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Recurrence and transience of contractive autoregressive processes and related Markov chains

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Dedicated to the memory of Prof. Dr. Dr. h.c. Herbert Heyer (1936–2018)

Abstract

We characterize recurrence and transience of nonnegative multivariate autoregressive processes of order one with random contractive coefficient matrix, of subcritical multitype Galton-Watson branching processes in random environment with immigration, and of the related max-autoregressive processes and general random exchange processes. Our criterion is given in terms of the maximal Lyapunov exponent of the coefficient matrix and the cumulative distribution function of the innovation/immigration component.

Keywords: autoregressive process; branching process; excited random walk; frog process; immigration; Lyapunov exponent; max-autoregressive process; product of random matrices; random affine recursion; random difference equation; random environment; random exchange process; recurrence; super-heavy tail; transience.
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1 Introduction

The classification of irreducible Markov chains as recurrent or transient is one of the fundamental objectives in the study of Markov chains. Scalar *nonnegative autoregressive processes* $(X_n)_{n\in\mathbb{N}_0}$ of the form

$$X_n = aX_{n-1} + Y_n, \qquad \text{where } 0 < a < 1 \text{ and } (Y_n)_{n\in\mathbb{N}} \text{ is i.i.d.},$$

and, closely related, *subcritical Galton-Watson processes* $(Z_n)_{n\in\mathbb{N}_0}$ *with immigration* $(Y_n)_{n\in\mathbb{N}}$ and average offspring $a \in (0, 1)$ are classical Markov chains. The study of these processes has a rich history which started more than half a century ago. However, most of the literature on these processes deals only with the positive recurrent case, i.e. the case where there exists a stationary probability distribution. To the best of our

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knowledge there is at present no complete classification in simple terms of recurrence versus transience of these processes although this problem has been investigated for several decades, see [30], [31], [21, Part I], [15, p. 1196], [39], [4], and the review below.

In the present article we characterize recurrence and transience of these processes in terms of a and the cumulative distribution function of Y_1. More precisely, we show that either process is

$$\text{recurrent} \qquad \text{iff} \qquad \sum_{n\geq 0}\prod_{m=0}^{n} P[Y_1 \leq ya^{-m}] = \infty \tag{1.1}$$

for some $y \in (0,\infty)$, see Theorem 3.1. (For the branching process we need to assume a certain moment condition on the offspring distribution.) Note that the divergence or convergence of the series in (1.1) can often be easily checked by ratio tests.

We also extend this result to certain multidimensional cases in random environment by classifying nonnegative multivariate autoregressive processes of order one with random contractive coefficient matrix and subcritical multitype Galton-Watson processes in random environment with immigration, see Theorem 4.2. The same criterion also applies to two other related processes, sometimes called *max-autoregressive process* and *general random exchange process*.

We first introduce these four processes and review the existing literature on the subject. (The precise definition of recurrence and transience is given in the next section.)

Autoregressive processes. Autoregressive models are among the most widely used stochastic models, see e.g. [23], [7], [29], [8]. We consider nonnegative multidimensional autoregressive processes $X = (X_n)_{n\geq 0}$ of order one (AR(1) processes) with random coefficient matrix, defined as follows. Fix a dimension $d \in \mathbb{N}$. Let $Y = (Y_n)_{n\geq 0}$ be a sequence of $[0,\infty)^d$-valued random vectors, called *innovations*, and let $(A_n)_{n\geq 1}$ be a sequence of $[0,\infty)^{d\times d}$-valued random matrices. Assume that $(A_n, Y_n)_{n\geq 1}$ is i.i.d. and independent of Y_0. To avoid cases which are not interesting in the present context we suppose that the support of the law of Y_1 is unbounded. Set $X_0 := Y_0$ and

$$X_n := A_n X_{n-1} + Y_n \qquad \text{for } n \geq 1. \tag{1.2}$$

Relation (1.2) is sometimes called a *random difference equation* or *random affine recursion*, see also [8, p. 1]. Solving this recursion we obtain the explicit expression

$$X_n = \sum_{m=0}^{n} A_n A_{n-1} \dots A_{m+1} Y_m \qquad \text{for } n \geq 0. \tag{1.3}$$

We only consider the subcritical (contractive) case where the maximal Lyapunov exponent of A_n is strictly less than 0. For convenience we phrase our statements in terms of the negative λ of the Lyapunov exponent, defined as

$$\lambda := \sup_{n\geq 1} \frac{E[S_n]}{n}, \quad \text{where} \quad S_n := -\ln \|A_1 \dots A_n\|, \tag{1.4}$$

see also (4.1) for alternative expressions for λ. It has been shown for the subcritical case ($\lambda > 0$) that under various conditions X is positive recurrent iff

$$E[\ln_+ \|Y_1\|] < \infty, \tag{1.5}$$

see e.g. [38, Theorem 1.6 (b)], [21, Part III, Theorem (8.5)], [15, Corollary 4.1 (b)], [8, Theorem 2.1.3] for $d = 1$ and [39, Proposition 2] for $d \geq 1$ and constant coefficient matrix $A = A_n$. For the multidimensional case with random A_n see [6] and [11]. (Note that the equivalence of positive recurrence and the existence of an invariant probability measure, which is well-known for countable state spaces, also holds in this setting, see e.g. [22, Section 6].) The case where (1.5) fails is sometimes referred to as *super-heavy tailed*, see [39].

Among the few works which deal with recurrence and transience of AR(1) processes are [39], [4], and the unpublished preprint [21, Part I]. All three articles deal with the one-dimensional case and both [21, Part I] and [39] consider only the case of constant coefficients $A_1 = a \in (0,1)$. In [21, Part I, Theorem (3.1)] Kellerer shows that X is

$$\text{transient} \quad \text{if} \quad \liminf_{t\to\infty} t \cdot P[\ln Y_1 > t] > -\ln a \quad \text{and} \tag{1.6}$$

$$\text{recurrent} \quad \text{if} \quad \limsup_{t\to\infty} t \cdot P[\ln Y_1 > t] < -\ln a. \tag{1.7}$$

Note that (1.6) and (1.7) follow from (1.1) and Raabe's test.

In [39, Theorem 1] Zeevi and Glynn consider log-Pareto distributed innovations Y_n, whose common distribution is given by $P[\ln(1+Y_1) > t] = (1+\beta t)^{-p}$ for some $\beta > 0$ and $p > 0$. For this case they completely characterize recurrence and transience by showing that X is positive recurrent if $p > 1$, null recurrent if $p = 1$ and $\beta \ln(1/a) \geq 1$, and transient otherwise. Note that in this example the distinction between recurrence and transience also follows from Kellerer's result (1.6) and (1.7) except in the critical case $p = 1, \beta \ln(1/a) = 1$. In the critical case the result follows from (1.1) and Raabe's test. Moreover, in [39, Lemma 1] the authors provide a (strictly) sufficient condition for recurrence in terms of the divergence of an integral, which resembles our criterion (1.1).

In the proof of [4, Theorem 2.2, pp. 642, 643] Bauernschubert shows that Kellerer's transience criterion (1.6) can be extended to the case of random coefficients by proving that under some moment conditions X is

$$\text{transient} \quad \text{if} \quad \liminf_{t\to\infty} t \cdot P[\ln Y_1 > t] > -E[\ln A_1]. \tag{1.8}$$

We postpone the discussion of the recent work [2], which appeared as a preprint several months after the present work, to Appendix B, where we also comment on assumption (BA) of our second main result, Theorem 4.2.

Branching processes with immigration. The classical Galton-Watson model as a basic model for branching populations, see e.g. [16] and [3], has been extended in various directions, for example by allowing finitely many different types of individuals with different offspring distributions [3, Chapter V], by letting the offspring distribution depend on time in a random way [3, Chapter VI.5], or by allowing immigration [3, Chapter VI.7]. Following e.g. [24], [35], and [36], we consider a combination of these three generalizations, namely multitype Galton-Watson branching processes $Z = (Z_n)_{n\geq 0}$ in random environment with immigration.

We postpone the precise definition of Z to the next section and first give an informal description of the model. Let $d \in \mathbb{N}$ and $(Y_n)_{n\geq 0}$ be as above and let us assume for the moment that all Y_n are $\mathbb{N}_0^d$-valued. There are d different types of individuals, enumerated by $1, \ldots, d$. The i-th component of the $\mathbb{N}_0^d$-valued random variable Z_n is the number of individuals of type i present in generation n. Given Z_{n-1}, the n-th generation is obtained as follows. Each member of generation $n-1$ gets independently of the other members of that generation a random number of children of the d different types. The distribution of the number of children of a certain type may depend on the type of the parent. It may also depend in an i.i.d. way, called the random environment, on the number $n-1$ of the generation. The n-th generation consists of the children of the individuals of the previous generation and additional immigrants of type $1, \ldots, d$, whose numbers are given by Y_n.

This process Z is closely related to AR(1) processes in the following way. Define the (i,j)-th entry of the matrix A_n as the conditional expectation of the number of children of type i in generation n of a parent of type j given the random environment. Then the conditional expected value of Z given the random environment and given the numbers of immigrants satisfies the recursion (1.2) and is therefore an AR(1) process, see (2.3) below.

The process Z is called subcritical iff $\lambda > 0$, where λ is defined as in (1.4). As for AR(1) processes, positive recurrence of a subcritical Z is related to the validity of (1.5), see e.g. [33], [12, Corollary 2], [31, Theorem A], [24, Theorem 3.3], [35], and [36].

In the one-dimensional case the results in the literature concerning the distinction between recurrence and transience of Z are more complete than the corresponding results for autoregressive processes. Pakes considers in [30] and [31] subcritical single-type processes with immigration in an environment which is constant in time. He gives several sufficient conditions for recurrence or transience in terms of generating functions and provides several examples.

For subcritical single-type branching processes in random environment Bauernschubert shows in [4, Theorem 2.2] that (1.8) also holds for Z and derives in [4, Theorem 2.3] the analogous condition for recurrence by showing that under suitable assumptions Z is

$$\text{recurrent} \quad \text{if} \quad \limsup_{t\to\infty} t \cdot P[\ln Y_1 > t] < -E[\ln A_1]. \tag{1.9}$$

For a different but similar model in continuous time, Li, Chen, and Pakes [26, Theorem 3.3 (ii)] give a necessary and sufficient criterion for recurrence and transience in terms of generating functions. Unfortunately, “it is not easily applicable in specific cases” [26, p. 136]. This raises the question whether a modification of our criterion (1.1) also holds for that model.

Max-autoregressive processes. By replacing the sum in (1.2) with the maximum we obtain the process $M = (M_n)_{n\geq 0}$ defined by $M_0 := Y_0$ and

$$M_n := \max\{A_n M_{n-1}, Y_n\}, \quad n \geq 1. \tag{1.10}$$

Here the maximum is taken for each coordinate of $\mathbb{R}^d$ separately. Such processes have been studied e.g. in [14] and [34] and are sometimes called *max-autoregressive*. They appear naturally in our proof. If $d = 1$ then similarly to (1.3), $M_n = \max_{m=0}^n A_n \dots A_{m+1} Y_m$ for all $n \geq 0$. For general dimension $d \geq 1$ we have

$$X_n \geq M_n \geq N_n := \max_{m=0}^{n} A_n \dots A_{m+1} Y_m \tag{1.11}$$

componentwise. We are not aware of any results in the literature on the classification of recurrence versus transience of max-autoregressive processes.

