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Numerical scheme for Dynkin games under model uncertainty

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Abstract

We introduce an efficient numerical scheme for continuous time Dynkin games under model uncertainty. We use the Skorokhod embedding in order to construct recombining tree approximations. This technique allows us to determine convergence rates and to construct numerically optimal stopping strategies. We apply our method to several examples of game options.

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1 Introduction

In this paper, we propose an efficient numerical scheme for the computations of values of Dynkin games under volatility uncertainty. We consider a finite maturity, continuous–time robust Dynkin game with respect to a non dominated set of mutually singular probabilities on the canonical space of continuous paths. In this game, Player 1 who negatively/conservatively thinks that the nature is also against him, will pay the following payment to Player 2 if the two players choose stopping strategies γ and τ respectively,

$$H(\gamma,\tau) := \mathbb{I}_{\gamma<\tau} X_\gamma + \mathbb{I}_{\tau\leq\gamma} Y_\tau + \int_0^{\gamma\wedge\tau} Z_u du. \tag{1.1}$$

We model uncertainty by assuming that the stochastic processes X, Y, Z are path–independent functions of an underlying asset S which is an exponential martingale with volatility in a given interval. Thus, our setup can be viewed as a Dynkin game variant of Peng's G–expectation (see [26]).

For finite maturity optimal stopping problems/games, there are no explicit solutions even in the relatively simple framework where the probabilistic setup is given and

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the payoffs are path–independent functions of the standard Brownian motion. Hence, numerical schemes come naturally into the picture.

In [1], the authors presented a recombining trinomial tree based approximations for what is now known as a G–expectation in the sense of Peng ([26]). However, they did not provide a rigorous proof for the convergence of their scheme and did not obtain error estimates. Moreover, a priori, it is not clear whether the tree approximations from [1] can be applied for optimal stopping problems/games.

In this paper, we modify slightly the trinomial trees from [1]. For the modified (recombining) trees we construct a discrete time version of the Dynkin game given by (1.1). The main idea is to apply the Skorokhod embedding technique in order to prove the existence of an exact scheme along stopping times with the required properties. More precisely, for any exponential martingale with volatility in a given interval we prove that there exists a sequence of stopping times such that the ratio of the martingale between two sequel times belongs to some fixed set of the form $\left\{\exp\left(-\bar{\sigma}\sqrt{\frac{T}{n}}\right), 1, \exp\left(\bar{\sigma}\sqrt{\frac{T}{n}}\right)\right\}$ and the expectation of the difference between two sequel times is approximately equal to $\frac{T}{n}$. Here $\bar{\sigma} > 0$ is the right endpoint of the volatility uncertainty interval, n is the number of time steps and T is the maturity date. This machinery also allows to go in the reverse direction, namely for a given distribution on the trinomial tree we can find a "close" distribution on the canonical space which lies in our set of model uncertainty.

We prove the convergence of the discrete time approximations to the original control problem. Moreover, we provide error estimates of order $O(n^{-1/4})$. The recombining structure of the trinomial trees allows to compute the corresponding value with complexity $O(n^2)$ where n is the number of time steps.

The idea of using the Skorokhod embedding technique in order to obtain an exact sequence along stopping times was also employed in a recent work [5] where the authors approximated a one dimensional time–homogeneous diffusion by recombining trinomial trees (and obtained the same error of order $O(n^{-1/4})$). In [5], the authors were able to construct explicitly the stopping times. The construction relies heavily on the well established theory for exit times of one dimensional time–homogeneous diffusion processes. This theory cannot be applied in the present work, since the martingales in the volatility uncertainty setup are not necessarily diffusions, or even Markov processes. Thus, the case of model uncertainty requires additional machinery which we develop in Section 3. Moreover, since the martingales may not be Markovian we cannot provide an explicit construction of the stopping times (as done in [5]), but only prove their existence.

Let us remark that the multidimensional version of the above described result is an open question which requires a completely different approach. In particular, it is not clear how to derive recombining tree models which will approximate volatility uncertainty in the multidimensional setup. We leave this challenging question for future research.

Since its introduction in [10], Dynkin games have been analyzed in discrete and continuous time models for decades (see, for instance, [6, 2, 21, 23, 25]). In Mathematical Finance, the theory of Dynkin games can be applied to pricing and hedging game options and their derivatives, see [9, 15, 17, 16, 22] and the references in the survey paper [19]. In particular, the nondominated version of the optional decomposition theorem developed in [24] provides a direct link (as we will see rigorously) between Dynkin games and pricing game options in the model uncertainty framework. In general, the theme of Dynkin games is a central topic in stochastic control.

In [8], the authors connected Dynkin games to backward stochastic differential equations (BSDEs) with two reflecting barriers. This link inspired a very active research in the field of Dynkin games in a Brownian framework, see e.g. [7, 3, 12, 13, 14, 27]. Motivated by Knightian uncertainty, recently there is also a growing interest in Dynkin games under model uncertainty, see [4, 9, 14, 29]). In [4] the authors analyzed a robust version of the Dynkin game over a set of mutually singular probabilities. They proved that the game admits a value. Moreover, they established submartingale properties of the value process. These results will be essential in the present work.

The rest of the paper is organized as follows. In the next section we formulate our main result (Theorem 2.2). In Section 3, we introduce our main tool which is Skorokhod embedding under model uncertainty. In Section 4, we complete the proof of the main result. Section 5 is devoted to some auxiliary estimates which are used in the proof of Theorem 2.2. In Section 6, we provide numerical analysis for several examples of game options. Moreover, we argue rigorously the link between Dynkin games and pricing of game options in the current setup of model uncertainty.

2 Preliminaries and main result

Let $\Omega := C(\mathbb{R}_+, \mathbb{R})$ be the space of continuous paths equipped with the topology of locally uniform convergence and the Borel σ–field $\mathcal{F} := \mathcal{B}(\Omega)$. We denote by $B = B_t$, $t \geq 0$ the canonical process $B_t(\omega) := \omega_t$ and by $\mathcal{F} = \mathcal{F}_t$, $t \geq 0$ the natural filtration generated by B. For any t, $\mathcal{T}_t$ denotes the set of all stopping times with values in $[0, t]$. We denote by $\mathcal{T}$ the set of all stopping times (we allow the stopping times to take the value ∞).

For a closed interval $I = [\underline{\sigma}, \overline{\sigma}] \subset \mathbb{R}_+$ and $s > 0$ let $\mathcal{P}_s^{(I)}$ be the set of all probability measures P on Ω under which the canonical process B is a strictly positive martingale such that $B_0 = s$ P–a.s., the quadratic variation $\langle B \rangle$ is absolutely continuous $dt \otimes P$ a.s. and $B_t^{-1}\sqrt{\frac{d\langle B\rangle_t}{dt}} \in I$ $dt \otimes P$ a.s. Observe that if we define the local martingale $M_t := \int_0^t \frac{dB_u}{B_u}$, then from Itô Isometry we get $\sqrt{\frac{d\langle M\rangle_t}{dt}} = B_t^{-1}\sqrt{\frac{d\langle B\rangle_t}{dt}} \in I$. Thus M is a true martingale and $B_t = \exp(M_t - \langle M \rangle_t/2)$, $t \geq 0$ is the Doléans–Dade exponential of M. In other words, the set $\mathcal{P}_s^{(I)}$ is the set of all probability measures (on the canonical space) such that the canonical process (which starts in s) is a Doléans–Dade exponential of a true martingale with volatility in the interval I.

From mathematical finance point of view, the set $\mathcal{P}_s^{(I)}$ describes the set of all possible distributions of the (discounted) stock price process. We assume that I is a finite interval, i.e. $\overline{\sigma} < \infty$. This implies that the set $\mathcal{P}_s^{(I)}$ is weakly compact and so we can apply the results form [4] related to the existence of the optimal strategy of the Dynkin game. Moreover, the assumption $\overline{\sigma} < \infty$ is essential for constructing an appropriate sequence of trinomial models . In addition, we assume that $\underline{\sigma} > 0$, in other words the model uncertainty setup is “noisy enough”. This assumption is technical and will be needed for obtaining uniform bounds on the expectation of the hitting times related to the canonical process.

We consider a Dynkin game with maturity date $T < \infty$ and a payoff given by (1.1) with $X_t = g(t, B_t)$, $Y_t = f(t, B_t)$, $Z_t = h(t, B_t)$ where $g, f, h : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ satisfy $g \geq f$ and the following Lipschitz condition

$$|f(t_1, x_1) - f(t_2, x_2)| + |g(t_1, x_1) - g(t_2, x_2)| + |h(t_1, x_1) - h(t_2, x_2)| \leq \\ L\left((1 + |x_1|)|t_2 - t_1| + |x_2 - x_1|\right), \ t_1, t_2 \in [0, T], \ x_1, x_2 \in \mathbb{R}_+ \tag{2.1}$$

for some constant L.

For any $(t, x) \in [0, T] \times \mathbb{R}_+$ define the lower value and the upper value of the game at time t given that the canonical process satisfies $B_t = x$

$$\underline{V}^{(I)}(t, x) := \sup_{P \in \mathcal{P}_x^{(I)}} \sup_{\tau \in \mathcal{T}_{T-t}} \inf_{\gamma \in \mathcal{T}_{T-t}} E_P[g(\gamma + t, B_\gamma)\mathbb{I}_{\gamma < \tau} + \\ f(\tau + t, B_\tau)\mathbb{I}_{\tau \leq \gamma} + \int_0^{\gamma \wedge \tau} h(u + t, B_u)du]$$

and

$$\overline{V}^{(I)}(t,x) := \inf_{\gamma\in\mathcal{T}_{T-t}} \sup_{P\in\mathcal{P}_x^{(I)}} \sup_{\tau\in\mathcal{T}_{T-t}} E_P[g(\gamma+t,B_\gamma)\mathbb{I}_{\gamma<\tau} + \\ +f(\tau+t,B_\tau)\mathbb{I}_{\tau\le\gamma} + \int_0^{\gamma\wedge\tau} h(u+t,B_u)du].$$

From Theorem 4.1 in [4] it follows that the lower value and the upper value coincide and thus the game has a value

$$V^{(I)}(t,x) := \overline{V}^{(I)}(t,x) = \underline{V}^{(I)}(t,x), \quad \forall (t,x)\in[0,T]\times\mathbb{R}_+. \tag{2.2}$$

Our goal is to calculate numerically the value $V^{(I)}(0,s)$. Moreover, from Theorem 4.1 in [4] it follows that the stopping time $\gamma^* := T\wedge\inf\{t : g(t,B_t)=V^{(I)}(t,B_t)\}$ is an optimal exercise time for Player 1. In Section 6, we use this formula for numerical calculations of Player 1's optimal strategy.

Remark 2.1. Our setup is slightly different from the one considered in [4]. If we use our notations, then the control problem studied in [4] is

$$\inf_{P\in\mathcal{P}_x^{(I)}} \inf_{\gamma\in\mathcal{T}_T} \sup_{\tau\in\mathcal{T}_T} E_P\left[\mathbb{I}_{\gamma<\tau}X_\gamma + \mathbb{I}_{\tau\le\gamma}Y_\tau + \int_0^{\gamma\wedge\tau} Z_u du\right]. \tag{2.3}$$

Theorem 4.1 in [4] shows that the above infimum and supremum can be exchanged. Furthermore, the authors showed that $\tau^* := T\wedge\inf\{t : Y_t = V^{(I)}(t,B_t)\}$ is an optimal stopping time for Player 2 which can be viewed as the holder of the corresponding game option. The term given in (2.3) is the lowest arbitrage free price of the corresponding game option.

