Supermartingale decomposition theorem under G-expectation[*]

Hanwu Li[†]   Shige Peng[‡]   Yongsheng Song[§]

Abstract

The objective of this paper is to establish the decomposition theorem for supermartin-gales under the G-framework. We first introduce a g-nonlinear expectation via a kind of G-BSDE and the associated supermartingales. We have shown that this kind of supermartingales has the decomposition similar to the classical case. The main ideas are to apply the property on uniform continuity of $S_G^\beta(0,T)$, the representation of the solution to G-BSDE and the approximation method via penalization.

Keywords: G-expectation; $\hat{\mathbb{E}}^g$-supermartingale; $\hat{\mathbb{E}}^g$-supermartingale decomposition theorem.
AMS MSC 2010: 60H10; 60H30.
Submitted to EJP on March 8, 2017, final version accepted on April 28, 2018.

1   Introduction

The classical Doob-Meyer decomposition theorem tells us that a large class of sub-martingales can be uniquely represented as the summation of a martingale and a predictable increasing process. This is one of the most fundamental results in the theory of stochastic analysis. This theorem was firstly proved in [9] for the discrete time case. Then [16, 17] proved this result for the continuous time case. This theorem is important for the optimal stopping problem used to solve the pricing for the American options (see [1],[14]). Besides, it can be applied to study the problem of hedging contingent claims by portfolios constraint to take values in a given closed, convex set (see [6]). A general case of Doob-Meyer decomposition theorem was introduced in [20] when the supermartingale Y is defined by a nonlinear operator. It was proved that the nonlinear version of Doob-Meyer decomposition theorem also holds.

[†]School of Mathematics, Shandong University, Jinan, China. E-mail: lihanwu@mail.sdu.edu.cn
[‡]School of Mathematics and Qilu Institute of Finance, Shandong University, Jinan, China. E-mail: peng@sdu.edu.cn
[§]RCSDS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, China. E-mail: yssong@amss.ac.cn

Supermartingale decomposition theorem under G-expectation

The objective of this paper is to solve the problem of decomposition theorem of Doob-Meyer's type for nonlinear supermartingales defined in the G-expectation space. In order to understand the motivation of this objective, let us recall its special linear case, namely, in a framework of Wiener probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, P)$ in which the canonical process $B_t(\omega) = \omega(t)$ for $\omega \in \Omega = C_0([0, \infty))$ is a d-dimensional standard Brownian motion. Given a function $g = g(s, \omega, y, z) : [0, \infty) \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ where $g(\cdot, y, z)$ satisfies the "usual Lipschitz conditions" in the framework of BSDE (see [18]), such that, for each $T \in [0, \infty)$, the following BSDE has a unique solution on $[0, T]$,

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds + (A_T - A_t) - \int_t^T z_s dB_s, \quad s \in [0, T],$$

where ξ is a given random variable in $L^2(\Omega, \mathcal{F}_T, P)$ and A is a given continuous and increasing process with $A_0 = 0$ and $A_t \in L^2(\Omega, \mathcal{F}_t, P)$ for each $t \in (0, T]$. We call y a g-supersolution. If $A \equiv 0$, then y is called a g-solution. For the latter case, since for each given $t \leq T$, the $\mathcal{F}_t$ measurable random variable y_t is uniquely determined by the terminal condition $y_T = \xi \in L^2(\Omega, \mathcal{F}_T, P)$, then we can define a backward semigroup [19, 21]

$$\mathcal{E}_{t,T}^g[\xi] := y_t, \quad 0 \leq t \leq T < \infty. \tag{a}$$

This semiproup gives us a generalized notion of nonlinear expectation with corresponding $\mathcal{F}_t$-conditional expectation, called g-expectation [19]. By the comparison theorem of BSDEs we know that any g-supersolution Y is also a g-supermartingale (i.e., we have $\mathcal{E}_{s,t}^g[Y_t] \leq Y_s$, for each $s \leq t$). But the proof of the inverse claim, namely, a g-supermartingale is a g-supersolution, is not at all trivial (we refer to [20] for detailed proof). In fact this is a generalization of the classical Doob-Meyer decomposition to the case of nonlinear expectations, and the linear situation corresponds to the case $g \equiv 0$.

Moreover, this nonlinear Doob-Meyer decomposition theorem plays a key role to obtain the following representation theorem of nonlinear expectations: for a given arbitrary $\mathcal{F}_t$-conditional nonlinear expectation $(\mathcal{E}_{s,t}[\xi])_{0 \leq s \leq t < \infty}$ with certain regularity, there exists a unique function $g = g(\cdot, y, z)$ satisfying the usual conditions of BSDE, such that,

$$\mathcal{E}_{t,T}[\xi] = \mathcal{E}_{t,T}^g[\xi], \quad \text{for all } 0 \leq t \leq T < \infty, \quad \text{and } \xi \in L^2(\Omega, \mathcal{F}_T, P).$$

We refer to [4], [21], [22] for the proof of this very deep result, also to [7] where a wide class of time consistent risk measures are identified to be g-expectations.

It is known that volatility model uncertainty (VMU) involves an essentially non-dominated family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{F})$. This is a main reason why many risk measures, and pricing operators cannot be well-defined within a framework of a single probability space such as Wiener space $(\Omega, \mathcal{F}_T, P)$. [23] introduced the framework of (fully nonlinear) time consistent G-expectation space $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$ such that all probability measures in $\mathcal{P}$ are dominated by this sublinear expectation and such that the canonical process $B.(\omega) = \omega(\cdot)$ becomes a nonlinear Brownian motion, called G-Brownian. Many random variables, negligible under the probability measure $P \in \mathcal{P}$, as well as under other measures in $\mathcal{P}$, can be clearly distinguished in this new framework. The corresponding theory of stochastic integration and stochastic calculus of Itô's type have been established in [23, 25]. In particular, the existence and uniqueness of BSDE driven by G-Brownian motion (G-BSDE) have been established in [10]. Roughly speaking (see next section for details), a G-BSDE is as follows

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s dB_s - (K_T - K_t), \quad t \in [0, T],$$

where $g(\cdot, y, z)$ and ξ satisfy very similar conditions with the classical case. The solution of this G-BSDE consists of a triplet of adapted processes (y, z, K) where K is a decreasing

G-martingale with $K_0 = 0$. We then call y a g-solution under $\hat{\mathbb{E}}$. From the existence and uniqueness of the G-BSDE, we can also define $\hat{\mathbb{E}}^g_{t,T}[\xi] = y_t$ which forms a time consistent nonlinear expectation. If K is just a decreasing process then we call y a g-supersolution under $\hat{\mathbb{E}}$.

By the comparison theorem of G-BSDE obtained in [11], we can prove that a g-supersolution under $\hat{\mathbb{E}}$ Y is also an $\hat{\mathbb{E}}^g$-supermartingale, i.e., we have $\hat{\mathbb{E}}^g_{s,t}[Y_t] \leq Y_s$, for each $s \leq t$. The objective of this paper is to prove its inverse property: a continuous $\hat{\mathbb{E}}^g$-supermartingale Y is also a g-supersolution under $\hat{\mathbb{E}}$. Namely, Y can be written as

$$Y_t = Y_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + (A_T - A_t), \quad t \in [0, T],$$

where A is a continuous increasing process. A special case of this result is when $g \equiv 0$. In this case Y is a G-supermartingale and it can be decomposed into the following

$$Y_t = Y_0 + \int_0^t Z_s dB_s - A_t,$$

where A is an increasing process. This is still a new and non-trivial result.

The proof of this decomposition theorem involves a penalization procedure,

$$y_s^n = Y_T + \int_t^T g(s, y_s^n, z_s^n)ds - \int_t^T z_s^n dB_s - (K_T^n - K_t^n) + (L_T^n - L_t^n), \quad t \in [0, T],$$

for $n = 1, 2, \cdots$, where $L_t^n = n \int_0^t (Y_s - y_s^n)ds$ and K^n is an decreasing martingale. In order to prove that $y^n \uparrow Y$, it is necessary to show that $y^n \leq Y$. A main problem is that the corresponding Doob's optional sampling is still an open problem. We overcome this difficulty by proving that, for each probability dominated by $\mathcal{P}$, we have $y^n \leq Y$. We also need to introduce some new methods, see Lemma 3.7 and Lemma 3.8, to prove the uniform convergence of y^n. Generally speaking, the well-known Fatou's lemma cannot be directly and automatically used in this sublinear expectation framework. Besides, a bounded subset in $M_G^\beta(0, T)$ is not necessarily weakly compact. Many proofs become more delicate and challenging.

We believe that the proof of our new decomposition theorem of Doob-Meyer's type under G-framework will play a key role for understanding and solving many important problems. Note that the Doob-Meyer decomposition theorem for g-supermartingale is the builing block to obtain the representation theorem for "enough regular" filtration consistent nonlinear expectations. Since our new result is the generalization of the g-expectation case which represents the drift uncertainty, the decomposition theorem under G-framework is a key step towards the understanding and solving a general representation theorem of dynamically consistent nonlinear expectations, as well as dynamic risk measures and pricing operators in the volatility uncertainty model.

The paper is organized as follows. In Section 2, we set up some notations and results as preliminaries for the later proofs. Section 3 is devoted to the study of the so-called $\hat{\mathbb{E}}^g$-supermartingales. The representation theorem is established with detailed proofs. In Section 4, we present the relationship between the $\hat{\mathbb{E}}^g$-supermartingales and the fully nonlinear parabolic PDEs.

2 Preliminaries

2.1 G-expectation and G-Itô's calculus

The main purpose of this section is to recall some basic notions and results of G-expectation, which are needed in the sequel. The readers may refer to [10], [11], [24], [25] for more details.