General random exchange processes. These are one-dimensional processes $R = (R_n)_{n\geq 0}$ which have been studied e.g. in [17] and are defined as follows. Let $(W_n)_{n\geq 0}$ be a sequence of nonnegative random variables with unbounded support and let $(T_n)_{n\geq 1}$ be a sequence of real-valued random variables such that $(T_n, W_n)_{n\geq 1}$ is i.i.d. and independent of W_0. Set $R_0 := W_0$ and

$$R_n := \max\{R_{n-1} - T_n, W_n\}, \quad n \geq 1. \tag{1.12}$$

The starting point of our investigation were the recurrence/transience conditions given by Lamperti and Kesten in [25] in the special case where T_n is a positive constant (random exchange process). Their results were phrased in terms of long range percolation. While Lamperti derived the counterparts of (1.6) and (1.7), Kesten gave in the appendix to [25] a necessary and sufficient criterion. Later Kesten’s criterion was stated in a more general form in terms of Markov chains by Kellerer [22, pp. 268, 269].

Proposition 1.1 (Random exchange process; Kesten, Kellerer)**.** *Let $W_n, n \geq 1$, be i.i.d. $\mathbb{N}_0$-valued random variables satisfying $P[W_1 = 0] > 0$. Then the state 0 is recurrent for the Markov chain R satisfying $R_n := \max\{R_{n-1} - 1, W_n\}$ if and only if*

$$\sum_{n \geq 0} \prod_{m=0}^{n} P[W_1 \leq m] = \infty. \tag{1.13}$$

Proof. It is well-known, see e.g. [9, Theorem 6.4.2], that 0 is recurrent iff $\sum_{n \geq 1} P[R_n = 0] = \infty$. Solving the recursion with initial state $R_0 = 0$ yields $R_n = \max_{m=1}^{n}(W_m - n + m)$ for all $n \geq 1$. Since $(W_n)_{n \geq 1}$ is i.i.d. we have for all $n \geq 1$,

$$P[R_n = 0] = \prod_{m=1}^{n} P[W_m \leq n - m] = \prod_{m=0}^{n-1} P[W_1 \leq m].$$

The claim follows. ☐

The significance of Proposition 1.1 in the present context is that on a heuristic level one can easily deduce from it in several steps the recurrence/transience criterion for our processes X, Z and M introduced above. However, our actual proof will not follow these steps.

Step 1. If R satisfies only the more general recursion $R_n = \max\{R_{n-1} - c, W_n\}$ for some constant $c \in \mathbb{N}$ then the event $\{W_1 \leq m\}$ in (1.13) has to be replaced by $\{W_1 \leq mc\}$.

Step 2. If we do not require the minimum y of the support of W_1 to be 0 then the event $\{W_1 \leq mc\}$ in Step 1 has to be replaced by $\{W_1 \leq y + mc\}$. In fact, one may choose any y satisfying $P[W_1 \leq y] > 0$.

Step 3. It is easy to guess but much harder to prove that for the general random exchange process satisfying (1.12) and $E[T_1] > 0$ the event $\{W_1 \leq y + mc\}$ from Step 2 has to be replaced by $\{W_1 \leq y + mE[T_1]\}$, see Corollary 4.4 below.

Step 4. The process e^R is a one-dimensional max-autoregressive process M which satisfies the recursion $M_n = \max\{A_n M_{n-1}, Y_n\}$ with $A_n = e^{-T_n}$ and $Y_n = e^{W_n}$. It follows from Step 3 that if y is such that $P[Y_1 \leq y] > 0$ then M should be recurrent iff $\sum_{n \geq 0} \prod_{m=0}^{n} P[Y_1 \leq y e^{-mE[\ln A_1]}]$ is infinite.

Step 5. By the strong law of large numbers, $\lambda = -E[\ln A_1]$ if $d = 1$. Thus for multi-dimensional max-autoregressive processes one should replace $-E[\ln A_1]$ in Step 4 by λ and get that for all y satisfying $P[\|Y_1\| \leq y] > 0$, M is recurrent iff

$$\sum_{n \geq 0} \prod_{m=0}^{n} P[\|Y_1\| \leq y e^{m\lambda}] = \infty. \tag{RR}$$

Step 6. It is a well-known phenomenon (max-sum-equivalence) that the sum of heavy tailed random variables tends to be comparable to the largest summand. Thus one might expect that the recurrence criterion for M derived in Step 5 also applies to X and, due to the relation between X and Z described above, to Z as well.

That the conclusion of this heuristics is indeed true under suitable conditions is the content of our main results, Theorem 3.1 and Theorem 4.2.

Remark 1.2. Since positive recurrence implies recurrence, (1.5) should imply (RR). This implication can easily be derived directly as follows. Note that (1.5) is equivalent to $\sum_{m \geq 0} P\left[\|Y_1\| > y e^{m\lambda}\right] < \infty$. This in turn is equivalent to $\prod_{m \geq 0} P\left[\|Y_1\| \leq y e^{m\lambda}\right] > 0$, which implies (RR).

Let us now describe how the remainder of the present article is organized. In the next section we provide additional definitions and collect some elementary statements. In Section 3 we first treat the special case of constant, deterministic environments because in this case our proof is shorter and requires weaker assumptions than in the genuinely random case. We also provide an application to so-called frog processes on the integers. The general case of random environments is dealt with in Section 4, where we also give an application to random walks in random environments perturbed by cookies. In Appendix A we collect some general bounds which we need for the multidimensional case with random environment. Appendix B comments on [2] and our condition (BA).

2 Preliminaries

2.1 Notation

The ℓ_p-vector norms ($1 \le p \le \infty$) on $\mathbb{R}^d$ and their associated matrix norms are denoted by $\|\cdot\|_p$. We abbreviate $\|\cdot\|_\infty$ by $\|\cdot\|$. Recall that for a matrix A, $\|A\|$ is the maximum row sum and $\|A\|_1$ is the maximum column sum. The i-th coordinate of a vector x is denoted by $[x]_i$ and the (i,j)-th entry of a matrix A by $[A]_{i,j}$. For $x, y \in [0,\infty)^d$ we write $x \le y$ (or $y \ge x$) iff $[x]_i \le [y]_i$ for all $i = 1, \dots, d$. By $c_1, c_2, \dots$ we mean suitable strictly positive and finite constants which may depend on other constants.

2.2 Branching processes

While branching processes are most often defined and studied in terms of generating functions we prefer to use a different, but equivalent definition which allows us to couple the branching process in a natural way to the AR(1) process introduced above, see (2.3) below.

Fix $d \ge 1$ and let D be the set of all cadlag functions $\psi : [0,1] \to \mathbb{N}_0^d$. Endow D with the σ-field generated by the Skorohod topology. An *environment* for a multitype Galton-Watson branching process is a sequence $(\psi_n)_{n\ge1} = ((\psi_n^j)_{j=1,\dots,d})_{n\ge1} \in (D^d)^{\mathbb{N}}$. Here ψ_n determines the reproduction behavior of the individuals in the $(n-1)$-st generation, namely, if U is distributed uniformly on $[0,1]$ then $P[\psi_n^j(U) = (x_1, \dots, x_d)]$, $j = 1, \dots, d$, is interpreted as the probability that an individual of type j in the $(n-1)$-st generation gets x_i children of type i, $i = 1, \dots, d$.

Let $\Psi = (\Psi_n)_{n\ge1} = ((\Psi_n^j)_{j=1,\dots,d})_{n\ge1}$ be an i.i.d. sequence of D^d-valued random variables, called the *random environment* for the branching process, and let $Y = (Y_n)_{n\ge0}$ be a sequence of $[0,\infty)^d$-valued random vectors such that $(\Psi_n, Y_n)_{n\ge1}$ is i.i.d. and independent of Y_0. The vector $\lfloor Y_n \rfloor$ of integer parts of the components of Y_n gives the numbers of *immigrants* of the d possible types who join the population at time n. Moreover, let $(U_{m,n,k}^j)_{0\le m<n;1\le k;1\le j\le d}$ be an i.i.d. family of random variables which are distributed uniformly on $[0,1]$. Assume that this family is independent of Ψ and Y. Set

$$\xi_{m,n,k}^{i,j} := [\Psi_n^j(U_{m,n,k}^j)]_i. \tag{2.1}$$

We interpret $\xi_{m,n,k}^{i,j}$ as the (random) number of children of type i of the k-th individual of type j in generation $n-1$ whose ancestors immigrated at time m, provided that there are at least k individuals of this kind. The descendants of the individuals who immigrated at time m constitute a Galton-Watson process $(B_{m,n})_{n\ge m}$ in random environment (without immigration), which is defined by $B_{m,m} := \lfloor Y_m \rfloor$ and

$$B_{m,n} := \left(\sum_{j=1}^{d} \sum_{k=1}^{[B_{m,n-1}]_j} \xi_{m,n,k}^{i,j} \right)_{i=1,\dots,d}, \quad n \ge m+1.$$

The process $(Z_n)_{n\ge0}$ defined by

$$Z_n := \sum_{m=0}^{n} B_{m,n}, \quad n \ge 0, \tag{2.2}$$

(and any other process with the same distribution) is called a *branching process with immigration* $\lfloor Y \rfloor$ *in the random environment* Ψ. Here $[Z_n]_j$ is the number of individuals of type j present at time n. The random matrix

$$A_n := \left(E\left[\xi^{i,j}_{0,n,1} \mid \Psi\right]\right)_{i,j=1,\dots,d}$$

contains at position (i,j) the expected value, given the environment, of the number of children of type i of a member of type j of the $(n-1)$-st generation. As above in the definition of the AR(1) process X, the sequence $(A_n, Y_n)_{n\geq 1}$ is i.i.d. and independent of Y_0. It is well-known that for all $0 \leq m \leq n$,

$$E[B_{m,n} \mid \Psi, Y] = A_n \dots A_{m+1}\lfloor Y_m \rfloor,$$

see for example [16, Chapter II, (4.1)]. It follows from (1.3) and (2.2) that for all $n \geq 0$ a.s.

$$E[Z_n \mid \Psi, Y] = X_n \qquad \text{if } Y_m \in \mathbb{N}_0^d \text{ a.s. for all } m \geq 0. \tag{2.3}$$

2.3 Recurrence and transience

We use the notion of recurrence and transience of $[0,\infty)^d$-valued Markov chains which was used by Kellerer in [22] in a more general setting. Let $\mathcal{H}$ be the set of continuous functions from $[0,\infty)^d$ to $[0,\infty)^d$ which are monotone with respect to the partial order $\leq$ and endow $\mathcal{H}$ with the σ-field generated by the topology of compact convergence. Then a $[0,\infty)^d$-valued Markov chain $(V_n)_{n\geq 0}$ is called *order-preserving* if it fulfills a recursion of the form $V_n = H_n(V_{n-1})$ for an i.i.d. sequence $(H_n)_{n\geq 1}$ of $\mathcal{H}$-valued random variables which is independent of the initial value V_0. Observe that all four processes X, Z, M, and R defined above are order-preserving Markov chains.

Let π be the transition kernel of such an order-preserving Markov chain. Then π (and any Markov chain with transition kernel π) is called *irreducible* for the state space $[0,\infty)^d$ iff for any $x \in [0,\infty)^d$ there is some $n \geq 0$ such that $P[V_n \geq x] > 0$, where $V = (V_n)_{n\geq 0}$ is a Markov chain with kernel π starting at 0, see [22, Definition 1.1].