Clearly, if we replace X,Y,Z by $-Y,-X,-Z$ and replace $\gamma\leftrightarrow\tau$, then the above control problem is equivalent to

$$\sup_{P\in\mathcal{P}_x^{(I)}} \sup_{\tau\in\mathcal{T}_T} \inf_{\gamma\in\mathcal{T}_T} E_P\left[\mathbb{I}_{\gamma\le\tau}X_\gamma + \mathbb{I}_{\tau<\gamma}Y_\tau + \int_0^{\gamma\wedge\tau} Z_u du\right]. \tag{2.4}$$

This is almost the same control problem as we consider, up to the following change. In our setup, on the event $\{\gamma=\tau\}$ Player 1 pays the low payoff $Y_\tau + \int_0^\tau Z_u du$ while in (2.4) Player 1 pays the high payoff $X_\gamma + \int_0^\gamma Z_u du$. Still, Theorem 4.1 in [4] can be extended to this setup as well by following the same proof. Furthermore, analogously, the optimal exercise time for Player 1 is given by $\gamma^* := T\wedge\inf\{t : X_t = V^{(I)}(t,B_t)\}$. Namely, Theorem 4.1 in [4] provides an optimal exercise time for the player which plays against nature. In our setup, this is Player 1 who can be seen as the seller of the game option. The term given by (2.2) is the highest arbitrage free price of the game option.

Next, we describe the trinomial models and the main result. Fix $n\in\mathbb{N}$. Let $\xi_1^{(n)},...,\xi_n^{(n)}$ be random variables with values in the set $\{-1,0,1\}$ and let $\mathcal{F}^{(n)} = \{\mathcal{F}_k^{(n)}\}_{k=0}^n$ be the filtration generated by $\xi_k^{(n)}$, $k=0,1,...,n$. Denote by $\mathcal{T}_n$ the set of all stopping times (with respect to the filtration $\mathcal{F}^{(n)}$) with values in the set $\{0,1,...,n\}$.

For a given $t\in[0,T]$ and $s\ge 0$ consider the geometric random walk

$$S_k^{t,s,n} := s\exp\left(\overline{\sigma}\sqrt{\frac{T-t}{n}}\sum_{i=1}^k \xi_i^{(n)}\right) \quad k=0,1,...,n.$$

Clearly, the process $\{S_k^{t,s,n}\}_{k=0}^n$ lies on the grid $s\exp\left(\overline{\sigma}\sqrt{\frac{T-t}{n}}i\right)$, $i=-n,1-n,...,0,1,...,n$.

Denote by $\mathcal{P}^{I,t,n}$ the set of all probability measures on $\mathcal{F}_n^{(n)}$ such that for any $k = 1, ..., n$

$$P(\xi_k^{(n)} = 1|\mathcal{F}_{k-1}^{(n)}) \in \frac{1}{1+\exp(\overline{\sigma}\sqrt{\frac{T-t}{n}})}\left[\exp\left(-4\overline{\sigma}\sqrt{\frac{T-t}{n}}\right)\underline{\sigma}^2/\overline{\sigma}^2, 1\right] \tag{2.5}$$

$$P(\xi_k^{(n)} = -1|\mathcal{F}_{k-1}^{(n)}) = \exp(\overline{\sigma}\sqrt{\tfrac{T-t}{n}})P(\xi_k^{(n)} = 1|\mathcal{F}_{k-1}^{(n)}) \tag{2.6}$$

$$P(\xi_k^{(n)} = 0|\mathcal{F}_{k-1}^{(n)}) = 1 - P(\xi_k^{(n)} = 1|\mathcal{F}_{k-1}^{(n)}) - P(\xi_k^{(n)} = -1|\mathcal{F}_{k-1}^{(n)}). \tag{2.7}$$

Let us explain the intuition behind the definition of the set $\mathcal{P}^{I,t,n}$. First, we observe that for any $P \in \mathcal{P}^{I,t,n}$ and $k \geq 1$, $P(\xi_k^{(n)} = 0|\mathcal{F}_{k-1}^{(n)}) \geq 0$, i.e. P is indeed a probability measure. Moreover, from (2.6)–(2.7) it follows that for any $k \geq 1$

$$E_P\left(\frac{S_k^{t,s,n}}{S_{k-1}^{t,s,n}}|\mathcal{F}_{k-1}^{(n)}\right) = \exp\left(\overline{\sigma}\sqrt{\frac{T-t}{n}}\right)P(\xi_k^{(n)} = 1|\mathcal{F}_{k-1}^{(n)}) +$$
$$\exp\left(-\overline{\sigma}\sqrt{\frac{T-t}{n}}\right)P(\xi_k^{(n)} = -1|\mathcal{F}_{k-1}^{(n)}) + P(\xi_k^{(n)} = 0|\mathcal{F}_{k-1}^{(n)}) = 1.$$

Hence, $\{S_k^{t,s,n}\}_{k=0}^n$ is a martingale with respect to any probability measure $P \in \mathcal{P}^{I,t,n}$. Finally, from (2.5)–(2.6) we have that for any $P \in \mathcal{P}^{I,t,n}$ and $k \geq 1$ the conditional expectation of the ratio of the square of the return and the time step satisfy

$$\frac{n}{T-t}E_P\left(\left(\ln S_k^{t,s,n} - \ln S_{k-1}^{t,s,n}\right)^2|\mathcal{F}_{k-1}^{(n)}\right) =$$
$$\overline{\sigma}^2\left(P(\xi_k^{(n)} = 1|\mathcal{F}_{k-1}^{(n)}) + P(\xi_k^{(n)} = -1|\mathcal{F}_{k-1}^{(n)})\right) =$$
$$\overline{\sigma}^2\left(1 + \exp\left(\overline{\sigma}\sqrt{\frac{T-t}{n}}\right)\right)P(\xi_k^{(n)} = 1|\mathcal{F}_{k-1}^{(n)}) \in \overline{\sigma}^2\left[\exp\left(-4\overline{\sigma}\sqrt{\frac{T-t}{n}}\right)\underline{\sigma}^2/\overline{\sigma}^2, 1\right]$$
$$= \left[\underline{\sigma}^2, \overline{\sigma}^2\right]\bigcup\underline{\sigma}^2\left[\exp\left(-4\overline{\sigma}\sqrt{\frac{T-t}{n}}\right), 1\right].$$

In the above union of intervals, the first interval is exactly the square of the model uncertainty interval I, and the second interval vanishing as $n \to \infty$. This is the reason that we expect that the set $\mathcal{P}^{I,t,n}$ will be a good approximation of the set $\mathcal{P}_s^{(I)}$ restricted to the interval $[0, T-t]$. We emphasis that although the interval $\left[\exp\left(-4\overline{\sigma}\sqrt{\frac{T-t}{n}}\right), 1\right]$ is vanishing, it will be essential for the Skorokhod embedding procedure.

Next, we define the corresponding Dynkin game under model uncertainty. Introduce the lower value and the upper value of the game

$$\underline{V}^{I,n}(t,s) :=$$
$$\sup_{P\in\mathcal{P}^{I,t,n}}\max_{\eta\in\mathcal{T}_n}\min_{\zeta\in\mathcal{T}_n}E_P[g(t+\zeta(T-t)/n, S_\zeta^{t,s,n})\mathbb{I}_{\zeta<\eta}$$
$$+f(t+\eta(T-t)/n, S_\eta^{t,s,n})\mathbb{I}_{\eta\leq\zeta} + \frac{T-t}{n}\sum_{k=0}^{\zeta\wedge\eta-1}h(t+k(T-t)/n, S_k^{t,s,n})]$$

and

$$\overline{V}^{I,n}(t,s) := \min_{\zeta\in\mathcal{T}_n}\sup_{P\in\mathcal{P}^{I,t,n}}\max_{\eta\in\mathcal{T}_n}E_P[g(t+\zeta(T-t)/n, S_\zeta^{t,s,n})\mathbb{I}_{\zeta<\eta}$$
$$+f(t+\eta(T-t)/n, S_\eta^{t,s,n})\mathbb{I}_{\eta\leq\zeta} + \frac{T-t}{n}\sum_{k=0}^{\zeta\wedge\eta-1}h(t+k(T-t)/n, S_k^{t,s,n})].$$

We argue that the above two values coincide. In [16], the authors proved a similar statement for the setup where the set of probability measures is the set of equivalent martingale measures. However, the only property that was used in their proof is that there exists a a reference measure. Namely, that there exists a measure Q such that all the probability measures in the model uncertainty set are absolutely continuous with respect to Q. In our case the probability measures in $\mathcal{P}^{I,t,n}$ are defined on a finite sample space which supports the random variables $\xi_1^{(n)},...,\xi_n^{(n)}$. Thus, there exists a reference measure Q for the set $\mathcal{P}^{I,t,n}$. For instance, take Q to be the probability measure for which $\xi_1^{(n)},...,\xi_n^{(n)}$ are i.i.d. and taking the values $-1,0,1$ with the same probability $1/3$. Following the proof of Theorem 2.2 in [16] we conclude that the lower value and the upper value coincide and so the game has a value

$$V^{I,n}(t,s) := \overline{V}^{I,n}(t,s) = \underline{V}^{I,n}(t,s) \ \ \forall t,s.$$

Moreover, by using standard dynamical programming for Dynkin games (see [25]) we can calculate $V^{I,n}(t,s)$ by the following backward recursion. Define the functions $J_k^{I,t,s,n} : \{-k,1-k,...,0,1,...,k\} \to \mathbb{R}, \ \ k=0,1,...,n.$

$$J_n^{I,t,s,n}(z) := f\left(T, s\exp\left(\overline{\sigma}\sqrt{\frac{T-t}{n}}z\right)\right). \tag{2.8}$$

For $k=0,1,...,n-1$

$$\begin{aligned} J_k^{I,t,s,n}(z) &:= \max\left(f\left(t+k(T-t)/n, s\exp\left(\overline{\sigma}\sqrt{\tfrac{T-t}{n}}z\right)\right),\right. \qquad (2.9)\\ &\min\left(g\left(t+k(T-t)/n, s\exp\left(\overline{\sigma}\sqrt{\tfrac{T-t}{n}}z\right)\right), \tfrac{T-t}{n}h\left(t+k(T-t)/n, S_k^{t,s,n}\right) + \right.\\ &\sup_{p\in\left[\exp\left(-4\overline{\sigma}\sqrt{\frac{T-t}{n}}\right)\underline{\sigma}^2/\overline{\sigma}^2,1\right]}\left((1-p)J_{k+1}^{I,t,s,n}(z) + \frac{p}{1+\exp\left(\overline{\sigma}\sqrt{\frac{T-t}{n}}\right)}J_{k+1}^{I,t,s,n}(z+1)\right.\\ &\left.\left.\left.+\frac{p\exp\left(\overline{\sigma}\sqrt{\frac{T-t}{n}}\right)}{1+\exp\left(\overline{\sigma}\sqrt{\frac{T-t}{n}}\right)}J_{k+1}^{I,t,s,n}(z-1)\right)\right)\right)\\ &= \max\left(f\left(t+k(T-t)/n, s\exp\left(\overline{\sigma}\sqrt{\tfrac{T-t}{n}}z\right)\right),\right.\\ &\min\left(g\left(t+k(T-t)/n, s\exp\left(\overline{\sigma}\sqrt{\tfrac{T-t}{n}}z\right)\right), \tfrac{T-t}{n}h\left(t+k(T-t)/n, S_k^{t,s,n}\right) + \right.\\ &\max_{p\in\left\{\exp\left(-4\overline{\sigma}\sqrt{\frac{T-t}{n}}\right)\underline{\sigma}^2/\overline{\sigma}^2,1\right\}}\left((1-p)J_{k+1}^{I,t,s,n}(z) + \frac{p}{1+\exp\left(\overline{\sigma}\sqrt{\frac{T-t}{n}}\right)}J_{k+1}^{I,t,s,n}(z+1)\right.\\ &\left.\left.\left.+\frac{p\exp\left(\overline{\sigma}\sqrt{\frac{T-t}{n}}\right)}{1+\exp\left(\overline{\sigma}\sqrt{\frac{T-t}{n}}\right)}J_{k+1}^{I,t,s,n}(z-1)\right)\right)\right), \end{aligned}$$

where the last equality follows from the fact that the supremum (maximum) on an interval of a linear function (with respect to p) is achieved at the end points. We get that

$$V^{I,n}(t,s) = J_0^{I,t,s,n}(0). \tag{2.10}$$

Hence, we see that the computation of $V^{I,n}$ is very simple and its complexity is $O(n^2)$. Next, we formulate our main result.

