MEAN GEOMETRY FOR 2D RANDOM FIELDS: LEVEL PERIMETER AND LEVEL TOTAL CURVATURE INTEGRALS

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We introduce the level perimeter integral and the total curvature integral associated with a real-valued function f defined on the plane $\mathbb{R}^2$, as integrals allowing to compute the perimeter of the excursion set of f above level t and the total (signed) curvature of its boundary for almost every level t. Thanks to the Gauss–Bonnet theorem, the total curvature is directly related to the Euler characteristic of the excursion set. We show that the level perimeter and the total curvature integrals can be computed in two different frameworks: smooth (at least C^2) functions and piecewise constant functions (also called here elementary functions). Considering 2D random fields (in particular shot noise random fields), we compute their mean perimeter and total curvature integrals, and this provides new "explicit" computations of the mean perimeter and Euler characteristic densities of excursion sets, beyond the Gaussian framework: for piecewise constant shot noise random fields, we give some examples of completely explicit formulas, and for smooth shot noise random fields the provided examples are only partly explicit, since the formulas are given under the form of integrals of some special functions.

1. Introduction. Considering a real-valued stationary 2-dimensional random field $X = (X(x))_{x \in \mathbb{R}^2}$, defined on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we are interested in statistically describing the geometry of its excursions sets, defined for $t \in \mathbb{R}$ by

$$E_X(t) := \{X \geq t\} \subset \mathbb{R}^2,$$

in a given bounded open subset of $\mathbb{R}^2$. In the following, U will denote an open bounded set or simply $\mathbb{R}^2$ when not bounded. We will focus on $E_X(t) \cap U = \{x \in U; X(x) \geq t\}$ as well as $\partial E_X(t) \cap U$ its boundary trace on U, where as usual $\partial E_X(t) = \overline{E_X(t)} \backslash \mathring{E}_X(t)$.

In view of the measurability of X, its excursion sets $E_X(t)$ are a.s. Borel sets for all level $t \in \mathbb{R}$. When moreover X is a.s. upper semicontinuous, these random sets are a.s. closed (see [30]) and, therefore, for bounded U, the set $E_X(t) \cap \overline{U}$ is a compact random set. In dimension 2, the geometry of a compact "nice" set $K \subset \mathbb{R}^2$ with piecewise C^2 boundary ∂K may be described by three functionals: its area $\mathcal{L}(K)$, where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^2$, its perimeter $\text{Per}(K) = \mathcal{H}^1(\partial K)$, where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, and its Euler characteristic $\chi(K)$ that counts the number of connected components minus the number of holes. According to the Gauss–Bonnet theorem, when ∂K is a disjoint finite union of closed curves, $\chi(K)$ is also equal to $\frac{1}{2\pi} \text{TC}(\partial K)$, with $\text{TC}(\partial K)$ the total curvature of the positively oriented curve ∂K (see precise definitions and statements in Definition 2 and Theorem 1). Let us notice that these geometrical features are also used with different conventions according to the setting. For instance, in convex geometry, for K a convex body, intrinsic volumes, respectively, Minkowski's functionals, are defined by $V_0(K) = \chi(K) = 1$,

$V_1(K) = \frac{1}{2}\mathcal{H}^1(\partial K)$ and $V_2(K) = \mathcal{L}(K)$, respectively $W_2(K) = \frac{1}{2}\text{TC}(\partial K) = \pi$, $W_1(K) = \frac{1}{2}\mathcal{H}^1(\partial K)$ and $W_0(K) = \mathcal{L}(K)$ (see [34]), while in differential geometry when K is a compact 2-dimensional submanifold with C^2 smooth boundary, Lipschitz Killing curvatures of K are defined by $C_0(K) = \frac{1}{2\pi}\text{TC}(\partial K)$, $C_1(K) = \frac{1}{2}\mathcal{H}^1(\partial K)$ and $C_2(K) = \mathcal{L}(K)$ and may be extended to sets with positive reach (see [35]).

When considering stationary random sets, it is natural to define corresponding mean density functionals (see Section 9.2 of [33], for instance). They are usually defined considering the limit behavior of a rescaled observation through a large window, say rU, for r large. Actually, this procedure allows to remove boundary effects. We adopt a similar point of view in this paper, by removing boundary effect, using a window U that is open. Hence, for a bounded open U we will focus on the mean area $\mathbb{E}(\mathcal{L}(E_X(t) \cap U))$, the mean perimeter $\mathbb{E}(\text{Per}(E_X(t), U))$, and on the mean total curvature $\mathbb{E}(\text{TC}(\partial E_X(t), U))$ of excursion sets. Densities will then clearly appear as

$$\mathbb{E}(\mathcal{L}(E_X(t) \cap U)) = \overline{\mathcal{L}(E_X(t))}\mathcal{L}(U),$$

$$\mathbb{E}(\text{Per}(E_X(t), U)) = \overline{\text{Per}(E_X(t))}\mathcal{L}(U) \quad \text{and}$$

$$\mathbb{E}(\text{TC}(\partial E_X(t), U)) = \overline{\text{TC}(E_X(t))}\mathcal{L}(U).$$

As far as stationarity is involved, the mean area is not hard to find since

$$\mathbb{E}(\mathcal{L}(E_X(t) \cap U)) = \int_U \mathbb{E}(\mathbf{1}_{X(x) \geq t}) \, dx$$

$$= \mathcal{L}(U)\mathbb{P}(X(0) \geq t).$$

It follows that an exact formula can be set up as soon as the distribution of $X(0)$ (that is the same as any $X(x)$ by stationarity) is known and $\overline{\mathcal{L}(E_X(t))} = \mathbb{P}(X(0) \geq t)$. Now establishing formulas for the mean perimeter and the mean total curvature or Euler characteristic is more difficult and requires additional assumptions on the field. Computing the Euler characteristic of excursion sets of random fields is a problem that has received much attention. Indeed, in many applications, the Euler characteristic is a very useful index of the geometry of the field, as explained for instance in the review paper of R. Adler [2], or in the papers of K. Worsley [37] or [38] where applications in astrophysics or in brain imaging are mentioned.

Despite its "global" definition (the number of connected components minus the number of holes), the Euler characteristic of an excursion set is in fact a purely local quantity related, by Morse theory, to the number of critical points of X in U, or by the Gauss–Bonnet theorem, to the total curvature of the boundary of the excursion set. Here, we will extensively use this second equivalence to obtain explicit computations of the mean Euler characteristic density of the excursion sets of some random fields.

In the framework of Gaussian random fields, the first equivalence is usually used. For stationary isotropic Gaussian random fields, an explicit formula for any level t may be obtained for the expectation of the Euler characteristic density, only depending on the variance and on the second spectral moment of the field. This is an important result with many statistical applications. In particular, for large levels t, the Euler characteristic gives a good approximation of the probability that the supremum of the field is greater than t and can therefore be used as a p-value: this is the Euler characteristic heuristic (see [5], for instance). In a "tour de force," a central limit theorem has recently been established in [19] that proves the accuracy of the estimation over only one sample path as the size of the observation is growing. There are also some interesting results apart from the Gaussian framework for χ^2, F and t-fields [37] as well as stable [3] or infinitely divisible random fields [4], for instance. A test of Gaussianity can therefore be set up using the Euler characteristic of the level sets as proposed in

[16]. However, most of the general results rely on strong smoothness regularity assumptions and on conditional distribution densities that are often difficult to evaluate for non-Gaussian fields.

Now, in this paper, we will be particularly interested in another family of infinitely divisible random fields, that are not Gaussian, namely the *shot noise random fields*. A shot noise random field is defined on $\mathbb{R}^2$ by

$$\forall x \in \mathbb{R}^2, \quad X(x) = \sum_i g_{m_i}(x - x_i),$$

where the x_i are the points of an homogeneous Poisson point process of intensity λ in $\mathbb{R}^2$, and the m_i are "marks," independent of the Poisson point process. Such fields allow explicit computations and may appear, in view of their asymptotic normality in high intensity [21], as a bridge between the Gaussian setting and the discrete models of stochastic geometry such as the Boolean ones [34]. Several results for the computation of the perimeter were obtained in our previous paper [11]. Since the study for all level t is often difficult, we extend here our point of view of working in a weak framework by considering the geometric quantities as functions of the level t. Hence quantities of interest will be given, when it makes sense, by the *mean level perimeter integral* $\mathbb{E}(\mathrm{LP}_X(h, U))$ and the *mean level total curvature integral* $\mathbb{E}(\mathrm{LTC}_X(h, U))$ of X, where the *level perimeter integral* and the *level total curvature integral* are defined, for h a continuous bounded function on $\mathbb{R}$, by

$$(1) \qquad \mathrm{LP}_X(h, U) := \int_{\mathbb{R}} h(t) \operatorname{Per}\big(E_X(t), U\big) \, dt$$

and

$$(2) \qquad \mathrm{LTC}_X(h, U) := \int_{\mathbb{R}} h(t) \operatorname{TC}\big(\partial E_X(t), U\big) \, dt.$$

Let us remark that this allows us to get information on the mean geometry of excursion sets for almost every level t, and hence to give insights on their evolution according to the level. We will in particular be interested in the case of functions $h_u : t \mapsto e^{iut}$ with $u \in \mathbb{R}$, computing then the Fourier transform of $t \mapsto \operatorname{Per}(E_X(t), U)$ and $t \mapsto \operatorname{TC}(\partial E_X(t), U)$. This setting will also allow us to recover and generalize some important results established in the two different frameworks of Boolean models in stochastic geometry and of smooth Gaussian random fields.

In Section 2, we propose a general definition of the level perimeter integral and of the level total curvature integral of a function, that allows to compute the perimeter and the total curvature (and, therefore, the Euler characteristic) of its excursion sets for almost every level. Section 3 is devoted to the results on smooth (random) functions. In particular, our weak framework allows us to get formulas for some isotropic fields, recovering known Gaussian results. We introduce then elementary functions as a particular case of piecewise constant functions (with piecewise smooth discontinuity set) in Section 4 and compute their level perimeter and total curvature integral. We give explicit expressions for elementary shot noise random fields, where the functions g_{m_i} are elementary functions. This allows us to generalize results of the literature (about the Boolean model [28], or about "random configurations" [15]).

Let us finally emphasize that we have made here the deliberate choice of not working in the weakest possible functional framework. Our goal is to work else with smooth or with piecewise constant functions (like the indicator function of a set having a piecewise C^2 boundary for instance). But we believe some of our results can be extended to functions with a weakest regularity. Let us also mention the recent work of R. Lachièze–Rey in [25] and [24] that relates the Euler characteristic to the three-point joint distribution of the random field. And also the paper [23] where R. Lachièze–Rey gives formulas for the Euler characteristic of isotropic shot noise random field that are a.s. Morse functions.

2. General framework.

2.1. *Sets of finite perimeter and finite total curvature.* We consider a Borel set E of $\mathbb{R}^2$ and an open set $U \subset \mathbb{R}^2$. We assume that the indicator function $\mathbf{1}_E$ is a function in $\text{SBV}(U)$. Let us recall (see [7]) that a function f belongs to $\text{SBV}(U)$, the space of *special functions of bounded variation in U* if $f \in L^1(U)$ and has its distributional derivative representable by a finite Radon measure in U, that is,

$$\int_U f(x) \frac{\partial \phi}{\partial x_l}(x)\, dx = - \int_U \phi(x) D_l f(dx) \quad \forall \phi \in C_c^1(U, \mathbb{R}), \forall l = 1, 2$$

for some $\mathbb{R}^2$-valued measure $Df = (D_1 f, D_2 f)$, such that

$$Df = \nabla f \mathcal{L} + (f^+ - f^-)\nu_f \mathcal{H}^1 \angle J_f,$$

where $\nabla f \mathcal{L}$ is the absolutely continuous part of the Radon measure Df with respect to the Lebesgue measure $\mathcal{L}$ and $(f^+ - f^-)\nu_f \mathcal{H}^1 \angle J_f$ is the *jump part* of Df, with J_f the set of *approximate jump points* of f. The set J_f is included in the *approximate discontinuity set $\mathcal{S}_f$* that is the set of points where f is not *approximately continuous* (see [7] Proposition 3.64, page 160 and Theorem 3.83, page 176 for more details).

This framework, used in our previous paper [11], is convenient to define the perimeter of a set E in U such that $\mathbf{1}_E \in \text{SBV}(U)$ as

$$\text{Per}(E, U) := \|D\mathbf{1}_E\|(U)$$
$$= \sup\left\{ \int_U \mathbf{1}_E \operatorname{div} \varphi\, dx \,|\, \varphi \in C_c^1(U, \mathbb{R}^2), \|\varphi\|_\infty \le 1 \right\} < +\infty.$$

It follows that, denoting by $\mathcal{S}_{\mathbf{1}_E}$ the approximate discontinuity set of $\mathbf{1}_E$, the set $\mathcal{S}_{\mathbf{1}_E} \cap U$ (its trace in U) is included in $\partial E \cap U$ the boundary of E in U. In order to get information linked with length and Euler characteristic we make the stronger assumption that $\partial E \cap U$ coincides with $\mathcal{S}_{\mathbf{1}_E} \cap U$ (which is equivalent to say that the discontinuity points are exactly the approximate discontinuity points in U), and is a piecewise C^2 plane curve.

Let us recall here some basic facts and definitions about plane curves, following [17]. When Γ is a piecewise C^2 simple oriented curve (possibly closed) we define a *regular point* or a *corner point* x of Γ by the following properties:

- *Regular point*: one can find an arc-length C^2 parametrization $\gamma : (0, \varepsilon) \to \Gamma$ with $x = \gamma(s)$ for some $s \in (0, \varepsilon)$, with $\varepsilon > 0$, and a normal vector $\nu_\Gamma(x) = \gamma'(s)^\perp \in S^1$ with $\gamma'(s)^\perp$ the $+\frac{\pi}{2}$ rotation of the tangent vector $\gamma'(s)$. The normal cone (defined for sets with positive reach [35]) of Γ at x is given by $\text{Nor}(\Gamma, x) = \{-\nu_\Gamma(x)\}$. The signed curvature $\kappa_\Gamma(x)$ of Γ at $x = \gamma(s)$ is then defined as

$$\kappa_\Gamma(x) = \langle \gamma''(s), \nu_\Gamma(x)\rangle,$$

  where $\langle \cdot, \cdot \rangle$ is the usual Euclidean scalar product on $\mathbb{R}^2$. Note that since γ is an arc-length parametrization we have $\mathcal{H}^1(\gamma(0, \varepsilon)) = \varepsilon$.

- *Corner point*: one can find a simple continuous arc-length parametrization $\gamma : (-\varepsilon, \varepsilon) \to \Gamma$ such that $x = \gamma(0)$ with γ being C^2 on $(-\varepsilon, \varepsilon)\backslash\{0\}$ and γ' admits limits $\gamma'(0^-) \in S^1$ and $\gamma'(0^+) \in S^1$ at 0, with $\nu_\Gamma^-(x) := \gamma'(0^-)^\perp$ and $\nu_\Gamma^+(x) := \gamma'(0^+)^\perp$ linearly independent (no "cusp") in S^1 such that the normal cone of Γ at x is given by $\text{Nor}(\Gamma, x) = \{-p\nu_\Gamma^-(x) - q\nu_\Gamma^+(x); p, q \ge 0\} \cap S^1$. We then define $\beta_\Gamma(x) \in (0, \pi)$ the angle of the cone $\text{Nor}(\Gamma, x)$, corresponding to the size of the jump of ν_Γ at point x and $\alpha_\Gamma(x) = \pm\beta_\Gamma(x) \in (-\pi, \pi)$ the *turning angle* at x, where the sign is given according to the orientation of the curve. Note that we also have $\mathcal{H}^1(\gamma((-\varepsilon, \varepsilon))) = 2\varepsilon$.

We denote by $\mathcal{R}_\Gamma$ the set of regular points and by $\mathcal{C}_\Gamma$ the set of corner points of the curve Γ. In the following, we will also denote by $\mathcal{H}^0$ the zero-dimensional Hausdorff measure, that is just the counting measure.

DEFINITION 1 (Piecewise regular curve). We say that Γ is a *piecewise regular* curve if it is a simple oriented curve given by a finite union of piecewise C^2 disjoint Jordan curves (i.e., simple closed curves) with a finite number of corner points such that $\Gamma = \mathcal{R}_\Gamma \cup \mathcal{C}_\Gamma$. It has a finite length given by $\mathcal{H}^1(\Gamma) = \mathcal{H}^1(\mathcal{R}_\Gamma)$ and a finite absolute total curvature on any Borel set $U \subset \mathbb{R}^2$, given by

$$\mathrm{TaC}(\Gamma, U) = \int_{\mathcal{R}_\Gamma \cap U} |\kappa_\Gamma(x)| \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_\Gamma \cap U} |\alpha_\Gamma(x)| < +\infty.$$

Notice that the definition of TaC is the same as the one introduced by Milnor in [29] for closed curves. But here, in this work, we will pay a particular attention to the signed total curvature, and not to its absolute value.

DEFINITION 2 (Elementary set, perimeter and total curvature). We say that a Borel set $E \subset \mathbb{R}^2$ is an elementary set if ∂E is a piecewise regular curve (Definition 1) positively oriented in such a way that the normals are oriented toward E. It follows that, for any open bounded set $U \subset \mathbb{R}^2$, then $\mathbf{1}_E \in \mathrm{SBV}(U)$, the length of the curve $\partial E \cap U$ is given by

$$\mathcal{H}^1(\partial E \cap U) = \mathcal{H}^1(\mathcal{R}_{\partial E} \cap U) = \mathrm{Per}(E, U) < +\infty,$$

and its total curvature in U is

$$\mathrm{TC}(\partial E, U) := \int_{\mathcal{R}_{\partial E} \cap U} \kappa_{\partial E}(x) \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_{\partial E} \cap U} \alpha_{\partial E}(x) \in \mathbb{R}.$$

The link between $\mathcal{H}^1(\partial E \cap U)$ and $\mathrm{Per}(E, U)$ follows from Gauss–Green theorem (see Section 3.3 of [7]). Our definition of total curvature is the same as the one of Santaló in [31], Chapter 7. The total curvature is intrinsic, it does not depend on the parametrization of the curve. But it depends on its orientation: if we reverse the orientation of the curve then its total curvature is changed into its opposite. In particular, when E is an elementary set, since $\partial E = \partial E^c$, its complement $E^c = \mathbb{R}^2 \backslash E$ is also an elementary set, with for all U open bounded set:

$$\mathrm{Per}(E^c, U) = \mathrm{Per}(E, U), \qquad \mathrm{TaC}(\partial E^c, U) = \mathrm{TaC}(\partial E, U) \quad \text{and}$$
$$\mathrm{TC}(\partial E^c, U) = -\mathrm{TC}(\partial E, U).$$

On Figure 1, we give examples of sets that are or are not elementary according to our definition.

The link between the total curvature and the Euler Characteristic is given by the Gauss–Bonnet theorem. Since the boundary of an elementary set is a finite disjoint union of Jordan piecewise C^2 curves we can apply Gauss–Bonnet theorem (as stated in [17], page 274) for a regular region.

THEOREM 1 (Gauss–Bonnet theorem). *Let E be a bounded elementary set that is a regular region* (*meaning that $E = \overline{\mathring{E}}$*), *then the Euler characteristic of E is given by*

$$\chi(E) = \frac{1}{2\pi} \mathrm{TC}(\partial E) = \frac{1}{2\pi} \left(\int_{\mathcal{R}_{\partial E}} \kappa_{\partial E}(x) \mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_{\partial E}} \alpha_{\partial E}(x) \right).$$
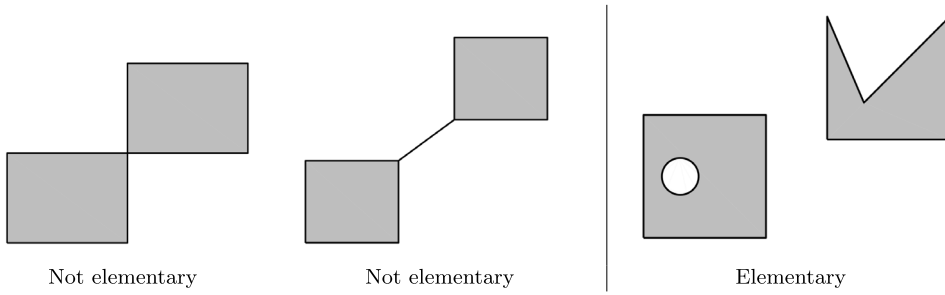
Not elementary   Not elementary   Elementary

FIG. 1.   *Left*: *two sets that are not elementary sets according to Definition* 2: *the boundary of the first set has one multiple point*; *for the second set, its boundary is not the union of disjoint piecewise* C^2 *Jordan curves. Right*: *an elementary set.*

REMARK 1.   It is a well-known result of differential geometry of plane curves that the total curvature of any regular simple closed curve is 2π or -2π (depending on the orientation of the curve). This result is sometimes called Hopf's Umlaufsatz, or also the theorem of turning tangents ([17], page 396).

REMARK 2.   Note that when E is a bounded elementary set and a regular region, then we have seen that $\partial E = \partial E^c$ and $\mathrm{TC}(\partial E^c, U) = -\mathrm{TC}(\partial E, U)$ for any open set U such that $E \subset U$. In contrast, for Euler characteristic, we have to consider a compact set. For instance, if V is a closed rectangle we can take $V \setminus \mathring{E}$, and that yields $\chi(V \setminus \mathring{E}) = 1 - \chi(E)$.

The notion of elementary set is stable under union and intersection provided that the sets are in "generic" position. Indeed, let us introduce the following definition.

DEFINITION 3 (Generic position).   Let E and F be two elementary sets. We say that E and F are in generic position if $\partial E \cap \partial F$ is a finite set, included in $\mathcal{R}_{\partial E} \cap \mathcal{R}_{\partial F}$ and if for $x \in \partial E \cap \partial F$, the normals $\nu_{\partial E}(x)$ and $\nu_{\partial F}(x)$ are not colinear.

When $n \geq 3$, we say that n elementary sets $E_1, \ldots, E_n$ are in generic position if for any subset $J \subset \{1, \ldots, n\}$ with cardinal greater than 3 we have $\bigcap_{j \in J} \partial E_j = \varnothing$ and if each pair of sets are in generic position.

It follows that if $E_1, \ldots, E_n$ are elementary sets in generic position, then $\bigcap_{i=1}^n E_i$ and $\bigcup_{i=1}^n E_i$ are also elementary sets.

2.2. *Level integrals for excursion sets.*

DEFINITION 4 (Level perimeter and total curvature integrals).   Let $U \subset \mathbb{R}^2$ be an open set and let $f \in \mathrm{SBV}(U)$ be a real-valued special function of bounded variation defined on U. For $t \in \mathbb{R}$, we define the excursion set of f for the level t as

$$E_f(t) := \{x \in U; f(x) \geq t\} \subset \mathbb{R}^2.$$

We assume that for almost every $t \in \mathbb{R}$, the set $E_f(t)$ is an elementary set in the sense of Definition 2 and that $t \mapsto \mathrm{TaC}(\partial E_f(t), U)$ is an integrable function on $\mathbb{R}$. We then say that the function f is of finite level total curvature integral on U. The level perimeter integral and the level total curvature integral of f are defined for any bounded continuous function h on $\mathbb{R}$ by

$$\mathrm{LP}_f(h, U) = \int_{\mathbb{R}} h(t)\,\mathrm{Per}(E_f(t), U)\,dt \quad \text{and}$$

(3)

$$\mathrm{LTC}_f(h, U) = \int_{\mathbb{R}} h(t)\,\mathrm{TC}(\partial E_f(t), U)\,dt.$$

We simply denote $V_f(U)$ for $\mathrm{LP}_f(1, U)$ ($=\|Df\|(U)$ by the co-area formula [7]) and $\mathrm{LTC}_f(U)$ for $\mathrm{LTC}_f(1, U)$.

