\alpha]_{(n)}}{[-\alpha]_{(n)}}\right| \\ &\le \left|\Gamma(-\alpha)\right|\sum_{i=2}^{n}t^i\frac{\Gamma(n-i\alpha)}{\Gamma(n-\alpha)} \le \left|\Gamma(-\alpha)\right|\frac{\max_{i=2,\dots,n}n^{i\alpha}\Gamma(n-i\alpha)}{\Gamma(n-\alpha)}\sum_{j=2}^{n}\left(\frac{t}{n^\alpha}\right)^j.\end{aligned}$$

The monotonic increasing character of the function $(0,n)\ni x\mapsto n^x\Gamma(n-x)$, due to $\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \le \log(z)$ for any $z>0$, entails that $\max_{i=2,\dots,n}n^{i\alpha}\Gamma(n-i\alpha) = n^{n\alpha}\Gamma(n-n\alpha)$. Thus, observe that $\frac{n^{n\alpha}\Gamma(n-n\alpha)}{\Gamma(n-\alpha)}\sim n^\alpha$ to conclude that

$$\frac{\max_{i=2,\dots,n}n^{i\alpha}\Gamma(n-i\alpha)}{\Gamma(n-\alpha)}\sum_{j=2}^{n}\left(\frac{t}{n^\alpha}\right)^j \sim \left(\frac{t}{n^\alpha}\right)^2\frac{(\frac{t}{n^\alpha})^{n-1}-1}{(\frac{t}{n^\alpha})-1}n^\alpha \sim \frac{1}{n^\alpha}$$

as $n\to+\infty$, completing the proof. □

REMARK 1. Let Π_n denote the random partition of the set $\{1,\dots,n\}$ induced by a random sample $(X_1,\dots,X_n)$ from $\tilde{\mathfrak{p}}_{\alpha,\theta}$, and let $M_{l,n}$ be the number of blocks with frequency l, that is, $M_{l,n} = \sum_{1\le j\le K_n}\mathbb{1}\{N_{j,n}=l\}$. It is not difficult to show that results analogous to Proposition 1 and Lemma 2 hold true for $M_{l,n}$. In particular, let $R_l(\alpha,n,z)$ be a random variable whose distribution coincides with the conditional distribution of the number of $Q_j(\alpha,q)$'s equal to l, given the size of the random sample $S(\alpha,q,z)=n$. It can be shown that: (i) for $l\ge 1$, $n\in\mathbb{N}$, $\alpha\in(0,1)$ and $\theta>-\alpha$ the distribution of $M_{l,n}$ coincides with the distribution of the random variable $R_l(\alpha,n,S_{\alpha,\theta}G^\alpha_{\theta+n,1})$; (ii) for $l\ge 1$, $\alpha\in(0,1)$ and $z>0$ the random variable $R_l(\alpha,n,z)$ converges weakly, as $n\to+\infty$, to a Poisson random variable with parameter $z\alpha[1-\alpha]_{(l-1)}/l!$.

2.2. *A quantitative Laplace method for* $\mathcal{I}_n(z)$. Here, we study the Laplace integral $\mathcal{I}_n(z) := \int_0^{+\infty} e^{-n\phi_z(y)} f_\alpha(y)\,\mathrm{d}y$ for $z > 0$, where $\phi_z(y) := zy - \log y$. This quantity is connected with (12) in view of the identity

$$d_n(x) := \sum_{j=1}^{n} \mathscr{C}(n, j; \alpha)\big(xn^\alpha\big)^j = e^{xn^\alpha} x^{n/\alpha} n^n \mathcal{I}_n\big(x^{1/\alpha}\big) \tag{13}$$

valid for all $x > 0$. As first step, after noticing that $\overline{y}(z) := 1/z$ is the only minimum point of $\phi_z(y)$, a direct application of the Laplace method (Section 7 in Chapter 3 of Olver [20]) shows that $\mathcal{I}_n(z) \sim (\frac{1}{z})^{n+1} f_\alpha(\frac{1}{z}) e^{-n}\sqrt{\frac{2\pi}{n}}$ as $n \to +\infty$. However, a more precise large n estimate is provided by the next lemma.

LEMMA 3. *For any* $n \in \mathbb{N}$, *there exists a continuous function* $\delta_n : (0, +\infty) \to (0, +\infty)$ *such that*

$$\mathcal{I}_n(z) = \left(\frac{1}{z}\right)^{n+1} f_\alpha\left(\frac{1}{z}\right) e^{-n}\sqrt{\frac{2\pi}{n}}\big[1 + \delta_n(z)\big] \tag{14}$$

and $|\delta_n(z)| \le \Delta(z)/n$ *for any* $z > 0$, *where* $\Delta : (0, +\infty) \to (0, +\infty)$ *is a suitable continuous function which is independent of* n. *Moreover,* Δ *can be chosen in such a way that* $\Delta(z) = O(1)$ *as* $z \to 0$, *and* $\Delta(z) f_\alpha(1/z) = O(z^{-\infty})$ *as* $z \to +\infty$.

Recall from Kanter [14] that the positive α-stable density function (3) can be written as

$$f_\alpha(z) = \frac{1}{\pi}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{z}\right)^{\frac{1}{1-\alpha}} \int_0^\pi A(\varphi) \exp\left\{-\left(\frac{1}{z}\right)^{\frac{\alpha}{1-\alpha}} A(\varphi)\right\} \mathrm{d}\varphi \tag{15}$$

holds for any $\alpha \in (0, 1)$ and $z > 0$, with $A(\varphi) := (\frac{\sin(\alpha\varphi)}{\sin(\varphi)})^{\frac{1}{1-\alpha}} (\frac{\sin((1-\alpha)\varphi)}{\sin(\alpha\varphi)})$. In particular, (15) entails

$$f_\alpha(z) \sim \frac{(\alpha/z)^{\frac{2-\alpha}{2(1-\alpha)}}}{\sqrt{2\pi\alpha(1-\alpha)}} \exp\left\{-(1-\alpha)\left(\frac{\alpha}{z}\right)^{\frac{\alpha}{1-\alpha}}\right\} \tag{16}$$

as $z \to 0$, the asymptotic relation remaining valid after differentiating both sides with respect to z as many times as needed. See, for example, Zolotarev [31] for details.

PROOF OF LEMMA 3. The change of variables $s = zy - 1$ gives $\mathcal{I}_n(z) = (\frac{1}{z})^{n+1} e^{-n} \times \int_{-1}^{+\infty} e^{-nh(s)} f_\alpha(\frac{s+1}{z})\,\mathrm{d}s$, with $h(s) := s - \log(s+1)$. In order to exploit the analyticity of the function h for $s \in (-1, 1)$, fix $\sigma \in (0, 1)$ and split the above integral into the regions $s \in (\sigma, +\infty)$, $s \in (0, \sigma)$, $s \in (-\sigma, 0)$ and $s \in (-1, -\sigma)$. The ensuing analysis will provide the desired bound on Δ by expressing it as a sum of three other functions called Δ_1, Δ_2 and Δ_3. First, write $h(s) \ge h'(\sigma)(s - \sigma) + h(\sigma)$ for every $s \in (\sigma, +\infty)$ by the convexity of h, yielding

$$\begin{aligned}
&\int_\sigma^{+\infty} e^{-nh(s)} f_\alpha\left(\frac{s+1}{z}\right)\mathrm{d}s \\
&\quad\le (\sigma+1)^n \exp\left\{-\frac{n\sigma}{\sigma+1}\right\} \int_\sigma^{+\infty} \exp\left\{-\frac{ns\sigma}{\sigma+1}\right\} f_\alpha\left(\frac{s+1}{z}\right)\mathrm{d}s \\
&\quad= (\sigma+1)^n \int_{\sigma+1}^{+\infty} \exp\left\{-\frac{nt\sigma}{\sigma+1}\right\} f_\alpha\left(\frac{t}{z}\right)\mathrm{d}t.
\end{aligned}$$

Observe that the analysis of this term leads to the study of

$$\Delta_1(z) := \frac{1}{f_\alpha(1/z)} \sup_{n\in\mathbb{N}} n^{3/2}(\sigma+1)^n \int_{\sigma+1}^{+\infty} \exp\left\{-\frac{nt\sigma}{\sigma+1}\right\} f_\alpha\left(\frac{t}{z}\right) \mathrm{d}t.$$

For small z, there holds $f_\alpha(1/z) = O(z^{1+\alpha})$ in view of (3), and the supremum turns out to be of the same order z, that is, $O(z^{1+\alpha})$, since $f_\alpha(x) \le C_\alpha x^{-(1+\alpha)}$. For large z, put $\lambda := \frac{\sigma}{2(\sigma+1)}$ and use the Cauchy–Schwarz inequality to obtain

$$\begin{aligned}
&\int_{\sigma+1}^{+\infty} \exp\left\{-\frac{nt\sigma}{\sigma+1} + \lambda t\right\} e^{-\lambda t} f_\alpha\left(\frac{t}{z}\right) \mathrm{d}t \\
&\quad \le \left(\int_{\sigma+1}^{+\infty} \exp\left\{-\frac{2nt\sigma}{\sigma+1} + 2\lambda t\right\} \mathrm{d}t\right)^{1/2} \left(\|f_\alpha\|_\infty \int_0^{+\infty} e^{-2\lambda t} f_\alpha\left(\frac{t}{z}\right) \mathrm{d}t\right)^{1/2} \\
&\quad = \frac{e^{\sigma/2}\sqrt{1+1/\sigma}}{\sqrt{2n-1}} e^{-n\sigma} \sqrt{z} \|f_\alpha\|_\infty^{1/2} \exp\left\{-\frac{1}{2}\left(\frac{\sigma z}{\sigma+1}\right)^\alpha\right\},
\end{aligned}$$

which entails the desired asymptotic behavior. Second, take into account the region $s \in (-1, -\sigma)$. Writing $h(s) \ge h'(-\sigma)(s+\sigma) + h(-\sigma)$ for every $s \in (-1, -\sigma)$, and still using the convexity of the function h, we can write the following:

$$\int_{-1}^{-\sigma} e^{-nh(s)} f_\alpha\left(\frac{s+1}{z}\right) \mathrm{d}s \le (1-\sigma)^n \int_0^{1-\sigma} \exp\left\{\frac{nt\sigma}{1-\sigma}\right\} f_\alpha\left(\frac{t}{z}\right) \mathrm{d}t.$$