Theorem 2.2. *There exists a constant $C>0$ such that for all $(t,s)\in[0,T]\times\mathbb{R}_+$,*

$$|V^{I,n}(t,s) - V^{(I)}(t,s)| \le C(1+s)n^{-1/4}.$$

From (2.8)–(2.9) and the backward induction it follows that for a fixed n the function $J_0^{I,\cdot,\cdot,n} : [0,T]\times\mathbb{R}_+ \to \mathbb{R}$ is continuous. This together with (2.10) and Theorem 2.2 gives immediately the following Corollary.

Corollary 2.3. *The function $V^{(I)}(t,s) : [0,T]\times\mathbb{R}_+ \to \mathbb{R}$ is continuous.*

3 Skorokhod embedding under model uncertainty

In this section we fix an arbitrary $n \in \mathbb{N}$. For any $A \in (0, \overline{\sigma}\sqrt{T/n}]$ and stopping time $\theta \in \mathcal{T}$ (recall that $\mathcal{T}$ is the set of all stopping times with respect to the canonical filtration) consider the stopping times

$$\rho_A^{(\theta)} := \inf\{t \geq \theta : |\ln B_t - \ln B_\theta| = A\} \text{ and} \tag{3.1}$$
$$\kappa_A^{(\theta)} := \infty \mathbb{I}_{\rho_A^{(\theta)}=\infty} + \textstyle\sum_{i=1}^2 (-1)^i \mathbb{I}_{\ln B_{\rho_A^{(\theta)}} = \ln B_\theta + (-1)^i A} \times$$
$$\inf\left\{t \geq \rho_A^{(\theta)} : \ln B_t = \ln B_\theta \text{ or } \ln B_t = \ln B_\theta + (-1)^i \overline{\sigma}\sqrt{T/n}\right\},$$

where the infimum over an empty set is equal to ∞. Set

$$z := z(n) = \exp(-2\overline{\sigma}\sqrt{T/n})\overline{\sigma}^{-2} \frac{\exp(2\overline{\sigma}\sqrt{T/n}) + \exp(-2\overline{\sigma}\sqrt{T/n}) - 2}{2 + \exp(\overline{\sigma}\sqrt{T/n}) + \exp(-\overline{\sigma}\sqrt{T/n})}.$$

Observe that $z = T/n + O(n^{-3/2})$. As usual, we use the convention $O(x)$ to denote a random variable ($z(n)$ is deterministic) that is uniformly (in time and space) bounded after dividing by x.

We start with the following lemma.

Lemma 3.1. *Let $P \in \mathcal{P}_s^{(I)}$ and let $\theta \in \mathcal{T}$ satisfy $E_P[\theta] < \infty$. There exists a stopping time $\mathcal{T} \ni \hat{\theta} \geq \theta$ such that P a.s. we have $\hat{\theta} < \infty$ and $\frac{B_{\hat{\theta}}}{B_\theta} \in \left\{\exp(-\overline{\sigma}\sqrt{T/n}), 0, \exp(\overline{\sigma}\sqrt{T/n})\right\}$. Furthermore, $E_P(\hat{\theta} - \theta | \mathcal{F}_\theta) = z$ and*

$$P\left(\tfrac{B_{\hat{\theta}}}{B_\theta} = \exp(\overline{\sigma}\sqrt{T/n}) | \mathcal{F}_\theta\right) \in \tfrac{1}{1+\exp(\overline{\sigma}T/n)} \left[\exp\left(-4\overline{\sigma}T/n\right) \underline{\sigma}^2/\overline{\sigma}^2, 1\right], \tag{3.2}$$

$$P\left(\tfrac{B_{\hat{\theta}}}{B_\theta} = \exp(-\overline{\sigma}\sqrt{T/n}) | \mathcal{F}_\theta\right) = \exp(\overline{\sigma}\sqrt{T/n}) P\left(\tfrac{B_{\hat{\theta}}}{B_\theta} = \exp(\overline{\sigma}\sqrt{T/n}) | \mathcal{F}_\theta\right), \tag{3.3}$$

$$P\left(B_{\hat{\theta}} = B_\theta | \mathcal{F}_\theta\right) = 1 - P\left(\tfrac{B_{\hat{\theta}}}{B_\theta} = \exp(\overline{\sigma}\sqrt{T/n}) | \mathcal{F}_\theta\right) \tag{3.4}$$
$$-P\left(\tfrac{B_{\hat{\theta}}}{B_\theta} = -\exp(\overline{\sigma}\sqrt{T/n}) | \mathcal{F}_\theta\right).$$

Notice the resemblance to the formulas (2.5)–(2.7). In particular, (3.2) gives the technical reason for the definition given by (2.5).

Proof. Denote $\rho := \rho^{(\theta)}_{\overline{\sigma}\sqrt{T/n}}$. From the fact that B is a P–martingale with volatility bonded away from zero, it follows that $E_P[\rho] < \infty$. Thus, $\frac{B_\rho}{B_\theta} = \exp(\pm\overline{\sigma}\sqrt{T/n})$, P–a.s., and from the martingale property we have

$$P\left(B_\rho = B_\theta \exp(\pm\overline{\sigma}\sqrt{T/n}) | \mathcal{F}_\theta\right) = \frac{1}{1 + \exp(\pm\overline{\sigma}\sqrt{T/n})}.$$

Hence,

$$E_P\left((B_\rho - B_\theta)^2 | \mathcal{F}_\theta\right) = \frac{\exp(2\overline{\sigma}\sqrt{T/n}) + \exp(-2\overline{\sigma}\sqrt{T/n}) - 2}{2 + \exp(\overline{\sigma}\sqrt{T/n}) + \exp(-\overline{\sigma}\sqrt{T/n})} B_\theta^2. \tag{3.5}$$

From the Itô isometry and the fact that under P, the process B is an exponential martingale with volatility less or equal then $\overline{\sigma}$ we obtain

$$E_P\left((B_\rho - B_\theta)^2 | \mathcal{F}_\theta\right) \leq E_P\left[\int_\theta^\rho B_t^2 \overline{\sigma}^2 dt | \mathcal{F}_\theta\right] \leq \overline{\sigma}^2 \exp(2\overline{\sigma}\sqrt{T/n}) B_\theta^2 E_P(\rho - \theta | \mathcal{F}_\theta),$$

where the last inequality follows from the fact that $B_t \leq \exp(\overline{\sigma}\sqrt{T/n}) B_\theta$ for $t \in [\theta, \rho]$. This together with (3.5) yields

$$E_P(\kappa^{(\theta)}_{\overline{\sigma}\sqrt{T/n}} - \theta | \mathcal{F}_\theta) = E_P(\rho - \theta | \mathcal{F}_\theta) \geq z. \tag{3.6}$$

Next, we notice that for $A_2 > A_1$ we have $\kappa^{(\theta)}_{A_2} > \kappa^{(\theta)}_{A_1}$, P a.s. Moreover, if $A_n \uparrow A$ then $\kappa^{(\theta)}_{A_n} \uparrow \kappa^{(\theta)}_{A}$ P a.s. Hence, from the Monotone Convergence Theorem

$$A_n \uparrow A \ \Rightarrow \ E_P(\kappa^{(\theta)}_A|\mathcal{F}_\theta) = \lim_{n\to\infty} E_P(\kappa^{(\theta)}_{A_n}|\mathcal{F}_\theta). \tag{3.7}$$

Let $\mathbb{Q}$ be the set of rational numbers. Define the random variable

$$\mathcal{Z} := \sup\{q \in \mathbb{Q}\cap(0,\overline{\sigma}\sqrt{T/n}] : E_P(\kappa^{(\theta)}_q|\mathcal{F}_\theta) \le z\}.$$

Clearly, $\mathcal{Z}$ is $\mathcal{F}_\theta$–measurable. Moreover, from the monotonicity property of $\kappa^{(\theta)}_A$ and (3.6)–(3.7), we obtain for the stopping time $\hat{\theta} := \kappa^{(\theta)}_{\mathcal{Z}}$ that $E_P(\hat{\theta}-\theta|\mathcal{F}_\theta) = z$.

Finally, from the fact that $\frac{B_{\hat{\theta}}}{B_\theta} \in \left\{\exp(-\overline{\sigma}\sqrt{T/n}), 0, \exp(\overline{\sigma}\sqrt{T/n})\right\}$ and $E_P\left(\frac{B_{\hat{\theta}}}{B_\theta}|\mathcal{F}_\theta\right) = 1$ we conclude that (3.3)–(3.4) hold true. Thus,

$$\begin{aligned} E_P\left(B^2_{\hat{\theta}}/B^2_\theta - 1|\mathcal{F}_\theta\right) = \left(\exp(2\overline{\sigma}\sqrt{T/n}) + \exp(-\overline{\sigma}\sqrt{T/n})\right) \times \\ P\left(\tfrac{B_{\hat{\theta}}}{B_\theta} = \exp(\overline{\sigma}\sqrt{T/n})|\mathcal{F}_\theta\right) - \left(1+\exp(\overline{\sigma}\sqrt{T/n})\right) P\left(\tfrac{B_{\hat{\theta}}}{B_\theta} = \exp(\overline{\sigma}\sqrt{T/n})|\mathcal{F}_\theta\right). \end{aligned} \tag{3.8}$$

By applying the Itô isometry, we obtain

$$E_P\left[\int_\theta^{\hat{\theta}} B_t^2\underline{\sigma}^2 dt|\mathcal{F}_\theta\right] \le E_P\left(B^2_{\hat{\theta}} - B^2_\theta|\mathcal{F}_\theta\right) \le E_P\left[\int_\theta^{\hat{\theta}} B_t^2\overline{\sigma}^2 dt|\mathcal{F}_\theta\right].$$