Let us remark that when $t > \sup_U f$, then $E_f(t) \cap U = \varnothing$ and, therefore, $\mathrm{Per}(E_f(t), U) = \mathrm{TC}(\partial E_f(t), U) = 0$. On the other hand, when $t \leq \inf_U f$, then $E_f(t) \cap U = U$, and thus $\partial E_f(t) \cap U = \varnothing$. Therefore, we also have $\mathrm{Per}(E_f(t), U) = \mathrm{TC}(\partial E_f(t), U) = 0$. This shows that the perimeter and total curvature are 0 for levels t outside the range of f.

Let us also notice that when H is a C^1 diffeomorphism on $\mathbb{R}$ with bounded nonnegative derivative $h = H'$, by a simple change of variable, the function $H \circ f$ is also of special variation and of finite level total curvature integral on U with $E_{H \circ f}(t) = E_f(H^{-1}(t))$ so that

$$V_{H \circ f}(U) = \mathrm{LP}_f(h, U) \quad \text{and} \quad \mathrm{LTC}_{H \circ f}(U) = \mathrm{LTC}_f(h, U).$$

Link with Euler integral. Due to the additivity property of the Euler characteristic (valid for instance on the class of polyconvex sets; see [14])

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B),$$

it is natural to set up an integration theory with respect to the Euler characteristic [26, 36]. However, since χ is only finitely additive, a careful choice of integrands must be done. This problem was tackled by defining the class of constructible functions [32], then extended by the class of "tame" real-valued functions in [9]. Following this framework, Bobrowski and Borman obtained in [13] the first probabilistic statement about the persistent homology generated by sublevel sets. We briefly recall the definitions used in [13] for comparison with our setting. When f is a real continuous function defined on a compact topological set S, it is said to be a tame function if the homotopy types of $E_f(t) = \{f \geq t\}$ and $\{f \leq t\}$ change only finitely many times as t varies over $\mathbb{R}$, and the Euler characteristic of each set is always finite. For such a function, a lower and upper Euler integrals are defined by

$$\int_S f \lfloor d\chi \rfloor = \int_0^{+\infty} \left(\chi(f \geq t) - \chi(f < -t) \right) dt,$$

$$\int_S f \lceil d\chi \rceil = \int_0^{+\infty} \left(\chi(f > t) - \chi(f \leq t) \right) dt,$$

where $\chi(f \geq t) = \chi(E_f(t))$, $\chi(f < t) = \chi(S) - \chi(f \geq t)$, etc. Note that when $U = (0, T)^2$ with some $T > 0$, and $S = \overline{U}$, we have $\chi(f \geq t) = \chi(S) = 1$ for any $t \leq \min_S f$, and thus $t \mapsto \chi(f \geq t)$ is not integrable on $\mathbb{R}$, explaining the above definition of Euler integrals.

In contrast, we can simply define $\mathrm{LTC}_f(U)$ for f a function of special bounded variation and finite level total curvature on U. Of course, $\frac{1}{2\pi} \mathrm{TC}(\partial E_f(t), U)$ will not coincide with $\chi(E_f(t) \cap U)$ when the boundary of the excursion set is not included in the observation window U. See also an illustration of this fact on Figure 2. But it can be seen as a "modified" Euler characteristic, in a sense very similar to the one used in the book of Adler and Taylor [6] or in the paper of Estrade and León [19], where critical points in U are only taken into account, and not the ones on the boundary of U. Moreover, when considering large domains (that is rU for r going to infinity) the total curvature (in expectation) will grow like $r^2 \mathcal{L}(U)$ whereas the sum of the turning angles on $\partial(rU)$ will (in expectation also) grow like $r\mathcal{H}^1(\partial U)$, being negligible for large r.

We will show how the perimeter and the level total curvature integrals can be computed in different situations and we will apply it for computing perimeter and Euler characteristic densities of some stationary fields. The first situation is the case of smooth (at least C^2) functions, and the second situation is the case of sums of piecewise constant functions (also called elementary functions).
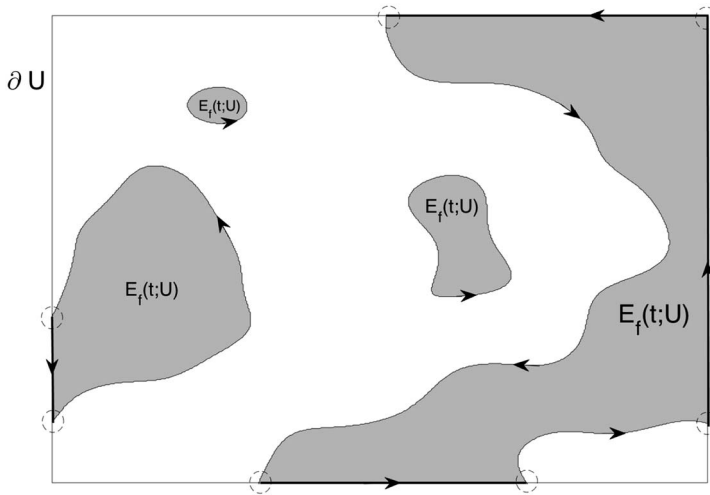
FIG. 2. *By the Gauss–Bonnet theorem, the Euler characteristic of the excursion set $E_f(t) \cap U$ (in gray) is equal to the total curvature of its boundary in the open rectangular domain U plus the turning angles at the points where $\partial(E_f(t) \cap U)$ meets ∂U and also plus $\pi/2$ for each of the corners of the rectangle U that are in $E_f(t)$. All these special points are the ones marked by the small dashed circles on the figure.*

3. Level integrals of smooth functions and random fields.

3.1. *The case of smooth functions.* We start by considering the case of smooth functions. In the sequel, for f a C^2 function we denote by ∇f its gradient vector and by $D^2 f$ its Hessian matrix.

PROPOSITION 1. *Let U be an open bounded subset of $\mathbb{R}^2$ such that its boundary is a piecewise regular curve. Let f be a C^2 function defined on an open set containing $\overline{U}$. Then f is of special bounded variation and of finite level total curvature integral on U with*

$$V_f(U) = \int_U \|\nabla f(x)\| \, dx \quad and \quad \mathrm{LTaC}_f(U) \le \int_U \|D^2 f(x)\|_2 \, dx,$$

where $\| \cdot \|_2$ is the matrix norm subordinated to the Euclidean norm. Moreover, for h a bounded continuous function on $\mathbb{R}$, the level perimeter and total curvature integrals of f are given by

$$\mathrm{LP}_f(h, U) = \int_U h(f(x)) \|\nabla f(x)\| \, dx \quad and$$

$$\mathrm{LTC}_f(h, U) = -\int_U h(f(x)) \frac{D^2 f(x) \cdot (\nabla f(x)^\perp, \nabla f(x)^\perp)}{\|\nabla f(x)\|^2} \mathbf{1}_{\|\nabla f(x)\| > 0} \, dx,$$

where if $A = (a_{ij})_{1 \le i, j \le 2}$ is a 2×2 symmetric matrix and $y = (y_1, y_2) \in \mathbb{R}^2$, we use the notation

$$A \cdot (y, y) = {}^t y A y = a_{11} y_1^2 + a_{22} y_2^2 + 2 a_{12} y_1 y_2.$$

PROOF. Note that since f is C^1 on an open set containing $\overline{U}$ we have that $f \in \mathrm{SBV}(U)$, and thus the set

$$E_f(t) := \{x \in U; f(x) \ge t\}$$

is of finite perimeter in U for a.e. $t \in \mathbb{R}$ by the co-area formula (see Theorem 3.40 [7]). Moreover, since f is actually C^2 on an open set containing $\overline{U}$, by Morse–Sard theorem (see

[22], page 69 for instance), the set of critical values of f and of critical values of f restricted to ∂U, denoted as $f_{/\partial U}$ has measure 0 in $\mathbb{R}$. Hence for a.e. $t \in \mathbb{R}$, for all points $x \in U$ such that $f(x) = t$ then $\nabla f(x) \neq 0$ and for all $x \in \mathcal{R}_{\partial U}$ such that $f(x) = t$ then $\langle \nabla f(x), n_{\partial U}(x)^{\perp} \rangle \neq 0$. Let t be such a noncritical value. Let us show that $E_f(t)$ is an elementary set. We first notice that

$$\partial E_f(t) = \{x \in U; f(x) = t\} \cup \{x \in \partial U; f(x) \geq t\}.$$

By the implicit function theorem, if $x \in \partial E_f(t) \cap U$, it is a regular point and one can find a parametrization γ given by an implicit form $f(\gamma(s)) = t$, with $x = \gamma(s)$, normal vector $\gamma'(s)^{\perp} = \nabla f(x)/\|\nabla f(x)\|$ and curvature given by

$$\kappa_f(x) := -\frac{D^2 f(x) \cdot (\nabla f^{\perp}(x), \nabla f^{\perp}(x))}{\|\nabla f(x)\|^3}.$$

From the compactness of $\partial E_f(t)$ and the fact that t is not a critical value for both f and $f_{/\partial U}$, we deduce that $\partial E_f(t)$ is the finite union of disjoint piecewise C^2 Jordan curves that have a finite number of corner points (more precisely this number is bounded by $\mathcal{H}^0(\mathcal{C}_{\partial U})$ plus the number of $x \in \partial U$ such that $f(x) = t$, which is finite). Therefore, $E_f(t)$ is an elementary set and we have that

$$\mathrm{TaC}(\partial E_f(t), U) = \int_{\partial E_f(t) \cap U} |\kappa_f(x)| \mathcal{H}^1(dx) < +\infty$$

$$\text{and} \quad \mathrm{TC}(\partial E_f(t), U) = \int_{\partial E_f(t) \cap U} \kappa_f(x) \mathcal{H}^1(dx).$$

Now let us define κ_f as a measurable function on U by setting for all $x \in U$

$$(4) \qquad \kappa_f(x) = -\frac{D^2 f(x) \cdot (\nabla f^{\perp}(x), \nabla f^{\perp}(x))}{\|\nabla f(x)\|^3} \mathbf{1}_{\|\nabla f(x)\| > 0}.$$

Let us recall the co-area formula for Lipschitz mappings (see [20], page 117, for instance): for any nonnegative measurable or $\mathcal{L}$-integrable function g, the function $t \mapsto \int_{\partial E_f(t) \cap U} g(x) \mathcal{H}^1(dx)$ is measurable and

$$\int_U g(x) \|\nabla f(x)\| dx = \int_{\mathbb{R}} \int_{\partial E_f(t) \cap U} g(x) \mathcal{H}^1(dx) dt.$$

Taking $g = 1$, we recover the co-area formula:

$$\int_U \|\nabla f(x)\| dx = \int_{\mathbb{R}} \mathcal{H}^1(\partial E_f(t) \cap U) dt = V_f(U),$$

while for $g = |\kappa_f|$,

$$\mathrm{LTaC}_f(U) = \int_{\mathbb{R}} \int_{\partial E_f(t) \cap U} |\kappa_f(x)| \mathcal{H}^1(dx) = \int_U |\kappa_f(x)| \|\nabla f(x)\| dx$$

$$\leq \int_U \|D^2 f(x)\|_2 dx$$

in view of (4). Therefore, $t \mapsto \mathrm{TaC}(\partial E_f(t), U)$ is integrable on $\mathbb{R}$ and f is of finite level total curvature integral on U. Moreover, for h a bounded continuous function on $\mathbb{R}$, using again twice the co-area formula but now with $\max(g, 0)$ and $-\min(g, 0)$ for $g = h \circ f$ or $g = (h \circ f)\kappa_f$, and subtracting we get

$$\mathrm{LP}_f(h, U) = \int_U h(f(x)) \|\nabla f(x)\| dx,$$

$$\text{LTC}_f(h, U) = \int_U h(f(x)) \kappa_f(x) \|\nabla f(x)\| \, dx$$

$$= -\int_U h(f(x)) \frac{D^2 f(x) \cdot (\nabla f(x)^\perp, \nabla f(x)^\perp)}{\|\nabla f(x)\|^2} \mathbf{1}_{\|\nabla f(x)\| > 0} \, dx.$$

This can clearly be extended to complex valued functions h by linearity. $\square$

3.2. *A general result for smooth stationary random fields.* In this section, we consider a smooth stationary random field X defined on $\mathbb{R}^2$. Let us introduce some notation for the derivatives of X. A point $x \in \mathbb{R}^2$ is defined by its two coordinates $x = (x_1, x_2)$ and we denote for $j, k = 1, 2$,

$$X_j := \frac{\partial X}{\partial x_j} \quad \text{and} \quad X_{jk} := \frac{\partial^2 X}{\partial x_j \partial x_k}.$$

With these notation, it follows that $\nabla X = \binom{X_1}{X_2}$ and $D^2 X = \left(\begin{smallmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{smallmatrix} \right)$. Note that since X is stationary, for any $x \in \mathbb{R}^2$,

$$(X(x), \nabla X(x), D^2 X(x)) \stackrel{d}{=} (X(0), \nabla X(0), D^2 X(0)).$$

When X, ∇X and $D^2 X$ have also finite second-order moment, $X(x)$ and $\nabla X(x)$ are not correlated, as well as $\nabla X(x)$ and $D^2 X(x)$ (see [1], page 31, for instance). This is very useful for Gaussian fields since it implies that $\nabla X(x)$ is independent from $(X(x), D^2 X(x))$.

Using the result of Section 3.1 and the stationarity of X, we have the following formula.

THEOREM 2. *Let U be an open bounded subset of $\mathbb{R}^2$ such that its boundary is a piecewise regular curve. Let X be a stationary C^2 random field on $\mathbb{R}^2$, such that $X(0)$, $X_j(0)$ and $X_{jk}(0)$ have finite expectations for all $j, k = 1, 2$. Then, a.s., $X \in \text{SBV}(U)$ with $DX = \nabla X \mathcal{L}$ and for a.e. $t \in \mathbb{R}$, the random variables $\text{Per}(E_X(t), U)$ and $\text{TC}(\partial E_X(t), U)$ have finite expectation such that for all h bounded continuous function on $\mathbb{R}$, one has*

$$\mathbb{E}(\text{LP}_X(h, U)) = \int_\mathbb{R} h(t) \mathbb{E}(\text{Per}(E_X(t), U)) \, dt = \mathcal{L}(U) \overline{\text{LP}_X}(h) \quad \text{and}$$

$$\mathbb{E}(\text{LTC}_X(h, U)) = \int_\mathbb{R} h(t) \mathbb{E}(\text{TC}(\partial E_X(t), U)) \, dt = \mathcal{L}(U) \overline{\text{LTC}_X}(h)$$

with

$$\overline{\text{LP}_X}(h) = \mathbb{E}(h(X(0)) \|\nabla X(0)\|) \quad \text{and}$$

$$\overline{\text{LTC}_X}(h) = -\mathbb{E}\left(h(X(0)) \frac{D^2 X(0) \cdot (\nabla X(0)^\perp, \nabla X(0)^\perp)}{\|\nabla X(0)\|^2} \mathbf{1}_{\|\nabla X(0)\| > 0} \right).$$

It follows that when the field X is isotropic, that is, $X \circ A \stackrel{fdd}{=} X$ for all orthogonal matrices A, then the above formulas reduce to, $\forall j = 1, 2$,

$$\overline{\text{LP}_X}(h) = \frac{\pi}{2} \mathbb{E}(h(X(0)) |X_j(0)|) \quad \text{and}$$

$$\overline{\text{LTC}_X}(h) = \overline{\text{LTC}_X^{(1)}}(h) + \overline{\text{LTC}_X^{(2)}}(h),$$

where

$$\overline{\text{LTC}_X^{(1)}}(h) = -\mathbb{E}(h(X(0)) X_{jj}(0) \mathbf{1}_{\|\nabla X(0)\| > 0}),$$

$$\overline{\text{LTC}_X^{(2)}}(h) = 4\mathbb{E}\left(h(X(0)) \frac{X_1(0) X_2(0)}{\|\nabla X(0)\|^2} X_{12}(0) \mathbf{1}_{\|\nabla X(0)\| > 0} \right).$$

It follows that when $X(0)$ admits a probability density $p_{X(0)}$, we get for the densities defined in the Introduction, for almost every $t \in \mathbb{R}$, $\forall j = 1, 2$,

$$\overline{\text{Per}(E_X(t))} = \frac{\pi}{2} \mathbb{E}(|X_j(0)| \mid X(0) = t) p_{X(0)}(t),$$

$$\overline{\text{TC}(E_X(t))} = \left[-\mathbb{E}(X_{jj}(0) \mathbf{1}_{\|\nabla X(0)\| > 0} \mid X(0) = t) \right.$$

$$\left. + 4\mathbb{E}\left(\frac{X_1(0) X_2(0)}{\|\nabla X(0)\|^2} X_{12}(0) \mathbf{1}_{\|\nabla X(0)\| > 0} \mid X(0) = t \right) \right] p_{X(0)}(t).$$

REMARK. Let us remark that assuming that the field X is isotropic and symmetric, that is, $X \overset{fdd}{=} -X$, we get $\overline{\text{LTC}_X}(h) = 0$ for any even bounded function h and in particular for $h = 1$. If moreover $(X(0), X_1(0), X_2(0), X_{12}(0)) \overset{d}{=} (X(0), -X_1(0), X_2(0), X_{12}(0))$, we get $\overline{\text{LTC}_X^{(2)}}(h) = 0$ for any bounded function h. This is in particular the case for isotropic fields satisfying $\nabla X(0)$ independent from $(X(0), X_{12}(0))$, as centered stationary isotropic Gaussian random fields.

PROOF OF THEOREM 2. According to Proposition 1, since X is a.s C^2, it is a.s. of special bounded variation and of finite level curvature integral on U with

$$V_X(U) = \int_U \|\nabla X(x)\| \, dx \quad \text{and} \quad \text{LTaC}_X(U) \leq \int_U \|D^2 X(x)\|_2 \, dx \quad \text{a.s.}$$

Since X is stationary, the finite expectation assumption implies that $V_X(U)$ and $\text{LTaC}_X(U)$ are nonnegative random variables with finite expectation. By Fubini's theorem, this implies that $(t, \omega) \mapsto \text{Per}(E_{X(\omega)}(t), U) \in L^1(\mathbb{R} \times \Omega)$ and $(t, \omega) \mapsto \text{TaC}(\partial E_{X(\omega)}(t), U) \in L^1(\mathbb{R} \times \Omega)$ so that we also have $(t, \omega) \mapsto \text{TC}(\partial E_{X(\omega)}(t), U) \in L^1(\mathbb{R} \times \Omega)$. Moreover, a.s., for any h bounded continuous function on $\mathbb{R}$,

$$\text{LP}_X(h, U) = \int_{\mathbb{R}} h(t) \text{Per}(E_X(t), U) \, dt \quad \text{and}$$

$$\text{LTC}_X(h, U) = \int_{\mathbb{R}} h(t) \text{TC}(\partial E_X(t), U) \, dt,$$

with

$$\text{LP}_X(h, U) = \int_U h(X(x)) \|\nabla X(x)\| \, dx \quad \text{and}$$

$$\text{LTC}_X(h, U) = -\int_U h(X(x)) \frac{D^2 X(x) \cdot (\nabla X(x)^\perp, \nabla X(x)^\perp)}{\|\nabla X(x)\|^2} \mathbf{1}_{\|\nabla X(x)\| > 0} \, dx.$$

Hence, taking the expectation, Fubini's theorem and the stationarity of X imply the results.

Under the assumption that the field is isotropic, we can exploit further on this formula. First, let us recall that by Taylor formula, since X is a.s. C^2 we have a.s. for all $x, z \in \mathbb{R}^2$,

$$X(x + z) = X(x) + \langle \nabla X(x), z \rangle + \frac{1}{2} D^2 X(x) \cdot (z, z) + o_{\|z\| \to 0}(\|z\|^2).$$

In particular, we obtain that, for any orthogonal matrix A,

$$\nabla(X \circ A)(x) = {}^t A \nabla X(Ax) \text{ and } D^2(X \circ A)(x) = {}^t A (D^2 X)(Ax) A.$$

Since $X \circ A \overset{fdd}{=} X$, we deduce that

$$(X(x), \nabla X(x), D^2 X(x)) \overset{d}{=} (X(Ax), {}^t A \nabla X(Ax), {}^t A (D^2 X)(Ax) A),$$

and specifying to $x = 0$, it follows that

(5) $$(X(0), \nabla X(0), D^2 X(0)) \overset{d}{=} (X(0), {}^t A \nabla X(0), {}^t A (D^2 X)(0) A).$$

Hence $(X(0), \nabla X(0)) \overset{d}{=} (X(0), {}^t A \nabla X(0))$ and for any $\theta \in [0, 2\pi)$, denoting $u(\theta) = (\cos\theta, \sin\theta) \in S^1$, one has $(X(0), X_j(0)) \overset{d}{=} (X(0), \langle u(\theta), \nabla X(0)\rangle)$, according to orthogonal invariance (see Proposition 4.8 of [12]). Moreover, $\|\nabla X(0)\| = \frac{1}{4} \int_0^{2\pi} |\langle u(\theta), \nabla X(0)\rangle| \, d\theta$ so that we deduce

$$\mathbb{E}(h(X(0))\|\nabla X(0)\|) = \frac{1}{4} \int_0^{2\pi} \mathbb{E}(h(X(0))|\langle u(\theta), \nabla X(0)\rangle|) \, d\theta$$
$$= \frac{\pi}{2} \mathbb{E}(h(X(0))|X_j(0)|),$$

and the result for $\overline{\mathrm{LP}_X}(h)$ follows.