Definition 2.1. *Let Ω be a given set and let $\mathcal{H}$ be a vector lattice of real valued functions defined on Ω, namely $c \in \mathcal{H}$ for each constant c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of random variables. A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have*

(a) *Monotonicity: If $X \geq Y$, then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;*

(b) *Constant preserving: $\hat{\mathbb{E}}[c] = c$;*

(c) *Sub-additivity: $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;*

(d) *Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for each $\lambda \geq 0$.*

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space. $X \in \mathcal{H}$ is called a random variable in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. In the following, unless otherwise stated, we consider the following sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$: if $Y_1, \ldots, Y_n \in \mathcal{H}$, then $\varphi(Y_1, \ldots, Y_n) \in \mathcal{H}$ for each $\varphi \in C_{Lip}(\mathbb{R}^n)$. We often call $Y = (Y_1, \ldots, Y_d), Y_i \in \mathcal{H}$ a d-dimensional random vector in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.

Definition 2.2. *Let X_1 and X_2 be two n-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)]$, for all $\varphi \in C_{Lip}(\mathbb{R}^n)$, where $C_{Lip}(\mathbb{R}^n)$ is the space of real continuous functions defined on $\mathbb{R}^n$ such that*

$$|\varphi(x) - \varphi(y)| \leq C|x - y| \ \text{ for all } x, y \in \mathbb{R}^n,$$

where C depends only on φ.

Definition 2.3. *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}[\cdot]$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$.*

Definition 2.4. *(G-normal distribution) A d-dimensional random vector $X = (X_1, \cdots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called G-normally distributed if for each $a, b \geq 0$ we have*

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2} X,$$

where $\bar{X}$ is an independent copy of X, i.e., $\bar{X} \overset{d}{=} X$ and $\bar{X} \perp X$. Here the letter G denotes the function

$$G(A) := \frac{1}{2} \hat{\mathbb{E}}[\langle AX, X \rangle] : \mathbb{S}_d \to \mathbb{R},$$

where $\mathbb{S}_d$ denotes the collection of $d \times d$ symmetric matrices.

It is proved in [24] that $X = (X_1, \cdots, X_d)$ is G-normally distributed if and only if for each $\varphi \in C_{Lip}(\mathbb{R}^d)$, $u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}^d$, is the solution of the following fully nonlinear parabolic equation:

$$\partial_t u - G(D_x^2 u) = 0, \ u(0, x) = \varphi(x).$$

where $D_x^2 u = \{\partial_{x_i x_j}^2 u\}_{i,j=1}^d$.

In the case $d = 1$, the function $G : \mathbb{R} \to \mathbb{R}$ is a given monotonic and sublinear function of the form

$$G(a) = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-), \ \ a \in \mathbb{R}, \tag{2.1}$$

where $\bar{\sigma}^2 = \hat{\mathbb{E}}[X^2]$ and $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-X^2]$. In this paper we only consider the non-degenerate G-normal distribution, i.e., $\underline{\sigma} > 0$ in the 1-dimensional case.

We present the notion of G-Brownian motion in a sublinear expectation space. For notational simplification, we only consider the case of 1-dimensional G-Brownian motion. But the methods of this paper can be directly applied to d-dimensional situations.

Let $\Omega = C_0([0, \infty); \mathbb{R})$ be the space of real valued continuous functions on $[0, \infty)$ with $\omega_0 = 0$ endowed with the following distance

$$\rho(\omega^1, \omega^2) := \sum_{N=1}^{\infty} 2^{-N}[(\max_{t \in [0,N]} |\omega_t^1 - \omega_t^2|) \wedge 1],$$

and $B_t(\omega) = \omega_t$, $t \geq 0$, $\omega \in \Omega$ be the canonical process. For each $T > 0$, set $\Omega_T = \{\omega(\cdot \wedge T), \ \omega \in \Omega\}$. We denote by $\mathcal{B}(\Omega)$ the collection of all Borel-measurable subsets of Ω.

Definition 2.5. *i)Set*

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, ..., B_{t_n}) : n \geq 1, t_1, ..., t_n \in [0, T], \varphi \in C_{Lip}(\mathbb{R}^n)\},$$
$$L_{ip}(\Omega) := \bigcup_{T>0} L_{ip}(\Omega_T).$$

Let $G : \mathbb{R} \to \mathbb{R}$ be a given monotonic and sublinear function of the form (2.1). G-expectation is a sublinear expectation defined on the space of the random variable $(\Omega, L_{ip}(\Omega))$ in the following way: for each $X \in L_{ip}(\Omega)$ in the form $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}}) \in L_{ip}(\Omega)$, with $t_0 < t_1 < \cdots < t_m$, we set

$$\hat{\mathbb{E}}[X] = \tilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_m - t_{m-1}}\xi_m)],$$

where $\xi_1, \cdots, \xi_n$ are identically distributed 1-dimensional G-normally distributed random vectors in a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ such that ξ_{i+1} is independent of $(\xi_1, \cdots, \xi_i)$ for every $i = 1, \cdots, m-1$.

The canonical process $B_t(\omega) = \omega_t$, $t \geq 0$, is called a G-Brownian motion on the sublinear expectation space $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}}[\cdot])$

ii) Let us define the conditional G-expectation $\hat{\mathbb{E}}_t$ of $\xi \in L_{ip}(\Omega_T)$ knowing $L_{ip}(\Omega_t)$, for $t \in [0, T]$. Without loss of generality we can assume that ξ has the representation $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$ with $t = t_i$, for some $1 \leq i \leq m$, and we put

$$\hat{\mathbb{E}}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]$$
$$= \tilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_i} - B_{t_{i-1}}),$$

where
$$\tilde{\varphi}(x_1, \cdots, x_i) = \hat{\mathbb{E}}[\varphi(x_1, \cdots, x_i, B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_m} - B_{t_{m-1}})].$$

Define $\|X\|_{L_G^p} = (\hat{\mathbb{E}}[|\xi|^p])^{1/p}$ for $X \in L_{ip}(\Omega)$ and $p \geq 1$. Then for all $t \in [0, T]$, $\hat{\mathbb{E}}_t[\cdot]$ is a continuous mapping on $L_{ip}(\Omega_T)$ w.r.t. the norm $\|\cdot\|_{L_G^p}$. Therefore it can be extended continuously to the completion $L_G^p(\Omega_T)$ of $L_{ip}(\Omega_T)$ under the norm $\|\cdot\|_{L_G^p}$. Denis et al. [8] proved that the completions of $C_b(\Omega_T)$ (the set of bounded continuous function on Ω_T) under the norm $\|\cdot\|_{L_G^p}$ coincides with $L_G^p(\Omega_T)$.

Let $\pi_t^N = \{t_0^N, \cdots, t_N^N\}$, $N = 1, 2, \cdots$, be a sequence of partitions of $[0, t]$ such that $\mu(\pi_t^N) = \max\{|t_{i+1}^N - t_i^N| : i = 0, \cdots, N-1\} \to 0$, the quadratic variation process of B is defined by

$$\langle B \rangle_t = L_G^2 - \lim_{\mu(\pi_t^N) \to 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2.$$

Let us denote the set of all probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$ by $\mathcal{M}_1(\Omega_T)$.

Theorem 2.6. *([8, 12]) There exists a tight set $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$ such that*

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X] \;\; for \; all \; X \in L_{ip}(\Omega_T).$$

$\mathcal{P}$ is called a set that represents $\hat{\mathbb{E}}$.

Let $\mathcal{P}$ be a tight set that represents $\hat{\mathbb{E}}$. For this $\mathcal{P}$, we define capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \;\; A \in \mathcal{B}(\Omega_T).$$

The set $A \subset \Omega_T$ is said to be polar if $c(A) = 0$. A property holds "quasi-surely" (q.s. for short) if it holds outside a polar set. In the following, we do not distinguish two random variables X and Y if $X = Y$ q.s..

Remark 2.7. Let $(\Omega, \mathcal{F}, P^0)$ be a probability space and $(W_t)_{t \geq 0}$ be a 1-dimensional Brownian motion under P^0. Let $\mathbb{F} = \{\mathcal{F}_t\}$ be the augmented filtration generated by W. [8] proved that

$$\mathcal{P}_M := \{P_h | P_h = P^0 \circ X^{-1}, X_t = \int_0^t h_s dW_s, h \in L_{\mathbb{F}}^2([0,T];[\underline{\sigma},\overline{\sigma}])\},$$

is a set that represents $\hat{\mathbb{E}}$, where $L_{\mathbb{F}}^2([0,T];[\underline{\sigma},\overline{\sigma}])$ is the collection of $\mathbb{F}$-adapted measurable processes with values in $[\underline{\sigma},\overline{\sigma}]$.

For $\xi \in L_{ip}(\Omega_T)$, let $\mathcal{E}(\xi) = \hat{\mathbb{E}}[\sup_{t \in [0,T]} \hat{\mathbb{E}}_t[\xi]]$, where $\hat{\mathbb{E}}$ is the G-expectation. For convenience, we call $\mathcal{E}$ the G-evaluation. For $p \geq 1$ and $\xi \in L_{ip}(\Omega_T)$, define $\|\xi\|_{p,\mathcal{E}} = [\mathcal{E}(|\xi|^p)]^{1/p}$. Let $L_{\mathcal{E}}^p(\Omega_T)$ denote the completion of $L_{ip}(\Omega_T)$ under $\| \cdot \|_{p,\mathcal{E}}$. We shall give an estimate between the two norms $\| \cdot \|_{L_G^p}$ and $\| \cdot \|_{p,\mathcal{E}}$.

Theorem 2.8 ([30]). *For any $\alpha \geq 1$ and $\delta > 0$, $L_G^{\alpha+\delta}(\Omega_T) \subset L_{\mathcal{E}}^{\alpha}(\Omega_T)$. More precisely, for any $1 < \gamma < \beta := (\alpha + \delta)/\alpha$, $\gamma \leq 2$, we have*

$$\|\xi\|_{\alpha,\mathcal{E}}^{\alpha} \leq \gamma^* \{\|\xi\|_{L_G^{\alpha+\delta}}^{\alpha} + 14^{1/\gamma} C_{\beta/\gamma} \|\xi\|_{L_G^{\alpha+\delta}}^{(\alpha+\delta)/\gamma}\}, \;\; for \; all \; \xi \in L_{ip}(\Omega_T),$$

where $C_{\beta/\gamma} = \sum_{i=1}^{\infty} i^{-\beta/\gamma}$, $\gamma^* = \gamma/(\gamma - 1)$.