This together with the equality $E_P(\hat{\theta}-\theta|\mathcal{F}_\theta) = z$ and the inequality $\exp(-\overline{\sigma}\sqrt{T/n})B_\theta \le B_t \le \exp(\overline{\sigma}\sqrt{T/n})B_\theta$ gives

$$E_P\left(B^2_{\hat{\theta}}/B^2_\theta - 1|\mathcal{F}_\theta\right) \in z[\underline{\sigma}^2\exp(-2\overline{\sigma}\sqrt{T/n}), \overline{\sigma}^2\exp(2\overline{\sigma}\sqrt{T/n})].$$

Hence, from (3.8) and the definition of z we conclude (3.2) and completes the proof. $\square$

Next, for a given initial stock price $s > 0$, we construct an embedding of probability measures $\Psi_n : \mathcal{P}^{I,0,n} \to \mathcal{P}^{(I)}_s$. Choose $P \in \mathcal{P}^{I,0,n}$. There exists functions

$$\phi_i : \{-1,0,1\}^i \to \frac{1}{1+\exp(\overline{\sigma}\sqrt{T/n})}\left[\exp\left(-4\overline{\sigma}\sqrt{T/n}\right)\underline{\sigma}^2/\overline{\sigma}^2, 1\right], \ \ i = 0,1,...,n-1$$

such that (2.5) holds true with

$$P(\xi^{(n)}_k = 1|\mathcal{F}^{(n)}_{k-1}) = \phi_{k-1}(\xi^{(n)}_1,...,\xi^{(n)}_{k-1}), \ \ k = 1,...,n.$$

Recall the canonical space $\Omega = C(\mathbb{R}_+,\mathbb{R})$. On this sample space we define a sequence of random variables $A_0,...,A_n,\theta_0,...,\theta_n$ by the following recursion. Let $\theta_0 := 0$ and $A_0 \in (0,\overline{\sigma}\sqrt{T/n}]$ be the unique solution of the equation

$$\frac{\exp(x)-1}{(1+\exp(x))(\exp(\overline{\sigma}\sqrt{T/n})-1)} = \phi_0.$$

Recall the definition given by (3.1). For $k = 1,...,n$ set $\theta_k := \kappa^{(\theta_{k-1})}_{A_{k-1}}$, and on the event $\{\theta_k < \infty\}$ define $A_k \in (0,\overline{\sigma}\sqrt{T/n}]$ to be the unique solution of the equation

$$\begin{aligned} \tfrac{\exp(x)-1}{(1+\exp(x))(\exp(\overline{\sigma}\sqrt{T/n})-1)} = \\ \phi_k\left(\overline{\sigma}^{-1}(T/n)^{-1/2}(\ln B_{\theta_1} - \ln B_{\theta_0}),...,\overline{\sigma}^{-1}(T/n)^{-1/2}(\ln B_{\theta_k} - \ln B_{\theta_{k-1}})\right). \end{aligned}$$

On the event $\{\theta_k = \infty\}$ we set $A_k = 0$. Define the random variables $\sigma_0, ..., \sigma_{n-1}$ by

$$\sigma_k := \mathbb{I}_{\theta_k < \infty} \max \left(\underline{\sigma}, \overline{\sigma} \sqrt{1 + \exp(\overline{\sigma}\sqrt{T/n})} \times \right. \tag{3.9}$$
$$\left. \left(\phi_k \left(\overline{\sigma}^{-1}(T/n)^{-1/2}(\ln B_{\theta_1} - \ln B_{\theta_0}), ..., \overline{\sigma}^{-1}(T/n)^{-1/2}(\ln B_{\theta_k} - \ln B_{\theta_{k-1}})\right)\right)^{1/2} \right).$$

Observe that on the event $\{\theta_k < \infty\}$ we have $\sigma_k \in I$. Thus, the fact that the volatility interval I is bounded away from zero implies that there exists a unique probability measure $\hat{P} := \Psi_n(\Pi) \in \mathcal{P}_s^{(I)}$ such that $E_{\hat{P}}[\theta_n] < \infty$, and that for any $k < n$, $B_t^{-1}\sqrt{\frac{d\langle B \rangle_t}{dt}} \equiv \sigma_k$ on the random interval $[\theta_k, \theta_{k+1})$ $\hat{P}$ a.s.

Lemma 3.2. *The joint distribution of* $\ln B_{\theta_1} - \ln B_{\theta_0}, ..., \ln B_{\theta_n} - \ln B_{\theta_{n-1}}$ *under* $\hat{P}$ *is equal to the joint distribution of* $\overline{\sigma}\sqrt{T/n}\xi_1^{(n)}, ..., \overline{\sigma}\sqrt{T/n}\xi_n^{(n)}$ *under* P. *Moreover, for any* $k < n$, $\hat{P}(B_{\theta_{k+1}} | \mathcal{F}_{\theta_k}) = \hat{P}(B_{\theta_{k+1}} | B_{\theta_1}, ..., B_{\theta_k})$ *and* $E_{\hat{P}}(\theta_{k+1} - \theta_k | \mathcal{F}_{\theta_k}) = T/n + O(n^{-3/2})$.

Proof. For any k we have $\frac{B_{\theta_{k+1}}}{B_{\theta_k}} \in \left\{\exp(-\overline{\sigma}\sqrt{T/n}), 0, \exp(\overline{\sigma}\sqrt{T/n})\right\}$ and $E_{\hat{P}}\left(\frac{B_{\theta_{k+1}}}{B_{\theta_k}} | \mathcal{F}_{\theta_k}\right) = 1$. Fix $k < n$. We argue that

$$\hat{P}\left(\frac{B_{\theta_{k+1}}}{B_{\theta_k}} = \exp(\overline{\sigma}\sqrt{T/n}) | \mathcal{F}_{\theta_k}\right) = \tag{3.10}$$
$$\phi_k \left(\overline{\sigma}^{-1}(T/n)^{-1/2}(\ln B_{\theta_1} - \ln B_{\theta_0}), ..., \overline{\sigma}^{-1}(T/n)^{-1/2}(\ln B_{\theta_k} - \ln B_{\theta_{k-1}})\right).$$

Indeed, from (3.1), the definition of A_k and the martingale property of B we get

$$\hat{P}\left(\frac{B_{\theta_{k+1}}}{B_{\theta_k}} = \exp(\overline{\sigma}\sqrt{T/n}) | \mathcal{F}_{\theta_k}\right) =$$
$$\hat{P}\left(B_{\rho^{(A_k)}_{\theta_k}} = \exp(A_k) B_{\theta_k} | \mathcal{F}_{\theta_k}\right) \times$$
$$\hat{P}\left(B_{\theta_{k+1}} = \exp(\overline{\sigma}\sqrt{T/n}) B_{\theta_k} | B_{\rho^{(A_k)}_{\theta_k}} = \exp(A_k) B_{\theta_k}, \mathcal{F}_{\theta_k}\right) =$$
$$\frac{1}{1+\exp(A_k)} \frac{\exp(A_k) - 1}{\exp(\overline{\sigma}\sqrt{T/n}) - 1} =$$
$$\phi_k \left(\overline{\sigma}^{-1}(T/n)^{-1/2}(\ln B_{\theta_1} - \ln B_{\theta_0}), ..., \overline{\sigma}^{-1}(T/n)^{-1/2}(\ln B_{\theta_k} - \ln B_{\theta_{k-1}})\right)$$

as required. In particular $\hat{P}(B_{\theta_{k+1}} | \mathcal{F}_{\theta_k}) = \hat{P}(B_{\theta_{k+1}} | B_{\theta_1}, ..., B_{\theta_k})$. Furthermore, from the definition of ϕ_k, $k = 0, 1, ..., n-1$ we conclude that the joint distribution of $\ln B_{\theta_1} - \ln B_{\theta_0}, ..., \ln B_{\theta_n} - \ln B_{\theta_{n-1}}$ is equal to the joint distribution of $\overline{\sigma}\sqrt{T/n}\xi_1^{(n)}, ..., \overline{\sigma}\sqrt{T/n}\xi_n^{(n)}$.

Finally, we estimate $E_{\hat{P}}(\theta_{k+1} - \theta_k | \mathcal{F}_{\theta_k})$. From (3.9) and the inequality

$$\phi_k \geq \frac{1}{1 + \exp(\overline{\sigma}\sqrt{T/n})} \exp\left(-4\overline{\sigma}\sqrt{T/n}\right) \underline{\sigma}^2 / \overline{\sigma}^2$$

we get

$$\phi_k \left(\overline{\sigma}^{-1}(T/n)^{-1/2}(\ln B_{\theta_1} - \ln B_{\theta_0}), ..., \overline{\sigma}^{-1}(T/n)^{-1/2}(\ln B_{\theta_k} - \ln B_{\theta_{k-1}})\right) =$$
$$\sigma_k^2 \left(\frac{1}{\overline{\sigma}^2(1 + \exp(\overline{\sigma}\sqrt{T/n}))} + O(\sqrt{T/n})\right).$$

This together with (3.3)–(3.4) and (3.10) yields

$$E_{\hat{P}}\left(B^2_{\theta_{k+1}} / B^2_{\theta_k} - 1 | \mathcal{F}_{\theta_k}\right) \tag{3.11}$$
$$= \left(\exp(2\overline{\sigma}\sqrt{T/n}) + \exp(-\overline{\sigma}\sqrt{T/n}) - 1 - \exp(\overline{\sigma}\sqrt{T/n})\right) \times$$
$$\sigma_k^2 \left(\frac{1}{\overline{\sigma}^2(1 + \exp(\overline{\sigma}\sqrt{T/n}))} + O(\sqrt{T/n})\right) = \sigma_k^2 \left(\frac{T}{n} + O(n^{-3/2})\right).$$

From the Itô isometry and the fact that (under the probability measure $\hat{P}$) the volatility of the canonical process B is constant (equal to σ_k) on the interval $[\theta_k, \theta_{k+1})$ we obtain

$$E_{\hat{P}}\left(B^2_{\theta_{k+1}}/B^2_{\theta_k} - 1|\mathcal{F}_{\theta_k}\right) \in \sigma_k^2 E_{\hat{P}}(\theta_{k+1}-\theta_k|\mathcal{F}_{\theta_k})[\exp(-2\overline{\sigma}\sqrt{T/n}), \exp(2\overline{\sigma}\sqrt{T/n})].$$

Thus, from (3.11) it follows that $E_{\hat{P}}(\theta_{k+1}-\theta_k|\mathcal{F}_{\theta_k}) = (1+O(1/\sqrt{n}))\frac{T}{n}$, and the proof is completed. □

4 Proof Theorem 2.2

For simplicity, we assume that the starting time is $t=0$. For a general $t \in [0,T]$ the proof is done in the same way. Denote by $s>0$ the initial stock price.

