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# CORRECTION NOTE: A STRONG ORDER 1/2 METHOD FOR MULTIDIMENSIONAL SDES WITH DISCONTINUOUS DRIFT

BY GUNTHER LEOBACHER[1] AND MICHAELA SZÖLGYENYI[2]

*University of Graz and University of Klagenfurt*

There is a gap in the proof of [3], Theorem 3.20. For closing this gap, weak additional assumptions on the regularity of the exceptional set $\Theta$ are needed. In this note, we close the gap and state the corrected version of the main theorems of [3]. The changes we state below only apply from Section 3 onward. The one-dimensional case in Section 2 is not affected.

For the multidimensional case, the function $\phi\colon \mathbb{R} \to \mathbb{R}$ defined in [3], equation (2), needs to be $C^3$; we define

$$\phi(u) = \begin{cases} (1+u)^4(1-u)^4 & \text{if } |u| \le 1, \\ 0 & \text{else}. \end{cases} \tag{1}$$

This function has the properties:

1. $\phi$ is $C^3$ on all of $\mathbb{R}$;
2. $\phi(0) = 1$, $\phi'(0) = 0$, $\phi''(0) = -8$;
3. $\phi(u) = \phi'(u) = \phi''(u) = \phi'''(u) = 0$ for all $|u| \ge 1$.

With this, we define for some $c \in (0, \operatorname{reach}(\Theta))$ and for all $x \in \Theta^c$,

$$G(x) := x + \big(x - p(x)\big) \cdot n\big(p(x)\big) \|x - p(x)\| \phi\left(\frac{\|x - p(x)\|}{c}\right) \alpha\big(p(x)\big), \tag{2}$$

where for all $\xi \in \Theta$,

$$\alpha(\xi) := \lim_{h \to 0+} \frac{\mu(\xi - hn(\xi)) - \mu(\xi + hn(\xi))}{2n(\xi)^\top \sigma(\xi)\sigma(\xi)^\top n(\xi)}. \tag{3}$$

Note that (3) replaces [3], equation (6), and $G$ has precisely the same form as in [3], equation (5), only now we use the new versions of $\alpha$, $\phi$.

Due to the change in the definition of $\phi$, the following lemma needs to be adapted.

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LEMMA 1 (Replaces [3], Lemma 3.18). *Assume* [3], *Assumptions* 3.1–3.4. *Fix* $\varkappa > 1$ *and let*

$$c_0 := \min\biggl(1, \frac{\varepsilon_0}{\varkappa \max(K,1)}, \\ \biggl(1 + \frac{d}{3}\sup_{\xi\in\Theta}\biggl(\max_{1\le i\le d}|\alpha_i(\xi)| + \frac{d}{4}\frac{\varkappa}{\varkappa-1}\max_{1\le i,j\le d}\biggl|\frac{\partial\alpha_i(\xi)}{\partial x_j}\biggr|\biggr)\biggr)^{-1}\biggr).$$

*Then for every choice of* $c \in (0, c_0)$ *we have that* $G'(x)$ *is invertible for every* $x \in \mathbb{R}^d$.

PROOF. Note that $c_0 > 0$, since $\alpha$ and $\alpha'$ are bounded by [3], Assumption 3.4. Let $x \in \mathbb{R}^d$ and recall equation (7) from the proof of [3], Theorem 3.14,

$$\begin{aligned} G'(x) &= \mathrm{id}_{\mathbb{R}^d} + \bar\phi'(\|x - p(x)\|)\alpha(p(x))n(p(x))^\top \\ &\quad + \bar\phi(\|x - p(x)\|)\alpha'(p(x))\mathcal{I}_\xi(\mathscr{T}^{-1}(x))(\mathrm{id}_{\mathbb{R}^d} - n(p(x))n(p(x))^\top) \\ &=: 1 + \mathcal{A}(x). \end{aligned}$$

We begin by estimating the operator norm of $\mathcal{A}(x)$ for given $c \in (0, c_0)$.