Therefore, the quantity to bound is now equal to

$$\Delta_2(z) := \frac{1}{f_\alpha(1/z)} \cdot \sup_{n\in\mathbb{N}} n^{3/2}(1-\sigma)^n \int_0^{1-\sigma} \exp\left\{\frac{nt\sigma}{1-\sigma}\right\} f_\alpha\left(\frac{t}{z}\right) \mathrm{d}t.$$

For small z, argue as above using that $f_\alpha(1/z) = O(z^{1+\alpha})$ and $f_\alpha(x) \le C_\alpha x^{-(1+\alpha)}$. For large values of z, exploit that $\sup_{t\in[0,1-\sigma]} \exp\{\frac{nt\sigma}{1-\sigma}\} = e^{n\sigma}$ and conclude by using that $\int_0^{1-\sigma} f_\alpha(\frac{t}{z})\,\mathrm{d}t \sim z^{\frac{1-3\alpha/2}{1-\alpha}} \exp\{-C_\alpha z^{\alpha/(1-\alpha)}\}$, as a consequence of (16). Third, to study the integral in the region $(0,\sigma)$, consider the inversion of the analytic function $h(s)$. Since $t = h(s) = \sum_{k=2}^\infty \frac{(-s)^k}{k}$ for $s \in (0,\sigma)$, by means of Lagrange's inversion formula $s = \sum_{k=1}^\infty \alpha_k t^{k/2}$. The coefficients α_k are given by $\alpha_1 = \sqrt{2}$, $\alpha_2 = 2/3$ and

$$\frac{k+2}{\sqrt{2}}\alpha_{k+1} = \alpha_k - \sum_{j=0}^{k-2} \frac{j+2}{2} \alpha_{j+2}\alpha_{k-j}$$

for $k = 2, 3, \ldots$. See, for example, Example 1 in Chapter 2 of Wong [30]. Thus, $h : (0,\sigma) \to (0, h(\sigma))$ is bijective, with inverse function given by $q(t) := \sum_{k=1}^\infty \alpha_k t^{k/2}$ for $t \in (0, h(\sigma))$. These facts guarantee the possibility to change the variable, to get

$$\int_0^\sigma e^{-nh(s)} f_\alpha\left(\frac{s+1}{z}\right) \mathrm{d}s = \int_0^{h(\sigma)} e^{-nt} f_\alpha\left(\frac{q(t)+1}{z}\right) q'(t)\,\mathrm{d}t.$$

At this stage, invoke the Taylor formula to show that, for all $s \in (0,\sigma)$ and $t \in (0, h(\sigma))$, there holds $|F(s;z)| \le \frac{1}{2}\sup_{y\in(0,\sigma)} |f_\alpha''(\frac{y+1}{z})|(\frac{s}{z})^2$, $|Q(t)| \le Ct^{3/2}$ and $|Q_1(t)| \le Ct^{1/2}$ with some numerical constant C, where we defined

$$\begin{aligned}
F(s;z) &:= f_\alpha\left(\frac{s+1}{z}\right) - \left[f_\alpha\left(\frac{1}{z}\right) + f_\alpha'\left(\frac{1}{z}\right)\frac{s}{z}\right], \\
Q(t) &:= q(t) - \left[\alpha_1 t^{1/2} + \alpha_2 t\right], \\
Q_1(t) &:= q'(t) - \left[\frac{1}{2}\alpha_1 t^{-1/2} + \alpha_2\right].
\end{aligned}$$

Finally, the study in the region $(-\sigma, 0)$ starts from the inversion of the analytic functions $h(-s)$ for $s \in (0, \sigma)$. Deduce that $s = \sum_{k=1}^{\infty} (-1)^{k+1} \alpha_k t^{k/2} =: \overline{q}(t)$ is the inverse of $t = h(-s)$ for $s \in (0, \sigma)$. Changing the variable yields

$$\int_{-\sigma}^{0} e^{-nh(s)} f_\alpha\biggl(\frac{s+1}{z}\biggr)\,\mathrm{d}s = \int_0^{h(-\sigma)} e^{-nt} f_\alpha\biggl(\frac{-\overline{q}(t)+1}{z}\biggr)\overline{q}'(t)\,\mathrm{d}t.$$

Again, the Taylor formula shows that, for all $s \in (0, \sigma)$ and $t \in (0, h(-\sigma))$, there holds $|\overline{F}(s; z)| \le \frac{1}{2} \sup_{y \in (0,\sigma)} |f_\alpha''(\frac{-y+1}{z})| (\frac{s}{z})^2$, $|\overline{Q}(t)| \le Ct^{3/2}$ and $|\overline{Q}_1(t)| \le Ct^{1/2}$ with some numerical constant C, where we defined

$$\begin{aligned}
\overline{F}(s; z) &:= f_\alpha\biggl(\frac{-s+1}{z}\biggr) - \biggl[f_\alpha\biggl(\frac{1}{z}\biggr) - f_\alpha'\biggl(\frac{1}{z}\biggr)\frac{s}{z}\biggr],\\
\overline{Q}(t) &:= \overline{q}(t) - \bigl[\alpha_1 t^{1/2} - \alpha_2 t\bigr],\\
\overline{Q}_1(t) &:= q'(t) - \biggl[\frac{1}{2}\alpha_1 t^{-1/2} - \alpha_2\biggr].
\end{aligned}$$

Then, the way is paved to study

$$\Delta_3(z) := \sup_{n \in \mathbb{N}} n \biggl| \frac{\int_0^{\sigma} e^{-nh(s)} f_\alpha(\frac{s+1}{z})\,\mathrm{d}s + \int_{-\sigma}^{0} e^{-nh(s)} f_\alpha(\frac{s+1}{z})\,\mathrm{d}s}{f_\alpha(1/z)\sqrt{\frac{2\pi}{n}}} - 1 \biggr|$$

by expanding the sum of the two integrals in the numerator. The multitude of the ensuing terms is then split into four groups, according to the dependence on the z-variable. The first group corresponds to

$$\begin{aligned}
f_\alpha\biggl(\frac{1}{z}\biggr)\biggl\{&\frac{\alpha_1}{2}\int_0^{h(\sigma)} e^{-nt} t^{-1/2}\,\mathrm{d}t + \alpha_2 \int_0^{h(\sigma)} e^{-nt}\,\mathrm{d}t + \int_0^{h(\sigma)} e^{-nt} Q_1(t)\,\mathrm{d}t\\
&+ \frac{\alpha_1}{2}\int_0^{h(-\sigma)} e^{-nt} t^{-1/2}\,\mathrm{d}t - \alpha_2 \int_0^{h(-\sigma)} e^{-nt}\,\mathrm{d}t + \int_0^{h(-\sigma)} e^{-nt}\overline{Q}_1(t)\,\mathrm{d}t\biggr\}.
\end{aligned}$$

The key remark rests on the identity $\alpha_1 \int_0^{+\infty} e^{-nt} t^{-1/2}\,\mathrm{d}t = \sqrt{\frac{2\pi}{n}}$, so that the term -1 that appears in the definition of $\Delta_3(z)$ cancels out. The other noteworthy simplification is obtained by considering the identity $\int_0^{h(\sigma)} e^{-nt}\,\mathrm{d}t - \int_0^{h(-\sigma)} e^{-nt}\,\mathrm{d}t = \frac{1}{n}[e^{-nh(-\sigma)} - e^{-nh(\sigma)}]$. The remaining terms are handled by means of the Watson lemma (Section 5.I of Wong [30]). Therefore, this first group of terms contributes as a bounded function of z, because of the simplification of the prefactor $f_\alpha(1/z)$ with the same term appearing in the denominator of the expression that defines Δ_3. The second group of terms is given by

$$\begin{aligned}
\frac{1}{z} f_\alpha'\biggl(\frac{1}{z}\biggr)\biggl\{&\frac{\alpha_1^2}{2}\int_0^{h(\sigma)} e^{-nt}\,\mathrm{d}t + \alpha_1 \int_0^{h(\sigma)} e^{-nt} t^{1/2}\bigl[\alpha_2 + Q_1(t)\bigr]\,\mathrm{d}t\\
&+ \int_0^{h(\sigma)} e^{-nt}\bigl[\alpha_2 t + Q(t)\bigr]\cdot\biggl[\frac{1}{2}\alpha_1 t^{-1/2} + \alpha_2 + Q_1(t)\biggr]\,\mathrm{d}t\\
&- \frac{\alpha_1^2}{2}\int_0^{h(-\sigma)} e^{-nt}\,\mathrm{d}t - \alpha_1 \int_0^{h(-\sigma)} e^{-nt} t^{1/2}\bigl[-\alpha_2 + \overline{Q}_1(t)\bigr]\,\mathrm{d}t\\
&+ \int_0^{h(-\sigma)} e^{-nt}\bigl[-\alpha_2 t + \overline{Q}(t)\bigr]\cdot\biggl[\frac{1}{2}\alpha_1 t^{-1/2} - \alpha_2 + \overline{Q}_1(t)\biggr]\,\mathrm{d}t\biggr\}.
\end{aligned}$$

