The Annals of Applied Probability
2019, Vol. 29, No. 4, 2594–2612
https://doi.org/10.1214/19-AAP1469

PROPAGATION OF CHAOS FOR TOPOLOGICAL INTERACTIONS

BY P. DEGOND[1] AND M. PULVIRENTI

Imperial College London and University of L'Aquila

We consider a N-particle model describing an alignment mechanism due to a topological interaction among the agents. We show that the kinetic equation, expected to hold in the mean-field limit $N \to \infty$, as following from the previous analysis in (*J. Stat. Phys.* **163** (2016) 41–60) can be rigorously derived. This means that the statistical independence (propagation of chaos) is indeed recovered in the limit, provided it is assumed at time zero.

1. Introduction. Propagation of chaos is a fundamental property in kinetic theory: it allows to pass from a N-particle description, which is usually intractable due to the huge number of particles to handle, to a single partial differential equation. Originally, it refers to deterministic particle systems and it has been introduced by Boltzmann in the formal derivation of his famous equation. From the mathematical side, we address the well-known paper by Lanford [25] (see also [6, 10, 12, 18, 19, 34, 35, 40, 42] for subsequent progresses) where the validity of the Boltzmann equation has been proved for a short time interval. On the other hand, other stochastic processes have been introduced to derive the Boltzmann equation and the most famous model is Kac's model [22, 23]. See also [28] and [32] for recent developments. Similar models of interest for the numerics have also been studied for instance in [24, 36, 37]. Nowadays, the methodology and techniques of kinetic theory have been applied also to mean-field limits of particle models in which interactions are averages of binary interactions and which, at the kinetic level, give rise to nonlinear Vlasov (in the deterministic case) or Fokker–Planck (in the stochastic case) equations; see, for example, [7, 11, 16, 20, 26, 30, 41]. For recent approaches to propagation of chaos, see [29].

In most mean-field models, binary interactions are weighted by a function of the relative distance between the two particles. However, recent observations [2, 9] have shown that interactions between animals in nature are weighted by a function of their rank, irrespective of the relative distance, meaning that the interaction probability of an individual with its kth nearest neighbor is the same whether

Received March 2018; revised November 2018.

[1]Supported by the Engineering and Physical Sciences Research Council (EPSRC) under grants EP/M006883/1 and EP/P013651/1 by the Royal Society and the Wolfson Foundation through a Royal Society Wolfson Research Merit Award no. WM130048 and by the National Science Foundation (NSF) under Grant RNMS11-07444 (KI-Net). PD is on leave from CNRS, Institut de Mathématiques de Toulouse, France.

MSC2010 subject classifications. 70K45, 92D50, 91C20.

Key words and phrases. Rank-based interactions, Boltzmann equation.

this individual is close or far. This new type of interaction has been called "topological," by contrast to the usual "metric" interaction which is a function of the subjects' relative distance. Numerical simulations of particle systems undergoing topological interactions seem to support the observations [5, 8, 13]. In the recent past, the literature on the applications of topological interactions to flocking has grown exponentially [17, 21, 31, 38]. On the mathematical side, flocking under topological interactions has been studied in [15, 27, 39, 43]. In [15], mean-field kinetic and fluid models for topological mean-field interactions are formally derived. Recently, [3] and [4] have formally derived kinetic models for jump processes ruled by topological interactions. In the former, the number of particles interacting with a given particle is unbounded in the large particle number limit, while in the latter, particles only interact with a fixed finite number of closest neighbors. In the large particle number limit, the former gives rise to an interaction operator in integral form, while the latter provides a diffusion-like interaction operator.

The goal of this paper is to give a rigorous proof of convergence for the jump process of [3] in the limit of the number of particles tending to infinity, that is, to prove that propagation of chaos holds for this system in this limit, providing a rigorous derivation of the kinetic equation.

Here, new difficulties arise. Indeed in usual metric models particles interact through two-body interactions which are averaged through weights that depend on the distance between the two interacting particles. This structure reflects in the system satisfied by the hierarchy of joint probability distributions (also known as the BBGKY hierarchy): the evolution of the sth marginal only depends on the $(s+1)$th marginal. This structure is lost with topological interactions as the rank of a particle neighbor depends on all the other particles. Now the study of the hierarchy usually describing the time evolution of the marginals is not possible anymore: the time evolution of the s-particle marginal depends on the full N-particle probability measure. Therefore, to prove propagation of chaos, we are facing new, previously unmet, problems.

Obviously, the hierarchical approach is not the only possible one. For instance, we quote [14] where Kac's model has been treated by a coupling technique, yielding by the way, optimal estimates. Such a technique is not easy to apply to the present context. First, the transition probability does not depend on the initial and final state of the jumping pair but on the whole configuration of the N-particle system. However, this is not the main obstruction. For instance, in [33] the coupling technique works in this case for a metric interaction. The difficulty we find here in applying these methods is mostly due to the topological nature of the interaction. All of these previous references use the total variation distance to control the coupled process. A weaker topology as the usual Wasserstein distance works well for the McKean–Vlasov diffusion processes but one can also include suitable jumps; see [1]. Unfortunately, this technique does not apply immediately to the present context due to the very special nature of the jumps considered in [1].

Therefore, our strategy is different. We assume the function that weights the interaction strength with the various partners to be real analytic. For such a kind of interactions, we can establish a new hierarchy for which the time evolution of the j-particle marginal f_j is expressed in terms of an infinite sequence of marginals f_m with $m > j$, with decreasing weight.

2. The model. Here, we recall the setting of [3]. We consider a N-particle system in $\mathbb{R}^d$, $d = 1, 2, 3, \dots$ (or in $\mathbb{T}^d$ the d-dimensional torus). Each particle, say particle i, has a position x_i and velocity v_i. The configuration of the system is denoted by

$$Z_N = \{z_i\}_{i=1}^N = \{x_i, v_i\}_{i=1}^N = (X_N, V_N).$$

Given the particle i, we order the remaining particles $j_1, j_2, \dots, j_{N-1}$ according their distance from i, namely by the following relation:

$$|x_i - x_{j_s}| \le |x_i - x_{j_{s+1}}|, \quad s = 1, 2, \dots, N-1.$$

The rank (with respect to i) of particle $k = j_s$ is s. The rank is denoted by $R(i,k)$.

