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Time and place of the maximum for one-dimensional diffusion bridges and meanders*

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Abstract: For three constrained Brownian motions, the excursion, the meander, and the reflected bridge, the densities of the maximum and of the time to reach it were expressed as double series by Majumdar, Randon-Furling, Kearney, and Yor (2008). Some of these series were regularized by Abel summation. Similar results for Bessel processes were obtained by Schehr and Le Doussal (2010) using the real space renormalization group method. Here this work is reviewed, and extended from the point of view of one-dimensional diffusion theory to some other diffusion processes including skew Brownian bridges and generalized Bessel meanders. We discuss the limits of the application of this method for other diffusion processes.

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Contents

*This is an original survey paper

1. Introduction

Studying the maximum and its hitting time for a random process, denoted (M, ρ), is an interest in numerous models in economics, biology, or data science. The law of the maximum of some standard diffusion processes is used in statistics to understand the quality of an estimator (such as the law of the maximum of a reflected Brownian bridge which is the basis of the Kolmogorov-Smirnov test). Moreover, a better understanding of these laws builds some connections between probabilities and complex analysis: for example the maximum of a Brownian excursion is closely related to the Riemann zeta function. See [3] for a review of various probabilistic interpretations of this function. Finally, studying the law of the time when various processes achieve their maximum yields to surprising and often counter-intuitive answers. The first of this kind of results is the case of the Brownian motion and the Brownian bridge on $[0, 1]$. These are the 'arcsine law' and the 'uniform law' found by Levy in [21].

$$\begin{aligned}\text{Brownian motion}: \frac{\mathbb{P}(\rho \in du)}{du} &= \frac{1}{\pi\sqrt{u(1-u)}}\\ \text{Brownian bridge}: \frac{\mathbb{P}(\rho \in du)}{du} &= 1\end{aligned}$$

See also [31] for more background and some generalizations.

In [25], Satya. N. Majumdar, Julien Randon-Furling, Michael J. Kearney, and Marc Yor have computed the joint density of the place and time of the maximum, (M, ρ), for three constrained Brownian motions: the standard excursion, the Brownian meander, and the standard reflected Brownian bridge. They have used two methods. The first one relies on a physical argument, based on the idea that the statistical weight of a path is proportional to a propagator defined with a Hamiltonian and a potential (describing the constraints). Shortly said, it is a path integral method. The second method uses some 'agreement formulas' and decomposition at the maximum (as discussed in Section 2). These formulas are identities in law with the following form, where U and V are independent positive random variables, c and μ some reals, and F an arbitrary non-negative measurable function.

$$\mathbb{E}[F(M^2, \rho)] = c\mathbb{E}\left[F\left(\frac{1}{U+V}, \frac{U}{U+V}\right)(U+V)^{\mu}\right] \tag{1}$$

The laws of U and V and the values of c and μ depend on the case according to the following table.

	Excursion	Meander	Reflected bridge
c	$\sqrt{\frac{\pi}{2}}$	$\sqrt{\frac{\pi}{8}}$	$\sqrt{\frac{\pi}{2}}$
μ	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\mathbb{E}(e^{-\lambda U})$	$\frac{\sqrt{2\lambda}}{\sinh(\sqrt{2\lambda})}$	$\frac{\sqrt{2\lambda}}{\sinh(\sqrt{2\lambda})}$	$\frac{1}{\cosh(\sqrt{2\lambda})}$
$\mathbb{E}(e^{-\lambda V})$	$\frac{\sqrt{2\lambda}}{\sinh(\sqrt{2\lambda})}$	$\frac{2}{\sqrt{2\lambda}} \tanh\left(\frac{\sqrt{2\lambda}}{2}\right)$	$\frac{1}{\cosh(\sqrt{2\lambda})}$

The authors of [25] verified that the two methods give the same expression for the density of (M, ρ) in each case, and confirmed their results numerically with high precision. The proofs of the above formulas can be found explicitly in the literature for the case of the excursion and reflected Brownian bridge, in [32], by Jim Pitman and Marc Yor, for example. However, in [25], it is pointed out the same can not be said for the case of the meander. Here we present a general theorem of decomposition at the maximum, then apply it explicitly to derive these formulas, and so we focus ourselves on the method based on the agreement formula.

In 2010, Schehr and Le Doussal also derived the joint density of (M, ρ) in [36] thanks to another method called the real space renormalization group. This method is powerful enough to study not only the Brownian excursion, meander, and reflected bridge, but also Bessel bridges. It also shares with the path integral method the quality to be more physically intuitive than the agreement formula method. However, both of these methods raise some issues of technical rigor. So we prefer to work here with the rigorous methods of one-dimensional diffusion theory.

Moreover, the authors of [25] and [36] deduced expressions for the densities of ρ and M for each case, as a double series. To do so, they integrated the joint density of (M, ρ). Especially in [25], it was obtained thanks to a version of the agreement formula (1) in the terms of densities and after expanding the densities of U and V as a series. This yielded the following expressions, with $z > 0$ and $0 < u < 1$.

Excursion

$$\frac{\mathbb{P}(M \in dz)}{dz} = \frac{2^{\frac{3}{2}}\pi^{\frac{5}{2}}}{z^4} \sum_{m,n=1}^{\infty} (-1)^{m+n} \frac{m^2 n^2}{m^2 - n^2} \left(e^{-\frac{n^2\pi^2}{2z^2}} - e^{-\frac{m^2\pi^2}{2z^2}} \right)$$

$$\frac{\mathbb{P}(\rho \in du)}{du} = 3 \sum_{m,n=1}^{\infty} (-1)^{m+n} \frac{m^2 n^2}{[n^2 u + m^2(1-u)]^{5/2}}$$

Meander

$$\frac{\mathbb{P}(M \in dz)}{dz} = \frac{2^{\frac{3}{2}}\pi^{\frac{1}{2}}}{z^2} \sum_{m,n=1}^{\infty} ((-1)^{m+n} - (-1)^n) \frac{n^2}{m^2 - n^2} \left(e^{-\frac{n^2\pi^2}{2z^2}} - e^{-\frac{m^2\pi^2}{2z^2}} \right)$$

$$\frac{\mathbb{P}(\rho \in du)}{du} = 2 \sum_{m=0,n=1}^{\infty} (-1)^{n+1} \frac{n^2}{[n^2 u + (2m+1)^2(1-u)]^{3/2}}$$

Reflected bridge

$$\frac{\mathbb{P}(M \in dz)}{dz} = \frac{2^{\frac{3}{2}}\pi^{\frac{1}{2}}}{z^2} \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{(m+\frac{1}{2})(n+\frac{1}{2})}{(m+\frac{1}{2})^2 - (n+\frac{1}{2})^2} \times \left(e^{-\frac{(n+1/2)^2\pi^2}{2z^2}} - e^{-\frac{(m+1/2)^2\pi^2}{2z^2}} \right)$$

$$\frac{\mathbb{P}(\rho \in du)}{du} = 2 \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{(2m+1)(2n+1)}{[(2n+1)^2 u + (2m+1)^2(1-u)]^{3/2}}$$

Bessel bridge of dimension $\delta > 0$

$$\frac{\mathbb{P}(M \in dz)}{dz} = \frac{4}{C_\nu z^{1+\delta}} \sum_{m,n\geq 1} \frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})} \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} \frac{e^{-\frac{j_{\nu,n}^2}{2z^2}} - e^{-\frac{j_{\nu,m}^2}{2z^2}}}{j_{\nu,m}^2 - j_{\nu,n}^2}$$

$$\frac{\mathbb{P}(\rho \in du)}{du} = 2\delta \sum_{m,n\geq 1} \frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})} \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} \frac{1}{[j_{\nu,n}^2 u + j_{\nu,m}^2(1-u)]^{\nu+2}}$$

The notations used for the case of Bessel bridges are precised in Section 4.1.

Some of these series are not absolutely convergent. In [25], it is explained that their meaning should be given by a regularization by adding some α^n terms, or said with other words, with the non-standard Abel summation. Abel summation can give values to some divergent series but in general divergent series are not Abel summable either. Even if those series were indeed Abel summable, their Abel summations do not need to be equal to the desired densities. As these points were not fully addressed in [25] nor [36], we shall make precise a sense of these summations before proving them.

First, in Section 2, we will present a family of known results, called agreement formulas, linked with a decomposition at the maximum for diffusion processes. We will give some examples in Section 3, including (1) as a particular case. In Section 4, from the agreement formula, we will rigorously derive the above expressions for the densities of M and ρ for all Bessel bridges. Those expressions have been properly proven in [33] (Section 11) by Pitman and Yor for M and Bessel bridges of index $\nu < -1/2$, but our results are a generalization, and our rigorous proof of the expression for the density for ρ seems to be new.

To deduce these results we will employ three tools already provided by others: the agreement formula, the Abel summation, and a series expansion of the density of the first hitting time of 1. We will discuss our use of these tools and under which conditions the method might or might not be extended to a wider class of processes. We also provide similar formulas for skew Brownian bridges in Section 5, for which the same method can not be applied. In Section 6, we provide some formulas for the case of generalized Bessel meanders.

2. Decomposition at the maximum and Agreement formula

We place ourselves within the framework of the theory of diffusion processes detailed by Itō and McKean in *Diffusion Processes and Their Sample Paths*, [12]. See [12] for the precise definitions of the following notions and the proofs of the following facts.

Let X be a one dimensional regular diffusion process taking values on a real interval I containing $[0, \infty)$, with infinite lifetime. Its infinitesimal generator is $A = D_m D_s$, with s the scale function and m the speed function. Then, for $x \in I$, the distribution of X_t given $X_0 = x$, which is denoted $\mathbb{P}_x$, admits a jointly continuous density according to m, denoted $p(t, x, y)$. In other words,

$$\mathbb{P}_x(X_t \in dy) = p(t, x, y)m(dy).$$

Several functions of the path of X interest us. For $z \in I$, we denote by

$$T_z = \inf\{t \geq 0,\ X_t = z\}$$

the first hitting time of z. It is known, and we will use, that T_z has a continuous $\mathbb{P}_x$-density relative to dt, denoted f, such that

$$\mathbb{P}_x(T_z \in dt) = f_{xz}(t)dt.$$

Let also $M_t = \sup\limits_{0 \leq u \leq t} X_u$ be the maximum of X on $[0, t]$, and

$$\rho_t = \inf\{u \geq 0,\ X_u = M_t\}$$

be the first time when it is reached. Almost surely, ρ_t is the unique time $s \in [0, t]$ such that $X_s = M_t$. In the following, we will often denote by T the random variable T_1 under $\mathbb{P}_0$, and by $f = f_{01}$ its density.

A result of decomposition at the maximum is an identity of law about the paths of a stochastic process. Such theorem first describes the joint distribution of (M_t, ρ_t, X_t) then states that conditionally given (M_t, ρ_t, X_t), the path fragments before and after the argmax are independent and follow some law. Levy, in [22], Louchard, in [23], and Vincze, in [39], provided the cases of Brownian motion and Brownian bridge. However, Williams first gave a general path decomposition on $[0, \infty)$ at the time when the maximum is reached for all diffusion processes verifying $M_\infty < \infty$ almost surely. Moreover, in those works, the joint distribution of respectively (M_t, ρ_t, X_t) and (M_∞, ρ_∞) were described by a factorization of their density.

This paved the way for various authors to extend those fundamental ideas to a broader context. We can cite for example Denisov [7], Millar [28], Le Gall [19], Pitman [30], Salminen [35]... About our needs, we will use the generalization of the path decomposition at the maximum of Williams to all diffusion processes on a finite time interval.

Theorem 1. *Let $0 < u < 1$, $x, y \le z < \infty$ and $x, y, z \in I$.*

(i)

$$\mathbb{P}_x(M_t \in dz, \rho_t \in du, X_t \in dy) = f_{xz}(u) f_{yz}(t-u) s(dz) m(dy) du \tag{2}$$

(ii)

$$\mathbb{P}_x(M_t \in dz, \rho_t \in du | X_t = y) = \frac{f_{xz}(u) f_{yz}(t-u)}{p(t, x, y)} s(dz) du \tag{3}$$

(iii) Under $\mathbb{P}_x$, conditionally given $M_t = z$, $\rho_t = u$, and $X_t = y$, the path fragments $(X_v, 0 \le v \le u)$ and $(X_{t-v}, 0 \le v \le t-u)$ are independent, and respectively of law $(X_v, 0 \le v \le T_z)$ under $\mathbb{P}_x$ given $T_z = u$, and $(X_v, 0 \le v \le T_z)$ under $\mathbb{P}_y$ given $T_z = t - u$.

This theorem has been stated exactly in the same form in [32] as the Theorem 2. The point *(i)* of the theorem is a density factorization for the joint law of (M_t, ρ_t, X_t) and will be the main tool of this paper. It was independently proved at the same time by Fitzsimmons in [9], a former unpublished manuscript, and by Csáki, Földes, and Salminen in [6]. This fundamental formula generalizes the results on the Brownian motion of Levy, Louchard, and Vincze. Moreover, Shepp has written (2) for the Brownian motion with drift in [37], while Imhof in [11] and Louchard in [24] have treated the cases of the killed Brownian motion and the tree-dimensional Bessel process.

By conditioning on X_t, the point *(ii)* is a direct consequence of (2). The point (iii) is shown by Fitzsimmons in [9], with a proof similar to that of Williams in [40]. Since (2) is a key formula, we will briefly recall one of its proofs later.

As said in [32], if $\forall z \ge x$, $T_z < \infty$ $\mathbb{P}_x$-a.s., then for any x, y, the theorem gives two equivalent definitions of a σ-finite measure on the space of continuous function on $[0, \infty)$ with compact support:

(I) pick t according to $p(t, x, y)dt$ and run an X-bridge of length t to x from y.

(II) pick z according to $s(dz)$ restricted to $(x \vee y, \infty)$, run two independent copies of X till T_z, one starting from x and the other from y, then put them back to back.

Remark. When X is a 3-dimensional Bessel process, this measure corresponds to Ito's excursion law. While (I) is conditioning on the length, (II) is conditioning on the maximum.

This 'agreement' between the two descriptions of the measure motivated Pitman and Yor to call the formula (3) an agreement formula, in [32]. We will do the same in this work and we will also call agreement formula any particular case or equivalent formulation of (3). In particular, in Section 3, we prove that (1) is indeed an agreement formula, as a particular version of (3).

2.1. Standard bridge of a diffusion process with a Brownian scaling

In the following, we fix $x = y = 0$ and $t = 1$. We suppose $\forall z > 0,\ T_z < \infty$ a.s. and we denote by X^{br} the X-bridge of length 1 from 0 to 0. Moreover, we assume that X enjoys a Brownian scaling. In more precise words, we suppose $(X_t) \stackrel{d}{=} (\sqrt{c}X_{t/c})$ for any $c > 0$. The scaling implies, for $z > y \geq 0$,

$$f_{yz}(u) = \frac{1}{z^2} f_{\frac{y}{z}1}\left(\frac{u}{z^2}\right) \text{ and } \mathbb{E}_y(e^{-\lambda T_z}) = E_{\frac{y}{z}}(e^{-\lambda z^2 T_1}).$$

Futhermore, we let the reader show that the scaling implies that s is differentiable and that there is a $\mu \in \mathbb{R}$ such that $s'(z) = s'(1)z^{\mu}$. The proof of those facts could also be found in [18]. Under those assumptions, the agreement formula (3) becomes

$$\mathbb{P}(M^{\mathrm{br}} \in dz, \rho^{\mathrm{br}} \in du) = \frac{1}{p(1,0,0)} f\left(\frac{u}{z^2}\right) f\left(\frac{1-u}{z^2}\right) \frac{s'(z)}{z^4} dz du. \tag{4}$$

Inspired by the equivalence of definitions of the measure of paths presented at the end of the previous section, we construct a process by re scaling the definition (II) and we compare it with the standard bridge. It is a direct generalization of a construction in [32] for Bessel bridges. In other words, the only useful hypotheses to adapt the proof of Pitman and Yor are the ones above. The case for Bessel bridges from [32] is also presented later.