Proposition 2.1. *Let $K \in \mathbb{N}$ be such that $P[A_1 \dots A_K \in (0,\infty)^{d\times d}] > 0$. Then the processes X, Z, and M are irreducible for the state space $[0,\infty)^d$.*

Proof. Without loss of generality we assume that a.s. $Y_1 \in \mathbb{N}_0^d$. Let μ be the minimum of the entries of the matrix $A_{K+1}A_K \dots A_2$ and choose $\varepsilon > 0$ such that $P[\mu \geq \varepsilon] > 0$. Then we have for all $x \in [0,\infty)^d$ due (2.3) and (1.11) that

$$\begin{aligned} P[E[Z_{K+1} \mid \Psi, Y] \geq x] &= P[X_{K+1} \geq x] \geq P[M_{K+1} \geq x] \;\geq\; P[N_{K+1} \geq x] \\ &\geq P[A_{K+1}A_K \dots A_2 Y_1 \geq x] \;\geq\; P\left[\mu\|Y_1\| \geq \|x\|, \mu \geq \varepsilon\right] \\ &\geq P[\|Y_1\| \geq \|x\|/\varepsilon] P[\mu \geq \varepsilon] \end{aligned}$$

by independence. Therefore, $P[V_{K+1} \geq x] > 0$ for all $V \in \{Z, X, M\}$. □

If π is irreducible then π (and any Markov chain with transition kernel π) is called *recurrent* iff there exists $b \in (0,\infty)$ such that

$$\sum_{n\geq 0} P[\|V_n\| \leq b] = \infty, \tag{2.4}$$

where V is a Markov chain with kernel π starting at 0. In fact, the initial state is not important here. Condition (2.4) holds either for all Markov chains with transition kernel π or for none, see [22, Definition 2.5]. A Markov chain V is recurrent iff there is a finite b such that a.s. $\|V_n\| \leq b$ infinitely often. If π is not recurrent then it is called *transient*. Transience is equivalent to the almost sure divergence of the Markov chain in all components to ∞, see [22, Section 2]. (For the definition and characterization of positive recurrence in this context see [22, Section 6].)

In order to deduce from the recurrence of one process the recurrence of another process we will need to infer from the divergence of a series of the form $\sum_{n\geq 0} a_n$ the divergence of another series $\sum_{n\geq 0} b_n$. Sometimes we will do this by showing either $\sup_n a_n/b_n < \infty$ or $\inf_n b_n/a_n > 0$. Sometimes we shall use the following lemma instead.

Lemma 2.2. *For $n \geq 0$ let U_n and V_n be $\mathbb{R}^d$-valued random variables. Assume that there are $b, c > 0$ such that $\sum_n P[\|U_n\| \leq b] = \infty$ and $E[\|V_n\|; \|U_n\| \leq b] \leq cP[\|U_n\| \leq b]$ for all $n \geq 0$. Then $\sum_n P[\|V_n\| \leq 2c] = \infty$.*

Proof. By Markov's inequality,

$$\begin{aligned} P[\|V_n\| \leq 2c] &\geq P[\|V_n\| \leq 2c, \|U_n\| \leq b] \\ &= P[\|U_n\| \leq b] - P[\|V_n\| > 2c, \|U_n\| \leq b] \\ &\geq P[\|U_n\| \leq b] - \frac{E[\|V_n\|; \|U_n\| \leq b]}{2c} \geq \frac{P[\|U_n\| \leq b]}{2}, \end{aligned}$$

which is not summable in n by assumption. □

3 Constant environment

Recall that a matrix $A \in [0,\infty)^{d\times d}$ is called *primitive* iff there is a $K \in \mathbb{N}$ such that $A^K \in (0,\infty)^{d\times d}$.

Theorem 3.1 (Subcritical case, constant environment)**.** *Assume that there is a primitive matrix A with spectral radius $\varrho < 1$ such that a.s. $A_n = A$ for all $n \geq 1$. Let $y \in (0,\infty)$ be such that $P[\|Y_1\| \leq y] > 0$. Then the following three assertions are equivalent.*

The autoregressive processes X is recurrent. (XR)

The max-autoregressive process M is recurrent. (MR)

$$\sum_{n\geq 0} \prod_{m=0}^{n} P\left[\|Y_1\| \leq y\varrho^{-m}\right] = \infty. \tag{RC}$$

If we assume in addition that there is a $\psi \in D^d$ such that a.s. $\Psi_n = \psi$ for all $n \geq 1$ and that $E[\xi^{i,j}_{0,1,1} \ln \xi^{i,j}_{0,1,1}] < \infty$ for all $i, j \in \{1,\dots,d\}$ then (XR), (MR), *and* (RC) *are equivalent to the following assertion.*

The branching process with immigration Z is recurrent. (ZR)

Proof. By Proposition 2.1, X, Z, and M are irreducible since A is primitive. When checking (XR), (MR), and (ZR) we assume without loss of generality that Y_0 has the same distribution as Y_n, $n \geq 1$, since the initial state does not matter, see Section 2.3. Recall from (1.11) that $N_n := \max_{m=0}^{n} A^{n-m} Y_m$ and note that

$$\|N_n\| = \max_{m=0}^{n} \|A^{n-m} Y_m\| \quad \text{for all } n \geq 0. \tag{3.1}$$

We consider the following auxiliary conditions.

(NR) There exists $b \in \mathbb{N}$ such that $\sum_{n\geq 0} P[\|N_n\| \leq b] = \infty$.

(RC') There exists $b \in \mathbb{N}$ such that $\sum_{n\geq 0} \prod_{m=0}^{n} P\left[\|Y_1\| \leq b\varrho^{-m}\right] = \infty$.

We shall prove the equivalence of the conditions (XR), (MR), (ZR), (RC), (NR), and (RC’) as indicated in the following diagram.

$$\begin{array}{ccccccc} & & \text{(MR)} & & & & \\ & \nearrow\!\!\!\nearrow & \Downarrow & & & & \\ \text{(XR)} & \Longleftarrow & \text{(NR)} & \Longleftrightarrow & \text{(RC')} & \Longleftrightarrow & \text{(RC)} \\ & \searrow\!\!\!\searrow & & \nearrow\!\!\!\nearrow & & & \\ & & \text{(ZR)} & & & & \end{array}$$

(XR)⇒(MR)⇒(NR): These implications follow from (1.11).

(NR)⇒(XR): Due to $\varrho < 1$,

$$\sigma := \sum_{m \ge 0} \|A^m\| < \infty. \tag{3.2}$$

In particular, there exists $b \in \mathbb{N}$ such that $P[\|A^m Y_1\| \le b] \ge 1/2$ for all $m \ge 0$ and, due to (NR), $\sum_{n \ge 0} P[\|N_n\| \le b] = \infty$. By (1.3), (3.1) and since Y is i.i.d. we have for all $n \ge 0$ that

$$\begin{aligned} E[\|X_n\| \mid \|N_n\| \le b] &\le \sum_{m=0}^{n} E\left[\|A^{n-m}Y_m\| \;\middle|\; \bigcap_{i=0}^{n}\{\|A^{n-i}Y_i\| \le b\}\right] \\ &= \sum_{m=0}^{n} E\left[\|A^{n-m}Y_m\| \mid \|A^{n-m}Y_m\| \le b\right] \;\le\; \sum_{m \ge 0} \frac{E[\|A^m Y_1\|; \|A^m Y_1\| \le b]}{P[\|A^m Y_1\| \le b]}. \end{aligned} \tag{3.3}$$

Note that $T := \inf\{m \ge 0 : \|A^m Y_1\| \le b\}$ is a.s. finite due to (3.2). Therefore and by our choice of b, the right-hand side of (3.3) can be bounded form above by

$$2E\left[\sum_{m \ge T} \|A^m Y_1\|\right] \;=\; 2E\left[\sum_{m \ge 0} \|A^{m+T} Y_1\|\right] \;\overset{(3.2)}{\le}\; 2E\left[\sigma \|A^T Y_1\|\right] \le 2\sigma b =: c < \infty.$$

Applying Lemma 2.2 to $(U, V) = (N, X)$ we obtain the claim (XR).

(NR)⇔(RC’): Since Y is i.i.d. and due to (3.1), (NR) is satisfied iff there is a $b \in \mathbb{N}$ such that $\sum_n \prod_{m=0}^n P[\|A^m Y_1\| \le b] = \infty$. Recall from Perron-Frobenius theory, see e.g. [20, Appendix, Theorem 2.3], that there is a matrix $H \in (0,\infty)^{d \times d}$ such that $\lim_{n\to\infty} \varrho^{-n} A^n = H$. Therefore, there exist $k, L \in \mathbb{N}$ such that $k^{-1}\varrho^m \le [A^m]_{i,j} \le k\varrho^m$ for all $m \ge L$ and all $i, j = 1, \ldots, d$. Hence, $k^{-1}\varrho^m \|Y_1\| \le \|A^m Y_1\| \le dk\varrho^m \|Y_1\|$ for all $m \ge L$. This implies the claim.

(RC’)⇔(RC): The implication ⇐ is trivial. For the reverse implication let $b \in (0, \infty)$ be according to (RC’) and $k \in \mathbb{N}_0$ be such that $y\varrho^{-k} \ge b$. Then

$$\begin{aligned} \sum_{n \ge 0} \prod_{m=0}^{n} P\left[\|Y_1\| \le y\varrho^{-m}\right] \;&\ge\; \sum_{n \ge k} \prod_{m=0}^{n} P\left[\|Y_1\| \le y\varrho^{-m}\right] \\ =\;& \left(\prod_{m=0}^{k-1} P\left[\|Y_1\| \le y\varrho^{-m}\right]\right) \sum_{n \ge 0} \prod_{m=0}^{n} P\left[\|Y_1\| \le \left(y\varrho^{-k}\right)\varrho^{-m}\right] = \infty. \end{aligned}$$

(XR)⇒(ZR): Let $b \in (0, \infty)$ be such that $\sum_n P[\|X_n\| \le b] = \infty$. Due to monotonicity we may assume without loss of generality that a.s. $Y_1 \in \mathbb{N}_0^d$. Note that for all $x \in [0, \infty)^d$, $\|x\| \le \|x\|_1 = x_1 + \ldots + x_n \le d\|x\|$. Therefore, by (2.3),

$$\begin{aligned} E[\|Z_n\|; \|X_n\| \le b] \;&\le\; E[\|Z_n\|_1; \|X_n\| \le b] = \|E[E[Z_n; \|X_n\| \le b \mid \Psi, Y]]\|_1 \\ &=\; \|E[E[Z_n \mid \Psi, Y]; \|X_n\| \le b]\|_1 \;\le\; \|E[X_n; \|X_n\| \le b]\|_1 \\ &\le\; E[d\|X_n\|; \|X_n\| \le b] \;\le\; bdP[\|X_n\| \le b]. \end{aligned}$$

Lemma 2.2 applied to $(U,V)=(X,Z)$ implies (ZR).