The normalized rank is defined as

$$r(i,k) = \frac{R(i,k)}{N-1} \in \left\{\frac{1}{N-1}, \frac{2}{N-1}, \dots\right\}.$$

Next, we introduce a (smooth) function

$$K : [0,1] \to \mathbb{R}^+ \quad \text{s.t.} \int_0^1 K(r)\,dr = 1,$$

and the following quantities:

$$\pi_{i,j} = \frac{K(r(i,j))}{\sum_s K(\frac{s}{N-1})}. \tag{2.1}$$

Clearly,

$$\sum_j \pi_{i,j} = 1.$$

We are now in the right position to introduce a stochastic process describing alignment via a topological interaction. The particles go freely, namely following the trajectory $x_i + v_i t$. At some random time dictated by a Poisson process of intensity N, a particle (say i) is chosen with probability $\frac{1}{N}$ and a partner particle, say j, with probability $\pi_{i,j}$. Then the transition

$$(v_i, v_j) \to (v_j, v_j),$$

is performed. After that, the system goes freely with the new velocities and so on.

The process is fully described by the continuous-time Markov generator given, for any $\Phi \in C_b^1(\mathbb{R}^{2dN})$ by

$$\begin{aligned}
&L_N \Phi(x_1, v_1, \dots, x_N, v_N) \\
&\quad = \sum_{i=1}^N v_i \cdot \nabla_{x_i} \Phi(x_1, v_1, \dots, x_N, v_N) \\
&\qquad + \sum_{i=1}^N \sum_{\substack{1 \le j \le N \\ i \neq j}} \pi_{i,j} \big[\Phi(x_1, v_1, \dots, x_i v_j \cdots x_j, v_j \cdots x_N, v_N) \\
&\qquad - \Phi(x_1, v_1, \dots, x_N, v_N) \big].
\end{aligned} \tag{2.2}$$

Note that $\pi_{i,j} = \pi_{i,j}^N$ depends not only on N but also on the whole configuration Z_N.

The law of the process $W^N(Z_N; t)$ is driven by the following evolution equation:

$$\begin{aligned}
&\partial_t \int W^N(t) \Phi \\
&\quad = \int W^N(t) \sum_{i=1}^N v_i \cdot \nabla_{x_i} \Phi \\
&\qquad + \int W^N(t) \sum_{i=1}^N \sum_{\substack{1 \le j \le N \\ i \neq j}} \pi_{i,j} \big[\Phi(x_1, v_1, \dots, x_i v_j \cdots x_j, v_j \cdots x_N, v_N) \\
&\qquad - \Phi(x_1, v_1, \dots, x_N, v_N) \big],
\end{aligned} \tag{2.3}$$

for any test function Φ.

We assume that the initial measure $W_0^N = W^N(0)$ factorizes, namely $W_0^N = f_0^{\otimes N}$ where f_0 is the initial datum for the limiting kinetic equation we are going to establish. Note also that $W^N(Z_N; t)$, for $t \geq 0$, is symmetric in the exchange of particles.

The strong form of equation (2.3) is

$$\left(\partial_t + \sum_{i=1}^N v_i \cdot \nabla_{x_i} \right) W^N(t) = -N W^N(t) + \mathcal{L}_N W^N(t), \tag{2.4}$$

where

$$\mathcal{L}_N W^N(X_N, V_N, t) = \sum_{i=1}^N \sum_{\substack{1 \le j \le N \\ i \neq j}} \int du \, \pi_{i,j} \, W^N\big(X_N, V_N^{(i)}(u)\big) \delta(v_i - v_j). \tag{2.5}$$

Here, $V_N^{(i)}(u) = (v_1 \cdots v_{i-1}, u, v_{i+1} \cdots v_N)$ if $V_N = (v_1 \cdots v_{i-1}, v_i, v_{i+1} \cdots v_N)$.

3. Kinetic description. Here, we present a heuristic derivation of the kinetic equation we expect to be valid in the limit $N \to \infty$. This derivation is slightly simpler than in [3].

We first compute explicitly the transition probability $\pi_{i,j}$. In general,

$$r(i,j) = \frac{1}{N-1} \sum_{\substack{1 \le k \le N \\ k \ne i}} \chi_{B(x_i, |x_i - x_j|)}(x_k),$$

where $\chi_{B(x_i,|x_i-x_j|)}$ is the characteristic function of the ball $\{y \mid |x_i - y| \le |x_i - x_j|\}$. Moreover, recalling that $\int K = 1$,

$$\begin{aligned}\sum_s K\left(\frac{s}{N-1}\right) &= (N-1)\left(1 - \int_0^1 K(x)\,dx + \frac{1}{N-1}\sum_s K\left(\frac{s}{N-1}\right)\right) \\ &= (N-1)\bigl(1 - e_K(N)\bigr),\end{aligned}$$

where the last identity defines $e_K(N)$. Note that e_K measures the difference between the integral and the Riemann sum of K.

Clearly,

$$\bigl|e_K(N)\bigr| \le \|K'\|_{L^\infty} \frac{1}{N-1}. \tag{3.1}$$

Therefore, by (2.1),

$$\pi_{i,j} = \alpha_N K\left(\frac{1}{N-1} \sum_{k \ne i} \chi_{B(x_i,|x_i-x_j|)}(x_k)\right), \tag{3.2}$$

where

$$\alpha_N = \frac{1}{(N-1)(1 - e_K(N))}. \tag{3.3}$$

Setting $\Phi(Z_N) = \varphi(z_1)$ in (2.3), we obtain

$$\partial_t \int f_1^N \varphi = \int f_1^N v \cdot \nabla_x \varphi - \int f_1^N \varphi + \int W^N \sum_{j \ne 1} \pi_{i,j} \varphi(x_1, v_j). \tag{3.4}$$

Here, f_1^N denotes the one-particle marginal of the measure W^N. We recall that the s-particle marginals are defined by

$$f_s^N(Z_s) = \int W^N(Z_s, z_{s+1} \cdots z_N)\, dz_{s+1} \cdots dz_N, \quad s = 1, 2, \dots, N,$$

and are the distribution of the first s particles (or of any group of s tagged particles).

In order to describe the system in terms of a single kinetic equation, we expect that chaos propagates. Actually, since W^N is initially factorizing, although the dynamics creates correlations, we hope that, due to the weakness of the interaction, factorization still holds approximately also at any positive time t, namely

$$f_s^N \approx f_1^{\otimes s}$$

for any fixed integer s. In this case, the strong law of large numbers does hold, that is for almost all i.i.d. variables $\{z_i(0)\}$ distributed according to $f_1(0) = f_0$, the random measure

$$\frac{1}{N}\sum_j \delta\big(z - z_j(t)\big)$$

approximates weakly $f_1^N(z,t)$. Then

$$\begin{aligned}\pi_{i,j} &\approx \frac{1}{N-1} K\biggl(\frac{1}{N-1}\sum_{k\neq i} \chi_{B(x_i,|x_i-x_j|)}(x_k)\biggr) \\ &\approx \frac{1}{N-1} K\big(M_\rho\big(x_i, |x_i - x_j|\big)\big),\end{aligned} \tag{3.5}$$

where

$$M_\rho(x,R) = \int_{B(x,R)} \rho(y)\,dy,$$

and where $\rho(x) = \int dv\, f_1^N(x,v)$ is the spatial density and $B(x,R)$ is the ball of center x and radius R.