Now, let us consider $\overline{\mathrm{LTC}_X}(h)$. We introduce the random variable Θ with values in $2\pi\mathbb{T}$ (identified with $[0, 2\pi)$), such that when $\|\nabla X(0)\| > 0$,

$$\nabla X(0) = \begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix} = \|\nabla X(0)\| \begin{pmatrix} \cos\Theta \\ \sin\Theta \end{pmatrix}.$$

For the sake of brevity, we drop the point notation (0) in the sequel of the proof. Then

$$\mathbb{E}\left(h(X)\frac{D^2 X \cdot (\nabla X^\perp, \nabla X^\perp)}{\|\nabla X\|^2}\mathbf{1}_{\|\nabla X\|>0}\right)$$
$$= \mathbb{E}(h(X)(X_{11}\sin^2\Theta + X_{22}\cos^2\Theta - 2X_{12}\sin\Theta\cos\Theta)\mathbf{1}_{\|\nabla X\|>0})$$
$$= \mathbb{E}\left(h(X)\left(\frac{X_{11} + X_{22}}{2} + \frac{X_{22} - X_{11} + 2iX_{12}}{4}e^{2i\Theta}\right.\right.$$
$$\left.\left. + \frac{X_{22} - X_{11} - 2iX_{12}}{4}e^{-2i\Theta}\right)\mathbf{1}_{\|\nabla X\|>0}\right).$$

Hence introducing the complex random variables $J = \|\nabla X\|e^{i\Theta}\mathbf{1}_{\|\nabla X\|>0}$ and $K = \frac{1}{4}(X_{22} - X_{11} - 2iX_{12})$, the rotation invariance (5) implies that for any $\theta \in [0, 2\pi)$,

(6) $$(X, J, K) \overset{d}{=} (X, e^{i\theta}J, e^{2i\theta}K).$$

Now, we remark that

$$\mathbb{E}\left(h(X)\frac{D^2 X \cdot (\nabla X^\perp, \nabla X^\perp)}{\|\nabla X\|^2}\mathbf{1}_{\|\nabla X\|>0}\right) = \alpha_0(h) + \alpha_2(h) + \overline{\alpha_2(\overline{h})},$$

where

$$\alpha_0(h) = \mathbb{E}\left(h(X)\left(\frac{X_{11} + X_{22}}{2}\right)\mathbf{1}_{\|\nabla X\|>0}\right) = \mathbb{E}(h(X)X_{jj}\mathbf{1}_{\|\nabla X\|>0})$$
$$= -\overline{\mathrm{LTC}_X^{(1)}}(h),$$

for any $j \in \{1, 2\}$, using the fact that $(X, X_{11}) \overset{d}{=} (X, X_{22})$ by (5), and

(7) $$\alpha_2(h) = \mathbb{E}(h(X)\overline{K}e^{2i\Theta}\mathbf{1}_{\|\nabla X\|>0}).$$

But (6) implies that for all $n \in \mathbb{Z}$ with $n \neq 2$ and $\theta \in [0, 2\pi)$, we have $\mathbb{E}(h(X)\overline{K}e^{in\Theta} \times \mathbf{1}_{\|\nabla X\|>0}) = e^{i(n-2)\theta}\mathbb{E}(h(X)\overline{K}e^{ni\Theta}\mathbf{1}_{\|\nabla X\|>0}) = 0$. It follows that

$$\alpha_2(h) = \mathbb{E}(h(X)\overline{K}(e^{2i\Theta} - e^{-2i\Theta})\mathbf{1}_{\|\nabla X\|>0}) = 2i\mathbb{E}(h(X)\overline{K}\sin(2\Theta)\mathbf{1}_{\|\nabla X\|>0}).$$

Finally, we deduce that when h is real-valued, denoting by $\Re$ the real part of a complex number, since $\Re(i\overline{K}) = -\frac{1}{2}X_{12}$,

$$2\Re\alpha_2(h) = -2\mathbb{E}\big(h(X)X_{12}\sin(2\Theta)\mathbf{1}_{\|\nabla X\|>0}\big) = -\overline{\mathrm{LTC}_X^{(2)}}(h).$$

This concludes the proof for real-valued functions h. The result clearly extends to complex-valued functions by linearity. Moreover, when $X(0)$ admits a density $p_{X(0)}$, we can further write

$$\overline{\mathrm{LP}_X}(h) = \int_{\mathbb{R}} h(t)\frac{\pi}{2}\mathbb{E}\big(|X_j(0)|\,|X(0)=t\big)p_{X(0)}(t)\,dt,$$

$$\overline{\mathrm{LTC}_X^{(1)}}(h) = -\int_{\mathbb{R}} h(t)\mathbb{E}\big(X_{jj}(0)\mathbf{1}_{\|\nabla X(0)\|>0}|X(0)=t\big)p_{X(0)}(t)\,dt,$$

$$\overline{\mathrm{LTC}_X^{(2)}}(h) = 4\int_{\mathbb{R}} h(t)\mathbb{E}\left(\frac{X_1(0)X_2(0)}{\|\nabla X(0)\|^2}X_{12}(0)\mathbf{1}_{\|\nabla X(0)\|>0}|X(0)=t\right)p_{X(0)}(t)\,dt,$$

such that for any continuous bounded function h,

$$\int_{\mathbb{R}} h(t)\overline{\mathrm{Per}(E_X(t))}\,dt = \int_{\mathbb{R}} h(t)\frac{\pi}{2}\mathbb{E}\big(|X_j(0)|\,|X(0)=t\big)p_{X(0)}(t)\,dt \quad \text{and}$$

$$\int_{\mathbb{R}} h(t)\overline{\mathrm{TC}(E_X(t))}\,dt$$

$$= \int_{\mathbb{R}} h(t)\bigg[-\mathbb{E}\big(X_{jj}(0)\mathbf{1}_{\|\nabla X(0)\|>0}|X(0)=t\big)$$

$$+ 4\mathbb{E}\left(\frac{X_1(0)X_2(0)}{\|\nabla X(0)\|^2}X_{12}(0)\mathbf{1}_{\|\nabla X(0)\|>0}|X(0)=t\right)\bigg]p_{X(0)}(t)\,dt,$$

implying the stated equalities for almost every $t \in \mathbb{R}$. $\quad\square$

EXAMPLE. Let X be a stationary centered C^2 isotropic Gaussian random field. Then, $X_1(0)$ being independent from $X(0)$, we get $\mathbb{E}(|X_1(0)|\,|X(0)) = \mathbb{E}(|X_1(0)|) = \sqrt{\frac{2\lambda_2}{\pi}}$, where $\lambda_2 = \mathrm{Var}(X_1(0)) > 0$ denotes the second spectral moment. Moreover,

$$\mathbb{E}\big(X_{11}(0)|X(0)\big) = \frac{\mathrm{Cov}(X(0), X_{11}(0))}{\mathrm{Var}(X(0))}X(0) = \frac{\partial_1^2\rho_X(0)}{\rho_X(0)}X(0) = \frac{-\lambda_2}{\sigma^2}X(0),$$

where $\rho_X(x) = \mathrm{Cov}(X(x), X(0))$ and $\sigma^2 = \rho_X(0)$, while by independence

$$\mathbb{E}\left(\frac{X_1(0)X_2(0)}{\|\nabla X(0)\|^2}X_{12}(0)|X(0)\right) = \mathbb{E}\left(\frac{X_1(0)X_2(0)}{\|\nabla X(0)\|^2}\right)\mathbb{E}\big(X_{12}(0)|X(0)\big) = 0.$$

Hence, since $\mathbb{P}(\|\nabla X(0)\| = 0) = 0$, we get in this case that, for almost every $t \in \mathbb{R}$,

$$\tag{8} \overline{\mathrm{Per}(E_X(t))} = \sqrt{\frac{\pi\lambda_2}{2}}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma^2}},$$

$$\tag{9} \overline{\mathrm{TC}(E_X(t))} = \frac{\lambda_2}{\sigma^2}t\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma^2}}.$$

Let us emphasize that this last expression yields exactly to the formula obtained for $2\pi\mathbb{E}(\chi(E_X(t)\cap\overline{U}))$, stated for all $t \in \mathbb{R}$, under additional assumptions on X (see (3.2.8) of [2], for instance), where χ denotes the Differential Topology (DT) characteristic of the set and, therefore, $\overline{\mathrm{TC}(E_X(t))}/2\pi$ corresponds to the Euler characteristic density of $E_X(t)$.

FIG. 3. *Gaussian random field with covariance* $\rho(x) = e^{-T^2\|x\|^2}$ *for* $T = 10$. *This sample has been obtained using Matlab, with a discretized domain of size* $2^{10} \times 2^{10}$ *pixels, using the circulant embedding matrix method. Top right figure*: *empirical Euler characteristic as a function of the level* t (*computed thanks to the Matlab function* `bweuler`), *compared with the theoretical value* (*red curve*) *of equation* (9). *Bottom line*: *Three excursion sets corresponding respectively from left to right to the level* $t = -1$, $t = 0$ (*i.e., the "critical level" where the Euler characteristic turns from negative to positive*), *and* $t = 1$.

Examples of such stationary isotropic random fields with comparisons between the theoretical values of $\overline{\mathrm{TC}(E_X(t))}/2\pi$ and an empirical estimate of Euler characteristic on the square of fixed size $[0, 1]$ are shown on Figures 3 and 4 (with $\sigma^2 = 1$ and $\lambda_2 = 2T^2$ for $T = 10$ and $T = 100$). The captions of the figures give the practical and technical details of the simulations. Note that in view of the covariance functions, a scaling relation may be set between T and the size of the square, explaining the convergence without boundary effects as T increases.

It is more difficult to compute exact formulas in the general case where the gradient $\nabla X(0)$ is not independent from $(X(0), D^2 X(0))$. However, we can use the following expressions that allow more tractable computations. The proofs are technical and postponed to the Appendix.

PROPOSITION 2. *Let* W, X_1, X_2 *be real random variables and let* $\varepsilon \in (0, 1]$.

1. *If* $\mathbb{E}(|WX_1|) < +\infty$ *and* $\mathbb{E}(|W||X_1|^{1+\varepsilon}) < +\infty$, *then*

$$\mathbb{E}(W|X_1|) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{u} \mathbb{E}(WX_1 \sin(uX_1)) \, du.$$

2. *If* $\mathbb{E}(|W|) < +\infty$ *and* $\mathbb{E}(|W||X_1X_2|^\varepsilon) < +\infty$, *then*

$$\mathbb{E}\left(W \frac{X_1X_2}{X_1^2 + X_2^2} \mathbf{1}_{X_1^2+X_2^2>0}\right)$$
$$= \frac{4}{\pi} \int_0^{+\infty} \int_0^{+\infty} \frac{v_1 v_2}{(v_1^2 + v_2^2)^2} \mathbb{E}(W \sin(v_1 X_1) \sin(v_2 X_2)) \, dv_1 \, dv_2.$$
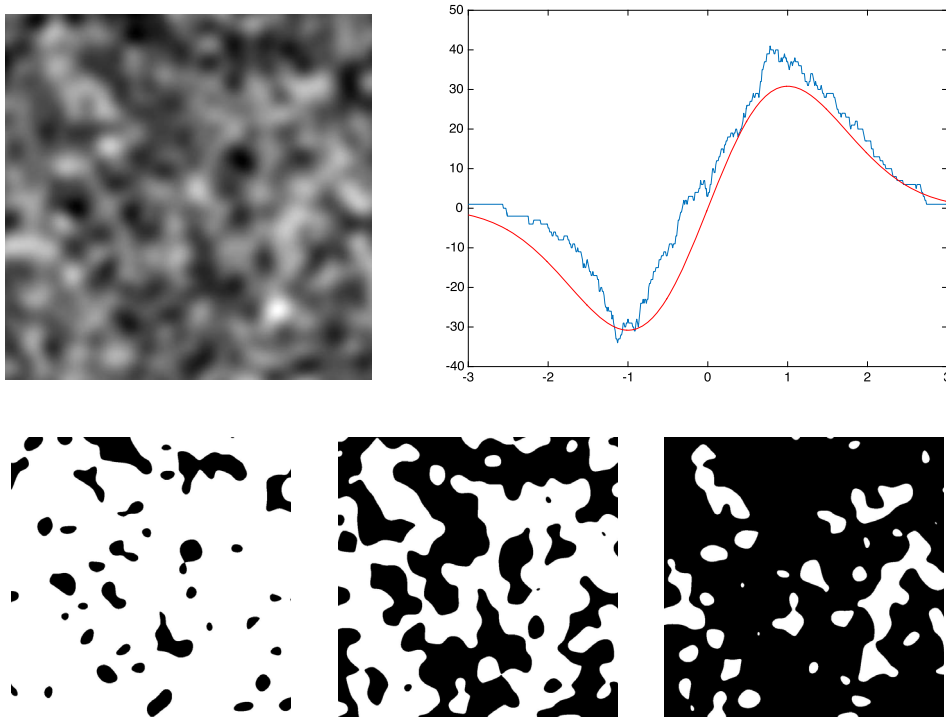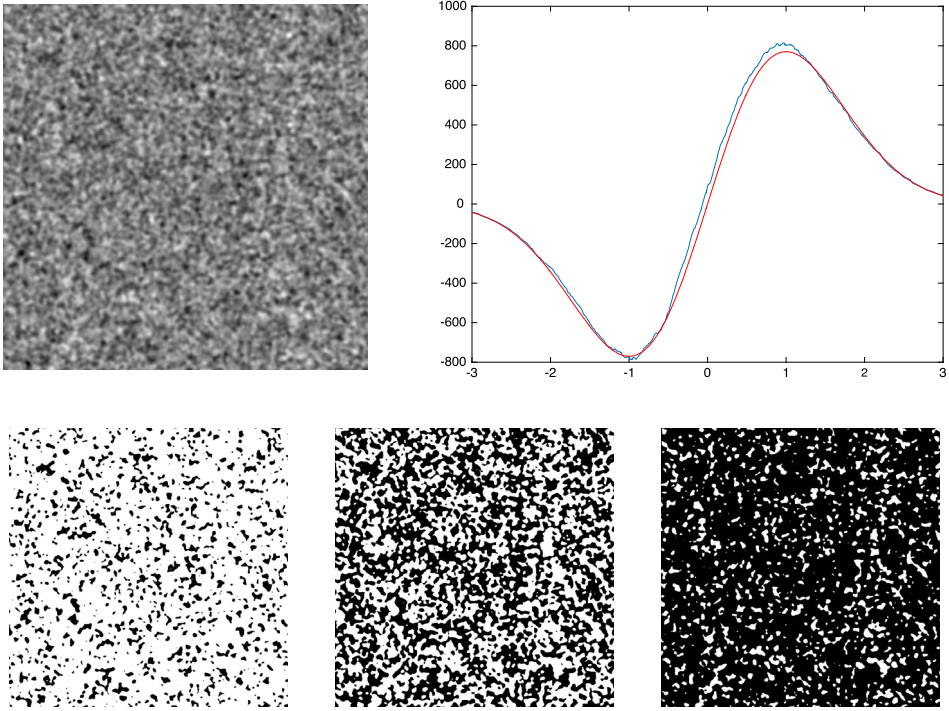
FIG. 4. *Gaussian random field with covariance $\rho(x) = e^{-T^2 \|x\|^2}$ for $T = 100$. This sample has been obtained using Matlab, with a discretized domain of size $2^{10} \times 2^{10}$ pixels, using the circulant embedding matrix method. Top right figure*: *empirical Euler characteristic as a function of the level t (computed thanks to the Matlab function* `bweuler`*), compared with the theoretical value (red curve) of equation* (9). *Bottom line*: *Three excursion sets corresponding respectively from left to right to the level $t = -1$, $t = 0$ (that is the "critical level" where the Euler characteristic turns from negative to positive), and $t = 1$.*

Here, the improper integrals $\int_0^{+\infty}$ are obtained as $\lim_{M \to +\infty} \int_0^M$.

Another useful result can be stated using the invariance property (6). The proof of this result is also postponed to the Appendix.

PROPOSITION 3. *Let W, J, K be complex random variables such that for any $\theta \in [0, 2\pi)$,*

$$(10) \qquad (W, J, K) \overset{d}{=} \left(W, e^{i\theta} J, e^{2i\theta} K\right),$$

and $\mathbb{E}(|WK|) < +\infty$. Then, writing $J = \mathrm{Re}^{i\Theta} \mathbf{1}_{|J|>0}$, on the one hand, for any $g : 2\pi \mathbb{T} \to \mathbb{R}$ continuous bounded 2π periodic function, one has

$$\mathbb{E}\left(W \overline{K} g(\Theta) \mathbf{1}_{|J|>0}\right) = c_2(g) \mathbb{E}\left(W \overline{K} e^{2i\Theta} \mathbf{1}_{|J|>0}\right),$$

with $c_2(g) = \frac{1}{2\pi} \int_0^{2\pi} e^{-2i\pi\theta} g(\theta) \, d\theta$. On the other hand, if there exists $\varepsilon > 0$ such that $\mathbb{E}(|WK||J|^\varepsilon) < +\infty$, then

$$\mathbb{E}\left(W \overline{K} \sin(2\Theta) \mathbf{1}_{|J|>0}\right) = i \int_0^{+\infty} \frac{1}{v} \mathbb{E}\left(W \overline{K} \cos(v \Re(J))\right) dv.$$

REMARK. A closely related result is given in Corollary 2.3 of [23], that should rewrite in our stationary setting, under additional assumption on X, as

$$\int_{\mathbb{R}} h(t) \overline{\chi(E_X(t))} \, dt$$

$$= -\mathbb{E}\left(h(X(0)) \left[\sum_{j=1}^{2} \mathbb{1}_{\nabla X(0) \in Q_j} X_{jj}(0) \right] \right.$$
$$\left. + h'(X(0)) \left[\sum_{j=1}^{2} \mathbb{1}_{\nabla X(0) \in Q_j} X_j(0)^2 \right] \right),$$

for $h : \mathbb{R} \to \mathbb{R}$ a C^1 function with compact support, $Q_1 = \{x = (x_1, x_2); x_2 < x_1 < 0\}$ and $Q_2 = \{x = (x_1, x_2); x_1 < x_2 < 0\}$, and where $\overline{\chi(E_X(t))}$ stands for an Euler characteristic density. Under the assumption that X is also isotropic, according to Remark 2.5 of [23],

$$\mathbb{E}(h'(X(0)) \mathbb{1}_{\nabla X(0) \in Q_j} X_j(0)^2) = \frac{\pi - 2}{16\pi} \mathbb{E}(h'(X(0)) \|\nabla X(0)\|^2).$$

Note that by stationarity, denoting $e_1 = (1, 0)$ and $e_2 = (0, 1)$, for any $j = 1, 2$,

$$\mathbb{E}(h'(X(0)) X_j(0)^2) = \int_0^1 \mathbb{E}(h'(X(te_j)) X_j(te_j)^2) \, dt$$
$$= \mathbb{E}\left(\int_0^1 h'(X(te_j)) X_j(te_j)^2 \, dt \right)$$
$$= -\mathbb{E}\left(\int_0^1 h(X(te_j)) X_{jj}(te_j) \, dt \right) = -\mathbb{E}(h(X(0)) X_{jj}(0)),$$

integrating by parts and using again the stationarity of X. Moreover,

$$\mathbb{E}(h(X(0)) \mathbb{1}_{\nabla X(0) \in Q_j} X_{jj}(0)) = \mathbb{E}(h(X(0)) g_j(\Theta) X_{jj}(0)),$$

with $g_j(\theta) = \mathbb{1}_{\theta \in \pi + \frac{\pi}{4}(2-j, 3-j)}$ that is bounded but not continuous. However, if we assume that $(X(0), \nabla X(0), X_{jj}(0))$ admits a density we can adapt the arguments of Proposition 3, to compute $\alpha_2(h) = \frac{1}{c_2(g_j)} \mathbb{E}(h(X(0)) g_j(\Theta) \overline{K})$, with $c_2(g_1) = -\frac{1}{4\pi}(1 + i)$ and $c_2(g_2) = -\frac{1}{4\pi}(-1 + i)$ and we obtain

$$\mathbb{E}\left(\int_{\mathbb{R}} h(t) \overline{\chi(E_X(t))} \right) dt = \frac{1}{2\pi} \int_{\mathbb{R}} h(t) \overline{\mathrm{TC}(\partial E_X(t))} \, dt,$$

by Theorem 2, since it is also assumed that $\|\nabla X(0)\| > 0$ a.s..

3.3. *Smooth shot noise random fields.* We consider here a shot-noise random field defined on $\mathbb{R}^2$ by

$$\forall x \in \mathbb{R}^2, \quad X_\Phi(x) = \sum_{i \in I} g_{m_i}(x - x_i),$$

where $\Phi = \{(x_i, m_i)\}_{i \in I}$ is a Poisson point process on $\mathbb{R}^2 \times \mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, of intensity $\lambda \mathcal{L} \times F$, with $\lambda > 0$ real, $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}^2$ and F a probability measure on $\mathbb{R}^d$. Note that equivalently, we may define Φ as an independently *marked Poisson point process* where $\{x_i\}_i$ is an homogeneous Poisson point process of intensity λ and the m_i are "marks," following a law $F(dm)$ on $\mathbb{R}^d$ (with $d \geq 1$) and independent of the Poisson point process $\{x_i\}_i$. Let $g : \mathbb{R}^2 \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that the functions $g_m := g(\cdot, m)$ satisfy

$$(11) \qquad \int_{\mathbb{R}^2 \times \mathbb{R}^d} |g_m(x)| \, dx \, F(dm) < +\infty.$$

Then the random field X_Φ is well-defined as an almost surely locally integrable function on $\mathbb{R}^2$ (see [11]). Moreover, the random field X_Φ is stationary. We will first give sufficient conditions to ensure smoothness properties and also isotropy, in order to obtain explicit formulas.

But then, since smooth shot noise random fields do not always admit a probability density (we have discussed this through several examples in our first paper [10]), we will have to work with their characteristic functions.

THEOREM 3. *Assume that $g : \mathbb{R}^2 \times \mathbb{R}^d \to \mathbb{R}$ is a measurable function such that for F-almost every $m \in \mathbb{R}^d$ the functions $g_m := g(\cdot, m)$ are C^3 on $\mathbb{R}^2$ satisfying*

$$(12) \qquad \int_{\mathbb{R}^2 \times \mathbb{R}^d} |D^{\mathbf{j}} g_m(x)| \, dx \, F(dm) < +\infty,$$

for all $\mathbf{j} = (j_1, j_2) \in \mathbb{N}^2$ with $|\mathbf{j}| = j_1 + j_2 \leq 3$ and where $D^{\mathbf{j}} g_m = \frac{\partial^{|\mathbf{j}|} g_m}{\partial x_1^{j_1} \partial x_2^{j_2}}$. Then X_Φ is a.s. a stationary C^2 field such that X, X_j, X_{jk} have finite expectation for $j, k = 1, 2$, ensuring the assumptions of Theorem 2.

If we assume moreover that for F-a.e. m, the function g_m is invariant under rotations, then X_Φ is isotropic. It follows that, when we also have

$$(13) \qquad \int_{\mathbb{R}^2 \times \mathbb{R}^d} |D^{\mathbf{j}} g_m(x)|^2 \, dx \, F(dm) < +\infty,$$

for $|\mathbf{j}| \in \{1, 2\}$, then for all $u \in \mathbb{R}$, denoting h_u the function $t \mapsto e^{iut}$,

$$\overline{LP_{X_\Phi}}(h_u) = \int_0^{+\infty} \frac{1}{v} \varphi(u, v) S_0(u, v) \, dv,$$

$$\overline{LTC_{X_\Phi}^{(1)}}(h_u) = \varphi(u, 0) S_1(u),$$

$$\overline{LTC_{X_\Phi}^{(2)}}(h_u) = \int_0^{+\infty} \frac{1}{v} \varphi(u, v) S_2(u, v) \, dv,$$

where φ is the characteristic function of $(X_\Phi(0), \partial_1 X_\Phi(0))$ given by

$$\varphi(u, v) = \mathbb{E}\left(e^{iuX_\Phi(0) + iv\partial_1 X_\Phi(0)}\right)$$

$$= \exp\left(\lambda \int_{\mathbb{R}^d \times \mathbb{R}^2} \left[e^{i[ug_m(x) + v\partial_1 g_m(x)]} - 1\right] F(dm) \, dx\right),$$

$$S_0(u, v) = -i\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \partial_1 g_m(x) e^{i[ug_m(x) + v\partial_1 g_m(x)]} \, dx \, F(dm),$$

$$S_1(u) = -\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \partial_1^2 g_m(x) e^{iug_m(x)} \, dx \, F(dm),$$

$$S_2(u, v) = \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \left[\partial_2^2 g_m(x) - \partial_1^2 g_m(x)\right] e^{i[ug_m(x) + v\partial_1 g_m(x)]} \, dx \, F(dm)$$

and the notation ∂_j, respectively, ∂_j^2, stands for $\frac{\partial}{\partial x_j}$, respectively $\frac{\partial^2}{\partial x_j^2}$ for $j \in \{1, 2\}$.

REMARK. Let us remark that these formulas allow to get an explicit expression for a.e. t as soon as one can identify the right members with some Fourier transforms (with respect to u).

PROOF OF THEOREM 3. Following similar arguments as in Proposition 3 of [10], Condition (12) will ensure that X_Φ is a.s. a stationary C^2 field such that X, X_j, X_{jk} have finite expectation for $j, k = 1, 2$, and we can differentiate under the sum. In particular, for all $j = 1, 2$,

$$\partial_j X_\Phi(x) = \sum_{i \in I} \partial_j g_{m_i}(x - x_i) \quad \text{and} \quad \partial_j^2 X_\Phi(x) = \sum_{i \in I} \partial_j^2 g_{m_i}(x - x_i).$$

Hence the general formula of Theorem 2 is valid for X_Φ.

Under the additional assumption on the kernel, we can prove isotropy. Actually, for any $k \geq 1$, $u_1, \ldots, u_k \in \mathbb{R}$ and $y_1, \ldots, y_k \in \mathbb{R}^2$, one has (see, for instance, [10])

$$\mathbb{E}\big(e^{i\sum_{j=1}^k u_j X_\Phi(y_j)}\big) = \exp\Big(\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \big(e^{i\sum_{j=1}^k u_j g_m(y_j-x)} - 1\big) \, dx \, F(dm)\Big).$$

Hence, for any orthogonal matrix A, by the change of variables $x = Ay$,

$$\mathbb{E}\big(e^{i\sum_{j=1}^k u_j X_\Phi(Ay_j)}\big) = \exp\Big(\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \big(e^{i\sum_{j=1}^k u_j g_m(A(y_j-y))} - 1\big) \, dy \, F(dm)\Big)$$

$$= \mathbb{E}\big(e^{i\sum_{j=1}^k u_j X_\Phi(y_j)}\big)$$

since $g_m \circ A = g_m$ for F-a.e. $m \in \mathbb{R}^d$.