(ZR)$\Rightarrow$(RC'): Since (RC') holds iff it holds for $\lfloor Y_1 \rfloor$ instead of Y_1 we assume without loss of generality that Y_1 is a.s. $\mathbb{N}_0^d$-valued. Denote by $q_{j,k}$ the probability that a given individual of type j does not have any descendants k generations later, i.e. with a slight abuse of notation, $q_{j,k} := P[B_{0,k}=0 \mid Y_0 = e_j]$, where $e_j \in \mathbb{Z}^d$ is the j-th standard unit vector. Due to the moment assumption on $\xi_{0,1,1}^{i,j}$ and [18, Theorems 2 (3.6) and 4], $\varrho^{-k}(1-q_{j,k})$ tends as $k\to\infty$ for all $j=1,\dots,d$, to a strictly positive and finite limit. In particular, there are $c,\ell\in\mathbb{N}$ such that

$$0 < q_{j,k} \le 1-\varrho^k/c \ \text{ for all } k\ge \ell \text{ and } j=1,\dots,d. \tag{3.4}$$

Set $\widetilde{Z}_n := \sum_{m=0}^{n-\ell} B_{m,n}$ for $n \ge \ell$. By (ZR) and subadditivity there is some $z\in\mathbb{N}_0^d$ such that $\sum_n P[Z_n = z]=\infty$. Then on the one hand by the Markov property and independence,

$$\sum_{n\ge \ell} P[\widetilde{Z}_n = 0] \ge \sum_{n\ge\ell} P[Z_{n-\ell}=z] \prod_{j=1}^d q_{j,\ell}^{[z]_j} = \infty \tag{3.5}$$

by our choice of z and (3.4). On the other hand, since the processes $(B_{m,m+n})_{n\ge 0}$, $m\ge 0$, are i.i.d., we have for all $n\ge \ell$,

$$\begin{aligned} P[\widetilde{Z}_n = 0] &= P\left[\bigcap_{m=0}^{n-\ell}\{B_{m,n}=0\}\right] = \prod_{k=\ell}^n E[P[B_{0,k}=0|Y_0]] \\ &= \prod_{k=\ell}^n E\left[\prod_{j=1}^d q_{j,k}^{[Y_0]_j}\right] \overset{(3.4)}{\le} \prod_{k=\ell}^n E\left[(1-\varrho^k/c)^{\|Y_0\|}\right] \\ &= \prod_{k=\ell}^n \int_0^1 P\left[(1-\varrho^k/c)^{\|Y_0\|} \ge t\right]\, dt \ \le\ \prod_{k=\ell}^n \int_0^1 G(c(-\ln t)\varrho^{-k})\, dt, \end{aligned}$$

where G is the cumulative distribution function of $\|Y_1\|$. Choose $b\in\mathbb{N}$ such that $G(b)>1/2$ and set $\bar{G}:=1-G$. Then by the above for all $n\ge\ell$,

$$\begin{aligned} \frac{P[\widetilde{Z}_n=0]}{\prod_{k=\ell}^n G(b\varrho^{-k})} &\le \prod_{k=\ell}^n \int_0^1 \frac{G\left(c(-\ln t)\varrho^{-k}\right)}{G(b\varrho^{-k})}\, dt \\ &= \prod_{k=\ell}^n \left(1+\int_0^1 \frac{\bar{G}(b\varrho^{-k})-\bar{G}\left(c(-\ln t)\varrho^{-k}\right)}{G(b\varrho^{-k})}\, dt\right) \\ &\le \exp\left(2\sum_{k=\ell}^n \int_0^1 \left(\bar{G}(b\varrho^{-k})-\bar{G}\left(c(-\ln t)\varrho^{-k}\right)\right)_+\, dt\right) \\ &= \exp\left(2\int_0^{\exp(-b/c)} \sum_{k=\ell}^n \bar{G}(b\varrho^{-k})-\bar{G}\left(c(-\ln t)\varrho^{-k}\right)\, dt\right). \end{aligned} \tag{3.6}$$

We set $f(t) := \left(\ln(c/b)+\ln(-\ln t)\right)/(-\ln\varrho)$ and use the telescopic form of the sum in (3.6) for estimating this sum for all $t\in(0,e^{-b/c})$ from above by

$$\begin{aligned} &\sum_{k=\ell}^{n\vee(\ell+\lceil f(t)\rceil)} \bar{G}(b\varrho^{-k})-\bar{G}\left(c(-\ln t)\varrho^{-k}\right) \\ \le\ & \lceil f(t)\rceil + \sum_{k=\ell}^{(n-\lceil f(t)\rceil)\vee\ell} \bar{G}(b\varrho^{-(k+\lceil f(t)\rceil)})-\bar{G}\left(c(-\ln t)\varrho^{-k}\right) \le f(t)+1 \end{aligned} \tag{3.7}$$

since all the differences in (3.7) are nonpositive. Since $\int_0^{\exp(-b/c)} f(t)\, dt<\infty$, the right-hand side of (3.6) is bounded from above uniformly in n. Therefore, (3.5) implies that $\sum_{n\ge\ell}\prod_{k=\ell}^n G(b\varrho^{-k})$ diverges. The claim (RC') follows. □

3.1 An application to frog processes

For a survey on frog processes we refer to [32]. The following application is related to [32, Theorem 4.3]. Let $(Y_n)_{n\geq 0}$ be an i.i.d. sequence of $\mathbb{N}_0$-valued random variables. Put on each $n \geq 0$ a number Y_n of sleeping frogs. Fix $r \in (0,1)$ and $p \in (0,1]$ such that $r < 1/2$ or $p < 1$ holds. Wake up the frogs at 0 (if there are any). Once woken up, every frog performs a nearest-neighbor random walk, jumping independently of everything else with probability r to the right and with probability $1-r$ to the left, until it dies after an independent number of steps which is geometrically distributed with parameter $1-p$ and may be 0. (If $p = 1$ then the frog never dies.) Whenever a frog visits a site with sleeping frogs those frogs are woken up as well and start their own independent lives.

Theorem 3.2. *Let $y \in (0,\infty)$ be such that $P[Y_0 \leq y] > 0$. Then the following statements are equivalent.*

Almost surely only finitely many different frogs visit 0. (3.8)

Almost surely only finitely many frogs are woken up. (3.9)

$$\sum_{n\geq 0}\prod_{m=0}^{n} P\left[Y_0 \leq y\varrho^{-m}\right] = \infty, \quad \textit{where} \quad \varrho := \frac{1-\sqrt{1-4p^2r(1-r)}}{2p(1-r)}. \tag{3.10}$$

Proof. Let $a_\pm \in (0,1)$ be the probability that a frog which starts at 0 ever hits ± 1 before it dies.

(3.9)⇒(3.8): This implication is obvious.

(3.9)⇔(3.10): By conditioning on the first step we see that a_+ satisfies $a_+ = pr + p(1-r)a_+^2$ and get $a_+ = \varrho < 1$. Assign to each frog the trajectory which the frog will follow once it has been woken up. For any $0 \leq m \leq n$ let $B_{m,n}$ be the number of frogs originally sleeping at m whose trajectories reach the site n. Then for all $m \geq 0$, $B_{m,m} = Y_m$ and $(B_{m,m+k})_{k\geq 0}$ is a Galton-Watson branching processes with offspring distribution Bernoulli(a_+). Moreover, the processes $(B_{m,m+k})_{k\geq 0}, m \geq 0$, are independent. Hence, if we denote by Z_n, $n \geq 0$, the total number of frogs originating in $\{0,1,\ldots,n\}$ whose trajectories visit n then $(Z_n)_{n\geq 0}$ is a subcritical branching process with immigration. By Theorem 3.1, $(Z_n)_{n\geq 0}$ is recurrent iff (3.10) holds. On the other hand, $(Z_n)_{n\geq 0}$ is recurrent iff there is a.s. an $n \geq 1$ such that $Z_n = Y_n$, i.e. iff there is a site n which is never visited, which is equivalent to (3.9).

(¬(3.9)∧¬(3.10))⇒ ¬(3.8): Since the frogs jump between nearest neighbors, ¬(3.9) implies that with positive probability all frogs are woken up. Moreover, as shown in Remark 1.2, ¬(3.10) implies $E[\ln_+ Y_0] = \infty$ and hence a.s. $\sum_{n\geq 0} Y_n a_-^n = \infty$, see e.g. [27, Theorem 5.4.1]. Since a_-^n is the probability that a frog starting at n ever reaches 0, ¬(3.8) follows from the Borel Cantelli lemma. □

4 Random environment

For the case of genuinely random environments we need the following bounds on the coefficient matrices A_n.

If $d = 1$ then $\ln A_1 - E[\ln A_1]$ is sub-Gaussian.
If $d \geq 2$ then there exist $K, \gamma \in \mathbb{N}$ and $\kappa > 0$ such that a.s. (BA)
$\|A_1\| \leq \gamma$ and $A_1 \ldots A_K \in [\kappa,\infty)^{d\times d}$.

For an introduction to sub-Gaussian random variables see e.g. [37] and [5]. If (BA) holds then λ defined in (1.4) can be expressed in various ways, as we state next. In fact, for this much weaker assumptions would suffice, see [13].

Proposition 4.1. *If* (BA) *holds then* $E[|S_n|] < \infty$ *for all* $n \geq 1$ *and*

$$\frac{S_n}{n} \xrightarrow[n\to\infty]{\text{a.s.}} \lambda := \sup_{n\geq 1} \frac{E\,[S_n]}{n} = \lim_{n\to\infty} \frac{E[S_n]}{n} \in \mathbb{R}. \tag{4.1}$$

Proof. The statement follows from Liggett's subadditive ergodic theorem, see e.g. [9, Theorem 7.4.1]. The only assumptions of this theorem which rely on (BA) are $E[\ln_+ \|A_1\|] < \infty$ and $\sup_n E[S_n/n] < \infty$. If $d = 1$ then this follows from $S_n = -\ln A_1 - \ldots - \ln A_n$, $(A_n)_{n\geq 1}$ being i.i.d. and $E[|\ln A_1|] < \infty$. For $d \geq 2$ denote by $\mathcal{A} \subseteq [0,\infty)^{d\times d}$ the support of A_1. Due to (BA) the assumptions of Lemma A.1 from Appendix A are satisfied. Thus, by (A.2), a.s. $\|A_1 \ldots A_n\| \geq c^n \|A_1\| \ldots \|A_n\|$ and hence $\sup_n E[S_n/n] \leq E[S_1] - \ln c$, which is finite since $\kappa \leq \|A_1^K\| \leq \|A_1\|^K$. $\square$

We also need the following mild regularity condition on the tail of the distribution of Y_1. Here λ is the constant defined in (1.4) or (4.1).

$$\lim_{x\to\infty} x^{2/3} (\ln x)^2 P[\|Y_1\| > e^x] = 0 \quad \text{or} \quad \liminf_{x\to\infty} x P[\|Y_1\| > e^x] > \lambda. \tag{REG}$$

Roughly speaking, condition (REG) requires that $P[\|Y_1\| > e^x]$ does not oscillate between decaying faster than x^{-1} and slower than $x^{-2/3}$. In particular, (REG) holds if $P[\|Y_1\| > e^x]$ varies regularly as $x \to \infty$.

The final condition concerns only branching processes. For $j = 1, \ldots, d$ we denote by $\mathcal{V}^j$ the covariance matrix of the vector $(\xi^{i,j}_{0,1,1})_{i=1,\ldots,d}$ given Ψ.

$$\text{There exists } C \in \mathbb{N} \text{ such that a.s. } \|\mathcal{V}^1\|, \ldots, \|\mathcal{V}^d\| \leq C\|A_1\|. \tag{BV}$$

Theorem 4.2 (Subcritical case, random environment)**.** *Assume* (BA)*,* (REG)*, and* $\lambda > 0$*. Let* $y \in (0,\infty)$ *be such that* $P[\|Y_1\| \leq y] > 0$*. Then the following three statements are equivalent.*

$$\textit{The autoregressive processes } X \textit{ is recurrent.} \tag{XR}$$

$$\textit{The max-autoregressive process } M \textit{ is recurrent.} \tag{MR}$$

$$\sum_{n\geq 0} \prod_{m=0}^{n} P[\|Y_1\| \leq y e^{m\lambda}] = \infty. \tag{RR}$$

If we assume in addition (BV) *then* (XR)*,* (MR)*, and* (RR) *are equivalent to the following statement.*

$$\textit{The branching process with immigration } Z \textit{ is recurrent.} \tag{ZR}$$

For the proof of Theorem 4.2 we denote the cumulative distribution function of $\ln \|Y_1\|$ by F and its tail by $\bar{F} := 1 - F$.