As before, it is worth remarking the simplification involving the quantity $\int_0^{h(\sigma)} e^{-nt}\,\mathrm{d}t - \int_0^{h(-\sigma)} e^{-nt}\,\mathrm{d}t = \frac{1}{n}[e^{-nh(-\sigma)} - e^{-nh(\sigma)}]$. Again, the remaining terms are handled by means

of the Watson lemma. As to the asymptotic behavior, recall (3) and (16) to get $f'_\alpha(z^{-1})/[z \times f_\alpha(z^{-1})] = O(1)$ and $f'_\alpha(z^{-1})/[z f_\alpha(z^{-1})] = O(z^{\frac{\alpha}{1-\alpha}})$ as $z \to 0$ and $z \to +\infty$, respectively. Finally, consider

$$\frac{1}{2z^2} \sup_{y\in(0,\sigma)} \left| f''_\alpha\left(\frac{y+1}{z}\right)\right| \int_0^{h(\sigma)} e^{-nt}\big[q(t)\big]^2 \cdot \left[\frac{1}{2}\alpha_1 t^{-1/2} + \alpha_2 + Q_1(t)\right] \mathrm{d}t$$

and

$$\frac{1}{2z^2} \sup_{y\in(0,\sigma)} \left| f''_\alpha\left(\frac{-y+1}{z}\right)\right| \int_0^{h(-\sigma)} e^{-nt}\big[\overline{q}(t)\big]^2 \cdot \left[\frac{1}{2}\alpha_1 t^{-1/2} - \alpha_2 + \overline{Q}_1(t)\right] \mathrm{d}t.$$

Thanks to equations (3) and (16), the asymptotic behavior of these expressions stems from $f''_\alpha(z^{-1})/[z^2 f_\alpha(z^{-1})] = O(1)$ and $f''_\alpha(z^{-1})/[z^2 f_\alpha(z^{-1})] = O(z^{\frac{2\alpha}{1-\alpha}})$ as $z \to 0$ and $z \to +\infty$, respectively. This completes the proof. □

2.3. *A quantitative Poisson approximation.* Our result improves Theorem 1 of Hwang [13] by reformulating it as a true inequality, that is, without "big O" terms. As to notation, for any $r > 0$ and $z_0 \in \mathbb{C}$, $D_r(z_0)$ ($\overline{D_r}(z_0)$, respectively) will denote the open (closed, respectively) disc in $\mathbb{C}$ of radius r, centered at z_0.

PROPOSITION 2. *Let G_{X_n} be holomorphic in $D_{\eta+\tau_n}(0)$ for every $n \in \mathbb{N}$, with $\eta > 3$ independent of n and $\tau_n > 0$. Moreover, suppose that G_{X_n} can be written as*

$$\mathsf{G}_{X_n}(s) = \exp\big\{\lambda_n(s-1)\big\} s^h \big[g(s) + \epsilon_n(s)\big] \tag{17}$$

for every $s \in D_{\eta+\tau_n}(0)$, where:

(i) *$\{\lambda_n\}_{n\geq 1}$ is a diverging sequence of positive numbers*;
(ii) *$h \in \mathbb{N}_0$ is a constant, independent of n*;
(iii) *the restriction of g to $D_\eta(0)$ is independent of n and holomorphic on that domain, with $g(1) = 1$ and $g(0) \neq 0$*;
(iv) *ϵ_n is holomorphic in $D_\eta(0)$ and satisfies $K(\eta - 1) < +\infty$, where*

$$K(\delta) := \sup_{n\in\mathbb{N}} \sup_{0<|s-1|<\delta} \lambda_n \left|\frac{\epsilon_n(s)}{s-1}\right| < +\infty.$$

Then there exist some $n_0 \in \mathbb{N}$ and $C(\eta) > 0$, independent of n, such that

$$\sum_{k\geq h} \left| \mathsf{P}[X_n = k] - \exp\big\{-\big[\lambda_n + g'(1)\big]\big\} \frac{[\lambda_n + g'(1)]^{k-h}}{(k-h)!} \right| \leq \frac{C(\eta)}{\lambda_n} \tag{18}$$

holds for every $n \geq n_0$.

The following proof also contains a quantification of $C(\eta)$. In addition, notice that $\lambda_n + g'(1) > 0$ holds eventually (with respect to n), and that the integer h is well defined in view of $g(0) \neq 0$ and the holomorphic character of g and ϵ_n about $s = 0$.

PROOF OF PROPOSITION 2. Set $\sigma_n := \lambda_n + g'(1)$. Write $\mathsf{P}[X_n = k + h] = I_1^{(k,n)}(\rho) + I_2^{(k,n)}(\rho)$, where, in view of (17) and the Cauchy formula,

$$I_1^{(k,n)}(\rho) = \frac{1}{2\pi i} \oint_{D_\rho(0)} g(s) s^{-(k+1)} e^{\lambda_n(s-1)}\, \mathrm{d}s,$$

$$I_2^{(k,n)}(\rho) = \frac{1}{2\pi i} \oint_{D_\rho(0)} \epsilon_n(s) s^{-(k+1)} e^{\lambda_n(s-1)}\, \mathrm{d}s$$

for any $\rho \in (0, \eta)$. Observe that the left-hand side of (18) is bounded by

$$\sum_{k\geq 0}\left[\left|I_1^{(k,n)}(\rho) - e^{-\sigma_n}\frac{\sigma_n^k}{k!}\right| + |I_2^{(k,n)}(\rho)|\right]. \tag{19}$$

Fix $\delta \in (1, \eta - 1)$ and define $M_1(n) := \lambda_n - \lambda_n^{1/2+1/7}$ and $M_2(n) := \lambda_n + \lambda_n^{1/2+1/7}$, yielding $M_1(n) < M_2(n) < \delta\lambda_n$ definitely with respect to n. Then, (19) is majorized by

$$\begin{aligned}
&\sum_{0\leq k<M_1(n)}\left[e^{-\sigma_n}\frac{\sigma_n^k}{k!} + |I_1^{(k,n)}(\rho)| + |I_2^{(k,n)}(\rho)|\right]\\
&\quad+\sum_{M_1(n)\leq k<M_2(n)}\left[\left|I_1^{(k,n)}(\rho) - e^{-\sigma_n}\frac{\sigma_n^k}{k!}\right| + |I_2^{(k,n)}(\rho)|\right]\\
&\quad+\sum_{k\geq M_2(n)} e^{-\sigma_n}\frac{\sigma_n^k}{k!} + \sum_{M_2(n)\leq k<\delta\lambda_n}\left[|I_1^{(k,n)}(\rho)| + |I_2^{(k,n)}(\rho)|\right]\\
&\quad+\sum_{k\geq\delta\lambda_n}\left[|I_1^{(k,n)}(\rho)| + |I_2^{(k,n)}(\rho)|\right].
\end{aligned} \tag{20}$$

For the last sum in (20), choose $\rho = \delta$ to obtain

$$|I_1^{(k,n)}(\delta)| + |I_2^{(k,n)}(\delta)| \leq \frac{1}{2}\sqrt{\frac{\pi}{2}}\Big[\eta K(\eta)\lambda_n^{-1} + \sup_{s\in D_\eta(0)}|g(s)|\Big]\lambda_n^{-1/2}\delta^{-k}e^{\lambda_n(\delta-1)}$$

for all $k \geq \delta\lambda_n$. Hence, noticing that $1 - \delta + \delta\log\delta > 0$, conclude that

$$\sum_{k\geq\delta\lambda_n}\left[|I_1^{(k,n)}(\delta)| + |I_2^{(k,n)}(\delta)|\right] \leq C_1(\eta,\delta,g)\lambda_n^{-1/2}e^{-\lambda_n(1-\delta+\delta\log\delta)}. \tag{21}$$

Then, if $k \in (0, M_1(n)) \cup [M_2(n), \delta\lambda_n)$, set $\rho := k/\lambda_n$ to get

$$|I_1^{(k,n)}(k/\lambda_n)| \leq \frac{e\sqrt{\pi}}{2\sqrt{2}}\Big[\sup_{s\in D_\eta(0)}|g(s)|\Big]e^{-\lambda_n}\frac{\lambda_n^k}{k!},$$

$$|I_2^{(k,n)}(k/\lambda_n)| \leq \frac{e\sqrt{\pi}}{2\sqrt{2}}K(\eta)\Big(\frac{1}{\lambda_n}\Big)\Big[\Big|\frac{k}{\lambda_n} - 1\Big| + \sqrt{\frac{\pi\delta}{2\lambda_n}}\Big]e^{-\lambda_n}\frac{\lambda_n^k}{k!}.$$

By Chernoff bounds for Poisson probabilities, for the fourth sum in (20) there holds

$$\sum_{M_2(n)\leq k<\delta\lambda_n}\left[|I_1^{(k,n)}(\rho)| + |I_2^{(k,n)}(\rho)|\right] \leq C_2(\eta,\delta,g)\exp\Big\{-\frac{1}{2}\lambda_n^{2/7}\Big\}$$

while, for the third sum in (20), there exists a constant C_3 such that

$$\sum_{0\leq k<M_1(n)} e^{-\sigma_n}\frac{\sigma_n^k}{k!} + \sum_{k\geq M_2(n)} e^{-\sigma_n}\frac{\sigma_n^k}{k!} \leq C_3\exp\Big\{-\frac{1}{2}\lambda_n^{2/7}\Big\}.$$

Furthermore, the same argument based on the Chernoff bounds for Poisson probabilities shows that

$$\sum_{0\leq k<M_1(n)}\left[|I_1^{(k,n)}(\rho)| + |I_2^{(k,n)}(\rho)|\right] \leq C_4(\eta,\delta,g)\exp\Big\{-\frac{1}{2}\lambda_n^{2/7}\Big\}.$$