Construction: Take X and $\hat{X}$ two independent copies starting at 0, respectively hitting 1 for the first time at T and $\hat{T}$. Define a new process by putting them back to back.

$$\tilde{X}_t = \begin{cases} X_t & \text{if } t \leq T \\ \hat{X}_{T+\hat{T}-t} & \text{if } T \leq t \leq T + \hat{T} \end{cases}$$

Finally, let $\tilde{X}^{\mathrm{br}}_u = \dfrac{1}{\sqrt{T+\hat{T}}} X_{u(T+\hat{T})}$ for $0 \leq u \leq 1$. Its maximum and its argmax are respectively

$$\tilde{M}^{\mathrm{br}} = \frac{1}{\sqrt{T+\hat{T}}} \text{ and } \tilde{\rho}^{\mathrm{br}} = \frac{T}{T+\hat{T}}.$$

Since the mutual density of T and $\hat{T}$ is f, the joint density of $(\tilde{M}^{\mathrm{br}}, \tilde{\rho}^{\mathrm{br}})$ is easily computed, thanks to change of variable $t = \dfrac{u}{z^2}$ and $\hat{t} = \dfrac{1-u}{z^2}$.

$$\mathbb{P}(\tilde{M}^{\mathrm{br}} \in dz, \tilde{\rho}^{\mathrm{br}} \in du) = 2f\left(\frac{u}{z^2}\right) f\left(\frac{1-u}{z^2}\right) \frac{1}{z^5} dz du$$

Comparing this formula with (4), we deduce a new form of the agreement formula for standard diffusion bridge with a Brownian scaling. If F is an arbitrary non-negative measurable function, then

$$E[F((M^{\mathrm{br}})^2,\rho^{\mathrm{br}})] = \frac{1}{2p(1,0,0)}\mathbb{E}\left[F\left(\frac{1}{T+\hat{T}},\frac{T}{T+\hat{T}}\right)\frac{1}{\sqrt{T+\hat{T}}}s'\left(\frac{1}{\sqrt{T+\hat{T}}}\right)\right]. \quad (5)$$

The point *(iii)* of the Theorem 1 extends (5) to the functions of the paths: if Ψ is an arbitrary non-negative measurable function, then

$$\mathbb{E}[\Psi(X^{\mathrm{br}})] = \frac{1}{2p(1,0,0)}\mathbb{E}[\Psi(\tilde{X}^{\mathrm{br}})\tilde{M}^{\mathrm{br}}s'(\tilde{M}^{\mathrm{br}})]. \quad (6)$$

Remark. Starting from the above equality of expectations, and assuming the scaling property, a simple change of variable gets us the agreement formula as in Theorem 1 for the case of bridges from 0 to 0, and reversely. The assumption $T_z < \infty$ is only needed to well define the construction, but not to have the equivalence between (4) and (5).

2.2. Proof of the agreement formula

We recall a computational but straightforward proof of (2) provided in [6]. A more probabilistic proof was also proposed in [6] and [9]. It is based on the Levy-Khintchine representation of Levy processes, on the fact that the hitting time process of a diffusion is a subordinator, and on the theory of the excursions below the maximum of a diffusion. If the reader wishes to know more about the excursions below the maximum, we advise to check [34].

Let $\alpha > 0$, and let $\phi^{\uparrow}$ and $\phi^{\downarrow}$ respectively be the increasing and the decreasing solution of

$$A\phi = \alpha\phi,$$

where we recall that A is the infinitesimal generator of X. We denote by $W = \phi^{\downarrow}\phi^{\uparrow +} - \phi^{\downarrow +}\phi^{\uparrow}$ the Wronskian, which is constant, where ϕ^{+} is the derivative with respect to s. For $x, z \in I$, let also g be the Green function:

$$g(x,z) = \int_0^\infty e^{-\alpha t}p(t,x,z)dt.$$

Ito and McKean, [12], give expressions for g and for the Laplace transform of T_z thanks to $\phi^{\uparrow}$ and $\phi^{\downarrow}$.

$$\mathbb{E}_x(e^{-\alpha T_z}) = \begin{cases} \frac{\phi^{\uparrow}(x)}{\phi^{\uparrow}(z)} & \text{if } x \le z \\ \frac{\phi^{\downarrow}(x)}{\phi^{\downarrow}(z)} & \text{if } x \ge z \end{cases}$$

$$g(x,z) = \begin{cases} \frac{1}{W}\phi^{\downarrow}(z)\phi^{\uparrow}(x) & \text{if } x \le z \\ \frac{1}{W}\phi^{\uparrow}(z)\phi^{\downarrow}(x) & \text{if } x \ge z \end{cases}$$

Because the Wronskian is constant, and since $\dfrac{\phi^{\downarrow}}{\phi^{\uparrow}}(\infty) = 0$ because the lifetime is infinite, we observe that

$$g(x,z) = \phi^{\uparrow}(z)\phi^{\uparrow}(x)\int_{x\vee z}^{\infty}\frac{s(dv)}{(\phi^{\uparrow}(v))^2}.$$

Now, let $u \in [0,t]$ and $x, y \le z$.

$$\begin{aligned}
&\int_0^{\infty} e^{-\alpha t}\mathbb{P}_x(X_t \in dy, M_t \ge z)dt \\
&\qquad = \int_0^{\infty} e^{-\alpha t}\int_0^t \mathbb{P}_x(T_z \in ds)\mathbb{P}_z(X_{t-s} \in dy)dt \\
&\qquad = \int_0^{\infty} e^{-\alpha s}\mathbb{P}_x(T_z \in ds)\int_0^{\infty} e^{-\alpha t}\mathbb{P}_z(X_t \in dy)dt \\
&\qquad = \mathbb{E}_x(e^{-\alpha T_z})g(z,y)m(dy) \\
&\qquad = \phi^{\uparrow}(x)\phi^{\uparrow}(y)\int_z^{\infty}\frac{s(dv)}{(\phi^{\uparrow}(v))^2}m(dy)
\end{aligned}$$

The first equality comes from the strong Markov property at T_z, while the last is an use of the formulas for g and for the Laplace transform of T_z. We differentiate with respect to z to found:

$$\begin{aligned}
\int_0^{\infty} e^{-\alpha t}\mathbb{P}_x(X_t \in dy, M_t \in dz)dt &= \frac{\phi^{\uparrow}(x)}{\phi^{\uparrow}(z)}\frac{\phi^{\uparrow}(y)}{\phi^{\uparrow}(z)}s(dz)m(dy) \\
&= \mathbb{E}_x(e^{-\alpha T_z})\mathbb{E}_y(e^{-\alpha T_z})s(dz)m(dy).
\end{aligned}$$

By uniqueness of the Laplace transform, we deduce the joint density of (X_t, M_t).

$$\mathbb{P}_x(X_t \in dy, M_t \in dz) = \left(\int_0^t f_{xz}(u)f_{yz}(t-u)du\right)s(dz)m(dy)$$

Eventually,

$$\begin{aligned}
&\mathbb{P}_x(X_t \in dy, M_t \in dz, \rho_t > u) \\
&\qquad = \mathbb{E}_x(\mathbb{1}[M_u < z]\mathbb{P}_{X_u}(X_{t-u} \in dy, M_{t-u} \in dz)) \\
&\qquad = \mathbb{E}_x\left(\mathbb{1}[M_u < z]\int_u^t f_{X_u z}(v-u)f_{yz}(t-v)dv\right)s(dz)m(dy) \\
&\qquad = \left(\int_u^t \mathbb{E}_x(\mathbb{1}[T_z > u]f_{X_u z}(v-u))f_{yz}(t-v)dv\right)s(dz)m(dy) \\
&\qquad = \left(\int_u^t f_{xz}(v)f_{yz}(t-v)dv\right)s(dz)m(dy).
\end{aligned}$$

Here, we have used the Markov property at u. The formula (2) follows by differentiating with respect to u.

3. Examples

We apply the previous section to write the agreement formula for three examples of Brownian-related processes, all enjoying a Brownian scaling, at least indirectly. In this section, F and Ψ are arbitrary non-negative measurable functions, $z > 0$, and $0 < u < 1$. Various examples of (2) or (4) can be found in the excellent *Handbook of Brownian Motion – Facts and Formulae* of Borodin and Salminen, [4]. Let us cite that 3.1.13.8 page 351 and 5.1.13.8 page 449 represent (2) respectively for the cases of reflected Brownian motion and 3-dimensional Bessel process.

3.1. Case of Bessel bridges

A Bessel process of parameter $\delta > 0$ is the solution of the stochastic differential equation

$$dX_t = dB_t + \frac{\delta - 1}{2} \frac{dt}{X_t}$$

where B is a one-dimensional Brownian motion. When δ is a positive integer, X has the law of the norm of a δ-dimensional Brownian motion. Precise definition, background, and more details about Bessel processes are provided in [32] by Pitman and Yor. Moreover, our application of the decomposition at the maximum and the agreement formula for the Bessel bridges is already presented more carefully in [32].

Let $\delta > 0$, $X \overset{d}{=} \mathrm{BES}(\delta)$ and $\delta = 2(\nu + 1)$, we can choose $s(dx) = x^{1-\delta}dx$, $m(dx) = 2x^{\delta-1}dx$. We have $p(t, 0, y) = (2t)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} \exp\left(-\frac{y^2}{2t}\right)$. We denote

$$C_\nu = \frac{1}{2p(1, 0, 0)} = 2^{\frac{\delta}{2}-1} \Gamma\left(\frac{\delta}{2}\right).$$

For the case of a (standard) Bessel bridge of dimension δ, we can respectively write (4), (5), and (6) as follows.

$$\mathbb{P}(M^{\mathrm{br}} \in dz, \rho^{\mathrm{br}} \in du) = \frac{2C_\nu}{z^{3+\delta}} f_\nu\left(\frac{u}{z^2}\right) f_\nu\left(\frac{1-u}{z^2}\right) dzdu \tag{7}$$

$$\mathbb{E}[F((M^{\mathrm{br}})^2, \rho^{\mathrm{br}})] = C_\nu \mathbb{E}\left[F\left(\frac{1}{T + \hat{T}}, \frac{T}{T + \hat{T}}\right) (T + \hat{T})^\nu\right]$$

$$\mathbb{E}[\Psi(X^{\mathrm{br}})] = C_\nu \mathbb{E}[\Psi(\tilde{X}^{\mathrm{br}}) (\tilde{M}^{\mathrm{br}})^{2-\delta}]$$

Moreover, we have the well-known Laplace transform

$$\mathbb{E}(e^{-\lambda T}) = \frac{(2\lambda)^{\frac{\nu}{2}}}{C_\nu I_\nu(\sqrt{2\lambda})}$$

where I_ν is the modified Bessel function of the first kind of index ν.

$$I_\nu(z) = \sum_{k \geq 0} \frac{1}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{2k+\nu}$$

For $\delta = 1$, the Bessel bridge is simply the reflected Brownian bridge. For $\delta = 3$, we have the case of the Brownian excursion. Respectively,

$$\mathbb{E}(e^{-\lambda T^{(1)}}) = \frac{1}{\cosh(\sqrt{2\lambda})}, \ \mathbb{E}(e^{-\lambda T^{(3)}}) = \frac{\sqrt{2\lambda}}{\sinh(\sqrt{2\lambda})},$$

and

$$C_{-1/2} = C_{1/2} = \sqrt{\frac{\pi}{2}}.$$

Observe that we have retrieved the formulas (1) of [25].

3.2. Case of skew Brownian bridges

Let $0 < \beta < 1$, the skew Brownian motion of parameter β is a diffusion process which behaves like the Brownian motion away from 0, but such that we have $\mathbb{P}(X_t > 0) = \beta$ for any $t > 0$. It can be constructed as a Brownian motion where the signs of the excursions away from 0 are independent and identically distributed. See the survey of Lejay [20] for definitions and more details on skew Brownian motions.

Let X be a skew Brownian motion of parameter β, we define

$$\sigma(x) = \begin{cases} \frac{1}{\beta} & \text{if } x \geq 0 \\ \frac{1}{1-\beta} & \text{if } x < 0 \end{cases}$$

and we can choose $s(x) = \sigma(x)x$ (then $s'(x) = \sigma(x)$) and $m(dx) = \dfrac{2}{\sigma(x)}dx$. For $a > 0$, $\mathbb{P}(X_1 > a) = 2\beta\mathbb{P}(B_1 > a)$ and $\mathbb{P}(X_1 < -a) = 2(1-\beta)\mathbb{P}(B_1 < -a)$, so, $p(1,0,y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$, which is the density of B_1 against dy. After application of the agreement formulas (4) and (5) for the case of a (standard) skew Brownian bridge, we find the following formulas.

$$\mathbb{P}(M^{\text{br}} \in dz, \rho^{\text{br}} \in du) = \frac{\sqrt{2\pi}}{\beta z^4} f_\beta\left(\frac{u}{z^2}\right) f_\beta\left(\frac{1-u}{z^2}\right) dzdu \tag{8}$$

$$\mathbb{E}[F((M^{\text{br}})^2, \rho^{\text{br}})] = \sqrt{\frac{\pi}{2}}\frac{1}{\beta}\mathbb{E}\left[F\left(\frac{1}{T+\hat{T}}, \frac{T}{T+\hat{T}}\right)\frac{1}{\sqrt{T+\hat{T}}}\right]$$

To compute the Laplace transform of T, we consider the successive excursions above 1 of a reflected Brownian motion and condition on which one is the first to remain positive for X.

$$\mathbb{E}(e^{-\lambda T}) = \frac{\beta}{\cosh(\sqrt{2\lambda}) - (1-\beta)e^{-\sqrt{2\lambda}}}$$

Remark. By applying the equality with $\beta = 1$, corresponding to the reflected Brownian bridge, we easily verify that we found the same formula as before. It is not surprising, as a consequence of a convergence in law.

The case $\beta = 1/2$ corresponds to the (non-reflected) Brownian bridge, and we have

$$\sqrt{\frac{\pi}{2}}\frac{1}{\beta} = \sqrt{2\pi} \text{ and } \mathbb{E}(e^{-\lambda T}) = e^{-\sqrt{2\lambda}}.$$

3.3. Case of generalized Brownian meanders

Let $k \in \mathbb{N}^*$. The kth Brownian meander is a continuous stochastic process $X^{k-\mathrm{me}}$ on $[0,1]$ such that $\mathbb{P}(X_1^{k-\mathrm{me}} \in dy) = \frac{2}{2^{k/2}\Gamma(k/2)} y^{k-1} e^{-\frac{y^2}{2}} dy$ for $y > 0$, and conditionally given $X_1^{k-\mathrm{me}} = y$, $X^{k-\mathrm{me}}$ is a Bessel bridge of dimension 3, from 0 to y, and of length 1. In fancier words $X_1^{k-\mathrm{me}}$ follows the chi distribution of dimension k. In particular, the cases $k = 1, 2, 3$ are respectively called the Brownian co-meander, the Brownian meander, and the 3-dimensional Bessel process started at 0 on $[0,1]$. Mansuy and Yor give some background, properties, and equivalent definitions of these processes in [26].