Lemma 4.3. *Assume* (REG)*. Suppose that for all* $\varepsilon > 0$ *there exists* $b_\varepsilon \in \mathbb{N}$ *such that* $\sum_{n\geq 0} \prod_{i=0}^{n} F(b_\varepsilon + (\lambda+\varepsilon)i) = \infty$*. Then* $\lim_{x\to\infty} x^{2/3}(\ln x)^2 \bar{F}(x) = 0$ *and therefore* $E[(\ln_+ \|Y_1\|)^{2/3}] < \infty$*.*

Proof. Raabe's test implies that for all $\mu > 1$ and $\varepsilon > 0$, $F(b_\varepsilon + (\lambda+\varepsilon)i) \geq 1 - \mu/i$ for infinitely many i. Therefore, $\liminf_x x\bar{F}(x) \leq (\lambda+\varepsilon)\mu$. Letting $\mu \searrow 1$ and $\varepsilon \searrow 0$ yields $\liminf_x x\bar{F}(x) \leq \lambda$. The statement now follows from (REG). $\square$

Proof of Theorem 4.2. By Proposition 2.1 and (BA), X, Z, and M are irreducible. As in the proof of Theorem 3.1 we assume that Y_0 has the same distribution as $Y_n,\ n \geq 1$.

Denote by $\mathcal{A} \subseteq [0,\infty)^{d\times d}$ the support of A_1. Due to (BA) the assumptions of Lemma A.1 from Appendix A are satisfied. In addition to the auxiliary condition (NR) introduced in the proof of Theorem 3.1 we need the following assertions.

(RR′) There exists $b \in \mathbb{N}$ such that $\sum_{n\ge 0}\prod_{m=0}^{n} P\left[\|Y_1\| \le b e^{m\lambda}\right] = \infty$.

(R+) $E[(\ln_+\|Y_1\|)^{2/3}] < \infty$ and there exists $b \in \mathbb{N}$ such that $\sum_{n\ge 0}\prod_{i=0}^{n} F(b+\lambda(i+i^{2/3})) = \infty$.

(R−) There exists $b \in \mathbb{N}$ such that $\sum_{n\ge 0}\prod_{i=0}^{n} F(b+\lambda(i-i^{2/3})) = \infty$.

We shall prove the equivalence of the conditions (XR), (MR), (ZR), (NR), (RR), (RR′), (R+), and (R−) as indicated in the following diagram.

$$
\begin{array}{ccccccccc}
 & & (\mathrm{MR}) & \Longrightarrow & (\mathrm{NR}) & & & & \\
 & \Nearrow & & & & \Searrow & & & \\
(\mathrm{XR}) & \Longrightarrow & (\mathrm{ZR}) & & \Longrightarrow & & (\mathrm{R}+) & & \\
 & \Nwarrow & & & \Swarrow & & \Uparrow & & \\
 & & (\mathrm{R}-) & & \Longrightarrow & & (\mathrm{RR}') & \Longleftrightarrow & (\mathrm{RR})
\end{array}
$$

The proofs of (XR)⇒(MR)⇒(NR), of (XR)⇒(ZR) and of (RR)⇔(RR′) are the same as for the corresponding statements of Theorem 3.1.

(ZR)⇒(R+): Since (R+) holds iff it holds for $\lfloor Y_1 \rfloor$ instead of Y_1 we assume without loss of generality that Y_1 is a.s. $\mathbb{N}_0^d$-valued. Set $P_\Psi[\cdot] := P[\,\cdot \mid \Psi]$. Denote by $q_{\Psi,j,m,n}$ the probability that in the environment Ψ a given individual of type j who immigrated at time m does not have any descendants at time n, i.e. with a slight abuse of notation, $q_{\Psi,j,m,n} = P_\Psi[B_{m,n} = 0 \mid Y_m = e_j]$. Proposition A.2 and (BV) yield that for all $j = 1, \dots, d$, a.s.

$$(1 - d\|A_n \dots A_{m+1}\|)_+ \le q_{\Psi,j,m,n} \le 1 - c_1 \frac{\|A_n \dots A_{m+1}\|}{\sum_{k=m+1}^{n}\|A_n \dots A_k\|} =: q_{\Psi,m,n}. \tag{4.2}$$

Since $\lambda > 0$, there exists $\ell \in \mathbb{N}$ with $P[d\|A_\ell \dots A_1\| < 1] > 0$. Set $\widetilde{Z}_n := \sum_{m=0}^{n-\ell} B_{m,n}$ for $n \ge \ell$. By (ZR) there exists $z \in \mathbb{N}_0^d$ such that $\sum_{n\ge 0} P[Z_n = z] = \infty$. Hence

$$P[\widetilde{Z}_n = 0] \ge P[\widetilde{Z}_n = 0, Z_{n-\ell} = z] = P[Z_{n-\ell} = z] E\left[\prod_{j=1}^{d} q_{\Psi,j,0,\ell}^{[z]_j}\right] \tag{4.3}$$

since Ψ is i.i.d.. Due to the lower bound in (4.2) and our choice of ℓ the expected value in (4.3) is strictly positive. Therefore,

$$\sum_{n\ge\ell} P[\widetilde{Z}_n = 0] = \infty. \tag{4.4}$$

For all $b \in \mathbb{N}$ with $F(b) \ge 1/2$ and all functions $g : \mathbb{N}_0 \to [0,\infty)$ with $\ln_+ n \le g(n) \le n$ for all $n \in \mathbb{N}_0$ let

$$q(b,g) := \sup_{n\ge 0} \frac{P[\widetilde{Z}_n = 0]}{\prod_{i=\ell}^{n} F(b+\lambda i + g(i))}. \tag{4.5}$$

Due to (4.4) it suffices to show for the proof of (R+) that there exists b such that

$$E[(\ln_+\|Y_1\|)^{2/3}] < \infty \quad \text{and} \quad q(b,h) < \infty, \quad \text{where } h : n \mapsto \lambda n^{2/3}. \tag{4.6}$$

To this end we first bound the enumerator in (4.5) by observing that for all $n \ge \ell$,

$$P[\widetilde{Z}_n = 0] = E\left[P_\Psi\left[\bigcap_{m=0}^{n-\ell}\{B_{m,n} = 0\}\right]\right] = E\left[\prod_{m=0}^{n-\ell} P_\Psi[B_{m,n} = 0]\right],$$

due to independence, see e.g. [19, Proposition 6.6 and Corollary 6.7 (i)]. Using independence once again we have

$$\begin{aligned} P_\Psi[B_{m,n}=0] &= E_\Psi\left[P[B_{m,n}=0 \mid \Psi, Y]\right] = E_\Psi\left[\prod_{j=1}^d q_{\Psi,j,m,n}^{[Y_m]_j}\right] \overset{(4.2)}{\leq} E_\Psi\left[q_{\Psi,m,n}^{\|Y_m\|}\right] \\ &= \int_0^1 P_\Psi\left[q_{\Psi,m,n}^{\|Y_m\|}\geq t\right]\,dt = \int_0^1 F\left(\ln\left(\frac{\ln t}{\ln q_{\Psi,m,n}}\right)\right)\,dt, \end{aligned}$$

see e.g. [19, Lemma 3.11]. Thus for all b and g as above,

$$q(b,g)\leq \sup_{n\geq 0} E\left[\prod_{m=0}^{n-\ell}\int_0^1 \frac{F\left(\ln\left(\frac{\ln t}{\ln q_{\Psi,m,n}}\right)\right)}{F(b+\lambda(n-m)+g(n-m))}\,dt\right]. \tag{4.7}$$

Since $(A_1,\ldots,A_n)$ has the same distribution as $(A_n,\ldots,A_1)$, $(q_{\Psi,0,n},\ldots,q_{\Psi,n-1,n})$ has the same distribution as $(q'_{\Psi,0,n},\ldots,q'_{\Psi,n-1,n})$, where

$$q'_{\Psi,m,n} := 1-c_1\frac{\|A_1\ldots A_{n-m}\|}{\sum_{k=m+1}^n \|A_1\ldots A_{n+1-k}\|}\leq 1-c_1\frac{\|A_1\ldots A_{n-m}\|}{\sigma} \tag{4.8}$$

and $\sigma := \sum_{k\geq 1}\|A_1\ldots A_k\|$. Let $r_{\Psi,m} := \exp\left(-c_1\|A_1\ldots A_m\|/\sigma\right)$, $f_{\Psi,m}(t) := \ln\left(\frac{\ln t}{\ln r_{\Psi,m}}\right)$, and $t_{\Psi,m} := r_{\Psi,m}^{\exp(b+\lambda m+g(m))}$. By (4.8), $q'_{\Psi,m,n}\leq r_{\Psi,n-m}$. Therefore, by (4.7),

$$\begin{aligned} q(b,g) &\leq \sup_{n\geq 0} E\left[\prod_{i=\ell}^n\int_0^1 \frac{F\left(f_{\Psi,i}(t)\right)}{F(b+\lambda i+g(i))}\,dt\right] \\ &= \sup_{n\geq 0} E\left[\prod_{i=\ell}^n\left(1+\int_0^1 \frac{\bar F(b+\lambda i+g(i))-\bar F\left(f_{\Psi,i}(t)\right)}{F(b+\lambda i+g(i))}\,dt\right)\right] \\ &\leq E\left[\exp\left(\sum_{i\geq 0}\int_0^1 \frac{\left(\bar F(b+\lambda i+g(i))-\bar F\left(f_{\Psi,i}(t)\right)\right)_+}{F(b+\lambda i+g(i))}\,dt\right)\right] \\ &= E\left[\exp\left(\sum_{i\geq 0}\int_0^{t_{\Psi,i}} \frac{\left(\bar F(b+\lambda i+g(i))-\bar F\left(f_{\Psi,i}(t)\right)\right)_+}{F(b+\lambda i+g(i))}\,dt\right)\right] \\ &\leq E\left[\exp\left(2\sum_{i\geq 0} t_{\Psi,i}\bar F(b+\lambda i)\right)\right]. \end{aligned}$$

Let $T_g := \inf\{n\geq 0 \mid \forall i>n : S_i\leq \lambda i+g(i)/2-\ln\sigma\}$. Then for all $i>T_g$, $t_{\Psi,i}\leq \exp\left(-c_1e^{b+g(i)/2}\right)\leq \exp\left(-c_1e^b\sqrt{i}\right)$ due to $g(i)\geq \ln_+ i$. Hence

$$\begin{aligned} q(b,g) &\leq E\left[\exp\left(2+2\sum_{i=1}^{T_g}\bar F(b+\lambda i)+2\sum_{i>T_g} t_{\Psi,i}\right)\right] \\ &\leq c_2 E\left[\exp\left(2\sum_{i=1}^{T_g}\bar F(b+\lambda i)\right)\right]. \end{aligned} \tag{4.9}$$

Next we show that there are numbers $c_3=c_3(g), c_4=c_4(g)\in(0,\infty)$ such that

$$P[T_g=n]\leq c_3\exp\left(-c_4 g(n)^2/n\right)\qquad \text{for all } n\geq 1. \tag{4.10}$$

To this end let $c_5 := \left(\sum_{i\geq 1} e^{-\lambda i/2}\right)^{-1}$. Then for all $t > c_5^{-1}$,

$$\begin{aligned}
P[\sigma \geq t] \quad &\leq \quad \sum_{i\geq 1} P\left[\|A_1\dots A_i\| \geq c_5 e^{-\lambda i/2} t\right] \;\leq\; \sum_{i\geq 1} P[|S_i - \lambda i| \geq \lambda i/2 + \ln(c_5 t)] \\
&\overset{(A.15)}{\leq} \quad \sum_{i\geq 1} c_6 e^{-c_7(\lambda i/2 + \ln(c_5 t))^2/i} \;\leq\; c_6 \sum_{i\geq 1} e^{-c_7\left(\lambda^2 i/4 + \lambda \ln(c_5 t)\right)} \\
&= \quad c_8 t^{-c_9}.
\end{aligned} \tag{4.11}$$

Therefore, for all n large enough such that $e^{g(n)/4} > c_5^{-1}$,

$$\begin{aligned}
P[T_g = n] \quad &\leq \quad P\left[\lambda n + g(n)/2 - \ln\sigma \leq S_n\right] \\
&\leq \quad P\left[\lambda n + g(n)/2 - \ln\sigma \leq S_n \leq \lambda n + g(n)/4\right] + P\left[S_n \geq \lambda n + g(n)/4\right] \\
&\leq \quad P[\sigma \geq e^{g(n)/4}] + P\left[|S_n - \lambda n| \geq g(n)/4\right] \\
&\leq \quad c_8 \exp(-c_{10} g(n)) + c_6 \exp(-c_7 g(n)^2/n)
\end{aligned}$$

due to (4.11) and (A.15). By using $g(n) \leq n$ and increasing constants we obtain (4.10).