In conclusion, we expect that, by (3.4), using the symmetry of W^N, $f_1^N \to f$ and $f_2^N \to f^{\otimes 2}$ in the limit $N \to \infty$, where f solves

$$\begin{aligned}\partial_t \int f\varphi &= \int f v\cdot\nabla_x\varphi - \int f\varphi \\ &\quad + \int f(z_1) f(z_2)\varphi(x_1,v_2) K\big(M_\rho\big(x_1, |x_1 - x_2|\big)\big),\end{aligned} \tag{3.6}$$

or, in strong form,

$$(\partial_t + v\cdot\nabla_x) f = -f + \rho(x)\int dy\, K\big(M_\rho\big(x, |x-y|\big)\big)\, f(y,v), \tag{3.7}$$

which is the equation we want to derive rigorously.

As regards existence and uniqueness of the solutions to equation (3.7), we can apply the Banach fixed-point theorem in find a unique solution for (3.7) in mild form, for a short time interval, provided that K has bounded derivative in $[0,1]$. Actually, we realize that the map

$$\begin{aligned} g(x,v,t) \to{}& e^{-t} f_0(x - vt, v) \\ &+ \int_0^t d\tau \int dy\, \rho_{g(\tau)}\big(x - v(t-\tau), v\big) e^{-(t-\tau)} \\ &\times K\big(M_{\rho_{g(\tau)}}\big(x - v(t-\tau), |x - v(t-\tau) - y|\big)\big) g(y,v,\tau),\end{aligned} \tag{3.8}$$

where $\rho_{g(\tau)} = \int dv\, g(\cdot, v, \tau)$, is a contraction in $C([0,T];L^1)$ provided that T is small enough.

The global solution is recovered by the conservation of the $L^1(x,v)$ norm. The method is classical and we leave the details to the reader.

4. Hierarchies. We assume the function K to be expressible in terms of a power series,

$$K(x) = \sum_{m=0}^{\infty} a_m x^m, \quad x \in [0, 1], \tag{4.1}$$

for some sequence of coefficients a_m. The normalization condition gives the constraint $a_0 + \sum_{m=1}^{\infty} \frac{1}{m+1} a_m = 1$. Note that the coefficients a_m are not necessarily positive.

We further assume that

$$A := \sum_{m=0}^{\infty} |a_m| 8^m < +\infty. \tag{4.2}$$

REMARK. An example of a function K satisfying the above hypotheses is, for $x \in (0, 1)$:

$$K(x) = \frac{e^{1-x} - 1}{e - 2} = \frac{1}{e - 2}\left(e - 1 + e \sum_{r \geq 1} \frac{(-1)^r x^r}{r!}\right).$$

To outline the behavior of the s-particle marginal f_s^N, we integrate (2.4) with respect to the last $N - s$ variables and compute preliminarily

$$\begin{aligned}&\sum_{i=s+1}^{N} \sum_{\substack{1 \leq j \leq N \\ i \neq j}} \int du \, \pi_{i,j} W^N\big(X_N, V_N^{(i)}(u)\big) \delta(v_i - v_j) \, dz_{s+1} \cdots dz_N \\ &\quad = (N - s) f_s^N(X_s, V_s),\end{aligned}$$

since the variable z_i is integrated. Therefore,

$$\begin{aligned}&\left(\partial_t + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i}\right) f_s^N(t) \\ &\quad = -s f_s^N(t) + E_s^1(t) \\ &\qquad + (N - s) \sum_{i=1}^{s} \int dz_{s+1} \cdots dz_N \, \pi_{i,s+1} W^N\big(X_N, V_N^{(i,s+1)}; t\big),\end{aligned} \tag{4.3}$$

where

$$V_N^{(i,s+1)} = \{v_1 \cdots v_{i-1}, v_{s+1}, v_{i+1} \cdots v_s, v_i, v_{s+2} \cdots v_N\},$$

namely the velocities of particles i and $s + 1$ exchange their positions in the sequence $V_N = \{v_1 \cdots v_N\}$, and

$$E_s^1(t) = \sum_{i=1}^{s} \sum_{\substack{1 \leq j \leq s \\ i \neq j}} \int du \, dz_{s+1} \cdots dz_N \, \pi_{i,j} W^N\big(X_N, V_N^{(i)}(u); t\big) \delta(v_i - v_j). \tag{4.4}$$

We expect E_s^1 to be $O(\frac{s^2}{N})$ since $\pi_{i,j} = O(\frac{1}{N})$ (see (3.2) and (3.3)). This is the first error term entering in the present analysis. A precise estimate of this term is forthcoming. Note also that we used the symmetry to deduce the last term in the right-hand side of (4.3).

Next, setting $\chi_{i,j} = \chi_{B(x_i,|x_i-x_j|)}$, we have from (3.2) and (4.1)

$$\pi_{i,j} = \alpha_N \sum_{r=0}^{\infty} a_r \frac{1}{(N-1)^r} \sum_{(k_1,k_2,\dots,k_r)\in(\{1,N\}\setminus\{i\})^r} \chi_{i,j}(x_{k_1})\cdots\chi_{i,j}(x_{k_r}). \tag{4.5}$$

Inserting this quantity into the last term of (4.3), we obtain

$$\begin{aligned}\left(\partial_t + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i}\right) f_s^N(t) &= -s f_s^N(t) + E_s^1(t) + E_s^2 \\ &\quad + (N-s)\alpha_N \sum_{r=0}^{\infty} a_r\, C_{s,s+r+1}^N\, f_{s+r+1}^N,\end{aligned} \tag{4.6}$$

where $C_{s,s+r+1}^N : L^1(\mathbb{R}^{2d(s+r+1)}) \to L^1(\mathbb{R}^{2ds})$ is a linear operator defined by

$$\begin{aligned} & C_{s,s+r+1}^N g_{s+r+1}(X_s, V_s) \\ &\quad = \frac{(N-s-1)\cdots(N-s-r)}{(N-1)^r} \\ &\qquad \times \sum_{i=1}^{s} \int dz_{s+1}\cdots dz_{s+r+1} \chi_{i,s+1}(x_{s+2})\cdots\chi_{i,s+1}(x_{s+r+1}) \\ &\qquad \times g_{s+r+1}\big(X_{s+r+1}, V_{s+r+1}^{(i,s+1)}\big).\end{aligned} \tag{4.7}$$