It remains the second sum in (20), which brings the main contribution. In particular, for the sum relative to ϵ_n, choose $\rho := k/\lambda_n$ and write

$$\begin{aligned}|I_2^{(k,n)}(k/\lambda_n)| &\le \left(e^{-\lambda_n}\frac{\lambda_n^k}{k!}\right)\left(\frac{k!e^k}{2\pi k^k\lambda_n}\right)\\ &\quad\times\left[K(\delta_n)\int_{|\theta|\le\lambda_n^{-5/14}}\left|\frac{k}{\lambda_n}e^{i\theta}-1\right|e^{-k(1-\cos\theta)}\,\mathrm{d}\theta\right.\\ &\quad\left.+K(\eta)\int_{|\theta|>\lambda_n^{-5/14}}\left|\frac{k}{\lambda_n}e^{i\theta}-1\right|e^{-k(1-\cos\theta)}\,\mathrm{d}\theta\right]\\ &\le C_5(\eta)\left(e^{-\lambda_n}\frac{\lambda_n^k}{k!}\right)\left[K(\delta_n)\exp\left\{-\frac{1}{5}\lambda_n^{2/7}\right\}+K(\delta_n)\lambda_n^{-19/14}\right],\end{aligned}$$

where $\delta_n := \sqrt{[1-(1-\lambda_n^{-5/14})\cos\lambda_n^{-5/14}]^2+[(1-\lambda_n^{-5/14})\sin\lambda_n^{-5/14}]^2} \sim \sqrt{2}\lambda_n^{-5/14}$. Therefore,

$$\sum_{M_1(n)\le k<M_2(n)}|I_2^{(k,n)}(\rho)| \le C_5(\eta)\left[K(\delta_n)\exp\left\{-\frac{1}{5}\lambda_n^{2/7}\right\}+K(\delta_n)\lambda_n^{-19/14}\right].$$

Finally, upon noticing that

$$I_1^{(k,n)}(\rho)-e^{-\sigma_n}\frac{\sigma_n^k}{k!}=\frac{1}{2\pi i}\oint_{D_\rho(0)}[g(s)-e^{g'(1)(s-1)}]s^{-(k+1)}e^{\lambda_n(s-1)}\,\mathrm{d}s$$

is valid, write the integral on the right-hand side as

$$\begin{aligned}&\frac{g''(1)-[g'(1)]^2}{4\pi i}\oint_{D_\rho(0)}(s-1)^2s^{-(k+1)}e^{\lambda_n(s-1)}\,\mathrm{d}s\\ &\quad+\sum_{m=3}^{\infty}\frac{1}{m!}\{g^{(m)}(1)-[g'(1)]^m\}\frac{1}{2\pi i}\oint_{D_\rho(0)}(s-1)^ms^{-(k+1)}e^{\lambda_n(s-1)}\,\mathrm{d}s.\end{aligned}$$

Choosing $\rho := k/\lambda_n$ once again, and taking account of the well-known Cauchy estimates for holomorphic functions, conclude that the modulus of the above sum is bounded by

$$\begin{aligned}&e^{-\lambda_n}\frac{\lambda_n^k}{k!}\left[\frac{1}{2}|g''(1)-[g'(1)]^2|\cdot|\mathcal{C}_2(\lambda_n,k)|\right.\\ &\quad\left.+\left(\frac{1}{\lambda_n}\right)^{\frac{15}{14}}\sum_{m=3}^{\infty}2^{m+1}\lambda_n^{-(m-3)/14}M_n(g)^m\right],\end{aligned}$$

where $\mathcal{C}_2(\lambda_n,k) := \frac{k^2-(2\lambda_n+1)k+\lambda_n^2}{\lambda_n^2}$ stands for the Poisson–Charlier polynomial of degree 2 and $M_n(g) := \sup_{|s-1|\le\lambda_n^{-2/7}}|g(s)|$. To conclude the proof, recall that $\sum_{k=0}^{\infty}e^{-\lambda_n}\frac{\lambda_n^k}{k!}|\mathcal{C}_2(\lambda_n,k)| \le C_6\lambda_n^{-1}$ holds with some numerical constant C_6 by virtue of Proposition 1 of Hwang [13], and observe that $2\lambda_n^{-1/14}M_n(g)\le\frac{1}{2}$ for all n sufficiently large. Hence, for all $n\ge n_0$, write

$$\sum_{M_1(n)\le k<M_2(n)}\left|I_1^{(k,n)}(\rho)-e^{-\sigma_n}\frac{\sigma_n^k}{k!}\right| \le \frac{C_7}{\lambda_n}\left[\frac{1}{2}|g''(1)-[g'(1)]^2|+1\right] \tag{22}$$

for some suitable numerical constant C_7. This fact completes the proof. □

A sharper bound than the one obtained in Proposition 2 can be obtained by strengthening the hypotheses on g. This is stated in the following.

COROLLARY 1. *In addition to the assumptions made in Proposition 2, suppose that*

$$\left|\sum_{l=0}^{r}\binom{r}{l}g^{(l)}(1)[-g'(1)]^{r-l}\right| \le \sqrt{r!}B^r \tag{23}$$

holds for every $r \in \mathbb{N}_0$ and some $B > 0$ independent of r. Then, it holds

$$\sum_{k\ge 0}\left|I_1^{(k,n)}(\rho) - \exp\{-[\lambda_n + g'(1)]\}\frac{[\lambda_n + g'(1)]^k}{k!}\right| \le \frac{C_*(\eta)}{\lambda_n}$$

for every $n \ge n_0$, $C_*(\eta)$ being a suitable constant depending solely on η.

PROOF. Rewrite $I_1^{(k,n)}(\rho)$ by means of an identity due to Ch. Jordan (see Kullback [16] and Uspensky [29]). For any $x > 0$ and $r \in \mathbb{N}_0$, define

$$L_r(x;n) := \frac{r!}{2\pi i}\oint_{D_{\overline{\rho}}(0)} g(1+s)s^{-(k+1)}e^{(\lambda_n - x)s}\,\mathrm{d}s = \sum_{l=0}^{r}\binom{r}{l}g^{(l)}(1)(\lambda_n - x)^{r-l}$$

the radius $\overline{\rho}$ being any number in $(1, \eta - 1)$. See formulae (6) and (2.3) in Uspensky [29] and Kullback [16]. Setting $x = \sigma_n := \lambda_n + g'(1)$, Jordan's identity reads

$$I_1^{(k,n)}(\rho) = \sum_{r=0}^{\infty}\frac{1}{r!}L_r(\sigma_n;n)\mathcal{C}_r(\sigma_n;k)e^{-\sigma_n}\frac{\sigma_n^k}{k!} \tag{24}$$

the symbol $\mathcal{C}_r(x;k) := \sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l}[k]_{[l]}x^{-l}$ standing for the Poisson–Charlier polynomial, where $[k]_{[j]}$ denotes the falling factorial, that is, $[k]_{[0]} := 1$ and $[k]_{[j]} := \prod_{m=0}^{j-1}(k-m)$ if $j \in \mathbb{N}$. At this stage, the argument used to prove Proposition 1 of Hwang [13] (see also Lemma 6 in Shorgin [28]) shows that, for any $x_0 > 1$, there exists a constant $M(x_0)$ depending solely by x_0 such that

$$\sum_{k=0}^{\infty}e^{-x}\frac{x^k}{k!}\left|\mathcal{C}_r(x;k)\right| \le [M(x_0)]^r\Gamma\left(\frac{r+1}{2}\right)x^{-r/2} \tag{25}$$

holds for every $x \ge x_0$ and $r \in \mathbb{N}_0$. Combination of (23)–(25) yields

$$\sum_{k\ge 0}\left|I_1^{(k,n)}(\rho) - e^{-\sigma_n}\frac{\sigma_n^k}{k!}\right| \le \frac{BM(x_0)}{\sigma_n}\sum_{r=2}^{\infty}\Gamma\left(\frac{r+1}{2}\right)\frac{1}{\sqrt{r!}}\left(\frac{BM(x_0)}{\sigma_n}\right)^{(r-2)/2} \tag{26}$$

whenever $\sigma_n \ge x_0 > 1$, since $L_0(\sigma_n;n) = 1$ and $L_1(\sigma_n;n) = 0$. Thus, if $\sigma_n \ge 2BM(x_0)$, the series on the right-hand side of (26) is bounded by a numerical constant, say c_1. Since $-2g'(1) \le \lambda_n$ holds eventually in n, entailing that $1/\sigma_n \le 2/\lambda_n$, the left-hand side of (26) is bounded by $2c_1BM(x_0)/\lambda_n$, for all n greater or equal than some n_0. This completes the proof. □

Our application of Proposition 2 starts from the evaluation of the probability generating function of the random variable $R(\alpha, n, tn^\alpha)$, which is playing the role of X_n. In fact, a combination of equations (7) and (13) with Lemma 3 yields

$$\mathsf{G}_{R(\alpha,n,tn^\alpha)}(s) = e^{tn^\alpha(s-1)}s\frac{(\frac{1}{s})^{\frac{1}{\alpha}+1}f_\alpha((\frac{1}{st})^{\frac{1}{\alpha}})[1+\delta_n((st)^{\frac{1}{\alpha}})]}{f_\alpha((\frac{1}{t})^{\frac{1}{\alpha}})[1+\delta_n(t^{\frac{1}{\alpha}})]}. \tag{27}$$

Currently, (27) holds for all $s, t > 0$, but it will be shown that the numerator on the right-hand side can be analytically continued, as a function of s, to the whole complex plane. To parallel (27) with (17), set $\lambda_n = tn^\alpha$, $h = 1$,

$$g(s) = \frac{(\frac{1}{s})^{\frac{1}{\alpha}+1} f_\alpha((\frac{1}{st})^{\frac{1}{\alpha}})}{f_\alpha((\frac{1}{t})^{\frac{1}{\alpha}})} \quad \text{and} \quad \epsilon_n(s) = g(s)\Big[\frac{1+\delta_n((st)^{\frac{1}{\alpha}})}{1+\delta_n(t^{\frac{1}{\alpha}})} - 1\Big].$$