It remains to show how (4.10) implies (4.6). For $\varepsilon > 0$ and $n \in \mathbb{N}_0$ let $g_\varepsilon(n) := \varepsilon n$. Then (4.9) yields $q(b, g_\varepsilon) \leq c_2 E\left[\exp\left(2T_{g_\varepsilon}\bar{F}(b)\right)\right]$, which is finite for some large $b = b_\varepsilon$ since T_{g_ε} has some finite exponential moment due to (4.10). Therefore, due to (4.4), the assumptions of Lemma 4.3 are satisfied. This lemma yields the first statement in (4.6) and the existence of b large enough such that $F(b) \geq 1/2$, $(\ln b)^{-2} \leq c_4\lambda^{8/3}/12$, and such that for all $i \geq 1$,

$$\bar{F}(b + \lambda i) \leq (b + \lambda i)^{-2/3}(\ln(b + \lambda i))^{-2} \leq \frac{c_4\lambda^{8/3}}{12}(b + \lambda i)^{-2/3}, \tag{4.12}$$

where $c_4 := c_4(h)$. We obtain from (4.9) and (4.12) that

$$\begin{aligned}
q(b,h) \leq c_2 E&\left[\exp\left(\frac{c_4\lambda^{8/3}}{6}\sum_{i=1}^{T_h}(b + \lambda i)^{-2/3}\right)\right] \\
\leq \quad c_2 E&\left[\exp\left(\frac{c_4\lambda^{8/3}}{6}\int_0^{T_h}(b + \lambda t)^{-2/3}\,dt\right)\right] \leq c_2 E\left[\exp\left(\frac{c_4\lambda^2}{2}{T_h}^{1/3}\right)\right] \\
\overset{(4.10)}{\leq} \quad c_2 + c_3 &\sum_{n\geq 1}\exp\left(\frac{c_4\lambda^2}{2}n^{1/3} - \frac{c_4 h(n)^2}{n}\right) = c_2 + c_3\sum_{n\geq 1}\exp\left(-\frac{c_4\lambda^2}{2}n^{1/3}\right) < \infty.
\end{aligned}$$

<u>(NR)⇒(R+):</u> Let $A := (A_k)_{k\geq 1}$. Set $N_n' := \max_{i=0}^n A_1\dots A_i Y_{i+1}$ for all $n \geq 0$ and note that N_n' has the same distribution as N_n since $(A_n, \dots, A_1, Y_n, \dots, Y_0)$ has the same distribution as $(A_1, \dots, A_n, Y_1, \dots, Y_{n+1})$. Therefore, by (NR) there is a c_{11} such that

$$\begin{aligned}
\infty \quad &= \quad \sum_{n\geq 0} P\left[\|N_n'\| \leq c_{11}\right] \;=\; \sum_{n\geq 0} E\left[P\left[\forall i = 0, \dots, n : \|A_1\dots A_i Y_{i+1}\| \leq c_{11} \mid A\right]\right] \\
&= \quad \sum_{n\geq 0} E\left[\prod_{i=0}^n P\left[\|A_1\dots A_i Y_{i+1}\| \leq c_{11} \,\middle|\, A\right]\right] \\
&\overset{(A.1)}{\leq} \quad \sum_{n\geq 0} E\left[\prod_{i=0}^n P\left[\|A_1\dots A_i\|\|Y_{i+1}\| \leq c_{12} \,\middle|\, A\right]\right] \;=\; R(c_{13}),
\end{aligned}$$

where $R(b) := \sum_{n\geq 0} E\left[\prod_{i=0}^n F(b + S_i)\right]$. For all functions $g : \mathbb{N}_0 \to [0, \infty)$ with $g(0) = 0$ let $T_g := \inf\{n \geq 0 \mid \forall i > n : S_i \leq \lambda i + g(i)\}$. Moreover, for all such functions g and all

$b \in \mathbb{N}$ with $F(b) \geq 1/2$ let

$$\begin{aligned} q(b,g) &:= \sup_{n\geq 0} \frac{E\left[\prod_{i=0}^n F(b+S_i)\right]}{\prod_{i=0}^n F(b+\lambda i+g(i))} \leq E\left[\prod_{i=1}^{T_g}\left(\frac{F(b+S_i)}{F(b+\lambda i+g(i))}\vee 1\right)\right] \\ &\leq E\left[\prod_{i=1}^{T_g}\frac{1}{F(b+\lambda i)}\right] = E\left[\prod_{i=1}^{T_g}\left(1+\frac{\bar F(b+\lambda i)}{F(b+\lambda i)}\right)\right] \\ &\leq E\left[\exp\left(\sum_{i=1}^{T_g}\frac{\bar F(b+\lambda i)}{F(b)}\right)\right] \leq E\left[\exp\left(2\sum_{i=1}^{T_g}\bar F(b+\lambda i)\right)\right]. \end{aligned} \tag{4.13}$$

Since $P[T_g = n] \leq P\left[S_n \geq \lambda n + g(n)\right]$ for all $n \geq 1$, Lemma A.3 implies that

$$P[T_g = n] \leq \ c_6 \exp\left(-c_7 g(n)^2/n\right) \qquad \text{for all } n \geq 1. \tag{4.14}$$

The claim (R+) follows now from (4.14) in exactly the same way as (4.6) follows from (4.10) in the proof of (ZR)⇒(R+).

(R+)⇒(R−): Choose b according to (R+). Define $g_\pm(t) := b + \lambda(t \pm t^{2/3})$ for $t \in [1,\infty)$. Note that both functions g_+ and g_- are strictly increasing. Denote by $g_\pm^{-1}$ their inverse functions. For all $x \geq b+2\lambda$ we have $x = g_\pm(g_\pm^{-1}(x))$, that is

$$g_\pm^{-1}(x) \pm \left(g_\pm^{-1}(x)\right)^{2/3} = \frac{x-b}{\lambda}. \tag{4.15}$$

For the proof of (R−) it suffices to show that the following quantities are finite.

$$\sup_{n\geq 1}\prod_{i=1}^n \frac{F(g_+(i))}{F(g_-(i))} = \prod_{i\geq 1}\left(1+\frac{F(g_+(i))-F(g_-(i))}{F(g_-(i))}\right) \leq \prod_{i\geq 1}\left(1+\frac{F(g_+(i))-F(g_-(i))}{F(b)}\right).$$

Therefore, it is enough to show that $F(g_+(i)) - F(g_-(i))$ is summable in i. Set $\eta := \ln\|Y_1\|$. Then

$$\begin{aligned} &\sum_{i\geq 8} F(g_+(i)) - F(g_-(i)) = \sum_{i\geq 8} E[\mathbf{1}_{(g_-(i),g_+(i)]}(\eta); \eta > g_-(8)] \\ = \quad & E\left[\sum_{i\geq 8}\mathbf{1}_{[g_+^{-1}(\eta),g_-^{-1}(\eta))}(i); \eta > g_-(8)\right] \leq E[g_-^{-1}(\eta) - g_+^{-1}(\eta); \eta > g_-(8)] + 1 \\ \overset{(4.15)}{=} \quad & E\left[\left(g_-^{-1}(\eta)\right)^{2/3} + \left(g_+^{-1}(\eta)\right)^{2/3}; g_-^{-1}(\eta) > 8\right] + 1 \\ \leq \quad & E\left[2\left(g_-^{-1}(\eta)\right)^{2/3}; g_-^{-1}(\eta) > 8\right] + 1 \\ \leq \quad & 2E\left[\left(2g_-^{-1}(\eta) - 2\left(g_-^{-1}(\eta)\right)^{2/3}\right)^{2/3}; \eta > g_-(8)\right] + 1 \\ \overset{(4.15)}{=} \quad & 2^{8/3}E\left[\left(\frac{\eta-b}{\lambda}\right)^{2/3}; \eta > g_-(8)\right] + 1, \end{aligned}$$

which is finite due to (R+).

(R−)⇒(RR'): This implication follows from monotonicity.

(RR')⇒(R+): The first part of (R+) follows from monotonicity and Lemma 4.3. The second part follows from monotonicity.

(R−)⇒(XR): Let b be according to (R−). Due to (4.1) and $\lambda > 0$, we have $(S_i - \lambda i^{2/3}/2) \to \infty$ a.s. as $i \to \infty$. Therefore, there is $b' \geq b$ such that $P[B] > 0$, where

$B := \{F(b' + S_i - \lambda i^{2/3}/2) > 1/2 \text{ for all } i \geq 0\}$. For $i \in \mathbb{N}_0$ set $\mu_i := \exp\left(b' - \lambda i^{2/3}/2\right)$. Then $\mu := \sum_{i\geq 0} \mu_i < \infty$. Recall (1.3) and set for all $n \geq 0$, $X'_n := \sum_{i=0}^n A_1 \ldots A_i Y_{i+1}$. Then for each n, X_n has the same distribution as X'_n. Therefore, it suffices to show that $\sum_{n\geq 0} P[\|X'_n\| < \mu] = \infty$. Hence we estimate

$$\begin{aligned} P[\|X'_n\| < \mu] &\geq P\left[\sum_{i=0}^n \|A_1 \ldots A_i\| \, \|Y_{i+1}\| < \mu\right] \\ &\geq E\left[P\left[\bigcap_{i=0}^n \{\ln \|Y_{i+1}\| < \ln \mu_i + S_i\} \middle| (A_k)_{k\geq 1}\right]\right] = E\left[\prod_{i=0}^n F(\ln \mu_i + S_i)\right]. \end{aligned}$$

By Lemma A.3, $T := \inf\{n \geq 0 \mid \forall i > n : S_i \geq \lambda(i - i^{2/3}/2)\}$ is a.s. finite. Therefore,

$$\begin{aligned} \inf_{n\geq 0} E\left[\prod_{i=0}^n \frac{F(\ln \mu_i + S_i)}{F(b' + \lambda(i - i^{2/3}))}\right] &\geq E\left[\prod_{i\geq 1}\left(\frac{F(\ln \mu_i + S_i)}{F(b' + \lambda(i - i^{2/3}))} \wedge 1\right); B\right] \\ = \quad E\left[\prod_{i=1}^T \left(\frac{F(b' + S_i - \lambda i^{2/3}/2)}{F(b' + \lambda(i - i^{2/3}))} \wedge 1\right); B\right] &\geq E[2^{-T}; B] > 0 \end{aligned}$$

since $P[B] > 0$. This implies (XR). ☐

By exponentiating R we obtain the following generalization of Proposition 1.1.