The form (4.7) of the operator $C_{s,s+r+1}^N$ comes from considering in the sum $\sum_{k_1,k_2,\dots,k_r}$ in (4.5), only the contributions given by

$$\sum_{\substack{k_1\neq k_2\neq\cdots\neq k_r \\ k_m > s+1; m=1,\dots,r}},$$

namely all the k_m are different and larger than $s+1$. Clearly, we also used the symmetry. The term E_s^2 is what remains, namely

$$\begin{aligned} E_s^2(Z_s) &= (N-s)\alpha_N \sum_{i=1}^{s}\sum_{r=0}^{\infty} a_r \left(\frac{1}{N-1}\right)^r \\ &\qquad \times \sum_{k_1,k_2,\dots,k_r}^{*} \int dz_{s+1}\cdots dz_N \chi_{i,s+1}(x_{k_1})\cdots\chi_{i,s+1}(x_{k_r}) \\ &\qquad \times W^N(Z_s, z_{s+1}\cdots z_N; t),\end{aligned} \tag{4.8}$$

with

$$\sum_{k_1,k_2,\dots,k_r}^{*} = \sum_{\substack{k_1,k_2,\dots,k_r\\ k_m\neq i, m=1,\dots,r}} - \sum_{\substack{k_1\neq k_2\neq\cdots\neq k_r\\ k_m>s+1, m=1,\dots,r}}.$$

Again we expect that E_s^2 is negligible in the limit as we shall see in a moment.

Note that for $s = N$ (4.6) becomes identical to equation (2.3) as the last two terms are equal to zero. We will also use the convention that $f_s^N(t) = 0$ if $s > N$.

We have to compare equation(4.6) with a similar hierarchy satisfied by the sequence of marginals $f_j(t) = f^{\otimes j}(t)$, where f solves the kinetic equation. Such a hierarchy is easily recovered. Indeed coming back to the kinetic equation (3.7) we observe that, by virtue of (4.1),

$$\begin{aligned} &K\big(M_\rho(x_i, |x_i - x_{s+1}|)\big)\\ &\quad = \sum_r a_r \int dz_{s+2}\cdots dz_{s+r+1}\chi_{i,s+1}(x_{s+2})\cdots\chi_{i,s+1}(x_{k_{s+r+1}})\\ &\qquad \times f^{\otimes r}(z_{s+2}\cdots z_{s+r+1}), \end{aligned} \tag{4.9}$$

and (3.7) becomes (recalling that $z_1 = (x_1, v_1)$):

$$\begin{aligned} &(\partial_t + v_1\cdot\nabla_{x_1}) f(z_1,t) + f(z_1,t)\\ &\quad = \sum_{r=0}^{\infty} a_r \int dz_2\cdots\int dz_{2+r}\chi_{1,2}(x_3)\cdots\chi_{1,2}(x_{2+r})\\ &\qquad \times f(x_1, v_2; t) f(x_2, v_1; t) f^{\otimes r}(z_3\cdots z_{2+r}; t). \end{aligned} \tag{4.10}$$

As a consequence, an easy computation shows that $f_s = f^{\otimes s}$ solves

$$\left(\partial_t + \sum_{i=1}^{s} v_i\cdot\nabla_{x_i}\right) f_s(t) = -s f_s(t) + \sum_{r=0}^{\infty} a_r\, C_{s,s+r+1}\, f_{s+r+1}, \tag{4.11}$$

where

$$\begin{aligned} &C_{s,s+r+1} f_{s+r+1}(X_s, V_s)\\ &\quad = \sum_{i=1}^{s}\int dz_{s+1}\cdots dz_{s+r+1}\chi_{i,s+1}(x_{s+2})\cdots\chi_{i,s+1}(x_{s+r+1})\\ &\qquad \times f_{s+r+1}\big(X_{s+r+1}, V_{s+r+1}^{(i,s+1)}\big). \end{aligned} \tag{4.12}$$

In view of the comparison of f_s^N with f_s, we rewrite (4.6) as

$$\left(\partial_t + \sum_{i=1}^{s} v_i\cdot\nabla_{x_i}\right) f_s^N(t) = -s f_s^N(t) + E_s(t) + \sum_{r=0}^{\infty} a_r\, C_{s,s+r+1}\, f_{s+r+1}^N, \tag{4.13}$$

where

$$E_s = E_s^1(t) + E_s^2(t) + E_s^3(t) \tag{4.14}$$

and

$$E_s^3(t) = (N-s)\alpha_N \sum_{r=0}^{\infty} a_r\, C_{s,s+r+1}^N\, f_{s+r+1}^N - \sum_{r=0}^{\infty} a_r\, C_{s,s+r+1}\, f_{s+r+1}^N. \tag{4.15}$$

The initial conditions for (4.13) and (4.11) are

$$f_s^N(0) = f_0^{\otimes s}\mathbf{1}_{\{s\le N\}},$$

where $\mathbf{1}_{\{s\le N\}}$ is the indicator of the set $\{s \le N\}$ and

$$f_s(0) = f_0^{\otimes s},$$

respectively. Here, $f_0 \in L^1$ is the initial datum of the kinetic equation.

5. Estimates of the error term. In this section, we establish some estimates of the error term E_s appearing in equation (4.13).

We observe preliminarily that, by the particular form of the function K given by (4.1), we have, $\|K'\|_{L^\infty} \le A$ and, using (3.1),

$$\left|e_K(N)\right| \le \frac{A}{N-1}. \tag{5.1}$$

Therefore,

$$\alpha_N = \frac{1}{(N-1)(1-e_K(N))} \le \frac{4e^{|e_K(N)|}}{N-1} \le \frac{4e^{\frac{A}{N-1}}}{N-1}, \tag{5.2}$$

for $N > 2A+1$. This follows by the obvious inequality

$$\frac{1}{1-x} \le 4e^x$$

valid for $x \in (0, \frac{1}{2})$

As a consequence, by (3.2) and from the fact that $\|K\|_{L^\infty} \le A$,

$$\pi_{i,j} \le \alpha_N A \le \frac{4Ae^{\frac{A}{N-1}}}{N-1}. \tag{5.3}$$

The operators C^N and C are easily estimated:

$$\max\left(\|C_{s,s+r+1}^N g_{s+r+1}\|_{L^1}, \|C_{s,s+r+1} g_{s+r+1}\|_{L^1}\right) \le s\|g_{s+r+1}\|_{L^1}, \tag{5.4}$$

due to the fact that $\chi \le 1$ and that the prefactor in formula (4.7) is less than unity.

As regards the error terms (4.4), we have by (5.3),

$$\|E_s^1(t)\|_{L^1} \le s^2 \frac{4Ae^{\frac{A}{N-1}}}{N-1}. \tag{5.5}$$

Strictly speaking here, we make a notational abuse. E^1 is a measure so that $\|E_s^1(t)\|_{L^1}$ has to be understood as the total variation norm. In other words, $\|\mu\|_{L^1}$ is the L^1 norm of the densities whenever μ is absolutely continuous. Otherwise, it is the total variation.