It follows that by Theorem 2, for any h bounded continuous function on $\mathbb{R}$, one has

$$\overline{\mathrm{LP}_{X_\Phi}}(h) = \frac{\pi}{2}\mathbb{E}\big(h\big(X_\Phi(0)\big)|\partial_1 X_\Phi(0)|\big),$$

$$\overline{\mathrm{LTC}_{X_\Phi}^{(1)}}(h) = -\mathbb{E}\big(h\big(X_\Phi(0)\big)\partial_1^2 X_\Phi(0)\mathbf{1}_{\|\nabla X_\Phi(0)\|>0}\big)$$

and for h with real values,

$$\overline{\mathrm{LTC}_{X_\Phi}^{(2)}}(h) = -2\Re\big(\alpha_2(h)\big),$$

where

$$\alpha_2(h) = \mathbb{E}\big(h\big(X_\Phi(0)\big)\overline{K_\Phi}e^{2i\Theta_\Phi}\mathbf{1}_{\|\nabla X_\Phi(0)\|>0}\big),$$

for $K_\Phi = \frac{1}{4}(\partial_2^2 X_\Phi(0) - \partial_1^2 X_\Phi(0) - 2i\partial_{12}^2 X_\Phi(0))$ and Θ_Φ the angle of $\nabla X_\Phi(0)$.

Assuming moreover (13) is enough to use Propositions 2 and 3 with $\varepsilon = 1$, and we can further write

$$\alpha_2(h) = 2i\mathbb{E}\big(h\big(X_\Phi(0)\big)\overline{K_\Phi}\sin(2\Theta_\Phi)\mathbf{1}_{\|\nabla X_\Phi(0)\|>0}\big)$$

$$= -2\int_0^{+\infty} \frac{1}{v}\mathbb{E}\big(h\big(X_\Phi(0)\big)\overline{K_\Phi}\cos(v\partial_1 X_\Phi(0))\big) \, dv$$

and

$$2\Re\alpha_2(h) = -4\int_0^{+\infty} \frac{1}{v}\mathbb{E}\big(h\big(X_\Phi(0)\big)\Re\overline{K_\Phi}\cos(v\partial_1 X_\Phi(0))\big) \, dv,$$

so that

(14)
$$\overline{\mathrm{LTC}_{X_\Phi}^{(2)}}(h) = \int_0^{+\infty} \frac{1}{v}\mathbb{E}\big(h\big(X_\Phi(0)\big)\big[\partial_2^2 X_\Phi(0)$$
$$- \partial_1^2 X_\Phi(0)\big]\cos(v\partial_1 X_\Phi(0))\big) \, dv,$$

that may be extended to complex-valued functions h. Hence, taking $h_u = e^{iu\cdot}$ for $u \in \mathbb{R}$ we remark that since $\varphi(u, v) = \varphi(u, -v) = \mathbb{E}(h_u(X_\Phi(0))\cos(v\partial_1 X_\Phi(0)))$,

$$\mathbb{E}\big(h_u\big(X_\Phi(0)\big)\partial_1 X_\Phi(0)\sin(v\partial_1 X_\Phi(0))\big) = -\frac{\partial\varphi}{\partial v}(u, v) = S_0(u, v)\varphi(u, v).$$

This leads to the formula for $\overline{\mathrm{LP}_{X_\Phi}}(h_u)$. Similarly, using the fact that

$$\tilde{\varphi}(u, v) := \mathbb{E}\big(e^{iuX_\Phi(0)+iv\partial_1^2 X_\Phi(0)}\big)$$

$$= \exp\Big(\lambda \int_{\mathbb{R}^d \times \mathbb{R}^2} \big[e^{i[ug_m(x)+v\partial_1^2 g_m(x)]} - 1\big]F(dm)\,dx\Big),$$

we can compute

$$\overline{\mathrm{LTC}_{X_\Phi}^{(1)}}(h_u) = i\frac{\partial\tilde\varphi}{\partial v}(u,0) = S_1(u)\tilde\varphi(u,0) = S_1(u)\varphi(u,0).$$

Finally, introducing

$$\tilde{\tilde\varphi}(u,v,w)$$
$$:= \mathbb{E}\big(e^{iuX_\Phi(0)+iv\partial_1 X_\Phi(0)+iw[\partial_2^2 X_\Phi(0)-\partial_1^2 X_\Phi(0)]}\big)$$
$$= \exp\Big(\lambda\int_{\mathbb{R}^d\times\mathbb{R}^2}\big(e^{i[ug_m(x)+v\partial_1 g_m(x)+w[\partial_2^2 g_m(x)-\partial_1^2 g_m(x)]]} - 1\big)F(dm)\,dx\Big),$$

since $\tilde{\tilde\varphi}(u,v,w) = \tilde{\tilde\varphi}(u,-v,w)$ we also have

$$\tilde{\tilde\varphi}(u,v,w) = \mathbb{E}\big(e^{iuX_\Phi(0)+iw[\partial_2^2 X_\Phi(0)-\partial_1^2 X_\Phi(0)]}\cos(\partial_1 X_\Phi(0)v)\big).$$

Hence,

$$\mathbb{E}\big(h_u\big(X_\Phi(0)\big)\big[\partial_2^2 X_\Phi(0) - \partial_1^2 X_\Phi(0)\big]\cos(\partial_1 X_\Phi(0)v)\big)$$
$$= -i\frac{\partial\tilde{\tilde\varphi}}{\partial w}(u,v,0) = S_2(u,v)\varphi(u,v). \qquad\square$$

EXAMPLE 1.   We will consider here the example of a smooth isotropic shot noise random field given by

$$\forall x\in\mathbb{R}^2, \quad X_\Phi(x) = \sum_{i\in I}\beta_i g(x-x_i),$$

where $\{x_i\}_{i\in I}$ is a Poisson point process of intensity λ on $\mathbb{R}^2$, the β_i are independent weights following an exponential distribution of parameter μ on $\mathbb{R}_+$ and g is the function given by $g(x) = \exp(-a\|x\|^2/2)$, with $a > 0$ a fixed number. As previously, for $u\in\mathbb{R}$, let h_u be the function defined on $\mathbb{R}$ by $t\mapsto e^{iut}$. When β follows an exponential distribution of parameter μ, we can compute its characteristic function

$$\widehat{F_\beta}(u) = \mathbb{E}(e^{iu\beta}) = \int_0^{+\infty}\mu e^{ius}e^{-\mu s}\,ds = \frac{\mu}{\mu - iu}.$$

And we also have that

$$\mathbb{E}(\beta e^{iu\beta}) = -i\frac{\partial}{\partial u}\mathbb{E}(e^{iu\beta}) = \frac{\mu}{(\mu - iu)^2}.$$

Then, denoting by $\varphi(u,v)$ the joint characteristic function of $X_\Phi(0)$ and $\partial_1 X_\Phi(0)$, we have

$$\varphi(u,v) = \exp\Big(\lambda\iint\big[e^{iu\beta g(x)+iv\beta\partial_1 g(x)} - 1\big]F(d\beta)\,dx\Big)$$
$$= \exp\Big(\lambda\int_0^{2\pi}\int_0^{+\infty}\frac{i(u - avr\cos\theta)e^{-ar^2/2}}{\mu - i(u - avr\cos\theta)e^{-ar^2/2}}r\,dr\,d\theta\Big).$$

This allows us to compute explicitly the law of $X_\Phi(0)$, since

$$\varphi(u,0) = \mathbb{E}(e^{iuX_\Phi(0)}) = \exp\Big(\lambda\int_0^{2\pi}\int_0^{+\infty}\frac{iue^{-ar^2/2}}{\mu - iue^{-ar^2/2}}r\,dr\,d\theta\Big)$$
$$= \Big(\frac{\mu}{\mu - iu}\Big)^{2\pi\lambda/a}.$$

This shows that $X_\Phi(0)$ follows a Gamma distribution of parameters μ and $2\pi\lambda/a$. We can also compute the level perimeter integral, and get, according to Theorem 3,

$$\overline{\mathrm{LP}_{X_\Phi}}(h_u) = \int_0^{+\infty} \frac{1}{v}\varphi(u,v)S_0(u,v)\,dv,$$

with

$$S_0(u,v) = -i\lambda \int_0^{+\infty}\int_0^{2\pi} \frac{\mu a r^2 \cos(\theta)e^{-ar^2/2}}{(\mu - i(u - avr\cos\theta)e^{-ar^2/2})^2}\,d\theta\,dr.$$

For the level total curvature integral, the first term can be made explicit, and it is given by (see again Theorem 3) $\overline{\mathrm{LTC}_{X_\Phi}^{(1)}}(h_u) = S_1(u)\varphi(u,0)$, where

$$S_1(u) = -\lambda \iint \beta\partial_1^2 g(x)e^{iu\beta g(x)}F(d\beta)\,dx =: -\lambda I_1.$$

The integral I_1 can be computed as follows:

$$\begin{aligned}
I_1 &:= \iint \beta\partial_1^2 g(x)e^{iu\beta g(x)}F(d\beta)\,dx \\
&= \int_{\mathbb{R}^2} a(ax_1^2 - 1)g(x_1,x_2)\frac{\mu}{(\mu - iug(x_1,x_2))^2}\,dx_1\,dx_2 \\
&= \int_0^{2\pi}\int_0^{+\infty} a(ar^2\cos^2\theta - 1)e^{-ar^2/2}\frac{\mu}{(\mu - iue^{-ar^2/2})^2}r\,dr\,d\theta \\
&= -2\pi\left(\frac{1}{iu}\log\frac{\mu - iu}{\mu} + \frac{1}{\mu - iu}\right).
\end{aligned}$$

This finally leads to

$$\overline{\mathrm{LTC}_{X_\Phi}^{(1)}}(h_u) = 2\pi\lambda\left(\frac{\mu}{\mu - iu}\right)^{2\pi\lambda/a}\left(\frac{1}{iu}\log\frac{\mu - iu}{\mu} + \frac{1}{\mu - iu}\right).$$

Let us note that, writing $v = 2\pi\lambda/a > 0$, this term corresponds to the Fourier transform of the function $2\pi\lambda f_v$, where

$$f_v(t) = \left(\frac{(\mu t)^v}{\Gamma(v+1)}e^{-\mu t} - \int_0^{\mu t}(\psi(v) - \log(s))\frac{1}{\Gamma(v)}s^{v-1}e^{-s}\,ds\right)\mathbb{1}_{t>0},$$

using the fact that the inverse Laplace transform of $p \mapsto p^{-v}\log(p)$ is given by $s \mapsto \frac{s^{v-1}}{\Gamma(v)}(\psi(v) - \log(s))$ (see [18], page 251), where ψ is the logarithmic derivative of the Γ function.

For the second term, $\overline{\mathrm{LTC}_{X_\Phi}^{(2)}}(h_u)$, we use the formula given by

$$\overline{\mathrm{LTC}_{X_\Phi}^{(2)}}(h_u) = \int_0^{+\infty} \frac{1}{v}\varphi(u,v)S_2(u,v)\,dv,$$

where $\varphi(u,v)$ was computed above, and S_2 is given here by

$$S_2(u,v) = -\lambda \int_0^{+\infty}\int_0^{2\pi} \frac{\mu a^2 r^3 \cos(2\theta)e^{-ar^2/2}}{(\mu - i(u - avr\cos\theta)e^{-ar^2/2})^2}\,d\theta\,dr.$$

All these integrals can be efficiently numerically computed (using Matlab, for instance). Some results, comparing empirical and theoretical curves, are shown on Figure 5.
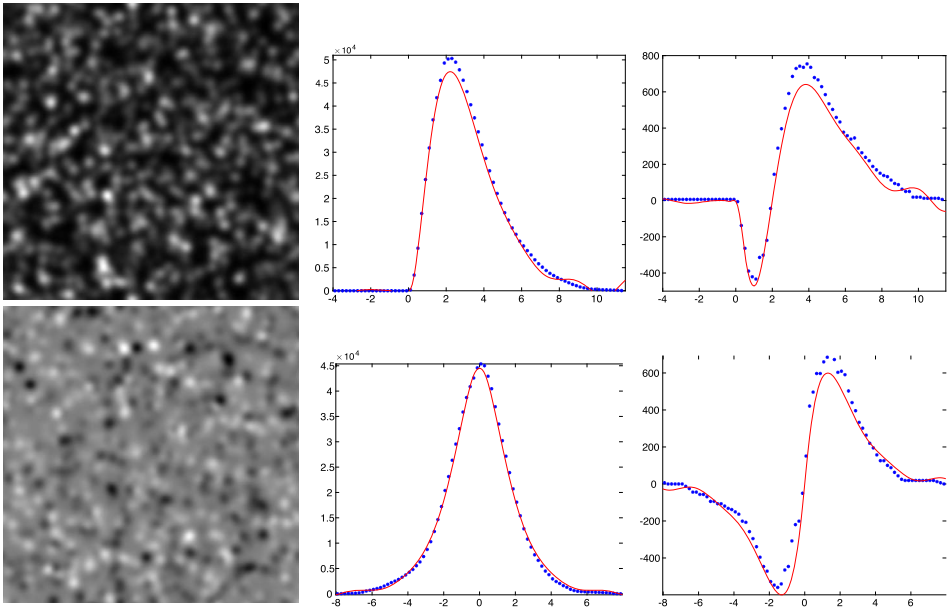
FIG. 5. *First line*: *on the left, a sample of the shot noise of Example* 1; *in the middle*: *its empirical perimeter* $t \mapsto \mathrm{Per}(E_X(t))$ (*blue stars*), *and the theoretical curve* $t \mapsto \overline{\mathrm{Per}(E_X(t))}$ (*red curve*) *obtained numerically by inverting the Fourier transform of* $u \mapsto \overline{LP_{X_\Phi}}(h_u)$; *on the right*: *the empirical total curvature* $t \mapsto \mathrm{TC}(E_X(t))$ (*blue stars*), *and the theoretical curve* $t \mapsto \overline{\mathrm{TC}(E_X(t))}$ (*red curve*) *obtained numerically by inverting the Fourier transform of* $u \mapsto \overline{\mathrm{LTC}_{X_\Phi}}(h_u)$. *Second line*: *same experiments but with the shot noise of Example* 2.

EXAMPLE 2. This example is the same as the previous one, except that the β_i follow now a Laplace distribution of parameter μ, and the shot noise random field has therefore the additional property of being symmetric. Here,

$$\widehat{F_\beta}(u) = \mathbb{E}(e^{iu\beta}) = \int_{-\infty}^{+\infty} \frac{1}{2}\mu e^{ius} e^{-\mu|s|} \, ds = \frac{\mu^2}{\mu^2 + u^2}.$$

And we also have that

$$\mathbb{E}(\beta e^{iu\beta}) = -i\frac{\partial}{\partial u}\mathbb{E}(e^{iu\beta}) = \frac{2iu\mu^2}{(\mu^2 + u^2)^2}.$$

Then, denoting again by $\varphi(u, v)$ the joint characteristic function of $X_\Phi(0)$ and $\partial_1 X_\Phi(0)$, we have

$$\varphi(u, v) = \exp\left(\lambda \iint \left[e^{iu\beta g(x) + iv\beta \partial_1 g(x)} - 1\right] F(d\beta) \, dx\right)$$

$$= \exp\left(-\lambda \int_0^{2\pi} \int_0^{+\infty} \frac{(u - avr\cos\theta)^2 e^{-ar^2}}{\mu^2 + (u - avr\cos\theta)^2 e^{-ar^2}} r \, dr \, d\theta\right).$$

This allows us to compute explicitly the law of $X_\Phi(0)$, since

$$\varphi(u, 0) = \mathbb{E}(e^{iuX_\Phi(0)})$$

$$= \exp\left(-\lambda \int_0^{2\pi} \int_0^{+\infty} \frac{u^2 e^{-ar^2}}{\mu^2 + u^2 e^{-ar^2}} r \, dr \, d\theta\right)$$

$$= \left(\frac{\mu^2}{\mu^2 + u^2}\right)^{\pi\lambda/a}.$$

This shows that $X_\Phi(0)$ follows a symmetric generalized Laplace distribution of parameters μ and $\pi\lambda/a$. We can also compute the level perimeter integral, and get according to Theorem 3,

$$\overline{LP_X}(h_u) = \int_0^{+\infty} \frac{1}{v}\varphi(u,v)S_0(u,v)\,dv,$$

with now

$$S_0(u,v) = \lambda \int_0^{+\infty} \int_0^{2\pi} \frac{2\mu^2 ar^2 \cos(\theta)(u - avr\cos\theta)e^{-ar^2}}{(\mu^2 + (u - avr\cos\theta)^2 e^{-ar^2})^2}\,d\theta\,dr.$$

For the level total curvature integral, the first term can also be made explicit, and it is given by (see again Theorem 3) $\overline{LTC_{X_\Phi}^{(1)}}(h_u) = S_1(u)\varphi(u,0)$ with

$$S_1(u) = -\lambda \iint \beta \partial_1^2 g(x)e^{iu\beta g(x)} F(d\beta)\,dx =: -\lambda I_1.$$

Here, the integral I_1 can be computed as follows:

$$
\begin{aligned}
I_1 &:= \iint \beta \partial_1^2 g(x)e^{iu\beta g(x)} F(d\beta)\,dx \\
&= \int_{\mathbb{R}^2} a(ax_1^2 - 1)g(x_1,x_2)\frac{2iug(x_1,x_2)\mu^2}{(\mu^2 + u^2 g(x_1,x_2)^2)^2}\,dx_1\,dx_2 \\
&= \int_0^{2\pi} \int_0^{+\infty} 2iua\mu^2 \frac{(ar^2\cos^2\theta - 1)e^{-ar^2}}{(\mu^2 + u^2 e^{-ar^2})^2} r\,dr\,d\theta \\
&= \frac{i\pi}{u}\left(\log\frac{\mu^2 + u^2}{\mu^2} - \frac{2u^2}{\mu^2 + u^2}\right).
\end{aligned}
$$

This finally leads to

$$\overline{LTC_{X_\Phi}^{(1)}}(h_u) = \frac{i\lambda\pi}{u}\left(\frac{\mu^2}{\mu^2 + u^2}\right)^{\pi\lambda/a}\left(\log\frac{\mu^2}{\mu^2 + u^2} + \frac{2u^2}{\mu^2 + u^2}\right).$$

For the second term, $\overline{LTC_{X_\Phi}^{(2)}}(h_u)$, we use the formula given by

$$\overline{LTC_{X_\Phi}^{(2)}}(h_u) = \int_0^{+\infty} \frac{1}{v}\varphi(u,v)S_2(u,v)\,dv,$$

where $\varphi(u,v)$ was computed above and S_2 is given here by

$$S_2(u,v) = i\lambda \int_0^{+\infty} \int_0^{2\pi} \frac{-2\mu^2 a^2 r^3 \cos(2\theta)(u - avr\cos\theta)e^{-ar^2}}{(\mu^2 + (u - avr\cos\theta)^2 e^{-ar^2})^2}\,d\theta\,dr.$$

Again, all these integrals can be efficiently numerically computed (using Matlab, for instance). Some results, comparing empirical and theoretical curves, are shown on Figure 5. Notice that the theoretical curves (red curves) present some oscillations that are due to the numerical approximations we have made (indeed to numerically compute the integrals, we had to truncate them, and since the Fourier transform is involved, this results in some oscillations).

4. Elementary functions and elementary shot noise random fields.

4.1. *Elementary functions.* We introduce now the class of elementary functions that are piecewise constant functions defined more precisely in the following way.

DEFINITION 5 (Elementary function). We say that a function f defined on $\mathbb{R}^2$ is an elementary function if there exist an integer n, nonzero real numbers $a_1, \ldots, a_n$ and n elementary sets (Definition 2) denoted $A_1, \ldots, A_n$ that are in generic position (Definition 3), such that

$$f = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}.$$

Let us remark that since $\mathbb{R}^2$ itself is an elementary set, constant functions on $\mathbb{R}^2$ are elementary functions.

In Section 2, we have seen the definitions of regular points and corner points of a curve. We here extend these definitions to elementary functions.

We first introduce some notation. For a point $x \in U$ and a real positive number ρ, we recall that $B_\rho(x)$ denotes the open ball of radius ρ and center x. For an oriented simple piecewise C^2 curve γ and a point x on γ, then for ρ small enough, $B_\rho(x) \setminus \gamma$ is made of two connected components. These two "half-balls" are respectively denoted by $B_\rho^+(x, \gamma)$ and $B_\rho^-(x, \gamma)$. The half-ball $B_\rho^+(x, \gamma)$ is the component that is on the side of the normal ν_γ to γ.

Let us make the important and useful remark that the discontinuity set $\mathcal{S}_f$ of an elementary function $f = \sum_i a_i \mathbb{1}_{A_i}$ is equal to the union of the ∂A_i:

$$\mathcal{S}_f = \bigcup_i \partial A_i.$$

Indeed the inclusion $\mathcal{S}_f \subset \bigcup_i \partial A_i$ is obvious, and the reverse inclusion holds since all the a_i are nonzero. Therefore, a point x on the discontinuity set belongs else to a single ∂A_i and it is then a regular discontinuity point or a corner discontinuity point, or it belongs to two boundaries ∂A_i and ∂A_j and it is then called an intersection discontinuity point. See Figure 6 for a schematic representation of the three types of points.

To be more rigorous, here are the definitions and some notation.

- A point x is a regular discontinuity point for f if there exists $\rho > 0$ such that $\mathcal{S}_f \cap B_\rho(x)$ is a simple C^2 oriented curve γ separating the ball $B_\rho(x)$ in two half-balls $B_\rho^+(x, \gamma)$ and $B_\rho^-(x, \gamma)$, and there are two real numbers $f^+(x) > f^-(x)$ such that $f(y) = f^+(x)$ for all $y \in B_\rho^+(x, \gamma)$ and $f(y) = f^-(x)$ for all $y \in B_\rho^-(x, \gamma)$. We will denote

$$\nu_f(x) = \nu_\gamma(x) \quad \text{and} \quad \kappa_f(x) = \kappa_\gamma(x).$$
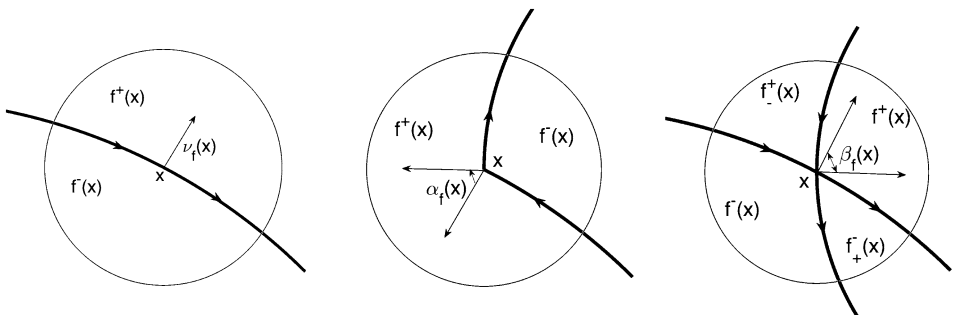

FIG. 6. *The three types of points of the discontinuity set of an elementary function. From left to right: a regular discontinuity point, a corner discontinuity point and an intersection discontinuity point.*

- A point x is a corner discontinuity point for f if there exists $\rho > 0$ such that $\mathcal{S}_f \cap B_\rho(x)$ is a simple piecewise C^2 oriented curve γ having only one corner at x. We write $\alpha_f(x) \in (-\pi, \pi)$ the turning angle of γ at x. As for regular discontinuity points, γ separates the ball $B_\rho(x)$ in two half-balls $B_\rho^+(x, \gamma)$ and $B_\rho^-(x, \gamma)$, and moreover, there exist two real numbers $f^+(x) > f^-(x)$ such that $f(y) = f^+(x)$ for all $y \in B_\rho^+(x, \gamma)$ and $f(y) = f^-(x)$ for all $y \in B_\rho^-(x, \gamma)$. The turning angle at such a corner point is denoted

$$\alpha_f(x) = \alpha_\gamma(x).$$

- A point x is an intersection discontinuity point for f if there exists $\rho > 0$ such that $\mathcal{S}_f \cap B_\rho(x)$ is the union of two different simple and oriented C^2 curves γ_1 and γ_2 in generic position and such that $\{x\} = \gamma_1 \cap \gamma_2$. Each curve separates the ball in two half-balls, and there exist 4 real numbers $f^-(x) \leq f_-^+(x)$, $f_+^-(x) \leq f^+(x)$ with at least 3 different values, such that $f = f^-(x)$ on $B_\rho^-(x, \gamma_1) \cap B_\rho^-(x, \gamma_2)$; $f = f_+^-(x)$ on $B_\rho^-(x, \gamma_1) \cap B_\rho^+(x, \gamma_2)$; $f = f_-^+(x)$ on $B_\rho^+(x, \gamma_1) \cap B_\rho^-(x, \gamma_2)$ and $f = f^+(x)$ on $B_\rho^+(x, \gamma_1) \cap B_\rho^+(x, \gamma_2)$. And we define

$$\begin{aligned}
\beta_f(x) &= d_{S^1}\left(v_{\gamma_1}(x), v_{\gamma_2}(x)\right) \\
&= \min\left(\left|\operatorname{Arg} v_{\gamma_1}(x) - \operatorname{Arg} v_{\gamma_2}(x)\right|, 2\pi - \left|\operatorname{Arg} v_{\gamma_1}(x) - \operatorname{Arg} v_{\gamma_2}(x)\right|\right) \\
&\in (0, \pi),
\end{aligned}$$

the geodesic distance between $v_{\gamma_1}(x)$ and $v_{\gamma_2}(x)$ on S^1.