Independently, [28] proved $L_G^{\alpha}(\Omega_T) \subset L_{\mathcal{E}}^2(\Omega_T)$ for $\alpha > 2$.

Definition 2.9. *Let $M_G^0(0,T)$ be the collection of processes in the following form: for a given partition $\{t_0, \cdots, t_N\} = \pi_T$ of $[0,T]$,*

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \cdots, N-1$. For each $p \geq 1$ and $\eta \in M_G^0(0,T)$, we denote by

$$\|\eta\|_{H_G^p} = \{\hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}, \quad \|\eta\|_{M_G^p} := (\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds])^{1/p}.$$

We use $H_G^p(0,T)$ and $M_G^p(0,T)$ to denote the completion of $M_G^0(0,T)$ under norms $\| \cdot \|_{H_G^p}$ and $\| \cdot \|_{M_G^p}$ respectively.

For two processes $\eta \in M_G^2(0,T)$ and $\xi \in M_G^1(0,T)$, the G-Itô integrals $(\int_0^t \eta_s dB_s)_{0 \leq t \leq T}$ and $(\int_0^t \xi_s d\langle B \rangle_s)_{0 \leq t \leq T}$ are well defined (see Li-Peng [15] and Peng [25]). Moreover, by Proposition 2.10 in [15] and the classical Burkholder-Davis-Gundy inequality, the following property holds.

Proposition 2.10. *If $\eta \in H_G^\alpha(0,T)$ with $\alpha \geq 1$, then we can get $\sup_{u \in [t,T]} |\int_t^u \eta_s dB_s|^p \in L_G^1(\Omega_T)$ and*

$$\underline{\sigma}^p c_p \hat{\mathbb{E}}_t[(\int_t^T |\eta_s|^2 ds)^{p/2}] \leq \hat{\mathbb{E}}_t[\sup_{u \in [t,T]} |\int_t^u \eta_s dB_s|^p] \leq \bar{\sigma}^p C_p \hat{\mathbb{E}}_t[(\int_t^T |\eta_s|^2 ds)^{p/2}],$$

where $p \in (0, \alpha]$, c_p and C_p are the classical B-D-G constants.

Let $S_G^0(0,T) = \{h(t, B_{t_1 \wedge t}, \ldots, B_{t_n \wedge t}) : t_1, \ldots, t_n \in [0,T], h \in C_{b,Lip}(\mathbb{R}^{n+1})\}$. For $p \geq 1$ and $\eta \in S_G^0(0,T)$, set $\|\eta\|_{S_G^p} = \{\hat{\mathbb{E}}[\sup_{t \in [0,T]} |\eta_t|^p]\}^{1/p}$. Let $S_G^p(0,T)$ denote the completion of $S_G^0(0,T)$ under the norm $\|\cdot\|_{S_G^p}$.

We consider the following type of G-BSDEs

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds + \int_t^T f(s, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad (2.2)$$

where g and f are given functions

$$g(t, \omega, y, z), f(t, \omega, y, z) : [0,T] \times \Omega_T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

satisfy the following properties:

(H1) There exists some $\beta > 1$ such that for any y, z, $g(\cdot, \cdot, y, z), f(\cdot, \cdot, y, z) \in M_G^\beta(0,T)$;

(H2) There exists some $L > 0$ such that

$$|g(t, y, z) - g(t, y', z')| + |f(t, y, z) - f(t, y', z')| \leq L(|y - y'| + |z - z'|).$$

For simplicity, we denote by $\mathfrak{S}_G^\alpha(0,T)$ the collection of processes (Y, Z, K) such that $Y \in S_G^\alpha(0,T)$, $Z \in H_G^\alpha(0,T)$, K is a decreasing G-martingale with $K_0 = 0$ and $K_T \in L_G^\alpha(\Omega_T)$.

Definition 2.11. *Let $\xi \in L_G^\beta(\Omega_T)$ and g and f satisfy (H1) and (H2) for some $\beta > 1$. A triplet of processes (Y, Z, K) is called a solution of Equation (2.2) if for some $1 < \alpha \leq \beta$ the following properties hold:*

(a) *$(Y, Z, K) \in \mathfrak{S}_G^\alpha(0,T)$;*

(b) *$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds + \int_t^T f(s, Y_s, Z_s) d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t)$.*

Theorem 2.12 ([10]). *Assume that $\xi \in L_G^\beta(\Omega_T)$ and g, f satisfy (H1) and (H2) for some $\beta > 1$. Then Equation (2.2) has a unique solution (Y, Z, K). Moreover, for any $1 < \alpha < \beta$ we have $Y \in S_G^\alpha(0,T)$, $Z \in H_G^\alpha(0,T)$ and $K_T \in L_G^\alpha(\Omega_T)$.*

We also have the comparison theorem for G-BSDEs.

Theorem 2.13 ([11]). *Let $(Y_t^i, Z_t^i, K_t^i)_{t \leq T}$, $i = 1, 2$, be the solutions of the following two G-BSDEs:*

$$Y_t^i = \xi^i + \int_t^T g_i(s) ds + \int_t^T f_i(s) d\langle B \rangle_s + V_T^i - V_t^i - \int_t^T Z_s^i dB_s - (K_T^i - K_t^i),$$

where $g_i(s) = g_i(s, Y_s^i, Z_s^i)$, $f_i(s) = f_i(s, Y_s^i, Z_s^i)$, $\xi^i \in L_G^\beta(\Omega_T)$, $\{V_t^i\}_{t \in [0,T]}$ are RCLL processes such that $\hat{\mathbb{E}}[\sup_{t \in [0,T]} |V_t^i|^\beta] < \infty$, g_i, f_i satisfy (H1) and (H2) with $\beta > 1$. Assume that $\xi^1 \geq \xi^2$, $f_1 \geq f_2$, $g_1 \geq g_2$ and $\{V_t^1 - V_t^2\}$ is a nondecreasing process, then $Y_t^1 \geq Y_t^2$.

2.2 Some results of classical penalized BSDEs

In this subsection, we will introduce some notions and results following Peng [20]. The probability space and filtration is given in Remark 2.7. For a given stopping time τ, we now consider the following classical BSDE:

$$y_t = \xi + \int_{t \wedge \tau}^{\tau} g(s, y_s, z_s) ds + (A_\tau - A_{t \wedge \tau}) - \int_{t \wedge \tau}^{\tau} z_s dW_s, \qquad (2.3)$$

where $\xi \in L^2(\Omega, \mathcal{F}_\tau)$ and g satisfies the following conditions:

(A1) $g(\cdot, y, z) \in L^2_{\mathbb{F}}(0, T; \mathbb{R})$, for each $(y, z) \in \mathbb{R}^2$;

(A2) There exists a constant $L > 0$ such that

$$|g(t, y, z) - g(t, y', z')| \leq L(|y - y'| + |z - z'|).$$

Here A is a given RCLL increasing process with $A_0 = 0$ and $E[A_\tau^2] < \infty$. We call (y_t) the g-supersolution on $[0, \tau]$ if (y, z) solves (2.3). In particular, when $A \equiv 0$, (y_t) is called a g-solution on $[0, \tau]$.

Definition 2.14. *An $\mathcal{F}_t$-progressively measurable real-valued process (Y_t) is called a g-supermartingale on $[0, T]$ in strong sense if, for each stopping time $\tau \leq T$, $E[|Y_\tau|^2] < \infty$, and the g-solution (y_t) on $[0, \tau]$ with terminal condition $y_\tau = Y_\tau$, satisfies $y_\sigma \leq Y_\sigma$ for all stopping time $\sigma \leq \tau$.*

Definition 2.15. *An $\mathcal{F}_t$-progressively measurable real-valued process (Y_t) is called a g-supermartingale on $[0, T]$ in weak sense if, for each deterministic time $t \leq T$, $E[|Y_t|^2] < \infty$, and the g-solution (y_t) on $[0, t]$ with terminal condition $y_t = Y_t$, satisfies $y_s \leq Y_s$ for all deterministic time $s \leq t$.*

It is obvious that a g-supermartingale in strong sense is also a g-supermartingale in weak sense. [3] proved that, under assumptions similar to the classical case, a g-supermartingale in weak sense coincides with a g-supermartingale in strong sense. This result is a generalization of the classical Optional Stopping Theorem. If (Y_t) is a g-supersolution on $[0, T]$, it follows from the comparison theorem that (Y_t) is a g-supermartingale. In fact, [20] proved that the inverse problem, i.e., nonlinear version of Doob-Meyer decomposition theorem, also holds. The method of proof is to apply the penalization approach and the first step is the following lemma.

Lemma 2.16 ([20])**.** *Let (Y_t) be a right-continuous g-supermartingale on $[0, T]$ in strong sense with $E[\sup_{0 \leq t \leq T} |Y_t|^2] \leq \infty$. Assume that g satisfies (A1) and (A2). For each $n = 1, 2, \cdots$, consider the following BSDEs:*

$$y_t^n = Y_T + \int_t^T g(s, y_s^n, z_s^n) ds + n \int_t^T (Y_s - y_s^n) ds - \int_t^T z_s^n dW_s.$$

Then, for each $n = 1, 2, \cdots$, $Y_t \geq y_t^n$.

Remark 2.17. Set $M_t = \int_0^t h_s dW_s$, where $h \in L^2_{\mathbb{F}}([0, T]; [\underline{\sigma}, \overline{\sigma}])$. If the BSDE (2.3) is driven by M,

$$y_t = \xi + \int_{t \wedge \tau}^{\tau} g(s, y_s, z_s) ds + (A_\tau - A_{t \wedge \tau}) - \int_{t \wedge \tau}^{\tau} z_s dM_s,$$

then, we can define a g_M-supersolution (also g_M-solution) and a g_M-supermartingale in strong sense (also in weak sense). Furthermore, we have a similar result as Lemma 2.16.