From (3), the Brownian scaling of a Bessel process, and after integrating with respect to y, we have

$$\mathbb{P}(M^{k-\mathrm{me}} \in dz, \rho^{k-\mathrm{me}} \in du) = 2c_k \frac{z^k}{z^6} f\left(\frac{u}{z^2}\right) \phi_k\left(\frac{1-u}{z^2}\right) dzdu,$$

where

$$c_k = \frac{1}{k} \times \frac{2^{3/2}\Gamma(3/2)}{2^{k/2}\Gamma(k/2)} \text{ and } \phi_k(t) = \int_0^1 kx^{k-1} f_{x1}(t)dx \geq 0.$$

Here, f_{x1} is the density of $T_1 = T$ for the 3-dimensional Bessel process started at x and $f = f_{01}$. The function ϕ_k is the density of some random time $S_k \geq 0$ because $\int_0^\infty \phi_k(t)dt = 1$. We can assume S_k to be independent from T and rewrite the identity with the densities as an equality of distribution.

$$\mathbb{E}[F((M^{k-\mathrm{me}})^2, \rho^{k-\mathrm{me}})] = c_k \mathbb{E}\left[F\left(\frac{1}{T+S_k}, \frac{T}{T+S_k}\right)\left(\frac{1}{\sqrt{T+S_k}}\right)^{k-1}\right]$$

We already know $\mathbb{E}(e^{-\lambda T}) = \frac{\sqrt{2\lambda}}{\sinh(\sqrt{2\lambda})}$. Let us compute the Laplace transform of S_k.

$$\begin{aligned}
\mathbb{E}(e^{-\lambda S_k}) &= \int_0^\infty e^{-\lambda t}\phi_k(t)dt \\
&= \int_0^1 kx^{k-1}\mathbb{E}_x(e^{-\lambda T_1})dx \\
&= \frac{k}{\sinh(\sqrt{2\lambda})}\int_0^1 x^{k-2}\sinh(\sqrt{2\lambda}x)dx
\end{aligned}$$

The last equality comes from the strong Markov property and the Brownian scaling of the 3-dimensional Bessel process:

$$\mathbb{E}_0(e^{-\lambda T_1}) = \mathbb{E}_0(e^{-\lambda T_x})\mathbb{E}_x(e^{-\lambda T_1}) = \mathbb{E}_0(e^{-\lambda x^2 T_1})\mathbb{E}_x(e^{-\lambda T_1}).$$

Moreover, for $\theta > 0$, $n \in \mathbb{N}$, it is easy to prove by induction the following identities.

$$\frac{\theta^{2n+1}}{(2n)!}\int_0^1 x^{2n}\sinh(\theta x)dx = \cosh(\theta)\sum_{i=0}^{n}\frac{\theta^{2i}}{(2i)!} - \sinh(\theta)\sum_{i=0}^{n-1}\frac{\theta^{2i+1}}{(2i+1)!} - 1$$
$$\frac{\theta^{2n+2}}{(2n+1)!}\int_0^1 x^{2n+1}\sinh(\theta x)dx = \cosh(\theta)\sum_{i=0}^{n}\frac{\theta^{2i+1}}{(2i+1)!} - \sinh(\theta)\sum_{i=0}^{n}\frac{\theta^{2i}}{(2i)!}$$

They can be written for any $m \in \mathbb{N}$ as

$$\frac{\theta^{m+1}}{m!}\int_0^1 x^m \sinh(\theta x)dx = \frac{e^{-\theta}}{2}\sum_{i=0}^{m}\frac{\theta^i}{i!} + (-1)^m\frac{e^{\theta}}{2}\sum_{i=0}^{m}\frac{(-\theta)^i}{i!} - \frac{1+(-1)^m}{2}.$$

For the case $k = 1$ (or $m = -1$) we use the function Shi, the hyperbolic sine integral.

$$\mathrm{Shi}(x) = \int_0^x \frac{\sinh(t)}{t}dt$$

To serve as an example, we explicitly give the expressions of c_k and $\mathbb{E}(e^{-\lambda S_k})$ for the cases $k = 1, 2, 3$, i.e the Brownian co-meander, the classical Brownian meander, and 3-dimensional Bessel process.

$$\begin{aligned}
c_1 &= 1 & \mathbb{E}(e^{-\lambda S_1}) &= \frac{\mathrm{Shi}(\sqrt{2\lambda})}{\sinh(\sqrt{2\lambda})} \\
c_2 &= \sqrt{\frac{\pi}{8}} & \mathbb{E}(e^{-\lambda S_2}) &= \frac{2}{\sqrt{2\lambda}}\tanh\left(\frac{\sqrt{2\lambda}}{2}\right) \\
c_3 &= \frac{1}{3} & \mathbb{E}(e^{-\lambda S_3}) &= \frac{3}{2\lambda\sinh(\sqrt{2\lambda})}(\sqrt{2\lambda}\cosh(\sqrt{2\lambda}) - \sinh(\sqrt{2\lambda}))
\end{aligned}$$

Remark. For $k = 2$, we have retrieved (1) for the case of the Brownian meander. Moreover, we recognize $4S_2$ has the same law as the last zero of a reflected Brownian motion started at 0 before hitting 1.

For $k = 3$, $\mathbb{E}(e^{-\lambda T^{(5)}})\mathbb{E}(e^{-\lambda S_3}) = \mathbb{E}(e^{-\lambda T^{(3)}})$ where $T^{(5)}$ or $T^{(3)}$ is the first hitting time of 1 respectively for the 5-dimensional or the 3-dimensional Bessel process started at 0.

4. Marginal densities for Bessel bridges

In this section, we deduce the marginal densities of M and ρ from the agreement formula for Bessel bridges. The expressions are already presented in [25] for the reflected Brownian bridge and the Brownian excursion, and in [3] for all Bessel bridges, after non-rigorous manipulations (see our introduction). To make rigorous the expressions of the densities of M and ρ from agreement formula we need to 'sum' divergent series.

4.1. Some background on divergent series

The addition of a finite number of terms enjoys very natural and pleasant properties, associativity and commutativity. Sadly, the situation is far less obvious when the numbers of terms is infinite. For example, if we try to consider $S = 1 - 1 + 1 - 1 + \dots$, then by analogy with the summation of a finite number of terms, we could be tempted to use the property of associativity: $S = (1-1)+(1-1)+(1-1)+(1-1)+\cdots = 0+0+0+0+\cdots = 0$. But on the other hand, $S = 1+(-1+1)+(-1+1)+(-1+1)+(-1+1)+\cdots = 1+0+0+0+0+\cdots = 1$. The core of the problem was to give a value of a not-defined-yet object. The formalism developed by Cauchy of convergent and absolute convergent series brought a satisfying answer to those concerns and provided a wonderful tool in analysis. However, it excluded the now-called divergent series such as S.

As noticed by Euler, Abel, Ramanujan and many others, careless manipulation of divergent series often leads to absurd and confusing observations. Nevertheless, sometimes a divergent series seems to have a 'natural' value, simply deduced by algebraic manipulation. Specifically, $S = 1-(1-1+1-1+\dots) = 1-S$, then we would like to say that $S = 1/2$, which is also the mean of 0 and 1, the two values guessed above. A method of summation should be a way to give some value to some series, should share as many properties as possible with classical summation, and should be consistent with the addition of a finite number of terms. In 1890, Cesàro gave the first rigorous definition of sum of some divergent series with its Cesàro summation. This opened the way for several other summation methods. However, more a method can be applied to a lot of series, more it lacks good properties. Even when two methods can be applied to the same series, they do not need to give the same value in general. The book *Divergent Series* of Hardy, [10], perfectly presents those years of work and gathers the most important results about divergent series. It is a wonderful and essential reference for the reader who would like to learn more about this subject.

Here, we will use the same non-standard summation as in [25], called the Abel summation in the literature. It is motivated by the Abel's Theorem on radius convergence of entire series and has inherited its name.

Definition. Abel summation

Let (u_n) be a sequence of complex numbers. If $\forall \alpha \in (0,1)$, $\sum_{n\geq 0} \alpha^n |u_n| < \infty$ and if there is $S \in \mathbb{C}$ such that

$$\sum_{n\geq 0} \alpha^n u_n \underset{\alpha\to 1}{\longrightarrow} S,$$

we say that S is the Abel summation of (u_n) and we denote

$$\sum_{n\geq 0} u_n = S\ (\mathcal{A}).$$

For example, $1-1+1-1+\cdots = \sum_{n\geq 0}(-1)^n = \frac{1}{2}\ (\mathcal{A})$. This summation has good properties and often gives a value for divergent series with alternating terms. See [10] for details and the proof of the proposition below.

Proposition 1. *The Abel summation enjoys these three properties.*

Linearity: If $\sum_{n\geq 0} u_n = S\ (\mathcal{A})$ *and* $\sum_{n\geq 0} v_n = T\ (\mathcal{A})$ *then* $\sum_{n\geq 0} u_n + \lambda v_n = S + \lambda T\ (\mathcal{A})$.

Stability: If $\sum_{n\geq 0} u_n = S\ (\mathcal{A})$ *then* $\sum_{n\geq 0} u_{n+1} = S - u_0\ (\mathcal{A})$.

Regularity [Abel's Theorem]: If $\sum_{n=0}^{N} u_n \underset{N\to\infty}{\longrightarrow} S$ *then* $\sum_{n\geq 0} u_n = S\ (\mathcal{A})$.

We will use a straight-forward generalization for double series that keeps the above properties.

Definition. Let $(u_{m,n})$ be a double sequence of complex numbers. If for all $\alpha \in (0,1)$, $\sum_{m,n\geq 0} \alpha^{m+n}|u_{m,n}| < \infty$ and if there is $S \in \mathbb{C}$ such that

$$\sum_{m,n\geq 0} \alpha_1^m \alpha_2^n u_{m,n} \underset{\substack{\alpha_1\to 1^- \\ \alpha_2 \to 1^-}}{\longrightarrow} S,$$

we write

$$\sum_{m,n\geq 0} u_{n,m} = S\ (\mathcal{A}).$$

In particular, in this case, $\sum_{k\geq 0}\left(\sum_{\substack{m,n\geq 0\\ m+n=k}} u_{m,n}\right) = S\ (\mathcal{A})$.

4.2. The formulas and their proof

In this subsection, $\delta = 2(\nu+1) > 0$ is fixed. Let us begin by defining some notations: $C_\nu = 2^\nu \Gamma(\nu+1)$ and $j_{\nu,n}$ are the strictly positive zeros in increasing order of J_ν, where J_ν is the Bessel function of the first kind of index ν defined by

$$J_\nu(z) = \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k+\nu}.$$

We recall the agreement formula (7) for a standard Bessel bridge of dimension δ: for $z>0$ and $0<u<1$,

$$\mathbb{P}(M\in dz,\rho\in du)=\frac{2C_\nu}{z^{3+\delta}}f_\nu\left(\frac{u}{z^2}\right)f_\nu\left(\frac{1-u}{z^2}\right)dzdu,$$

where f_ν is the density of the first hitting time of 1 of a Bessel process of dimension δ started from 0 (previously denoted T). From [15] and [33], we know a series expansion formula for f_ν.

$$f_\nu(t)=\frac{1}{C_\nu}\sum_{n\geq 1}\frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})}e^{-\frac{j_{\nu,n}^2}{2}t}=\frac{1}{C_\nu}\sum_{n\geq 1}(-1)^{n-1}g_{\nu,n}e^{-\frac{j_{\nu,n}^2}{2}t} \tag{9}$$

This series expansion is a key-point for the following work. We will discuss its basis later. We precise some notations and asymptotic equivalents (see [29] pages 237-242 and 247-248). Plus, most of useful facts about Bessel functions and their zeros can be read in [8].

$$\begin{aligned} g_{\nu,n}&=(-1)^{n-1}\frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})}>0\\ j_{\nu,n}&\sim\pi n\\ g_{\nu,n}&\sim\sqrt{\frac{\pi}{2}}(\pi n)^{\nu+3/2}\end{aligned}$$

Theorem 2. *The density of the max of a standard Bessel bridge is equal to*

$$\begin{aligned}\frac{\mathbb{P}(M\in dz)}{dz}&=\frac{2}{C_\nu z^{3+\delta}}\sum_{n\geq 1}\left(\frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})}\right)^2e^{-\frac{j_{\nu,n}^2}{2z^2}}\\ &\quad+\frac{4}{C_\nu z^{1+\delta}}\sum_{\substack{m,n\geq 1\\ m\neq n}}\frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})}\frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})}\frac{e^{-\frac{j_{\nu,n}^2}{2z^2}}-e^{-\frac{j_{\nu,m}^2}{2z^2}}}{j_{\nu,m}^2-j_{\nu,n}^2}\ (\mathcal{A}).\end{aligned} \tag{10}$$

The convergence in $(\alpha_1,\alpha_2)\to(1^-,1^-)$ is uniform on $(0,\infty)$.

Theorem 3. *The density of the argmax of a standard Bessel bridge is equal to*

$$\frac{\mathbb{P}(\rho\in du)}{du}=2\delta\sum_{m,n\geq 1}\frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})}\frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})}\frac{1}{[j_{\nu,n}^2u+j_{\nu,m}^2(1-u)]^{\nu+2}}\ (\mathcal{A}). \tag{11}$$

The convergence in $(\alpha_1,\alpha_2)\to(1^-,1^-)$ is uniform on every segment of $(0,1)$.

Remark. We would like to integrate the agreement formula with respect to z or u to find the marginal densities. However, for example with $\delta=3$,

$$\pi^2\sum_{n\geq 1}(-1)^{n-1}n^2\int_0^\infty\exp\left(-\frac{n^2\pi^2}{2}t\right)dt=2\sum_{n\geq 0}(-1)^n\neq 1=\int_0^\infty f_{\frac{1}{2}}(t)dt\ ?!$$

The integration erases the small exponential term so the series can not commute with the integral. It is why the Abel summation would be so useful because it adds new small terms independent of the integration and allows the commutation. We can remark also that with $\displaystyle\sum_{n\geq 0}(-1)^n = \frac{1}{2}$ $(\mathcal{A})$ the previous calculation works just fine.

Proof. For this proof, to simplify the notations, we could forget the indexes ν. Let us define for $0<\alpha<1$ and $t\geq 0$,

$$f^\alpha(t) = \frac{1}{C_\nu}\sum_{n\geq 1}\alpha^{n-1}\frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})}e^{-\frac{j_{\nu,n}^2}{2}t} = \frac{1}{C_\nu}\sum_{n\geq 1}(-1)^{n-1}\alpha^{n-1}g_n e^{-\frac{j_n^2}{2}t}.$$

By normal convergence, f^α is continuous on $[0,\infty)$. We denote naturally $f^1 = f$.

If $t>0$ and $0<\alpha\leq 1$, $|f^\alpha(t)| \leq \displaystyle\frac{1}{C_\nu}\sum_{n\geq 1}g_n e^{-\frac{j_n^2}{2}t}$. Thus, by dominated convergence, there is $t_0>0$ such that for any $0<\alpha\leq 1$:

$$\text{If } t>t_0, \text{ then } |f^\alpha(t)| \leq e^{-\frac{j_0^2}{4}t} \underset{t\to+\infty}{\longrightarrow} 0. \tag{12}$$

First, we are going to show

$$\frac{1}{z^{3+\delta}}\int_0^1 f^{\alpha_1}\left(\frac{u}{z^2}\right)f^{\alpha_2}\left(\frac{1-u}{z^2}\right)du \underset{\substack{\alpha_1\to 1^-\\ \alpha_2\to 1^-}}{\longrightarrow} \frac{1}{z^{3+\delta}}\int_0^1 f\left(\frac{u}{z^2}\right)f\left(\frac{1-u}{z^2}\right)du$$

uniformly for $z\in(0,\infty)$, then we will compute the LHS by inverting the series and the integral. We begin by a simple estimate for $t>0$.

$$|f(t)-f^\alpha(t)| \leq \frac{1}{C_\nu}\sum_{n\geq 1}|1-\alpha^{n-1}|g_n e^{-\frac{j_n^2}{2}t} \leq |1-\alpha|\times\frac{1}{C_\nu}\sum_{n\geq 1}n g_n e^{-\frac{j_n^2}{2}t} \tag{13}$$

So for any $r>0$, $||f-f^\alpha||_{\infty,[r,\infty)} \leq |1-\alpha|\times\displaystyle\frac{1}{C_\nu}\sum_{n\geq 1}n g_n e^{-\frac{j_n^2}{2}r} \underset{\alpha\to 1}{\longrightarrow} 0.$

Let us first admit a lemma we will prove later.