Moreover, by (4.8) and (5.2),

$$\|E_s^2(t)\|_{L^1} \leq 4e^{\frac{A}{N-1}}\left(\frac{N-s}{N-1}\right)\sum_{i=1}^{s}\sum_{r=0}^{\infty}|a_r|\left(\frac{1}{N-1}\right)^r \sum_{k_1,k_2,\dots,k_r}^{*} 1. \tag{5.6}$$

But

$$\sum_{k_1,k_2,\dots,k_r}^{*} 1 \leq \sum_{k_1,k_2,\dots,k_r}^{**} 1 + \sum_{k_1,k_2,\dots,k_r}^{***} 1,$$

where $\sum_{k_1,k_2,\dots,k_r}^{**} 1$ means that $k_m \leq s+1$ for at least one $m = 1, 2, \dots, r$, while $\sum_{k_1,k_2,\dots,k_r}^{***}$ means that all the k_m are larger than $s+1$ but $k_\ell = k_m$ for at least one couple ℓ, m in $1, 2, \dots, r$.

Moreover, denoting by ℓ the number of indices m for which $k_m \leq s+1$, we have

$$\begin{aligned}\sum_{k_1,k_2,\dots,k_r}^{**} 1 &= \sum_{\ell=1}^{r}\binom{r}{\ell}s^\ell(N-s-1)^{r-\ell}\\ &= (N-1)^r - (N-s-1)^r \leq rs\,(N-1)^{r-1},\end{aligned}$$

where in the last step we used the Taylor expansion of the function x^r with initial point $N-s-1$.

Furthermore,

$$\sum_{k_1,k_2,\dots,k_r}^{***} 1 \leq \frac{r(r-1)}{2}(N-s-1)^{r-1}.$$

Therefore,

$$\begin{aligned}\|E_s^2(t)\|_{L^1} &\leq 4e^{\frac{A}{N-1}} s\sum_{r=0}^{\infty}|a_r|\\ &\quad\times\frac{1}{(N-1)^r}\left(rs(N-1)^{r-1} + \frac{r(r-1)}{2}(N-s-1)^{r-1}\right)\\ &\leq 8e^{\frac{A}{N-1}}\frac{s^2}{N-1}\sum_{r=0}^{\infty}|a_r|r^2 \leq 8Ae^{\frac{A}{N-1}}\frac{s^2}{N-1},\end{aligned} \tag{5.7}$$

where we used that the sum in the second inequality is bounded by A due to (4.2) and the fact that $r^2 \leq 8^r$.

To estimate E_s^3, we have

$$E_s^3 = E_s^{3,1} + E_s^{3,2},$$

where

$$E_s^{3,1}(t) = -T_1 \sum_{r=0}^{\infty} a_r \, C_{s,s+r+1}^N \, f_{s+r+1}^N \tag{5.8}$$

and

$$E_s^{3,2}(t) = T_2 \sum_{r=0}^{\infty} a_r \, C_{s,s+r+1} \, f_{s+r+1}^N, \tag{5.9}$$

where

$$T_1 := 1 - (N-s)\alpha_N$$

and

$$T_2 := \frac{(N-s-1)\cdots(N-s-r)}{(N-1)^r} - 1.$$

Moreover,

$$\begin{aligned} T_1 &= 1 - \frac{N-s}{(N-1)(1-e_K(N))} \\ &= \frac{s-1}{(N-1)(1-e_K(N))} - \frac{e_K(N)}{(1-e_k(N))}. \end{aligned}$$

Therefore, since $A > 1$, using (5.1) and (5.2), we obtain

$$\begin{aligned} |T_1| &\le \frac{s-1}{(N-1)} 4e^{|e_K(N)|} + 4\frac{A}{N-1} e^{|e_K(N)|} \\ &\le 4e^{\frac{A}{N-1}} \left(\frac{s-1}{N-1} + \frac{A}{N-1} \right) \\ &\le 8Ae^{\frac{A}{N-1}} \frac{s}{N-1}. \end{aligned} \tag{5.10}$$

Finally,

$$\begin{aligned} |T_2| &\le \left| \frac{(N-s-1)\cdots(N-s-r)}{(N-1)^r} - 1 \right| \\ &\le \left| \frac{(N-s-r)^r - (N-1)^r}{(N-1)^r} \right| \\ &\le \frac{r(s+r)(N-1)^{r-1}}{(N-1)^r} \le \frac{2r^2 s}{N-1}. \end{aligned} \tag{5.11}$$

As matter of facts by using (5.4), we conclude that

$$\left\| E_s^3(t) \right\|_{L^1} \le 10A^2 e^{\frac{A}{N-1}} \frac{s^2}{N-1}. \tag{5.12}$$

Summarizing, we have the following.

PROPOSITION 1. *We have*

$$\|E_s(t)\|_{L^1} \le 22A^2 e^{\frac{A}{N-1}} \frac{s^2}{N-1}. \tag{5.13}$$

6. Convergence. In this section, we estimate the quantity

$$\Delta_s^N(t) = f_s^N(t) - f_s(t), \tag{6.1}$$

where $f_s^N(t)$ and $f_s(t)$ solve the initial value problems (4.13) and (4.11), respectively. Taking the difference between (4.13) and (4.11), we have

$$\left(\partial_t + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i}\right)\Delta_s^N(t) = -s\Delta_s^N(t) + E_s(t) + \sum_{r=0}^{\infty} a_r\, C_{s,s+r+1}\, \Delta_{s+r+1}^N, \tag{6.2}$$

with initial datum

$$\Delta_s^N(0) = -f_0^{\otimes s}\mathbf{1}_{\{s>N\}},$$

where C and E are given by (4.12) and (4.14).

We define the operator $S_j(t) : L^1(X_j, V_j) \to L^1(X_j, V_j)$ by

$$\big(S_j(t) f_j\big)(X_j, V_j) = e^{-jt} f_j(X_j - V_j t, V_j),$$

and notice that

$$\|S_j(t)\|_{L^1\to L^1} \le 1, \tag{6.3}$$

where $\|\cdot\|_{L^1\to L^1}$ denotes the operator norm.

We can express (6.2) in integral form

$$\begin{aligned} \Delta_j^N(t) &= S_j(t-t_1)\Delta_j^N(t_1) \\ &\quad + \int_{t_1}^{t} d\tau\, S_j(t-\tau) \sum_{r=0}^{\infty} a_r\, C_{j,j+r+1}\, \Delta_{j+r+1}^N(\tau) \\ &\quad + \int_{t_1}^{t} d\tau\, S_j(t-\tau) E_j(\tau). \end{aligned} \tag{6.4}$$

for any $t_1 \in [0, t)$.