4.1 Proof of the inequality $V^{(I)}(0,s) \le V^{I,n}(0,s) + C(1+s)n^{-1/4}$

Proof. Fix $n \in \mathbb{N}$ and choose $\epsilon > 0$. There exists a probability measure $P^* \in \mathcal{P}_s^{(I)}$ and a stopping time $\tau^* \in \mathcal{T}_T$ such that

$$V^{(I)}(0,s) \le \epsilon + \inf_{\gamma \in \mathcal{T}_T} E_{P^*}\left[g(\gamma, B_\gamma)\mathbb{I}_{\gamma<\tau^*} + f(\tau^*, B_{\tau^*})\mathbb{I}_{\tau^*\le\gamma} + \int_0^{\gamma\wedge\tau^*} h(u,B_u)du\right]. \quad (4.1)$$

From Lemma 3.1 it follows that we can choose a sequence of stopping times $0=\theta_0 < \theta_1 < \theta_2 < ... < \theta_n$ such that P^* a.s., for any $i=1,...,n$

$$\frac{B_{\theta_i}}{B_{\theta_{i-1}}} \in \left\{\exp(-\overline{\sigma}\sqrt{T/n}), 0, \exp(\overline{\sigma}\sqrt{T/n})\right\},$$

$$\begin{aligned}
P^*\left(\tfrac{B_{\theta_i}}{B_{\theta_{i-1}}} = \exp(\overline{\sigma}\sqrt{T/n})|\mathcal{F}_{\theta_{i-1}}\right) &\in \tfrac{1}{1+\exp(\overline{\sigma}\sqrt{T/n})}\left[\exp\left(-4\overline{\sigma}\sqrt{T/n}\right)\underline{\sigma}^2/\overline{\sigma}^2, 1\right],\\
P^*\left(\tfrac{B_{\theta_i}}{B_{\theta_{i-1}}} = \exp(-\overline{\sigma}\sqrt{T/n})|\mathcal{F}_{\theta_{i-1}}\right) &= \exp(\overline{\sigma}\sqrt{T/n})P^*\left(\tfrac{B_{\theta_i}}{B_{\theta_{i-1}}} = \exp(\overline{\sigma}\sqrt{T/n})|\mathcal{F}_{\theta_{i-1}}\right),\\
P^*\left(B_{\theta_i} = B_{\theta_{i-1}}|\mathcal{F}_{\theta_{i-1}}\right) &= 1 - P^*\left(\tfrac{B_{\theta_i}}{B_{\theta_{i-1}}} = \exp(\overline{\sigma}\sqrt{T/n})|\mathcal{F}_{\theta_{i-1}}\right)\\
&\quad -P^*\left(\tfrac{B_{\theta_i}}{B_{\theta_{i-1}}} = -\exp(\overline{\sigma}\sqrt{T/n})|\mathcal{F}_{\theta_{i-1}}\right),
\end{aligned}$$

and $E_{P^*}(\theta_i - \theta_{i-1}|\mathcal{F}_{\theta_{i-1}}) = z$ where $z=z(n)$ is given before Lemma 3.1. In words, we apply the Skorokhod embedding technique given by Lemma 3.1 in order to construct a sequence of stopping times such that the ratio of B between two sequel times belongs to $\left\{\exp\left(-\bar{\sigma}\sqrt{\frac{T}{n}}\right), 1, \exp\left(\bar{\sigma}\sqrt{\frac{T}{n}}\right)\right\}$. Moreover, the expectation of the difference between two sequel times is $\frac{T}{n} + O(n^{-3/2})$. The last fact will be used via the Auxiliary Lemmas 5.3–5.4.

Now, comes the main idea of the proof. Recall the geometric random walk $\{S_k^{0,s,n}\}_{k=0}^n$ and the trinomial models given by the set of probability measures $\mathcal{P}^{I,0,n}$. From (2.5)–(2.7) and the above properties of the probability measure P^* it follows that there exists a probability measure $\tilde{P} \in \mathcal{P}^{I,0,n}$ such that the distribution of $\{B_{\theta_i}\}_{i=0}^n$ under P^* equals to the distribution of $\{S_k^{0,s,n}\}_{k=0}^n$ under $\tilde{P}$. Moreover, using similar arguments as in Lemma 3.2 we obtain that for any $k<n$, $P^*(B_{\theta_{k+1}}|\mathcal{F}_{\theta_k}) = P^*(B_{\theta_{k+1}}|B_{\theta_1},...,B_{\theta_k})$. The above two properties give

$$\begin{gathered}
\max_{\eta\in\mathcal{T}_n}\min_{\zeta\in\mathcal{T}_n} E_{\tilde{P}}[g(\zeta T/n, S_\zeta^{0,s,n})\mathbb{I}_{\zeta<\eta}\\
+f(\eta T/n, S_\eta^{0,s,n})\mathbb{I}_{\eta\le\zeta} + \tfrac{T}{n}\sum_{k=0}^{\zeta\wedge\eta-1} h(kT/n, S_k^{0,s,n})] =\\
\sup_{\eta\in\mathcal{S}_n}\inf_{\zeta\in\mathcal{S}_n} E_{P^*}[g(\zeta T/n, B_{\theta_\zeta})\mathbb{I}_{\zeta<\eta}\\
+f(\eta T/n, B_{\theta_\eta})\mathbb{I}_{\eta\le\zeta} + \tfrac{T}{n}\sum_{k=0}^{\zeta\wedge\eta-1} h(kT/n, B_{\theta_k})].
\end{gathered}$$

Hence, we conclude

$$V^{I,n}(0,s) \geq \sup_{\eta\in\mathcal{S}_n} \inf_{\zeta\in\mathcal{S}_n} E_{P^*}[g(\zeta T/n, B_{\theta_\zeta})\mathbb{I}_{\zeta<\eta} + f(\eta T/n, B_{\theta_\eta})\mathbb{I}_{\eta\leq\zeta} + \frac{T}{n}\sum_{k=0}^{\zeta\wedge\eta-1} h(kT/n, B_{\theta_k})]. \tag{4.2}$$

The final step is technical. We are using (4.1)–(4.2) in order to bound from above the difference $V^{(I)}(0,s) - V^{I,n}(0,s)$.

Introduce the stopping time $\eta^* := n \wedge \min\{k : \theta_k \geq \tau^*\} \in \mathcal{S}_n$. In view of (4.2) there exists a stopping time $\zeta^* \in \mathcal{S}_n$ such that

$$V^{I,n}(0,s) \geq E_{P^*}\left[g(\zeta^* T/n, B_{\theta_{\zeta^*}})\mathbb{I}_{\zeta^*<\eta^*} + f(\eta^* T/n, B_{\theta_{\eta^*}})\mathbb{I}_{\eta^*\leq\zeta^*} + \frac{T}{n}\sum_{k=0}^{\zeta^*\wedge\eta^*-1} h(kT/n, B_{\theta_k})\right] - \epsilon. \tag{4.3}$$

Define the stopping time $\gamma^* := (T \wedge \theta^{(n)}_{\zeta^*})\mathbb{I}_{\zeta^*<n} + T\mathbb{I}_{\zeta^*=n} \in \mathcal{T}_T$. From (4.1) and (4.3) we obtain that

$$\begin{aligned} V^{(I)}(0,s) \leq{}& V^{(I,n)}(0,s) + 2\epsilon + \\ & E_{P^*}[g(\gamma^*, B_{\gamma^*})\mathbb{I}_{\gamma^*<\tau^*} - g(\zeta^* T/n, B_{\theta_{\zeta^*}})\mathbb{I}_{\zeta^*<\eta^*}] \\ & + E_{P^*}[f(\tau^*, B_{\tau^*})\mathbb{I}_{\tau^*\leq\gamma^*} - f(\eta^* T/n, B_{\theta_{\eta^*}})\mathbb{I}_{\eta^*\leq\zeta^*}] \\ & + E_{P^*}[\int_0^{\gamma^*\wedge\tau^*} h(u, B_u)du - \frac{T}{n}\sum_{k=0}^{\zeta^*\wedge\eta^*-1} h(kT/n, B_{\theta_k})]. \end{aligned} \tag{4.4}$$

From technical reasons we extend the function h to the domain $\mathbb{R}^2$ by $h(t,x) := h(t\wedge T, x)$. Clearly, the extended h is satisfying the Lipschitz condition given by (2.1) on the domain $\mathbb{R}^2$. We observe that if $\gamma^* < \tau^*$, then $\zeta^* < \eta^*$. This together with (2.1), which in particular implies that $h(t,x) = O(1)(1+|x|)(1+t)$, and (4.4) gives

$$\begin{aligned} V^{(I)}(0,s) \leq{}& V^{(I,n)}(0,s) + 2\epsilon + O(1)E_{P^*}|B_{\gamma^*\wedge\tau^*} - B_{\theta_{\zeta^*\wedge\eta^*}}| + \\ & O(1)E_{P^*}\left[(1+\sup_{0\leq t\leq\theta_n\vee T} B_t)(1+\theta_n\vee T)(|\gamma^*\wedge\tau^* - \zeta^*\wedge\eta^*\tfrac{T}{n}| + |\gamma^*\wedge\tau^* - \theta_{\zeta^*\wedge\eta^*}|)\right] \\ & + E_{P^*}\left(\max_{1\leq k\leq n}\left|\int_0^{\theta_k} h(t,B_t)dt - \sum_{i=0}^{k-1} h(iT/n, B_{\theta_i})\right|\right). \end{aligned} \tag{4.5}$$

From the definition of the stopping times η^* and γ^* it follows that $|\gamma^*\wedge\tau^* - \zeta^*\wedge\eta^*\frac{T}{n}| \leq \max_{1\leq k\leq n}|\theta_k - kT/n| + T/n$ and

$$|\gamma^*\wedge\tau^* - \theta_{\zeta^*\wedge\eta^*}| \leq |T-\theta_n| + \max_{1\leq k\leq n}\theta_k - \theta_{k-1} \leq 3\max_{1\leq k\leq n}|\theta_k - kT/n| + T/n.$$

Hence, from the Cauchy–Schwarz inequality, the Jensen inequality, Lemma 5.1 and Lemma 5.3 it follows that

$$\begin{aligned} & E_{P^*}\left[(1+\sup_{0\leq t\leq\theta_n\vee T} B_t)(1+\theta_n\vee T)(|\gamma^*\wedge\tau^* - \zeta^*\wedge\eta^*\tfrac{T}{n}| + |\gamma^*\wedge\tau^* - \theta_{\zeta^*\wedge\eta^*}|)\right] \\ & \leq \left(E_{P^*}((1+\sup_{0\leq t\leq\theta_n\vee T} B_t)^4)\right)^{1/4}\left(E_{P^*}((1+\theta_n\vee T)^4)\right)^{1/4} \times \\ & \left(E_{P^*}((4\max_{1\leq k\leq n}|\theta_k - kT/n| + 2T/n)^2)\right)^{1/2} = O((1+s)n^{-1/2}). \end{aligned} \tag{4.6}$$

Similarly, from the Itô isometry we obtain

$$E_{P^*}((B_{\gamma^*\wedge\tau^*} - B_{\theta_{\zeta^*\wedge\eta^*}})^2) \leq E_{P^*}[\overline{\sigma}^2 \max_{0\leq t\leq\theta_n\vee T} B_t^2|\gamma^*\wedge\tau^* - \theta_{\zeta^*\wedge\eta^*}|] = O(s^2 n^{-1/2}).$$