As η, it can be chosen as any number strictly greater than 3, since $\mathsf{G}_{R(\alpha,n,tn^\alpha)}$ is an entire function. Apropos of the analytic continuation of the right-hand side of (27), recall the definition of the Wright–Mainardi function $M_\alpha(z) := \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!}\Gamma(\alpha n)\sin(\pi\alpha n)$, for $z \in \mathbb{C}$ and $\alpha \in (0,1)$, which coincides with $\frac{1}{\alpha z^{1+1/\alpha}} f_\alpha(\frac{1}{z^{1/\alpha}})$ if $\Im(z) = 0$ and $\Re(z) > 0$ (Mainardi et al. [19]). It turns out that g can be re-written as $g(s) = M_\alpha(st)/M_\alpha(t)$, for all $s \in \mathbb{C}$, and (27) reads

$$\mathsf{G}_{R(\alpha,n,tn^\alpha)}(s) = e^{tn^\alpha(s-1)} s \frac{\int_{-1}^{+\infty} e^{nh(\sigma)}(1+\sigma)^{-\alpha} M_\alpha(st(1+\sigma)^{-\alpha})\,\mathrm{d}\sigma}{\int_{-1}^{+\infty} e^{nh(\sigma)}(1+\sigma)^{-\alpha} M_\alpha(t(1+\sigma)^{-\alpha})\,\mathrm{d}\sigma}.$$

Furthermore, the rate $\lambda_n + g'(1)$ of the shifted Poisson distribution in (18) becomes

$$\omega(t; n, \alpha) := \lambda_n + g'(1) = tn^\alpha + \frac{tM'_\alpha(t)}{M_\alpha(t)},$$

which is positive whenever n is sufficiently large and $t \le T(n,\alpha)$. The determination of the asymptotic behavior of $T(n,\alpha)$ is determined by the behavior of M_α for large arguments. In fact, as shown in the work of Mainardi et al. [19], $M_\alpha(t/\alpha) \sim (2\pi(1-\alpha))^{-1/2} t^{\frac{\alpha-1/2}{1-\alpha}} \exp\{-\frac{1-\alpha}{\alpha} t^{\frac{1}{1-\alpha}}\}$ as $t \to +\infty$, in agreement with (16).

At this stage, this subsection is complete by stating a corollary that matches Proposition 2 with Lemma 3. After recalling that the symbol N_λ denotes a Poisson random variable with parameter λ, we have the following.

COROLLARY 2. *There exists n_0 independent of t such that, for all $n \ge n_0$ and $t \le T(n,\alpha)$, there holds*

$$\sum_{k=1}^{\infty} \big|\mathsf{P}\big[R(\alpha,n,tn^\alpha) = k\big] - \mathsf{P}[1 + N_{\omega(t;n,\alpha)} = k]\big| \le \frac{\Upsilon(t)}{tn^\alpha},$$

where $T(n,\alpha) \sim n^{\kappa\alpha}$ as $n \to +\infty$ for some $\kappa \in (0,1)$, and $\Upsilon : (0,+\infty) \to (0,+\infty)$ is a suitable continuous function which is independent of n. Moreover, Υ can be chosen in such a way that $\Upsilon(t) = O(1)$ as $t \to 0$, and $\Upsilon(t) \cdot f_\alpha(\frac{1}{t^{1/\alpha}}) = O(t^{-\infty})$ as $t \to +\infty$.

PROOF. It is enough to recover the bounds (21)–(22). After they are rewritten in terms of t, it is enough to exploit the properties of the function Δ in Lemma 3. □

2.4. *Conclusion.* This subsection contains the heart of the proof of Theorem 1, whose strategy consists in four main steps. Maintaining the notation used in Sections 2.1 and 2.3, these steps can be summarized as follows:

(A) since the strong law of large numbers hints that $G^\alpha_{\theta+n,1} \sim n^\alpha$ as $n \to +\infty$, with probability 1, one tries to rigorously prove that the probability laws of K_n and $R(\alpha, n, S_{\alpha,\theta} \cdot n^\alpha)$ are close, even in total variation;

(B) after conditioning on the hypothesis that $S_{\alpha,\theta} = t$, one invokes Corollary 2 to prove that the probability laws of $R(\alpha, n, tn^\alpha)$ and $1 + N_{\omega(t,n,\alpha)}$ are close in total variation;

(C) by resorting to well-known results about Poisson mixtures contained in [17] (see also Proposition 4 below), one shows that the probability laws of $N_{\omega(S_{\alpha,\theta},n,\alpha)}$ and $N_{S_{\alpha,\theta}\cdot n^\alpha}$ are close in total variation, the random variable $S_{\alpha,\theta}$ being thought of as independent of the family of random variables $\{N_\lambda\}_{\lambda>0}$;

(D) in view of a quantitative law of large numbers for the Poisson process stated in Adell and de la Cal [1], one concludes by checking that the probability laws of $(1+N_{S_{\alpha,\theta}\cdot n^\alpha})/n^\alpha$ and $S_{\alpha,\theta}$ are close in the Kolmogorov metric.

To grasp our strategy, write $\mathrm{d}_K(X;Y)$ to denote the Kolmogorov distance between the distribution functions of X and Y, and define $\Omega_{n,\alpha,\theta} := \omega(S_{\alpha,\theta};n,\alpha)\mathbb{1}\{S_{\alpha,\theta}\le T(n,\alpha)\}\ge 0$, $T(n,\alpha)$ being the same as in Corollary 2, to get

$$\begin{aligned}\mathrm{d}_K\big(K_n/n^\alpha;S_{\alpha,\theta}\big) &\le \mathrm{d}_K\big(K_n/n^\alpha;R\big(\alpha,n,n^\alpha S_{\alpha,\theta}\big)/n^\alpha\big)\\ &\quad+\mathrm{d}_K\big(R\big(\alpha,n,n^\alpha S_{\alpha,\theta}\big)/n^\alpha;(1+N_{\Omega_{n,\alpha,\theta}})/n^\alpha\big)\\ &\quad+\mathrm{d}_K\big((1+N_{\Omega_{n,\alpha,\theta}})/n^\alpha;(1+N_{S_{\alpha,\theta}n^\alpha})/n^\alpha\big)\\ &\quad+\mathrm{d}_K\big((1+N_{S_{\alpha,\theta}n^\alpha})/n^\alpha;S_{\alpha,\theta}\big),\end{aligned}$$

where $S_{\alpha,\theta}$ is independent of both $\{R(\alpha,n,z)\}_{z>0}$ and $\{N_\lambda\}_{\lambda>0}$. Moreover, N_0 is intended, henceforth, as the degenerate random variable equal a.s. to 0. After recalling that $\mathrm{d}_K(aX+b;aY+b)=\mathrm{d}_K(X;Y)$ for any $a>0$ and $b\in\mathbb{R}$, and that $\mathrm{d}_K(X;Y)\le \mathrm{d}_{\mathrm{TV}}(X;Y):=\sup_{B\in\mathscr{B}(\mathbb{R})}|\mathsf{P}[X\in B]-\mathsf{P}[Y\in B]|$, d_{TV} being the total variation distance, one can write

$$\begin{aligned}&\mathrm{d}_K\big(K_n/n^\alpha;S_{\alpha,\theta}\big)\\ &\quad\le \mathrm{d}_{\mathrm{TV}}\big(K_n;R\big(\alpha,n,n^\alpha S_{\alpha,\theta}\big)\big)\\ &\qquad+\int_0^{T(n,\alpha)}\mathrm{d}_{\mathrm{TV}}\big(R\big(\alpha,n,tn^\alpha\big);1+N_{\omega(t;n,\alpha,\theta)}\big)f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t\\ &\qquad+\mathsf{P}\big[S_{\alpha,\theta}>T(n,\alpha)\big]\\ &\qquad+\mathrm{d}_K\big(N_{\Omega_{n,\alpha,\theta}};N_{S_{\alpha,\theta}\cdot n^\alpha}\big)+\mathrm{d}_K\big((1+N_{S_{\alpha,\theta}\cdot n^\alpha})/n^\alpha;S_{\alpha,\theta}\big).\end{aligned}\tag{28}$$

Hence, one starts by studying the first term on the right-hand side of (28).

PROPOSITION 3. *For fixed $\alpha\in(0,1)$ and $\theta>0$, there exists a positive constant $C_1(\alpha,\theta)$ such that*

$$\begin{aligned}&2\,\mathrm{d}_{\mathrm{TV}}\big(K_n;R\big(\alpha,n,n^\alpha S_{\alpha,\theta}\big)\big)\\ &\quad=\sum_{k=1}^n\big|\mathsf{P}[K_n=k]-\mathsf{P}\big[R\big(\alpha,n,n^\alpha S_{\alpha,\theta}\big)=k\big]\big|\le C_1(\alpha,\theta)/n\end{aligned}\tag{29}$$

holds for all $n\in\mathbb{N}$.