Lemma 1. *For any $q>0$, the family $(f^\alpha)_{0<\alpha<1}$ is bounded in $L^2(\mathbb{R}_+, t^{q-1}dt)$ by a constant $C(q)>0$.*

The reader could remark that this lemma (with the inequality (13)) implies $||f||_{L^2(\mathbb{R}_+,t^{q-1}dt)} \leq C(q)$. Since we will often apply the lemma with $q=1$, i.e. the family $(f^\alpha)_{0<\alpha\leq 1}$ is bounded in $L^2(\mathbb{R}_+)$, we will simply write $C(1)=C$.

Let $\varepsilon>0$. For any $0<\alpha_1,\alpha_2\leq 1$ and $z>0$, by using (12) and the lemma,

$$\left|\frac{1}{z^{3+\delta}}\int_0^{\frac{1}{2}} f^{\alpha_1}\left(\frac{u}{z^2}\right)f^{\alpha_2}\left(\frac{1-u}{z^2}\right)du\right| \leq \frac{e^{-\frac{j_0^2}{8z^2}}}{z^{3+\delta}}\int_0^{\frac{1}{2}}\left|f^{\alpha_1}\left(\frac{u}{z^2}\right)\right|du \text{ for small } z$$

$$
\begin{aligned}
&\leq \frac{e^{-\frac{j_0^2}{8z^2}}}{\sqrt{2}z^{3+\delta}} z||f^{\alpha_1}||_{L^2(\mathbb{R}_+)} \\
&\leq C \frac{e^{-\frac{j_0^2}{8z^2}}}{\sqrt{2}z^{2+\delta}} \\
&\leq \varepsilon \text{ for small } z.
\end{aligned}
$$

For big z, we can provide an other inequality,

$$
\begin{aligned}
&\left| \frac{1}{z^{3+\delta}} \int_0^1 f^{\alpha_1}\left(\frac{u}{z^2}\right) f^{\alpha_2}\left(\frac{1-u}{z^2}\right) du \right| \\
&\qquad \leq \frac{1}{z^{3+\delta}} \sqrt{\int_0^1 f^{\alpha_1}\left(\frac{u}{z^2}\right)^2 du} \sqrt{\int_0^1 f^{\alpha_2}\left(\frac{1-u}{z^2}\right)^2 du} \\
&\qquad \leq \frac{1}{z^{3+\delta}} z^2 ||f^{\alpha_1}||_{L^2(\mathbb{R}_+)} ||f^{\alpha_2}||_{L^2(\mathbb{R}_+)} \\
&\qquad \leq \frac{C^2}{z^{1+\delta}} \\
&\qquad \leq \varepsilon \text{ for big } z \text{ because } 1+\delta > 0.
\end{aligned}
$$

So we can choose a small $r > 0$ and a big $R > r$ such that

$$
\sup_{\substack{0<\alpha_1,\alpha_2\leq 1 \\ z\in(0,r)\cup(R,\infty)}} \left| \frac{1}{z^{3+\delta}} \int_0^1 f^{\alpha_1}\left(\frac{u}{z^2}\right) f^{\alpha_2}\left(\frac{1-u}{z^2}\right) du \right| \leq \varepsilon.
$$

Moreover, thanks to the uniform convergence deduced from (13), for any $\eta > 0$ we know that

$$
\frac{1}{z^{3+\delta}} \int_\eta^{1-\eta} f^{\alpha_1}\left(\frac{u}{z^2}\right) f^{\alpha_2}\left(\frac{1-u}{z^2}\right) du \underset{\substack{\alpha_1\to 1^- \\ \alpha_2\to 1^-}}{\longrightarrow} \frac{1}{z^{3+\delta}} \int_\eta^{1-\eta} f\left(\frac{u}{z^2}\right) f\left(\frac{1-u}{z^2}\right) du.
$$

uniformly on $[r, R]$.

Now, to obtain the desired convergence, we only need a small $\eta > 0$ such that

$$
\sup_{\substack{0<\alpha_1,\alpha_2\leq 1 \\ z\in[r,R]}} \left| \int_0^\eta f^{\alpha_1}\left(\frac{u}{z^2}\right) f^{\alpha_2}\left(\frac{1-u}{z^2}\right) du \right| \leq \varepsilon.
$$

Using (12), there is a constant $c > 0$ such that for all $0 < \alpha \leq 1$, we have $||f^\alpha||_{\infty,[1/2R^2,\infty)} \leq c$. For any small $\eta > 0$ and $z \in [r, R]$,

$$
\begin{aligned}
\left| \int_0^\eta f^{\alpha_1}\left(\frac{u}{z^2}\right) f^{\alpha_2}\left(\frac{1-u}{z^2}\right) du \right| &\leq \sqrt{\int_0^\eta f^{\alpha_1}\left(\frac{u}{z^2}\right)^2 du} \sqrt{\int_0^\eta f^{\alpha_2}\left(\frac{1-u}{z^2}\right)^2 du} \\
&\leq z||f^{\alpha_1}||_{L^2(\mathbb{R}_+)} \times ||f^{\alpha_2}||_{\infty,[1/2R^2,\infty)} \sqrt{\eta} \\
&\leq RCc\sqrt{\eta}
\end{aligned}
$$

$$\leq \varepsilon \text{ for small } \eta.$$

Finally, we only need to compute the integral depending on α. Notice all equalities are justified thanks to absolute convergence coming from α^n.

$$\begin{aligned}
&\alpha_1\alpha_2 \int_0^1 f^{\alpha_1}\left(\frac{u}{z^2}\right) f^{\alpha_2}\left(\frac{1-u}{z^2}\right) du \\
&\qquad = \frac{1}{C_\nu^2} \sum_{m,n\geq 1} (-1)^{m+n} \alpha_1^n \alpha_2^m g_m g_n \exp\left(-\frac{j_m^2}{2z^2}\right) \int_0^1 \exp\left(\frac{j_m^2 - j_n^2}{2z^2} u\right) du \\
&\qquad = \frac{1}{C_\nu^2} \sum_{n\geq 1} (\alpha_1\alpha_2)^n g_n^2 \exp\left(-\frac{j_n^2}{2z^2}\right) \\
&\qquad\quad + \frac{2z^2}{C_\nu^2} \sum_{\substack{m,n\geq 1 \\ m\neq n}} (-1)^{m+n} \alpha_1^n \alpha_2^m \frac{g_m g_n}{j_m^2 - j_n^2} \left(e^{-\frac{j_n^2}{2z^2}} - e^{-\frac{j_m^2}{2z^2}}\right)
\end{aligned}$$

The convergence

$$\sum_{n\geq 1} (\alpha_1\alpha_2)^n g_n^2 \exp\left(-\frac{j_n^2}{2z^2}\right) \underset{\substack{\alpha_1\to 1^- \\ \alpha_2 \to 1^-}}{\longrightarrow} \sum_{n\geq 1} g_n^2 \exp\left(-\frac{j_n^2}{2z^2}\right) < \infty$$

is ensured by monotone convergence. From the equality

$$\begin{aligned}
\int_0^1 f\left(\frac{u}{z^2}\right) f\left(\frac{1-u}{z^2}\right) du = \frac{1}{C_\nu^2} \sum_{n\geq 1} g_n^2 e^{-\frac{j_n^2}{2z^2}} \\
+ \frac{2z^2}{C_\nu^2} \sum_{\substack{m,n\geq 1 \\ m\neq n}} (-1)^{m+n} \frac{g_m g_n}{j_m^2 - j_n^2} \left(e^{-\frac{j_n^2}{2z^2}} - e^{-\frac{j_m^2}{2z^2}}\right) \quad (\mathcal{A})
\end{aligned}$$

and the formula (7), we deduce the first theorem (10).

The proof of the second theorem is also based on the lemma 1 and various control inequalities. Let $U \in (0, 1/2)$, $u \in [U, 1-U]$ and $\varepsilon > 0$.

$$\begin{aligned}
\int_0^\infty \frac{2}{z^{3+\delta}} f\left(\frac{u}{z^2}\right) f\left(\frac{1-u}{z^2}\right) dz &= 2\int_0^\infty f(x^2 u) f(x^2(1-u)) x^{1+\delta} dx \\
&= \int_0^\infty f(xu) f(x(1-u)) x^{\frac{\delta}{2}} dx < \infty \quad (14)
\end{aligned}$$

To control what happens near 0 (for x), let us take a small r, for any $0 < \alpha_1, \alpha_2 \leq 1$,

$$\begin{aligned}
&\left|\int_0^r f^{\alpha_1}(xu) f^{\alpha_2}(x(1-u)) x^{\frac{\delta}{2}} dx\right| \\
&\qquad \leq r^{\frac{1+\delta}{2}} \int_0^\infty \frac{1}{\sqrt{x}} |f^{\alpha_1}(xu) f^{\alpha_2}(x(1-u))| dx
\end{aligned}$$

$$\begin{aligned}
&\leq r^{\frac{1+\delta}{2}}\sqrt{\int_0^\infty \frac{f^{\alpha_1}(xu)^2}{\sqrt{x}}dx}\sqrt{\int_0^\infty \frac{f^{\alpha_2}(x(1-u))^2}{\sqrt{x}}dx}\\
&\leq \frac{C(1/2)^2}{\sqrt[4]{U(1-U)}}r^{\frac{1+\delta}{2}}\\
&\leq \varepsilon \text{ for small } r \text{ because } 1+\delta>0.
\end{aligned}$$

To control what happens near ∞ (for x), notice that thanks to (12), for any t big enough, $\forall\alpha\in(0,1]$, $|f^\alpha(t)|\leq e^{-\frac{j_0^2}{4}t}$, and for R big enough,

$$\begin{aligned}
\left|\int_R^\infty f^{\alpha_1}(xu)f^{\alpha_2}(x(1-u))x^{\frac{\delta}{2}}dx\right| &\leq \int_R^\infty |f^{\alpha_1}(xu)||f^{\alpha_2}(x(1-u))|x^{\frac{\delta}{2}}dx\\
&\leq \int_R^\infty x^{\frac{\delta}{2}}e^{-\frac{j_0^2}{4}x}dx\\
&\leq \varepsilon \text{ for big } R\text{, by dominated convergence.}
\end{aligned}$$

After choosing a small r and a big R, we have that for all $<\alpha_1,\alpha_2\leq 1$ and $u\in[U,1-U]$,

$$\left|\int_0^r f^{\alpha_1}(xu)f^{\alpha_2}(x(1-u))x^{\frac{\delta}{2}}dx\right| + \left|\int_R^\infty f^{\alpha_1}(xu)f^{\alpha_2}(x(1-u))x^{\frac{\delta}{2}}dx\right| \leq \varepsilon.$$

Moreover, by (13), uniformly on $[U,1-U]$,

$$\int_r^R f^{\alpha_1}(xu)f^{\alpha_2}(x(1-u))x^{\frac{\delta}{2}}dx \underset{\substack{\alpha_1\to1^-\\ \alpha_2\to1^-}}{\longrightarrow} \int_r^R f(xu)f(x(1-u))x^{\frac{\delta}{2}}dx.$$

Eventually,

$$\int_0^\infty f^{\alpha_1}(xu)f^{\alpha_2}(x(1-u))x^{\frac{\delta}{2}}dx \underset{\substack{\alpha_1\to1^-\\ \alpha_2\to1^-}}{\longrightarrow} \int_0^\infty f(xu)f(x(1-u))x^{\frac{\delta}{2}}dx$$

uniformly on $[U,1-U]$.

The proof ends with some easy computations.

$$\begin{aligned}
&\alpha_1\alpha_2\int_0^\infty f^{\alpha_1}(xu)f^{\alpha_2}(x(1-u))x^{\frac{\delta}{2}}dx\\
&\quad=\frac{1}{C_\nu^2}\sum_{m,n\geq1}(-1)^{m+n}\alpha_1^n\alpha_2^m g_m g_n\int_0^\infty x^{\nu+1}\exp\left(-\frac{j_n^2u+j_m^2(1-u)}{2}x\right)dx\\
&\quad=\frac{1}{C_\nu^2}\sum_{m,n\geq1}(-1)^{m+n}\alpha_1^n\alpha_2^m g_m g_n\left(\frac{2}{j_n^2u+j_m^2(1-u)}\right)^{\nu+2}\int_0^\infty x^{\nu+1}e^{-x}dx\\
&\quad=\frac{2^{\nu+2}\Gamma(\nu+2)}{C_\nu^2}\sum_{m,n\geq1}(-1)^{m+n}\alpha_1^n\alpha_2^m\frac{g_m g_n}{[j_n^2u+j_m^2(1-u)]^{\nu+2}}
\end{aligned}$$

$$\frac{2^{\nu+2}\Gamma(\nu+2)}{C_\nu^2} = 4(\nu+1) \times \frac{2^\nu \Gamma(\nu+1)}{C_\nu^2} = \frac{2\delta}{C_\nu}$$

After using (7), we find the second theorem (11).

$$\frac{\mathbb{P}(\rho \in du)}{du} = 2\delta \sum_{m,n\geq 1} (-1)^{m+n} \frac{g_m g_n}{[j_n^2 u + j_m^2(1-u)]^{\nu+2}} \ (\mathcal{A}) \qquad \square$$

For the Brownian excursion and the reflected Brownian bridge, there are explicit expressions for j_n and g_n, and the formulas (10) and (11) can be written as follows.

Theorem 4. *For the Brownian excursion, i.e. when $\nu = 1/2$, we have*

$$\frac{\mathbb{P}(M \in dz)}{dz} = \frac{\sqrt{2\pi}\pi^4}{z^6} \sum_{n\geq 1} n^4 e^{-\frac{n^2\pi^2}{2z^2}} + \frac{2\sqrt{2\pi}\pi^2}{z^4} \sum_{\substack{m,n\geq 1\\ m\neq n}} (-1)^{m+n} \frac{m^2 n^2}{m^2-n^2} \left(e^{-\frac{n^2\pi^2}{2z^2}} - e^{-\frac{m^2\pi^2}{2z^2}}\right) \ (\mathcal{A}), \tag{15}$$

and

$$\frac{\mathbb{P}(\rho \in du)}{du} = 3 \sum_{m,n\geq 1} (-1)^{m+n} \frac{m^2 n^2}{[n^2 u + m^2(1-u)]^{5/2}} \ (\mathcal{A}). \tag{16}$$

Theorem 5. *For the reflected Brownian bridge, i.e. when $\nu = -1/2$, we have*

$$\frac{\mathbb{P}(M \in dz)}{dz} = \frac{\sqrt{2\pi}\pi^2}{z^4} \sum_{n\geq 1} \left(n + \frac{1}{2}\right)^2 e^{-\frac{(n+1/2)^2\pi^2}{2z^2}} + \frac{2\sqrt{2\pi}}{z^2} \sum_{\substack{m,n\geq 1\\ m\neq n}} (-1)^{m+n} \frac{(m+1/2)(n+1/2)}{(m+1/2)^2-(n+1/2)^2} \times \left(e^{-\frac{(n+1/2)^2\pi^2}{2z^2}} - e^{-\frac{(m+1/2)^2\pi^2}{2z^2}}\right) \ (\mathcal{A}), \tag{17}$$

and

$$\frac{\mathbb{P}(\rho \in du)}{du} = 2 \sum_{m,n\geq 0} (-1)^{m+n} \frac{(2m+1)(2n+1)}{[(2n+1)^2 u + (2m+1)^2(1-u)]^{3/2}} \ (\mathcal{A}). \tag{18}$$

Remark. The use of the Abel summation is not arbitrary. First, (10) becomes false with the standard summation because they are an infinity of not-small terms, as said in [33]. However, Pitman and Yor pointed out the formula could become correct for $\nu = \frac{1}{2}$ (the case of the excursion) when you use the equality $\sum_{n\geq 1}(-1)^{n-1} = 1/2$, as it happens with Abel summation. Furthermore, in [25], the authors interpret their numerical calculations with a regularization $\alpha \to 1$. The summation in (10) is absolutely convergent iff $\nu < -\frac{1}{2}$ while the summation in (11) is never absolutely convergent.