This together with the Jensen inequality, (4.5)–(4.6) and Lemma 5.4 gives that

$$V^{(I)}(0,s) \leq V^{I,n}(0,s) + 2\epsilon + O((1+s)n^{-1/4})$$

and by letting $\epsilon \downarrow 0$ we complete the proof. □

4.2 Proof of the inequality $V^{I,n}(0,s) \le V^{(I)}(0,s) + C(1+s)n^{-1/4}$

Proof. The proof is very similar to the proof of the first inequality. Fix $n \in \mathbb{N}$ and choose $\epsilon > 0$. We abuse notations and denote by P^* a probability measure in $\mathcal{P}^{I,0,n}$ which satisfy

$$V^{I,n}(0,s) \le \epsilon + \max_{\eta \in \mathcal{T}_n} \min_{\zeta \in \mathcal{T}_n} E_{P^*}[g(\zeta T/n, S^{t,s,n}_\zeta)\mathbb{I}_{\zeta<\eta} \\ + f(\eta T/n, S^{t,s,n}_\eta)\mathbb{I}_{\eta \le \zeta} + \frac{T}{n}\sum_{k=0}^{\zeta\wedge\eta-1} h(kT/n, S^{t,s,n}_k)]. \tag{4.7}$$

Recall the definition of $\hat{P}^* := \Psi_n(P^*)$ and the stopping times $0 = \theta_0 < \theta_1 < ... < \theta_n$ given before Lemma 3.2. Denote by $\mathcal{S}_n$ the set of all stopping times with respect to the filtration $\{\mathcal{F}_{\theta_i}\}_{i=0}^n$ with values in the set $\{0, 1, ..., n\}$. By applying Lemma 3.2 and the same arguments as before (4.2) it follows that

$$\max_{\eta \in \mathcal{T}_n} \min_{\zeta \in \mathcal{T}_n} E_{P^*}[g(\zeta T/n, S^{t,s,n}_\zeta)\mathbb{I}_{\zeta<\eta} \\ + f(\eta T/n, S^{t,s,n}_\eta)\mathbb{I}_{\eta \le \zeta} + \frac{T}{n}\sum_{k=0}^{\zeta\wedge\eta-1} h(kT/n, S^{t,s,n}_k)] = \\ \sup_{\eta \in \mathcal{S}_n} \inf_{\zeta \in \mathcal{S}_n} E_{\hat{P}^*}[g(\zeta T/n, B_{\theta_\zeta})\mathbb{I}_{\zeta<\eta} \\ + f(\eta T/n, B_{\theta_\eta})\mathbb{I}_{\eta \le \zeta} + \frac{T}{n}\sum_{k=0}^{\zeta\wedge\eta-1} h(kT/n, B_{\theta_k})]. \tag{4.8}$$

The equality (4.8) is the cornerstone of the proof. The remaining part is technical, and we use (4.7)–(4.8) to estimate from above the difference $V^{I,n}(0,s) - V^{(I)}(0,s)$. Indeed, from (4.7)–(4.8) it follows that there exists $\eta^* \in \mathcal{S}_n$ (again we abuse notations) such that

$$V^{I,n}(0,s) \le 2\epsilon + \inf_{\zeta \in \mathcal{S}_n} E_{\hat{P}^*}[g(\zeta T/n, B_{\theta_\zeta})\mathbb{I}_{\zeta<\eta^*} \\ + f(\eta^* T/n, B_{\theta_{\eta^*}})\mathbb{I}_{\eta^* \le \zeta} + \frac{T}{n}\sum_{k=0}^{\zeta\wedge\eta^*-1} h(kT/n, B_{\theta_k})].$$

Define the stopping time $\tau^* := \theta_{\eta^*} \wedge T \in \mathcal{T}_T$. Clearly, there exists a stopping time $\gamma^* \in \mathcal{T}_T$ such that

$$V^{(I)}(0,s) \ge E_{\hat{P}^*}\left[g(\gamma^*, B_{\gamma^*})\mathbb{I}_{\gamma^*<\tau^*} + f(\tau^*, B_{\tau^*})\mathbb{I}_{\tau^* \le \gamma^*} + \int_0^{\gamma^*\wedge\tau^*} h(u, B_u)du\right] - \epsilon.$$

Next, introduce the stopping time $\zeta^* := n \wedge \min\{k : \theta_k \ge \gamma^*\}\mathbb{I}_{\gamma^*<T} + n\mathbb{I}_{\gamma^*=T} \in \mathcal{S}_n$. We observe that if $\zeta^* < \eta^*$ then $\gamma^* < \tau^*$. Thus, similarly to (4.5) we get

$$V^{I,n}(0,s) \le V^{(I)}(0,s) + 3\epsilon + O(1)E_{\hat{P}^*}|B_{\gamma^*\wedge\tau^*} - B_{\theta_{\zeta^*\wedge\eta^*}}| + \\ O(1)E_{\hat{P}^*}\left[(1 + \sup_{0\le t\le\theta_n\vee T} B_t)(1 + \theta_n \vee T)(|\gamma^* \wedge \tau^* - \zeta^* \wedge \eta^* \tfrac{T}{n}| + |\gamma^* \wedge \tau^* - \theta_{\zeta^*\wedge\eta^*}|)\right] \\ + E_{\hat{P}^*}\left(\max_{1\le k\le n}\left|\int_0^{\theta_k} h(t, B_t)dt - \sum_{i=0}^{k-1} h(iT/n, B_{\theta_i})\right|\right).$$

Finally, by using the same estimates as in Section 4.1, we obtain that

$$V^{I,n}(0,s) \le V^{(I)}(0,s) + 3\epsilon + O((1+s)n^{-1/4})$$

and by letting $\epsilon \downarrow 0$ we complete the proof. □

Remark 4.1. Let us notice that in the present setup of model uncertainty we get the same error estimates as in the case with no uncertainty which was studied in [5]. The main reason is that Lemma 5.3 which is essential for the proof cannot be improved even for the most simple case where the canonical process is a geometric Brownian motion with constant volatility. Namely, the Skorokhod embedding technique cannot provide error estimates of order better than $O(n^{-1/4})$ even for the approximations of American or game options in the Black–Scholes model. For details, see [18]. Fortunately, same estimates can be obtained for the volatility uncertainty setup.

5 Auxiliary lemmas

In this section we derive the estimates that we used in Section 4. We fix $n \in \mathbb{N}$ and a probability measure $P \in \mathcal{P}_s^{(I)}$. Furthermore, we fix a sequence of stopping times $0 = \theta_0 < \theta_1 < ... < \theta_n$ for which we assume that for any $i < n$, $\frac{B_{\theta_i}}{B_{\theta_{i-1}}} \in \left\{\exp(-\overline{\sigma}\sqrt{T/n}), 0, \exp(\overline{\sigma}\sqrt{T/n})\right\}$ P–a.s.,

$$P\left(\frac{B_{\theta_i}}{B_{\theta_{i-1}}} = \exp(\overline{\sigma}\sqrt{T/n})|\mathcal{F}_{\theta_{i-1}}\right) \in \frac{1}{1+\exp(\overline{\sigma}\sqrt{T/n})}\left[\exp\left(-4\overline{\sigma}\sqrt{T/n}\right)\underline{\sigma}^2/\overline{\sigma}^2, 1\right],$$
$$P\left(\frac{B_{\theta_i}}{B_{\theta_{i-1}}} = \exp(-\overline{\sigma}\sqrt{T/n})|\mathcal{F}_{\theta_{i-1}}\right) = \exp(\overline{\sigma}\sqrt{T/n})P\left(\frac{B_{\theta_i}}{B_{\theta_{i-1}}} = \exp(\overline{\sigma}\sqrt{T/n})|\mathcal{F}_{\theta_{i-1}}\right),$$
$$P\left(B_{\theta_i} = B_{\theta_{i-1}}|\mathcal{F}_{\theta_{i-1}}\right) = 1 - P\left(\frac{B_{\theta_i}}{B_{\theta_{i-1}}} = \exp(\overline{\sigma}\sqrt{T/n})|\mathcal{F}_{\theta_{i-1}}\right)$$
$$-P\left(\frac{B_{\theta_i}}{B_{\theta_{i-1}}} = -\exp(\overline{\sigma}\sqrt{T/n})|\mathcal{F}_{\theta_{i-1}}\right),$$

and $E_P(\theta_{i+1} - \theta_i|\mathcal{F}_{\theta_i}) = T/n + O(n^{-3/2})$. Observe that the stopping times $0 = \theta_0 < \theta_1 < ... < \theta_n$ from both Section 4.1 and Section 4.2 satisfy the above conditions.

We start with proving the following bound.

Lemma 5.1.
$$E_P\left(\sup_{0 \le t \le T \vee \theta_n} B_t^4\right) = O(1)s^4.$$

Proof. Clearly, for any $i < n$,

$$E_P(B_{\theta_{i+1}}^4 - B_{\theta_i}^4|\mathcal{F}_{\theta_i}) = B_{\theta_i}^4 P\left(\frac{B_{\theta_{i+1}}}{B_{\theta_i}} = \exp(\overline{\sigma}\sqrt{T/n})|\mathcal{F}_{\theta_i}\right) \times$$
$$\left(\exp(4\overline{\sigma}\sqrt{T/n}) - 1 + \exp(\overline{\sigma}\sqrt{T/n})(\exp(-4\overline{\sigma}\sqrt{T/n}) - 1)\right) \le B_{\theta_i}^4 O(1/n).$$

Hence, $E_P(B_{\theta_n}^4) \le s^4(1 + O(1/n))^n = O(1)s^4$. This together with the Doob inequality gives that

$$E_P\left(\sup_{0 \le t \le \theta_n} B_t^4\right) = O(1)s^4. \tag{5.1}$$

Next, we notice that the inequality $B_t^{-1}\sqrt{\frac{d\langle B\rangle_t}{dt}} \le \overline{\sigma}$ together with the Itô formula implies that $\exp(-6\overline{\sigma}^2 t)B_t^4$, $t \ge 0$ is a super–martingale. In particular, $E_P B_T^4 \le \exp(6\overline{\sigma}^2 T)s^4$. Thus, from the Doob inequality and (5.1) we obtain

$$E_P\left(\sup_{0 \le t \le T \vee \theta_n} B_t^4\right) \le E_P\left(\sup_{0 \le t \le T} B_t^4\right) + E_P\left(\sup_{0 \le t \le \theta_n} B_t^4\right) = O(1)s^4$$

and the proof is completed. □

Next, we prove the following.

Lemma 5.2. *For any* $i = 0, 1, ..., n-1$, $E_P((\theta_{i+1} - \theta_i)^4|\mathcal{F}_{\theta_i}) = O(n^{-4})$.

Proof. Choose $i < n$. From the Burkholder–Davis–Gundy inequality, the inequality $\frac{d\langle B\rangle_t}{dt} \ge \underline{\sigma}^2 B_t^2$ and the fact that $\frac{B_t}{B_{\theta_i}} \in [\exp(-\overline{\sigma}\sqrt{T/n}), \exp(\overline{\sigma}\sqrt{T/n})]$ for $t \in [\theta_i, \theta_{i+1}]$ it follows that

$$\underline{\sigma}^8 \exp(-8\overline{\sigma}\sqrt{T/n})B_{\theta_i}^8 E_P((\theta_{i+1} - \theta_i)^4|\mathcal{F}_{\theta_i}) \le$$
$$E_P\left((\langle B\rangle_{\theta_{i+1}} - \langle B\rangle_{\theta_i})^4|\mathcal{F}_{\theta_i}\right) = O(1)E_P((B_{\theta_{i+1}} - B_{\theta_i})^8|\mathcal{F}_{\theta_i}) = O(n^{-4})B_{\theta_i}^8$$

and the result follows. □

We arrive to our next estimate.