Let us remark that for an elementary function, at an intersection discontinuity point x, we must also have $f_-^+(x) + f_+^-(x) = f^-(x) + f^+(x)$. Indeed one can write locally $f(x) = f_1(x) + f_2(x)$ with x a regular discontinuity point for both f_1 and f_2. Hence $f^+(x) = f_1^+(x) + f_2^+(x)$ and $f^-(x) = f_1^-(x) + f_2^-(x)$ so that $f^-(x) + f^+(x) = f_-^+(x) + f_+^-(x)$.

DEFINITION 6 (Elementary function on U). Let U be an open set of $\mathbb{R}^2$. We say that a function f defined on U is an elementary function on U if there exists $\tilde{f}$ an elementary function such that for all $x \in U$ we have $f(x) = \tilde{f}(x)$.

PROPOSITION 4. *Let f be an elementary function, and let $\mathcal{R}_f$, $\mathcal{C}_f$ and $\mathcal{I}_f$ denote respectively the set of regular, corner and intersection discontinuity points of f. Then for all $t \in \mathbb{R}$, $E_f(t)$ is an elementary set (in the sense of Definition* 2). *Moreover, if U is an open subset of $\mathbb{R}^2$ and f is an elementary function on U, then f is of finite total variation and finite level total curvature integral on U with*

$$(15) \qquad V_f(U) = \|Df(U)\| = \int_{\mathcal{R}_f \cap U} [f^+(x) - f^-(x)] \mathcal{H}^1(dx), \quad and$$

$$\begin{aligned}
(16) \qquad \text{LTaC}_f(U) = &\int_{\mathcal{R}_f \cap U} [f^+(x) - f^-(x)] |\kappa_f(x)| \mathcal{H}^1(dx) \\
&+ \sum_{x \in \mathcal{C}_f \cap U} [f^+(x) - f^-(x)] |\alpha_f(x)| \\
&+ \sum_{x \in \mathcal{I}_f \cap U} [f^+(x) - \max(f_-^+(x), f_+^-(x)) \\
&\quad + \min(f_-^+(x), f_+^-(x)) - f^-(x)] \beta_f(x) < +\infty.
\end{aligned}$$

If h is a bounded continuous function on $\mathbb{R}$, and H is a primitive of h (for instance, $H(t) = \int_0^t h(u)\,du$), then the level perimeter integral and the total curvature integral of f are given by

$$\text{(17)} \qquad \text{LP}_f(h, U) = \int_{\mathcal{R}_f \cap U} [H(f^+(x)) - H(f^-(x))]\mathcal{H}^1(dx),$$

$$\text{LTC}_f(h, U) = \int_{\mathcal{R}_f \cap U} [H(f^+(x)) - H(f^-(x))]\kappa_f(x)\mathcal{H}^1(dx)$$

$$+ \sum_{x \in \mathcal{C}_f \cap U} [H(f^+(x)) - H(f^-(x))]\alpha_f(x)$$

(18)

$$+ \sum_{x \in \mathcal{I}_f \cap U} [H(f^+(x)) + H(f^-(x))$$

$$- H(f_-^+(x)) - H(f_+^-(x))]\beta_f(x).$$

In particular, when $h = 1$, we get $\text{LP}_f(1, U) = V_f(U)$ and

$$\text{LTC}_f(U) = \int_{\mathcal{R}_f \cap U} [f^+(x) - f^-(x)]\kappa_f(x)\mathcal{H}^1(dx)$$

(19)

$$+ \sum_{x \in \mathcal{C}_f \cap U} [f^+(x) - f^-(x)]\alpha_f(x).$$

REMARK. Notice that equation (19) does not involve the intersection discontinuity points of the function f. In particular, this implies that if f and g are two elementary functions such that their elementary sets are all in generic position (which is equivalent to say that their discontinuity sets $\mathcal{S}_f$ and $\mathcal{S}_g$ are in generic position), then $f + g$ is an elementary function and we have

$$V_{f+g}(U) = V_f(U) + V_g(U) \quad \text{and} \quad \text{LTC}_{f+g}(U) = \text{LTC}_f(U) + \text{LTC}_g(U),$$

for any bounded open set U. This result is quite striking, but we have to underline that it does not hold in general for other classes of functions. Indeed, in the previous section we have considered smooth functions f and the formula for the level perimeter and the total curvature integral were obviously not linear in f. Here, the main point is that when f and g are elementary functions in a generic position, then the two measures Df and Dg are mutually singular, which is not the case in general for any two functions f and g.

PROOF OF PROPOSITION 4. Since f is an elementary function, we can write $f = \sum_{j=1}^n a_j \mathbb{1}_{A_j}$ with $a_1, \ldots, a_n$ real numbers and $A_1, \ldots, A_n$ elementary sets in generic position. Let us assume that $m = \text{Card}(f(\mathbb{R}^2)) \geq 2$. Otherwise, if $m = 1$, then for all $t \in \mathbb{R}$, $E_f(t) = \varnothing$ or $\mathbb{R}^2$ and, therefore, $\text{Per}(E_f(t), U) = \text{TaC}(\partial E_f(t), U) = \text{TC}(\partial E_f(t), U) = 0$. In the following, we denote the values of f by $v_1 < \cdots < v_m$ and set $v_0 = -\infty$. We first remark that $E_f(t) = \varnothing$ for $t > v_m$, $E_f(t) = \mathbb{R}^2$ for $t \leq v_1$ and $E_f(t) = E_f(v_i)$ for $v_{i-1} < t \leq v_i$ and $2 \leq i \leq m$. The set of discontinuity points is given by $\mathcal{S}_f = \bigcup_{i=2}^m \partial E_f(v_i) = \bigcup_{j=1}^n \partial A_j$. Since the sets $(A_j)_{1 \leq j \leq n}$ are in generic position, each $\partial E_f(v_i)$ is a piecewise regular curve and, therefore, $E_f(v_i)$ is an elementary set with $\text{Per}(E_f(v_i), U) = \mathcal{H}^1(\mathcal{R}_f \cap \Gamma_i)$ for $\Gamma_i := \partial E_f(v_i) \cap U$. Now, for $x \in \Gamma_i$ we must have $f^+(x) \geq v_i$ and $f^-(x) < v_i$ and since $\Gamma_i \subset \mathcal{S}_f$, we can write

$$\Gamma_i = (\Gamma_i \cap \mathcal{R}_f) \cup (\Gamma_i \cap \mathcal{C}_f) \cup (\Gamma_i \cap \mathcal{I}_f).$$

If $x \in \mathcal{R}_f$ or $x \in \mathcal{C}_f$, then there exists a unique j such that $x \in \partial A_j$ is a regular or corner point of ∂A_j with $\kappa_{\Gamma_i}(x) = \kappa_f(x)$ or $\alpha_{\Gamma_i}(x) = \alpha_f(x)$. Otherwise, if $x \in \mathcal{I}_f$ there exist $j \neq j'$ such that $x \in \partial A_j \cap \partial A_{j'}$ and x may become a corner point or a regular point for Γ_i. Indeed, if $v_i \leq \min(f_-^+(x), f_+^-(x))$, then $x \in \mathcal{C}_{\Gamma_i}$ with $\alpha_{\Gamma_i}(x) = -\beta_f(x)$. Without loss of generality, we may assume that $f_-^+(x) \leq f_+^-(x)$. If $f_-^+(x) < v_i \leq f_+^-(x)$ then x is not a corner point of Γ_i but a regular point. If $v_i > \max(f_-^+(x), f_-^-(x))$, then x is again a corner point of Γ_i with $\alpha_{\Gamma_i}(x) = \beta_f(x)$. Therefore,

$$\mathrm{TaC}(\Gamma_i, U) = \int_{\Gamma_i \cap \mathcal{R}_f} |\kappa_f(x)| \mathcal{H}^1(dx) + \sum_{x \in \Gamma_i \cap \mathcal{C}_f} |\alpha_f(x)|$$

$$+ \sum_{x \in \Gamma_i \cap \mathcal{I}_f} \beta_f(x) (\mathbb{1}_{v_i > \max(f_-^+(x), f_-^-(x))} + \mathbb{1}_{v_i \leq \min(f_-^+(x), f_-^-(x))}),$$

while

$$\mathrm{TC}(\Gamma_i, U) = \int_{\Gamma_i \cap \mathcal{R}_f} \kappa_f(x) \mathcal{H}^1(dx) + \sum_{x \in \Gamma_i \cap \mathcal{C}_f} \alpha_f(x)$$

$$+ \sum_{x \in \Gamma_i \cap \mathcal{I}_f} \beta_f(x) (\mathbb{1}_{v_i > \max(f_-^+(x), f_-^-(x))} - \mathbb{1}_{v_i \leq \min(f_-^+(x), f_-^-(x))}),$$

and

$$\mathrm{Per}(\Gamma_i, U) = \int_{\Gamma_i \cap \mathcal{R}_f} \mathcal{H}^1(dx).$$

Now, let h be a continuous bounded function on $\mathbb{R}$ and H a primitive of h. For the level perimeter integral, we have

$$\mathrm{LP}_f(h, U) = \int_{\mathbb{R}} h(t) \mathrm{Per}(\partial E_f(t), U) \, dt = \sum_{i=2}^{m} \int_{v_{i-1}}^{v_i} h(t) \mathrm{Per}(\Gamma_i, U) \, dt$$

$$= \sum_{i=2}^{m} [H(v_i) - H(v_{i-1})] \mathrm{Per}(\Gamma_i, U)$$

$$= \int_{\mathcal{R}_f} \sum_{i=2}^{m} [H(v_i) - H(v_{i-1})] \mathbb{1}_{\Gamma_i}(x) \mathcal{H}^1(dx).$$

When $x \in \mathcal{R}_f \cap U = \bigcup_{i=2}^{m} \Gamma_i$, we denote by $i(x)$, (resp., $j(x) \geq i(x)$), the minimal (resp., maximal) index $i = 2, \dots, m$ such that $x \in \Gamma_i$ and then $f^-(x) = v_{i(x)-1}$, (resp., $f^+(x) = v_{j(x)}$). It follows that

(20)
$$\sum_{i=2}^{m} [H(v_i) - H(v_{i-1})] \mathbb{1}_{\Gamma_i}(x) = \sum_{i=i(x)}^{j(x)} [H(v_i) - H(v_{i-1})]$$

$$= H(v_{j(x)}) - H(v_{i(x)-1})$$

$$= H(f^+(x)) - H(f^-(x)).$$

Therefore, we get

$$\mathrm{LP}_f(h, U) = \int_{\mathcal{R}_f} (H(f^+(x)) - H(f^-(x))) \mathcal{H}^1(dx).$$

Now, for the total absolute curvature, we have

$$\int_{\mathbb{R}} h(t) \operatorname{TaC}(\partial E_f(t), U) \, dt = \sum_{i=2}^{m} \int_{v_{i-1}}^{v_i} h(t) \operatorname{TaC}(\Gamma_i, U) \, dt$$

$$= \sum_{i=2}^{m} [H(v_i) - H(v_{i-1})] \operatorname{TaC}(\Gamma_i, U).$$

Then, using the above formula for $\operatorname{TaC}(\Gamma_i, U)$, we get the sum of three terms. The first one is given by

$$\sum_{i=2}^{m} [H(v_i) - H(v_{i-1})] \int_{\mathcal{R}_f \cap \Gamma_i} |\kappa_f(x)| \mathcal{H}^1(dx)$$

$$= \int_{\mathcal{R}_f} |\kappa_f(x)| \sum_{i=2}^{m} [H(v_i) - H(v_{i-1})] \mathbb{1}_{\Gamma_i}(x) \mathcal{H}^1(dx).$$

Using equation (20), this first term is equal to

$$\int_{\mathcal{R}_f \cap U} [H(f^+(x)) - H(f^-(x))] |\kappa_f(x)| \mathcal{H}^1(dx).$$

Similarly, the second term is equal to

$$\sum_{i=2}^{m} [H(v_i) - H(v_{i-1})] \sum_{x \in \Gamma_i \cap \mathcal{C}_f} |\alpha_f(x)|$$

$$= \sum_{x \in \mathcal{C}_f \cap U} [H(f^+(x)) - H(f^-(x))] |\alpha_f(x)|.$$

Finally, the third one is given by

$$\sum_{i=2}^{m} [H(v_i) - H(v_{i-1})]$$

$$\times \sum_{x \in \Gamma_i \cap \mathcal{I}_f} |\beta_f(x)| (\mathbb{1}_{v_i > \max(f_-^+(x), f_-^-(x))} + \mathbb{1}_{v_i \leq \min(f_-^+(x), f_-^-(x))})$$

$$= \sum_{x \in \mathcal{I}_f \cap U} \beta_f(x) \left(\sum_{i=l(x)+1}^{j(x)} [H(v_i) - H(v_{i-1})] + \sum_{i=i(x)}^{k(x)} [H(v_i) - H(v_{i-1})] \right),$$

where we have introduced $k(x)$ and $l(x)$ with $i(x) - 1 \leq k(x) \leq l(x) \leq j(x)$ such that $\min(f_-^+(x), f_-^-(x)) = v_{k(x)}$ and $\max(f_-^+(x), f_-^-(x)) = v_{l(x)}$, with the convention that $\sum_{i=l(x)+1}^{j(x)} = 0$ if $l(x) = j(x)$ and $\sum_{i=i(x)}^{k(x)} = 0$ if $k(x) = i(x) - 1$. It follows that this third term is equal to

$$\sum_{x \in \mathcal{I}_f \cap U} \beta_f(x) \big(H(v_{j(x)}) - H(v_{l(x)}) + (H(v_{k(x)}) - H(v_{i(x)-1})) \big)$$

$$= \sum_{x \in \mathcal{I}_f} \beta_f(x) \big(H(f^+(x)) - H(\max(f_-^+(x), f_-^-(x)))$$

$$+ H(\min(f_-^+(x), f_-^-(x))) - H(f^-(x)) \big).$$

In particular, for $h = 1$, we obtain formula (16). The same computations as above give the result (18) for $\mathrm{LTC}_f(h, U)$. Indeed now, the third term is equal to

$$\sum_{i=2}^{m} [H(v_i) - H(v_{i-1})]$$

$$\times \sum_{x \in \Gamma_i \cap \mathcal{I}_f} \beta_f(x)(\mathbb{1}_{v_i > \max(f_-^+(x), f_-^-(x))} - \mathbb{1}_{v_i \le \min(f_-^+(x), f_-^-(x))})$$

$$= \sum_{x \in \mathcal{I}_f} \beta_f(x)\left(\sum_{i=l(x)+1}^{j(x)} [H(v_i) - H(v_{i-1})] - \sum_{i=i(x)}^{k(x)} [H(v_i) - H(v_{i-1})] \right)$$

$$= \sum_{x \in \mathcal{I}_f} \beta_f(x)(H(f^+(x)) + H(f^-(x)) - H(f_-^+(x)) - H(f_+^-(x))).$$

Taking again $h = 1$, we obtain (19). Finally, let us remark that since the above results depend only on the values of f in U, they also hold when f is an elementary function on U in the sense of Definition 6. □

4.2. *Level perimeter and level total curvature of an elementary shot noise random field.* As in Section 3.3, we consider here a shot noise random field defined on $\mathbb{R}^2$ by

$$\forall x \in \mathbb{R}^2, \quad X_\Phi(x) = \sum_{i \in I} g_{m_i}(x - x_i),$$

where $\Phi = \{(x_i, m_i)\}_{i \in I}$ is a Poisson point process on $\mathbb{R}^2 \times \mathbb{R}^d$ of intensity $\lambda \mathcal{L} \times F$, with $\lambda > 0$ real, $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}^2$ and F a probability measure on $\mathbb{R}^d$.

We will first give formulas for the level perimeter and total curvature integrals of X_Φ on an open bounded set U in the case where the g_m are elementary functions on $\mathbb{R}^2$, then we will compute their expectation. Finally we will give explicit results in the case of weighted indicator functions of random sets, obtained from a deterministic compact elementary set and regular region D, by random rotation and dilation. Specific computations for disks with $D = D(0, 1)$, and for squares with $D = [0, 1]^2$ are linked with some recent results on Boolean models.

Throughout the rest of this section, we also assume that for F-almost every $m \in \mathbb{R}^d$, the g_m are elementary functions on $\mathbb{R}^2$, with compact support, satisfying (11) and such that

$$(21) \qquad \int_{\mathbb{R}^d} V_{g_m}(\mathbb{R}^2) F(dm) < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} \mathrm{LTaC}_{g_m}(\mathbb{R}^2) F(dm) < +\infty,$$

where $V_{g_m}(\mathbb{R}^2)$ and $\mathrm{LTaC}_{g_m}(\mathbb{R}^2)$ are defined by (15) and (16) choosing $U = \mathbb{R}^2$. For F-almost every m, g_m is assumed to have a compact support that can be included in a square $[-T_m, T_m]^2$ with $T_m \in \mathbb{R}_+$, and its maximal value $\|g_m\|_\infty = \max_{[-T_m, T_m]^2} |g_m|$ is finite. We will assume moreover that

$$(22) \qquad \int_{\mathbb{R}^d} T_m^2 F(dm) < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} \|g_m\|_\infty F(dm) < +\infty.$$

Note that the first assumption of (22) implies that there is a.s. only a finite random number of g_m, denoted by $N(U)$, contributing to the values of X_Φ on the bounded open set $U \subset (-T, T)^2$, for some $T > 0$. Indeed, it is clear that

$$N(U) \le \#\{i; \overline{U} \cap (x_i + [-T_{m_i}, T_{m_i}]^2) \ne \varnothing\} \le \#\{i; \|x_i\|_\infty \le (T_{m_i} + T)\},$$

with $\|x\|_\infty = \max(|x_1|, |x_2|)$, for $x \in \mathbb{R}^2$. It follows that $\mathbb{E}(N(U)) \le \lambda \int_{\mathbb{R}^d} 4(T_m + T)^2 F(dm)$. Since F is a finite measure, under (22), we get

$$(23) \qquad\qquad\qquad \mathbb{E}(N(U)) < +\infty.$$

In the following, we will use the notation τ_x to denote the translation of x in $\mathbb{R}^2$ (i.e., $\tau_x y = y + x$ for all $y \in \mathbb{R}^2$). We will also denote $\Phi_i = \Phi \setminus \{(x_i, m_i)\}$ for $i \in I$, $\Phi_{ij} = \Phi \setminus \{(x_i, m_i), (x_j, m_j)\}$ for $i \neq j$ in I, and their associated shot noise random fields

$$\forall i, \quad X_{\Phi_i}(x) = \sum_{k; k \neq i} g_{m_k}(x - x_k) \quad \text{and}$$

$$\forall j \neq i, \quad X_{\Phi_{ij}}(x) = \sum_{k; k \neq i, k \neq j} g_{m_k}(x - x_k).$$

THEOREM 4. *Assume that for F-almost every $m \in \mathbb{R}^d$, the function g_m is an elementary function on $\mathbb{R}^2$ (Definition 5) satisfying (11), (21) and (22), and such that*

$$(24) \qquad \int_{\mathbb{R}^d} \mathcal{H}^0(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}) F(dm) < +\infty,$$

$$(25) \qquad \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) \, dx \, F(dm) F(dm') < +\infty,$$

$$(26) \qquad \begin{aligned} &\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\{y \in \mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}; \\ &\quad \nu_{g_{m'}}(y) = \pm \nu_{g_m}(y - x)\}) \, dx \, F(dm) F(dm') = 0. \end{aligned}$$

Then, almost surely, for all bounded open set $U \subset \mathbb{R}^2$, X_Φ is an elementary function on U and its discontinuity set on U is given by $\mathcal{S}_{X_\Phi} \cap U$ where $\mathcal{S}_{X_\Phi} = \bigcup_i \tau_{x_i} g_{m_i}$ may be written as $\mathcal{S}_{X_\Phi} = \mathcal{R}_{X_\Phi} \cup \mathcal{C}_{X_\Phi} \cup \mathcal{I}_{X_\Phi}$, with:

- $\mathcal{R}_{X_\Phi} = (\bigcup_i \tau_{x_i} \mathcal{R}_{g_{m_i}}) \setminus (\bigcup_{i,j \neq} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}})$, *and if $x \in \mathcal{R}_{X_\Phi} \cap U$, there exists a unique i such that $x \in \tau_{x_i} \mathcal{R}_{g_{m_i}}$.*
- $\mathcal{C}_{X_\Phi} = \bigcup_i \tau_{x_i} \mathcal{C}_{g_{m_i}}$, *and if $x \in \mathcal{C}_{X_\Phi} \cap U$, there exists a unique i such that $x \in \tau_{x_i} \mathcal{C}_{g_{m_i}}$.*
- $\mathcal{I}_{X_\Phi} = (\bigcup_i \tau_{x_i} \mathcal{I}_{g_{m_i}}) \cup (\bigcup_{i,j \neq} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}})$ *and if $x \in \mathcal{I}_{X_\Phi} \cap U$, only two situations occur:*
  - *there exists a unique i such that $x \in \tau_{x_i} \mathcal{I}_{g_{m_i}}$.*
  - *or there exists a unique pair $\{i, j\}$ with $i \neq j$ such that $x \in \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}}$.*

In particular, a.s.

$$V_{X_\Phi}(U) = \sum_i V_{g_{m_i}}(\tau_{-x_i} U) \quad \text{and} \quad \text{LTC}_{X_\Phi}(U) = \sum_i \text{LTC}_{g_{m_i}}(\tau_{-x_i} U).$$

PROOF. Since it is sufficient to prove the result for all rectangles $U = (a_1, b_1) \times (a_2, b_2)$ with $a_1, a_2, b_1, b_2 \in \mathbb{Q}$, we only have to prove it holds almost surely on some fixed rectangle $U = (a_1, b_1) \times (a_2, b_2)$, with $a_1 < b_1$ and $a_2 < b_2$. Let us notice that we already have proven that $X_\Phi \in \text{SBV}(U)$ in Theorem 2 of [11] in a more general framework. However, we need here to be more precise on its discontinuity set $\mathcal{S}_{X_\Phi} \subset \bigcup_i \tau_{x_i} \mathcal{S}_{g_{m_i}}$.