$$\begin{aligned} &\|\mathcal{A}(x)\| \\ &\quad\le \|\bar\phi'(\|x - p(x)\|)\| d \max_{1\le i\le d}|\alpha_i(p(x))| \\ &\qquad + \bar\phi(\|x - p(x)\|)\|\mathcal{I}_\xi\|\|\mathrm{id}_{\mathbb{R}^d} - n(p(x))n(p(x))^\top\| d^2 \max_{1\le i,j\le d}\biggl|\frac{\partial\alpha_i(p(x))}{\partial x_j}\biggr| \\ &\quad\le \frac{cd}{3}\max_{1\le i\le d}|\alpha_i(p(x))| + \frac{c^2d^2}{12}\frac{1}{1 - |y_1|\|n'\|}\max_{1\le i,j\le d}\biggl|\frac{\partial\alpha_i(p(x))}{\partial x_j}\biggr|, \end{aligned}$$

where we used that $\|\bar\phi'(\|x - p(x)\|)\| \le \frac{c}{3}$ and $|\bar\phi(\|x - p(x)\|)| \le \frac{c^2}{12}$ for $x \in \Theta^c$ (by estimating the maxima), and that $\|\mathrm{id}_{\mathbb{R}^d} - n(p(x))n(p(x))^\top\| \le 1$. Furthermore, $\|\mathcal{I}_\xi\| \le \frac{1}{1-|y_1|\|n'\|}$, since $\|y_1 n'\| < \frac{1}{\varkappa} < 1$ by $c < \frac{\varepsilon_0}{\varkappa\max(K,1)}$, [3], Lemma 3.17, and [3], Remark 3.16. Hence

$$\frac{1}{1 - |y_1|\|n'\|} \le \frac{\varkappa}{\varkappa - 1}.$$

Therefore, $\|\mathcal{A}(x)\| \le \frac{cd}{3}(\max_{1\le i\le d}|\alpha_i(p(x))| + \frac{cd}{4}\frac{\varkappa}{\varkappa-1}\max_{1\le i,j\le d}|\frac{\partial\alpha_i(p(x))}{\partial x_j}|)$.

We want $c$ small enough to have $\|\mathcal{A}(x)\| < 1$ and to that end we choose $c < 1$ and

$$c < \biggl(1 + \frac{d}{3}\biggl(\max_{1\le i\le d}|\alpha_i(p(x))| + \frac{d}{4}\frac{\varkappa}{\varkappa-1}\max_{1\le i,j\le d}\biggl|\frac{\partial\alpha_i(p(x))}{\partial x_j}\biggr|\biggr)\biggr)^{-1}.$$

Hence $G'(x)$ is invertible for $x \in \Theta^c$ by [3], Lemma 3.17. For $x \in \mathbb{R}^d \setminus \Theta^c$, $G'(x) = \mathrm{id}_{\mathbb{R}^d}$. □

We will need the following additional assumption.

ASSUMPTION 1. The exceptional set $\Theta$ of $\mu$ is $C^4$. Every unit normal vector $n$ of $\Theta$ has a bounded second and third derivative.

LEMMA 2. *Assume* [3], *Assumptions* 3.1 *and* 3.2, *and Assumption* 1. *Let* $c \in (0, \varepsilon_0)$.

*Then the function* $\tilde{\phi} \colon \Theta^c \setminus \Theta \to \mathbb{R}$ *with* $\tilde{\phi}(x) = (x - p(x)) \cdot n(p(x)) \|x - p(x)\| \phi(\frac{\|x-p(x)\|}{c})$ *is three times differentiable with a bounded first*, *second and third derivative*.

PROOF. For $x \in \Theta^c \setminus \Theta$, we have $(x - p(x)) \cdot n(p(x)) \|x - p(x)\| = s d(x, \Theta)^2$ with $s \in \{-1, 1\}$. By [1], Corollary 4.5, $d(\cdot, \Theta)$ is $C^4$ on $\Theta^c \setminus \Theta$.

Since $p'(x)$ maps into the tangent space of $\Theta$ in $p(x)$, it holds that $(x - p(x))^\top p'(x) = 0$. Thus we have $(d(x, \Theta)^2)' = (\|x - p(x)\|^2)' = 2(x - p(x))^\top \times (\mathrm{id}_{\mathbb{R}^d} - p'(x)) = 2(x - p(x))^\top$. Note that $(x - p(x))^\top$ is bounded by $c$ on $\Theta^c$.

The function $p \colon \Theta^c \to \Theta$ is $C^3$ by Assumption 1, [3], Assumptions 3.1 and 3.2, and [1], Theorem 4.1.

By [3], Assumptions 3.1 and 3.2, and [3], Lemma 3.10, the first derivative of every unit normal vector $n$ is bounded, and by Assumption 1 the second and third derivative of $n$ are bounded. Now [2], Corollary 4, implies that $p'$, $p''$, and $p'''$ are bounded on $\Theta^c$.