PROOF. The law of $R(\alpha,n,S_{\alpha,\theta}\cdot n^\alpha)$ is given by

$$\mathsf{P}\big[R\big(\alpha,n,S_{\alpha,\theta}\cdot n^\alpha\big)=k\big]=\int_0^{+\infty}\bigg[\frac{\mathscr{C}(n,k;\alpha)(tn^\alpha)^k}{\sum_{j=1}^n\mathscr{C}(n,j;\alpha)(tn^\alpha)^j}\bigg]f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t\tag{30}$$

for $k=1,\dots,n$. Combining (10), (13) and (30), the left-hand side of (29) is equal to

$$\sum_{k=1}^n\bigg|\mathscr{C}(n,k;\alpha)\frac{\Gamma(k+\theta/\alpha)}{\Gamma(\theta/\alpha)}\frac{\Gamma(\theta)}{\Gamma(n+\theta)}-\int_0^{+\infty}\frac{\mathscr{C}(n,k;\alpha)(tn^\alpha)^k}{d_n(t)}f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t\bigg|.$$

Set $d_n^*(t) := e^{tn^\alpha}(n-1)!\frac{1}{t^{1/\alpha}} f_\alpha(\frac{1}{t^{1/\alpha}})$ and majorize the above quantity by

$$\sum_{k=1}^{n}\left|\mathscr{C}(n,k;\alpha)\frac{\Gamma(k+\theta/\alpha)}{\Gamma(\theta/\alpha)}\frac{\Gamma(\theta)}{\Gamma(n+\theta)} - \int_0^{+\infty}\frac{\mathscr{C}(n,k;\alpha)(tn^\alpha)^k}{d_n^*(t)} f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t\right| + \int_0^{+\infty}\frac{|d_n^*(t)-d_n(t)|}{d_n^*(t)} f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t.$$

Since $\int_0^{+\infty}\frac{(tn^\alpha)^k}{d_n^*(t)} f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t = \frac{1}{(n-1)!}\frac{\Gamma(k+\theta/\alpha)}{n^\theta}\frac{\Gamma(\theta)}{\Gamma(\theta/\alpha)}$ holds after taking account of (4), one has

$$\sum_{k=1}^{n}\left|\mathscr{C}(n,k;\alpha)\frac{\Gamma(k+\theta/\alpha)}{\Gamma(\theta/\alpha)}\frac{\Gamma(\theta)}{\Gamma(n+\theta)} - \int_0^{+\infty}\frac{\mathscr{C}(n,k;\alpha)(tn^\alpha)^k}{d_n^*(t)} f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t\right| = \left|1-\frac{\Gamma(\theta+n)}{\Gamma(n)n^\theta}\right| \le \frac{([\theta]+1)!-1}{n},$$

where: (i) the identity is a consequence of (10); (ii) the inequality, in which $[\theta]$ denotes the integral part of θ, follows from the well-known Tricomi–Erdélyi expansion of the gamma ratio. Finally, by resorting to (13)–(14), one has

$$\int_0^{+\infty}\frac{|d_n^*(t)-d_n(t)|}{d_n^*(t)} f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t \le \left|\frac{(n/e)^n\sqrt{2\pi n}}{n!}-1\right| + \left(\frac{(n/e)^n\sqrt{2\pi n}}{n!}\right)\frac{1}{n}\int_0^{+\infty}\Delta\big(t^{1/\alpha}\big) f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t,$$

which leads to the desired conclusion, in view of the well-known Stirling approximation and the fact that $\int_0^{+\infty}\Delta(t^{1/\alpha}) f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t < +\infty$, by virtue of Lemma 3. Indeed, suffice it to recall that $\Delta(t^{1/\alpha}) = O(1)$ and $f_{S_{\alpha,\theta}}(t) \sim t^{\theta/\alpha}$ as $t \to 0$, whereas the condition $\Delta(z) f_\alpha(1/z) = O(z^{-\infty})$, valid as $z \to +\infty$, entails the integrability of $\Delta(t^{1/\alpha}) f_{S_{\alpha,\theta}}(t)$ at infinity. □

The conclusion of the proof of Theorem 1 requires to combine results of Section 2.3 with other known results. According to point (B), invoke Corollary 2 to obtain

$$\int_0^{T(n,\alpha)}\left[\sum_{k=1}^{\infty}\left|\mathsf{P}\big[R(\alpha,n,tn^\alpha)=k\big]-\mathsf{P}[1+N_{\omega(t;n,\alpha)}=k]\right|\right] f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t \le \frac{C_2(\alpha,\theta)}{n^\alpha},$$

where $C_2(\alpha,\theta) := \int_0^{+\infty} t^{-1}\Upsilon(t) f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t < +\infty$. The finiteness of this last integral follows from the asymptotic properties of Υ stated in Corollary 2 and (4), which, in turn, entails $f_{S_{\alpha,\theta}}(t) \sim t^{\theta/\alpha}$ as $t \to 0$.

In order to proceed with point (C), notice that

$$\mathsf{P}\big[\omega(S_{\alpha,\theta};n,\alpha)<0\big] = \int_{T(n,\alpha)}^{+\infty} f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t \sim \frac{1}{n^\alpha}$$

as $n \to \infty$.

To bound the fourth term on the right-hand side of (28), note that this is the Kolmogorov distance between two Poisson mixtures. This problem is tackled in Theorem 1 of Le Cam [17], which is restated for the reader's ease.

PROPOSITION 4 (Le Cam). *Given two probability measures γ_1, γ_2 on $\mathscr{B}([0,+\infty))$, one gets*

$$\sup_{x\geq 0}\left|\int_0^{+\infty} \mathsf{P}[N_\lambda \leq x]\gamma_1(\mathrm{d}x) - \int_0^{+\infty} \mathsf{P}[N_\lambda \leq x]\gamma_2(\mathrm{d}x)\right| \leq 2\big(1+e^2\big) \sup_{B\in\mathscr{B}([0,+\infty))} \big|\gamma_1(B) - \gamma_2(B)\big|.$$

Therefore, a direct application of Proposition 4 yields

$$\sum_{k=0}^{\infty}\left|\int_0^{T(n,\alpha)} \big[\mathsf{P}[N_{\omega(t;n,\alpha)} = k] - \mathsf{P}[N_{tn^\alpha} = k]\big] f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t\right| \leq 2\big(1+e^2\big)\int_0^{T(n,\alpha)} \big|f_{S_{\alpha,\theta}}(t) - n^\alpha f_{S_{\alpha,\theta}}\big(\psi_n\big(tn^\alpha\big)\big)\psi_n'\big(tn^\alpha\big)\big|\,\mathrm{d}t \sim \frac{1}{n^\alpha}$$

the function ψ_n denoting the inverse of $(0, T(n,\alpha)) \ni t \mapsto \omega(t;n,\alpha)$.

Finally, according to point (D), first get rid of the shift $+1$ relative to the Poisson mixture. In fact, this yields an extra term which goes to zero as $1/n^\alpha$, since $\int_0^{1/n^\alpha} f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t \sim 1/n^\alpha$. Then it remains to consider the following term:

$$\sup_{x\geq 0}\big|\mathsf{P}\big[N_{S_{\alpha,\theta}\cdot n^\alpha} \leq xn^\alpha\big] - \mathsf{P}[S_{\alpha,\theta} \leq x]\big| = \sup_{x\geq 0}\left|\int_0^{\infty} \big\{\mathsf{P}\big[N_{t\cdot n^\alpha} \leq xn^\alpha\big] - \mathbb{1}\{t\leq x\}\big\} f_{S_{\alpha,\theta}}(t)\,\mathrm{d}t\right|.$$

This quantity is bounded by $C_3(\alpha,\theta)/n^\alpha$ for a suitable $C_3(\alpha,\theta) > 0$, as a direct application of Theorem 1 of Adell and de la Cal [1], which is here restated for the reader's ease.

PROPOSITION 5 (Adell–de la Cal). *Consider a probability density $h : [0,+\infty) \to [0,+\infty)$ satisfying the following conditions*:

(i) $h \in \mathrm{C}_b^1([0,+\infty))$, *the space of* C^1 *functions on* $[0,+\infty)$ *which are bounded together with their first derivatives*;

(ii) $\sup_{x\geq 0}|h(x) + \frac{1}{2}xh'(x)| < +\infty$:

(iii) *there exist $b_0 \geq 0$ and $0 < \gamma < \frac{1}{2}$ such that*

$$\lim_{s\to+\infty} \sup_{b\geq b_0} b \sup_{\substack{x\geq 0,\\ |x-b|\leq b^{1-\gamma}/s^\gamma}} \big|h'(x) - h'(b)\big| = 0.$$

Then there exists a constant $C(h)$, depending on the above analytical properties of h, for which

$$\sup_{x\geq 0}\left|\int_0^{\infty} \big\{\mathsf{P}[N_{tm} \leq xm] - \mathbb{1}\{t\leq x\}\big\}h(t)\,\mathrm{d}t\right| \leq \frac{C(h)}{m}$$

holds for all $m > 0$.

To apply Proposition 5 it is enough to set $m = n^\alpha$ and $h = f_{S_{\alpha,\theta}}$. Checking conditions (i) and (ii) is obvious, while the condition (iii) can be verified as in Section 3 of the work of Adell and de la Cal [1]. The proof Theorem 1 is complete.

REMARK 2. According to Remark 1, a natural extension of Theorem 1 may be to consider the random number $M_{l,n}$ of blocks in the random partition Π_n induced by a random sample from $\tilde{\mathfrak{p}}_{\alpha,\theta}$. In particular, Pitman [25] showed that

$$\frac{M_{l,n}}{n^{\alpha}} \xrightarrow{\text{a.s.}} \frac{\alpha[1-\alpha]_{(l-1)}}{l!} S_{\alpha,\theta}, \tag{31}$$

as $n \to +\infty$. Then one may apply the results in Section 2.2 and Section 2.3 to obtain a Berry–Esseen theorem for (31). A more challenging task is to prove Theorem 1 for the class of α-stable Poisson–Kingman processes (Pitman [25]).