Let us first remark that when A_Φ is a finite set of points of $\mathbb{R}^2$ depending on the marked Poisson point process $\Phi = \{(x_i, m_i)\}$, as soon as $\mathbb{E}(\mathcal{H}^0(A_\Phi)) < +\infty$, one has

$$\bigcup_j A_{\Phi_j} \cap \tau_{x_j} \mathcal{S}_{g_{m_j}} = \varnothing \quad \text{a.s.}$$

This follows from the fact that, by the Slivnyak–Mecke formula (see [8] Theorem 1.4.5),

$$\mathbb{E}\left(\mathcal{H}^0\left(\bigcup_j A_{\Phi_j} \cap \tau_{x_j} \mathcal{S}_{g_{m_j}}\right)\right) \leq \lambda \int_{\mathbb{R}^2 \times \mathbb{R}^d} \mathbb{E}(\mathcal{H}^0(A_\Phi \cap \tau_x \mathcal{S}_{g_m})) \, dx \, F(dm)$$

$$\leq \lambda \mathbb{E}(\mathcal{H}^0(A_\Phi)) \int_{\mathbb{R}^d} \mathcal{L}(\mathcal{S}_{g_m}) F(dm) = 0,$$

since $\mathcal{L}(\mathcal{S}_{g_m}) = 0$, using Fubini theorem and the translation invariance of both $\mathcal{H}^0$ and $\mathcal{L}$.

Our first assumption (24) implies that

(27) $$\bigcup_{i,j\neq} \tau_{x_i}(\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}) \cap \tau_{x_j}\mathcal{S}_{g_{m_j}} \cap U = \varnothing \quad \text{a.s.}$$

and as a consequence we have $\bigcup_{i,j\neq} \tau_{x_i}\mathcal{S}_{g_{m_i}} \cap \tau_{x_j}\mathcal{S}_{g_{m_j}} \cap U = \bigcup_{i,j\neq} \tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap U$
a.s. Indeed, taking $A_\Phi = \bigcup_i \tau_{x_i}(\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}) \cap U$, Campbell formula (see [8] Theorem 1.4.3)
ensures that

$$\mathbb{E}(\mathcal{H}^0(A_\Phi)) \leq \lambda \int_{\mathbb{R}^2 \times \mathbb{R}^d} \mathcal{H}^0\big(\tau_x(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}) \cap U\big) dx\, F(dm)$$

$$\leq \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \mathcal{H}^0(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}) F(dm) < +\infty.$$

Then (27) follows from the preceding remark since

$$\bigcup_{i,j\neq} \tau_{x_i}(\mathcal{S}_{g_{m_i}} \setminus \mathcal{R}_{g_{m_i}}) \cap \tau_{x_j}\mathcal{S}_{g_{m_j}} \cap U = \bigcup_j A_{\Phi_j} \cap \tau_{x_j}\mathcal{S}_{g_{m_j}}.$$

The second assumption (25) will ensure both that

(28) $$\mathcal{H}^0\bigg(\bigcup_{i,j\neq} \tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap U\bigg) < \infty \quad \text{a.s.}$$

and

(29) $$\bigcup_{i,j,k\neq} \tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap \tau_{x_k}\mathcal{S}_{g_{m_k}} \cap U = \varnothing \quad \text{a.s.}$$

Here, we set $A_\Phi = \bigcup_{i,j\neq} \tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap U$. Using again the Slivnyak–Mecke formula
and Campbell formula we obtain that

$$\mathbb{E}(\mathcal{H}^0(A_\Phi))$$

$$\leq \lambda^2 \int_{\mathbb{R}^2 \times \mathbb{R}^d} \int_{\mathbb{R}^2 \times \mathbb{R}^d} \mathcal{H}^0(\tau_x\mathcal{R}_{g_m} \cap \tau_{x'}\mathcal{R}_{g_{m'}} \cap U)\, dx\, F(dm)\, dx'\, F(dm')$$

$$\leq \lambda^2 \mathcal{L}(U) \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x\mathcal{R}_{g_m})\, dx\, F(dm)F(dm') < +\infty.$$

It follows that $\mathcal{H}^0(A_\Phi) < +\infty$ a.s. and $\bigcup_k A_{\Phi_k} \cap \tau_{x_k}\mathcal{S}_{g_{m_k}} = \varnothing$ a.s.

Finally and similarly, the last assumption ensures that

(30) $$\left\{ y \in \bigcup_{i,j\neq} \tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap U; \right.$$

$$\left. v_{g_{m_i}}(y - x_i) = \pm v_{g_{m_j}}(y - x_j) \right\} = \varnothing \quad \text{a.s.}$$

This follows from the fact that the expected $\mathcal{H}^0$ measure of this set is zero, according to the
Slivnyak–Mecke formula and Campbell formula, Fubini theorem and translation invariance.

Now, let us remark that X_Φ coincides on U with $X_{\tilde{\Phi}}$ where $\tilde{\Phi} = \{(x_i, m_i) \in \Phi; \tau_{x_i}[-T_{m_i},$
$T_{m_i}] \cap U \neq \varnothing\}$ and $N(U) = \#\tilde{\Phi}$ is a.s. finite, as a consequence of (23) under the assumption
that $\int_{\mathbb{R}^d} T_m^2 F(dm) < +\infty$. Moreover, $X_{\tilde{\Phi}}$ is an elementary function (as a finite sum of ele-
mentary functions in generic position thanks to (28), (29) and (30)) and, therefore, X_Φ is an
elementary function on U. $\square$

THEOREM 5. *Under the hypothesis of Theorem* 4, *assuming moreover that*

$$
\text{(31)} \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} (\|g_m\|_\infty + \|g_{m'}\|_\infty)
$$
$$
\times \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) \, dx \, F(dm) F(dm') < +\infty,
$$

then the random variables $V_{X_\Phi}(U)$, $\mathrm{LTaC}_{X_\Phi}(U)$ *and* $\mathrm{LTC}_{X_\Phi}(U)$ *have finite expectation for any bounded open set* U. *Moreover, one has*

$$
\mathbb{E}\big(V_{X_\Phi}(U)\big) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} V_{g_m}(\mathbb{R}^2) F(dm) \quad and
$$

$$
\mathbb{E}(\mathrm{LTC}_{X_\Phi}(U)) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \mathrm{LTC}_{g_m}(\mathbb{R}^2) F(dm).
$$

It follows that for a.e. $t \in \mathbb{R}$, *the random variables* $\mathrm{Per}(E_{X_\Phi}(t), U)$ *and* $\mathrm{TC}(\partial E_{X_\Phi}(t), U)$ *have also finite expectation such that for any* h *continuous bounded function, denoting by* $\overline{h}_{X_\Phi(0)}$ *the function* $s \mapsto \mathbb{E}(h(X_\Phi(0) + s))$,

$$
\mathbb{E}(\mathrm{LP}_{X_\Phi}(h, U)) = \int_{\mathbb{R}} h(t) \mathbb{E}(\mathrm{Per}(E_{X_\Phi}(t), U)) \, dt = \mathcal{L}(U) \overline{\mathrm{LP}_{X_\Phi}}(h),
$$

and

$$
\mathbb{E}(\mathrm{LTC}_{X_\Phi}(h, U)) = \int_{\mathbb{R}} h(t) \mathbb{E}(\mathrm{TC}(\partial E_{X_\Phi}(t), U)) \, dt = \mathcal{L}(U) \overline{\mathrm{LTC}_{X_\Phi}}(h),
$$

where

$$
\overline{\mathrm{LP}_{X_\Phi}}(h) = \lambda \int_{\mathbb{R}^d} \mathrm{LP}_{g_m}(\overline{h}_{X_\Phi(0)}, \mathbb{R}^2) F(dm),
$$

$$
\overline{\mathrm{LTC}_{X_\Phi}}(h) = \lambda \int_{\mathbb{R}^d} (\mathrm{LTC}_{g_m}(\overline{h}_{X_\Phi(0)}, \mathbb{R}^2) + \lambda I(\overline{h}_{X_\Phi(0)}, m)) F(dm),
$$

with

$$
I(\overline{h}_{X_\Phi(0)}, m)
$$
$$
= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \mathcal{R}_{g_{m'}}} d_{S^1}(v_{g_m}(z - x), v_{g_{m'}}(z))
$$
$$
\times \int_{g_{m'}^-(z)}^{g_{m'}^+(z)} (\overline{h}_{X_\Phi(0)}(s + g_m^+(z - x))
$$
$$
- \overline{h}_{X_\Phi(0)}(s + g_m^-(z - x))) \, ds \, dx \, F(dm').
$$

PROOF. First note that the results on the perimeter follow from Theorem 2 and Proposition 1 of [11] since in view of (11) and (21) we have $\int_{\mathbb{R}^d} \|g_m\|_{\mathrm{BV}(\mathbb{R}^d)} F(dm) < +\infty$. Actually, it yields that

$$
\mathbb{E}\big(\mathrm{LP}_{X_\Phi}(h, U)\big)
$$

$$
= \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \int_{\mathcal{R}_{g_m}} \Big(\int_{g_m^-(z)}^{g_m^+(z)} \mathbb{E}(h(s + X_\Phi(0)) \, ds \Big) \mathcal{H}^1(dz) F(dm)
$$

$$
= \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \int_{\mathcal{R}_{g_m}} (\overline{H}_{X_\Phi(0)}(g_m^+(z)) - \overline{H}_{X_\Phi(0)}(g_m^-(z))) \mathcal{H}^1(dz) F(dm)
$$

$$
= \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \mathrm{LP}_{g_m}(\overline{h}_{X_\Phi(0)}, \mathbb{R}^2) F(dm),
$$

where we introduced $\overline{H}_{X_\Phi(0)}$ to denote a primitive of the function $\overline{h}_{X_\Phi(0)}$.

Moreover, according to Proposition 4, one has a.s.

$$\text{LTaC}_{X_\Phi}(U)$$

$$\leq \sum_i \text{LTaC}_{g_{m_i}}(\tau_{-x_i}U)$$

$$+ 2\pi \sum_{i,j\neq} \mathcal{H}^0(\tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap U)(\|g_{m_i}\|_\infty + \|g_{m_j}\|_\infty).$$

By the Campbell formula,

$$\mathbb{E}\left(\sum_i \text{LTaC}_{g_{m_i}}(\tau_{-x_i}U)\right) = \int_{\mathbb{R}^2 \times \mathbb{R}^d} \text{LTaC}_{g_m}(\tau_{-x}U)\lambda\,dx\,F(dm).$$

Hence, by the Fubini theorem,

$$\mathbb{E}\left(\sum_i \text{LTaC}_{g_{m_i}}(\tau_{-x_i}U)\right) = \lambda\mathcal{L}(U)\int_{\mathbb{R}^d} \text{LTaC}_{g_m}(\mathbb{R}^2)F(dm) < +\infty.$$

Moreover, by the Slivnyak–Mecke formula,

$$\mathbb{E}\left(\sum_{i,j\neq} \mathcal{H}^0(\tau_{x_i}\mathcal{R}_{g_{m_i}} \cap \tau_{x_j}\mathcal{R}_{g_{m_j}} \cap U)(\|g_{m_i}\|_\infty + \|g_{m_j}\|_\infty)\right)$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathcal{H}^0(\tau_x\mathcal{R}_{g_m} \cap \tau_{x'}\mathcal{R}_{g_{m'}} \cap U)$$

$$\times (\|g_m\|_\infty + \|g_{m'}\|_\infty)\lambda^2\,dx\,dx'\,F(dm)F(dm')$$

$$= \lambda^2\mathcal{L}(U)\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbb{1}_{z\in\mathcal{R}_{g_{m'}}\cap\tau_x\mathcal{R}_{g_m}}(\|g_m\|_\infty + \|g_{m'}\|_\infty)$$

$$\times \mathcal{H}^0(dz)\,dx\,F(dm)F(dm')$$

$$= \lambda^2\mathcal{L}(U)\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2}(\|g_m\|_\infty + \|g_{m'}\|_\infty)$$

$$\times \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x\mathcal{R}_{g_m})\,dx\,F(dm)F(dm') < +\infty,$$

by assumption (31). It follows that $\mathbb{E}(\text{LTaC}_{X_\Phi}(U)) < +\infty$ and, therefore,

$$\mathbb{E}\left(\int_{\mathbb{R}}|\text{TC}(\partial E_{X_\Phi}(t), U)|\,dt\right) < +\infty.$$

So, for a.e. $t \in \mathbb{R}$, the random variable $\text{TC}(\partial E_{X_\Phi}(t), U)$ has a finite expectation and $t \mapsto \mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), U))$ is a function in $L^1(\mathbb{R})$. Note that since $\text{LTC}_{X_\Phi}(U) = \sum_i \text{LTC}_{g_{m_i}}(\tau_{-x_i}U)$ we simply get by Campbell formula and Fubini theorem that

$$\mathbb{E}(\text{LTC}_{X_\Phi}(U)) = \lambda\mathcal{L}(U)\int_{\mathbb{R}^d} \text{LTC}_{g_m}(\mathbb{R}^2)F(dm).$$

Now, let h be a continuous bounded function with primitive denoted by H. We already know that a.s. $\text{LTC}_{X_\Phi}(h, U) = \int_{\mathbb{R}} h(t)\,\text{TC}(\partial E_{X_\Phi}(t), U)\,dt$ may be written as the sum of three terms $R_h + C_h + I_h$ with finite expectation under our assumptions. By the Fubini theorem, it follows that

$$\mathbb{E}(\text{LTC}_{X_\Phi}(h, U)) = \int_{\mathbb{R}} h(t)\mathbb{E}(\text{TC}(\partial E_{X_\Phi}(t), U))\,dt = \mathbb{E}(R_h) + \mathbb{E}(C_h) + \mathbb{E}(I_h).$$

For the first term, we have

$$\mathbb{E}(R_h) = \mathbb{E}\left(\int_{\mathcal{R}_{X_\Phi} \cap U} [H(X_\Phi^+(z)) - H(X_\Phi^-(x))] \kappa_{X_\Phi}(z) \mathcal{H}^1(dz)\right)$$

$$= \mathbb{E}\left(\sum_i \int_{\tau_{x_i} \mathcal{R}_{gm_i} \cap U} [H(X_{\Phi_i}(z) + g_{m_i}^+(z - x_i))\right.$$

$$\left. - H(X_{\Phi_i}(z) + g_{m_i}^-(z - x_i))] \kappa_{gm_i}(z - x_i) \mathcal{H}^1(dz)\right)$$

$$= \int_{\mathbb{R}^2 \times \mathbb{R}^d} \int_{\tau_x \mathcal{R}_{gm} \cap U} \mathbb{E}\big([H(X_\Phi(z) + g_m^+(z - x))$$

$$- H(X_\Phi(z) + g_m^-(z - x))]\big) \kappa_{gm}(z - x) \mathcal{H}^1(dz) \lambda \, dx \, F(dm),$$

by the Slivnyak–Mecke formula. Using the translation invariance of both $\mathcal{H}^1$ and $\mathcal{L}$ and the stationarity of X_Φ, we get that

$$\mathbb{E}(R_h) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \int_{\mathcal{R}_{gm}} \int_{g_m^-(z)}^{g_m^+(z)} \mathbb{E}\big(h(X_\Phi(0) + s)\big) \kappa_{gm}(z) \, ds \, \mathcal{H}^1(dz) F(dm)$$

(32)
$$= \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \int_{\mathcal{R}_{gm}} \big(\overline{H}_{X_\Phi(0)}(g_m^+(z))$$

$$- \overline{H}_{X_\Phi(0)}(g_m^-(z))\big) \kappa_{gm}(z) \mathcal{H}^1(dz) F(dm).$$

Similarly, for the second term we have

$$\mathbb{E}(C_h) = \mathbb{E}\left(\sum_{z \in \mathcal{C}_{X_\Phi} \cap U} [H(X_\Phi^+(z)) - H(X_\Phi^-(z))] \alpha_{X_\Phi}(z)\right)$$

$$= \mathbb{E}\left(\sum_i \mathbb{1}_{z \in \tau_{x_i} \mathcal{C}_{gm_i} \cap U} [H(X_{\Phi_i}(z) + g_{m_i}^+(z - x_i))\right.$$

$$\left. - H(X_{\Phi_i}(z) + g_{m_i}^-(z - x_i))] \alpha_{gm_i}(z - x_i) \mathcal{H}^1(dz)\right)$$

(33)
$$= \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \sum_{z \in \mathcal{C}_{gm}} \mathbb{E}\big([H(X_\Phi(0) + g_m^+(z))$$

$$- H(X_\Phi(0) + g_m^-(z))]\big) \alpha_{gm}(z) F(dm)$$

$$= \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \sum_{z \in \mathcal{C}_{gm}} \big(\overline{H}_{X_\Phi(0)}(g_m^+(z))$$

$$- \overline{H}_{X_\Phi(0)}(g_m^-(z))\big) \alpha_{gm}(z) F(dm).$$

Finally, the last term may be itself decomposed in two terms, say $I_h = I_h^{(1)} + I_h^{(2)}$. With similar computations, we get that $\mathbb{E}(I_h^{(1)})$ is equal to

$$\lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \sum_{z \in \mathcal{I}_{gm}} \left(\int_{g_m^{-}\,_+(z)}^{g_m^{+}(z)} \mathbb{E}\big(h(X_\Phi(0) + s)\big)\, ds \right.$$

(34)
$$\left. - \int_{g_m^{-}(z)}^{g_m^{-}\,_+(z)} \mathbb{E}\big(h(X_\Phi(0) + s)\big)\, ds \right) \beta_{gm}(z) F(dm).$$

$$= \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \sum_{z \in \mathcal{I}_{gm}} \big[\overline{H}_{X_\Phi(0)}(g_m^+(z)) + \overline{H}_{X_\Phi(0)}(g_m^-(z))$$

$$- \overline{H}_{X_\Phi(0)}(g_m^{-}\,_+(z)) - \overline{H}_{X_\Phi(0)}(g_m^{-}\,_+(z))\big] F(dm).$$

Adding (32), (33) and (34), we recognize

$$\lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \mathrm{LTC}_{gm}\big(\overline{h}_{X_\Phi(0)}, \mathbb{R}^2\big) F(dm).$$

For the remaining term, let us introduce

$$\Delta_{m_i, m_j} h(t, z - x_i, z - x_j)$$

$$= \int_{g_{m_j}^-(z - x_j)}^{g_{m_j}^+(z - x_j)} \big[h\big(t + g_{m_i}^+(z - x_i) + s\big)$$

$$- h\big(t + g_{m_i}^-(z - x_i) + s\big)\big]\, ds,$$

for $z \in \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}}$. Since $\tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} = \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap \tau_{x_i} \mathcal{R}_{g_{m_i}}$, the term $\mathbb{E}(I_h^{(2)})$ is equal to

$$\mathbb{E}\bigg(\frac{1}{2} \sum_{i \neq j} \sum_{z \in \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \cap U} \Delta_{m_i, m_j} h\big(X_{\Phi_{ij}}(z), z - x_i, z - x_j\big)$$

$$\times d_{S^1}\big(v_{g_{m_i}}(z - x_i), v_{g_{m_j}}(z - x_j)\big) \bigg)$$

$$= \frac{1}{2} \iint \sum_{z \in \tau_x \mathcal{R}_{gm} \cap \tau_{x'} \mathcal{R}_{g_{m'}} \cap U} \mathbb{E}\big(\Delta_{m, m'} h\big(X_\Phi(z), z - x, z - x'\big)\big)$$

$$\times d_{S^1}\big(v_{gm}(z - x), v_{g_{m'}}(z - x')\big) \lambda^2\, dx\, F(dm)\, dx'\, F(dm'),$$

by the Slivnyak–Mecke formula. By change of variables, translation invariance of $\mathcal{H}^0$ and $\mathcal{L}$, and stationarity of X_Φ, we get that $\mathbb{E}(I_h^{(2)})$ is equal to

$$\frac{\lambda^2}{2} \mathcal{L}(U) \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \sum_{z \in \tau_x \mathcal{R}_{gm} \cap \mathcal{R}_{g_{m'}}} \mathbb{E}\big(\Delta_{m, m'} \overline{h}_{X_\Phi(0)}(0, z - x, z)\big)$$

$$\times d_{S^1}\big(v_{gm}(z - x), v_{g_{m'}}(z)\big)\, dx\, F(dm)\, F(dm')$$

$$= \lambda^2 \mathcal{L}(U) \int_{\mathbb{R}^d} I\big(\overline{h}_{X_\Phi(0)}, m\big). \qquad \square$$

REMARK.   Let us notice that choosing $h_u(s) = e^{ius}$ for $u, s \in \mathbb{R}$ we have $\overline{h}_{u, X_\Phi(0)} = \mathbb{E}(e^{iuX_\Phi(0)}) \times h_u$ and by linearity

$$\mathrm{LP}_{gm}\big(\overline{h}_{u, X_\Phi(0)}, \mathbb{R}^2\big) = \mathbb{E}\big(e^{iuX_\Phi(0)}\big) \mathrm{LP}_{gm}(h_u, \mathbb{R}^2) \quad \text{and}$$

$$\text{LTC}_{g_m}\bigl(\overline{h}_{u, X_\Phi(0)}, \mathbb{R}^2\bigr) = \mathbb{E}\bigl(e^{iuX_\Phi(0)}\bigr)\,\text{LTC}_{g_m}\bigl(h_u, \mathbb{R}^2\bigr).$$

It follows that Fourier transforms of $t \mapsto \overline{\text{Per}(E_X(t))}$ and $t \mapsto \overline{\text{TC}(\partial E_X(t))}$ are given for $u \in \mathbb{R}$ by

(35)
$$\overline{\text{LP}_{X_\Phi}}(h_u) = \lambda \mathbb{E}\bigl(e^{iuX_\Phi(0)}\bigr) \int_{\mathbb{R}^d} \text{LP}_{g_m}\bigl(h_u, \mathbb{R}^2\bigr) F(dm)$$

and

$$\overline{\text{LTC}_{X_\Phi}}(h_u)$$

(36)
$$= \lambda \mathbb{E}\bigl(e^{iuX_\Phi(0)}\bigr) \int_{\mathbb{R}^d} \Biggl[\text{LTC}_{g_m}\bigl(h_u, \mathbb{R}^2\bigr)$$
$$+ \frac{\lambda}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \mathcal{R}_{g_{m'}}} \bigl(e^{iug_{m'}^+(z)} - e^{iug_{m'}^-(z)}\bigr)$$
$$\times \int_{g_m^-(z-x)}^{g_m^+(z-x)} e^{ius} d_{S^1}\bigl(\nu_{g_m}(z-x), \nu_{g_{m'}}(z)\bigr) ds\, dx\, F(dm') \Biggr] F(dm).$$

4.3. *Explicit computations.* In this section, we will give some explicit computations of the mean level total curvature function of elementary shot noise random fields. These results generalize the results of Decreusefond et al. [15] obtained for indicator functions of a square, and also the known results on the Boolean model (that correspond to the excursion set of level $t = 1$). We will also show some numerical simulations.

Let us first recall, as already used in Theorem 3, that for shot noise random fields, the characteristic function of $X_\Phi(0)$ is explicit and given by

$$\forall u \in \mathbb{R}, \quad \mathbb{E}\bigl(e^{iuX_\Phi(0)}\bigr) = \exp\Bigl(\lambda \int_{\mathbb{R}^2 \times \mathbb{R}^d} \bigl(e^{iug_m(x)} - 1\bigr) dx\, F(dm)\Bigr).$$

We will consider here D an elementary compact subset of $\mathbb{R}^2$. Its boundary $\Gamma = \partial D$ is a finite union of positively oriented, closed simple curves, piecewise C^2 and of finite total curvature, that is, $\text{TaC}(\Gamma, \mathbb{R}^2) < +\infty$. Note that by the Gauss–Bonnet theorem, we have

$$\text{TC}(\Gamma, \mathbb{R}^2) = 2\pi \chi(D).$$

We will focus on the case where the marks are of the form $m = (b, r, \theta) \in \mathbb{R} \times [0, +\infty) \times [0, 2\pi] \subset \mathbb{R}^d$ with $d = 3$, with distribution $F(dm) = F_B(db) F_R(dr) \times F_\Theta(d\theta)$ and functions g_m given by

$$\forall x \in \mathbb{R}^2, \quad g_m(x) = b \mathbb{1}_{R_\theta r D}(x),$$

where rD is the dilation of D by the factor r, and R_θ denotes the rotation of angle θ. We will denote $b^+ = \max(b, 0)$, $b^- = \min(0, b)$ such that for $x \in \partial R_\theta r D$, we have $g_m^+(x) = b^+$ and $g_{m-}(x) = b^-$ with $g_m^+(x) - g_{m-}(x) = b^+ - b^- = |b|$ and $b = (b^+ - b^-)\,\text{sgn}(b)$.