Now it follows from the chain and product rule that the function $x \mapsto d(x, \Theta)^2$ and its derivatives up to order 4 are bounded on $\Theta^c \setminus \Theta$.

Note further that

$$\phi\left(\frac{\|x - p(x)\|}{c}\right) = \begin{cases} \left(1 - \dfrac{d(x, \Theta)^2}{c^2}\right)^4 & d(x, \Theta) < c, \\ 0 & \text{else.} \end{cases}$$

In total, by the chain and product rule, the first three derivatives of $\tilde{\phi}$ are bounded. □

LEMMA 3. *Assume* [3], *Assumptions* 3.1, 3.2 *and* 3.4, *and Assumption* 1. *Let* $c \in (0, \varepsilon_0)$.

*Then the function* $\alpha \circ p \colon \Theta^c \setminus \Theta \to \mathbb{R}^d$ *is three times differentiable with a bounded first*, *second and third derivative*.

PROOF. By [3], Assumption 3.4, $\alpha$ is three times differentiable with a bounded first, second and third derivative. As shown in the proof of Lemma 2, $p \colon \Theta^c \to \Theta$

is $C^3$ and $p'$, $p''$ and $p'''$ are bounded on $\Theta^c$. The chain and product rules now assure that $(\alpha \circ p)'$, $(\alpha \circ p)''$, $(\alpha \circ p)'''$ are bounded. □

From now on, choose $c$ as in Lemma 1.

LEMMA 4. *Let* [3], *Assumptions* 3.1–3.5, *and Assumption* 1 *be satisfied. Then* $G''$ *is bounded and it is differentiable with bounded derivative on* $\Theta^c \backslash \Theta$.

PROOF. A sufficient condition for this is, by the definition of $G$ and the product rule, that the functions $x \mapsto \tilde{\phi}(x)$ and $x \mapsto \alpha(p(x))$ have this property. This is guaranteed by Lemmas 2 and 3. □

In the proof of [3], Theorem 3.20, we write "in the same way we see that $G''$ is differentiable with bounded derivative on $\Theta^c \backslash \Theta$ and is therefore intrinsic Lipschitz by [3], Lemma 3.8. Moreover, both $G''$ and $\sigma$ are bounded on $\Theta^c \setminus \Theta$." This statement holds under the additional Assumption 1 and is proven in Lemma 4.

THEOREM 5 (Replaces [3], Theorem 3.20). *Let* [3], *Assumptions* 3.1–3.5, *be satisfied. In addition, let Assumption* 1 *hold.*

*Then the SDE for* $G(X)$ *has Lipschitz coefficients.*

THEOREM 6 (Replaces [3], Theorem 3.21). *Let* [3], *Assumptions* 3.1–3.5, *be satisfied. In addition, let Assumption* 1 *hold.*

*Then the* $d$*-dimensional SDE* (1) *has a unique global strong solution.*

THEOREM 7 (Replaces [3], Theorem 3.23). *Let* [3], *Assumptions* 3.1–3.5, *be satisfied. In addition, let Assumption* 1 *hold.*

*Then* [3], *Algorithm* 3.22, *converges with strong order* 1/2 *to the solution* $X$ *of the* $d$*-dimensional SDE* (1).

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## REFERENCES

[1] DUDEK, E. and HOLLY, K. (1994). Nonlinear orthogonal projection. *Ann. Polon. Math.* **59** 1–31. MR1270298

[2] LEOBACHER, G. and STEINICKE, A. (2018). Existence, uniqueness and regularity of the projection onto differentiable manifolds. Available at arXiv:1811.10578.

[3] LEOBACHER, G. and SZÖLGYENYI, M. (2017). A strong order 1/2 method for multidimensional SDEs with discontinuous drift. *Ann. Appl. Probab.* **27** 2383–2418. MR3693529

INSTITUTE OF MATHEMATICS
AND SCIENTIFIC COMPUTING
UNIVERSITY OF GRAZ
HEINRICHSTRASSE 36
8010 GRAZ
AUSTRIA
E-MAIL: gunther.leobacher@uni-graz.at

DEPARTMENT OF STATISTICS
UNIVERSITY OF KLAGENFURT
UNIVERSITÄTSSTRASSE 65-67
9020 KLAGENFURT
AUSTRIA
E-MAIL: michaela.szoelgyenyi@aau.at