Corollary 4.4 (General random exchange process)**.** *Let* $E[T_1] > 0$ *and let* $T_1 - E[T_1]$ *be sub-Gaussian. Suppose that* $\lim_{x\to\infty} x^{2/3}(\ln x)^2 P[W_1 > x] = 0$ *or* $\liminf_{x\to\infty} xP[W_1 > x] > E[T_1]$. *Let* $y \in (0, \infty)$ *be such that* $P[W_1 \leq y] > 0$. *Then* R *is recurrent if and only if*

$$\sum_{n\geq 0} \prod_{m=0}^n P\left[W_1 \leq y + mE[T_1]\right] = \infty.$$

4.1 An application to random walks in random environments perturbed by cookies of maximal strength

We consider the same version of excited random walks in random environment as Bauernschubert in [4]. Let $\omega = (\omega_x)_{x\in\mathbb{Z}}$ be an i.i.d. family of $(0, 1)$-valued random variables and $Y = (Y_x)_{x\in\mathbb{Z}}$ be an i.i.d. family of $\mathbb{N}_0$-valued random variables such that $P[Y_0 = 0] > 0$. We call ω_x the environment at x and Y_x the number of cookies at x. The random walk $\xi = (\xi_n)_{n\geq 0}$ in the random environment ω perturbed by the cookies Y is defined as follows. The walk starts at $\xi_0 = 0$. Upon any of the first Y_x many visits to a site x the walker reduces the number of cookies at that site by one and then moves in the next step deterministically to $x + 1$. Upon the $(Y_x + 1)$-st or any later visit to x, i.e. when there are no cookies left at x, the walker jumps independently of everything else with probability ω_x to $x + 1$ and with probability $1 - \omega_x$ to $x - 1$. More formally, for all $n \geq 0$ and $z = \pm 1$ a.s.

$$P[\xi_{n+1} = \xi_n + z \mid (\xi_k)_{0\leq k\leq n}, Y, \omega] = \begin{cases} 1 & \text{if } z = 1, \#\{k \leq n \mid \xi_k = \xi_n\} \leq Y_{\xi_n} \\ \omega_x & \text{if } z = 1, \#\{k \leq n \mid \xi_k = \xi_n\} > Y_{\xi_n} \\ 1 - \omega_x & \text{if } z = -1, \#\{k \leq n \mid \xi_k = \xi_n\} > Y_{\xi_n}. \end{cases}$$

The random walk ξ is called transient to the right if $\xi_n \to \infty$ as $n \to \infty$, transient to the left if $\xi_n \to -\infty$ as $n \to \infty$, and recurrent if $\xi_n = 0$ for infinitely many n. In the case without cookies, i.e. where $P[Y_0 = 0] = 1$, we retrieve the classical one-dimensional random walk in random environment (RWRE). It is known that that under mild assumptions RWRE is a.s. recurrent iff $E[\ln \rho_0] = 0$, where $\rho_0 := (1-\omega_0)/\omega_0$, and a.s. transient to the right (resp. left) iff $E[\ln \rho_0] < 0$ (resp. > 0), see e.g. [40, Theorem 2.1.2].

We consider the case $E[\ln \rho_0] > 0$ in which the underlying RWRE is transient to the left and ask how many cookies are needed in order to make this walk recurrent or even transient to the right. Using (1.8), (1.9), and a well-known relationship between excursions of random walks and branching processes, Bauernschubert obtained in [4, Theorem 1.1] the following result.

Theorem 4.5 (Excited random walks; Bauernschubert)**.** *Assume that the two i.i.d. families $(\omega_x)_{x\in\mathbb{Z}}$ and $(Y_x)_{x\in\mathbb{Z}}$ are independent of each other and let $E[|\ln \rho_0|] < \infty$, $E[\ln \rho_0] > 0$, and $E[\omega_0^{-1}] < \infty$.*

(a) If $E[\ln_+ Y_1] < \infty$ then ξ is a.s. transient to the left.
(b) If $E[\ln_+ Y_1] = \infty$ and if $\limsup_{t\to\infty} t \cdot P[\ln Y_1 > t] < E[\ln \rho_0]$, then ξ is a.s. recurrent.
(c) If $\limsup_{t\to\infty} t \cdot P[\ln Y_1 > t] > E[\ln \rho_0]$ then ξ is a.s. transient to the right.

Replacing [4, (8),(9)] by Theorem 4.2 we obtain the following characterization of recurrence/transience of ξ.

Corollary 4.6. *Assume that $(\omega_x, Y_x)_{x\in\mathbb{Z}}$ is i.i.d., $E[\ln \rho_0] > 0$, that $\ln \rho_0 - E[\ln \rho_0]$ is sub-Gaussian, and* (REG)*.*

(a) If $E[\ln_+ Y_0] < \infty$ then ξ is a.s. transient to the left.
(b) If $E[\ln_+ Y_0] = \infty$ and if

$$\sum_{n\geq 0} \prod_{m=0}^{n} P\left[Y_0 \leq \exp\left(mE[\ln \rho_0]\right)\right] \tag{4.16}$$

is infinite then ξ is a.s. recurrent.
(c) If the series in (4.16) *is finite then ξ is a.s. transient to the right.*

A Appendix: bounds for the case of random environment

Lemma A.1. *Let $K, \gamma \in \mathbb{N}$, $\kappa > 0$ and $\mathcal{A} \subseteq [0,\infty)^{d\times d}$. For $n \in \mathbb{N}_0$ set $\mathcal{G}_n := \{A_1 \dots A_n : A_1, \dots, A_n \in \mathcal{A}\}$ and $\mathcal{G} := \bigcup_{n\geq 0} \mathcal{G}_n$. If $d \geq 2$ then assume that $\|A\| \leq \gamma$ for all $A \in \mathcal{A}$ and $\mathcal{G}_K \subseteq [\kappa,\infty)^{d\times d}$. Then there is a constant $c = c(K,\gamma,\kappa,d)$ such that*

$$\|A\|\|x\| \leq c\|Ax\| \qquad \text{for all } A \in \mathcal{G}, x \in [0,\infty)^d, \tag{A.1}$$

$$\|A\|\|B\| \leq c\|AB\| \qquad \text{for all } A \in \mathcal{G}, B \in [0,\infty)^{d\times d}, \text{ and} \tag{A.2}$$

$$\|A\| \leq c[A]_{1,1} \qquad \text{for all } n \geq K, A \in \mathcal{G}_n. \tag{A.3}$$

Proof. If $d = 1$ then (A.1)-(A.3) hold trivially with $c = 1$.

Consider now $d \geq 2$. For any matrix A let $\mu(A) := \min_j \max_i [A]_{i,j}$. The following two quantities are used to measure the variation among the entries of A.

$$\delta_A := \|A\|_1/\mu(A) \in [1,\infty] \qquad \text{for } A \in [0,\infty)^{d\times d}\backslash\{0\} \text{ and}$$

$$\Delta_A := \max\left\{\frac{[A]_{i,j}}{[A]_{i,k}}, \frac{[A]_{i,j}}{[A]_{k,j}} : i,j,k \in \{1,\dots,d\}\right\} \in [1,\infty) \qquad \text{for } A \in (0,\infty)^{d\times d}.$$

We first show the following relations.

$$\Delta_{AB} \leq \max\{\Delta_A, \Delta_B\} \qquad \text{for all } A, B \in (0,\infty)^{d\times d}. \tag{A.4}$$

$$\Delta_{AB} \leq \Delta_A \delta_B \qquad \text{for all } A \in (0,\infty)^{d\times d}, B \in [0,\infty)^{d\times d}\backslash\{0\}. \tag{A.5}$$

$$\delta_{AB} \leq \delta_A \delta_B \qquad \text{for all } A, B \in [0,\infty)^{d\times d}\backslash\{0\}. \tag{A.6}$$

$$\delta_A \leq d\Delta_A \qquad \text{for all } A \in (0,\infty)^{d\times d}. \tag{A.7}$$

Statement (A.4) follows from the fact that for all $i, j, k \in \{1, \dots, d\}$,

$$\begin{aligned} \frac{[AB]_{i,j}}{[AB]_{i,k}} &= \frac{\sum_n [A]_{i,n}[B]_{n,j}}{\sum_n [A]_{i,n}[B]_{n,k}} \le \frac{\sum_n [A]_{i,n}\Delta_B[B]_{n,k}}{\sum_n [A]_{i,n}[B]_{n,k}} = \Delta_B \quad \text{and similarly} \\ \frac{[AB]_{i,j}}{[AB]_{k,j}} &\le \Delta_A. \end{aligned} \tag{A.8}$$

To show (A.5) let m and k be such that $[B]_{m,k} = \max_n [B]_{n,k} = \mu(B)$. Then

$$\frac{[AB]_{i,j}}{[AB]_{i,k}} \le \frac{\sum_n \Delta_A [A]_{i,m}[B]_{n,j}}{[A]_{i,m}[B]_{m,k}} \le \Delta_A \delta_B.$$

Together with (A.8) this proves (A.5).

For the proof of (A.6) it suffices to show that $\mu(AB) \ge \mu(A)\mu(B)$ since $\|AB\|_1 \le \|A\|_1\|B\|_1$. To this end, fix $1 \le j \le d$, choose k such that $[B]_{k,j} \ge \mu(B)$ and m such that $[A]_{m,k} \ge \mu(A)$. Then $\max_i [AB]_{i,j} \ge [AB]_{m,j} \ge [A]_{m,k}[B]_{k,j} \ge \mu(A)\mu(B)$. Taking the minimum over j yields (A.6).

To prove (A.7) let k be such that $\max_i [A]_{i,k} = \mu(A)$. Then

$$\delta_A = \frac{\max_j \sum_i [A]_{i,j}}{\mu(A)} \le \frac{\Delta_A \sum_i [A]_{i,k}}{\max_i [A]_{i,k}} \le d\Delta_A.$$

This concludes to proof of (A.4)–(A.7). Next we show that

$$\sup\{\Delta_A : A \in \mathcal{G}_n, n \ge K\} < \infty \quad \text{and} \tag{A.9}$$

$$\sup\{\delta_A : A \in \mathcal{G}\} < \infty. \tag{A.10}$$

First note that $c_{14} := \sup\{\delta_A : A \in \mathcal{A}\} < \infty$. Indeed, let $A \in \mathcal{A}$ and $B \in \mathcal{G}_{K-1}$. Choose j such that $\max_i [A]_{i,j} = \mu(A)$. Since $BA \in \mathcal{G}_K$ we have $\kappa \le [BA]_{1,j} = \sum_i [B]_{1,i}[A]_{i,j} \le \|B\|\mu(A)$ and consequently, $\delta_A \le \|A\|_1\|B\|/\kappa \le d\gamma^K/\kappa$.

Second, due to $\mathcal{G}_K \subseteq [\kappa, \infty)^{d\times d}$, no element of $\mathcal{A}$ has a column of zeros. Hence, $\mathcal{G}_n \subseteq (0,\infty)^{d\times d}$ for all $n \ge K$. Therefore, if we let $K \le n = mK + r$ with $m \ge 1$ and $0 \le r < K$ then for all $A_1, \dots, A_n \in \mathcal{A}$, by (A.5),

$$\begin{aligned} \Delta_{A_1\dots A_n} &\le \Delta_{A_1\dots A_{mK}} \delta_{A_{mK+1}\dots A_n} \\ &\overset{(A.4),(A.6)}{\le} \max_{i=0}^{m-1} \Delta_{A_{iK+1}\dots A_{(i+1)K}} \delta_{A_{mK+1}} \dots \delta_{A_n} \le \gamma^K \kappa^{-1} c^K =: c_{15}, \end{aligned} \tag{A.11}$$

where we used in the last step that $\Delta_B \le \|B\|/\kappa \le \gamma^K/\kappa$ for all $B \in \mathcal{G}_K$. This implies (A.9). Moreover, (A.6) implies $\delta_A \le c_{14}^K$ for all $A \in \mathcal{G}_n, n \le K$, and (A.7) and (A.11) imply $\delta_A \le dc_{15}$ for all $A \in \mathcal{G}_n, n \ge K$. Together this yields (A.10).