Therefore, we can represent the solution $\Delta_j^N(t)$ as a series expansion in terms of the initial datum $\Delta_j^N(t_1)$ and $E_j(s)$. To this end, we define the operator $\mathcal{T}_n(t, t_1)$ by recurrence. For any sequence $F = \{F_j\}_{j=1}^{\infty}$, $F_j \in L^1(X_j, V_j)$, set

$$\big(\mathcal{T}_0(t, t_1)F\big)_j = S_j(t-t_1)F_j$$

and

$$\big(\mathcal{T}_n(t, t_1)F\big)_j = \int_{t_1}^{t} d\tau\, S_j(t-\tau) \sum_{r=0}^{\infty} a_r\, C_{j,j+r+1}\, \big(\mathcal{T}_{n-1}(\tau, t_1)F\big)_{j+r+1}.$$

Therefore, denoting by Δ^N and E the sequences $\{\Delta_j\}_{j=1}^\infty$ and $\{E_j\}_{j=1}^\infty$, respectively, by a standard computation we have

$$\Delta^N(t) = \sum_{n\geq 0} \mathcal{T}_n(t,t_1)\Delta^N(t_1) + \sum_{n\geq 0}\int_{t_1}^t ds\, \mathcal{T}_n(t,\tau)E(\tau). \tag{6.5}$$

We are now in position to establish the main result of the present paper.

THEOREM 1. *For any $T>0$ and $\alpha > \log 2$, there exists $N(T,\alpha)$ such that for any $t\in(0,T)$, any $j\in\mathbb{N}$ and for any $N > N(T,\alpha)$, we have*

$$\|\Delta_j^N(t)\|_{L^1} \leq 2^j\left(\frac{1}{N-1}\right)^{e^{-\alpha(8At+1)}}. \tag{6.6}$$

REMARK. Note that according to (6.6) the quality of the order of convergence rate deteriorates with increasing time. Note also that the magnitude of the error increases exponentially with the order j of the marginals. In particular, if j increases with N too fast, correlations are persistent in the limit $N\to\infty$.

PROOF. The proof follows two steps. First, we estimate $\mathcal{T}_n(t,t_1)$, and hence $\Delta^N(t)$ for a short time interval $\delta = t - t_1$. Then we split the time interval $(0,t)$ into m intervals of length δ, with δ small enough, to obtain the result inductively.

6.1. *Short time estimate.* We first observe, using (6.3), that

$$\|(\mathcal{T}_n(t,t_1)F)_j\|_{L^1} \leq j\sum_{r=0}^\infty |a_r|\int_{t_1}^t d\tau\, \|(\mathcal{T}_{n-1}(\tau,t_1)F)_{j+r+1}\|_{L^1}. \tag{6.7}$$

Iterating this inequality and using, for $t > t_1$,

$$\int_{t_1}^t d\tau_1 \int_{t_1}^{\tau_1} d\tau_2 \cdots \int_{t_1}^{\tau_{n-1}} d\tau_n = \frac{(t-t_1)^n}{n!},$$

we obtain, for any $F = \{F_j\}_{j=1}^\infty$, setting $\delta = \frac{1}{8A}$ and $R = \sum_{i=1}^{n-1} r_i$,

$$\begin{aligned}
&\|(\mathcal{T}_n(t,t-\delta)F)_j\|_{L^1}\\
&\quad\leq \frac{\delta^n}{n!}\sum_{r_1\cdots r_n}|a_{r_1}|\cdots|a_{r_n}|\\
&\quad\leq j(j+r_1+1)\cdots(j+R+n-1)\sup_{\tau\in(t-\delta,t)}\|F_{j+R+n}(\tau)\|_{L^1}\\
&\quad\leq \sum_{r_1\cdots r_n}|a_{r_1}|\cdots|a_{r_n}|2^{j+R-1}(2\delta)^n \sup_{\tau\in(t-\delta,t)}\|F_{j+R+n}(\tau)\|_{L^1}.
\end{aligned} \tag{6.8}$$

In the last step, we used that

$$\frac{j(j+r_1+1)\cdots(j+R+n-1)}{n!} \le \frac{(j+R)(j+R+1)\cdots(j+R+n-1)}{n!} \le \frac{(j+R+n-1)!}{n!(j+R-1)!} \le 2^{j+R+n-1}.$$

Applying (6.8) when $F = E$ with $t-\delta$ replaced by s, we get, by Proposition 1,

$$(6.9)\qquad \begin{aligned} &\int_{t-\delta}^{t} ds\, \|(\mathcal{T}_n(t,s)E(s))_j\|_{L^1} \\ &\quad \le CA^2 e^{\frac{A}{N-1}} \delta \sum_{r_1\cdots r_n} |a_{r_1}|\cdots|a_{r_n}| 2^{j+R-1}(2\delta)^n \frac{(j+R+n)^2}{N-1}, \end{aligned}$$

where from now on C will denote a positive numerical constant. Moreover,

$$(j+R+n)^2 < 3n^2 + 3j^2 + 3R^2$$

so that

$$(6.10)\qquad 2^{j-1} \sum_{r_1\cdots r_n} |a_{r_1}|\cdots|a_{r_n}| 2^R (R+j+n)^2 \le C 2^j A^n (j^2+n^2).$$

Here and in the sequel, we use systematically

$$\sum_{r_1\cdots r_n} |a_{r_1}|\cdots|a_{r_n}| 8^{(r_1+r_2+\cdots+r_n)} \le A^n.$$

Finally, summing over n, using that, for $x \in (0,1)$,

$$\sum_{n=0}^{\infty} (j^2+n^2)x^n = \frac{j^2}{1-x} + \frac{1-3(1-x)+(1-x)^2}{(1-x)^3} \le \frac{4j^2}{(1-x)^3}$$

we conclude that, recalling that $\delta = \frac{1}{8A}$,

$$(6.11)\qquad \sum_{n\ge 0} \int_{t-\delta}^{t} ds\, \|(\mathcal{T}_n(t,s)E(s))_j\|_{L^1} \le C(A) 2^j j^2 \frac{1}{N-1},$$

where $C(A)$ is a constant depending only on A.

6.2. *Iteration.* Given an arbitrary $t > 0$, we split the time interval $(0,t)$ in intervals $(k\delta, (k+1)\delta)$, $k = 1, \dots, m$ where m is an integer for which $t \in ((m-1)\delta, m\delta]$.