We denote by B (resp., $B^+ = \max(B, 0)$ and $B^- = \min(B, 0)$), R and Θ independent random variables with distributions F_B, F_R and F_Θ. We will mainly focus on the case where Θ is uniform on $[0, 2\pi]$, that is, $F_\Theta(d\theta) = \frac{1}{2\pi}\mathbb{1}_{[0, 2\pi]}\, d\theta$ for random shapes with uniform rotation; or on the case where $\Theta = 0$ a.s., that is, $F_\Theta = \delta_0$ corresponding to simpler marks $m = (b, r)$.

THEOREM 6. *We assume that*

$$\mathbb{E}(|B|) < +\infty \quad \text{and} \quad \mathbb{E}(R^2) < +\infty.$$

We denote the mean perimeter and the mean area of $R_\theta r D$ by

$$\overline{p} := \int_{\mathbb{R}^2} \mathrm{Per}(R_\theta r D) F_R(dr) F_\Theta(d\theta) = \mathrm{Per}(D)\mathbb{E}(R) \quad and$$

$$\overline{a} := \int_{\mathbb{R}^2} \mathcal{L}(R_\theta r D) F_R(dr) F_\Theta(d\theta) = \mathcal{L}(D)\mathbb{E}(R^2).$$

Then X_Φ satisfies the assumptions of Theorem 4 and 5 and

$$\mathbb{E}(\mathrm{LP}_{X_\Phi}(U)) = \lambda \mathcal{L}(U)\mathbb{E}(|B|)\overline{p} \quad and \quad \mathbb{E}(\mathrm{LTC}_{X_\Phi}(U)) = 2\pi \lambda \mathcal{L}(U)\mathbb{E}(B)\chi(D).$$

Moreover, when F_Θ is the uniform law on $[0, 2\pi]$, then, for all $u \neq 0$,

$$\int_{\mathbb{R}} e^{iut} \overline{\mathrm{Per}(E_{X_\Phi}(t))} \, dt = \lambda \mathbb{E}(e^{iuX_\Phi(0)}) \frac{\widehat{F_{B^+}}(u) - \widehat{F_{B^-}}(u)}{iu} \overline{p},$$

$$\int_{\mathbb{R}} e^{iut} \overline{\mathrm{TC}(\partial E_{X_\Phi}(t))} \, dt$$

$$= \lambda \mathbb{E}(e^{iuX_\Phi(0)})$$

$$\times \frac{1}{iu} \left(2\pi (\widehat{F_B}(u) - 1)\chi(D) + \frac{\lambda}{2} (\widehat{F_{B^+}}(u) - \widehat{F_{B^-}}(u))^2 \overline{p}^2 \right),$$

where $\widehat{F_B}$, respectively $\widehat{F_{B^+}}$ or $\widehat{F_{B^-}}$, denotes the characteristic function of B, respectively, B^+ or B^-. In the case where $B = 1 \, (=B^+)$ a.s., we deduce that $\forall k \in \mathbb{N}$, $\forall t \in (k, k+1]$,

$$\overline{\mathcal{L}(E_{X_\Phi}(t))} = \mathbb{P}(X_\Phi(0) \geq k+1) = 1 - e^{-\lambda \overline{a}} \sum_{l=0}^{k} \frac{(\lambda \overline{a})^l}{l!},$$

while

(37)
$$\overline{\mathrm{Per}(E_{X_\Phi}(t))} = \lambda e^{-\lambda \overline{a}} \frac{(\lambda \overline{a})^k}{k!} \overline{p} \quad and$$

$$\overline{\mathrm{TC}(\partial E_{X_\Phi}(t))} = 2\pi \lambda e^{-\lambda \overline{a}} \frac{(\lambda \overline{a})^k}{k!} \left(\chi(D) - \frac{\lambda}{4\pi} \overline{p}^2 + \frac{\overline{p}^2}{4\pi \overline{a}} k \right).$$

PROOF. Since $\mathcal{H}^1(R_\theta r \Gamma) = r\mathcal{H}^1(\Gamma)$ and $\mathrm{TaC}(R_\theta r \Gamma, \mathbb{R}^2) = \mathrm{TaC}(\Gamma, \mathbb{R}^2)$, the g_m's are elementary functions, and for $F(dm)$ almost every $m = (b, r, \theta)$, writing $L = \mathcal{H}^1(\Gamma) = \mathrm{Per}(D)$, we have

$$V_{g_m}(\mathbb{R}^2) = |b|rL \quad and \quad \mathrm{LTaC}(g_m, \mathbb{R}^2) = |b|\mathrm{TaC}(\Gamma, \mathbb{R}^2).$$

Let us remark that equation (11) becomes

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^2} |g_m(x)| \, dx F(dm) = \mathbb{E}(|B|)\overline{a} < +\infty.$$

Equation (21) is easily checked since

$$\int_{\mathbb{R}^d} V_{g_m}(\mathbb{R}^2) F(dm) = \mathbb{E}(|B|)\mathbb{E}(R)L < +\infty \quad and$$

$$\int_{\mathbb{R}^d} \mathrm{LTaC}_{g_m}(\mathbb{R}^2) F(dm) = \mathbb{E}(|B|)\mathrm{TaC}(\Gamma, \mathbb{R}^2) < +\infty.$$

Assumption (22) also follows from the fact that $\overline{a} < +\infty$ and $\mathbb{E}(|B|) < +\infty$. For (24), let us remark that $\mathcal{I}_{g_m} = \varnothing$ and $\mathcal{C}_{g_m} = R_\theta r \mathcal{C}_\Gamma$ so that $\mathcal{H}^0(\mathcal{S}_{g_m} \setminus \mathcal{R}_{g_m}) = \mathcal{H}^0(\mathcal{C}_\Gamma)$, ensuring (24), by assumption on Γ.

Now, in order to check (25) and (26) we need a kind of *kinematic formula*.

LEMMA 1 (Kinematic formula). *Let f be a measurable nonnegative function on $[-\pi, \pi)^2$, periodically extended to $\mathbb{R}^2$, and let Γ_1, Γ_2 be two piecewise regular Jordan (simple closed) curves. Let $I_{\Gamma_1, \Gamma_2}(f)$ be defined by*

$$I_{\Gamma_1, \Gamma_2}(f) := \int_{\mathbb{R}^2} \sum_{z \in \Gamma_1 \cap \tau_x \Gamma_2} f\left(\mathrm{Arg}\, \nu_{\Gamma_1}(z), \mathrm{Arg}\, \nu_{\Gamma_2}(z - x)\right) dx.$$

Then

$$(38) \qquad I_{\Gamma_1, \Gamma_2}(f) = \int_0^{L_1} \int_0^{L_2} f\left(\theta_1(s_1) + \frac{\pi}{2}, \theta_2(s_2) + \frac{\pi}{2}\right)$$
$$\times \left|\sin(\theta_1(s_1) - \theta_2(s_2))\right| ds_1\, ds_2,$$

where the curve Γ_1 (resp. Γ_2) of length $L_1 = \mathcal{H}^1(\Gamma_1)$ (resp., $L_2 = \mathcal{H}^1(\Gamma_2)$), is parametrized by $s_1 \mapsto \gamma_1(s_1)$ (resp., by $s_2 \mapsto \gamma_2(s_2)$) where s_1 (resp., s_2) is arc length, and $\theta_1(s_1) = \mathrm{Arg}\, \gamma_1'(s_1)$ (resp., $\theta_2(s_2) = \mathrm{Arg}\, \gamma_2'(s_2)$).

PROOF. A heuristic proof of this result is given in the book of Santaló [31]. But it can be made fully rigorous using the co-area formula for Lipschitz mappings. Indeed, if $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is Lipschitz, then for any nonnegative measurable function v on $\mathbb{R}^2$, the co-area formula (see [20] or [7]) states that

$$\int_{\mathbb{R}^2} v(y) |\det(DF(y))|\, dy = \int_{\mathbb{R}^2} \left(\int_{F^{-1}(x)} v(t) \mathcal{H}^0(dt)\right) dx = \int_{\mathbb{R}^2} \sum_{t \in F^{-1}(x)} v(t)\, dx.$$

Now, we use this formula with F given by $F(s_1, s_2) = \gamma_1(s_1) - \gamma_2(s_2)$ for $(s_1, s_2) \in [0, L_1] \times [0, L_2]$ (and extended to $\mathbb{R}^2$ by periodicity, for instance), and v given by $v(s_1, s_2) = f(\theta_1(s_1) + \frac{\pi}{2}, \theta_2(s_2) + \frac{\pi}{2}) \mathbb{1}_{(s_1, s_2) \in [0, L_1] \times [0, L_2]}$. Since Γ_1 and Γ_2 are piecewise regular Jordan curves, they have only a finite number of corner points and, therefore, γ_1 and γ_2 are differentiable except at a finite number of points, and we can compute at a regular point $y = (s_1, s_2)$

$$|\det(DF(y))| = |\det(\gamma_1'(s_1), \gamma_2'(s_2))| = |\sin(\theta_1(s_1) - \theta_2(s_2))|.$$

And we also notice that if $x \in \mathbb{R}^2$ and $t = (s_1, s_2) \in F^{-1}(x)$, then $x = \gamma_1(s_1) - \gamma_2(s_2)$ and therefore $z = \gamma_1(s_1)$ belongs to both Γ_1 and $\tau_x \Gamma_2$, and $\gamma_2(s_2) = z - x$. This concludes the proof of the lemma. $\square$

In particular, taking $f = 1$, it follows that $I_{\Gamma_1, \Gamma_2}(1) \leq L_1 L_2$. Note also that we moreover have the exact formula

$$\int_0^{2\pi} \int_0^{2\pi} I_{R_\theta \Gamma_1, R_{\theta'} \Gamma_2}(1)\, d\theta\, d\theta' = 2\pi \times \int_0^{2\pi} |\sin(\theta)|\, d\theta L_1 L_2 = 2\pi \times 4 L_1 L_2,$$

according to a generalization of Poincaré's formula (see [31], for instance). Using the fact that Γ is a finite disjoint union of Jordan curves, we obtain that $I_{R_\theta r \Gamma, R_{\theta'} r' \Gamma}(1) \leq L^2 r r'$, since the length of $R_\theta r \Gamma$, resp. $R_{\theta'} r' \Gamma$, is rL, resp. $r'L$, with $L = \mathcal{H}^1(\Gamma)$ the length of Γ and

$$\int_0^{2\pi} \int_0^{2\pi} I_{R_\theta r \Gamma, R_{\theta'} r' \Gamma}(1)\, d\theta\, d\theta' = 2\pi \times 4 L^2 r r'.$$

It follows that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m})\, dx\, F(dm) F(dm')$$

$$= \int_{\mathbb{R}_+^2} \int_{[0, 2\pi]^2} I_{R_\theta r \Gamma, R_{\theta'} r' \Gamma}(1) F_\Theta(d\theta) F_\Theta(d\theta') F_R(dr) F_R(dr')$$

$$\leq 4 L^2 \left(\int_{\mathbb{R}^+} r F_r(dr)\right)^2 = 4 \overline{p}^2,$$

with $\overline{p} = L\mathbb{E}(R) = \mathcal{H}^1(\Gamma)\mathbb{E}(R)$ the mean perimeter, proving (25).

Moreover, for $f(\theta_1, \theta_2) = \mathbb{1}_{\theta_1 \equiv \theta_2} + \mathbb{1}_{\theta_1 \equiv \theta_2 + \pi}$, where $\equiv$ stands for equality modulo 2π, we clearly have

$$I_{\Gamma_1, \Gamma_2}(f) = 0,$$

in view of (38). Since Γ is a finite disjoint union of Jordan curves, it follows that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} \mathcal{H}^0(\{y \in \mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m};$$
$$v_{g_{m'}}(y) = \pm v_{g_m}(y - x)\}) \, dx \, F(dm) F(dm')$$
$$= \int_{\mathbb{R}_+^2} \int_{[0,2\pi]^2} I_{R_\theta r \Gamma, R_{\theta'} r' \Gamma}(f) F_\Theta(d\theta) F_\Theta(d\theta') F_R(dr) F_R(dr') = 0,$$

so that (26) holds. Therefore, we get the statement of Theorem 4. Moreover, we also get

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^2} (\|g_m\|_\infty + \|g_{m'}\|_\infty) \mathcal{H}^0(\mathcal{R}_{g_{m'}} \cap \tau_x \mathcal{R}_{g_m}) \, dx \, F(dm) F(dm')$$
$$\leq 4\overline{p}^2 \int_{\mathbb{R} \times \mathbb{R}} (|b| + |b'|) F_B(db) F_B(db') = 8\overline{p}^2 \mathbb{E}(|B|) < +\infty,$$

and (31) is also satisfied so that Theorem 5 holds.

Note that, we simply have

$$\mathbb{E}(V_{X_\Phi}(U)) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} V_{g_m}(\mathbb{R}^2) F(dm) = \lambda \mathcal{L}(U) \mathbb{E}(|B|) \overline{p}$$

and

$$\mathbb{E}(\mathrm{LTC}_{X_\Phi}(U)) = \lambda \mathcal{L}(U) \int_{\mathbb{R}^d} \mathrm{LTC}_{g_m}(\mathbb{R}^2) F(dm) = \lambda \mathcal{L}(U) \mathbb{E}(B) \, \mathrm{TC}(\Gamma, \mathbb{R}^2),$$

using the fact that $\mathrm{LTC}_{g_m}(\mathbb{R}^2) = (b^+ - b^-) \times \mathrm{sgn}(b) \times \mathrm{TC}(\Gamma, \mathbb{R}^2) = b \, \mathrm{TC}(\Gamma, \mathbb{R}^2)$. Since $\mathrm{TC}(\Gamma, \mathbb{R}^2) = 2\pi \chi(D)$, we obtain the first general statement. For $u \neq 0$, we can explicitly compute the characteristic function of $X_\Phi(0)$, given by

$$\mathbb{E}(e^{iuX_\Phi(0)}) = \exp\left(\lambda \int_{\mathbb{R}^d \times \mathbb{R}^2} (e^{iug_m(x)} - 1) F(dm) \, dx\right) = e^{\lambda \overline{a}(\widehat{F}_B(u) - 1)},$$

where $\widehat{F}_B(u) = \mathbb{E}(e^{iuB}) = \int_{\mathbb{R}} e^{iub} F_B(db)$ is the characteristic function of B. We also have, writing h_u as previously,

$$\mathrm{LP}_{g_m}(h_u, \mathbb{R}^2) = \int_{b^-}^{b^+} e^{iut} \mathcal{H}^1(R_\theta r \Gamma) \, dt = r \mathcal{H}^1(\Gamma) \frac{e^{iub^+} - e^{iub^-}}{iu}$$

$$\mathrm{LTC}_{g_m}(h_u, \mathbb{R}^2) = \int_{b^-}^{b^+} e^{iut} \, \mathrm{sgn}(b) \, \mathrm{TC}(R_\theta r \Gamma, \mathbb{R}^2) \, dt$$

$$= \mathrm{TC}(\Gamma, \mathbb{R}^2) \frac{e^{iub^+} - e^{iub^-}}{iu} \, \mathrm{sgn}(b),$$

so that

$$\int_{\mathbb{R}^d} \mathrm{LP}_{g_m}(h_u, \mathbb{R}^2) F(dm) = \overline{p} \frac{\widehat{F_{B^+}}(u) - \widehat{F_{B^-}}(u)}{iu}$$

and

$$\int_{\mathbb{R}^d} \mathrm{LTC}_{g_m}(h_u, \mathbb{R}^2) F(dm) = \mathrm{TC}(\Gamma, \mathbb{R}^2) \frac{\widehat{F_B}(u) - 1}{iu}.$$

Therefore

$$\overline{LP_{X_\Phi}}(h_u) = \lambda \mathbb{E}(e^{iuX_\Phi(0)})\overline{p}\frac{\widehat{F_{B^+}}(u) - \widehat{F_{B^-}}(u)}{iu}.$$

Moreover, the last term in (36) may be expressed as

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^2 \times \mathbb{R}^d} (e^{iub^+} - e^{iub^-})\left(\int_{b^-}^{b^+} e^{ius}\, ds\right)$$

$$\times \sum_{z\in\tau_x\, \mathcal{R}_{g_m}\cap\mathcal{R}_{g_{m'}}} d_{S^1}\big(v_{g_m}(z-x), v_{g_{m'}}(z)\big)\, dx\, F(dm)\, F(dm')$$

(39)
$$= \frac{(\int_0^{+\infty}(e^{iub}-1)F_B(db))^2 + (\int_{-\infty}^0 (1-e^{iub})F_B(db))^2}{iu}$$

$$\times \int_{\mathbb{R}_+^2} \int_{[0,2\pi]^2} I_{R_\theta r\Gamma, R_{\theta'}r'\Gamma}(f)F_\Theta(d\theta)F_\Theta(d\theta')F_R(dr)F_R(dr')$$

$$+ 2\frac{(\int_0^{+\infty}(e^{iub}-1)F_B(db))(\int_{-\infty}^0(1-e^{iub'})F_B(db'))}{iu}$$

$$\times \int_{\mathbb{R}_+^2} \int_{[0,2\pi]^2} I_{R_\theta r\Gamma, R_{\theta'}r'\Gamma}(f_\pi)F_\Theta(d\theta)F_\Theta(d\theta')F_R(dr)F_R(dr'),$$

where $f(\theta_1, \theta_2) = \min(|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|)$ (distance between two angles) and $f_\pi(\theta_1, \theta_2) = f(\pi + \theta_1, \theta_2)$.

This last expression may not be simple to compute. However, assuming from now on that $F_\Theta(d\theta) = \frac{1}{2\pi}\mathbb{1}_{[0,2\pi]}\, d\theta$, we obtain by (38), for two Jordan curves Γ_1, Γ_2,

$$\int_{[0,2\pi]^2} I_{R_\theta\Gamma_1, R_{\theta'}\Gamma_2}(f)F_\Theta(d\theta)F_\Theta(d\theta')$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{L_1}\int_0^{L_2}\big|\sin(\theta_1(s_1) - \theta_2(s_2) - \theta)\big|$$

$$\times \min\big(|\theta_1(s_1) - \theta_2(s_2) - \theta|, 2\pi - |\theta_1(s_1) - \theta_2(s_2) - \theta|\big)\, ds_1\, ds_2\, d\theta$$

$$= L_1 L_2.$$

Similarly, we have

$$\int_{[0,2\pi]^2} I_{R_\theta\Gamma_1, R_{\theta'}\Gamma_2}(f_\pi)F_\Theta(d\theta)F_\Theta(d\theta') = L_1 L_2.$$

Since Γ is a finite disjoint union of such Jordan curves, we get

$$\int_{[0,2\pi]^2} I_{R_\theta r\Gamma, R_{\theta'}r'\Gamma}(f)F_\Theta(d\theta)F_\Theta(d\theta')$$

$$= \int_{[0,2\pi]^2} I_{R_\theta r\Gamma, R_{\theta'}r'\Gamma}(f_\pi)F_\Theta(d\theta)F_\Theta(d\theta') = L^2 rr'.$$

It follows that for uniform rotations, we have

$$\overline{LTC_{X_\Phi}}(h_u)$$

$$= \lambda\mathbb{E}(e^{iuX_\Phi(0)})\frac{1}{iu}\left((\widehat{F_B}(u) - 1)TC(\Gamma, \mathbb{R}^2) + \frac{\lambda}{2}(\widehat{F_{B^+}}(u) - \widehat{F_{B^-}}(u))^2\overline{p}^2\right).$$

When moreover $B = 1$ a.s., $B^+ = 1$ and $B^- = 0$ a.s., we can deduce an exact formula for the area, perimeter and total curvature densities of the excursions of X_Φ. Actually, in this

case $X_\Phi(0)$ follows a Poisson law of parameter $\lambda\overline{a}$, which is enough to compute $\overline{\mathcal{L}(E_{X_\Phi}(t))} = \mathbb{P}(X_\Phi(0) \geq t)$ for any $t \in \mathbb{R}$, and $\frac{\widehat{F_{B^+}}(u) - \widehat{F_{B^-}}(u)}{iu} = \frac{\widehat{F_B}(u) - 1}{iu}$ is the characteristic function of a uniform random variable Z on $[0, 1]$, while $\widehat{F_B}(u)\frac{\widehat{F_B}(u) - 1}{iu}$ is the characteristic function of $Z + 1$. Then, considering Z independent from $X_\Phi(0)$, we can write

$$\overline{LP_{X_\Phi}}(h_u) = \lambda\mathbb{E}(e^{iu[X_\Phi(0)+Z]})\overline{p}, \quad \text{and}$$

$$\overline{LTC_{X_\Phi}}(h_u) = \lambda\left[\mathbb{E}(e^{iu[X_\Phi(0)+Z]})\left(TC(\Gamma, \mathbb{R}^2) - \frac{\lambda}{2}\overline{p}^2\right)\right.$$
$$\left. + \mathbb{E}(e^{iu[X_\Phi(0)+Z+1]})\frac{\lambda}{2}\overline{p}^2\right].$$

Note also that since here the random field X_Φ has integer values, then $\forall k \in \mathbb{N}$,

$$\forall t \in (k, k+1], \quad \{x \in U; X_\Phi(x) \geq t\} = \{x \in U; X_\Phi(x) \geq k+1\}$$

and, therefore,

$$\overline{Per(E_{X_\Phi}(t))} = \overline{Per(E_{X_\Phi}(k+1))} \quad \text{and} \quad \overline{TC(\partial E_{X_\Phi}(t))} = \overline{TC(\partial E_{X_\Phi}(k+1))}.$$

Hence we can conclude that $\forall k \in \mathbb{N}$, $\forall t \in (k, k+1]$,

(40)
$$\overline{Per(E_{X_\Phi}(t))} = \lambda e^{-\lambda\overline{a}}\frac{(\lambda\overline{a})^k}{k!}\overline{p} \quad \text{and}$$

$$\overline{TC(\partial E_{X_\Phi}(t))} = \lambda e^{-\lambda\overline{a}}\frac{(\lambda\overline{a})^k}{k!}\left(TC(\Gamma, \mathbb{R}^2) - \frac{\lambda}{2}\overline{p}^2 + \frac{\overline{p}^2}{2\overline{a}}k\right).$$

Note that, thanks to the Gauss–Bonnet theorem, we have $TC(\Gamma, \mathbb{R}^2) = 2\pi\chi(D)$, so that we can rewrite this as in (37). $\square$

Let us remark that formula (37) only involves the Euler characteristic, the mean perimeter and the mean area of the shapes. When $k = 0$, we find the formula of the mean Euler characteristic density of a rotation invariant Boolean model as obtained by Mecke and Wagner in [28] and by Mecke in [27] stating that

$$\lim_{r \to +\infty}\frac{\mathbb{E}(\chi(\{x \in rU; X_\Phi(x) \geq 1\}))}{\pi\mathcal{L}(rU)} = e^{-\lambda m_0(D)}(\lambda m_2(D) - \lambda^2 m_1(D)^2),$$

with $m_0(D) = \overline{a}$, $m_1(D) = \overline{p}/2\pi$ and $m_2(D) = \chi(D)/\pi$. Actually, defining the Euler characteristic density as $\overline{\chi(E_{X_\Phi}(t))} = \overline{TC(\partial E_{X_\Phi}(t))}/2\pi$ we get, for all $k \in \mathbb{N}$ and $t \in (k, k+1]$,

$$\overline{\chi(E_{X_\Phi}(t))} = \lambda e^{-\lambda\overline{a}}\frac{(\lambda\overline{a})^k}{k!}\left(\chi(D) - \frac{\lambda}{4\pi}\overline{p}^2 + \frac{\overline{p}^2}{4\pi\overline{a}}k\right),$$

recovering the results of Boolean model for $k = 0$ in dimension 2 (see page 389 of [33]).