3 Nonlinear expectations generated by G-BSDEs and the associated supermartingales

For simplicity, we only consider the following G-BSDE driven by 1-dimensional G-Brownian motion. The result still holds for multi-dimensional cases.

$$Y_t^{T,\xi} = \xi + \int_t^T g(s, Y_s^{T,\xi}, Z_s^{T,\xi})ds - \int_t^T Z_s^{T,\xi}dB_s - (K_T^{T,\xi} - K_t^{T,\xi}), \tag{3.1}$$

where g satisfies the following conditions:

(H1') There exists some $\beta > 2$ such that for any y, z, $g(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$;

(H2) There exists some $L > 0$ such that

$$|g(t, y, z) - g(t, y', z')| \leq L(|y - y'| + |z - z'|).$$

For each $\xi \in L_G^\beta(\Omega_T)$ with $\beta > 2$, we define

$$\hat{\mathbb{E}}_{t,T}^g[\xi] := Y_t^{T,\xi}.$$

Definition 3.1. *A process $\{Y_t\}_{t \in [0,T]}$ is called an $\hat{\mathbb{E}}^g$-supermartingale if, for each $t \leq T$, $Y_t \in L_G^\beta(\Omega_t)$ with $\beta > 2$ and $\hat{\mathbb{E}}_{s,t}^g[Y_t] \leq Y_s$, $\forall 0 \leq s \leq t \leq T$.*

Remark 3.2. (i)If $g = 0$, the $\hat{\mathbb{E}}^g$-supermartingale (Y_t) is in fact a G-supermartingale.
(ii)If the decreasing G-martingale (K_t) in (3.1) is replaced by a continuous decreasing process A with $A_0 = 0$, $\hat{\mathbb{E}}[A_T^2] < \infty$, then (Y_t) is called a g-supersolution under $\hat{\mathbb{E}}$ on $[0, T]$. It follows from the comparison theorem of G-BSDE that a g-supersolution under $\hat{\mathbb{E}}$ is also an $\hat{\mathbb{E}}^g$-supermaringale.
(iii)If there exists a generator f corresponding to the $d\langle B \rangle$ term in (3.1), we can define the operator $\hat{\mathbb{E}}_{t,T}^{g,f}[\cdot]$ and the associated $\hat{\mathbb{E}}^{g,f}$-supermartingales.

The following theorem, which is a main result of this paper, tells us that an $\hat{\mathbb{E}}^g$-supermartingale is also a g-supersolution under $\hat{\mathbb{E}}$. It generalizes the well-known decomposition theorem of Doob-Meyer's type to a framework of fully nonlinear expectation–G-expectation.

Theorem 3.3. *Let $Y = (Y_t)_{t \in [0,T]} \in S_G^\beta(0, T)$ be an $\hat{\mathbb{E}}^g$-supermartingale with $\beta > 2$. Suppose that g satisfies (H1') and (H2). Then (Y_t) has the following decomposition*

$$Y_t = Y_0 - \int_0^t g(s, Y_s, Z_s)ds + \int_0^t Z_s dB_s - A_t, \quad q.s., \tag{3.2}$$

where $\{Z_t\} \in M_G^2(0, T)$ and $\{A_t\}$ is a continuous nondecreasing process with $A_0 = 0$ and $A_T \in L_G^2(\Omega_T)$. Furthermore, the above decomposition is unique.

We divide the proof into a sequence of lemmas. For $P \in \mathcal{P}_M$, $\mathbb{F}$-stopping time τ, and $\mathcal{F}_\tau$-measurable random variable $\eta \in L^2(P)$, let (Y^P, Z^P) denote the solution to the following standard BSDE:

$$Y_s^P = \eta + \int_s^\tau g(r, Y_r^P, Z_r^P)dr - \int_s^\tau Z_r^P dB_r, \quad 0 \leq s \leq \tau, \quad P\text{-}a.s..$$

We recall from [29] that every $P \in \mathcal{P}_M$ satisfies the martingale representation property. Then there exists a unique adapted solution (Y^P, Z^P) of the above equation. We define $E_{t,\tau}^{g,P}[\eta] := Y_t^P$. For $P \in \mathcal{P}_M$ and $t \in [0, T]$, set $\mathcal{P}(t, P) := \{Q \in \mathcal{P}_M | Q|_{\mathcal{F}_t} = P|_{\mathcal{F}_t}\}$. The following lemma provides a representation for solution $Y^{T,\xi}$ of Equation (3.1).

Lemma 3.4 ([29]). *For each $\xi \in L_G^\beta(\Omega_T)$ with $\beta > 2$, we have, for $P \in \mathcal{P}_M$ and $t \in [0,T]$*

$$\hat{\mathbb{E}}_{t,T}^g[\xi] = \operatorname*{ess\,sup}_{Q \in \mathcal{P}(t,P)}{}^P E_{t,T}^{g,Q}[\xi], \ \ P\text{-a.s.},$$

where $\operatorname{ess\,sup}^P$ *means that the essential supremum is taken under the probability P.*

For reader's convenience, we give a brief proof here.

Proof. By the comparison theorem of classical BSDE, for $Q \in \mathcal{P}(t,P)$, we have $E_{t,T}^{g,Q}[\xi] \leq \hat{\mathbb{E}}_{t,T}^g[\xi]$, Q-a.s.. Consequently, we have $E_{t,T}^{g,Q}[\xi] \leq \hat{\mathbb{E}}_{t,T}^g[\xi]$, P-a.s.. Besides, by Theorem 16 in [13] (see also Proposition 3.4 in [28]) and noting that $(K_t^{T,\xi})$ is a decreasing G-martingale, we have

$$0 = \hat{\mathbb{E}}_t[K_T^{T,\xi} - K_t^{T,\xi}] = \operatorname*{ess\,sup}_{Q \in \overline{\mathcal{P}(t,P)}}{}^P E_t^Q[K_T^{T,\xi} - K_t^{T,\xi}], \ \ P\text{-a.s.},$$

where $\overline{\mathcal{P}(t,P)}$ is the closure of $\mathcal{P}(t,P)$ with respect to the weak topology. Then there exists $Q \in \overline{\mathcal{P}(t,P)}$, such that $E^Q[K_T^{T,\xi} - K_t^{T,\xi}] = 0$. Choose $\{Q_n\} \subset \mathcal{P}(t,P)$ such that $Q_n \to Q$ weakly, by Lemma 29 in [8], then we obtain

$$E^{Q_n}[|K_T^{T,\xi} - K_t^{T,\xi}|^{1+\alpha}] \leq \{E^{Q_n}[|K_T^{T,\xi} - K_t^{T,\xi}|^{\frac{1}{1-\alpha}}]\}^{1-\alpha}\{E^{Q_n}[|K_T^{T,\xi} - K_t^{T,\xi}|]\}^\alpha \to 0,$$

where $0 < \alpha < 1 - \frac{1}{\beta}$. By Proposition 3.2 in [2], we derive that

$$|\hat{\mathbb{E}}_{t,T}^g[\xi] - E_{t,T}^{g,Q_n}[\xi]|^{1+\alpha} \leq C_\alpha E_t^{Q_n}[|K_T^{T,\xi} - K_t^{T,\xi}|^{1+\alpha}], Q_n\text{-a.s.}.$$

Consequently, the above inequality holds P-a.s.. Then we have

$$E^P[|\hat{\mathbb{E}}_{t,T}^g[\xi] - E_{t,T}^{g,Q_n}[\xi]|^{1+\alpha}] \leq C_\alpha E^{Q_n}[|K_T^{T,\xi} - K_t^{T,\xi}|^{1+\alpha}] \to 0.$$

The proof is complete. $\qquad\square$

Lemma 3.5. *Let $Y = (Y_t)_{t \in [0,T]} \in S_G^\beta(0,T)$ be an $\hat{\mathbb{E}}^g$-supermartingale with $\beta > 2$. Suppose that g satisfies (H1') and (H2). For each $n = 1, 2, \cdots$, consider the following G-BSDEs:*

$$y_t^n = Y_T + \int_t^T g(s, y_s^n, z_s^n)ds + n\int_t^T (Y_s - y_s^n)ds - \int_t^T z_s^n dB_s - (K_T^n - K_t^n). \tag{3.3}$$

Then, for $n = 1, 2, \cdots$, $Y_t \geq y_t^n$, q.s..

Proof. Suppose the lemma were false. Then we could find some $t \in [0,T]$ and $P^* \in \mathcal{P}_M$ such that $P^*(y_t^n > Y_t) > 0$.

Applying Lemma 3.4 and the definition of $\hat{\mathbb{E}}^g$-supermartingales, we have for any $P \in \mathcal{P}_M$ and $s \leq t$,

$$E_{s,t}^{g,P}[Y_t] \leq \operatorname*{ess\,sup}_{P' \in \mathcal{P}(s,P)}{}^P E_{s,t}^{g,P'}[Y_t] = \hat{\mathbb{E}}_{s,t}^g[Y_t] \leq Y_s, P\text{-a.s.}.$$

This shows that, under the measure $P \in \mathcal{P}_M$, (Y_t) can be seen as an g_B-supermartingale in weak sense (see Remark 2.17). Since $(Y_t) \in S_G^\beta(0,T)$ is continuous, it is an g_B-supermartingale in strong sense. For any $Q \in \mathcal{P}(t, P^*)$, let $(\bar{Y}^Q, \bar{Z}^Q)$ denote the solution to the following standard BSDE:

$$\bar{Y}_s^Q = Y_T + \int_s^T g_n(r, \bar{Y}_r^Q, \bar{Z}_r^Q)dr - \int_s^T \bar{Z}_r^Q dB_r, \ \ Q\text{-a.s.}.$$

where $g_n(s, y, z) = f(s, y, z) + n(Y_s - y)$. Since (Y_t) is an g_B-supermartingale and g satisfies the assumptions in Lemma 2.16, then it is easy to check that $Y_t \geq E_{t,T}^{g_n,Q}[Y_T](= \bar{Y}_t^Q)$, Q-a.s.. By the definition of $\mathcal{P}(t, P^*)$, we obtain that $Y_t \geq E_{t,T}^{g_n,Q}[Y_T]$, P^*-a.s.. Again by Lemma 3.4, we have $\operatorname{ess\,sup}_{Q \in \mathcal{P}(t,P^*)}^{P^*} E_{t,T}^{g_n,Q}[Y_T] = y_t^n$, P^*-a.s.. This leads to a contradiction. $\quad\square$