Let us point out there exists simpler formulas than (15) and (17) to express the density of the maximum for the Brownian excursion and for the reflected Brownian bridge. For the excursion,

$$\frac{\mathbb{P}(M \in dz)}{dz} = 8z \sum_{n \geq 1} (4n^2 z^2 - 3) n^2 e^{-2n^2 z^2}$$

was found by Chung [5] and by Kennedy [14]. For the reflected bridge, the famous formula

$$\frac{\mathbb{P}(M \in dz)}{dz} = 8z \sum_{n \geq 1} (-1)^{n-1} n^2 e^{-2n^2 z^2}$$

appears within the study of the statistical Kolmogorov-Smirnov test, and was first demonstrated by Kolmogorov in [17]. The reader could also check [27] for more information about this test.

In fact, these two formulas are the cases $\delta = 3$ and $\delta = 1$ of an expression proven in [33] by Pitman and Yor, called the Gikhman-Kiefer Formula, for the density of the maximum of the δ-dimensional Bessel bridge.

$$\frac{\mathbb{P}(M_\delta \in dz)}{dz} = \frac{2}{C_\nu z^\delta} \sum_{n \geq 1} \frac{j_{\nu,n}^{2\nu}}{J_{\nu+1}^2(j_{\nu,n})} \left(\frac{j_{\nu,n}^2}{z^3} - \frac{\delta}{z} \right) e^{-\frac{j_{\nu,n}^2}{2z^2}} \tag{19}$$

The cases $\delta = 3$ and $\delta = 1$ of (19) can be also found in [4], respectively as 5.1.1.8 page 435 and 3.1.1.8 page 339, but in a different form from the two above formulas. More precisely, it is possible to obtain them from (19) by using the functional relation of Jacobi's theta function (see [33] for more details).

The formula (10) has also been found in [33] (Section 11) for $\nu < -\frac{1}{2}$, but with the standard summation. Moreover, in this article, Pitman and Yor deduced formulas on the zeros of the Bessel functions, equating their double series with the Gikhman-Kiefer Formula. This led to interesting identities since (10) is quite different from other representations in the literature. Indeed, Pitman, Yor, and us were not able to prove directly that the RHS of (10) and (19) give the same result. Even for the particular cases $\delta = 3$ or $\delta = 1$, this respectively leads in [33] to the noteworthy identities for any $n \in \mathbb{N}^*$,

$$\sum_{\substack{m \geq 1 \\ m \neq n}} \frac{(-1)^m m^2}{m^2 - n^2} = \frac{3}{4}(-1)^{n-1} \ (\mathcal{A}), \text{ and}$$

$$\sum_{\substack{m \geq 1 \\ m \neq n}} \frac{(-1)^m (2m-1)}{(2m-1)^2 - (2n-1)^2} = \frac{(-1)^{n-1}}{4(2n-1)}.$$

We will unsuccessfully try to derive more identities in the same way later in Section 4.4. Justifying this method seems theoretically promising and extending it to ρ may be the goal of another publication.

The formulas (10) and (11) could also be applied for numerical simulations. We advise the reader to check [25] for a very nice numerical analysis of (16).

Moreover, they could be useful to study more easily the tails of the distributions of M and ρ. Eventually, we will show in Section 4.4 that the Abel summation in (10) and (11) commutes with the integration when we compute the expected value of a bounded function of M or ρ. Once again, this could lead to some theoretical identities of interest or facilitate numerical approximations of the law of M or ρ.

4.3. Proof of the lemma

We keep the same notations as before. First,

$$\begin{aligned}\int_0^\infty f^\alpha(t)^2 t^{q-1}dt &= \int_0^\infty \frac{1}{C_\nu^2}\sum_{m,n\geq 1}(-\alpha)^{m+n-2}g_m g_n \exp\left(-\frac{j_m^2+j_n^2}{2}t\right)t^{q-1}dt\\ &= \frac{1}{C_\nu^2}\sum_{m,n\geq 1}(-\alpha)^{m+n-2}g_m g_n\int_0^\infty t^{q-1}\exp\left(-\frac{j_m^2+j_n^2}{2}t\right)dt\\ &= \frac{2^q\Gamma(q)}{C_\nu^2}\sum_{m,n\geq 1}(-\alpha)^{m+n-2}\frac{g_m g_n}{(j_m^2+j_n^2)^q}\\ &= \frac{2^q\Gamma(q)}{C_\nu^2}\sum_{n\geq 2}(-\alpha)^{n-2}u_n \text{ with } u_n=\sum_{k=1}^{n-1}\frac{g_k g_{n-k}}{(j_k^2+j_{n-k}^2)^q}.\end{aligned}$$

Begin by observing that (u_n) has a polynomial growth. We only need to show $((-1)^n u_n)$ is Abel summable. The strategy of the proof is to provide an asymptotic expansion of u_n where all the terms are either of the form constant $\times\, n^a$ or absolutely summable. For this, we use the existence of asymptotic expansions of the following form at any order for j_n^2 and g_n: for all $N\in\mathbb{N}^*$,

$$\begin{aligned}\frac{j_n^2}{(\pi n)^2} &= 1+\sum_{i=1}^{N-1}\text{constant(i)}\times n^{-i}+O(n^{-N}),\ \text{ and}\\ \sqrt{\frac{2}{\pi}}\frac{g_n}{(\pi n)^{\nu+3/2}} &= 1+\sum_{i=1}^{N-1}\text{constant}'(i)\times n^{-i}+O(n^{-N}).\end{aligned}$$

It comes (after some manipulations) from Hankel's expansion of $J_{\nu+1}(z)$ for large z and McMahon's expansion of j_n for large n (see [29] pages 237-242 and 247-248 or more simply [8]). Then, we will conclude thanks to the following fact.

Proposition 2. *$\forall s\in\mathbb{C}$, $\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^s}x^{n-1}$ is absolutely convergent for $x\in\mathbb{C}$, $|x|<1$ and can be continuously extended to 1. In particular, $((-1)^n n^{-s})$ is Abel summable.*

Remark (A short digression about η and ζ functions)**.** In fact,

$$\eta(s)=\sum_{n\geq 1}(-1)^{n-1}n^{-s}\ (\mathcal{A})$$

defines an entire function. Since the series is absolutely convergent for $\Re s > 1$, we can easily verify the relation $\eta(s) = (1 - 2^{1-s})\zeta(s)$ where ζ is the Riemann zeta function. Then, we can use the relation to extend ζ to $\mathbb{C}$ except to the points $s_n = 1 + 2in\pi/\ln(2)$, $n \in \mathbb{Z}$. More work is required to show $\eta(s_n) = 0$ for $n \neq 0$ and to give an analytic definition of ζ on $\mathbb{C}\backslash\{1\}$. Moreover, the well-known value of the alternating harmonic series gives $\eta(1) = \ln(2)$ and so $(s-1)\zeta(s) \underset{s\to 1}{\longrightarrow} 1$. While (n) is not Abel summable, it is not difficult to prove $\eta(-1) = 1/4$ to assign to $\displaystyle\sum_{n\geq 1} n$ the famous value $\zeta(-1) = -1/12$. This is a particular case of another non-standard summation called the method of zeta function regularization. See [13] for more details and a proof of the proposition. The proof of the Abel summability of $((-1)^n n^{-s})$ is also done in [10], Section 6.10. [38] solves the problem of the vanishing of η at the s_n.

We use the symbol $\preceq$ when an inequality is true up to a multiplicative constant independent of k or n. We denote

$$x(k,n) = \frac{j_k^2 - \pi^2 k^2 + j_{n-k}^2 - \pi^2(n-k)^2}{\pi^2 k^2 + \pi^2 (n-k)^2}, \text{ and}$$

$$\frac{1}{j_k^2 + j_{n-k}^2} = \frac{1}{\pi^2(k^2 + (n-k)^2)} \frac{1}{1 + x(k,n)}.$$

Observe that

$$|x(k,n)| \preceq \frac{k + (n-k)}{k^2 + (n-k)^2} \preceq \frac{n}{n^2} \preceq \frac{1}{n}.$$

By classic Taylor expansion, we have a polynomial P which does not depend on our problem such that, for $x \to 0$,

$$\frac{1}{(1+x)^q} - P(x) = O(x^{\delta+4}).$$

It follows that

$$\begin{aligned}
\left| u_n - \sum_{k=1}^{n-1} \frac{g_k g_{n-k}}{\pi^{2q}(k^2 + (n-k)^2)^q} P(x(k,n)) \right| &\preceq \sum_{k=1}^{n-1} \frac{|g_k g_{n-k}|}{\pi^{2q}(k^2 + (n-k)^2)^q} x(k,n)^{\delta+4} \\
&\preceq \sum_{k=1}^{n-1} \frac{k^{\nu+3/2}(n-k)^{\nu+3/2}}{\pi^{2q}(k^2 + (n-k)^2)^q} \frac{1}{n^{\delta+4}} \\
&\preceq \sum_{k=1}^{n-1} \frac{n^{2\nu+3}}{n^{\delta+4}} \\
&\preceq \frac{1}{n^2}.
\end{aligned}$$

We only need the Abel summability of $(-1)^n \displaystyle\sum_{k=1}^{n-1} \frac{g_k g_{n-k}}{(k^2 + (n-k)^2)^q} x(k,n)^d$ where $d \in \mathbb{N}$, by linearity. We write the asymptotic expansion for j_k^2 and g_k with order $O\left(\frac{1}{k^{d+\nu+3/2+2}}\right)$ and after expanding $\frac{g_k g_{n-k}}{(k^2+(n-k)^2)^q}x(k,n)^d$, without precising the constant coefficients, we have a finite number of terms of the form $\frac{\text{function}(k)\times \hat{\text{function}}(n-k)}{(k^2+(n-k)^2)^{q+d}}$. The functions do not depend on k or n and can be either explicit, with the form function$(k) = k^a$ where $a \le d+\nu+3/2$, or implicit, with the form function$(k) = \frac{\mu(k)}{k^2}$ where μ is a bounded function.

We now want to show

$$\lim_{\alpha\to 1}\sum_{m,n\ge 1}\alpha^{m-1}\alpha^{n-1}(-1)^{m+n}\frac{\text{function}(m)\times\hat{\text{function}}(n)}{(m^2+n^2)^{q+d}}$$

exists for each type of terms.

Lemma 2. *Let $q>0$. If $f(x,y)=\sum_{m,n\ge 1}x^{m-1}y^{n-1}a_{m,n}$ absolutely converges for $|x|,|y|<1$ and $x,y\in\mathbb{C}$, and if it can be continuously extended to $(1,1)$, then the function g defined by*

$$g(x,y)=\sum_{m,n\ge 1}x^{m-1}y^{n-1}\frac{a_{m,n}}{(m^2+n^2)^q}$$

satisfies the same properties.

It is simply because for $|x|,|y|<1$ we have the identity

$$g(x,y)=\frac{2^q}{\Gamma(q)^2}\int_0^1\int_0^1 |\ln t|^{q-1}|\ln s|^{q-1}tsf(t^{1+i}s^{1-i}x, t^{1-i}s^{1+i}y)dtds$$

that extends continuously g to $(1,1)$ (by dominated convergence).

Since $q>0$ and $d\ge 0$, this lemma allows us to just do the case $q+d=0$ where the double series can be factorized. We just have to show

$$\sum_{n\ge 1}x^{n-1}(-1)^{n-1}\text{function}(n)$$

absolutely converges for $|x|<1$ and can be continuously extended to 1. If the function is implicit, function $(k)=\mu(k)/k^2$, it is true by dominated convergence. If the function is explicit, function $(k)=k^a$, it is the proposition 2.

Our proof works because g_n and j_n have a very regular growth. They can be asymptotically expanded at any order, with only sequences k_n such that $((-1)^n k_n)$ are Abel summable. We have also used the convergence of j_n/n^2.

4.4. Complements for double series formulas

Theorem 6. *If φ a bounded measurable function then*

$$\int_0^1\int_0^\infty \varphi(z,u)\times\frac{2C_\nu}{z^{3+\delta}}f^{\alpha_1}\left(\frac{u}{z^2}\right)f^{\alpha_2}\left(\frac{1-u}{z^2}\right)dzdu \underset{\substack{\alpha_1\to 1^-\\ \alpha_2\to 1^-}}{\longrightarrow} \mathbb{E}[\varphi(M,\rho)].$$

In particular, the Abel summation in (10) and (11) commutes with the integration when we compute the expected value of a bounded function of M or ρ.

Proof. We want to show

$$\int_0^1 \int_0^\infty \varphi(u,x) \times f^\alpha(xu) f^\alpha(x(1-u)) x^{\frac{\delta}{2}} dx du \underset{\substack{\alpha_1 \to 1^- \\ \alpha_2 \to 1^-}}{\longrightarrow} \int_0^1 \int_0^\infty \varphi(u,x) \times f(xu) f(x(1-u)) x^{\frac{\delta}{2}} dx du.$$

Thanks to the uniform convergence of f^α on any segment of $(0,\infty)$ (see (13)), we just have to dominate the behavior of $|f^{\alpha_1}(xu)f^{\alpha_2}(x(1-u))x^{\frac{\delta}{2}}|$ at the boundaries of $(0,1)\times(0,\infty)$. On $(0,1)\times[R,\infty)$, we write

$$\begin{aligned}
&\int_0^1 \int_R^\infty |f^{\alpha_1}(xu) f^{\alpha_2}(x(1-u)) x^{\frac{\delta}{2}}| dx du \\
&\quad = 2 \int_0^{\frac{1}{2}} \int_R^\infty |f^{\alpha_1}(xu) f^{\alpha_2}(x(1-u)) x^{\frac{\delta}{2}}| dx du \\
&\quad \leq 2 \int_0^{\frac{1}{2}} \int_R^\infty x^{\frac{\delta}{2}} e^{-\frac{j_0^2}{4}x} dx du \text{ for big } R \\
&\quad \leq \varepsilon \text{ for bigger } R \text{, by dominated convergence.}
\end{aligned}$$

On $[0,1]\times[0,r]$, we write

$$\int_0^1 \int_0^r |f^{\alpha_1}(xu) f^{\alpha_2}(x(1-u)) x^{\frac{\delta}{2}}| dx du \leq r^{\frac{\delta}{2}} \int_0^1 \frac{C^2}{\sqrt{u(1-u)}} du \leq \varepsilon \text{ for small } r.$$

We fix $r > 0$ small enough and $R > r$ big enough. Thanks to the inequality $||f^\alpha||_{\infty,[r/2,\infty)} \leq c < \infty$ for all $0 < \alpha \leq 1$ given by (12), for a small $\eta > 0$, we can conclude.