The typical behavior of $\overline{\chi(E_{X_\Phi}(k))}$, as a function of $k \in \mathbb{N}$, is the following:

• It starts, when k is small, by being negative. This is explained by the fact that $\{X_\Phi \geq k\}$ is essentially made of one big connected component with many small holes in it. In particular, the minimum value of $\overline{\chi(E_{X_\Phi}(k))}$ is achieved for an integer denoted k_-. The explicit value of k_- can be computed from equation (40). The formula is not very nice, but it has a simple asymptotic behavior when λ is large, since then we have

$$k_- = \lambda\overline{a} - \sqrt{\lambda\overline{a}} + O(1).$$

FIG. 7. *Shot noise random field with indicator functions of random disks. This sample has been obtained using Matlab, with a domain of size 2000×2000 pixels, a Poisson point process of intensity $\lambda = 0.001$, and random disks of radius $R = 50$ or $R = 100$ (each with probability 0.5). Top middle and right figure: empirical perimeter and Euler characteristic as a function of the level t (computed thanks to the Matlab functions* bwperim *and* bweuler), *compared with the theoretical values (red stars) of equation* (37). *Bottom line: Three excursion sets corresponding respectively from left to right to the level $t = 15$, $t = 19$ (i.e., the "critical level" where the Euler characteristic turns from negative to positive) and $t = 25$.*

- Then, after k_-, the density $\overline{\chi(E_{X_\Phi}(k))}$ increases and it crosses 0 in the interval that contains k_0 where

$$k_0 = \lambda \overline{a} - \frac{2\pi \overline{a}}{\overline{p}^2} \chi(D) = \lambda \overline{a} + O(1).$$

For this level, there are as many connected components as holes.

- After k_0, the density $\overline{\chi(E_{X_\Phi}(k))}$ is positive and it increases till a value k_+ and afterwards it decreases and goes to 0 as k goes to infinity. As for k_-, the value of k_+ is explicit, and its asymptotic behavior when λ is large is

$$k_+ = \lambda \overline{a} + \sqrt{\lambda \overline{a}} + O(1).$$

EXAMPLE 1 (Random disks). We assume here that $D = D(0, 1)$ is a disk of radius 1, and that $B = 1$ a.s. In this case, we have $\mathrm{TC}(\partial D) = 2\pi \chi(D) = 2\pi$, $\overline{p} = 2\pi \mathbb{E}(R)$ and $\overline{a} = \pi \mathbb{E}(R^2)$. Note also that since $R_\theta r D = r D$ for all θ, whatever F_Θ is, the shot noise random field has the same law as the one with marks given by $\tilde{m} = (b, r) \in [0, +\infty)^2 \subset \mathbb{R}^2$, with distribution $G(d\tilde{m}) = F_B(db) F_R(dr)$. An example of such a random field with comparisons between the theoretical value of $\mathbb{E}(\mathrm{TC}(\partial E_{X_\phi}(t), U))/2\pi$ of Euler characteristic and an empirical estimate on a large domain are shown on Figure 7. The caption of the figure gives the practical and technical details of the simulation.

Let us remark that we can also compute the mean level total curvature for a nonisotropic shape. This is the case of squares, for instance, as developed in the following example.

EXAMPLE 2 (Random squares). We assume here that D is a square of side length 1 and $\Theta = 0$ a.s. with $F(dm) = F_B(db) F_R(dr) \delta_0(d\theta)$ or equivalently that marks are given by

$\tilde{m} = (b, r) \in \mathbb{R} \times [0, +\infty) \subset \mathbb{R}^2$, with distribution $G(d\tilde{m}) = F_B(db) F_R(dr)$. In this case, $\Gamma = \partial D$ is made of four line segments, with $\mathrm{TC}(\Gamma, \mathbb{R}^2) = 2\pi \chi(D) = 2\pi$, $\overline{p} = 4\mathbb{E}(R)$ and $\overline{a} = \mathbb{E}(R^2)$. On the boundary of a square, the curvature is 0, and it has four corner points with a turning angle equal to $\pi/2$. Now, considering the first term of (36), we get for $u \neq 0$ and $h_u(t) = e^{iut}$,

$$\int_{\mathbb{R}^d} \mathrm{LTC}_{g_m}(h_u) F(dm) = \mathrm{TC}(\Gamma, \mathbb{R}^2) \frac{\widehat{F_B}(u) - 1}{iu}.$$

For the second term, using (39) and according to the kinematic formula (38), the only remaining terms are for $\theta_1(s_1) = \theta_2(s_2) \pm \frac{\pi}{2}$ for which

$$f(\theta_1(s_1), \theta_2(s_2)) = \min(|\theta_1(s_1) - \theta_2(s_2)|, 2\pi - |\theta_1(s_1) - \theta_2(s_2)|)$$
$$= \frac{\pi}{2} = f_\pi(\theta_1(s_1), \theta_2(s_2)).$$

It follows that

$$I_{\Gamma_1, \Gamma_2}(f) = \frac{\pi}{2} \times 8 r_1 \times r_2 = I_{\Gamma_1, \Gamma_2}(f_\pi),$$

and note that

$$\int_{\mathbb{R}^2} \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \mathcal{R}_{g_{m'}}} d_{S^1}(\nu_{g_m}(z - x), \nu_{g_{m'}}(z)) dx = \frac{\pi}{2} \times 8 r \times r'.$$

Hence, the second term in (36) becomes

$$\frac{\lambda}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^2 \times \mathbb{R}^d} \sum_{z \in \tau_x \mathcal{R}_{g_m} \cap \mathcal{R}_{g_{m'}}} (e^{iu g_{m'}^+(z)} - e^{iu g_{m'}^-(z)})$$

$$\times \left(\int_{g_m^-(z-x)}^{g_m^+(z-x)} e^{ius} ds \right) d_{S^1}(\nu_{g_m}(z - x), \nu_{g_{m'}}(z)) dx F(dm) F(dm')$$

$$= \lambda \frac{(\widehat{F_{B^+}}(u) - \widehat{F_{B^-}}(u))^2}{iu} 2\pi \int_{\mathbb{R}^2_+} r r' F_R(dr) F_R(dr')$$

$$= 2\pi \lambda \frac{(\widehat{F_{B^+}}(u) - \widehat{F_{B^-}}(u))^2}{iu} \frac{\overline{p}^2}{16}.$$

Finally, in this case

$$\overline{\mathrm{LTC}_{X_\Phi}}(h_u)$$

$$= \lambda \mathbb{E}(e^{iu X_\Phi(0)}) \frac{1}{iu} \left((\widehat{F_B}(u) - 1) \mathrm{TC}(\Gamma, \mathbb{R}^2) + 2\pi \lambda (\widehat{F_{B^+}}(u) - \widehat{F_{B^-}}(u))^2 \frac{\overline{p}^2}{16} \right),$$

and it corresponds to $\int_{\mathbb{R}} e^{iut} \overline{\mathrm{TC}(\partial E_{X_\Phi}(t))} dt$.

For $B = 1$ a.s., taking the inverse Fourier transform as previously, we obtain $\forall k \in \mathbb{N}$, $\forall t \in (k, k + 1]$,

$$(41) \qquad \overline{\chi(E_{X_\Phi}(t))} = \frac{1}{2\pi} \overline{\mathrm{TC}(\partial E_{X_\Phi}(t))} = \lambda e^{-\lambda \overline{a}} \frac{(\lambda \overline{a})^k}{k!} \left(1 - \frac{\lambda}{16} \overline{p}^2 + \frac{\overline{p}^2}{16\overline{a}} k \right).$$

It is illustrated on Figure 8. This formula generalizes one of the results of Decreusefond et al. [15]. Indeed, considering the Boolean model made of squares of constant size $R = 2\varepsilon$ a.s. for some $\varepsilon > 0$, we get for $k = 0$, and $a > 0$,

$$\forall t \in (0, 1], \quad \frac{1}{2\pi} \mathbb{E}(\mathrm{TC}(\partial E_{X_\Phi}(t), (0, a)^2)) = a^2 \overline{\chi(E_{X_\Phi}(t))}$$

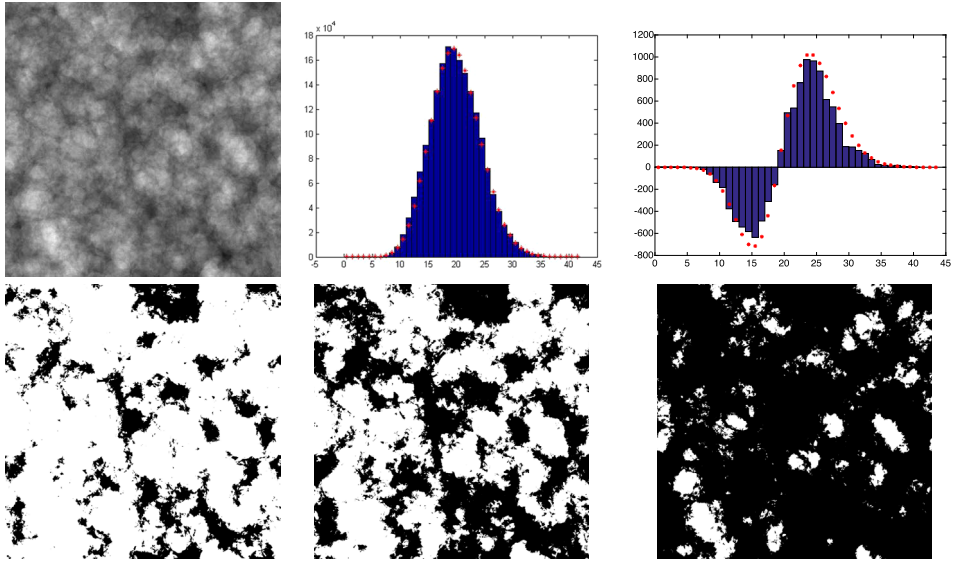$$= \lambda a^2 e^{-\lambda (2\varepsilon)^2} (1 - \lambda (2\varepsilon)^2),$$
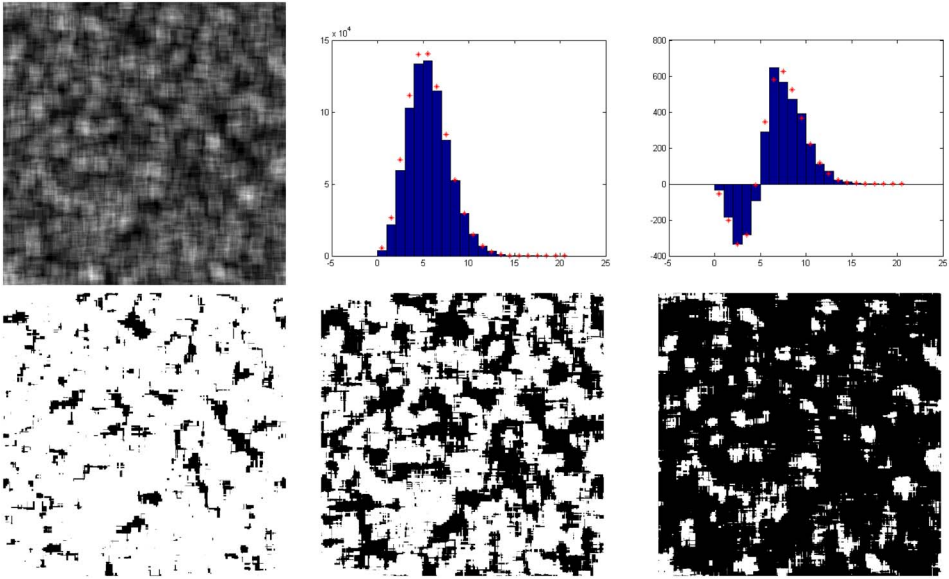
FIG. 8. *Shot noise random field with indicator functions of random squares. This sample has been obtained on a domain of size* 2000×2000 *pixels, with a Poisson point process of intensity* $\lambda = 0.005$, *and random squares of fixed side length* $R = 100$. *Top middle and right figures: empirical perimeter and Euler characteristic as functions of the level* t, *compared with the theoretical value (red stars) of equation* (41). *Bottom line: Three excursion sets corresponding respectively from left to right to the level* $t = 3$, $t = 5$ *(i.e., the "critical level" where the Euler characteristic turns from negative to positive) and* $t = 8$.

that corresponds to the mean Euler characteristic of the Boolean model in dimension 2, considered in the torus of size $a > 0$ in Theorem 11 of [15].

APPENDIX

A.1. Proof of Proposition 2. For the first point, we will use the fact that for all $y \in \mathbb{R}$,

$$\frac{2}{\pi} \int_0^{+\infty} \frac{\sin(uy)}{u} \, du := \lim_{M \to +\infty} \frac{2}{\pi} \int_0^M \frac{\sin(uy)}{u} \, du = \text{sgn}(y),$$

with $\text{sgn}(y) = 1$ if $y > 0$, $\text{sgn}(y) = -1$ if $y < 0$ and $\text{sgn}(y) = 0$ for $y = 0$. We also remark that $C = \sup_{A>0} \frac{2}{\pi} |\int_0^A \frac{\sin(u)}{u} \, du| < +\infty$. Since for $\varepsilon \in (0, 1]$, we have $|\sin(uX_1)| \le |uX_1|^\varepsilon$ and $\mathbb{E}(|W||X_1|^{1+\varepsilon}) < +\infty$, by the Fubini theorem,

$$\frac{2}{\pi} \int_0^M \frac{1}{u} \mathbb{E}(WX_1 \sin(uX_1)) \, du = \mathbb{E}\left(WX_1 \frac{2}{\pi} \int_0^M \frac{\sin(uX_1)}{u} \, du\right).$$

But, as M goes to $+\infty$, $\frac{2}{\pi} \int_0^M \frac{\sin(uX_1)}{u} \, du$ converges to $\text{sgn}(X_1)$ a.s. and, since $|\int_0^M \frac{\sin(uX_1)}{u} \, du| = |\int_0^{M|X_1|} \frac{\sin(u)}{u} \, du|$, it is uniformly bounded by C. Hence the dominated convergence theorem allows to conclude that

$$\lim_{M \to +\infty} \frac{2}{\pi} \int_0^M \frac{1}{u} \mathbb{E}(WX_1 \sin(uX_1)) \, du = \mathbb{E}(WX_1 \, \text{sgn}(X_1)) = \mathbb{E}(W|X_1|).$$

For the second point, we start with the following identity:

$$\frac{x_1 x_2}{x_1^2 + x_2^2} \mathbf{1}_{x_1^2 + x_2^2 > 0} = \frac{4}{\pi} \int_0^{+\infty} \int_0^{+\infty} \frac{v_1 v_2}{(v_1^2 + v_2^2)^2} \sin(v_1 x_1) \sin(v_2 x_2) \, dv_1 \, dv_2.$$

This identity can be proved using the fact that the Fourier transform (or more precisely the sine transform) of a Cauchy–Lorentz function (function of the form $x \mapsto 2a/(x^2 + a^2)$) is a

two-sided exponential function (function of the form $y \mapsto \exp(-a|y|)$). Moreover, we have the following uniform bound.

LEMMA 2. *For $M, M' > 0$ and $x_1, x_2 \in \mathbb{R}$, let us define*

$$F(M, M', x_1, x_2) = \int_0^M \int_0^{M'} \frac{v_1 v_2}{(v_1^2 + v_2^2)^2} \sin(v_1 x_1) \sin(v_2 x_2) \, dv_1 \, dv_2.$$

Then there exists a finite constant $C > 0$ such that

$$\forall M, M' > 0, \quad \forall x_1, x_2 \in \mathbb{R}, \quad |F(M, M', x_1, x_2)| < C.$$

PROOF. Let $x_1, x_2 \in \mathbb{R}$, and let us denote $x_1 = \rho \cos\theta$ and $x_2 = \rho \sin\theta$, with $\rho > 0$ and $\theta \in [0, 2\pi)$. In the integral that defines $F(M, M', x_1, x_2)$, we make a change of variable to use polar coordinates. For $\alpha \in [0, \frac{\pi}{2}]$, we denote $R_\alpha^{M,M'} = \frac{M}{\cos\alpha}$ if $\alpha \leq \arccos\frac{M}{\sqrt{M^2+M'^2}}$, and $R_\alpha^{M,M'} = \frac{M'}{\sin\alpha}$ if $\alpha \geq \arccos\frac{M}{\sqrt{M^2+M'^2}}$. We then get

$$F(M, M', x_1, x_2)$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{R_\alpha^{M,M'}} \frac{r^2 \cos\alpha \sin\alpha}{r^4} \sin(r\rho \cos\theta \cos\alpha) \sin(r\rho \cos\theta \cos\alpha) r \, dr \, d\alpha$$

$$= \int_0^{\frac{\pi}{2}} \cos\alpha \sin\alpha \int_0^{\rho R_\alpha^{M,M'}} \frac{1}{r} \sin(r \cos\theta \cos\alpha) \sin(r \cos\theta \cos\alpha) \, dr \, d\alpha$$

$$= \int_0^{\frac{\pi}{2}} \cos\alpha \sin\alpha \int_0^{\rho R_\alpha^{M,M'}} \frac{1}{2r} \big[\cos(r \cos(\theta + \alpha)) - \cos(r \cos(\theta - \alpha))\big] \, dr \, d\alpha.$$

Now, let $A > 0$, and let $t \in [-1, 1]$. We have, by an integration by part,

$$\int_1^A \frac{1}{r} \cos(rt) \, dr = \frac{\sin(At)}{At} - \frac{\sin(t)}{t} + \int_1^A \frac{\sin(rt)}{r^2 t} \, dr.$$

Using the change of variable $(r \mapsto r|t|)$ and the fact that $|\sin(x)| \leq \min(|x|, 1)$, we can bound the above right-hand side integral by

$$\left| \int_1^A \frac{\sin(rt)}{r^2 t} \, dr \right| \leq \int_{|t|}^{|t|A} \frac{|\sin(r)|}{r^2} \, dr \leq \int_{|t|}^1 \frac{1}{r} \, dr + \int_1^{+\infty} \frac{1}{r^2} \, dr = 1 - \log|t|.$$

If t' is also in $[-1, 1]$, using that, for all $r > 0$, $|\cos(rt) - \cos(rt')| \leq r|t - t'| \leq 2r$, we get

$$\int_0^1 \frac{1}{2r} |\cos(rt) - \cos(rt')| \, dr \leq 1.$$

Putting all this together, we finally have

$$|F(M, M', x_1, x_2)| \leq \int_0^{\frac{\pi}{2}} \left(3 - \frac{1}{2}\log|\cos(\theta + \alpha)| - \frac{1}{2}\log|\cos(\theta - \alpha)|\right) d\alpha$$

$$\leq \frac{3\pi}{2} - \frac{1}{2} \int_0^\pi \log|\sin(x)| \, dx < +\infty. \qquad \square$$

Hence, for M, M' positive real numbers, let

$$Y_{M,M'} := \frac{4}{\pi} \int_0^M \int_0^{M'} \frac{v_1 v_2}{(v_1^2 + v_2^2)^2} \sin(v_1 X_1) \sin(v_2 X_2) \, dv_1 \, dv_2,$$

that is bounded by C according to the previous lemma. Since $\mathbb{E}(|W||X_1 X_2|^{\varepsilon}) < +\infty$, by the Fubini theorem,

$$\frac{4}{\pi} \int_0^M \int_0^{M'} \frac{v_1 v_2}{(v_1^2 + v_2^2)^2} \mathbb{E}\big(W \sin(v_1 X_1) \sin(v_2 X_2)\big) \, dv_1 \, dv_2 = \mathbb{E}(\mathrm{W} \mathrm{Y}_{M,M'}).$$

We have that $Y_{M,M'}$ converges a.s., as M, M' go to infinity, to

$$Y := \frac{4}{\pi} \int_0^{+\infty} \int_0^{+\infty} \frac{v_1 v_2}{(v_1^2 + v_2^2)^2} \sin(v_1 X_1) \sin(v_2 X_2) \, dv_1 \, dv_2$$

$$= \frac{X_1 X_2}{X_1^2 + X_2^2} \mathbf{1}_{X_1^2 + X_2^2 > 0}.$$

Hence the result follows again from the dominated convergence theorem.

A.2. Proof of Proposition 3. As in the proof of Theorem 2, our assumption implies that $\mathbb{E}(W\overline{K}e^{in\Theta} \mathbf{1}_{|J|>0}) = 0$ for all $n \neq 2$. Let $g : 2\pi\mathbb{T} \to \mathbb{R}$ be a continuous bounded 2π periodic function. For $N \geq 1$, we denote by $S_N(g)(\theta) = \sum_{n=-N}^{N} (1 - \frac{|n|}{N}) c_n(g) e^{in\theta}$ the Fejer sum of order N of g, where the Fourier coefficients are given by $c_n(g) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} g(\theta) \, d\theta$. We obtain that

$$\mathbb{E}(W\overline{K} S_N(g)(\Theta) \mathbf{1}_{|J|>0}) = \left(1 - \frac{2}{N}\right) c_2(g) \mathbb{E}(W\overline{K} e^{2i\Theta} \mathbf{1}_{|J|>0}).$$

Since $(S_N(g))_N$ converges uniformly to g that is bounded and $\mathbb{E}(|WK|) < +\infty$, it follows that $\mathbb{E}(W\overline{K} g(\Theta) \mathbf{1}_{|J|>0}) = c_2(g) \mathbb{E}(W\overline{K} e^{2i\Theta} \mathbf{1}_{|J|>0})$, by the dominated convergence theorem. Now for the second point, we also introduce the real and the imaginary part of J, $J_1 = \Re(J)$ and $J_2 = \Im(J)$, so that

$$\sin(2\Theta) \mathbf{1}_{|J|>0} = 2 \frac{J_1 J_2}{J_1^2 + J_2^2} \mathbf{1}_{J_1^2 + J_2^2 > 0}.$$

According to the second point of Proposition 2, we have

$$\mathbb{E}\big(W\overline{K} \sin(2\Theta) \mathbf{1}_{|J|>0}\big)$$

$$= \frac{8}{\pi} \int_0^{+\infty} \int_0^{+\infty} \frac{v_1 v_2}{(v_1^2 + v_2^2)^2} \mathbb{E}\big(W\overline{K} \sin(v_1 J_1) \sin(v_2 J_2)\big) \, dv_1 \, dv_2$$

$$= -\frac{8}{\pi} \int_0^{+\infty} \int_0^{+\infty} \frac{v_1 v_2}{(v_1^2 + v_2^2)^2} \mathbb{E}\big(W\overline{K} \cos(v_1 J_1 + v_2 J_2)\big) \, dv_1 \, dv_2,$$

using the fact that $\mathbb{E}(W\overline{K} \cos(v_1 J_1) \cos(v_2 J_2)) = \mathbb{E}(W(-\overline{K}) \cos(v_1 J_2) \cos(v_2 J_1))$ by (10) for $\theta = \pi/2$. By a change of variables in polar coordinates, we obtain

$$\mathbb{E}(W\overline{K} \sin 2\Theta \mathbf{1}_{|J|>0})$$

$$= -\frac{4}{\pi} \int_0^{+\infty} \int_0^{\pi/2} \mathbb{E}(W\overline{K} \cos(v \Re(Je^{-i\alpha}))) \sin(2\alpha) \, d\alpha \, \frac{dv}{v}$$

$$= -\frac{4}{\pi} \int_0^{+\infty} \int_0^{\pi/2} \mathbb{E}(W\overline{K} e^{-2i\alpha} \cos(v \Re(J))) \sin(2\alpha) \, d\alpha \, \frac{dv}{v}$$

$$= i \int_0^{+\infty} \mathbb{E}(W\overline{K} \cos(v \Re(J))) \, \frac{dv}{v}.$$

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