It follows from the comparison theorem that $y_t^n \leq y_t^{n+1}$. Thus for all $n = 1, 2, \cdots$, $|y_t^n|$ is dominated by $|y_t^1| \vee |Y_t|$. Then we can find a constant C independent of n, such that for $1 < \alpha < \beta$, and for all $n = 1, 2, \cdots$,

$$\hat{\mathbb{E}}[\sup_{t \in [0,T]} |y_t^n|^\alpha] \leq \hat{\mathbb{E}}[\sup_{t \in [0,T]} (|y_t^1| \vee |Y_t|)^\alpha] \leq C. \tag{3.4}$$

Now let $L_t^n = n \int_0^t (Y_s - y_s^n) ds$, then $(L_t^n)_{t \in [0,T]}$ is an increasing process. We can rewrite G-BSDE (3.3) as

$$y_t^n = Y_T - \int_t^T z_s^n dB_s + \int_t^T g(s, y_s^n, z_s^n) ds - (K_T^n - K_t^n) + (L_T^n - L_t^n).$$

Lemma 3.6. *There exists a constant C independent of n, such that for $1 < \alpha < \beta$,*

$$\hat{\mathbb{E}}[(\int_0^T (z_t^n)^2 dt)^{\frac{\alpha}{2}}] \leq C, \ \hat{\mathbb{E}}[|K_T^n|^\alpha] \leq C, \ \hat{\mathbb{E}}[|L_T^n|^\alpha] = n^\alpha \hat{\mathbb{E}}[(\int_0^T |Y_s - y_s^n| ds)^\alpha] \leq C.$$

Proof. By a similar analysis as Proposition 3.5 in [10], we have

$$\hat{\mathbb{E}}[(\int_0^T |z_s^n|^2 ds)^{\frac{\alpha}{2}}] \leq C_\alpha \{\hat{\mathbb{E}}[\sup_{t \in [0,T]} |y_t^n|^\alpha] + (\hat{\mathbb{E}}[\sup_{t \in [0,T]} |y_t^n|^\alpha])^{\frac{1}{2}} (\hat{\mathbb{E}}[(\int_0^T |g(s,0,0)| ds)^\alpha])^{\frac{1}{2}}\},$$

$$\hat{\mathbb{E}}[|L_T^n - K_T^n|^\alpha] \leq C_\alpha \{\hat{\mathbb{E}}[\sup_{t \in [0,T]} |y_t^n|^\alpha] + \hat{\mathbb{E}}[(\int_0^T |g(s,0,0)| ds)^\alpha]\},$$

where the constant C_α depends on α, T, G and L. Thus we conclude that there exists a constant C independent of n, such that for $1 < \alpha < \beta$,

$$\hat{\mathbb{E}}[(\int_0^T |z_t^n|^2 dt)^{\frac{\alpha}{2}}] \leq C, \quad \hat{\mathbb{E}}[|L_T^n - K_T^n|^\alpha] \leq C.$$

Since L_T^n and $-K_T^n$ are nonnegative, we get

$$\hat{\mathbb{E}}[|K_T^n|^\alpha] \leq C, \quad \hat{\mathbb{E}}[|L_T^n|^\alpha] = n^\alpha \hat{\mathbb{E}}[(\int_0^T |Y_s - y_s^n| ds)^\alpha] \leq C. \qquad \square$$

For $1 < \alpha < \beta$, we obtain the following inequality.

$$n \hat{\mathbb{E}}[\int_0^T (Y_s - y_s^n)^\alpha ds]$$

$$\leq C n \hat{\mathbb{E}}[\int_0^T (Y_s - y_s^n)|Y_s|^{\alpha-1} ds] + C n \hat{\mathbb{E}}[\int_0^T (Y_s - y_s^n)|y_s^n|^{\alpha-1} ds]$$

$$\leq C n (\hat{\mathbb{E}}[\sup_{s \in [0,T]} |Y_s|^{(\alpha-1)p}])^{1/p} (\hat{\mathbb{E}}[(\int_0^T (Y_s - y_s^n) ds)^q])^{1/q}$$

$$+ C n (\hat{\mathbb{E}}[\sup_{s \in [0,T]} |y_s^n|^{(\alpha-1)p}])^{1/p} (\hat{\mathbb{E}}[(\int_0^T (Y_s - y_s^n) ds)^q])^{1/q},$$

where $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, $(\alpha - 1)p < \beta$ and $q < \beta$. By Estimate (3.4) and Lemma 3.6, there exists a constant C independent of n, such that

$$n \hat{\mathbb{E}}[\int_0^T (Y_s - y_s^n)^\alpha ds] \leq C.$$

This implies that y^n converges to Y in $M_G^\alpha(0, T)$. In fact, this convergence holds in $S_G^\alpha(0, T)$. In order to prove this conclusion, we need the following property on uniform continuity for any $Y \in S_G^p(0, T)$ with $p > 1$.

Lemma 3.7. *For $Y \in S_G^p(0,T)$ with $p > 1$, we have, by setting $Y_s := Y_T$ for $s > T$,*

$$F(Y) := \lim_{\varepsilon \to 0} \sup(\hat{\mathbb{E}}[\sup_{t \in [0,T]} \sup_{s \in [t,t+\varepsilon]} |Y_t - Y_s|^p])^{\frac{1}{p}} = 0.$$

Proof. For $Y \in S_G^0(0,T)$, the conclusion is obvious. Noting that for $Y, Y' \in S_G^p(0,T)$ we have

$$|F(Y) - F(Y')| \leq C \|Y - Y'\|_{S_G^p},$$

which implies that $F(Y) = 0$ for any $Y \in S_G^p(0,T)$. $\qquad\square$

Lemma 3.8. *For some $1 < \alpha < \beta$, we have*

$$\lim_{n \to \infty} \hat{\mathbb{E}}[\sup_{t \in [0,T]} |Y_t - y_t^n|^\alpha] = 0.$$

Proof. By applying G-Itô's formula to $e^{-nt} y_t^n$, we get

$$y_t^n = e^{nt} \hat{\mathbb{E}}_t[e^{-nT} Y_T + \int_t^T n e^{-ns} Y_s ds + \int_t^T e^{-ns} g(s, y_s^n, z_s^n) ds].$$

Then we obtain

$$0 \leq Y_t - y_t^n \leq \hat{\mathbb{E}}_t[\tilde{Y}_t^n - \int_t^T e^{n(t-s)} g(s, y_s^n, z_s^n) ds],$$

where $\tilde{Y}_t^n = e^{n(t-T)}(Y_t - Y_T) + \int_t^T n e^{n(t-s)}(Y_t - Y_s) ds$. By Hölder's inequality, it follows that

$$\begin{aligned}
|\int_t^T e^{n(t-s)} g(s, y_s^n, z_s^n) ds| &\leq \frac{1}{\sqrt{2n}} (\int_0^T g^2(s, y_s^n, z_s^n) ds)^{1/2} \\
&\leq \frac{C}{\sqrt{n}} (\sup_{s \in [0,T]} |y_s^n|^2 + \int_0^T (g^2(s,0,0) + |z_s^n|^2) ds)^{1/2}.
\end{aligned}$$

Then for $1 < \alpha < \beta$, we have

$$\hat{\mathbb{E}}[\sup_{t \in [0,T]} |\int_t^T e^{n(t-s)} g(s, y_s^n, z_s^n) ds|^\alpha] \to 0, \text{ as } n \to \infty. \tag{3.5}$$

For $\varepsilon > 0$, it is simple to show that

$$\begin{aligned}
|\tilde{Y}_t^n| &= |e^{n(t-T)}(Y_t - Y_T) + \int_{t+\varepsilon}^T n e^{n(t-s)}(Y_t - Y_s) ds + \int_t^{t+\varepsilon} n e^{n(t-s)}(Y_t - Y_s) ds| \\
&\leq e^{n(t-T)} |Y_t - Y_T| + e^{-n\varepsilon} \sup_{s \in [t+\varepsilon,T]} |Y_t - Y_s| + \sup_{s \in [t,t+\varepsilon]} |Y_s - Y_t|.
\end{aligned}$$

For $T > \delta > 0$, from the above inequality we obtain

$$\begin{aligned}
\sup_{t \in [0,T-\delta]} |\tilde{Y}_t^n| &\leq e^{-n\delta} \sup_{t \in [0,T-\delta]} |Y_t - Y_T| + \sup_{t \in [0,T-\delta]} \sup_{s \in [t,t+\varepsilon]} |Y_s - Y_t| \\
&\quad + e^{-n\varepsilon} \sup_{t \in [0,T-\delta]} \sup_{s \in [t+\varepsilon,T]} |Y_t - Y_s| \\
&\leq 2 \sup_{t \in [0,T]} |Y_t| (e^{-n\varepsilon} + e^{-n\delta}) + \sup_{t \in [0,T]} \sup_{s \in [t,t+\varepsilon]} |Y_s - Y_t|.
\end{aligned}$$

It is easy to check that for each fixed $\varepsilon, \delta > 0$,

$$\begin{aligned}
\hat{\mathbb{E}}[\sup_{t \in [0,T-\delta]} |\tilde{Y}_t^n|^\beta] &\leq C[(e^{-n\beta\varepsilon} + e^{-n\beta\delta}) \hat{\mathbb{E}}[\sup_{t \in [0,T]} |Y_t|^\beta] + \hat{\mathbb{E}}[\sup_{t \in [0,T]} \sup_{s \in [t,t+\varepsilon]} |Y_s - Y_t|^\beta]] \\
&\to C\hat{\mathbb{E}}[\sup_{t \in [0,T]} \sup_{s \in [t,t+\varepsilon]} |Y_s - Y_t|^\beta], \text{ as } n \to \infty.
\end{aligned} \tag{3.6}$$