Lemma 5.3. $E_P\left(\max_{0\le k\le n}|\theta_k-kT/n|^4\right)=O(n^{-2})$.

Proof. Set $Z_i:=\theta_i-\theta_{i-1}-E_P(\theta_i-\theta_{i-1}|\mathcal{F}_{\theta_{i-1}})$, $i=1,...,n$. We use the fact that the expectation of the difference between two sequel times equals approximately to the time step. Formally, for any i, we have $E_P(\theta_i-\theta_{i-1}-T/n|\mathcal{F}_{\theta_{i-1}})=O(n^{-3/2})$. Hence,

$$\max_{0\le k\le n}|\theta_k-kT/n|=O(n^{-1/2})+\max_{1\le k\le n}|\sum_{i=1}^k Z_i|.$$

In view of the inequality $(a+b)^4\le 8(a^4+b^4)$, $a,b\ge 0$ it remains to prove that $E_P\left(\left(\max_{1\le k\le n}|\sum_{i=1}^k Z_i|\right)^4\right)=O(n^{-2})$. From the Jensen inequality and Lemma 5.2 it follows that $E_P\left((E_P(\theta_i-\theta_{i-1}|\mathcal{F}_{\theta_{i-1}}))^4\right)=O(n^{-4})$ for all i. This together with the inequality $(a-b)^4\le a^4+b^4$, $a,b\ge 0$ implies that $E_P[Z_i^4]=O(n^{-4})$ for all i. Thus, from the Burkholder–Davis–Gundy inequality applied to the martingale $\sum_{i=1}^k Z_i$, $k=1,...,n$, and the inequality $\left(\sum_{i=1}^n a_i\right)^2\le n\left(\sum_{i=1}^n a_i^2\right)$, $a_1,...,a_n\ge 0$, we obtain

$$E_P\left(\left(\max_{1\le k\le n}|\sum_{i=1}^k Z_i|\right)^4\right)=O(1)E_P\left(\left(\sum_{i=1}^n Z_i^2\right)^2\right)=O(n)\sum_{i=1}^n E_P Z_i^4=O(n^{-2})$$

as required. □

We end this section with proving the next estimate.

Lemma 5.4.

$$E_P\left(\max_{0\le k\le n}\left|\int_0^{\theta_k}h(t,B_t)dt-\frac{T}{n}\sum_{i=0}^{k-1}h(iT/n,B_{\theta_i})\right|\right)=O((1+s)n^{-1/2}).$$

Proof. Clearly,

$$\max_{0\le k\le n}\left|\int_0^{\theta_k}h(t,B_t)dt-\frac{T}{n}\sum_{i=0}^{k-1}h(iT/n,B_{\theta_i})\right|\le J_1+J_2+\theta_n J_3$$

where

$$J_1:=\max_{1\le k\le n}\left|\sum_{i=0}^{k-1}h(iT/n,B_{\theta_i})\left(E_P(\theta_{i+1}-\theta_i|\mathcal{F}_{\theta_i})-T/n\right)\right|,$$
$$J_2:=\max_{1\le k\le n}\left|\sum_{i=0}^{k-1}h(iT/n,B_{\theta_i})\left(\theta_{i+1}-\theta_i-E_P(\theta_{i+1}-\theta_i|\mathcal{F}_{\theta_i})\right)\right|,$$
$$\text{and } J_3:=\left(\max_{0\le k\le n-1}\sup_{\theta_k\le t\le\theta_{k+1}}|h(t,B_t)-h(kT/n,B_{\theta_k}|\right).$$

We have $E_P(\theta_{i+1}-\theta_i|\mathcal{F}_{\theta_i})=T/n+O(n^{-3/2})$. Hence, from the bound $h(t,x)=O(1)(1+|x|)(1+t)$, Lemma 5.1 and the Jensen inequality it follows that

$$E_P[J_1]=O(n^{-1/2})E_P(1+\max_{0\le k\le n-1}B_{\theta_k})=O((1+s)n^{-1/2}).$$

Next, we estimate J_2. We observe that the stochastic process

$$\sum_{i=0}^{k-1}h(iT/n,B_{\theta_i})\left(\theta_{i+1}-\theta_i-E_P(\theta_{i+1}-\theta_i|\mathcal{F}_{\theta_i})\right),\quad k=1,...,n$$

is a martingale. Thus, from the Doob inequality, the Cauchy–Schwarz inequality, Lemmas 5.1–5.2 and the above bound on h we obtain

$$E_P[J_2^2] = O(1)\sum_{i=0}^{n-1} E_P\left(h^2(iT/n, B^2_{\theta_i})\,(\theta_{i+1}-\theta_i - E_P(\theta_{i+1}-\theta_i|\mathcal{F}_{\theta_i}))^2\right)$$
$$= O(1)\sum_{i=0}^{n-1}\left(E_P\left(h^4(iT/n, B^2_{\theta_i})\right)\right)^{1/2}\left(E_P\left((\theta_{i+1}-\theta_i - E_P(\theta_{i+1}-\theta_i|\mathcal{F}_{\theta_i}))^4\right)\right)^{1/2}$$
$$= O((1+s)^2 n^{-1}).$$

From the Jensen inequality we conclude that $E_P[J_2] = O((1+s)n^{-1/2})$.

Finally, we estimate $E_P[\theta_n J_3]$. From (2.1) and the fact that $\frac{B_t}{B_{\theta_k}} = 1 + O(1/\sqrt{n})$ for $t \in [\theta_k, \theta_{k+1}]$ it follows that

$$J_3 \le O(n^{-1/2}) \max_{0\le k\le n-1} B_{\theta_k} + O(1) \max_{0\le k\le n-1}[(1+B_{\theta_k}) \sup_{\theta_k\le t\le\theta_{k+1}} |t - kT/n|].$$

Observe that $\max_{0\le k\le n-1}\sup_{\theta_k\le t\le\theta_{k+1}}|t-kT/n| \le T/n + \max_{1\le k\le n}|\theta_k - kT/n|$. This together with the Cauchy–Schwarz inequality, Lemma 5.1 and Lemma 5.3 gives

$$E_P[\theta_n J_3] = O(n^{-1/2})\left(E_P\left[\theta_n^2\right]\right)^{1/2}\left(E_P\left(\max_{0\le k\le n-1} B^2_{\theta_k}\right)\right)^{1/2} +$$
$$O(1)\frac{T}{n}\left(E_P\left[\theta_n^2\right]\right)^{1/2}\left(E_P\left(\max_{0\le k\le n-1}(1+B_{\theta_k})^2\right)\right)^{1/2} +$$
$$O(1)\left(E_P\left[\theta_n^2\right]\right)^{1/2}\left(E_P\left(\max_{0\le k\le n-1}(1+B_{\theta_k})^4\right)\right)^{1/4}\left(E_P\left(\max_{1\le k\le n}|\theta_k - kT/n|^4\right)\right)^{1/4}$$
$$= O\left((1+s)n^{-1/2}\right)$$

and the proof is completed. □

6 Game options and numerical results

In this section we apply Theorem 2.2 and provide numerical analysis for path–independent game options with the payoffs $Y_t = f(t, B_t)$ and $X_t = f(t, B_t)$, $t \in [0, T]$, and we set $Z \equiv 0$. First (for the above payoffs), we establish the connection between the super–hedging price of game options and Dynkin games, in the model uncertainty setup.

6.1 Game options

A game contingent claim (GCC) or game option, which was introduced in [17], is defined as a contract between the seller and the buyer of the option such that both have the right to exercise it at any time up to a maturity date (horizon) T. We consider the following GCC with Markovian payoffs. If the buyer exercises the contract at time t then he receives the payment $Y_t = f(t, B_t)$, but if the seller exercises (cancels) the contract before the buyer then the latter receives $X_t = g(t, B_t)$. The difference $X_t - Y_t$ is the penalty which the seller pays to the buyer for the contract cancellation. In short, if the seller will exercise at a stopping time $\gamma \le T$ and the buyer at a stopping time $\tau \le T$ then the former pays to the latter the amount $H(\gamma, \tau)$ given by (1.1).

Next, we introduce the setup of super–hedging for the seller (the buyer setup is symmetrical). Recall the natural filtration, $\mathcal{F} = \mathcal{F}_t$, $t \ge 0$. We denote by $L(B, \mathcal{P}_s^{(I)})$ the set of all $\mathcal{F}$–predictable processes $\Delta = \{\Delta_t\}_{t=0}^T$ such that for any $P \in \mathcal{P}_s^{(I)}$, the stochastic (Itô) integral $\int_0^t \Delta_u dB_u$, $t \in [0, T]$ is well defined and a super–martingale with respect to $\mathcal{F}$. We define a hedge for the seller as a triplet $(x, \Delta, \gamma) \in \mathbb{R} \times L(B, \mathcal{P}_s^{(I)}) \times \mathcal{T}_T$ which consists of an initial capital x, a trading strategy $\Delta = \{\Delta_t\}_{t=0}^T$ and a stopping time γ. A hedge (x, Δ, γ) is perfect if for any stopping time (for the buyer) $\tau \in \mathcal{T}_T$ we have the inequality

$$x + \int_0^{\gamma\wedge\tau} \Delta_u dB_u \ge H(\gamma, \tau) \;\; P-\text{a.s. for all } \; P \in \mathcal{P}_s^{(I)}.$$

The super–hedging price is defined by

$$\mathbf{V} := \inf\{x \in \mathbb{R} : \exists(\Delta, \gamma) \text{ such that } (x, \Delta, \gamma) \text{ is a perfect hedge}\}.$$

Lemma 6.1. *The super–hedging price is given by* $\mathbf{V} = V^{(I)}(0, s)$. *Moreover, there exists a perfect hedge with initial capital* $V^{(I)}(0, s)$.

Proof. As usual, the inequality $\mathbf{V} \geq V^{(I)}(0, s)$ is immediate. Indeed if (x, Δ, γ) is a perfect hedge then from the super–martingale property of $\int_0^t \Delta_u dB_u$, $t \in [0, T]$ we obtain that for any $\tau \in \mathcal{T}_T$ and $P \in \mathcal{P}_s^{(I)}$

$$x \geq E_P \left[x + \int_0^{\gamma \wedge \tau} \Delta_u dB_u \right] \geq E_P[H(\gamma, \tau)].$$

Thus $x \geq V^{(I)}(0, s)$ as required.

It remains to show that there exists a perfect hedge with initial capital $V^{(I)}(0, s)$. We apply Theorem 4.1 in [4] which not only gives the optimal stopping time for the player which plays against nature but also a sub–martingale property up to the optimal time. Once again taking Remark 2.1 into account, for our setup the sub–martingale property becomes a super–martingale property. More precisely, Theorem 4.1 in [4] implies that for the stopping time $\gamma^* := T \wedge \inf\{t : X_t = V^{(I)}(t, B_t)\}$ we have the following property. For any $P \in \mathcal{P}_s^{(I)}$, the process $V^{(I)}(t \wedge \gamma^*, B_{t \wedge \gamma^*})$, $t \in [0, T]$ is a P–super–martingale with respect to the natural filtration $\mathcal{F}_t$, $t \geq 0$.