For the proof of the first claim of the Lemma, (A.1), let k be such that $\|x\| = x_k$. Then for all $A \in \mathcal{G}$,

$$\begin{aligned} \|Ax\| &= \max_i \sum_j [A]_{i,j} x_j \ge \max_i [A]_{i,k} x_k = \|x\| \max_i [A]_{i,k} \\ &\ge \|x\| \min_j \max_i [A]_{i,j} = \frac{\|A\|_1 \|x\|}{\delta_A} \ge \frac{\|A\|\|x\|}{d\delta_A}. \end{aligned}$$

Along with (A.10) this implies (A.1). The second claim, (A.2), follows from (A.1) and the definition of the matrix norm $\|\cdot\|$. For the proof of (A.3) let $n \ge K$ and $A \in \mathcal{G}_n$. Then

$$\|A\| = \max_k \sum_\ell [A]_{k,\ell} \le \Delta_A \sum_\ell [A]_{1,\ell} \le \Delta_A^2 \sum_\ell [A]_{1,1} = d\Delta_A^2 [A]_{1,1}.$$

This together with (A.9) implies (A.3). □

The following result provides bounds on the extinction time of multitype branching process in varying environment. The easy bound is standard and based on a first moment method, i.e. Markov's inequality. To the best of our knowledge the opposite bound appeared first in a similar form in [1, Theorem 1]. We prove it by the second moment method. For precise asymptotics under different assumptions see e.g. [18], [10].

Proposition A.2 (Bounds on extinction time of multitype branching process in varying environment)**.** *Fix $\psi = (\psi_n)_{n\geq 1} = ((\psi_n^j)_{j=1,\ldots,d})_{n\geq 1} \in (D^d)^{\mathbb{N}}$. Let $(U_{n,k}^j)_{n,k\geq 1; j\in\{1,\ldots,d\}}$ be an i.i.d. family of random variables which are distributed uniformly on $[0,1]$. Let $\xi_{n,k}^{i,j} := [\psi_n^j(U_{n,k}^j)]_i$. Fix $s \in \{1,\ldots,d\}$ and define the branching process $(B_n)_{n\geq 0}$ in the environment ψ starting at time 0 with one individual of type s as follows. Set $B_0 := e_s$ and define recursively for all $n \geq 1$,*

$$B_n := \left(\sum_{j=1}^{d} \sum_{k=1}^{[B_{n-1}]_j} \xi_{n,k}^{i,j} \right)_{i=1,\ldots,d}. \tag{A.12}$$

Define the matrices $A_n := \left(E\left[\xi_{n,1}^{i,j}\right]\right)_{i,j=1,\ldots,d}$, $n \geq 1$, and suppose that $\gamma, K \in \mathbb{N}$, $\kappa > 0$, and $\mathcal{A} := \{A_n \mid n \geq 1\}$ satisfy the assumptions of Lemma A.1. Denote for $n \geq 1$ and $j = 1,\ldots,d$ by $\mathcal{V}_n^j$ the covariance matrix of the vector $(\xi_{n,1}^{i,j})_{i=1,\ldots,d}$ and suppose that

$$c_{16} := \sup_{n\geq 1, j=1,\ldots,d} \frac{\|\mathcal{V}_n^j\|}{\|A_n\|} < \infty. \tag{A.13}$$

Then there is a constant $c_1 = c_1(\gamma, K, \kappa, d, c_{16})$ such that for all $n \geq 1$,

$$c_1 \frac{\|A_n \ldots A_1\|}{\sum_{k=1}^{n} \|A_n \ldots A_k\|} \leq P[B_n \neq 0] \leq d\|A_n \ldots A_1\|.$$

Proof. It follows from (A.12) that $E[B_n] = A_n E[B_{n-1}]$ for all $n \geq 1$, see e.g. [16, Chapter II, (4.1)]. Therefore, $E[B_n] = A_n \ldots A_1 e_s$ for all $n \geq 0$. Thus

$$P[B_n \neq 0] = P[\|B_n\|_1 \geq 1] \leq E[\|B_n\|_1] = \|A_n \ldots A_1 e_s\|_1 \leq d\|A_n \ldots A_1\|.$$

For the lower bound set $\mathbf{C}_n := (E[[B_n]_i [B_n]_j])_{i,j=1,\ldots,d}$. By the second moment method

$$\begin{aligned} P[B_n \neq 0] &= P\left[\|B_n\| > 0\right] \geq \frac{(E[\|B_n\|])^2}{E\left[\|B_n\|^2\right]} \geq \frac{\|E[B_n]\|^2}{E\left[\max_i [B_n]_i^2\right]} \\ &\geq \frac{\|A_n \ldots A_1 e_s\|^2}{\sum_{i=1}^{d} E\left[[B_n]_i^2\right]} \overset{(A.1)}{\geq} c_{17} \frac{(\|A_n \ldots A_1\| \|e_s\|)^2}{\max_{i=1}^{d} E\left[[B_n]_i^2\right]} \\ &\geq c_{17} \frac{\|A_n \ldots A_1\|^2}{\|\mathbf{C}_n\|}. \end{aligned} \tag{A.14}$$

By [16, Chapter II, (4.2)] for all $n \geq 1$,

$$\begin{aligned} \mathbf{C}_n &= A_n \mathbf{C}_{n-1} A_n^T + \sum_{j=1}^{d} E\left[[B_{n-1}]_j\right] \mathcal{V}_n^j \\ &= A_n \ldots A_1 \mathbf{C}_0 A_1^T \ldots A_n^T + \sum_{k=1}^{n} A_n \ldots A_{k+1} \left(\sum_{j=1}^{d} E[[B_{k-1}]_j] \mathcal{V}_k^j \right) A_{k+1}^T \ldots A_n^T \end{aligned}$$

by induction. Consequently,

$$\begin{aligned}
\|\mathbf{C}_n\| &\leq c_{18}\|A_n\dots A_1\|^2 + \sum_{k=1}^n \|A_n\dots A_{k+1}\| \left(\sum_{j=1}^d E[[B_{k-1}]_j]\|\mathcal{V}_k^j\|\right) \|(A_n\dots A_{k+1})^T\| \\
&\overset{(A.13)}{\leq} c_{18}\|A_n\dots A_1\|^2 + c_{19}\sum_{k=1}^n \|A_n\dots A_{k+1}\|^2\|A_k\|\|E[B_{k-1}]\|_1 \\
&\overset{(A.1)}{\leq} c_{18}\|A_n\dots A_1\|^2 + c_{20}\sum_{k=1}^n \|A_n\dots A_{k+1}\|\|A_n\dots A_k E[B_{k-1}]\| \\
&\leq c_{18}\|A_n\dots A_1\|^2 + c_{20}\|A_n\dots A_1\|\|E[B_0]\|\sum_{k=1}^n \|A_n\dots A_{k+1}\| \\
&\leq c_{21}\|A_n\dots A_1\|\sum_{k=1}^n \|A_n\dots A_k\|.
\end{aligned}$$

Substituting this into (A.14) yields the claim. □

Lemma A.3 (Sub-Gaussian concentration inequality)**.** *Assume* (BA) *and let K,γ,κ be accordingly. Then there are constants c_6 and c_7 depending on (d,K,γ,κ) such that for all $n\in\mathbb{N}$ and $t\in(0,\infty)$,*

$$P\left[|S_n-\lambda n|\geq t\right]\leq c_6\exp\left(-c_7t^2/n\right). \tag{A.15}$$

Proof. If $d=1$ then S_n is the sum of independent random variables and (A.15) is a Hoeffding-type inequality, which is usually stated for bounded increments $S_{n+1}-S_n$, see e.g. [5, Theorem 2.8], but also holds for increments which are sub-Gaussian after centering, see e.g. [37, Proposition 5.10].

Now let $d\geq 2$. Denote by $\mathcal{A}\subseteq[0,\infty)^{d\times d}$ the support of A_1. Due to (BA) the assumptions of Lemma A.1 are satisfied. First we show the existence of $c_{22},c_{23}\in(0,\infty)$ such that for all $n\geq 0$ and $t>0$,

$$P\left[|S_n-E[S_n]|\geq t\right]\leq c_{22}\exp\left(-c_{23}t^2/n\right). \tag{A.16}$$

Let $n\geq 1$ and $f(B):=-\ln\|B_1\dots B_n\|$ for $B=(B_1,\dots,B_n)\in\mathcal{A}^n$. Suppose $B,B'\in\mathcal{A}^n$ differ only in a single coordinate, say the i-th one. Then by submultiplicativity and (A.2),

$$\begin{aligned}
f(B)-f(B') &\leq \ln\left(\|B_1\dots B_{i-1}\|\|B_i'\|\|B_{i+1}\dots B_n\|\right) \\
&\quad -\ln\left(c^2\|B_1\dots B_{i-1}\|\|B_i\|\|B_{i+1}\dots B_n\|\right) \\
&\leq \ln\gamma-\ln c^2\kappa^{1/K}=:c_{24}
\end{aligned}$$

due to $\kappa\leq\|B_i^K\|\leq\|B_i\|^K$. By symmetry, $|f(B)-f(B')|\leq c_{24}$. Now (A.16) follows from McDiarmid's inequality, see [28, Lemma (1.2)] or [5, Theorem 6.2].

For the proof of (A.15) observe that due to (4.1),

$$\sup_{n\geq 1}\frac{E\left[S_n\right]}{n} \;=\; \lambda \;=\; \lim_{n\to\infty}\frac{E[S_{nK}]}{nK} \;\leq\; \liminf_{n\to\infty}\frac{E[-\ln[A_1\dots A_{nK}]_{1,1}]}{nK}. \tag{A.17}$$

Since $[AB]_{1,1}\leq[A]_{1,1}[B]_{1,1}$ for any $A,B\in[0,\infty)^{d\times d}$, the subadditive ergodic theorem yields that the right most side of (A.17) is equal to

$$\inf_{n\geq 1}\frac{E[-\ln[A_1\dots A_{nK}]_{1,1}]}{nK}\overset{(A.3)}{\leq}\inf_{n\geq 1}\frac{c_{25}+E[S_{nK}]}{nK}. \tag{A.18}$$

By submultiplicativity, for all $0 \leq r < K$ and $n \geq 1$, $E[S_{nK+r}] \geq E[S_{nK}] + rE[S_1] \geq E[S_{nK}] - KE[|S_1|]$. Consequently, the right hand side of (A.18) is at most

$$\inf_{0\leq r<K} \inf_{n\geq 1} \frac{c_{26} + E[S_{nK+r}]}{nK} \leq \inf_{n>K} \frac{c_{26} + E[S_n]}{n-K}.$$

Together with (A.17) this implies that $|\lambda n - E[S_n]| \leq \lambda K + c_{26}$ for all $n > K$. The claim now follows from (A.16). □

B Appendix

Several months after the first two versions of the present paper appeared on arxiv.org, Alsmeyer, Buraczewski, and Iksanov posted a preprint of [2] in the same repository. In [2, Remark 3.3] they claim our introductory criterion (1.1), but provide neither a proof nor a reference. Their first main result, [2, Theorem 3.1], states that (1.8) and (1.9) hold for general nonnegative subcritical autoregressive processes X with random coefficients as defined in (1.2) in one dimension. This removes the moment assumptions from Bauernschubert's extension (1.8) of Kellerer's result (1.6) concerning transience of X and extends Kellerer's result (1.7) about recurrence of X to the case of random coefficients. Note that under the additional assumption (BA) that $\ln A_1 - E[\ln A_1]$ is sub-Gaussian (resp. bounded according to the first two versions of the present paper) this result also follows from our Theorem 4.2 and Raabe's test, see [2, Remark 3.1].

This raises the question to what extent condition (BA) in Theorem 4.2 can be relaxed, especially for $d \geq 2$. We are grateful to an anonymous referee for pointing out that for $d \geq 2$ bounds similar to those in Lemma A.1 were proved under weak moment conditions by Kesten [23, (2.19) and p. 225]. Unfortunately, we were not able to use these bounds to weaken assumption (BA) in Theorem 4.2 since we need (BA) also for the proof of the concentration inequality Lemma A.3, which relies for $d \geq 2$ on McDiarmid's inequality.

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