For $1 < \alpha < \beta$ and $0 < \delta < T$, noting that $y_t^1 \le y_t^n \le Y_t$, $n = 1, 2, \cdots$, we have

$$\hat{\mathbb{E}}[\sup_{t \in [0,T]} |Y_t - y_t^n|^\alpha]$$

$$\le \hat{\mathbb{E}}[\sup_{t \in [0,T-\delta]} |Y_t - y_t^n|^\alpha] + \hat{\mathbb{E}}[\sup_{t \in [T-\delta,T]} |Y_t - y_t^n|^\alpha]$$

$$\le \hat{\mathbb{E}}[\sup_{t \in [0,T-\delta]} \hat{\mathbb{E}}_t[|\tilde{Y}_t^n - \int_t^T e^{n(t-s)} g(s, y_s^n, z_s^n) ds|^\alpha]] + \hat{\mathbb{E}}[\sup_{t \in [T-\delta,T]} |Y_t - y_t^1|^\alpha] \qquad (3.7)$$

$$\le \hat{\mathbb{E}}[\sup_{t \in [0,T-\delta]} \hat{\mathbb{E}}_t[\sup_{u \in [0,T-\delta]} |\tilde{Y}_u^n|^\alpha]] + \hat{\mathbb{E}}[\sup_{t \in [T-\delta,T]} |Y_t - y_t^1|^\alpha]$$

$$+ \hat{\mathbb{E}}[\sup_{t \in [0,T-\delta]} \hat{\mathbb{E}}_t[\sup_{u \in [0,T]} |\int_u^T e^{n(t-s)} g(s, y_s^n, z_s^n) ds|^\alpha]] =: I + II + III.$$

Theorem 2.8 and (3.5) yield that $III \to 0$, as $n \to \infty$. Note that $Y - y^1 \in S_G^\alpha(0, T)$ and $Y_T - y_T^1 = 0$. By Lemma 3.7, we obtain $II \to 0$, as $\delta \to 0$. Then by applying Theorem 2.8 agian and (3.6), we derive that

$$I \le C\{\hat{\mathbb{E}}[\sup_{t \in [0,T]} \sup_{s \in [t,t+\varepsilon]} |Y_s - Y_t|^\beta] + (\hat{\mathbb{E}}[\sup_{t \in [0,T]} \sup_{s \in [t,t+\varepsilon]} |Y_s - Y_t|^\beta])^{\alpha/\beta}\}, \text{ as } n \to \infty.$$

First let $n \to \infty$ and then send $\varepsilon, \delta \to 0$ in (3.7). The above analysis proves that for $1 < \alpha < \beta$,

$$\lim_{n \to \infty} \hat{\mathbb{E}}[\sup_{t \in [0,T]} |Y_t - y_t^n|^\alpha] = 0. \qquad \square$$

Lemma 3.9. *The sequence $\{y^n, z^n, A^n\}_{n=1}^\infty$ of the solutions of G-BSDE (3.3) satisfies the following properties:*

$$\lim_{m,n \to \infty} \hat{\mathbb{E}}[\sup_{t \in [0,T]} |y_t^n - y_t^m|^\alpha] = 0, \quad \text{for } 1 < \alpha < \beta, \qquad (3.8)$$

$$\lim_{m,n \to \infty} \hat{\mathbb{E}}[\int_0^T |z_s^m - z_s^n|^2 ds] = 0, \quad \lim_{m,n \to \infty} \hat{\mathbb{E}}[\sup_{t \in [0,T]} |A_t^m - A_t^n|^2] = 0, \qquad (3.9)$$

where we set $A_t^n = n \int_0^t (Y_s - y_s^n) ds - K_t^n$.

Proof. By Lemma 3.8, it is easy to check (3.8). Set $\hat{y}_t = y_t^n - y_t^m$, $\hat{z}_t = z_t^n - z_t^m$, $\hat{K}_t = K_t^n - K_t^m$, $\hat{L}_t = L_t^n - L_t^m$ and $\hat{g}_t = g(t, y_t^n, z_t^n) - g(t, y_t^m, z_t^m)$. Applying Itô's formula to $|\hat{y}_t|^2$, we get

$$|\hat{y}_t|^2 + \int_t^T |\hat{z}_s|^2 d\langle B \rangle_s$$

$$= \int_t^T 2\hat{y}_s \hat{g}_s ds - \int_t^T 2\hat{y}_s d\hat{K}_s + \int_t^T 2\hat{y}_s d\hat{L}_s - \int_t^T 2\hat{y}_s \hat{z}_s dB_s \qquad (3.10)$$

$$\le 2L \int_t^T |\hat{y}_s|^2 + |\hat{y}_s||\hat{z}_s| ds - \int_t^T 2\hat{y}_s d\hat{K}_s + \int_t^T 2\hat{y}_s d\hat{L}_s - \int_t^T 2\hat{y}_s \hat{z}_s dB_s.$$

Note that for each $\varepsilon > 0$,

$$2L \int_t^T |\hat{y}_s||\hat{z}_s| ds \le L^2/\varepsilon \int_t^T |\hat{y}_s|^2 ds + \varepsilon \int_t^T |\hat{z}_s|^2 ds.$$

By simple calculation, we have

$$\int_t^T \hat{y}_s d\hat{L}_s = \int_t^T (y_s^n - y_s^m)[n(Y_s - y_s^n) - m(Y_s - y_s^m)]ds$$
$$\leq \int_t^T (m+n)(Y_s - y_s^m)(Y_s - y_s^n)ds.$$

Choosing $\varepsilon < \underline{\sigma}^2$ and taking expectations on both sides of (3.10), we get

$$\hat{\mathbb{E}}[\int_0^T |\hat{z}_s|^2 ds] \leq C\hat{\mathbb{E}}[\int_0^T (m+n)(Y_s - y_s^n)(Y_s - y_s^m)ds + \int_0^T |\hat{y}_s|^2 ds - \int_0^T \hat{y}_s d\hat{K}_s]$$
$$\leq C\hat{\mathbb{E}}[\sup_{s\in[0,T]} |Y_s - y_s^n||L_T^m| + \sup_{s\in[0,T]} |Y_s - y_s^m||L_T^n|$$
$$+ \int_0^T |\hat{y}_s|^2 ds + \sup_{t\in[0,T]} |\hat{y}_s|(|K_T^n| + |K_T^m|)].$$

By Lemma 3.6 and Lemma 3.8, we obtain the first convergence of (3.9). For the second one, we observe that, for each n, the process A_t^n is nondecreasing in t, and

$$A_t^m - A_t^n = (y_0^m - y_0^n) - (y_t^m - y_t^n) + \int_0^t (z_s^m - z_s^n)dB_s - \int_0^t (g(s, y_s^m, z_s^m) - g(s, y_s^n, z_s^n))ds.$$

It follows from the generalized Burkholder-Davis-Gundy inequality in Proposition 2.10 and Hölder's inequality that

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |A_t^n - A_t^m|^2] \leq C(\hat{\mathbb{E}}[\sup_{t\in[0,T]} |y_t^n - y_t^m|^2] + \hat{\mathbb{E}}[\int_0^T (z_s^n - z_s^m)^2 ds]) \to 0. \qquad \square$$

We are now in the final position to prove Theorem 3.3:

Proof of Theorem 3.3. From (3.8) and (3.9), the sequences of $\{y^n\}_{n=1}^\infty$ converges to $Y \in S_G^\alpha(0,T)$, $\{z^n\}$ converges to a process $Z \in M_G^2(0,T)$ and $\{A^n\}$ converges to a nondecreasing process $A \in S_G^2(0,T)$. Thus we obtain the Decomposition (3.2) by letting $n \to \infty$ in (3.3).

To prove the uniqueness, let $Z, Z' \in M_G^2(0,T)$ and $A, A' \in S_G^2(0,T)$ be such that

$$Y_t = Y_0 - \int_0^t g(s, Y_s, Z_s)ds + \int_0^t Z_s dB_s - A_t = Y_0 - \int_0^t g(s, Y_s, Z_s')ds + \int_0^t Z_s' dB_s - A_t',$$

where A, A' are nondecreasing processes with $A_0 = A_0' = 0$. By applying Itô's formula to $(Y_t - Y_t)^2 (\equiv 0)$ on $[0,T]$ and taking expectation, we get

$$\hat{\mathbb{E}}[\int_0^T (Z_s - Z_s')^2 d\langle B\rangle_s] = 0.$$

Therefore $Z_t \equiv Z_t'$. From this it follows that $A_t \equiv A_t'$. $\qquad \square$

Remark 3.10. If $g = 0$, then the $\hat{\mathbb{E}}^g$-supermartingale $\{Y_t\}_{t\in[0,T]}$ is a G-supermartingale. Theorem 3.3 also holds for this special case which is in fact the Doob-Meyer decomposition theorem for G-supermartinales. The penalized G-BSDE is of the following form, $n = 1, 2, \cdots$,

$$y_t^n = Y_T + n\int_t^T (Y_s - y_s^n)ds - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n).$$

We can show Lemma 3.5 in a simple way. Since the above G-BSDE is linear, we can solve it explicitly by applying Itô's formula to $e^{-nt}y_t^n$,

$$e^{-nt}y_t^n + \int_t^T e^{-ns}Z_s^n dB_s + \int_t^T e^{-ns}dK_s^n = e^{-nT} + \int_t^T ne^{-ns}Y_s^n ds.$$

According to Lemma 3.4 in [10], $\{\int_0^t e^{-ns}dK_s^n\}_{t\in[0,T]}$ is a G-martingle. Thus we get

$$\begin{aligned}
y_t^n &= e^{nt}\hat{\mathbb{E}}_t[e^{-nT}Y_T + \int_t^T ne^{-ns}Y_s ds] \\
&\leq e^{n(t-T)}\hat{\mathbb{E}}_t[Y_T] + \int_t^T ne^{n(t-s)}\hat{\mathbb{E}}_t[Y_s]ds \\
&\leq e^{n(t-T)}Y_t + \int_t^T ne^{n(t-s)}Y_t ds = Y_t.
\end{aligned}$$

Furthermore, if g is a linear function, the proof is similar.