On $[0,\eta]\times[r,R]$, we write

$$\begin{aligned}
&\int_0^\eta \int_r^R |f^{\alpha_1}(xu) f^{\alpha_2}(x(1-u)) x^{\frac{\delta}{2}}| dx du \\
&\quad \leq \int_r^R x^{\frac{\delta}{2}} \int_0^\eta |f^{\alpha_1}(xu) f^{\alpha_2}(x(1-u))| du dx \\
&\quad \leq \sqrt{\eta} c C \int_r^R x^{\frac{\delta-1}{2}} dx \\
&\quad \leq \varepsilon \text{ for small } \eta.
\end{aligned}$$

□

Conjecture 1. $\forall n \in \mathbb{N}^*,\ \forall \delta > 0,$

$$\frac{\delta}{4} \frac{j_{\nu,n}^{\nu-1}}{J_{\nu+1}(j_{\nu,n})} = \sum_{\substack{m\geq 1 \\ m\neq n}} \frac{1}{j_{\nu,n}^2 - j_{\nu,m}^2} \times \frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})} \ (\mathcal{A}). \tag{20}$$

Remark. This statement was proven in [33] for $-1 < \nu \leq -1/2$ with absolute convergence. Pitman and Yor have also remarked it becomes true for $\nu = 1/2$ if $\sum_{n\geq 0}(-1)^2 = 1/2$, that is the case with the Abel's summation. We would like to adapt their method for the Abel summation.

Recall Pitman and Yor have derived in [33] another expression for the density of M called the Gikhman-Kiefer formula, (19).

$$\frac{\mathbb{P}(M_\delta \in dz)}{dz} = \frac{2}{C_\nu z^\delta} \sum_{n\geq 1} \frac{j_{\nu,n}^{2\nu}}{J_{\nu+1}^2(j_{\nu,n})} \left(\frac{j_{\nu,n}^2}{z^3} - \frac{\delta}{z} \right) e^{-\frac{j_{\nu,n}^2}{2z^2}}$$

Combining this with (10), and after a change of variable $t = 1/z^2$ we obtain that for all $t > 0$,

$$-\frac{\delta}{2} \sum_{n\geq 1} \frac{j_{\nu,n}^{2\nu}}{J_{\nu+1}^2(j_{\nu,n})} e^{-\frac{j_{\nu,n}^2}{2}t} = \sum_{\substack{m,n\geq 1\\ m\neq n}} \frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})} \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} \frac{e^{-\frac{j_{\nu,n}^2}{2}t} - e^{-\frac{j_{\nu,m}^2}{2}t}}{j_{\nu,m}^2 - j_{\nu,n}^2} \;(\mathcal{A}).$$

Moreover, we can use the Fubini theorem on the RHS to find

$$\begin{aligned} &\sum_{\substack{m,n\geq 1\\ m\neq n}} \alpha^{m+n} \frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})} \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} \frac{e^{-\frac{j_{\nu,n}^2}{2}t} - e^{-\frac{j_{\nu,m}^2}{2}t}}{j_{\nu,m}^2 - j_{\nu,n}^2} \\ &\qquad = -2 \sum_{n\geq 1} \alpha^n \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} e^{-\frac{j_{\nu,n}^2}{2}t} \sum_{\substack{m\geq 1\\ m\neq n}} \frac{\alpha^m}{j_{\nu,n}^2 - j_{\nu,m}^2} \frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})}. \end{aligned}$$

We would like to identify the coefficients of $e^{-\frac{j_{\nu,n}^2}{2}t}$ but we lack a stronger convergence to do this. We can also say that Pitman and Yor showed how (20) can be deduced from the Mittag-Leffler expansion below (true when $-1 < \nu < -1/2$).

$$\frac{x^\nu}{J_\nu(x)} = 2 \sum_{m\geq 1} \frac{1}{j_{\nu,m}^2 - x^2} \times \frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})}$$

4.5. A key point of the proof: the series expansion of f

Our proof of (10) and (11) lies on 3 (4 for ρ) key points: the **agreement formula**, the **series expansion** of f, and the **lemma** 1 (and the scaling property for ρ). The agreement formula was already presented in Section 2, and the proof of the lemma only depends on the coefficients of the series expansion of f and their regularity. Let us see where does this expansion come from, and if it could exist for other processes than Bessel processes.

In [15], Kent applied Sturm-Liouville theory of second order differential equations and complex analysis to study the distribution of diffusion hitting times.

For a regular diffusion on $[r_0, r_1]$, the differential generator A of the diffusion is a generalized second order linear differential operator. If $r_0 < a < b < r_1$, $\lambda > 0$, there is a unique solution $v(x)$ up to a multiplicative constant of $Av = \lambda v$ together with a boundary condition at r_0 (depending on the boundary behavior of the scale function and the speed measure). Moreover, there is the relation $\mathbb{E}_a(\exp(-\lambda T_b)) = v(a)/v(b)$.

If r_0 is not a natural boundary (the diffusion cannot start at r_0 nor reach r_0 in finite time with positive probability, for example $r_0 = -\infty$), we even have a unique joint solution $u(x, \lambda)$ satisfying some initial and boundary conditions at r_0, and we know this solution is entire in λ. Plus, its zeros correspond to the eigenvalues of a Sturm-Liouville problem defined by A. Thanks to the Sturm-Liouville theory, Kent concluded the zeros $-\lambda_{x,n}$ are negative, simple and go to $-\infty$ fast enough. So, he could apply Hadamard's factorization theorem to express $\mathbb{E}_a(\exp(-\lambda T_b))$ as a infinite product indexed by the zeros, which could be view as a formal infinite partial expansion.

Then, Kent showed that under some mild regularity conditions, and again if r_0 is not a natural boundary and if $r_0 \leq a < b < r_1$, $\mathbb{E}_a(\exp(-\lambda T_b))$ could be inverted to give the $\mathbb{P}_a$ density of T_b as a series of the form

$$f_{ab}(t) = \sum d_k e^{-\lambda_{k,b} t} \text{ where } d_k = -u(a, \lambda_{k,b})/u'(b, \lambda_{k,b}),$$

and $\forall \varepsilon > 0,\ d_k = O(e^{\lambda_{k,b}\varepsilon})$ and $\sum \lambda_{k,b}^{-1} < \infty$.

Bessel processes satisfy all those hypotheses and (9) comes from this method. It was even the example chosen [15] by Kent to illustrate its method, and so (9) is explicitly written there. However, it is not the case for many diffusion processes. For example, $r_0 = -\infty$ for skew Brownian motions or the Ornstein-Uhlenbeck process. While such series expansion exists for the Ornstein-Uhlenbeck process anyway (see [1] for a complete study), it is impossible to find one for skew Brownian motions. A simple way to see it is to recall that $\mathbb{E}_a(T_b) = \infty$ for skew Brownian motions, while such formula necessarily implies $\mathbb{E}_a(T_b) < \infty$.

$$\mathbb{E}_a(T_b) \leq 1 + \mathbb{E}_a(T_b 1[T_b > 1]) \leq 1 + \frac{4}{\lambda_{1,k}} \sum \frac{d_k}{\lambda_{k,b}} e^{-\lambda_{k,b}/2} < \infty$$

More essentially, the distribution of a diffusion hitting time is intimately connected with the spectral measure of the differential generator. In particular, the existence of a series expansion for its density is closely related with the discrete nature of the spectrum of the differential generator of the killed diffusion. For a precise discussion about these deep links, the reader could check another paper of Kent, [16].

4.6. How could we generalize those formulas?

We consider a continuous stochastic process defined on $[0, U]$. Let us suppose the 3 key points of our proof: the **agreement formula** of course, but also the **series expansion of** f and the **lemma** 1. We verify which complementary hypothesis we need to make our proof work. More precisely, we suppose $\exists C > 0$ such that $\forall z > 0,\ u \in [0, U],\ t > 0,\ i = 1$ or 2, we have

$$\mathbb{P}(M \in dz, \rho \in du) = c\phi_1(z,u)\phi_2(z,U-u)s(dz)du, \tag{21}$$

$$\phi_i(z,t) = \sum_{n\geq 1} d_{i,n}(z)e^{-\lambda_{i,n}(z)t}, \tag{22}$$

$$C \geq ||\phi_1^\alpha(z,.)||_{L^2(\mathbb{R}_+)} + ||\phi_2^\alpha(z,.)||_{L^2(\mathbb{R}_+)}, \tag{23}$$

where $\phi_i^\alpha(z,t) = \sum_{n\geq 1} \alpha^n d_{i,n}(z)e^{-\lambda_{i,n}(z)t}$.

In the case of a diffusion bridge, $\phi_1(z,u) = f_{xz}(u)$ and $\phi_2(z,u) = f_{yz}(u)$ where x and y are some fixed points. To have an expression of the form of (10) for $\mathbb{P}(M \in dz)$, without asking for the uniform convergence, our reader can remark that we used that for any $z > 0$ and $i = 1$ or 2,

$$\begin{aligned}
&0 < \lambda_{i,n}(z) \underset{n\to\infty}{\longrightarrow} +\infty, \\
&\sum_{n\geq 1} |d_{i,n}(z)| n e^{-\lambda_{i,n}(z)t} < \infty, \\
&\sum_{m,n\geq 1} \alpha^{m+n}|d_{1,m}(z)d_{2,n}(z)|e^{-\lambda_{1,m}(z)\wedge\lambda_{2,n}(z)U} \\
&\qquad \leq \sum_{m,n\geq 1} \alpha^{m+n}|d_{1,m}(z)d_{2,n}(z)| < \infty.
\end{aligned}$$

The conditions only depend on the coefficients d_n and λ_n, and we do not need any assumptions on s or the Brownian scaling, because we integrate the agreement formula with respect to u. They are satisfied for example when for any $z > 0$ and $i = 1$ or 2,

$$0 < \frac{\lambda_{i,n}(z)}{n} \underset{n\to\infty}{\longrightarrow} +\infty \text{ and } \forall\varepsilon > 0,\ d_{i,n}(z) = O(e^{\varepsilon n}),$$

that is the case for the Bessel bridges.

Then, we can use the same proof as the Section 4.2 to derive

$$\frac{\mathbb{P}(M \in dz)}{s(dz)} = c\sum_{m,n\geq 1} d_{1,m}(z)d_{2,n}(z)\frac{e^{-\lambda_{2,n}(z)U} - e^{-\lambda_{1,m}(z)U}}{\lambda_{1,m}(z) - \lambda_{2,n}(z)} \ (\mathcal{A})$$

with the convention $\dfrac{e^{-\lambda_{2,n}(z)U} - e^{-\lambda_{1,m}(z)U}}{\lambda_{1,m}(z) - \lambda_{2,n}(z)} = Ue^{-\lambda_{2,n}(z)U}$ if $\lambda_{1,m}(z) = \lambda_{2,n}(z)$.

It seems rather easy to derive such expressions for more processes. If you have the agreement formula, and could apply Kent's theory to obtain some series expansions, you just have to study the asymptotic behavior of d_n and λ_n to prove the lemma 1 and the above conditions. Even if it does not work well, another possibility of generalization is to replace α^n by an other regularizing sequence in the definition of ϕ^α. For example, if we take $e^{-\lambda_n\varepsilon}$, $\phi^\varepsilon(t) = \phi(t+\varepsilon)$.

However, to derive an expression of the form of (11) for $\mathbb{P}(\rho \in du)$, our proof also asks to have a **scaling property** such that the agreement formula could be written, for all $z > 0$ and $u \in [0, U]$,

$$\mathbb{P}(M \in dz, \rho \in du) = c\phi_1\left(\frac{u}{w^{-1}(z)}\right)\phi_2\left(\frac{U-u}{w^{-1}(z)}\right)\frac{s(dz)}{w^{-1}(z)^2}du$$

where w is a continuous strictly increasing function on $[0, \infty)$ such that $w(0) = 0$ and $w(\infty) = \infty$ (with the Brownian scaling $w(z) = \sqrt{z}$). It is necessary because we want to integrate with respect to z and so to remove the dependance in z of $d_n(z)$ and $\lambda_n(z)$. In the beginning of this paper, such simplification was possible because we were in the case of a diffusion bridge from 0 to 0, with a **self-similarity property**. More precisely, $\phi_1(z,t) = \phi_2(z,t) = f_{0z}(t)$ and $(X_{at})_{t\geq 0} \stackrel{d}{=} (w(a)X_t)_{t\geq 0}$, so $f_{0z}(t) = f_{01}(t/w^{-1}(z))/w^{-1}(z)$. Note this does not work for other bridges from x to y, when $x \neq 0$ or $y \neq 0$. We precise the definition of a self-similar process.

Definition. A stochastic process $(X_t)_{t\geq 0}$, continuous in probability with $X_0 = 0$, is self-similar (or semi-stable) when there is a function w on $(0, \infty)$ such that $\forall a > 0,\ (X_{at})_{t\geq 0} \stackrel{d}{=} (w(a)X_t)_{t\geq 0}$.

It is easy to see that if X has a non-degenerate distribution then $w(a) = a^q$ for some $q \geq 0$. We call q the order of X.

Remark. Bessel processes and skew Brownian motions are self-similar of order 1/2.

In [18], Lamperti classified the self-similar Markov processes and in particular, the self-similar diffusion processes. His results imply any non-degenerate semi-stable diffusion on $[0, \infty)$ such that 0 is not absorbing (with other words $\mathbb{P}_0(T_1 < \infty) > 0$) is of the form

$$X = cR^{2q} \text{ with } c > 0 \text{ and } R \stackrel{d}{=} \mathrm{BES}(\delta) \text{ for some } \delta > 0. \tag{24}$$

Moreover, the left boundary of a self-similar diffusion can only be $-\infty$ or 0, so the series expansion theorem of Kent (see previous section) could only be applied if the diffusion is on $[0, \infty)$. In fact, if X is also regular on $\mathbb{R}$, it is possible to show $\mathbb{E}_0(T_1) = \infty$ thanks to the above description. So, we cannot hope for a series expansion like (9) if the left boundary is $-\infty$.

In short, to be able to reproduce the exact same proof to find an expression of the density of the argmax like (11) for an X-bridge from x to y, we need X to be a self-similar diffusion on $[0, \infty)$, and $x = y = 0$. But, in this case, the density of ρ can simply be deduced from (11) and (24). Thus, our method to prove the formula (11) can not be generalized to essentially different diffusion bridges without being modified.

Nevertheless, one has to be aware they are the conditions we used in our proof and not necessary conditions. Some expressions like (11) may be found for other processes by using different series expansions or other simplifications of the agreement formula. It is what happens for the next example of skew Brownian bridges.

5. Marginal densities for skew Brownian bridges

Recently, in [2], Appuhamillage and Sheldon derived a series expansion for f_β, the density of T for the case of a skew Brownian motion such that $P(X_1 > 0) = \beta \in (0,1)$.

$$f_\beta(t) = \frac{2\beta}{\sqrt{2\pi t^3}} \sum_{n\geq 1} (1-2\beta)^{n-1}(2n-1)\exp\left(-\frac{(2n-1)^2}{2t}\right) \tag{25}$$

Notice this expression is different from the one for the Bessel processes. Although (9) comes from the Sturm-Liouville theory and the method of Kent (see Section 4.5 and [15]), (25) comes from conditioning on the signs of the excursions away from 0 of the skew Brownian motion (details in [2]). The two methods are essentially different and can not be applied to the other family of processes. However, it is possible to use (25) instead of (9) with the agreement formula to derive an expression for the density of ρ looking like (11).