We apply the nondominated version of the optional decomposition theorem . Since quadratic variation can be defined in a pathwise form then the condition $B^{-1}\sqrt{\frac{d\langle B \rangle}{dt}} \in I$ is invariant under equivalent change of measure. Hence the set $P_s^{(I)}$ is a saturated set (using [24] terminology) of martingale measures. Namely, if $P \in \mathcal{P}_s^{(I)}$ and $Q \sim P$ is a martingale measure on the canonical space then $Q \in \mathcal{P}_s^{(I)}$. Thus, from Theorem 2.4 in [24] it follows that there exists a process $\Delta^* \in L(B, \mathcal{P}_s^{(I)})$ such that for any probability measure $P \in \mathcal{P}_s^{(I)}$

$$P\left(V^{(I)}(0, s) + \int_0^t \Delta_u^* dB_u - V^{(I)}(t \wedge \gamma^*, B_{t \wedge \gamma^*}) \geq 0, \ \ \forall t \in [0, T] \right) = 1. \tag{6.1}$$

We claim that $(V^{(I)}(0, s), \Delta^*, \gamma^*)$ is a perfect hedge. Indeed, let $\tau \in \mathcal{T}_T$ be a stopping time for the buyer and $P \in \mathcal{P}_s^{(I)}$. First consider the event $\{\gamma^* < \tau\}$. On this event we have (recall the definition of γ^*) $V^{(I)}(\tau \wedge \gamma^*, B_{\tau \wedge \gamma^*}) = X_{\gamma^*} = H(\gamma^*, \tau)$ and so from (6.1)

$$V^{(I)}(0, s) + \int_0^{\tau \wedge \gamma^*} \Delta_u^* dB_u \geq H(\gamma^*, \tau) \ \ P - \text{a.s.}$$

Finally, we consider the event $\{\gamma^* \geq \tau\}$. Applying (6.1) and the trivial inequality $V^{(I)}(t, x) \geq f(t, x)$ for all t, x we obtain

$$V^{(I)}(0, s) + \int_0^{\tau \wedge \gamma^*} \Delta_u^* dB_u \geq Y_\tau = H(\gamma^*, \tau) \ \ P - \text{a.s.}$$

and the proof is completed. □

Remark 6.2. It seems that by applying Theorem 4.1 in [4] and the optional decomposition Theorem 2.4 in [24], Lemma 6.1 can be extended to path dependent options as long as the regularity assumptions from [4] are satisfied. Since we are motivated by numerical applications, then for simplicity we considered path–independent payoffs. Still, a challenging open question, is whether Lemma 6.1 can be obtained under weaker (than Lipschitz or uniform type of continuity) regularity conditions.

Table 1: In this table we take the parameters $r = 0.06$, $T = 0.5$, $K = 100$, $\delta = 5$ and provide numerical results for game put options under model uncertainty given by the interval $I = [0, 0.4]$. We compare our results to previous numerical results (see [20]) for game put options in the Black–Scholes model with volatility $\sigma = 0.4$.

	Values obtained with				
S_0	n = 200	n = 400	n = 700	n = 1200	Black–Scholes with $\sigma = \overline{\sigma}$
80	20.7003	20.6719	20.6593	20.6532	20.6
90	12.4932	12.4787	12.4938	12.4683	12.4
100	5.00	5.00	5.00	5.00	5.00
110	3.7609	3.7240	3.6862	3.6916	3.64
120	2.6169	2.5897	2.5822	2.5729	2.54

6.2 Numerical results

In view of Lemma 6.1 we use Theorem 2.2 and provide a numerical analysis for the super–hedging price of path–independent game options. We assume that the interest rate in the market is a constant $r > 0$, and so the stock price before discounting is given by $S_t = e^{rt} B_t$, where, recall that B is the canonical process. The payoffs before discounting are of the form $\hat{X}_t = \hat{g}(S_t)$, $\hat{Y}_t = \hat{f}(S_t)$ where $\hat{g} \geq \hat{f}$. In order to compute the game option price we need to consider the discounted payoffs and so during this section we put $g(t,x) := e^{-rt}\hat{g}(e^{rt}x)$, $f(t,x) := e^{-rt}\hat{f}(e^{rt}x)$ and $h \equiv 0$.

In [11] (see Section 4), the author proved that for game options (with finite or infinite maturity) with continuous path–independent payoffs $\hat{g}, \hat{f}$ satisfying

$$\frac{\hat{g}(x)}{x}, \frac{\hat{f}(x)}{x} \text{ are non increasing for } x > 0 \tag{6.2}$$

the price is non decreasing in the volatility. Thus, (if the above assumption is satisfied) the price under volatility uncertainty which is given by the interval $I = [\underline{\sigma}, \overline{\sigma}]$ is the same as the price in the complete Black–Scholes market with a constant volatility $\overline{\sigma}$. The later value can be approximated by the standard binomial models (see [18]). In particular, this is the case for game put options given by

$$\hat{g}(x) = C(K-x)^+ + \delta \ \text{ and } \ \hat{f}(x) = (K-x)^+, \ \ C \geq 1, \ K, \delta > 0.$$

In Table 1, we test numerically the above statement from [11] for game put options. This is done by comparing our numerical results with previous numerics which was obtained in [20] for game put options in the Black–Scholes model.

Game call options

Next, we deal with game call options given by

$$\hat{g}(x) = C(x-K)^+ + \delta \ \text{ and } \ \hat{f}(x) = (x-K)^+, \ \ C \geq 1, \ K, \delta > 0.$$

We observe that in this case (6.2) is not satisfied and so we expect that the price for the model uncertainty interval $I = [\underline{\sigma}, \overline{\sigma}]$ will be strictly bigger than the game call option price in the Black–Scholes model with volatility $\overline{\sigma}$. We take $C = 1$, namely we consider game call options with constant penalty.

First, we compare (Table 2) the option prices under model uncertainty with the prices in the Black–Scholes model (with the highest volatility). Since we could not find previous numerical results for finite maturity game call options in the Black–Scholes model, we

Table 2: We take the same parameters as in Table 1 and provide numerical results for game call options under model uncertainty given by the interval $I = [0, 0.4]$. We compare our results to binomial approximations for the Black–Scholes model with $\sigma = 0.4$.

	Values obtained under model uncertainty			
S_0	n = 200	n = 400	n = 700	n = 1200
80	2.0805	2.0893	2.0847	2.0948
85	2.8138	2.7964	2.8055	2.8018
90	3.6553	3.5966	3.6241	3.6064
95	4.5827	4.4682	4.5050	4.4874
105	5.00	5.00	5.00	5.00
110	10.00	10.00	10.00	10.00
115	15.00	15.00	15.00	15.00
120	20.00	20.00	20.00	20.00
	Values obtained for Black–Scholes			
S_0	n = 200	n = 400	n = 700	n = 1200
80	2.0625	2.0359	2.0244	2.0210
85	2.7706	2.7301	2.7274	2.7143
90	3.5066	3.4889	3.4968	3.4798
95	4.3497	4.3124	4.3056	4.2481
105	5.00	5.00	5.00	5.00
110	10.00	10.00	10.00	10.00
115	14.9355	14.9304	14.9275	14.9260
120	19.7812	19.7735	19.7691	19.7669

compute it by applying the binomial trees from [18]. These trees are “almost” the same as our trees for the case where the volatility uncertainty interval I contains only one point. We observe that for call options the prices in general should not coincide.

Finally, we calculate numerically the stopping regions. We observe that the discounted payoff $f(t, B_t) = (B_t - Ke^{-rt})^+$, $t \geq 0$ is a sub–martingale with respect to any probability measure in the set $\mathcal{P}_s^{(I)}$. Thus, the buyer's optimal stopping time is just $\tau \equiv T$.

For the seller, the optimal stopping time is (see Theorem 4.1 in [4])

$$\gamma^* = T \wedge \inf\{t : g(t, B_t) = V^{(I)}(t, B_t)\}.$$

Introduce the function

$$\tilde{V}(u, x) := \sup_{P \in \mathcal{P}_x^{(I)}} \sup_{\tau \in \mathcal{T}_u} \inf_{\gamma \in \mathcal{T}_u} E_P \left[e^{-r(\tau \wedge \gamma)} \left((S_{\tau \wedge \gamma} - K)^+ + \delta \mathbb{I}_{\gamma < \tau} \right) \right]$$

where as before $S_t = e^{rt} B_t$, $t \geq 0$ is the stock price. The term $\tilde{V}(u, x)$ is the price of a game call option with maturity date u and initial stock price $S_0 = x$. We observe that $\gamma^* = T \wedge \inf\{t : S_t \in D\}$, where $D = D(T)$ is the stopping region (of course it depends on the maturity date T) given by

$$D = \{(t, x) : \tilde{V}(T - t, x) = (x - K)^+ + \delta\}.$$

In [28], the authors studied the structure of the stopping region D for game call options in the complete Black–Scholes market. They proved (see Theorem 4.2) that the stopping region D is of the form

$$D = \{(t, x) : t \in [0, T_1],\ K \leq x \leq b(t)\} \bigcup \{[T_1, T_2] \times \{K\}\}$$

where $T_1 < T_2 < T$ and $b : [0, T_1] \to [K, \infty)$ can be computed numerically.

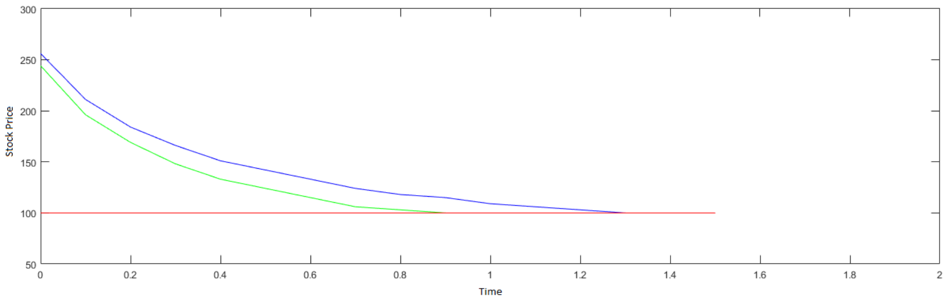

Figure 1: We consider a game call option with maturity date $T = 2$, a constant penalty $\delta = 12$ and a strike price $K = 100$. As before the interest rate is $r = 0.06$. We take $n = 1200$ and compute numerically the stopping regions for the seller. For the model uncertainty given by the interval $I = [0, 0.4]$ we get that for $t \in [0, 1.3]$ the seller should exercise at the first moment when the stock price is between the strike price and the value given by the blue curve. For $t \in [1.3, 1.5]$ the seller stops at the first moment the stock price equals to the strike price. After the time $t = 1.5$ the investor should not exercise (before the maturity date). For the Black–Scholes model with volatility $\sigma = 0.4$ we get that for $t \in [0, 0.9]$ the seller should exercise at the first moment when the stock price is between the strike price and the value given by the green curve. For $t \in [0.9, 1.5]$ the seller stops at the first moment the stock price equals to the strike price. After the time $t = 1.5$ the investor should not exercise (before the maturity date).

In Figure 1 we calculate numerically the stopping regions (for the seller) for game call options both in the model uncertainty setup given by the interval $I = [0, 0.4]$ and in the complete Black–Scholes model with volatility $\sigma = 0.4$. We obtain that the structure from [28] is valid for the model uncertainty case as well. Furthermore, for both cases T_2 is the same, while T_1 and b are different. Up to date, there is no theoretical results related to the explicit structure of stopping regions for game options under model uncertainty.

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