Remark 3.11. By Theorem 4.5 in [30] (see also Theorem 5.1 in [28]), for a G-martingale $X_t = \hat{\mathbb{E}}_t[\xi]$, $t \in [0, T]$, where $\xi \in L_G^\beta(\Omega_T)$ with $\beta > 1$, we have

$$X_t = X_0 + \int_0^t Z_s dB_s + K_t,$$

here $\{K_t\}$ is a decreasing G-martingale. Similar to the classical case, given a G-supermartingale Y, one may conjecture that

$$Y_t = Y_0 + \int_0^t Z_s dB_s + K_t - L_t, \tag{3.11}$$

where $\{K_t\}$ is a decreasing G-martingale and $\{L_t\}$ is a nondecreasing process with $L_0 = 0$. The problem is that the above representation is not unique unless $K \equiv 0$: $\widetilde{K} \equiv 0$, $\widetilde{L} = L - K$ is a different decomposition. That is why we put the increasing process $\{L_t - K_t\}$ as an integral.

It is worth pointing out that unlike with the classical case, considering the decomposition theorem for G-submartingales is fundamentally different from that for G-supermartingales. Indeed, if Y is a G-submartingale, $\{L_t\}$ in (3.11) should be a nonincreasing process. Therefore $\{L_t - K_t\}$ ends up with a finite variation process. Then this situation becomes much more complicated. We would like to refer the reader to [27] which defines a new norm for G-submartingales. As a byproduct, the decomposition is unique.

Then we establish the decomposition theorem for $\hat{\mathbb{E}}^{g,f}$-supermartingales.

Theorem 3.12. Let $Y = (Y_t)_{t\in[0,T]} \in S_G^\beta(0,T)$ be an $\hat{\mathbb{E}}^{g,f}$-supermartingale under with $\beta > 2$. Suppose that f and g satisfy (H1') and (H2). Then (Y_t) has the following decomposition

$$Y_t = Y_0 - \int_0^t g(s, Y_s, Z_s)ds - \int_0^t f(s, Y_s, Z_s)d\langle B\rangle_s + \int_0^t Z_s dB_s - A_t, \quad q.s., \tag{3.12}$$

where $\{Z_t\} \in M_G^2(0,T)$ and $\{A_t\}$ is a continuous nondecreasing process with $A_0 = 0$ and $A_T \in L_G^2(\Omega_T)$. Furthermore, the above decomposition is unique.

4 $\hat{\mathbb{E}}^g$-supermartingales and related PDEs

In this section, we present the relationship between the $\hat{\mathbb{E}}^g$-supermartingales and the fully nonlinear parabolic PDEs. For this purpose, we will put the $\hat{\mathbb{E}}^g$-supermartingales in a Markovian framework.

We will make the following assumptions throughout this section. Let b, h, σ : $[0,T] \times \mathbb{R} \to \mathbb{R}$ and $g : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ be deterministic functions and satisfy the following conditions:

(H4.1) b, h, σ, g are continuous in t;

(H4.2) There exists a constant $L > 0$, such that

$$|b(t,x) - b(t,x')| + |h(t,x) - h(t,x')| + |\sigma(t,x) - \sigma(t,x')| \leq L|x - x'|,$$
$$|g(t,x,y,z) - g(t,x',y',z')| \leq L(|x - x'| + |y - y'| + |z - z'|).$$

For each $t \in [0,T]$ and $\xi \in L^2_G(\Omega_t)$, we consider the following type of SDE driven by 1-dimensional G-Brownian motion:

$$dX^{t,\xi}_s = b(s, X^{t,\xi}_s)ds + h(s, X^{t,\xi}_s)d\langle B \rangle_s + \sigma(s, X^{t,\xi}_s)dB_s, \quad X^{t,\xi}_t = \xi. \tag{4.1}$$

We have the following estimates which can be found in Chapter V in [25].

Proposition 4.1. *Let $\xi, \xi' \in L^p_G(\Omega_t, \mathbb{R})$ with $p \geq 2$. Then we have, for each $\delta \in [0, T - t]$*

$$\hat{\mathbb{E}}_t[|X^{t,\xi}_{t+\delta} - X^{t,\xi'}_{t+\delta}|^p] \leq C|\xi - \xi'|^p,$$
$$\hat{\mathbb{E}}_t[\sup_{s \in [t,t+\delta]} |X^{t,\xi}_s - \xi|^p] \leq C(1 + |\xi|^p)\delta^{p/2},$$
$$\hat{\mathbb{E}}_t[\sup_{s \in [t,T]} |X^{t,\xi}_s|^p] \leq C(1 + |\xi|^p),$$

where the constant C depends on L, p, T and the function G. Consequently, for each $(x, y, z) \in \mathbb{R}^3$ and $p \geq 2$, we have $\{X^{t,x}_s\}_{s \in [t,T]}$, $\{g(s, X^{t,x}_s, y, z)\}_{s \in [t,T]} \in M^p_G(0, T)$.

Consider the following type of PDE:

$$\partial_t u + F(D^2_x u, D_x u, u, x, t) = 0, \tag{4.2}$$

where $G(a) = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$ and

$$F(D^2_x u, D_x u, u, x, t) = G(H(D^2_x u, D_x u, u, x, t)) + b(t,x)D_x u + g(t, x, u, \sigma(t,x)D_x u),$$
$$H(D^2_x u, D_x u, u, x, t) = \sigma^2(t,x)D^2_x u + 2h(t,x)D_x u.$$

Now we shall recall the definition of viscosity solution to Equation (4.2), which is introduced in [5]. Let $u \in C((0,T) \times \mathbb{R})$ and $(t,x) \in (0,T) \times \mathbb{R}$. Denote by $\mathcal{P}^{2,-}u(t,x)$ (the "parabolic subjet" of u at (t,x)) the set of triples $(a, p, X) \in \mathbb{R}^3$ such that

$$u(s,y) \geq u(t,x) + a(s - t) + p(y - x) + \frac{1}{2}X(y-x)^2 + o(|s - t| + |y - x|^2).$$

Similarly, we define $\mathcal{P}^{2,+}u(t,x)$ (the "parabolic superjet" of u at (t,x)) by $\mathcal{P}^{2,+}u(t,x) := -\mathcal{P}^{2,-}(-u)(t,x)$.

Definition 4.2. *$u \in C((0,T) \times \mathbb{R})$ is called a viscosity supersolution (resp. subsolution) of (4.2) on $(0,T) \times \mathbb{R}$ if at any point $(t,x) \in (0,T) \times \mathbb{R}$, for any $(a, p, X) \in \mathcal{P}^{2,-}u(t,x)$ (resp. $\in \mathcal{P}^{2,+}u(t,x)$)*

$$a + F(X, p, u, x, t) \leq 0 \ (resp. \geq 0).$$

$u \in C((0,T) \times \mathbb{R})$ is said to be a viscosity solution of (4.2) if it is both a viscosity supersolution and a viscosity subsolution.

Remark 4.3. We then give the following equivalent definition (see [5]). $u \in C((0, T) \times \mathbb{R})$ is called a viscosity supersolution (resp. subsolution) of (4.2) on $(0, T) \times \mathbb{R}$ if for each fixed $(t, x) \in (0, T) \times \mathbb{R}$, $v \in C^{1,2}((0, T) \times \mathbb{R})$ such that $u(t, x) = v(t, x)$ and $v \leq u$ (resp. $v \geq u$) on $(0, T) \times \mathbb{R}$, we have

$$\partial_t v(t, x) + F(D_x^2 v(t, x), D_x v(t, x), v(t, x), x, t) \leq 0 \ (resp. \geq 0).$$

We state the main result of this section.

Theorem 4.4. *Assume (H4.1) and (H4.2) hold. Let $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ be uniformly continuous with respect to (t, x) and satisfy*

$$|u(t, x)| \leq C(1 + |x|^k), \ \ t \in [0, T], x \in \mathbb{R},$$

where k is a positive integer. Then u is a viscosity supersolution of Equation (4.2), if and only if $\{Y_s^{t,x}\}_{s \in [t,T]} := \{u(s, X_s^{t,x})\}_{s \in [t,T]}$ is an $\hat{\mathbb{E}}^{g^{t,x}}$-supermartingale, for each fixed $(t, x) \in (0, T) \times \mathbb{R}$, where $g^{t,x} = g(s, X_s^{t,x}, y, z)$ and $\{X_s^{t,x}\}_{s \in [t,T]}$ is given by (4.1).

To prove this theorem, we introduce the following lemma.

Lemma 4.5. *We have, for each $p > 2$ and $(t, x) \in [0, T) \times \mathbb{R}$, $\{Y_s^{t,x}\}_{s \in [t,T]} \in S_G^p(0, T)$.*

Proof. Note that

$$\sup_{s \in [t,T]} |Y_s^{t,x}|^p \leq C \sup_{s \in [t,T]} (1 + |X_s^{t,x}|^{kp}).$$

By Proposition 4.1, we have $\hat{\mathbb{E}}[\sup_{s \in [t,T]} |Y_s^{t,x}|^p] < \infty$. Since u is uniformly continuous, we get the desired result. $\qquad\square$

Proof of Theorem 4.4. For a given function u satisfying the conditions in Theorem 4.4 and for each $n = 1, 2, \cdots, (t, x) \in [0, T] \times \mathbb{R}$, let us consider the following G-BSDEs:

$$y_s^{n,t,x} = Y_T^{t,x} + \int_s^T g(r, X_r^{t,x}, y_r^{n,t,x}, z_r^{n,t,x})dr + n \int_s^T (Y_r^{t,x} - y_r^{n,t,x})dr$$
$$- \int_s^T z_r^{n,t,x}dB_r - (K_T^{n,t,x} - K_s^{n,t,x}),$$

and, correspondingly, the following viscosity solution of PDEs:

$$\partial_t v^n(t, x) + F(D_x^2 v^n(t, x), D_x v^n(t, x), v^n(t, x), x, t) + n(u(t, x) - v^n(t, x)) = 0,$$

defined on $(0, T) \times \mathbb{R}$ with the Cauchy condition

$$v^n(T, x) = u(T, x).$$

From the nonlinear Feynman-Kac formula obtained in [11] (i.e., Theorem 4.5 in [11]), it follows that $y_s^{n,t,x} = v^n(s, X_s^{t,x})$, $s \in [t, T]$.