We also recall the agreement formula (8) for a (standard) skew Brownian bridge: for $z > 0$ and $0 < u < 1$,

$$\mathbb{P}(M \in dz, \rho \in du) = \frac{\sqrt{2\pi}}{\beta z^4} f_\beta\left(\frac{u}{z^2}\right) f_\beta\left(\frac{1-u}{z^2}\right) dzdu.$$

Theorem 7. *The density of the argmax of a standard skew Brownian bridge is equal to*

$$\frac{\mathbb{P}(\rho \in du)}{du} = 2\beta \sum_{m,n\geq 1} (1-2\beta)^{m+n-2} \frac{(2m-1)(2n-1)}{[(2n-1)^2 u + (2m-1)^2(1-u)]^{3/2}}. \tag{26}$$

Proof. The following equalities are justified by absolute convergence because $|1-2\beta| < 1$.

$$\begin{aligned}
&\frac{\mathbb{P}(\rho \in du)}{du} \\
&= \frac{\sqrt{2\pi}}{\beta} \times \frac{(2\beta)^2}{2\pi} \int_0^\infty \frac{1}{z^4} \times \frac{z^6}{[u(1-u)]^{3/2}} \sum_{m,n\geq 1} (1-2\beta)^{m+n-2}(2m-1)(2n-1) \\
&\qquad \times \exp\left(-\frac{(2n-1)^2 u + (2m-1)^2(1-u)}{2u(1-u)} z^2\right) dz \\
&= \frac{4\beta}{\sqrt{2\pi}} \times [u(1-u)]^{-3/2} \sum_{m,n\geq 1} (1-2\beta)^{m+n-2}(2m-1)(2n-1) \\
&\qquad \times \int_0^\infty z^2 \exp\left(-\frac{(2n-1)^2 u + (2m-1)^2(1-u)}{2u(1-u)} z^2\right) dz
\end{aligned}$$

$$= \frac{4\beta}{\sqrt{2\pi}} \times [u(1-u)]^{-3/2} \sum_{m,n\geq 1} (1-2\beta)^{m+n-2}(2m-1)(2n-1)$$
$$\times \left[\frac{2u(1-u)}{(2n-1)^2 u + (2m-1)^2(1-u)}\right]^{3/2} \int_0^\infty z^2 e^{-z^2} dz$$
$$= \frac{2\beta}{\sqrt{\pi}} \times \frac{\sqrt{\pi}\times 4}{4} \sum_{m,n\geq 1} (1-2\beta)^{m+n-2} \frac{(2m-1)(2n-1)}{[(2n-1)^2 u + (2m-1)^2(1-u)]^{3/2}}.$$

□

For $\beta = 1/2$, the case of the (non-reflected) Brownian bridge, (25) and (26) simply become the classical formulas

$$f_{1/2}(t) = \frac{1}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}\right), \text{ and}$$
$$\frac{\mathbb{P}(\rho \in du)}{du} = 1.$$

If we denote $\alpha = 2\beta - 1 < 1$, $\alpha \longrightarrow 1$ when $\beta \longrightarrow 1$, or in other words when skew Brownian motion(β) converges in law to the reflected Brownian motion, which is also the Bessel process of dimension $\delta = 1$. Remarkably, the convergence $\alpha \longrightarrow 1$ in (26) is exactly the convergence in the definition of the Abel summation used in (18) when $\alpha_1 = \alpha_2 = \alpha$. (18) is almost a direct consequence of (26).

It shows the strong links between (26) and (11), even if there is not any obvious connections between (25) and (9). It is very interesting and strange, mainly because the same thing does not happen for M.

Theorem 8. *The density of the max of a standard skew Brownian bridge is equal to*

$$\frac{\mathbb{P}(M \in dz)}{dz} = 8\beta z \sum_{k\geq 1} (1-2\beta)^{k-1} k^2 e^{-2k^2 z^2} \tag{27}$$

Proof. First, we recall an integral equality for $a, b > 0$, which comes from the stability of the laws of Levy.

$$\int_0^1 \sqrt{\frac{a}{2\pi}} \frac{e^{-\frac{a}{2u}}}{u^{3/2}} \times \sqrt{\frac{b}{2\pi}} \frac{e^{-\frac{b}{2(1-u)}}}{(1-u)^{3/2}} du = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2\pi}} e^{-\frac{(\sqrt{a}+\sqrt{b})^2}{2}}$$

By absolute convergence,

$$\frac{\mathbb{P}(M \in dz)}{dz}$$
$$= \frac{\sqrt{2\pi}}{\beta z^4} \sum_{m,n\geq 1} \frac{(2\beta)^2}{2\pi} (1-2\beta)^{m+n-2}(2m-1)(2n-1) z^6$$
$$\times \int_0^1 \frac{1}{[u(1-u)]^{3/2}} \exp\left(-\frac{(2n-1)^2 z^2}{2u} - \frac{(2m-1)^2 z^2}{2(1-u)}\right) du$$

$$
\begin{aligned}
&= \frac{4\beta}{\sqrt{2\pi}} z^2 \sum_{m,n\geq 1} (1-2\beta)^{m+n-2}(2m-1)(2n-1) \\
&\qquad\qquad \times \frac{\sqrt{2\pi}}{z} \frac{(2m-1)+(2n-1)}{(2m-1)(2n-1)} \exp\left(-\frac{((2m-1)+(2n-1))^2}{2} z^2\right) \\
&= 8\beta z \sum_{m,n\geq 1} (1-2\beta)^{m+n-2}(m+n-1)e^{-2(m+n-1)^2 z^2} \\
&= 8\beta z \sum_{k\geq 1} (1-2\beta)^{k-1} k^2 e^{-2k^2 z^2}. \qquad \square
\end{aligned}
$$

For $\beta = 1/2$, the case of the (non-reflected) Brownian bridge, (27) is again a classical formula.

$$
\frac{\mathbb{P}(M \in dz)}{dz} = 4ze^{-2z^2}
$$

For $\beta = 1$, the absolute convergence also holds and (27) becomes the well known formula used for the Kolmogorov-Smirnov test, already mentioned at the end of Section 4.2.

$$
\frac{\mathbb{P}(M \in dz)}{dz} = 8z \sum_{k\geq 1} (-1)^{k-1} k^2 e^{-2k^2 z^2}
$$

This expression belongs to the family of the Gikhman-Kiefer formulas (19) for all Bessel bridges of dimension $\delta > 0$. More precisely, it is the case $\delta = 1$, after using a functional relation for a Jacobi's theta function (see [33] for more details). The essential links between this formula and (27) with (17) and (10) are not obvious to us at all.

6. Marginal densities for generalized Bessel meanders

In Section 4.6, we discussed the importance of enjoying a scaling property to find a double series expression for the density of ρ with the method of Section 4.2. For a bridge where the end point is fixed, this one can only be 0. However, if the end point has a continuous distribution, we can obtain different processes for which the agreement formula can be simplified by scaling.

For $k = 2(\mu+1) > 0$ and $\delta = 2(\nu+1) > 0$, the k-generalized Bessel meander of dimension δ is a continuous stochastic process X on $[0,1]$ such that $\mathbb{P}(X_1 \in dy) = \dfrac{2}{2^{k/2}\Gamma(k/2)} y^{k-1} e^{-\frac{y^2}{2}} \mathbb{1}_{y>0} dy$, i.e. X_1 follows the chi distribution of parameter k, and conditionally given $X_1 = y$, X is a Bessel bridge of dimension δ, from 0 to y, and of length 1.

The k-generalized Bessel meander of dimension δ gives a generalization of several processes. When $k = 2$, it is the classical Bessel meander of dimension δ. When $\delta = 3$, it is the k-generalized Brownian meander (seen in Section 3.3 when k is an integer). When $k = 2$ and $\delta = 3$, it is the classical Brownian meander. When $k = \delta$, it is the Bessel process of dimension δ on $[0,1]$. See [26] for details, background, and properties about this class of processes.

We fix k and δ and use the notations of Sections 3.1 and 4.2: $C_\nu = 2^\nu \Gamma(\nu+1)$, f_{x1} the density of the first hitting time of 1 for the Bessel process of dimension δ started at $x \in [0,1]$, and $f = f_{01}$. As mentioned in Section 4.5, the method of Kent developed in [15] provides series expression for f_{x1} when $x \in [0,1]$. In fact, the case of the Bessel processes was done explicitly in [15].

$$f(t) = \frac{1}{C_\nu} \sum_{n\geq 1} \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} e^{-\frac{j_{\nu,n}^2}{2}t} = \frac{1}{C_\nu} \sum_{n\geq 1} (-1)^{n-1} g_{\nu,n} e^{-\frac{j_{\nu,n}^2}{2}t}$$

$$f_{x1}(t) = \sum_{n\geq 1} (-1)^{n-1} g_{\nu,n} \frac{J_\nu(x j_{\nu,n})}{(x j_{\nu,n})^\nu} e^{-\frac{j_{\nu,n}^2}{2}t} \text{ for } 0 < x < 1$$

We could and would forget the indexes ν of $j_{\nu,n}$ and $g_{\nu,n}$. In this case, the agreement formula becomes

$$\mathbb{P}(M \in dz, \rho \in du) = \frac{2C_\nu}{kC_\mu} \times \frac{1}{z^{3+(\delta-k)}} f\left(\frac{u}{z^2}\right) \phi_k\left(\frac{1-u}{z^2}\right) dzdu, \tag{28}$$

where $\phi_k(t) = \int_0^1 kx^{k-1} f_{x1}(t)dx \geq 0$.

Remark that ϕ_k is the density of some random time $S_k \geq 0$ because we have $\int_0^\infty \phi_k(t)dt = 1$. The chi distribution of X_1 allows us to simplify the agreement formula thanks to the scaling

By making a change of variable in the integral with respect to y, we find $f_{yz}(u) = f_{\frac{y}{z}1}(u/z^2)/z^2$, which makes appear ϕ_k. Then, the series expansion of f_{x1}, $x \in (0,1)$, induces a series expansion for ϕ_k.

$$\phi_k(t) = \sum_{n\geq 1} (-1)^{n-1} \left(\int_0^1 kx^{k-1} \frac{J_\nu(x j_n)}{(x j_n)^\nu} dx \right) g_n e^{-\frac{j_n^2}{2}t} \tag{29}$$

For the particular cases of the classical Bessel meander, the generalized Brownian meander, and the Bessel process, the formula (29) can be reduced by using classical identities for J_ν and $j_{\nu,n}$. Once again, we advise to check [8] if one is not familiar with Bessel functions.

- For the classical Bessel meander, $k = 2$, we compute

$$\begin{aligned} \int_0^{j_n} x^{1-\nu} J_\nu(x)dx &= [-x^{1-\nu} J_{\nu-1}(x)]_0^{j_n} \\ &= \frac{1}{2^{\nu-1}\Gamma(\nu-1+1)} - j_n^{1-\nu} J_{\nu-1}(j_n) \\ &= \frac{1}{C_{\nu-1}} + j_n^{1-\nu} J_{\nu+1}(j_n), \end{aligned}$$

  then deduce

$$\phi_2(t) = 2\sum_{n\geq 1} e^{-\frac{j_n^2}{2}t} + \frac{2}{C_{\nu-1}} \sum_{n\geq 1} \frac{j_n^{\nu-1}}{J_{\nu+1}(j_n)} e^{-\frac{j_n^2}{2}t}. \tag{30}$$

- For the generalized Brownian meander, $\delta = 3$, we compute

$$g_{\frac{1}{2},n} \int_0^1 kx^{k-1} \frac{J_{\frac{1}{2}}(xj_{\frac{1}{2},n})}{\sqrt{xj_{\frac{1}{2},n}}} dx = \sqrt{\frac{\pi}{2}} \pi^2 n^2 k \sqrt{\frac{2}{\pi}} \frac{1}{\pi n} \int_0^1 x^{k-2} \sin(x\pi n) dx$$
$$= \pi k n \int_0^1 x^{k-2} \sin(x\pi n) dx,$$

then deduce

$$\phi_k(t) = \pi k \sum_{n\geq 1} (-1)^{n-1} n \left(\int_0^1 x^{k-2} \sin(x\pi n) dx \right) e^{-\frac{\pi^2 n^2}{2} t}. \qquad (31)$$

- For the classical Brownian meander, $\delta = 3$ and $k = 2$, we directly find

$$\phi_2(t) = 2 \sum_{n\geq 1} (1 + (-1)^{n-1}) e^{-\frac{\pi^2 n^2}{2} t} = 4 \sum_{n\geq 0} e^{-\frac{\pi^2 (2n+1)^2}{2} t}. \qquad (32)$$

- For the Bessel process, $k = \delta$, we compute

$$\int_0^{j_n} x^{1+\nu} J_\nu(x) dx = [x^{1+\nu} J_{\nu+1}(x)]_0^{j_n}$$
$$= j_n^{1+\nu} J_{\nu+1}(j_n),$$

then deduce

$$\phi_\delta(t) = \delta \sum_{n\geq 1} e^{-\frac{j_n^2}{2} t}. \qquad (33)$$

We can observe $||\phi_\delta||_{L^2(\mathbb{R}_+)} = \infty$, which means the ϕ_k do not satisfy the lemma 1 in general, so we will need to slightly adapt the proof of the Section 4.1 to prove the following theorem.

Theorem 9. *If $0 < k < 1 + \delta$, then*

- *The density of the max of the k-generalized Bessel meander is equal to*

$$\frac{\mathbb{P}(M \in dz)}{dz}$$
$$= \frac{2}{kC_\mu z^{3+\delta-k}} \sum_{n\geq 1} \left(\frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} \right)^2 \left(\int_0^1 kx^{k-1} \frac{J_\nu(xj_{\nu,n})}{(xj_{\nu,n})^\nu} dx \right) e^{-\frac{j_{\nu,n}^2}{2z^2}}$$
$$+ \frac{4}{kC_\mu z^{1+\delta-k}} \sum_{\substack{m,n\geq 1 \\ m\neq n}} \frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})} \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})}$$
$$\times \left(\int_0^1 kx^{k-1} \frac{J_\nu(xj_{\nu,n})}{(xj_{\nu,n})^\nu} dx \right) \frac{e^{-\frac{j_{\nu,n}^2}{2z^2}} - e^{-\frac{j_{\nu,m}^2}{2z^2}}}{j_{\nu,m}^2 - j_{\nu,n}^2} \ (\mathcal{A}). \qquad (34)$$

The convergence in $(\alpha_1, \alpha_2) \to (1^-, 1^-)$ is uniform on every segment of $(0, \infty)$.

- *The density of the argmax of the k-generalized Bessel meander is equal to*

$$\frac{\mathbb{P}(\rho \in du)}{du} = \frac{2C_{\nu-\mu}}{kC_\mu} \sum_{m,n\geq 1} \frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})} \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} \times \left(\int_0^1 k x^{k-1} \frac{J_\nu(x j_{\nu,n})}{(x j_{\nu,n})^\nu} dx \right) \frac{1}{[j_{\nu,n}^2 u + j_{\nu,m}^2 (1-u)]^{\nu-\mu+1}} \ (\mathcal{A}). \quad (35)$$

The convergence in $(\alpha_1, \alpha_2) \to (1^-, 1^-)$ is uniform on every segment of $(0,1)$.

Proof. As for the proof of Section 4.2, we will compute those densities using (28) and the series expansions of f, (9), and of ϕ_k, (29). We want to show that for any $0 < r < R$ and $0 < U < 1/2$,

$$\frac{1}{z^{3+\delta-k}} \int_0^1 f^{\alpha_1}\left(\frac{u}{z^2}\right) \phi_k^{\alpha_2}\left(\frac{1-u}{z^2}\right) du \underset{\substack{\alpha_1 \to 1^- \\ \alpha_2 \to 1^-}}{\longrightarrow} \frac{1}{z^{3+\delta-k}} \int_0^1 f\left(\frac{u}{z^2}\right) \phi_k\left(\frac{1-u}{z^2}\right) du$$

$$\int_0^\infty f^{\alpha_1}(xu) \phi_k^{\alpha_2}(x(1-u)) x^{\frac{\delta-k}{2}} dx \underset{\substack{\alpha_1 \to 1^- \\ \alpha_2 \to 1^-}}{\longrightarrow} \int_0^\infty f(xu)\phi_k(x(1-u)) x^{\frac{\delta-k}{2}} dx$$

uniformly respectively for $z \in [r, R]$ and for $u \in [U, 1-U]$.