Denoting

$$(6.12)\qquad D_j(k) = \sup_{s\in((k-1)\delta, k\delta)} \|\Delta_j^N(s)\|_{L^1}, \quad k = 1, \dots, m,$$

with $D_j(0) = \Delta_j^N(0) = -f_0^{\otimes j}\mathbf{1}_{j>N}$, we assume inductively that, for α to be fixed later

$$D_j(k-1) \le 2^j \varphi(k-1, N) \quad \text{with } \varphi(k, N) = \frac{1}{(N-1)^{e^{-\alpha k}}}. \tag{6.13}$$

We want to prove that the same holds for k, namely

$$D_j(k) \le 2^j \varphi(k, N). \tag{6.14}$$

Note that the proof of the theorem is easily achieved once (6.14) is proven.

Equation (6.14) is trivially true for $k = 0$ since

$$D_j(0) \le 2^j 2^{-N}.$$

Assuming (6.13) and applying (6.8) and (6.11) to (6.5), with $t \in ((k-1)\delta, k\delta)$, $t_1 = (k-1)\delta$ and $F = \Delta^N((k-1)\delta)$, we have

$$\begin{aligned} D_j(k) \le {} & \sum_{n\ge 0} \sum_{r_1 \cdots r_n} |a_{r_1}| \cdots |a_{r_n}| 2^{j+R-1} (2\delta)^n 2^{j+R+n} \varphi(k-1, N) \\ & + j^2 2^j \frac{C(A)}{N-1}. \end{aligned} \tag{6.15}$$

Now observe that $D_j(k) \le 2$ so that (6.14) holds true whenever j is so large to satisfy

$$2^j \varphi(k, N) > 2. \tag{6.16}$$

Otherwise,

$$2^j \le \frac{2}{\varphi(k, N)} \tag{6.17}$$

or, equivalently

$$j \le 1 + \frac{e^{-\alpha k}}{\log 2} \log(N-1). \tag{6.18}$$

Using (6.18), we control the second term in the right-hand side of (6.15) by

$$2^j \varphi(k, N) \left\{ C(A) \left(1 + \frac{e^{-\alpha k}}{\log 2} \log(N-1)\right)^2 \left(\frac{1}{N-1}\right)^{1-e^{-\alpha k}} \right\}.$$

Now it is clear that

$$\{\cdots\} \le \frac{1}{2}$$

provided that N is sufficiently large depending on α, A and k (and hence on t).

On the other hand, the first term in the right-hand side of (6.15) is bounded by (using (6.17))

$$
\begin{aligned}
&\sum_{n\geq 0} A^n 2^j 2^{j-1}(4\delta)^n \varphi(k-1,N)\\
&\leq 2^j \frac{1}{1-4A\delta}\varphi(k-1,N)(N-1)^{e^{-\alpha k}}\\
&\leq \frac{1}{2}2^j\varphi(k,N).
\end{aligned}
\tag{6.19}
$$

The last step follows from the fact that

$$
\begin{aligned}
(N-1)^{e^{-\alpha k}}\left(\frac{1}{N-1}\right)^{e^{-\alpha(k-1)}} &= \left(\frac{1}{N-1}\right)^{e^{-\alpha k}}\left(\frac{1}{N-1}\right)^{e^{-\alpha k}(e^{\alpha}-2)}\\
&\leq \frac{1}{4}\left(\frac{1}{N-1}\right)^{e^{-\alpha k}}
\end{aligned}
$$

for $\alpha > \log 2$ and N sufficiently large, namely such that

$$
\left(\frac{1}{n-1}\right)^{\beta(T,\alpha)} < \frac{1}{4},
$$

where $\beta = e^{-\frac{\alpha T}{\delta}}(e^{\alpha}-2)$.

This concludes the proof. □

Data statement. No new data were collected in the course of this research.

Acknowledgments. We thank the referee for a careful reading of our manuscript and his useful comments.

REFERENCES

[1] ANDREIS, L., DAI PRA, P. and FISCHER, M. (2018). McKean–Vlasov limit for interacting systems with simultaneous jumps. *Stoch. Anal. Appl.* **36** 960–995. MR3925147

[2] BALLERINI, M., CABIBBO, N., CANDELIER, R., CAVAGNA, A., CISBANI, E., GIARDINA, I., LECOMTE, V., ORLANDI, A., PARISI, G. et al. (2008). Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study. *Proc. Natl. Acad. Sci. USA* **105** 1232–1237.

[3] BLANCHET, A. and DEGOND, P. (2016). Topological interactions in a Boltzmann-type framework. *J. Stat. Phys.* **163** 41–60. MR3472093

[4] BLANCHET, A. and DEGOND, P. (2017). Kinetic models for topological nearest-neighbor interactions. *J. Stat. Phys.* **169** 929–950. MR3719633

[5] BODE, N. W., FRANKS, D. W. and WOOD, A. J. (2010). Limited interactions in flocks: Relating model simulations to empirical data. *J. R. Soc. Interface* **8** rsif20100397.

[6] BODINEAU, T., GALLAGHER, I., SAINT-RAYMOND, L. and SIMONELLA, S. (2016). One-sided convergence in the Boltzmann–Grad limit. Preprint. Available at arXiv:1612.03722.

[7] BOLLEY, F., CAÑIZO, J. A. and CARRILLO, J. A. (2011). Stochastic mean-field limit: Non-Lipschitz forces and swarming. *Math*. *Models Methods Appl*. *Sci*. **21** 2179–2210. MR2860672

[8] CAMPERI, M., CAVAGNA, A., GIARDINA, I., PARISI, G. and SILVESTRI, E. (2012). Spatially balanced topological interaction grants optimal cohesion in flocking models. *Interface Focus* **2** 715–725.

[9] CAVAGNA, A., CIMARELLI, A., GIARDINA, I., PARISI, G., SANTAGATI, R., STEFANINI, F. and TAVARONE, R. (2010). From empirical data to inter-individual interactions: Unveiling the rules of collective animal behavior. *Math*. *Models Methods Appl*. *Sci*. **20** 1491–1510. MR3090590

[10] CERCIGNANI, C., ILLNER, R. and PULVIRENTI, M. (1994). *The Mathematical Theory of Dilute Gases*. *Applied Mathematical Sciences* **106**. Springer, New York. MR1307620

[11] DOBRUŠIN, R. L. (1979). Vlasov equations. *Funct*. *Anal*. *Appl*. **13** 115–123.

[12] GALLAGHER, I., SAINT-RAYMOND, L. and TEXIER, B. (2013). *From Newton to Boltzmann*: *Hard Spheres and Short-Range Potentials*. *Zurich Lectures in Advanced Mathematics*. European Mathematical Society (EMS), Zürich; Erratum to Chapter 5. MR3157048

[13] GINELLI, F. and CHATÉ, H. (2010). Relevance of metric-free interactions in flocking phenomena. *Phys*. *Rev*. *Lett*. **105** 168103.