3. Pitman's posterior α-diversity. Pitman's posterior α-diversity (5) was first introduced in Favaro et al. [10] for Bayesian nonparametric inference of the number of new species in m additional samples, given n initial observed samples. Samples are modeled by the random vectors $(X_1, \ldots, X_n)$ and $(X_{n+1}, \ldots, X_{n+m})$, which are subsequent segments of the exchangeable sequence $\{X_i\}_{i\geq 1}$ with $\tilde{\mathfrak{p}}_{\alpha,\theta}$ as directing measure. If $\text{card}(\cdot)$ denotes the cardinality of a set, then $K_m^{(n)} = K_m^{(n)}(X_1, \ldots, X_n; X_{n+1}, \ldots, X_{n+m}) := \text{card}(\{X_{n+1}, \ldots, X_{n+m}\} - \{X_1, \ldots, X_n\})$ is the number of new species in the additional sample $(X_{n+1}, \ldots, X_{n+m})$. Lijoi et al. [18] obtained the posterior distribution of $K_m^{(n)}$ given $(X_1, \ldots, X_n)$. In particular, for $(X_1, \ldots, X_n) = (x_1, \ldots, x_n) \in \mathbb{N}^n$, $K_n = j \in \{1, \ldots, n\}$ such that $\{X_1 = x_1, \ldots, X_n = x_n\} \cap \{K_n = j\} \neq \varnothing$, and $\mathbf{N}_n = (n_1, \ldots, n_j) \in \mathbb{N}^j$ such that $\sum_{1\leq i\leq j} n_i = n$, they showed that

$$\begin{aligned} &\mathsf{P}\big[K_m^{(n)} = k | X_1 = x_1, \ldots, X_n = x_n, K_n = j\big] \\ &\quad = \mathsf{P}\big[K_m^{(n)} = k | K_n = j, \mathbf{N}_n = (n_1, \ldots, n_j)\big] \\ &\quad = \mathsf{P}\big[K_m^{(n)} = k | K_n = j\big] = \frac{[\frac{\theta}{\alpha} + j]_{(k)}}{[\theta + n]_{(m)}} \mathscr{C}(m, k; \alpha, -n + j\alpha) \end{aligned} \tag{32}$$

for any $k \in \{0, 1, \ldots, m\}$, where $\mathscr{C}(n, k; s, r)$ is the noncentral generalized factorial coefficient, that is, $\mathscr{C}(n, k; s, r) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}[-is - r]_{(m)}$ (Charalambides [3]). The posterior distribution (32) is at the basis of Bayesian nonparametric inference for $K_m^{(n)}$, for example, estimation and uncertainty quantification (Lijoi et al. [18]). However, since the computational burden for evaluating the noncentral generalized factorial coefficient becomes overwhelming for large m, the evaluation of (32) is practically impossible for large m. To overcome this drawback, Proposition 2 of Favaro et al. [10], recalled in (5), introduced Pitman's posterior α-diversity $S_{\alpha,\theta}(n, j)$ to obtain large m approximated posterior inferences for $K_m^{(n)}$ via straightforward Monte Carlo sampling from $S_{\alpha,\theta}(n, j)$. Hereafter, we present a quantitative version of Proposition 2 of Favaro et al. [10]. That is, we formulate a Berry–Esseen theorem for Pitman's posterior α-diversity, thus quantifying the error of the approximated inference.

We start with a novel representation of the posterior distribution (32) in terms of the distribution of a compound sum of independent Bernoulli random variables with random parameter. For any $n \in \mathbb{N}$ and $p \in [0, 1]$, we denote by $Z(n, p)$ a Binomial random variable with parameters n, p. Also, we use the symbol $B_{a,b}$ to denote a Beta random variable with parameters $a, b > 0$.

LEMMA 4. *Let $n, m \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$. For any $\alpha \in (0, 1)$ and $\theta > -\alpha$, let K_m^* be the number of blocks of the random partition Π_m induced by a random sample $(X_1^*, \ldots, X_m^*)$ from $\tilde{\mathfrak{p}}_{\alpha,\theta+n}$. Then, for any $k \in \{0, 1, \ldots, m\}$, there holds*

$$\mathsf{P}\big[K_m^{(n)} = k | K_n = j\big] = \mathsf{P}\big[Z(K_m^*, B_{\theta/\alpha+j,n/\alpha-j}) = k\big],$$

where the random variables Z, K_m^* and $B_{\theta/\alpha+j,n/\alpha-j}$ are mutually independent under P.

PROOF. Let $[x]_{[n,a]}$ be the falling factorial of x of order n and decrement a, that is, $[x]_{[n,a]} = \prod_{0\le i\le n-1}(x - ia)$, let $S(n,k)$ be the Stirling number of the second kind, and let $S(n,k;x) := \sum_{k\le t\le n}\binom{t}{i}[x]_{[i-k,1]}S(n,i)$ be the noncentral Stirling number of the second kind. Furthermore, let $s(n,k)$ be the Stirling number of the first kind and recall that $[x]_{[n,1]} = \sum_{i=1}^{n} s(n,i)x^i$. We combine the definition of $S(n,k;x)$ with Proposition 1 in Favaro et al. [10] to write, for any $r \in \mathbb{N}$,

$$
\begin{aligned}
&\mathsf{E}\big[(K_m^{(n)})^r \,|\, K_n = j\big] \\
&\quad= \sum_{t=0}^{r} S(r,t)\frac{[j+\frac{\theta}{\alpha}]_{(t,1)}}{[\frac{\theta+n}{\alpha}]_{(t)}}\left[\frac{\theta+n}{\alpha}\right]_{(t)}\sum_{i=0}^{t}(-1)^{t-i}\binom{t}{i}\frac{[\theta+n+i\alpha]_{(m)}}{[\theta+n]_{(m)}} \\
&\quad= \sum_{t=0}^{r} S(r,t)\frac{[j+\frac{\theta}{\alpha}]_{(t,1)}}{[\frac{\theta+n}{\alpha}]_{(t)}}\mathsf{E}\big[[K_m^*]_{[t,1]}\big] \\
&\quad= \sum_{t=0}^{r} S(r,t)\mathsf{E}\big[[K_m^*]_{[t,1]}\big]\frac{\Gamma(\frac{\theta+n}{\alpha})}{\Gamma(\frac{\theta}{\alpha}+j)\Gamma(\frac{n}{\alpha}-j)}\int_0^1 x^{t+\frac{\theta}{\alpha}+j-1}(1-x)^{\frac{n}{\alpha}-j-1}\,\mathrm{d}x \\
&\quad= \sum_{t=0}^{r} S(r,t)\mathsf{E}\big[[K_m^*]_{[t,1]}\big]\mathsf{E}\big[(B_{\theta/\alpha+j,n/\alpha-j})^t\big] \\
&\quad= \mathsf{E}\big[\big(Z(K_m^*, B_{\theta/\alpha+j,n/\alpha-j})\big)^r\big],
\end{aligned}
$$

where the last identity follows by $\mathsf{E}[(Z(n,p))^r] = \sum_{0\le t\le r} S(r,t)[n]_{[t,1]}p^t$. Then the proof is complete by resorting to the one-to-one correspondence between the conditional law of $K_m^{(n)}$ and the sequence of its conditional moments. □

Let $\mathsf{F}_m(n,j)$ and $\mathsf{F}_{\alpha,\theta}(n,j)$ stand for the distribution functions of $K_m^{(n)}/m^\alpha$ and $S_{\alpha,\theta}(n,j)$, respectively, conditioned on the event $\{K_n = j\}$. Then the followiong theorem may be interpreted as the natural posterior counterpart of Theorem 1, namely a Berry–Esseen theorem for Pitman's posterior α-diversity.

THEOREM 2. *Let $n \in \mathbb{N}$ and $j \in \{1,\dots,n\}$. For any $\theta > 0$ and $\alpha \in (0,1)$ such that $\frac{n}{\alpha} - j \ge 1$, there exists a positive constant $C_{\alpha,\theta}(n,j)$, depending solely on n, j, α and θ, such that $\mathrm{d}_K(\mathsf{F}_m(n,j);\mathsf{F}_{\alpha,\theta}(n,j)) \le m^{-\alpha}C_{\alpha,\theta}(n,j)$ holds for every $m \in \mathbb{N}$.*

PROOF. For a generic random variable X, let F_X denote its distribution function. Then notice that $\mathsf{F}_{\alpha,\theta}(n,j)(x) = \mathsf{E}[F_{B_{\theta/\alpha+j,n/\alpha-j}}(x/S_{\alpha,\theta+n})]$. Hereafter, it will be shown there exists a suitable constant $C^*_{\alpha,\theta}(n,j)$ such that

$$
\mathrm{d}_K\big(\mathsf{F}_m(n,j);\mathsf{F}_{\alpha,\theta}(n,j)\big) \le C^*_{\alpha,\theta}(n,j)\mathsf{E}\left[\frac{1}{K_m^*+1}\right] + \mathrm{d}_K\big(\mathsf{F}_{K_m^*/m^\alpha};\mathsf{F}_{S_{\alpha,\theta+n}}\big), \tag{33}
$$

is valid, along with

$$
\mathsf{E}\left[\frac{1}{K_m^*+1}\right] \le \frac{1}{m^\alpha}\mathsf{E}\left[\frac{1}{S_{\alpha,\theta+n}}\right] + \mathrm{d}_K\big(\mathsf{F}_{K_m^*/m^\alpha};\mathsf{F}_{S_{\alpha,\theta+n}}\big). \tag{34}
$$