Since

$$\int_0^1 x^{k-1} \frac{J_\nu(x j_n)}{(x j_n)^\nu} dx = \frac{1}{j_n^k} \int_0^{j_n} x^{k-1} \frac{J_\nu(x)}{x^\nu} dx = O(1),$$

the function ϕ_k satisfies the inequalities (12) and (13). We study the asymptotic behavior of this integral by an integration by parts.

$$\int_1^{j_n} x^{k-1} \frac{J_\nu(x)}{x^\nu} dx = \left[-x^{k-2} \frac{J_{\nu-1}(x)}{x^{\nu-1}} \right]_1^{j_n} + (k-2) \int_1^{j_n} x^{k-3} \frac{J_{\nu-1}(x)}{x^{\nu-1}} dx$$

After such successive integrations by parts, we have a finite linear combination of terms of the form $\left[-x^{k-2i} \frac{J_{\nu-i}(x)}{x^{\nu-i}} \right]_1^{j_n}$ where $i \geq 1$, plus the following integral with some multiplicative constant:

$$\begin{aligned} &\int_1^{j_n} x^{k-\nu-\lfloor k \rfloor -3} J_{\nu-\lfloor k \rfloor -2}(x) dx \\ &\quad = \int_1^\infty x^{k-\nu-\lfloor k \rfloor -3} J_{\nu-\lfloor k \rfloor -2}(x) dx - \int_{j_n}^\infty x^{k-\nu-\lfloor k \rfloor -3} J_{\nu-\lfloor k \rfloor -2}(x) dx \\ &\quad = \int_1^\infty x^{k-\nu-\lfloor k \rfloor -3} J_{\nu-\lfloor k \rfloor -2}(x) dx + O\left(n^{k-\nu-\lfloor k \rfloor -2}\right) \end{aligned}$$

because $k - \nu - \lfloor k \rfloor - 3 < -1$ and $J_{\nu-\lfloor k \rfloor -2}$ is bounded on $(1, \infty)$. Moreover, for $i \geq 1$, we know

$$j_n^{k-i-\nu} J_{\nu-i}(j_n) = O(n^{k-(\nu+1+1/2)}) \text{ and } \frac{g_n}{j_n^k} = O(n^{\nu+1+1/2-k}).$$

So,

$$\left(\int_0^{j_n} kx^{k-1}\frac{J_\nu(x)}{x^\nu}dx\right)\frac{g_n}{j_n^k} = \text{constant} \times \frac{g_n}{j_n^k} + O(1).$$

Observe that $\phi_k(t)$ is the sum of two terms. The first one is

$$\tilde{\phi}_k(t) = \sum_{n\geq 1}(-1)^{n-1}\frac{g_n}{j_n^k}e^{-\frac{j_n^2}{2}t}$$

up to a multiplicative constant and the second one is dominated by $\phi_\delta(t)$. Because $\forall N \in \mathbb{N}^*$, $\dfrac{(\pi n)^k}{j_n^k} = 1 + \sum_{i=1}^{N-1} \text{constant(i)} \times n^{-i} + O(n^{-N})$, the proof of the Section 4.3 directly shows that $\tilde{\phi}_k$ verifies the lemma 1. We let the reader check that we can reuse the proof of Section 4.2 for $\tilde{\phi}_k$ because $\delta - k > 0$. Thus, to deduce the desired convergences, we only have to control the integrals of $\dfrac{1}{z^{3+(\delta-k)}} f\left(\dfrac{u}{z^2}\right)\phi_\delta\left(\dfrac{1-u}{z^2}\right)$ near the boundaries. Let $\varepsilon > 0$.

Note that if $q > 0$,

$$||\phi_\delta||_{L^2(\mathbb{R}_+,t^q dt)} \leq \delta\sqrt{2^{q+1}\Gamma(q+1)}\sum_{n\geq 1}\frac{1}{j_n^{1+q}} < \infty.$$

Let $0 < r < R$. For any small $\eta > 0$, $z \in [r, R]$, and $0 < \alpha_1, \alpha_2 \leq 1$, we have

$$\begin{aligned}
\left|\int_0^\eta f^{\alpha_1}\left(\frac{u}{z^2}\right)\phi_\delta^{\alpha_2}\left(\frac{1-u}{z^2}\right)du\right| &\leq \sqrt{\int_0^\eta f^{\alpha_1}\left(\frac{u}{z^2}\right)^2 du}\sqrt{\int_0^\eta \phi_\delta^{\alpha_2}\left(\frac{1-u}{z^2}\right)^2 du}\\
&\leq z||f^{\alpha_1}||_{L^2(\mathbb{R}_+)} \times ||\phi_\delta||_{\infty,[1/2R^2,\infty)}\sqrt{\eta}\\
&\leq RC||\phi_\delta||_{\infty,[1/2R^2,\infty)}\sqrt{\eta}\\
&\leq \varepsilon \text{ for small } \eta.,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^\eta f^{\alpha_1}\left(\frac{1-u}{z^2}\right)\ \phi_\delta^{\alpha_2}\left(\frac{u}{z^2}\right)du\right|\\
&\qquad \leq \sqrt{\int_0^\eta \phi_\delta^{\alpha_2}\left(\frac{u}{z^2}\right)^2\sqrt{u}du}\sqrt{\int_0^\eta f^{\alpha_1}\left(\frac{1-u}{z^2}\right)^2\frac{1}{\sqrt{u}}du}\\
&\qquad \leq z^{3/2}||\phi_\delta||_{L^2(\mathbb{R}_+,\sqrt{t}dt)} \times ||f^{\alpha_1}||_{\infty,[1/2R^2,\infty)}\sqrt[8]{\eta}\sqrt{\int_0^1\frac{du}{u^{3/4}}}\\
&\qquad \leq 2R^{3/2}||\phi_\delta||_{L^2(\mathbb{R}_+,\sqrt{t}dt)}\sup_{0<\alpha\leq 1}||f^\alpha||_{\infty,[1/2R^2,\infty)} \times \sqrt[8]{\eta}\\
&\qquad \leq \varepsilon \text{ for small } \eta.
\end{aligned}$$

Let $0 < U < 1/2$. For any small $r > 0$ and big $R > 0$, $u \in [U, 1-U]$, and $0 < \alpha_1, \alpha_2, \leq 1$, we have

$$\begin{aligned}\left|\int_R^\infty f^{\alpha_1}(xu)\phi_\delta^{\alpha_2}(x(1-u))x^{\frac{\delta}{2}}dx\right| &\leq \int_R^\infty |f^{\alpha_1}(xu)||\phi_\delta(x(1-u))|x^{\frac{\delta}{2}}dx \\ &\leq \int_R^\infty x^{\frac{\delta}{2}}e^{-\frac{j_0^2}{4}x}dx \\ &\leq \varepsilon \text{ for big } R\text{, by dominated convergence,}\end{aligned}$$

and

$$\begin{aligned}&\left|\int_0^r f^{\alpha_1}(xu)\ \phi_\delta^{\alpha_2}(x(1-u))x^{\frac{\delta-k}{2}}dx\right| \\ &\quad \leq r^{\frac{1+\delta-k}{8}}\int_0^r |f^{\alpha_1}(xu)|x^{\frac{\delta-k-1}{4}} \times |\phi_\delta^{\alpha_2}(x(1-u))|x^{\frac{1+\delta-k}{8}}dx \\ &\quad \leq r^{\frac{1+\delta-k}{8}}\sqrt{\int_0^r f^{\alpha_1}(xu)^2x^{\frac{\delta-k-1}{2}}dx}\sqrt{\int_0^r \phi_\delta(x(1-u))^2x^{\frac{1+\delta-k}{4}}dx} \\ &\quad \leq \frac{r^{\frac{1+\delta-k}{8}}}{u^{\frac{1+\delta-k}{4}}(1-u)^{1+\frac{1+\delta-k}{8}}}C\left(\frac{1+\delta-k}{2}\right)\sqrt{\int_0^\infty \phi_\delta(x)^2x^{\frac{1+\delta-k}{4}}dx} \\ &\quad \leq \frac{C((1+\delta-k)/2) \times ||\phi_\delta||_{L^2(\mathbb{R}_+, t^{\frac{1+\delta-k}{4}}dt)}}{[U(1-U)]^{1+\frac{1+\delta-k}{4}}} \times r^{\frac{1+\delta-k}{8}} \\ &\quad \leq \varepsilon \text{ for small } r \text{ because } 1+\delta-k > 0.\end{aligned}$$

□

Eventually, we simplify the expressions (34) and (35) for some particular cases.

Theorem 10. *If $1 < \delta$, then for the classical Bessel meander, i.e. $k = 2$, we have*

$$\frac{\mathbb{P}(M \in dz)}{dz} = \frac{2}{z^{\delta+1}}\sum_{n\geq 1}\left(\frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})}\right)\left(1 + \frac{1}{C_{\nu-1}}\frac{j_{\nu,n}^{\nu-1}}{J_{\nu+1}(j_{\nu,n})}\right)e^{-\frac{j_{\nu,n}^2}{2z^2}}$$
$$+ \frac{4}{z^{\delta-1}}\sum_{\substack{m,n\geq 1\\ m\neq n}}\frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})}\left(1 + \frac{1}{C_{\nu-1}}\frac{j_{\nu,n}^{\nu-1}}{J_{\nu+1}(j_{\nu,n})}\right)\frac{e^{-\frac{j_{\nu,n}^2}{2z^2}} - e^{-\frac{j_{\nu,m}^2}{2z^2}}}{j_{\nu,m}^2 - j_{\nu,n}^2}\ (\mathcal{A}), \quad (36)$$

and

$$\frac{\mathbb{P}(\rho \in du)}{du} = 2C_\nu\sum_{m,n\geq 1}\frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})}\left(1 + \frac{1}{C_{\nu-1}}\frac{j_{\nu,n}^{\nu-1}}{J_{\nu+1}(j_{\nu,n})}\right)$$
$$\times \frac{1}{[j_{\nu,n}^2u + j_{\nu,m}^2(1-u)]^{\nu+1}}\ (\mathcal{A}). \quad (37)$$

Theorem 11. *If $0 < k < 4$, then for the k-generalized Brownian meander, i.e. $\delta = 3$, we have*

$$\frac{\mathbb{P}(M \in dz)}{dz} = \frac{\pi^3\sqrt{2\pi}}{C_\mu z^{6-k}} \sum_{n\geq 1} n^3 \left(\int_0^1 x^{k-2}\sin(x\pi n)dx\right) e^{-\frac{n^2\pi^2}{2z^2}}$$
$$+ \frac{2\pi\sqrt{2\pi}}{C_\mu z^{4-k}} \sum_{\substack{m,n\geq 1\\ m\neq n}} (-1)^{m+n} m^2 n \left(\int_0^1 x^{k-2}\sin(x\pi n)dx\right) \frac{e^{-\frac{n^2\pi^2}{2z^2}} - e^{-\frac{m^2\pi^2}{2z^2}}}{m^2-n^2} \ (\mathcal{A}), \tag{38}$$

and

$$\frac{\mathbb{P}(\rho \in du)}{du} = \frac{\pi^{k-2}\sqrt{2\pi}C_{1/2-\mu}}{C_\mu} \sum_{m,n\geq 1} (-1)^{m+n} m^2 n \left(\int_0^1 x^{k-2}\sin(x\pi n)dx\right)$$
$$\times \frac{1}{[n^2u + m^2(1-u)]^{3/2-\mu}} \ (\mathcal{A}). \tag{39}$$

Theorem 12. *For the classical Brownian meander, i.e. $k = 2$ and $\delta = 3$, we have*

$$\frac{\mathbb{P}(M \in dz)}{dz} = \frac{\pi^2\sqrt{2\pi}}{z^4} \sum_{n\geq 1} (1-(-1)^n) m^2 e^{-\frac{n^2\pi^2}{2z^2}}$$
$$+ \frac{2\sqrt{2\pi}}{z^2} \sum_{\substack{m,n\geq 1\\ m\neq n}} ((-1)^{m+n} - (-1)^m) \times m^2 \frac{e^{-\frac{n^2\pi^2}{2z^2}} - e^{-\frac{m^2\pi^2}{2z^2}}}{m^2-n^2} \ (\mathcal{A}), \tag{40}$$

and

$$\frac{\mathbb{P}(\rho \in du)}{du} = 2\sum_{m,n\geq 1} (-1)^{m-1} \frac{m^2}{[(2n-1)^2u + m^2(1-u)]^{3/2}} \ (\mathcal{A}). \tag{41}$$

Theorem 13. *For the Bessel process on $[0,1]$, i.e. $0 < k = \delta$, we have*

$$\frac{\mathbb{P}(M \in dz)}{dz} = \frac{2}{C_\nu z^3} \sum_{n\geq 1} \left(\frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})}\right) e^{-\frac{j_{\nu,n}^2}{2z^2}}$$
$$+ \frac{4}{C_\nu z} \sum_{\substack{m,n\geq 1\\ m\neq n}} \frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})} \frac{e^{-\frac{j_{\nu,n}^2}{2z^2}} - e^{-\frac{j_{\nu,m}^2}{2z^2}}}{j_{\nu,m}^2 - j_{\nu,n}^2} \ (\mathcal{A}), \tag{42}$$

and

$$\frac{\mathbb{P}(\rho \in du)}{du} = \frac{2}{C_\nu} \sum_{m,n\geq 1} \frac{j_{\nu,m}^{\nu+1}}{J_{\nu+1}(j_{\nu,m})} \frac{1}{j_{\nu,n}^2 u + j_{\nu,m}^2(1-u)} \ (\mathcal{A}). \tag{43}$$

Remark. The formulas (40) and (41) have been found in [25] after some non-rigorous manipulations. The formula (42) has been discussed in [33]. But for the case of the maximum of a Bessel process, we can compute the density much more simply. Indeed,

$$\mathbb{P}(M \le z) = \mathbb{P}(T_z \ge 1) = \mathbb{P}\left(T \ge \frac{1}{z^2}\right),$$

hence, $\mathbb{P}(M \in dz) = \dfrac{2}{z^3} f\left(\dfrac{1}{z^2}\right) dz$.

7. Conclusion and perspectives

Thanks to the decomposition at the maximum, it is possible to give the densities of M and ρ for Bessel bridges and skew Brownian bridges as double Abel summation. While the law of M has often been studied (in [33] for Bessel bridges for example), much less is known about the law of ρ. More details such as its moments or its Mellin transform could be examined thanks to the provided density. As discussed at the end of Section 4.2, our formulas seem promising to find some interesting identities or to facilitate some numerical approximations of the law of M or ρ.

For the bridges, the decomposition at the maximum is very satisfying because it makes appear two independent parts with almost the same law of the path. In particular, the lengths of the two parts are i.i.d and are almost explicit in the decomposition. Their law is just a first hitting time of 1, up to a scaling of the total length.

The case of the classical Brownian meander is quite appealing because it shows how things are worse when the end point is not fixed. The two parts of the decomposition are again independent but obviously not of the same law. More disturbing, the second part can be viewed as if its starting point is chosen before running it. So the length of this part is not explicit and its density is an integral of densities corresponding to the possible starting points.

However, in the case of the meander, the law of S is surprisingly not too exotic. In fact, $4S$ has the same law as the last zero of a Brownian motion before hitting 1 or -1. We do not know if it is only a pretty coincidence, or if it comes from something deeper. Reformulating the question, could we find another decomposition for a class of processes making the end point or the time between the end of the path and the time of the maximum explicit?

A further interest could be finding different ways to deduce the joint density of (M, ρ) from decomposition at the maximum. For example, the articles [25] and [36] present how the path-integral method and the real space renormalization group lead to that density, even if they lack some technical arguments. A stronger approach may be used to unify the methods used for Bessel bridges and for skew Brownian bridges in our work, and even to generalize those results to other processes.

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