[14] GRAHAM, C. and MÉLÉARD, S. (1997). Stochastic particle approximations for generalized Boltzmann models and convergence estimates. *Ann*. *Probab*. **25** 115–132. MR1428502

[15] HASKOVEC, J. (2013). Flocking dynamics and mean-field limit in the Cucker–Smale-type model with topological interactions. *Phys*. *D* **261** 42–51. MR3144007

[16] HAURAY, M. and JABIN, P.-E. (2015). Particle approximation of Vlasov equations with singular forces: Propagation of chaos. *Ann*. *Sci*. *Éc*. *Norm*. *Supér*. (4) **48** 891–940. MR3377068

[17] HEMELRIJK, C. K. and HILDENBRANDT, H. (2011). Some causes of the variable shape of flocks of birds. *PLoS ONE* **6** e22479.

[18] ILLNER, R. and PULVIRENTI, M. (1986). Global validity of the Boltzmann equation for a two-dimensional rare gas in vacuum. *Comm*. *Math*. *Phys*. **105** 189–203. MR0849204

[19] ILLNER, R. and PULVIRENTI, M. (1989). Global validity of the Boltzmann equation for two- and three-dimensional rare gas in vacuum. Erratum and improved result: "Global validity of the Boltzmann equation for a two-dimensional rare gas in vacuum" [Comm. Math. Phys. **105** (1986), no. 2, 189–203; MR0849204 (88d:82061)] and "Global validity of the Boltzmann equation for a three-dimensional rare gas in vacuum" [ibid. **113** (1987), no. 1, 79–85; MR0918406 (89b:82052)] by Pulvirenti. *Comm*. *Math*. *Phys*. **121** 143–146. MR0985619

[20] JABIN, P.-E. and WANG, Z. (2016). Mean field limit and propagation of chaos for Vlasov systems with bounded forces. *J*. *Funct*. *Anal*. **271** 3588–3627. MR3558251

[21] JIAN, M., WEI-GUO, S. and GUANG-XUAN, L. (2010). Multi-grid simulation of pedestrian counter flow with topological interaction. *Chin*. *Phys*. *B* **19** 128901.

[22] KAC, M. (1956). Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1954–1955, *Vol*. *III* 171–197. Univ. California Press, Berkeley. MR0084985

[23] KAC, M. (1959). *Probability and Related Topics in Physical Sciences*. Interscience, London.

[24] LACHOWICZ, M. and PULVIRENTI, M. (1990). A stochastic system of particles modelling the Euler equations. *Arch*. *Ration*. *Mech*. *Anal*. **109** 81–93. MR1019171

[25] LANFORD, O. E. III (1975). Time evolution of large classical systems. In *Dynamical Systems*, *Theory and Applications* (*Rencontres*, *Battelle Res*. *Inst*., *Seattle*, *Wash*., 1974). *Lecture Notes in Phys*. **38** 1–111. Springer, Berlin. MR0479206

[26] LIONS, P.-L. and SZNITMAN, A.-S. (1984). Stochastic differential equations with reflecting boundary conditions. *Comm*. *Pure Appl*. *Math*. **37** 511–537. MR0745330

[27] MARTIN, S. (2014). Multi-agent flocking under topological interactions. *Systems Control Lett.* **69** 53–61. MR3212821

[28] MISCHLER, S. and MOUHOT, C. (2013). Kac's program in kinetic theory. *Invent. Math.* **193** 1–147. MR3069113

[29] MISCHLER, S., MOUHOT, C. and WENNBERG, B. (2015). A new approach to quantitative propagation of chaos for drift, diffusion and jump processes. *Probab. Theory Related Fields* **161** 1–59. MR3304746

[30] NEUNZERT, H. and WICK, J. (1974). Die Approximation der Lösung von Integro-Differentialgleichungen durch endliche Punktmengen. In *Numerische Behandlung nichtlinearer Integrodifferential- und Differentialgleichungen* (*Tagung*, *Math. Forschungsinst.*, *Oberwolfach*, 1973). *Lecture Notes in Math.* **395** 275–290. Springer, Berlin. MR0371338

[31] NIIZATO, T., MURAKAMI, H. and GUNJI, Y. P. (2014). Emergence of the scale-invariant proportion in a flock from the metric-topological interaction. *Biosystems* **119** 62–68.

[32] PAUL, T., PULVIRENTI, M. and SIMONELLA, S. (2019). On the size of chaos in the mean field dynamics. *Arch. Ration. Mech. Anal.* **231** 285–317. MR3894552

[33] PERTHAME, B. and PULVIRENTI, M. (1995). On some large systems of random particles which approximate scalar conservation laws. *Asymptot. Anal.* **10** 263–278. MR1332384

[34] PULVIRENTI, M., SAFFIRIO, C. and SIMONELLA, S. (2014). On the validity of the Boltzmann equation for short range potentials. *Rev. Math. Phys.* **26** 1450001, 64. MR3190204

[35] PULVIRENTI, M. and SIMONELLA, S. (2017). The Boltzmann–Grad limit of a hard sphere system: Analysis of the correlation error. *Invent. Math.* **207** 1135–1237. MR3608289

[36] PULVIRENTI, M., WAGNER, W. and ZAVELANI ROSSI, M. B. (1994). Convergence of particle schemes for the Boltzmann equation. *Eur. J. Mech. B Fluids* **13** 339–351. MR1284825

[37] RJASANOW, S. and WAGNER, W. (2005). *Stochastic Numerics for the Boltzmann Equation*. *Springer Series in Computational Mathematics* **37**. Springer, Berlin. MR2219702

[38] SHANG, Y. and BOUFFANAIS, R. (2014). Consensus reaching in swarms ruled by a hybrid metric-topological distance. *Eur. Phys. J. B* **87** Art. 294, 7. MR3284898

[39] SHANG, Y. and BOUFFANAIS, R. (2014). Influence of the number of topologically interacting neighbors on swarm dynamics. *Sci. Rep.* **4** 4184.

[40] SPOHN, H. (1980). Kinetic equations from Hamiltonian dynamics: Markovian limits. *Rev. Modern Phys.* **52** 569–615. MR0578142

[41] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185

[42] UKAI, S. (2001). The Boltzmann–Grad limit and Cauchy–Kovalevskaya theorem. *Jpn. J. Ind. Appl. Math.* **18** 383–392. MR1842918

[43] WANG, L. and CHEN, G. (2016). Synchronization of multi-agent systems with metric-topological interactions. *Chaos* **26** 094809, 11. MR3518988

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
LONDON SW7 2AZ
UNITED KINGDOM
E-MAIL: pdegond@imperial.ac.uk

INTERNATIONAL RESEARCH CENTER ON THE MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS MEMOCS
UNIVERSITY OF L'AQUILA
67100 L'AQUILA
ITALY
E-MAIL: pulviren@mat.uniroma1.it