With regards to (33), write

$$
\begin{aligned}
&\mathrm{d}_K\big(\mathsf{F}_m(n,j);\mathsf{F}_{\alpha,\theta}(n,j)\big) \\
&\quad= \sup_{x\ge 0}\left|\mathsf{F}_{K_m^{(n)}}\big(m^\alpha x\big) - \mathsf{E}\left[\mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}\left(\frac{x}{S_{\alpha,\theta+n}}\right)\right]\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sup_{x\geq 0}\left|\mathsf{F}_{K_m^{(n)}}(m^\alpha x) - \mathsf{E}\left[\mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}\left(\frac{m^\alpha x}{K_m^*}\right)\right]\right| \\
&\quad + \sup_{x\geq 0}\left|\mathsf{E}\left[\mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}\left(\frac{m^\alpha x}{K_m^*}\right)\right] - \mathsf{E}\left[\mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}\left(\frac{x}{S_{\alpha,\theta+n}}\right)\right]\right| \\
&= \sup_{x\geq 0}\left|\mathsf{F}_{K_m^{(n)}}(x) - \mathsf{E}\left[\mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}\left(\frac{x}{K_m^*}\right)\right]\right| \\
&\quad + \sup_{x\geq 0}\left|\mathsf{E}\left[\mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}\left(\frac{m^\alpha x}{K_m^*}\right)\right] - \mathsf{E}\left[\mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}\left(\frac{x}{S_{\alpha,\theta+n}}\right)\right]\right|
\end{aligned} \tag{35}
$$

and we treat separately the terms in (35). With regard to the first term,

$$
\begin{aligned}
&\sup_{x\geq 0}\left|\mathsf{F}_{K_m^{(n)}}(x) - \mathsf{E}\left[\mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}\left(\frac{x}{K_m^*}\right)\right]\right| \\
&\quad \leq \mathsf{E}\left[\sup_{x\geq 0}\left|\mathsf{F}_{K_m^{(n)}}(x) - \mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}\left(\frac{x}{K_m^*}\right)\right|\right] \\
&\quad = \mathsf{E}\left[\sup_{x\geq 0}\left|\mathsf{F}_{K_m^{(n)}}(K_m^* x) - \mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}(x)\right|\right] \\
&\quad = \mathsf{E}\left[\sup_{x\geq 0}\left|\mathsf{P}\left[\frac{1}{K_m^*}\sum_{i=1}^{K_m^*} Y_i \leq x \,\middle|\, K_m^*\right] - \mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}(x)\right|\right],
\end{aligned}
$$

where in the last identity we used Lemma 4. Thus, given $B_{\theta/\alpha+j,n/\alpha-j}$, the Y_i's are (conditionally) independent and identically distributed Bernoulli variables with parameter $B_{\theta/\alpha+j,n/\alpha-j}$. Moreover, K_m^*, $B_{\theta/\alpha+j,n/\alpha-j}$ and the sequence $\{Y_i\}_{i\geq 1}$ are mutually independent under P. Since $\frac{\theta}{\alpha}+j\geq 1$ and $\frac{n}{\alpha}-j\geq 1$, invoke Corollary 1.1 in Dolera and Favaro [6] to conclude that

$$
\sup_{x\geq 0}\left|\mathsf{P}\left[\frac{1}{K_m^*}\sum_{i=1}^{K_m^*} Y_i \leq x \,\middle|\, K_m^*\right] - \mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}(x)\right| \leq \frac{C^*_{\alpha,\theta}(n,j)}{K_m^*+1}, \tag{36}
$$

with $C^*_{\alpha,\theta}(n,j) := \frac{1}{2}\sup_{x\in[0,1]}|\mathsf{F}''_{B_{\theta/\alpha+j,n/\alpha-j}}(x)|$, exploiting that $1/K_m^* \leq 2/(K_m^*+1)$. Now, taking the expectation of both sides of (36) yields the first term on the right-hand side of (33). With regard to the second term in (35), we notice that, for any random variable X supported in $(0,+\infty)$ there holds $\mathsf{E}[\mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}(x/X)] = \int_0^1 \mathsf{F}_X(x/t)\,\mathrm{d}\mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}(t)$. Thus,

$$
\begin{aligned}
&\sup_{x\geq 0}\left|\mathsf{E}\left[\mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}\left(\frac{m^\alpha x}{K_m^*}\right)\right] - \mathsf{E}\left[\mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}\left(\frac{x}{S_{\alpha,\theta+n}}\right)\right]\right| \\
&\quad \leq \int_0^1 \sup_{x\geq 0}\left|\mathsf{F}_{K_m^*/m^\alpha}\left(\frac{x}{t}\right) - \mathsf{F}_{S_{\alpha,\theta+n}}\left(\frac{x}{t}\right)\right| \mathrm{d}\mathsf{F}_{B_{\theta/\alpha+j,n/\alpha-j}}(t) \\
&\quad \leq \mathrm{d}_K(\mathsf{F}_{K_m^*/m^\alpha}; \mathsf{F}_{S_{\alpha,\theta+n}}),
\end{aligned}
$$

yielding the second term on the right-hand side of (33). With regards to (34),

$$
\mathsf{E}\left[\frac{1}{K_m^*+1}\right] \leq \left|\mathsf{E}\left[\frac{1}{K_m^*+1}\right] - \mathsf{E}\left[\frac{1}{m^\alpha S_{\alpha,\theta+n}+1}\right]\right| + \frac{1}{m^\alpha}\mathsf{E}\left[\frac{1}{S_{\alpha,\theta+n}}\right]
$$

and we deal separately with the two expectations inside the modulus, taking advantage that we have expressions of the form $\mathsf{E}[1/(1+X)]$. Indeed, setting $\varphi(x) = (1+x)^{-1}$, we have

$\int_0^{+\infty} \varphi(x)\,\mathrm{d}\mathsf{F}(x) = \varphi(0) + \int_0^{+\infty} \varphi'(x)[1-\mathsf{F}(x)]\,\mathrm{d}x$ for every distribution function F supported in $[0,+\infty)$. From $\int_0^{+\infty} |\varphi'(x)|\,\mathrm{d}x = 1$, we get

$$\begin{aligned}
&\left|\mathsf{E}\left[\frac{1}{K_m^*+1}\right] - \mathsf{E}\left[\frac{1}{m^\alpha S_{\alpha,\theta+n}+1}\right]\right| \\
&\quad\le \int_0^{+\infty} \left|\varphi'(x)\right|\mathrm{d}x \cdot \sup_{x\ge 0}\left|\mathsf{F}_{K_m^*}(x) - \mathsf{F}_{m^\alpha S_{\alpha,\theta+n}}(x)\right| \\
&\quad= \mathrm{d}_K\left(\mathsf{F}_{K_m^*}; \mathsf{F}_{m^\alpha S_{\alpha,\theta+n}}\right) = \mathrm{d}_K\left(\mathsf{F}_{K_m^*/m^\alpha}; \mathsf{F}_{S_{\alpha,\theta+n}}\right).
\end{aligned}$$

To conclude, note that $\mathsf{E}[1/S_{\alpha,\theta+n}] = \frac{\alpha\Gamma(\theta+n+1)\Gamma(\alpha(\theta+n-\alpha))}{\Gamma(\frac{\theta+n}{\alpha}+1)\Gamma(\theta+n-\alpha)}$ is finite for all $n\in\mathbb{N}$ whenever $\theta > -\alpha$, so that (34) holds true. The proof is complete by combining (33)–(34), and then by a direct application of Theorem 1. □

As a direct consequence of Theorem 2, it is possible to construct credible intervals containing high posterior probability for the Bayesian nonparametric estimator $\hat{K}_m^{(n)}$. Due to the definition of the Kolmogorov distance, for every $a < b$,

$$\mathsf{P}\left[\frac{K_m^{(n)}}{m^\alpha} \in [a,b] | K_n = j\right] \ge \mathsf{P}\left[S_{\alpha,\theta}(n,j) \in [a,b] | K_n = j\right] - 2m^{-\alpha} C_{\alpha,\theta}(n,j).$$

Therefore, after fixing any confidence level, say $1-\gamma$, it is possible to choose a, b so that

$$\begin{cases} \text{length of } (a,b) \text{ is minimum,} \\ \mathsf{P}\left[S_{\alpha,\theta}(n,j) \in [a,b] | K_n = j\right] \ge 1-\gamma+2m^{-\alpha}C_{\alpha,\theta}(n,j). \end{cases}$$

This provides with a practical, simple tool for assessing uncertainty quantification in the context of the Bayesian nonparametric estimation of $K_m^{(n)}$.

We conclude this section by pointing out a useful generalization of Theorem 2, which follows by the invariance under scaling of the Kolmogorov distance. Let $\lambda(m)$ be an arbitrary function of m such that $\lambda(m)/m^\alpha \to 1$ as $m\to+\infty$. That is, $\lambda(m)$ is asymptotically equivalent to m^α for large m. Moreover, let $\mathsf{F}_m^{(\lambda)}(n,j)$ denote the posterior distribution function of $K_m^{(n)}/\lambda(m)$, given $(X_1,\dots,X_n)$ featuring $K_n = j \le n$ distinct species. For any $\alpha\in(0,1)$, $\theta > -\alpha$ and $n\in\mathbb{N}$ such that $\frac{n}{\alpha} - j \ge 1$, Theorem (2) implies that there exists a constant $C_{\alpha,\theta}^{(\lambda)}(n,j)$, depending solely on n, j, α and θ, such that

$$\mathrm{d}_K\left(\mathsf{F}_m^{(\lambda)}(n,j); \mathsf{F}_{\alpha,\theta}(n,j)\right) \le \frac{C_{\alpha,\theta}^{(\lambda)}(n,j)}{\lambda(m)} \tag{37}$$

for every $m\in\mathbb{N}$. One may apply (37) to identify a function λ that leads to an upper bound for $\mathrm{d}_K(\mathsf{F}_m^{(\lambda)}(n,j); \mathsf{F}_{\alpha,\theta}(n,j))$ which is smaller than the upper bound of Theorem 2. For instance, one may consider $\lambda(m) = (c+m)^\alpha - c^\alpha$, with c being a positive parameter, and then optimize the upper bound in (37) with respect to c.

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