To prove the "if" part of the Theorem, we assume that, for each (t, x), $\{Y_\cdot^{t,x}\}$ is an $\hat{\mathbb{E}}^{g^{t,x}}$-supermartingale on $[t, T]$. Observing that $(y_s^{n,t,x}, z_s^{n,t,x}, K_s^{n,t,x})_{s \in [t,T]}$ is a special case of (3.3), we can apply Lemma 3.5 and Lemma 3.9 to prove that $y_s^{n,t,x} \leq Y_s^{t,x}$ and then to get the convergence of $\{y_\cdot^{n,t,x}\}$ to $\{Y_\cdot^{t,x}\}$ on $[t, T]$, similar to (3.8). By the proof of Theorem 3.3, for any $(t, x) \in [0, T] \times \mathbb{R}$, we have

$$v^n(s, X_s^{t,x}) = y_s^{n,t,x} \leq Y_s^{t,x} = u(s, X_s^{t,x}),$$

and $v^n \uparrow u$. Since u is uniformly continuous on $[0, T] \times \mathbb{R}$, the convergence is also locally uniform. By Theorem 4.5 in [11] and noting that $v^n \leq u$, v^n is a viscosity supersolution of

PDE (4.2). It follows from the stability theorem of the viscosity solutions (see Proposition 4.3 in [5]) that the limit function u is also a viscosity supersolution of PDE (4.2).

Now we prove the "only if" part of the Theorem. For each $t_1 \in (0, T)$, let $v^{t_1, u(t_1, \cdot)}$ be the viscosity solution of PDE (4.2) on $(0, t_1) \times \mathbb{R}$ with Cauchy condition $v^{t_1, u(t_1, \cdot)}(t_1, x) = u(t_1, x)$. By the comparison theorem for viscosity solutions, for each $(s, x) \in [0, t_1] \times \mathbb{R}$, it is easy to check that $v^{t_1, u(t_1, \cdot)}(s, x) \le u(s, x)$. For any $t \le s \le r \le T$, by the nonlinear Feymann-Kac formula in [11], we have

$$
\begin{aligned}
\hat{\mathbb{E}}_{s,r}^{g^{t,x}}[Y_r^{t,x}] &= \hat{\mathbb{E}}_{s,r}^{g^{t,x}}[u(r, X_r^{t,x})] = \hat{\mathbb{E}}_{s,r}^{g^{t,x}}[v^{r, u(r, \cdot)}(r, X_r^{t,x})] \\
&= v^{r, u(r, \cdot)}(s, X_s^{t,x}) \le u(s, X_s^{t,x}) = Y_s^{t,x},
\end{aligned}
\tag{4.3}
$$

which implies that $\{Y_s^{t,x}\}_{s \in [t,T]} := \{u(s, X_s^{t,x})\}_{s \in [t,T]}$ is an $\hat{\mathbb{E}}^{g^{t,x}}$-supermartingale. The proof is complete. □

Remark 4.6. It is worth pointing out that under the Assumptions (H4.1) and (H4.2), the PDE (4.2) has at most one viscosity solution in the class of continuous functions satisfying polynomial growth at infinity.

The following result can be considered as the "inverse" comparison theorem for viscosity solutions of PDEs.

Corollary 4.7. *Let $V : [0, T] \times \mathbb{R} \to \mathbb{R}$ be uniformly continuous with respect to (t, x) and satisfy*

$$
|V(t, x)| \le C(1 + |x|^k), \quad t \in [0, T], x \in \mathbb{R},
$$

where k is a positive integer. Assume that

$$
V(t, x) \ge u^{t_1, V(t_1, \cdot)}(t, x), \quad \forall (t, x) \in [0, t_1] \times \mathbb{R}, \ t_1 \in [0, T],
$$

where $u^{t_1, V(t_1, \cdot)}$ denotes the viscosity solution of PDE (4.2) on $(0, t_1) \times \mathbb{R}$ with Cauchy condition $u^{t_1, V(t_1, \cdot)}(t_1, x) = V(t_1, x)$. Then V is a viscosity supersolution of PDE (4.2) on $(0, T) \times \mathbb{R}$.

Proof. For each fixed (t, x), set $\{Y_s^{t,x}\}_{s \in [t,T]} := \{V(s, X_s^{t,x})\}_{s \in [t,T]}$. Similar with (4.3), $\{Y_s^{t,x}\}_{s \in [t,T]}$ is an $\hat{\mathbb{E}}^{g^{t,x}}$-supermartingale. It follows from Theorem 4.4 that V is a viscosity supersolution of PDE (4.2). □

Conclusion

We obtain the decomposition theorem of Doob-Meyer's type for $\hat{\mathbb{E}}^g$-supermartingales, which is a generalization of the results of Peng [20]. Our theorem provides the first step for solving the representation theorem of dynamically consistent nonlinear expectations. Different from the classical case, the decomposition theorem for $\hat{\mathbb{E}}^g$-submartingales remains open.

References

[1] Bensoussan, A.: On the theory of option pricing. *Acta. Appl. Math.* **2**, (1984), 139–158. MR-0748007

[2] Briand, P., Delyon, B., Hu, Y., Pardoux, E. and Stoica, L.: L^p solutions of backward stochastic differential equations. *Stochastic Processes and their Applications* **108**, (2003), 109–129. MR-2008603

[3] Chen, Z. and Peng, S.: Continuous properties of g-martingales. *Chin. Ann. of Math.* **22**B, (2001), 115–128. MR-1823135

[4] Coquet, F., Hu, Y., Mémin, J. and Peng, S.: Filtration-consistent nonlinear expectations and related g-expectations. *Probab. Theory Relat. Fields* **123**, (2002), 1–27. MR-1906435

[5] Crandall, M. G., Ishii, H. and Lions, P. L.: User's guide to viscosity solutions of second order partial differential equations. *Bulletin of The American Mathematical Society* **27**(1), (1992), 1–67. MR-1118699

[6] Cvitanic, J. and Karatzas, I.: Hedging contingent claims with constrained portfolios. *Ann. of Appl. Proba.* **3**(3), (1993), 652–681. MR-1233619

[7] Delbaen, F., Peng, S. and Rosazza, G. E.: Representation of the penalty term of dynamic concave utilities. *Finance and Stochastics* **14**, (2010), 449–472. MR-2670421

[8] Denis, L., Hu, M. and Peng S.: Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion pathes. *Potential Anal.* **34**, (2011), 139–161. MR-2754968

[9] Doob, L.: Stochastic Process. *John Wiley & Sons*, New York, NY, USA, (1953).

[10] Hu, M., Ji, S., Peng, S. and Song, Y.: Backward stochastic differential equations driven by G-Brownian motion. *Stochastic Processes and their Applications* **124**, (2014), 759–784. MR-3131313

[11] Hu, M., Ji, S., Peng, S. and Song, Y.: Comparison theorem, Feynman-Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion. *Stochastic Processes and their Applications* **124**, (2014), 1170–1195. MR-3138611

[12] Hu, M. and Peng, S.: On representation theorem of G-expectations and paths of G-Brownian motion. *Acta Math. Appl. Sin. Engl. Ser.* **25**(3), (2009), 539–546. MR-2506990

[13] Hu, M. and Peng, S.: Extended conditional G-expectations and related stopping times, arXiv:1309.3829v1

[14] Karatzas, I.: On the pricing of American options. *Applied Mathematics and Optimization* **17**, (1988), 37–60. MR-0908938

[15] Li, X. and Peng, S.: Stopping times and related Itô's calculus with G-Brownian motion. *Stochastic Processes and their Applications* **121**, (2011), 1492–1508. MR-2802462

[16] Meyer, P. A.: A decomposition theorem for supermartingales. *Illinois J. Math.* **6**, (1962), 193–205. MR-0159359

[17] Meyer, P. A.: Decomposition for supermartingales: the uniqueness theorem. *Illinois J. Math.* **7**, (1963), 1–17. MR-0144382

[18] Pardoux, E. and Peng, S.: Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14**, (1990), 55–61. MR-1037747

[19] Peng, S.: BSDE and related g-expectations. In El Karoui, N., Mazliak, L. (eds.) Backward Stochastic Differential Equations, No. 364 in Pitman Research Notes in Mathematics Series, *Addison Wesley*, Longman, London, (1997), 141–159. MR-1752680

[20] Peng, S.: Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer's type. *Probab. Theory Relat. Fields.* **113**, (1999), 473–499. MR-1717527

[21] Peng, S.: Nonlinear expectations, nonlinear evaluations and risk measures, in: Lectures Notes in CIME-EMS Summer School, 2003, Bressanone, in: Springer's Lecture Notes in Mathematics, vol. 1856. MR-2113723

[22] Peng, S.: Dynamically consistent nonlinear evaluations and expectations, arXiv:math/0501415v1

[23] Peng, S.: G-expectation, G-Brownian Motion and Related Stochastic Calculus of Itô type. *Stochastic analysis and applications* (2007) 541–567, Abel Symp., 2, Springer, Berlin. MR-2397805

[24] Peng, S.: Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation. *Stochastic Processes and their Applications* **118**(12), (2008), 2223–2253. MR-2474349

[25] Peng, S.: Nonlinear expectations and stochastic calculus under uncertainty, arXiv:1002.4546v1

[26] Peng, S., Song, Y. and Zhang, J. A complete representation theorem for G-martingales. *Stochastics* **86**(4), (2014), 609–631. MR-3230070

[27] Pham, T. and Zhang, J.: Some norm estimates for semimaringales. *Electron. J. Probab.* **18**, (2013), 1–25. MR-3151729

[28] Soner, M., Touzi, N. and Zhang, J.: Martingale representation theorem under G-expectation. *Stochastic Processes and Their Applications.* **121**, (2011), 265–287. MR-2746175

[29] Soner, M., Touzi, N. and Zhang, J.: Wellposedness of second order backward SDEs. *Probab. Theory Related Fields* **153**, (2012), 149–190. MR-2925572

[30] Song, Y.: Some properties on G-evaluation and its applications to G-martingale decomposition. *Science China Mathematics* **54**, (2011), 287–300. MR-2771205

[31] Song, Y.: Uniqueness of the representation for G-martingales with finite variation. *Electron. J. Probab* **17**, (2012), 1-15. MR-2900465